Transformation

What is a Transformation

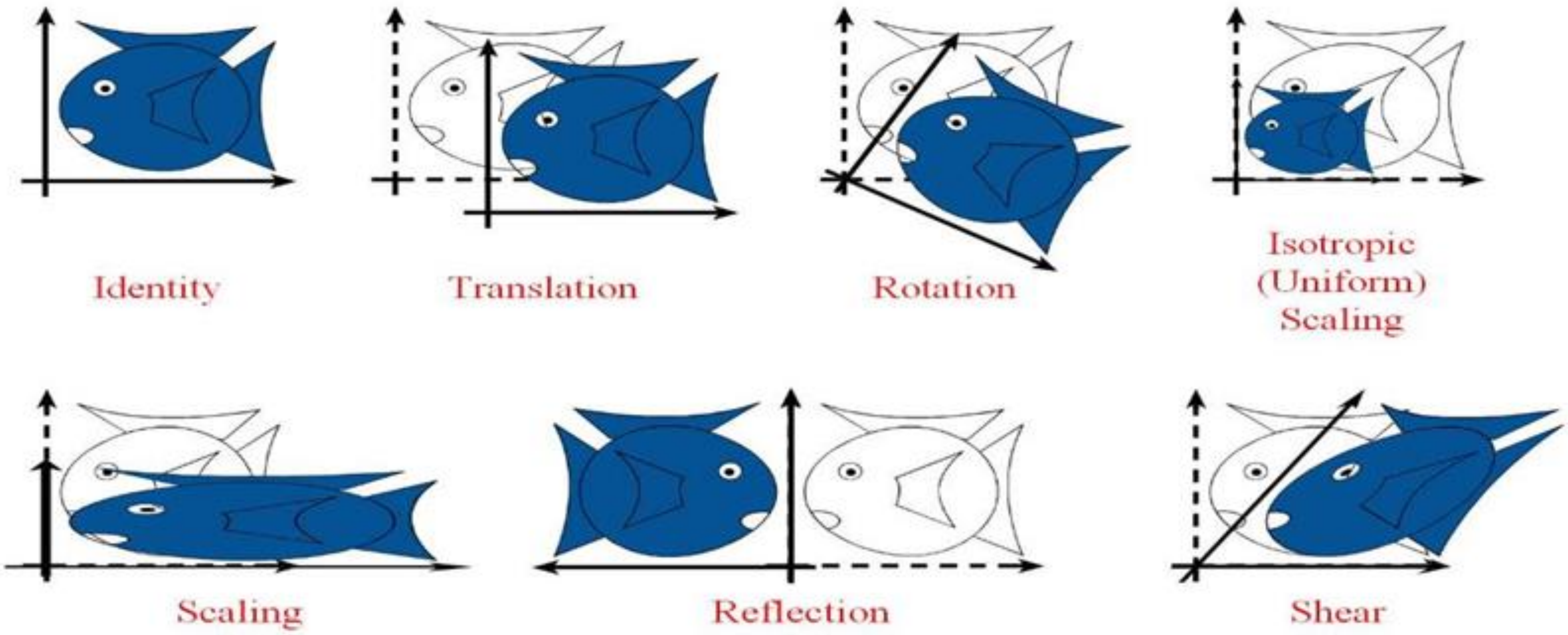
- Transformation:
  - An operation that changes one configuration into another
- For images, shapes, *etc*.
  - A geometric transformation maps positions that define the object to other positions
  - Linear transformation means the transformation is defined by a linear function... which is what matrices are good for.

What is a Transformation?

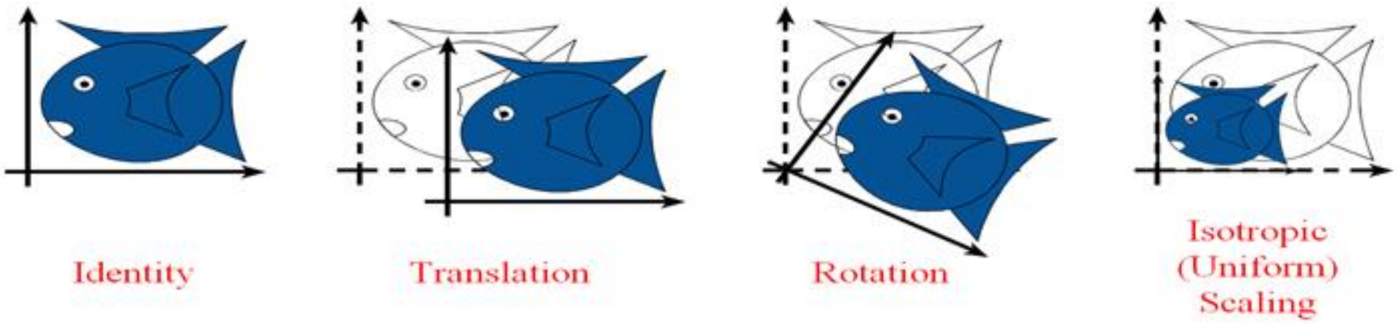
- A function that maps points x to points x':

Applications: animation, deformation, viewing, projection, real-time shadows, …

Examples of Transformation



Simple Transformations



- Can be combined
- Are these operations invertible?

Yes, except scale = 0

Rigid-Body / Euclidean Transforms

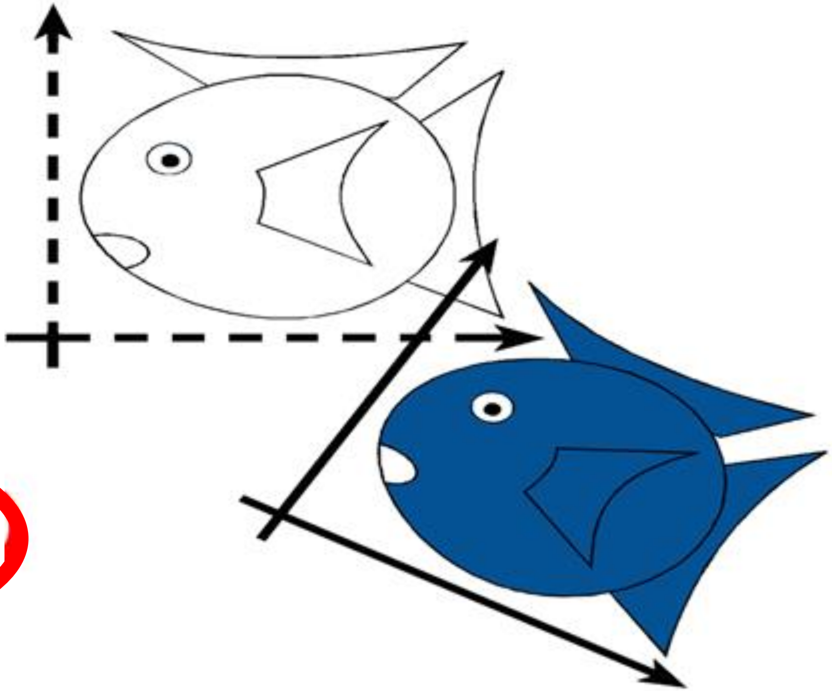
- Preserves distances
- Preserves angles

Rigid / Euclidean

Translation

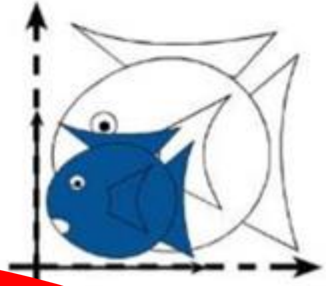
Identity

Rotation



Similitudes / Similarity Transforms

- Preserves angles



Similitudes

Rigid / Euclidean

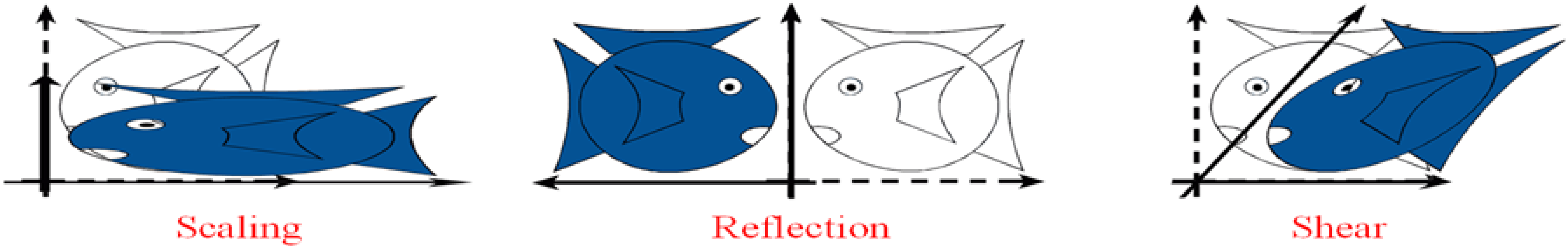
Translation

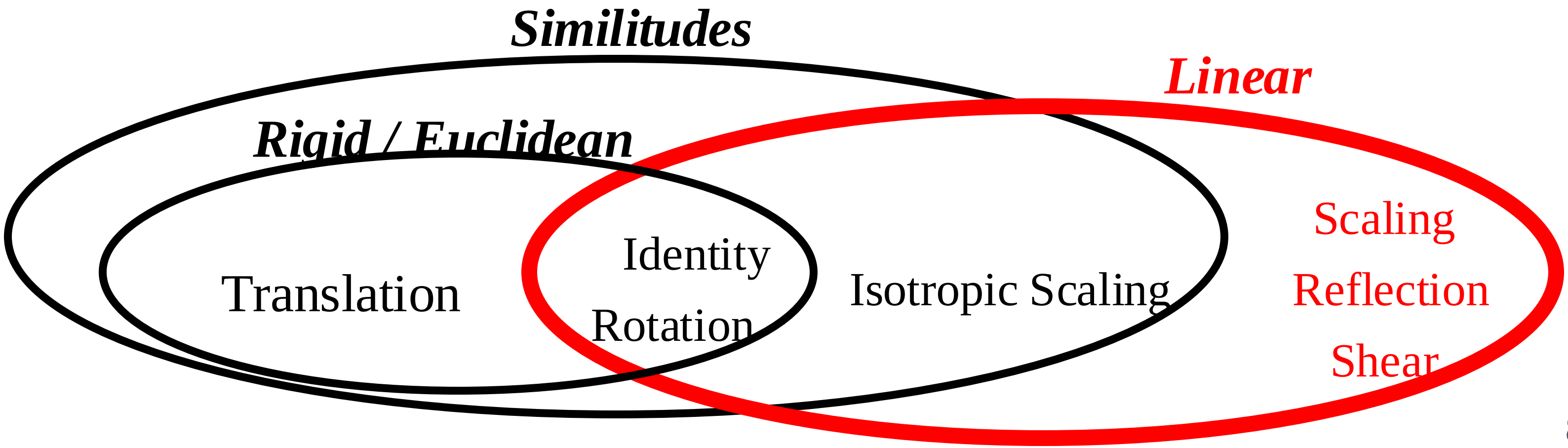
Identity

Rotation

Isotropic Scaling

Linear Transformations

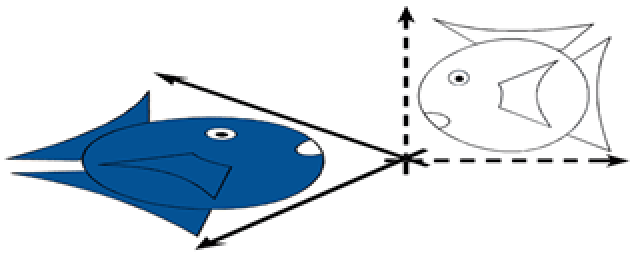


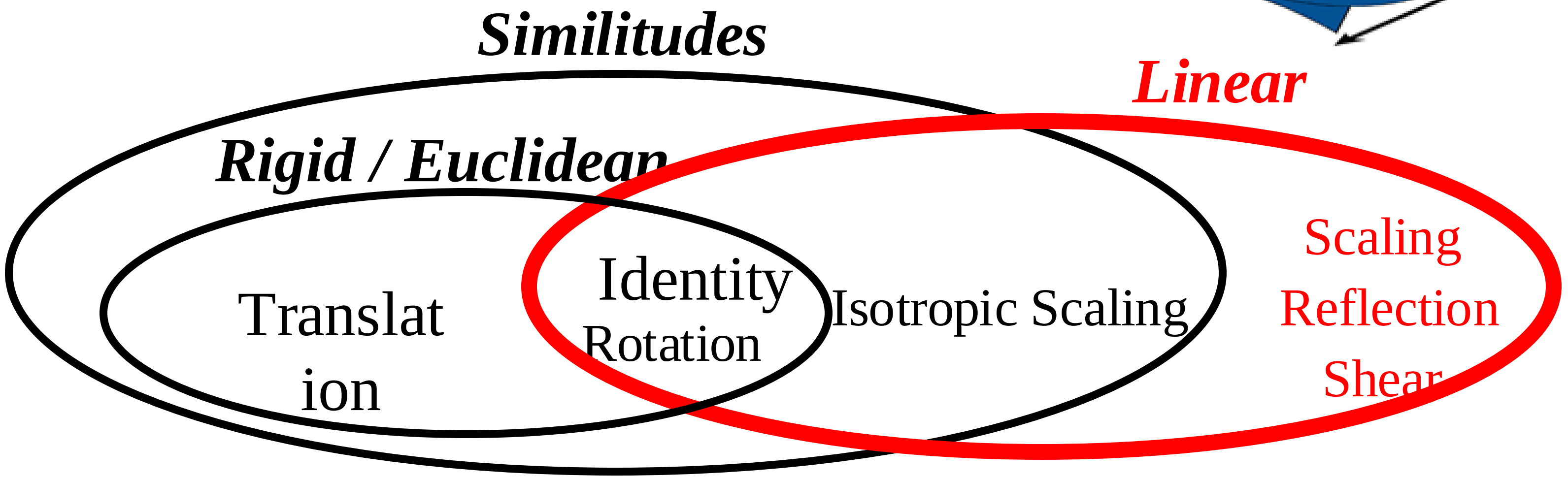


Linear Transformations

Additivity: $L(p + q) = L(p) + L(q)$

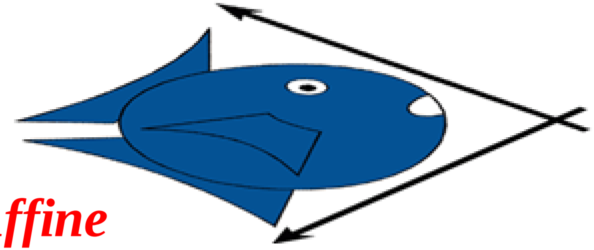
Homogeneity: $L(ap) = a\,L(p)$

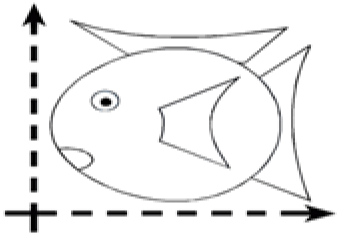


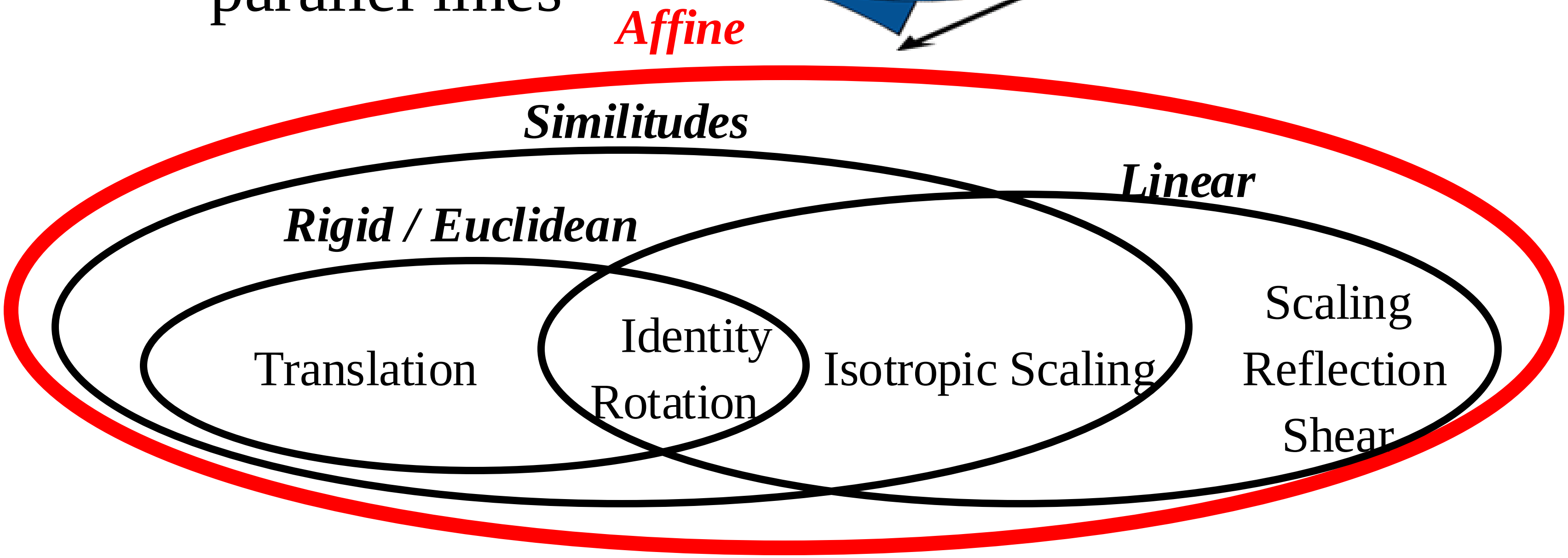


Affine Transformations

- preserves parallel lines

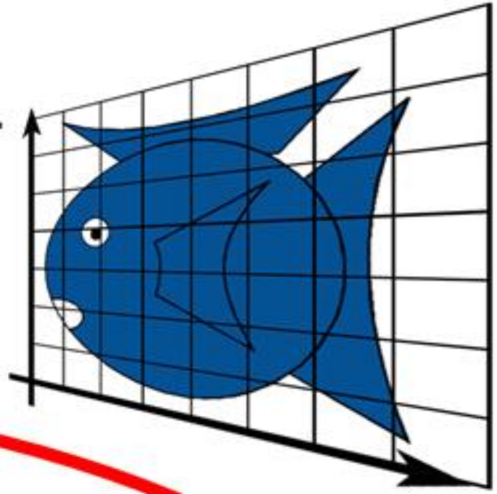


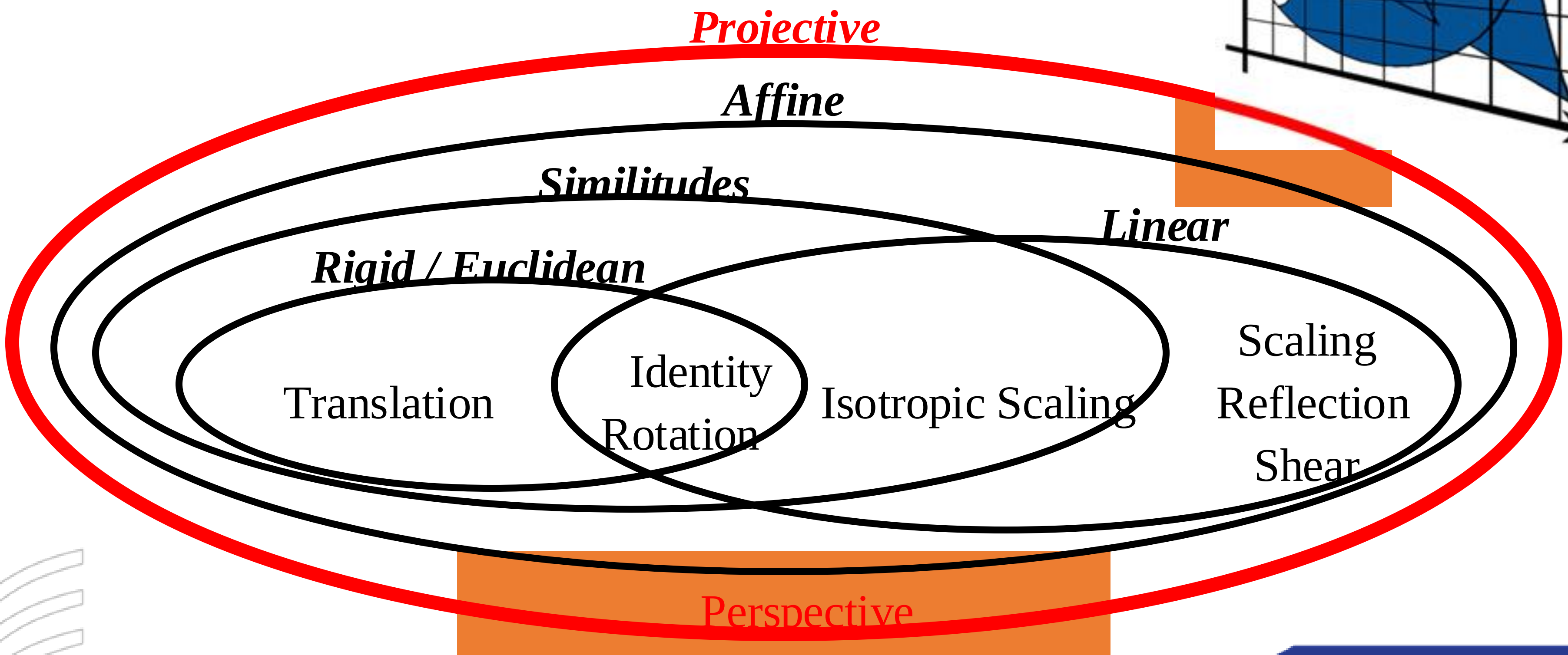




Projective Transformations

✶ preserves lines





How are Transforms Represented?

$$x' = ax + by + c$$

$$y' = dx + ey + f$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix}$$

$$p' = M p + t$$

Translation in homogeneous coordinates

$$x' = ax + by + c$$

$$y' = dx + ey + f$$

Cartesian formulation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix}$$

$$p' = M\,p + t$$

Homogeneous formulation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$p' = M\,p$$

0

Homogeneous Coordinates

- Translation, scaling and rotation are expressed (non-homogeneously) as:
  - translation: $P' = P + T$
  - Scale: $P' = S \cdot P$
  - Rotate: $P' = R \cdot P$
- Composition is difficult to express, since translation not expressed as a matrix multiplication
- Homogeneous coordinates allow all three to be expressed homogeneously, using multiplication by 3×3 matrices
- W is 1 for affine transformations in graphics

Homogeneous Coordinates

- Add an extra dimension
  - in 2D, we use 3 x 3 matrices
  - In 3D, we use 4 x 4 matrices
- Each point has an extra value, *w*

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$p' = {\color{red}M}\, p$$

Homogeneous Coordinates

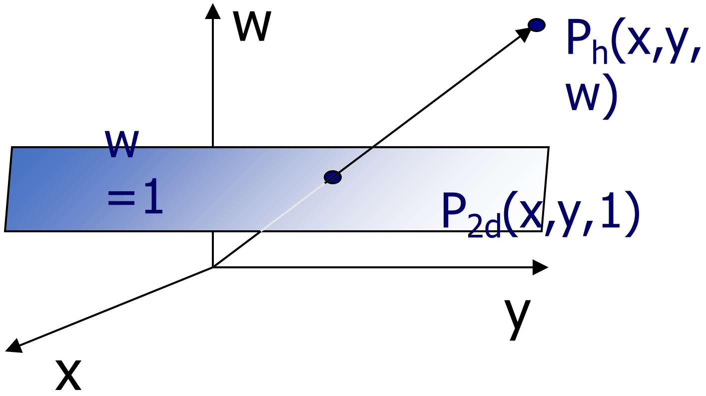
- Most of the time w = 1, and we can ignore it

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- If we multiply a homogeneous coordinate by an *affine matrix*, w is unchanged

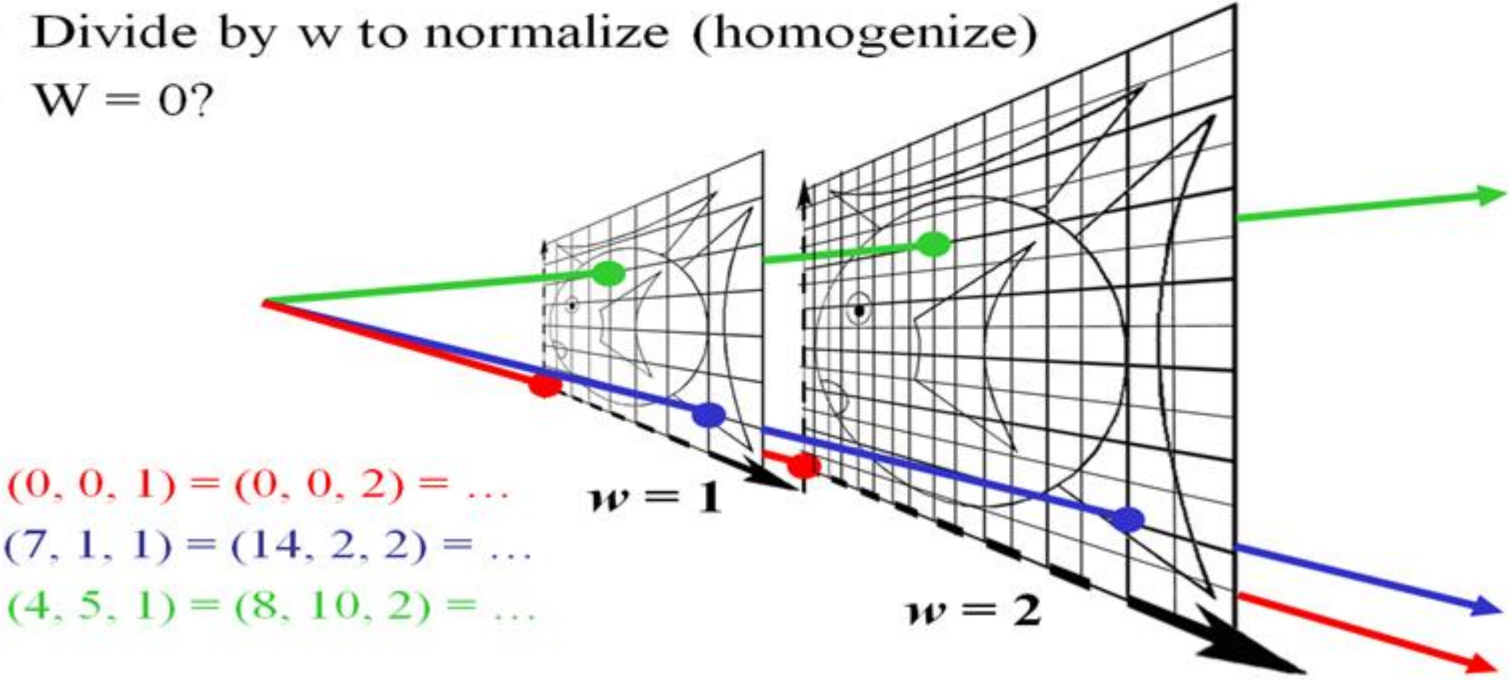
Homogeneous Coordinates

- P_{2d} is a projection of P_h onto the w = 1 plane
- So an infinite number of points correspond to : they constitute the whole line (tx, ty, tw)



Homogeneous Visualization

- Divide by w to normalize (homogenize)
- W = 0?



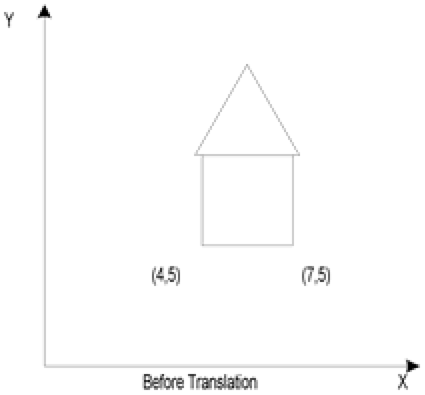
Mechanics of Rigid Transformations

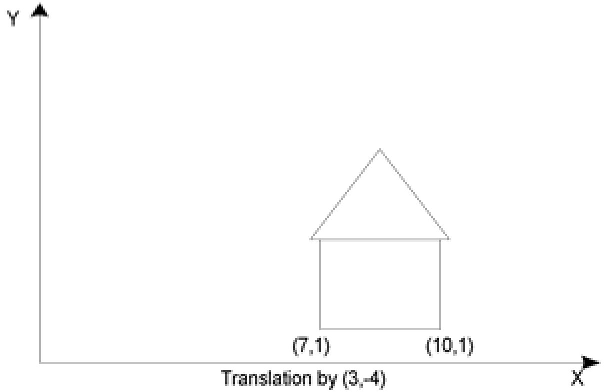
Translate

Rotate

Scale

Translation – 2D





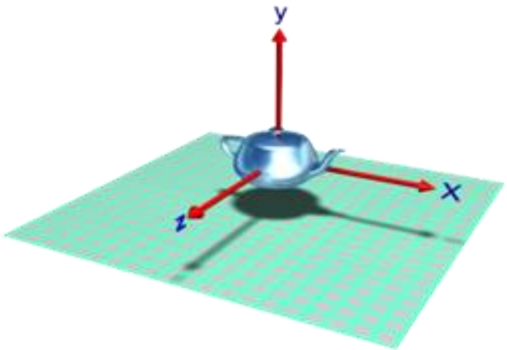
$x' = x + d_x$
$y' = y + d_y$

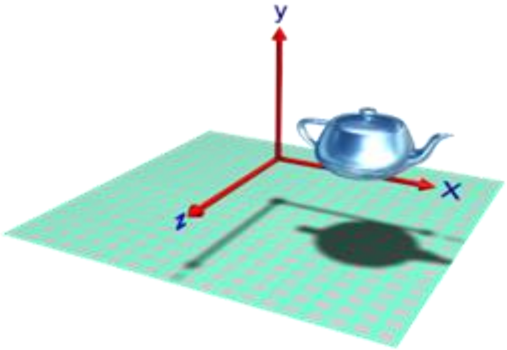
$$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad T = \begin{bmatrix} d_x \\ d_y \end{bmatrix} \qquad P' = P + T$$

Homogeniou s Form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation – 3D





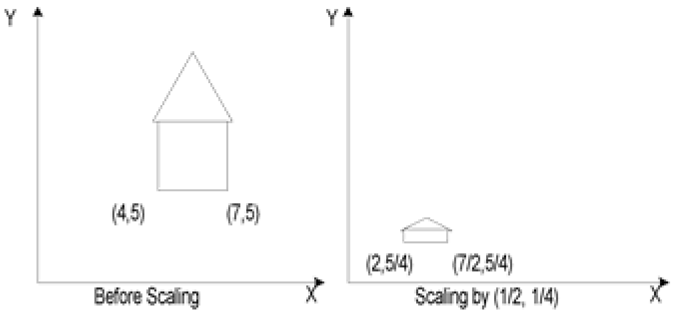
$$x' = x + d_x$$
$$y' = y + d_y$$
$$z' = z + d_z$$

$$\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+d_x \\ y+d_y \\ z+d_z \\ 1 \end{bmatrix}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$T(d_x, d_y, d_z) \quad * \quad P \quad = \quad P'$$

Scaling – 2D



Types of Scaling:

- Differential (s_x != s_y)
- Uniform (s_x = s_y)

$$x' = s_x * x$$

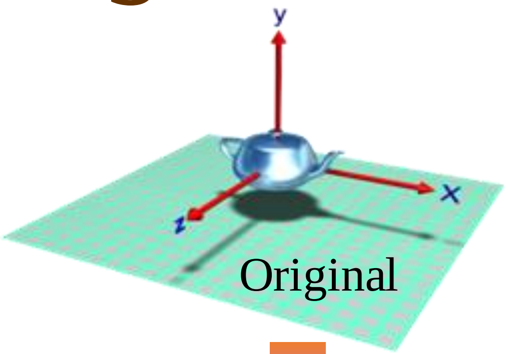
$$y' = s_y * y$$

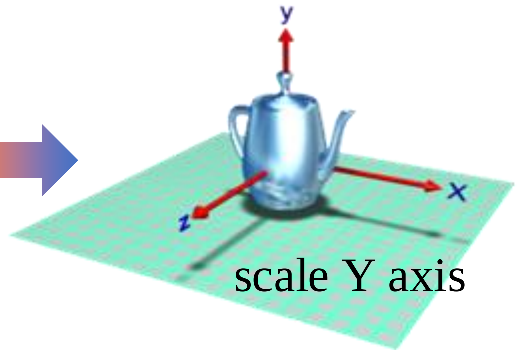
$$\begin{array}{ccccc} S & * & P & = & P' \\ \Downarrow & & \Downarrow & & \Downarrow \\ \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} & * & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} x * s_x \\ y * s_y \end{bmatrix} \end{array}$$

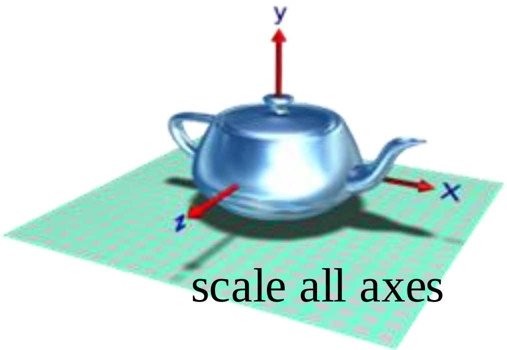
Homogenious Form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling – 3D







$$x' = s_x * x$$
$$y' = s_y * y$$
$$z' = s_z * z$$

$$S(s_x, s_y, s_z) \quad * \quad P \quad = \quad P'$$

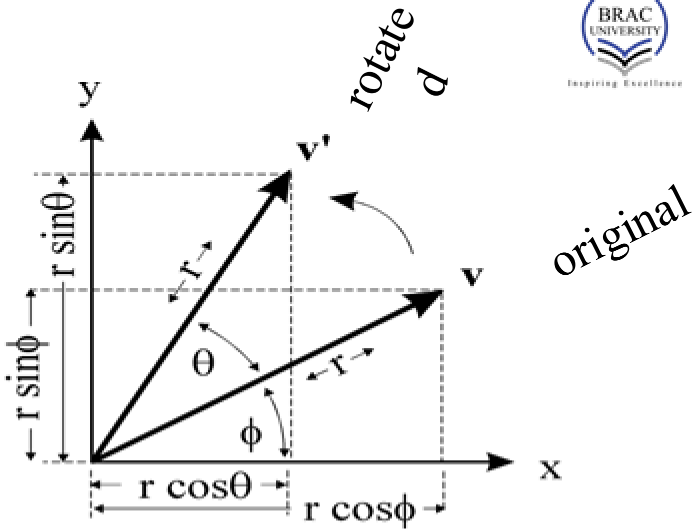
$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x * s_x \\ y * s_y \\ z * s_z \\ 1 \end{bmatrix}$$

Rotation – 2D

$$\mathbf{v} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$$

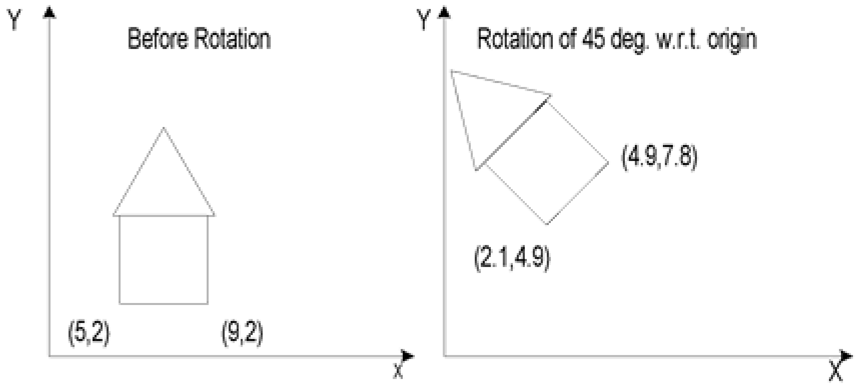
$$\mathbf{v}' = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}$$



$$\text{expand}\,(\phi+\theta) \Rightarrow \begin{cases} x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{cases}$$

$$\text{but} \quad \begin{matrix} x = r\cos\phi \\ y = r\sin\phi \end{matrix} \Rightarrow \begin{matrix} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{matrix}$$

Rotation – 2D



$$x * \cos\theta - y * \sin\theta = x'$$
$$x * \sin\theta + y * \cos\theta = y'$$

$$R \quad * \quad P = \quad P'$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x * \cos\theta - y * \sin\theta \\ x * \sin\theta + y * \cos\theta \end{bmatrix}$$

Homogenious Form

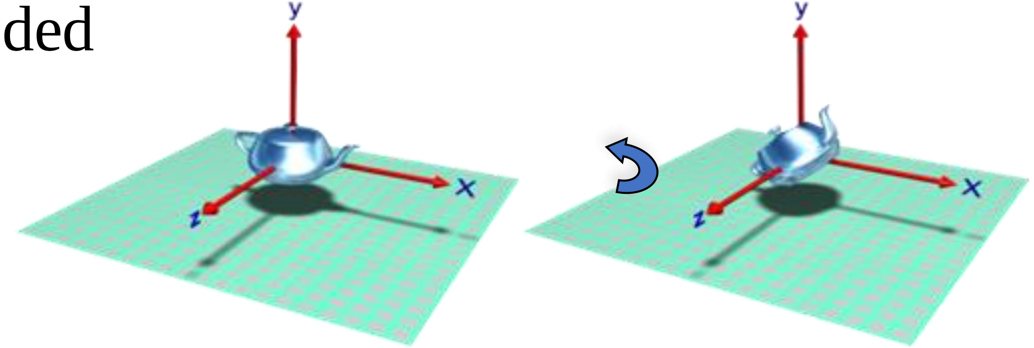
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation – 3D

For 3D-Rotation 2 parameters are needed

- ✶ Angle of rotation
- ✶ Axis of rotation

Rotation about z-axis:

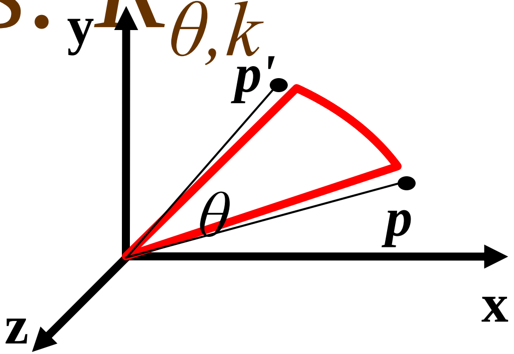


$$
\begin{array}{ccccc}
R_{\theta,k} & * & P & = & P' \\
\Downarrow & & \Downarrow & & \Downarrow
\end{array}
$$

$$
\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x * \cos\theta - y * \sin\theta \\ x * \sin\theta + y * \cos\theta \\ z \\ 1 \end{bmatrix}
$$

Rotation about Z axis: $R_{\theta,k}$

- About z axis



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation about Y-axis & X-axis

About y-axis

$$\begin{array}{ccccc} R_{\theta,j} & * & P & = & P' \\ \Downarrow & & \Downarrow & & \Downarrow \end{array}$$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x*\cos\theta + z*\sin\theta \\ y \\ -x*\sin\theta + z*\cos\theta \\ 1 \end{bmatrix}$$

About x-axis

$$\begin{array}{ccccc} R_{\theta,i} & * & P & = & P' \\ \Downarrow & & \Downarrow & & \Downarrow \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y*\cos\theta - z*\sin\theta \\ y*\sin\theta + z*\cos\theta \\ 1 \end{bmatrix}$$

Properties of rotation matrix

- The columns of rotation matrix are unit vectors perpendicular to each other
- The column vectors indicate where the unit vectors along the principal axes are transformed
- The rows of rotation matrix are unit vectors perpendicular to each other
- The row vectors indicate the vectors that are transformed into the unit vectors along the principal axes
- The inverse of rotation matrix is its transpose

Rotation

- How to rotate around (k_x, k_y, k_z), a unit vector on an arbitrary axis …
- Example : Rotate 30 degree around vector 2i+1j+3k
  – Can it be found from some rotation around x axis, then some rotation around y axis, then z axis ?

Finding the Rotation Matrix

Our previous method

- **Step 1,2** Perform two rotations so that **a** becomes aligned with the z-axis (two rotations are necessary)
- **Step 3** Do the required θ rotation around z-axis
- **Step 4,5** Undo the alignment rotations to restore **a** to its original direction

Trivial but you should be careful when doing in hands

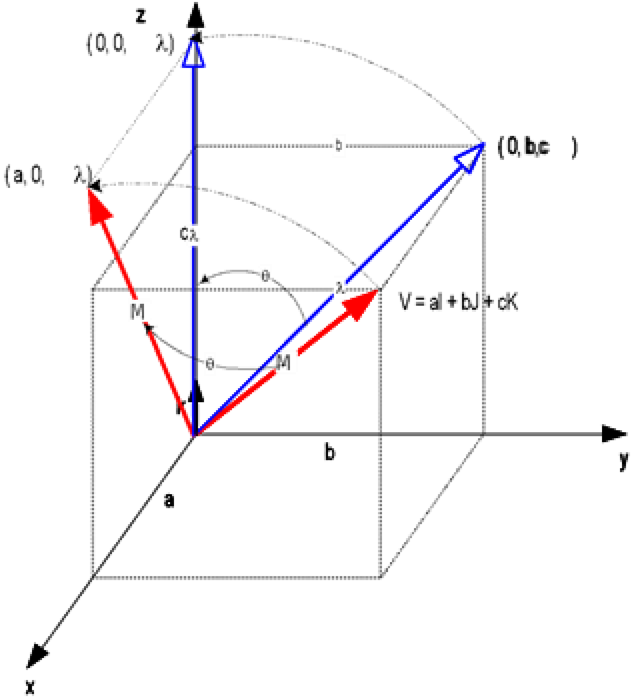
Step 1,2

- We now study a composite transformation
- $A_{\mathbf{V},\mathbf{N}}$ = aligning a vector **V** with a vector **N**
- To find the rotation matrix, we need to find $A_{v,k}$

A_V : *aligning vector* $\boldsymbol{V}$ *with* $\boldsymbol{k}$

$$\text{Step 1 : Rotate about x - axis by } \theta$$

$$\left.\begin{aligned} \sin\theta &= \frac{b}{\lambda} \\ \cos\theta &= \frac{c}{\lambda} \end{aligned}\right\} \quad \lambda = \sqrt{b^2 + c^2}$$



A_v $\qquad$ $R_{\theta,i}$

A_V : *aligning vector* **V** *with* ***k***

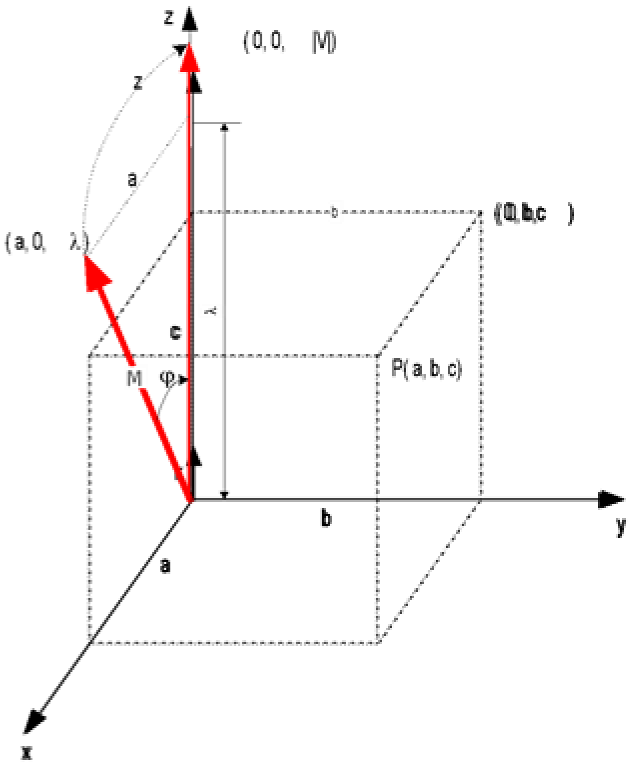
Step 1 : Rotate about x - axis by θ

$$\left.\begin{aligned} \sin\theta &= \frac{b}{\lambda} \\ \cos\theta &= \frac{c}{\lambda} \end{aligned}\right\} \quad \lambda = \sqrt{b^2 + c^2}$$

Step 2 : Rotate V about y - axis by $-\phi$

$$\left.\begin{aligned} \sin(-\phi) &= \frac{-a}{|V|} \\ \cos(-\phi) &= \frac{\lambda}{|V|} \end{aligned}\right\} \quad |V| = \sqrt{a^2 + b^2 + c^2}$$

$$A_v \qquad R_{-\phi,j} * R_{\theta,i}$$



A_V : *aligning vector* **V** *with* ***k***

- $A_V^{-1} = A_V^T$
- $A_{V,N} = A_N^{-1} * A_V$

$$A_V = \begin{bmatrix} \frac{\lambda}{|V|} & \frac{-ab}{\lambda|V|} & \frac{-ac}{\lambda|V|} & 0 \\ 0 & \frac{c}{\lambda} & \frac{-b}{\lambda} & 0 \\ \frac{a}{|V|} & \frac{b}{|V|} & \frac{c}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finding the rotation matrix

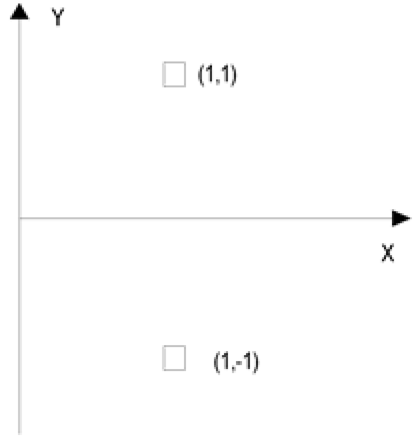
- Now we have done step 1,2 : A_V
- For step 3, we have to rotate theta angle around z axis
- Then take the inverse of step 1,2,

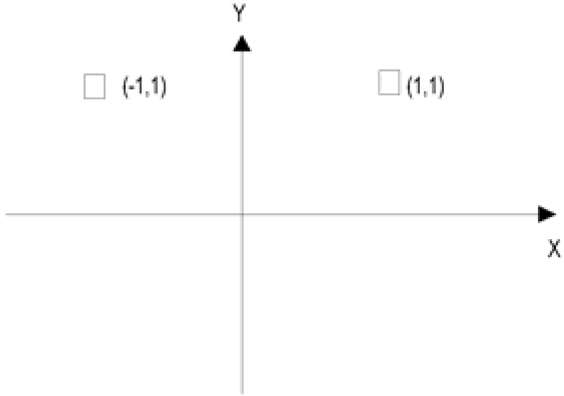
  $A_V^{-1} = A_V^T$

$$R = A_v^T R_z(\theta) A_v$$

- There are actually 5 rotations here.

Mirror Reflection





Reflection about X - axis

$$x' = x \quad y' = -y$$

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about Y - axis

$$x' = -x \quad y' = y$$

$$M_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection 3D

Reflection

- Reflection through the *xy*-plane:

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Reflection through the *yz*-plane:

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Reflection through the *xz*-plane:

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation(d_x, d_y)

$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Scale(s_x, s_y)

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation(θ)

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection$(x\ axis)$

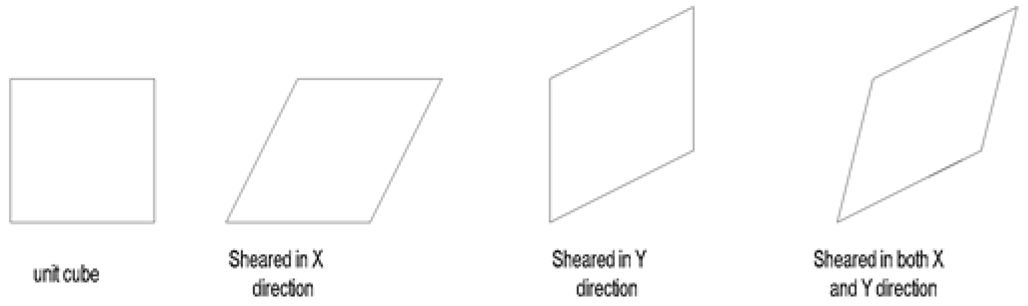
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection$(y\ axis)$

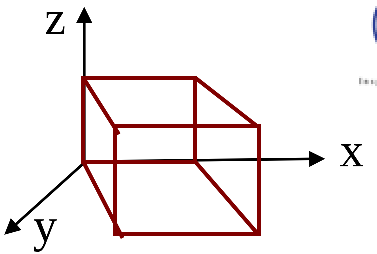
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shearing Transformation

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad SH_{xy} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Shear along Z-axis

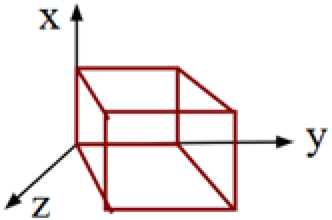


$$SH_{xy}(sh_x, sh_y) \quad * \quad P \quad = \quad P'$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + z * sh_x \\ y + z * sh_y \\ z \\ 1 \end{bmatrix}$$

Shear along X-axis

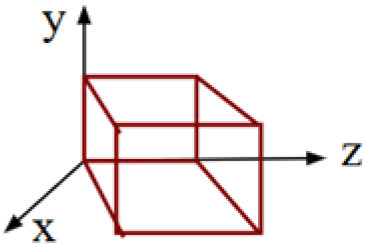


$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ Sh_y & 1 & 0 & 0 \\ Sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix

(In X axis)

Shear along Y-axis



$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & Sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Shearing Matrix

(In Y axis)

Inverse Transforms

- In general: $\mathbf{A}^{-1}$ undoes effect of $\mathbf{A}$
- Special cases:
  - Translation: negate t_x and t_y
  - Rotation: transpose
  - Scale: invert diagonal (axis-aligned scales)
- Others:
  - Invert matrix
  - Invert SVD matrices

Inverse Transformations

Translaiton : $T^{-1}_{(dx,dy)} = T_{(-dx,-dy)}$

Rotation : $R^{-1}_{(\theta)} = R_{(-\theta)} = R^{T}_{(\theta)}$

Sclaing : $S^{-1}_{(sx,sy)} = S_{(1/sx,\, 1/sy)}$

Mirror Ref : $M^{-1}_{x} = M_{x}$

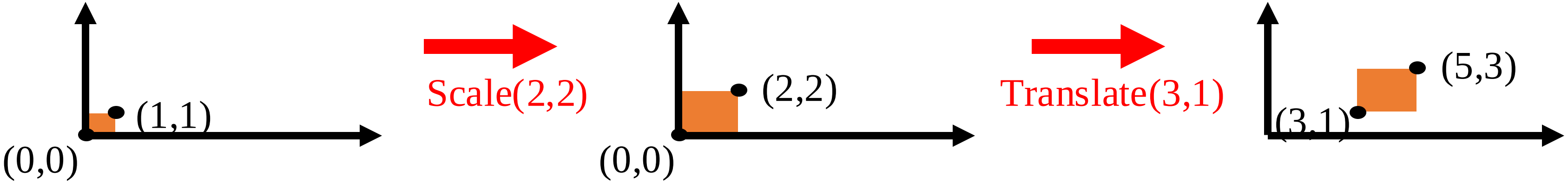
$M^{-1}_{y} = M_{y}$

Shear : $Sh^{-1}_{x} = Sh_{-x}$

Composing Transformations

How are transforms combined?

Scale then Translate



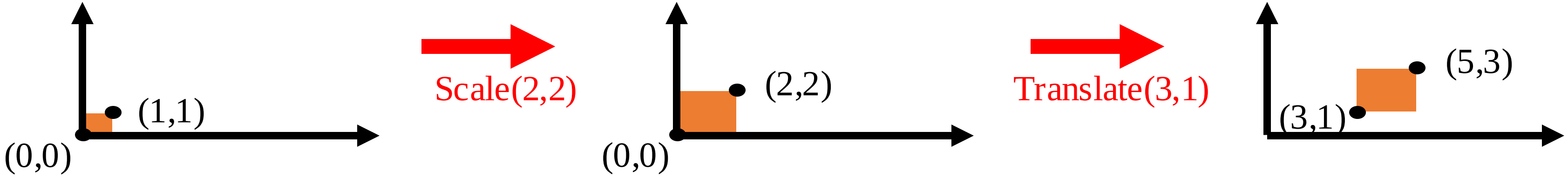
Use matrix multiplication: p' = T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

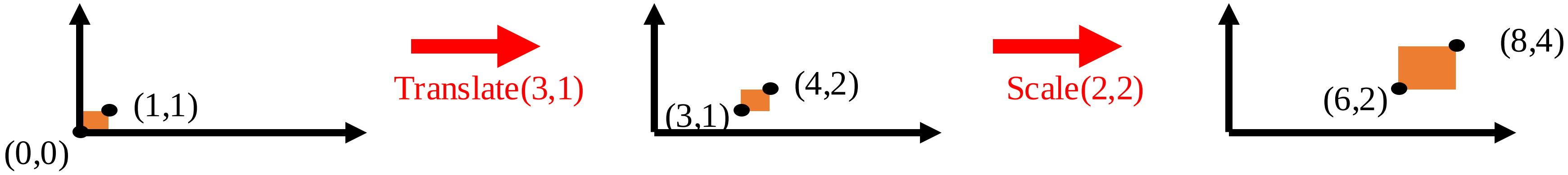
Caution: matrix multiplication is NOT commutative!

Non-commutative Composition

Scale then Translate: p' = T (S p) = TS p



Translate then Scale: p' = S (T p) = ST p



Non-commutative Composition

Scale then Translate: p' = T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Translate then Scale: p' = S (T p) = ST p

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Combining Translations, Rotations

- Order matters!! TR is not the same as RT (demo)
- General form for rigid body transforms
- We show rotation first, then translation (commonly used to position objects) on next slide. Slide after that works it out the other way

Rotate then Translate

$$P' = (TR)P = MP = RP + T$$

$$M = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{ccc|c} R_{11} & R_{12} & R_{13} & T_x \\ R_{21} & R_{22} & R_{23} & T_y \\ R_{31} & R_{32} & R_{33} & T_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$$

Translate then Rotate

$$P' = (RT)P = MP = R(P+T) = RP + RT$$

$$M = \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{3\times3} & R_{3\times3}T_{3\times1} \\ 0_{1\times3} & 1 \end{pmatrix}$$

Associativity of Matrix Multiplication

Create new affine transformations by multiplying sequences of the above basic transformations.

$$\mathbf{q} = \mathbf{CBAp}$$

$$\mathbf{q} = ((\mathbf{CB})\,\mathbf{A})\,\mathbf{p} = (\mathbf{C}\,(\mathbf{B}\,\mathbf{A}))\mathbf{p} = \mathbf{C}\,(\mathbf{B}\,(\mathbf{Ap}))\text{ etc.}$$

matrix multiplication is associative.

To transform just a point, better to do $\mathbf{q} = \mathbf{C}(\mathbf{B}(\mathbf{Ap}))$

But to transform many points, best to do

$$\mathbf{M} = \mathbf{CBA}$$

then do $\mathbf{q} = \mathbf{Mp}$ for any point **p** to be rendered.

For geometric pipeline transformation, define **M** and set it up with the model-view matrix and apply it to any vertex subsequently defined to its setting.

Example Composite Transforms

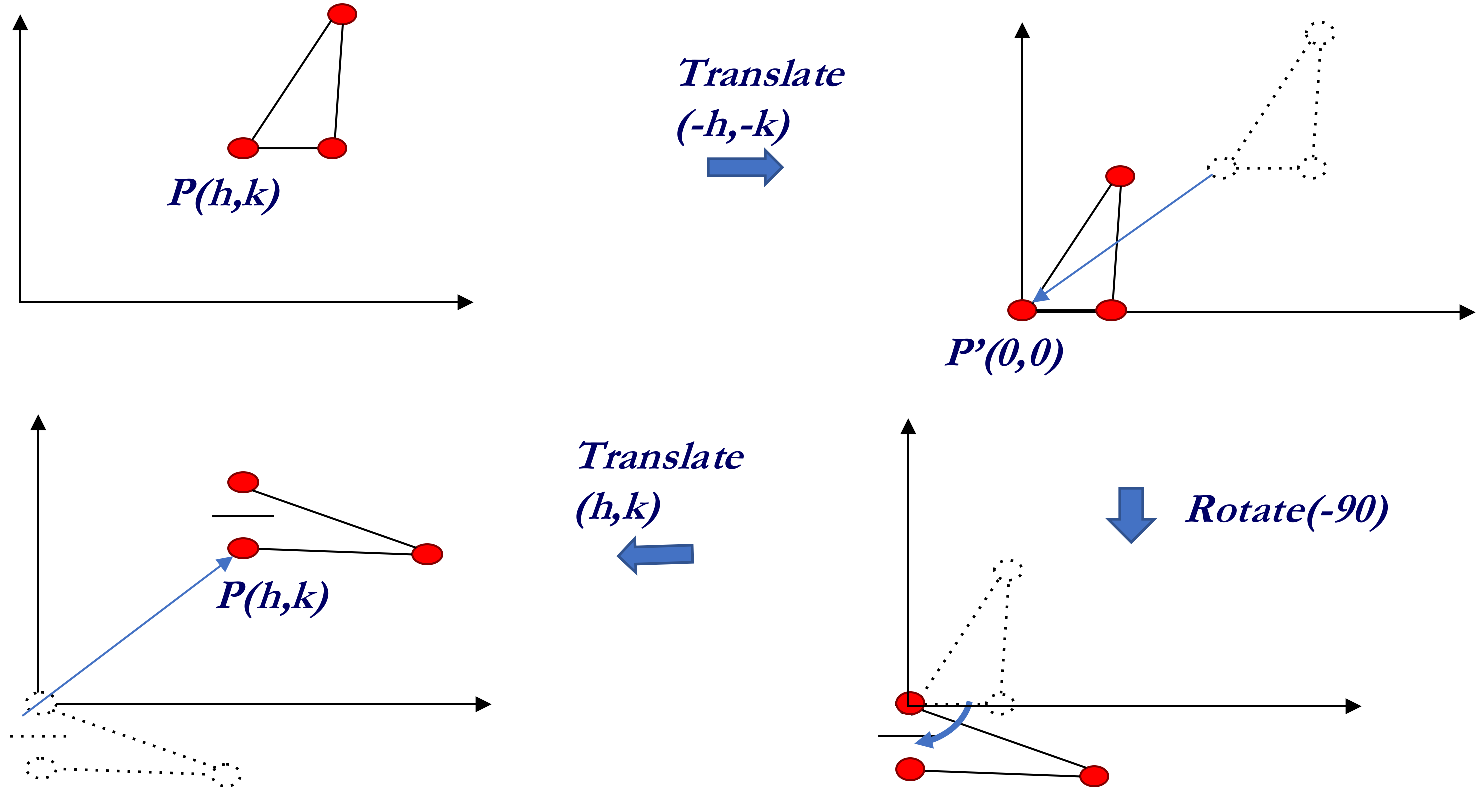
Rotation of θ about P(h,k): $R_{\theta,P}$

Step 1: Translate P(h,k) to origin

Step 2: Rotate θ w.r.t to origin

Step 3: Translate (0,0) to P(h,k0)

$\mathbf{R_{\theta,P} = T(h,k) * R_{\theta} * T(-h,-k)}$


P(h,k)
Translate
(-h,-k)
P'(0,0)
Translate
(h,k)
Rotate(-90)
P(h,k)

$$P' = T_{(h,k)} \times R_{(-90)} \times T_{(-h,-k)} \times P$$

$$Composite\ matrix, M = T_{(h,k)} \times R_{(-90)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90) & -\sin(-90) & 0 \\ \sin(-90) & \cos(-90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = T_{(h,k)} \times R_{(\theta)} \times T_{(-h,-k)} \times P$$

$$Composite\, matrix, M = T_{(h,k)} \times R_{(\theta)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose, a composite transformation is defined as rotating 90 degree counterclockwise with respect to point (5, 5). Calculate the composite transformation matrix in homogeneous form. Then, find the new coordinates of the point (10, 10) after transformation.

$$Composite\ matrix, M = T_{(5,5)} \times R_{(90)} \times T_{(-5,-5)}$$

$$Composite\ matrix, M = T_{(5,5)} \times R_{(90)} \times T_{(-5,-5)}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = M \times P$$

$$= \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix}$$

So, the new coordinate of P is (0, 10)

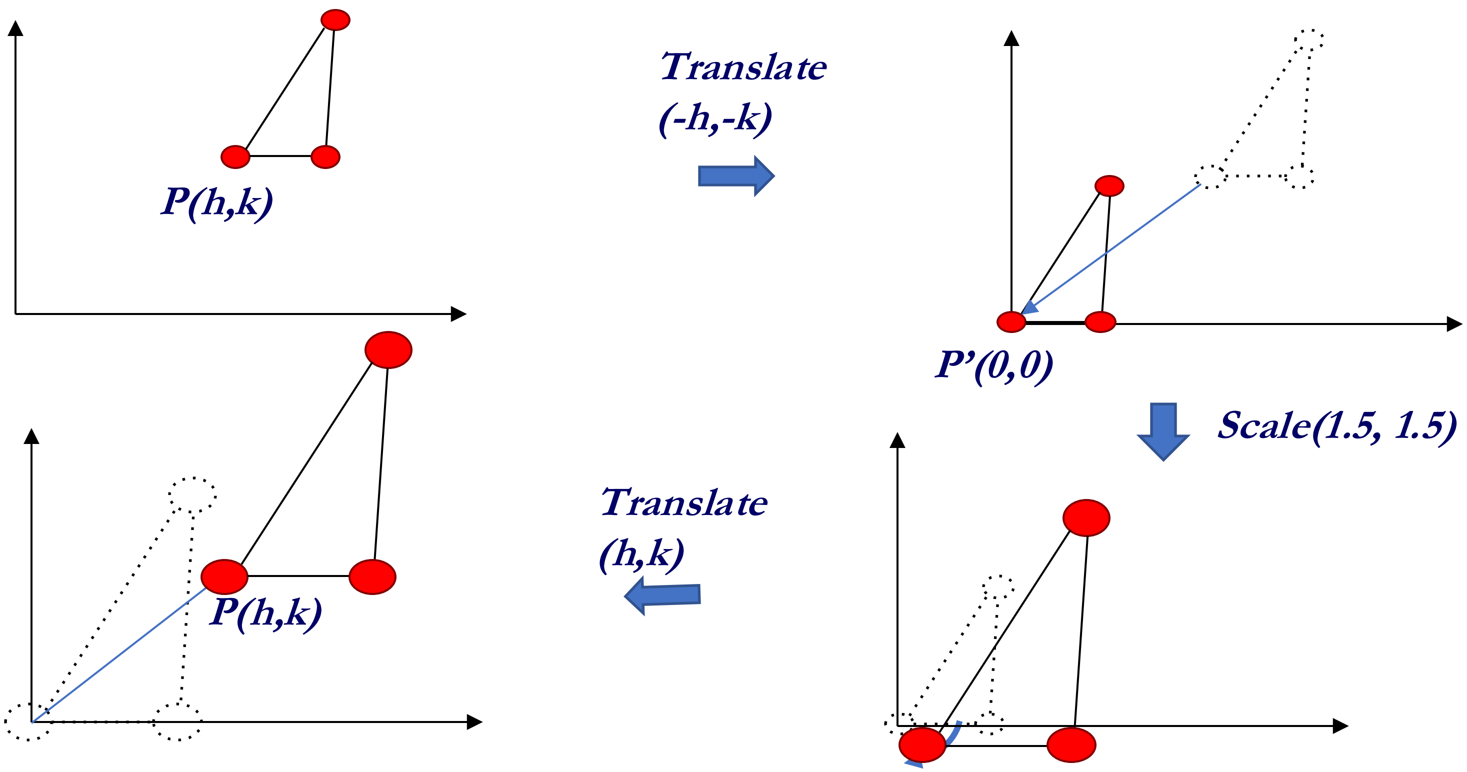
Scaling of $S_{a,b}$ about P(h,k): $R_{S,P}$

Step 1: Translate P(h,k) to origin

Step 2: Scale $S_{a,b}$ w.r.t to origin

Step 3: Translate (0,0) to P(h,k0)

$$\mathbf{R_{S,P} = T(h\ ,k) * S(a,b) * T(-h\ ,-k)}$$



P(h,k)
Translate
(-h,-k)
P'(0,0)
Scale(1.5, 1.5)
Translate
(h,k)
P(h,k)

$$P' = T_{(h,k)} \times S_{(a,b)} \times T_{(-h,-k)} \times P$$

$$Composite\, matrix, M = T_{(h,k)} \times S_{(a,b)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose, a composite transformation is defined as scaling 2 times in both axis with respect to point (5, 5). Calculate the composite transformation matrix in homogeneous form. Then, find the new coordinates of the point (10, 10) after transformation.

$$Composite\ matrix, M = T_{(5,5)} \times S_{(2,2)} \times T_{(-5,-5)}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

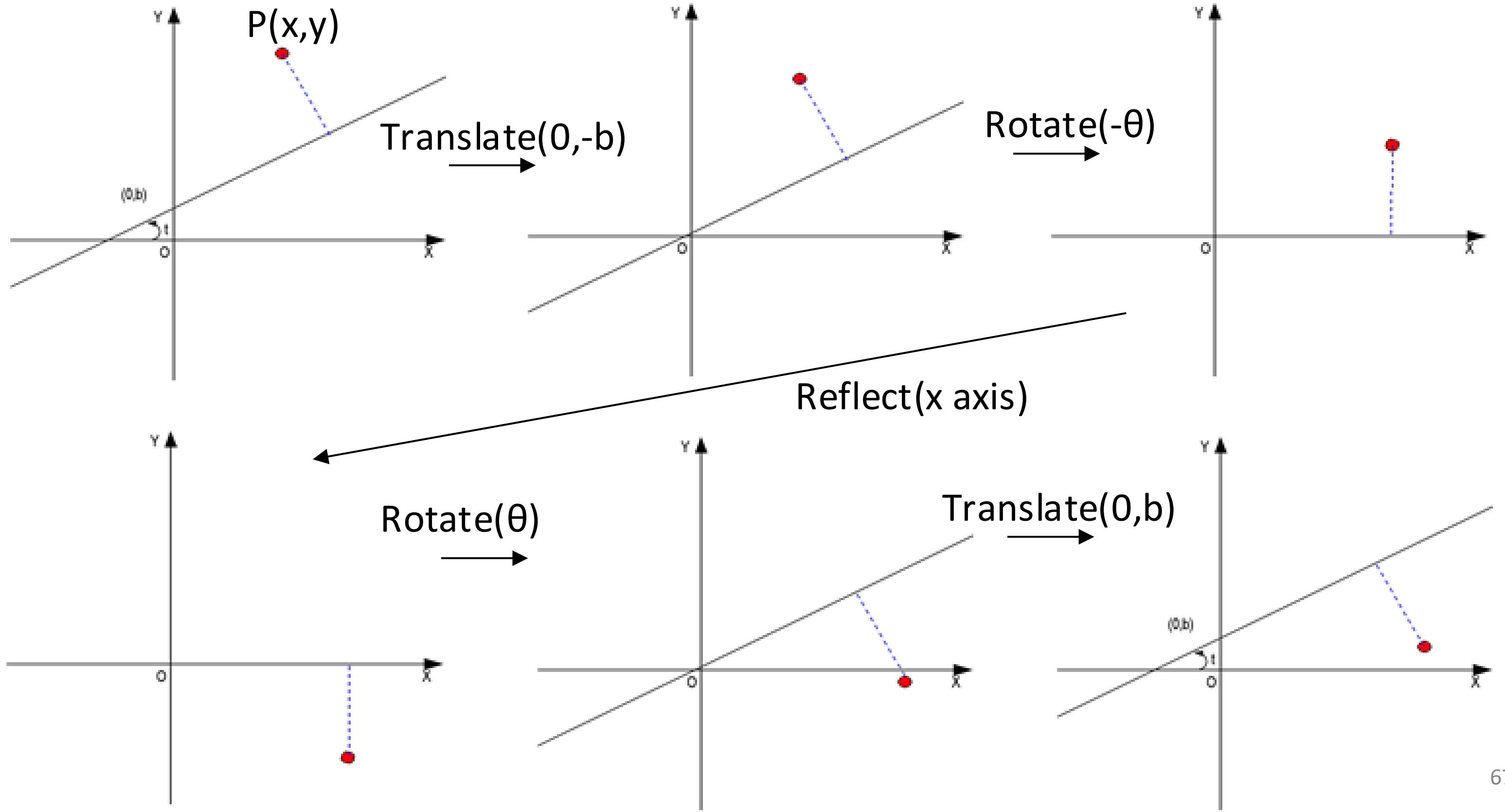
$$P' = M \times P$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 15 \\ 1 \end{bmatrix}$$

So, the new coordinate of P is (15, 15)

Reflection about line L, M_L



Reflection about line L, M_L

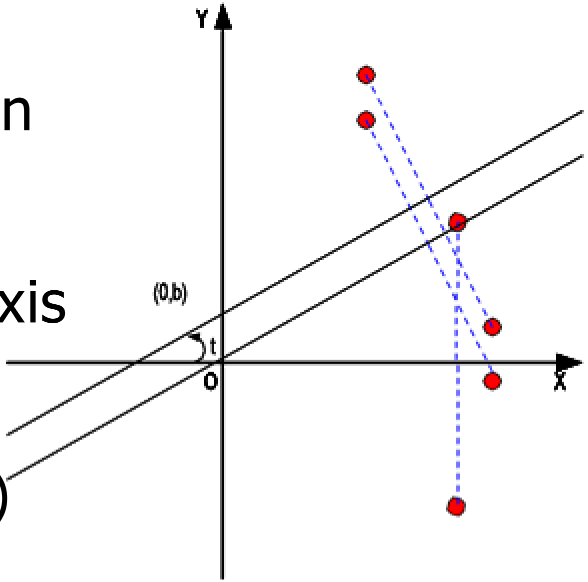
Step 1: Translate (0,-b) to origin

Step 2: Rotate -θ degrees

Step 3: Mirror reflect about X-axis

Step 4: Rotate θ degrees

Step 5: Translate origin to (0,b)



$$M_L = T(0\ ,b) * R(\theta) * M_x * R(-\theta) * T(0\ ,-b)$$

Reflect the point (10, 5) with respect to the line y=x+2.

$$b = 2, m = 1, \theta = \tan^{-1} 1 = 45$$

$$M_{composite} = T_{(0,2)} \times R_{(45)} \times Refl_{(x\ axis)} \times R_{(-45)} \times T_{(0,-2)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 1 \end{bmatrix}$$

New coordinate (3, 12)

Axis Angle Notation

Rodrigues Formula

There is another way that is both easier to understand and provides you with more insights into what rotation is really about. Instead of specifying a rotation by a series of canonical angles, we will specify an arbitrary axis of rotation and an angle. We will also first consider rotating vectors, and come back to points shortly.

$$R(a,\theta) = \cos\theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos\theta) \begin{pmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \text{Symmetric} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} (1-\cos\theta) + \text{Skew} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \sin\theta + \mathbf{I}\cos\theta \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

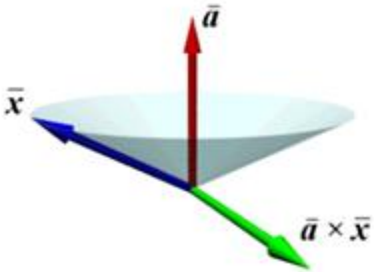
The vector *a* specifies the axis of rotation. *This axis vector must be normalized.* The rotation angle is given by θ.

You might ask "How am I going to remember **this** equation?". However, once you understand the geometry of rotation, the equation will seem obvious.

The basic idea is that *any rotation can be decomposed into weighted contributions from three different vectors*.

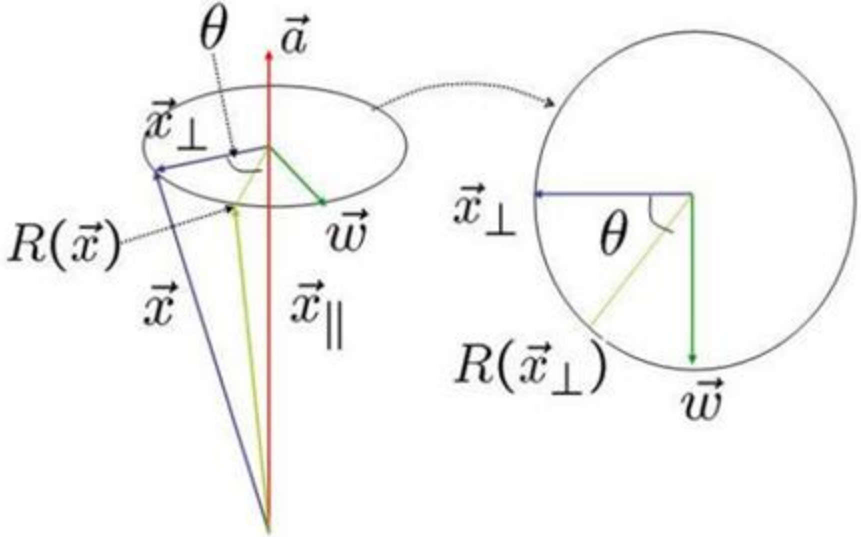
The Geometry of a Rotation

We can actually define a *natural* basis for rotation in terms of three defining vectors. These vectors are the rotation axis, a vector perpendicular to both the rotation axis and the vector being rotated, and the vector itself. These vectors correspond to the each respective term in the expression.

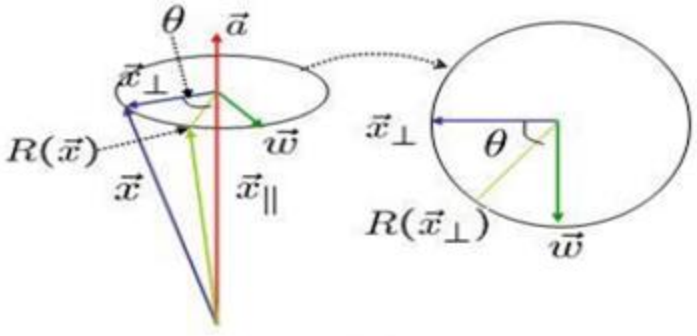


Let's look at this in greater detail

The Geometry of a Rotation



The Geometry of Rotation



$$\begin{aligned}\vec{w} &= \vec{a} \times \vec{x}_\perp \\ &= \vec{a} \times (\vec{x} - \vec{x}_{||}) \\ &= (\vec{a} \times \vec{x}) - (\vec{a} \times \vec{x}_{||}) \\ &= \vec{a} \times \vec{x}\end{aligned}$$

$$R(\vec{x}_\perp) = \cos\theta \vec{x}_\perp + \sin\theta \vec{w}$$

$$\vec{x}_{||} = (\vec{a} \cdot \vec{x})\vec{a}$$
$$\vec{x}_\perp = \vec{x} - \vec{x}_{||} = \vec{x} - (\vec{a} \cdot \vec{x})\vec{a}$$

$$\begin{aligned}R(\vec{x}) &= R(\vec{x}_{||}) + R(\vec{x}_\perp) \\ &= R(\vec{x}_{||}) + \cos\theta \vec{x}_\perp + \sin\theta \vec{w} \\ &= (\vec{a} \cdot \vec{x})\vec{a} + \cos\theta(\vec{x} - (\vec{a} \cdot \vec{x})\vec{a}) + \sin\theta \vec{w} \\ &= \cos\theta \vec{x} + (1 - \cos\theta)(\vec{a} \cdot \vec{x})\vec{a} + \sin\theta(\vec{a} \times \vec{x})\end{aligned}$$

The Geometry of Rotation

- So, we find the rotation vector is $R(\vec{x}) = \cos\theta\vec{x} + (1 - \cos\theta)(\vec{a}\cdot\vec{x})\vec{a} + \sin\theta(\vec{a}\times\vec{x})$

Vector Form

- This is a vector equation, which is good
- But, we need a matrix form also, to work ...ly.

$$R(k,\theta) = \cos\theta\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos\theta)\begin{pmatrix} a_x^2 & a_x a_y & a_x a_z \\ a_x a_y & a_y^2 & a_y a_z \\ a_x a_z & a_y a_z & a_z^2 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

Rodrigues Formula

Matrix Form

Rodrigues Formula

- Vector a $R(\vec{x}) = \cos\theta\vec{x} + (1-\cos\theta)(\vec{a}\cdot\vec{x})\vec{a} + \sin\theta(\vec{a}\times\vec{x})$

$$X = [x \quad y \quad z]^T$$

$$R(x) = R(k,\theta)X$$

$$R(k,\theta) = \cos\theta\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + (1-\cos\theta)\begin{pmatrix}a_x^2 & a_xa_y & a_xa_z\\ a_xa_y & a_y^2 & a_ya_z\\ a_xa_z & a_ya_z & a_z^2\end{pmatrix} + \sin\theta\begin{pmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{pmatrix}$$

- How we find these equivalent matrix and vector forms ?

Vector and Matrix algebra

You've probably been exposed to vector algebra in previous courses. These include *vector addition*, the *vector dot product*, and the *vector cross product*. Let's take a minute to discuss an equivalent set of matrix operators.

We begin with the *dot product*. This operation acts on two vectors and returns a scalar. Geometrically, this operation can be described as a projection of one vector onto another. The dot product has the following matrix formulation.

$$\bar{a} \cdot \bar{b} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \alpha$$

Cross product in Matrix Form

The vector cross product also acts on two vectors and returns a third vector. Geometrically, this new vector is constructed such that its projection onto either of the two input vectors is zero.

$$\bar{a} \times \bar{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \bar{c} \qquad \begin{matrix} \bar{a} \cdot \bar{c} = 0 \\ \bar{b} \cdot \bar{c} = 0 \end{matrix}$$

In order for one vector to project onto another with a length of zero, it must either have a length of zero, or be *perpendicular* to the second vector. Yet the vector generated by the cross-product operator is perpendicular to two vectors. Since the two input vectors define a plane in space, the vector that results from the cross product operation is perpendicular, or *normal* to this plane.

For any plane there are two possible choices of a normal vector, one on each side of a plane. The cross product is defined to be the one of these two vectors where the motion from the tip of the first input vector to the tip of the second input vector is in a counter-clockwise direction when observed from the side of the normal. This is just a restatement of the right-hand rule that you are familiar with.

Finding rotations from a rotation matrix

- Given, R is a pure rotation matrix
- Find **axis of rotation *K(k_x,k_y,k_z)* and angle theta from R**

$$\cos\theta = \left(\frac{Trace(R)-1}{2}\right)$$

- Recall rodrigues formula, it gives R
- Check that :
- Trace(R) = S
  rotation matr

$$R = \begin{pmatrix} \boxed{\begin{matrix} k_xk_x(1-c)+c & k_yk_x(1-c)-k_zs & k_xk_z(1-c)+k_ys \\ k_yk_x(1-c)+k_zs & k_yk_y(1-c)+c & k_yk_z(1-c)-k_xs \\ k_zk_x(1-c)-k_ys & k_zk_y(1-c)+k_xs & k_zk_z(1-c)+c \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & 1 \end{pmatrix}$$

Finding rotations from a rotation matrix

- Check that :

$$\begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} = \frac{1}{2\sin(\theta)}\left(R - R^T\right)$$

$$\longrightarrow \quad k_x = \frac{1}{2\sin(\theta)}\left(R_{3,2} - R_{2,3}\right)$$

$$k_y = \frac{1}{2\sin(\theta)}\left(R_{1,3} - R_{3,1}\right)$$

$$k_z = \frac{1}{2\sin(\theta)}\left(R_{2,1} - R_{1,2}\right)$$

$$R = \begin{pmatrix} k_x k_x(1-c)+c & k_y k_x(1-c)-k_z s & k_x k_z(1-c)+k_y s & 0 \\ k_y k_x(1-c)+k_z s & k_y k_y(1-c)+c & k_y k_z(1-c)-k_x s & 0 \\ k_z k_x(1-c)-k_y s & k_z k_y(1-c)+k_x s & k_z k_z(1-c)+c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

References

- Chapter 4 and Chapter 6, Schaum's Outline of Computer Graphics (2nd Edition) by Zhigang Xiang, Roy A. Plastock
- Chapter 5, Computer Graphics: Principles and Practice in C (2nd Edition) by James D. Foley, Andries van Dam, Steven K. Feiner, John F. Hughes
- Chapter 6, Fundamentals of Computer Graphics, by Peter Shirley