Transformation Background

What is Transformation ?

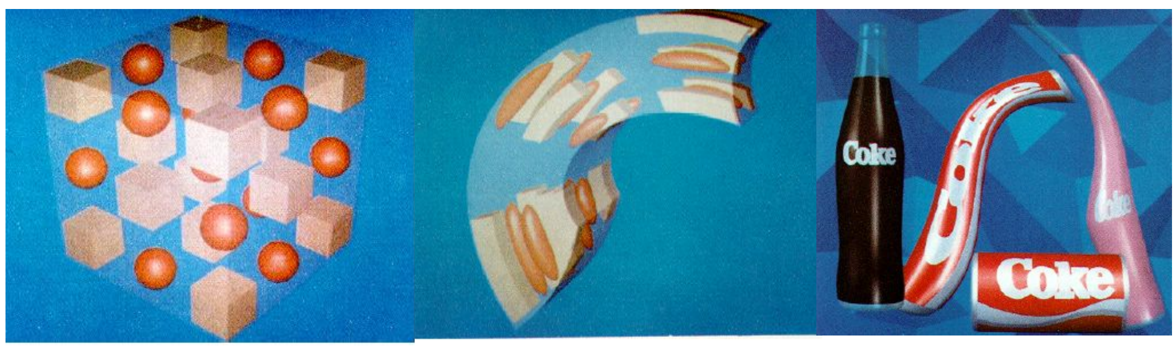
What is a Transformation

- Transformation:
  - An operation that changes one configuration into another
- For images, shapes, *etc.*
  - A geometric transformation maps positions that define the object to other positions
  - Linear transformation means the transformation is defined by a linear function... which is what matrices are good for.

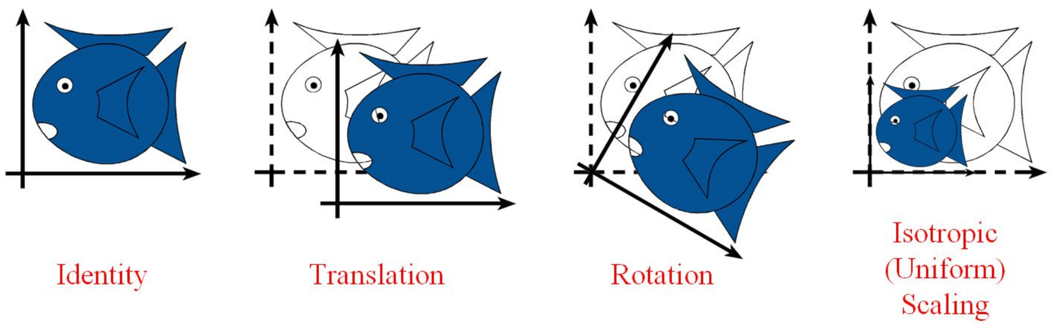
What is a Transformation?

- A function that maps points *x* to points *x'*:

  Applications: animation, deformation, viewing, projection, real-time shadows, …



Simple Transformations

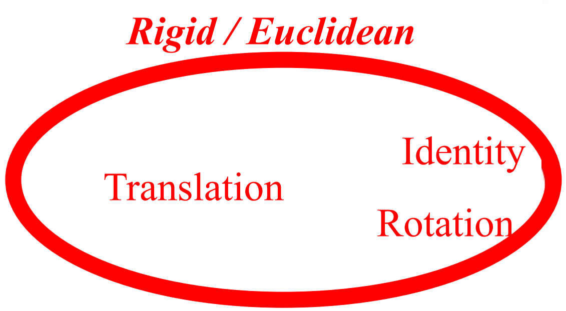


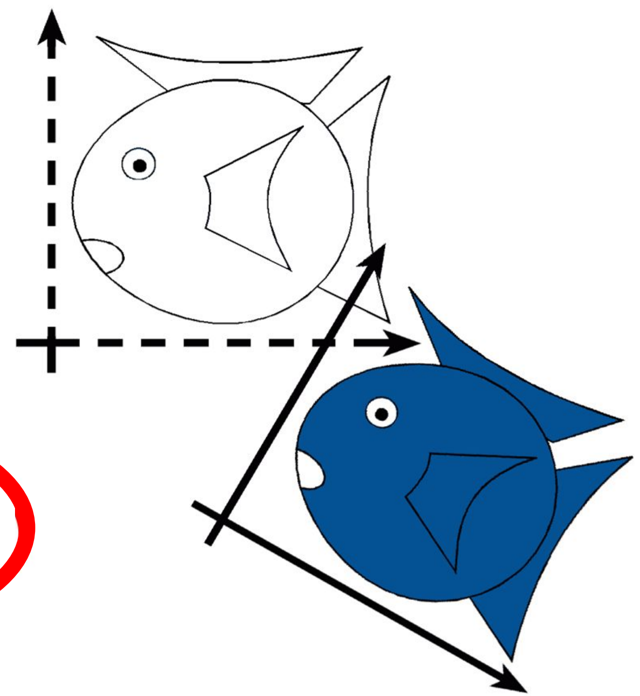
- Can be combined
- Are these operations invertible?

Yes, except scale = 0

Rigid-Body / Euclidean Transforms

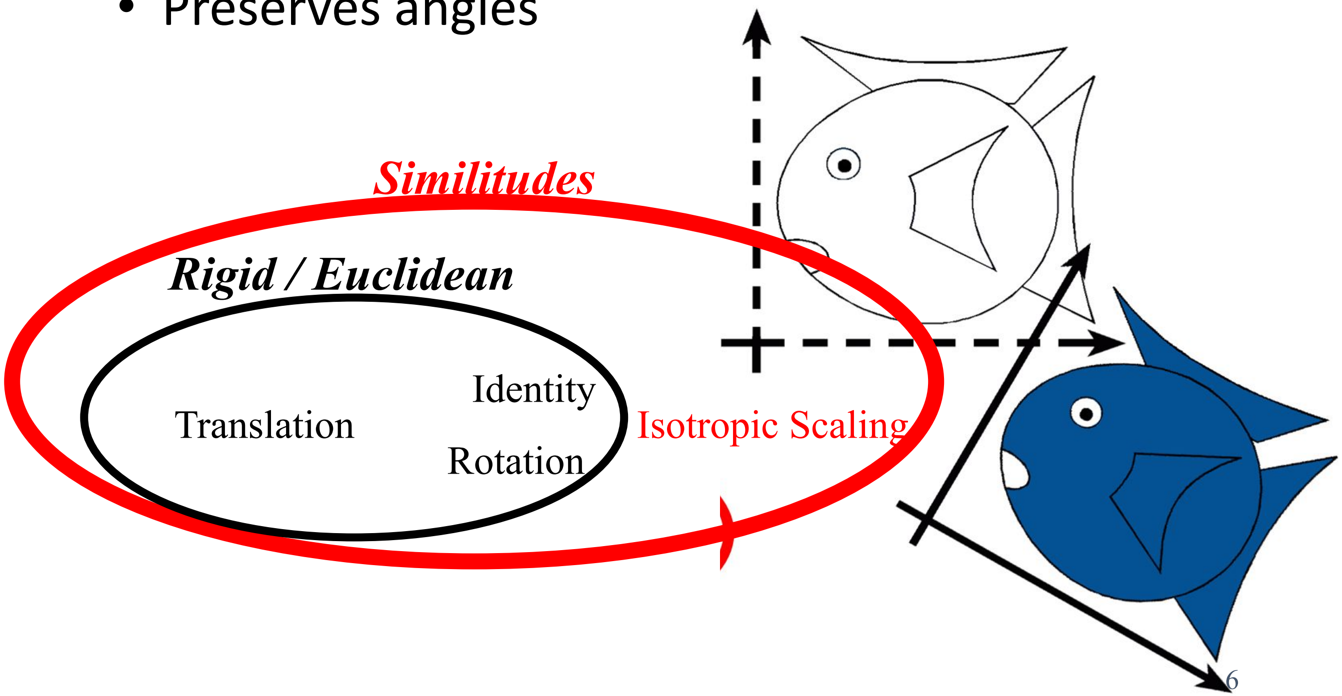
- Preserves distances
- Preserves angles



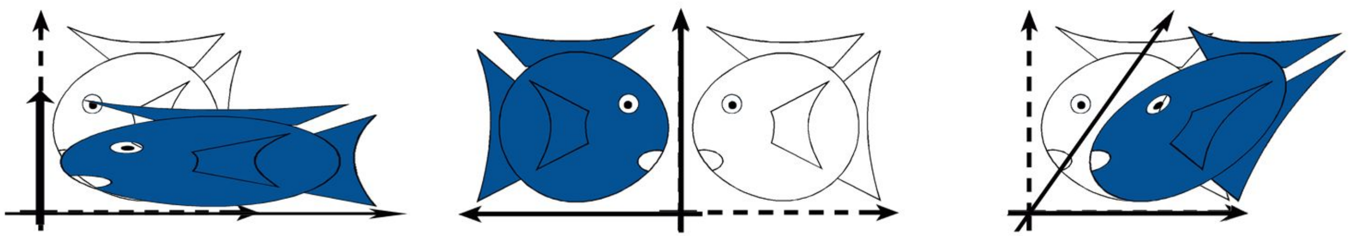


Similitudes / Similarity Transforms

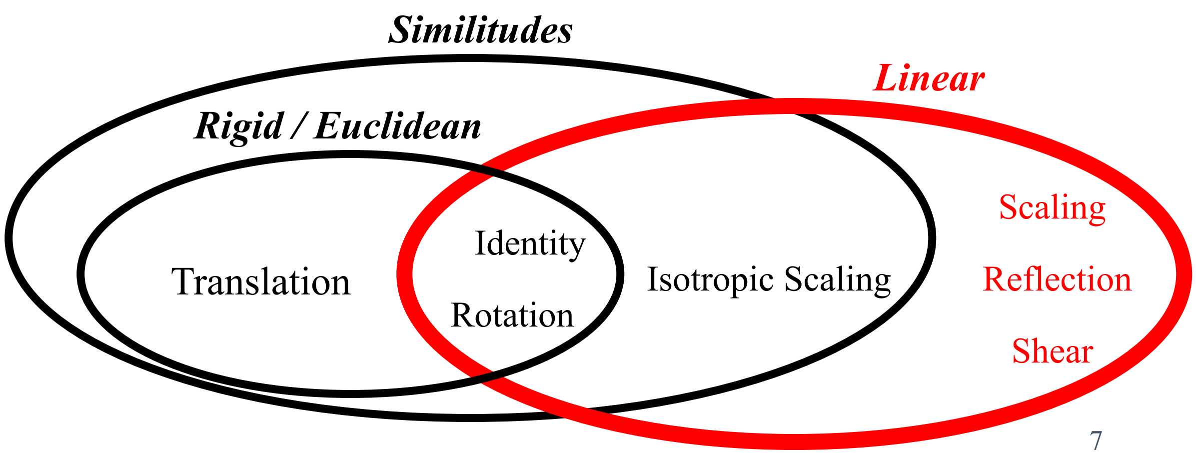
- Preserves angles



Linear Transformations

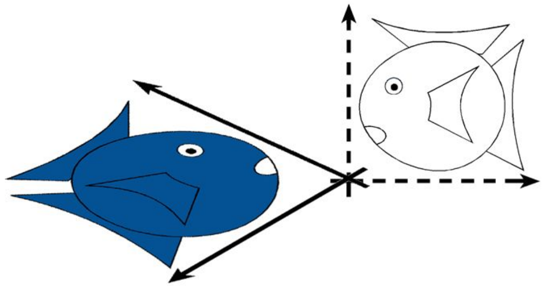


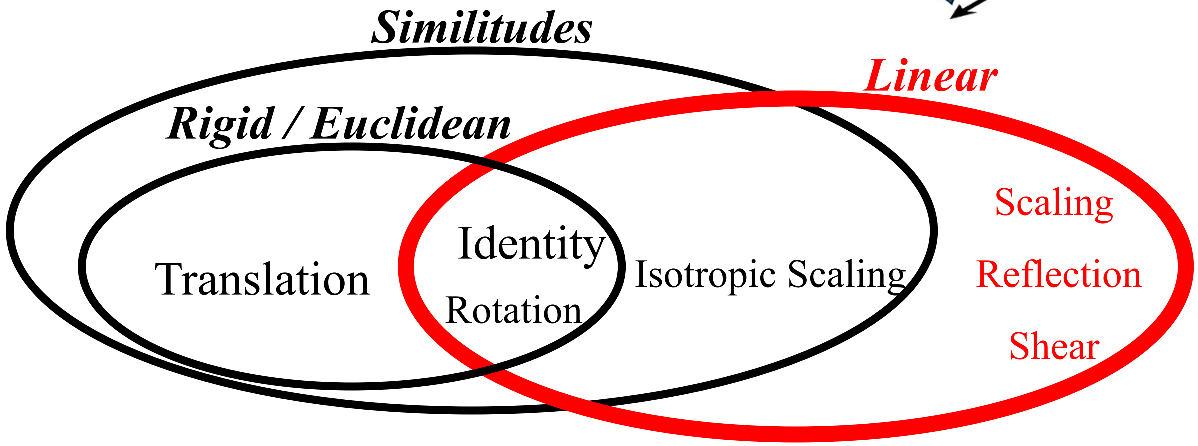
Scaling Reflection Shear



Linear Transformations

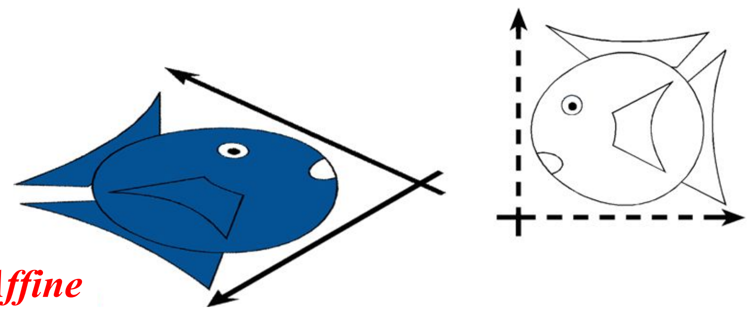
- $L(p + q) = L(p) + L(q)$
- $L(ap) = a\, L(p)$

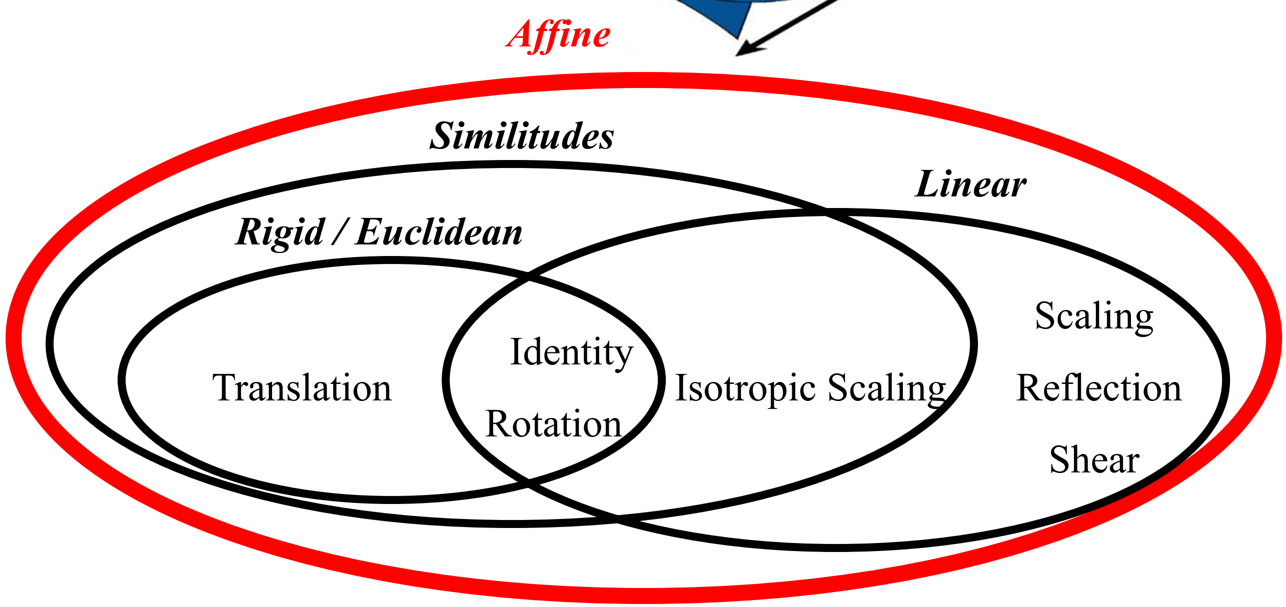




Affine Transformations

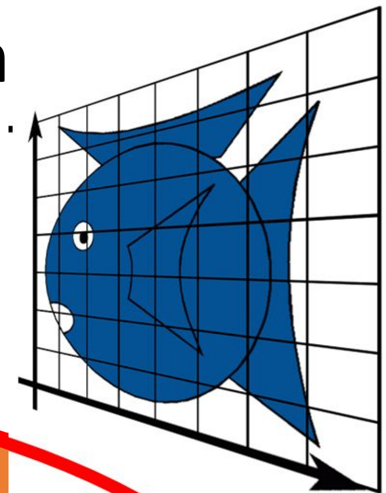
- preserves parallel lines

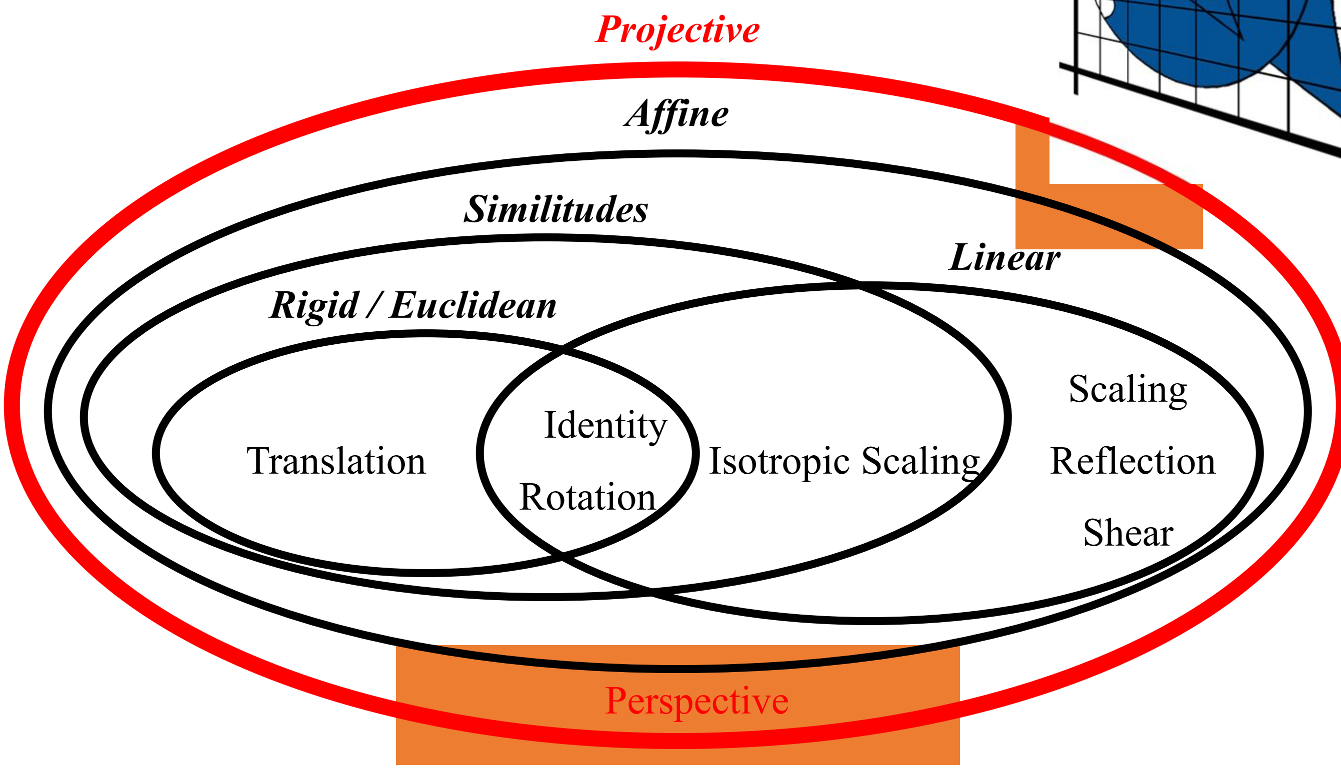




Projective Transformation

- preserves lines





How are Transforms Represented?

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = Mp + t$$

Translation in homogenous coordinates

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

Cartesian formulation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = Mp + t$$

Homogeneous formulation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underset{0}{\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p' = Mp$$

Homogeneous Co-ordinates

- Translation, scaling and rotation are expressed (non-homogeneously) as:
  - translation: $P' = P + T$
  - Scale: $P' = S \cdot P$
  - Rotate: $P' = R \cdot P$
- Composition is difficult to express, since translation not expressed as a matrix multiplication
- Homogeneous coordinates allow all three to be expressed homogeneously, using multiplication by 3×3 matrices
- W is 1 for affine transformations in graphics

Homogeneous Coordinates

- Add an extra dimension
  - in 2D, we use 3 x 3 matrices
  - In 3D, we use 4 x 4 matrices
- Each point has an extra value, *w*

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$p' = \color{red}{M}\color{black}{p}$$

Homogeneous Coordinates

- Most of the time w = 1, and we can ignore it

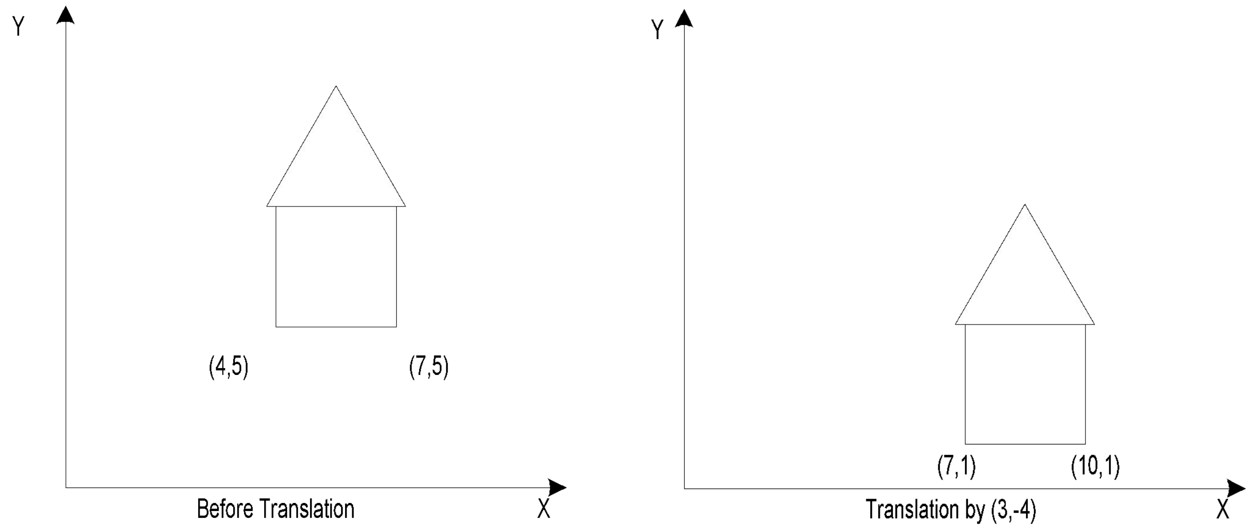
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- If we multiply a homogeneous coordinate by an *affine matrix*, w is unchanged

Mechanics of Rigid Transformations

- Translate
- Rotate
- Scale

Translation – 2D



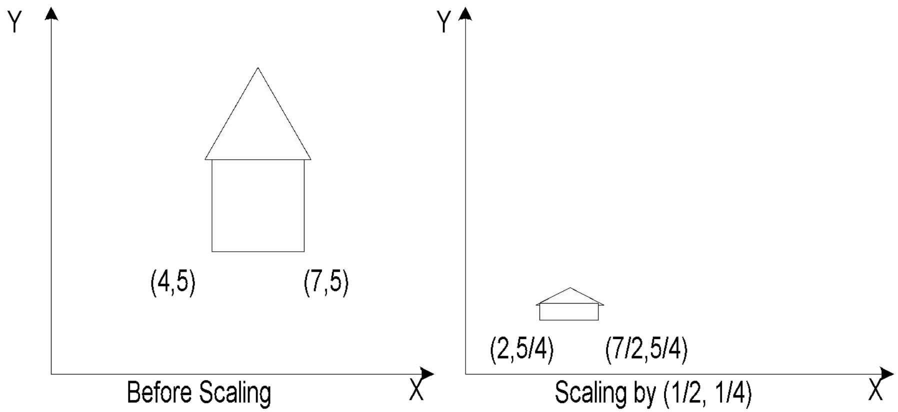
x′ = x + d_x
y′ = y + d_y

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad T = \begin{bmatrix} d_x \\ d_y \end{bmatrix} \qquad P' = P + T$$

Homogenious Form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling – 2D



Types of Scaling:

- Differential (s_x != s_y)
- Uniform (s_x = s_y)

$$x' = s_x * x$$

$$y' = s_y * y$$

$$\begin{array}{ccccc} S & * & P & = & P' \\ \Downarrow & & \Downarrow & & \Downarrow \\ \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} & * & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} x * s_x \\ y * s_y \end{bmatrix} \end{array}$$

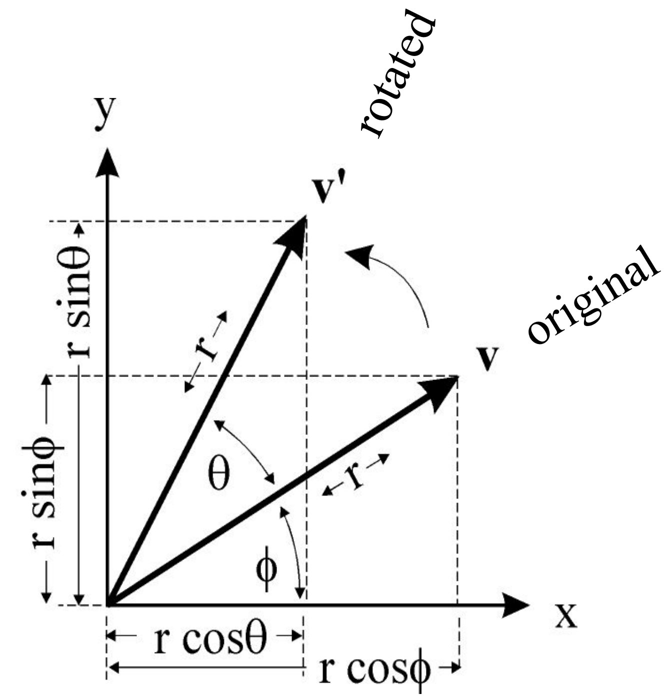
Homogenious Form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation – 2D

$$\mathbf{v} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$$

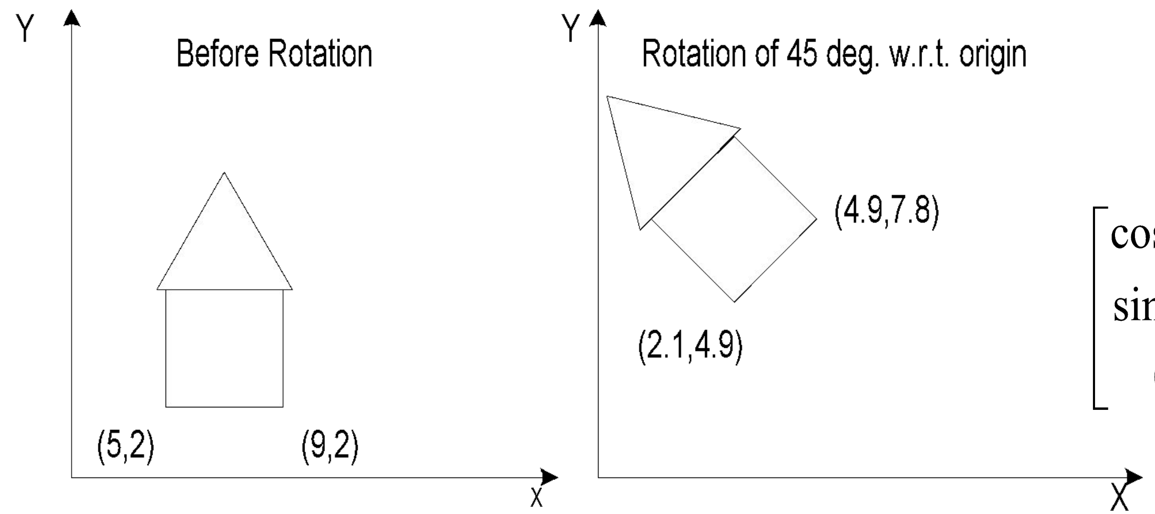
$$\mathbf{v}' = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}$$



$$\text{expand}\,(\phi+\theta) \Rightarrow \begin{cases} x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{cases}$$

$$\text{but} \quad \begin{matrix} x = r\cos\phi \\ y = r\sin\phi \end{matrix} \Rightarrow \begin{matrix} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{matrix}$$

Rotation – 2D



$$\begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x * \cos\theta - y * \sin\theta = x'$$
$$x * \sin\theta + y * \cos\theta = y'$$

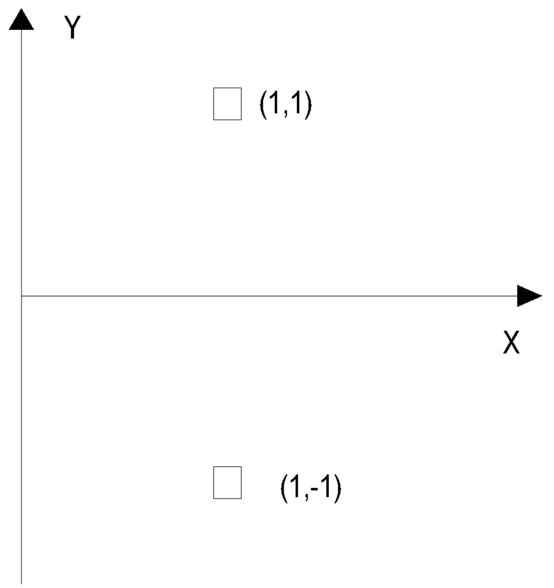
$$R \quad * \quad P \quad = \quad P'$$

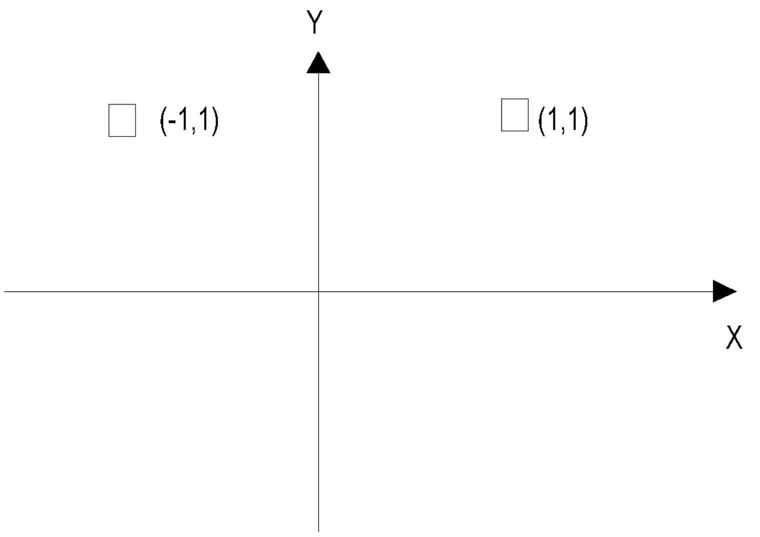
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x * \cos\theta - y * \sin\theta \\ x * \sin\theta + y * \cos\theta \end{bmatrix}$$

Homogenous Form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Mirror Reflection





Reflection about X - axis

$$x' = x \quad y' = -y$$

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about Y - axis

$$x' = -x \quad y' = y$$

$$M_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation(d_x, d_y)

$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Scale(s_x, s_y)

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation(θ)

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection(x $axis$)

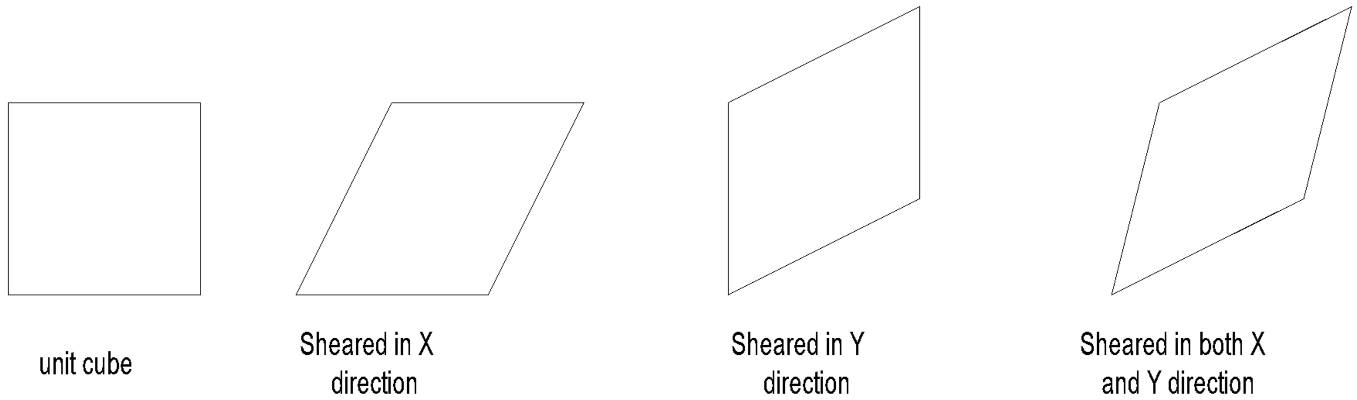
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection(y $axis$)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shearing Transformation

$$SH_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad SH_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad SH_{xy} = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Inverse Transforms

- In general: $\mathbf{A}^{-1}$ undoes effect of $\mathbf{A}$
- Special cases:
  - Translation: negate t_x and t_y
  - Rotation: transpose
  - Scale: invert diagonal (axis-aligned scales)
- Others:
  - Invert matrix
  - Invert SVD matrices

Inverse Transformations

$$\text{Translation}: \quad T^{-1}_{(dx,dy)} = T_{(-dx,-dy)}$$

$$\text{Rotation} \quad : \quad R^{-1}_{(\theta)} = R_{(-\theta)} = R^{T}_{(\theta)}$$

$$\text{Scaling} \quad : \quad S^{-1}_{(sx,sy)} = S_{(1/sx,\, 1/sy)}$$

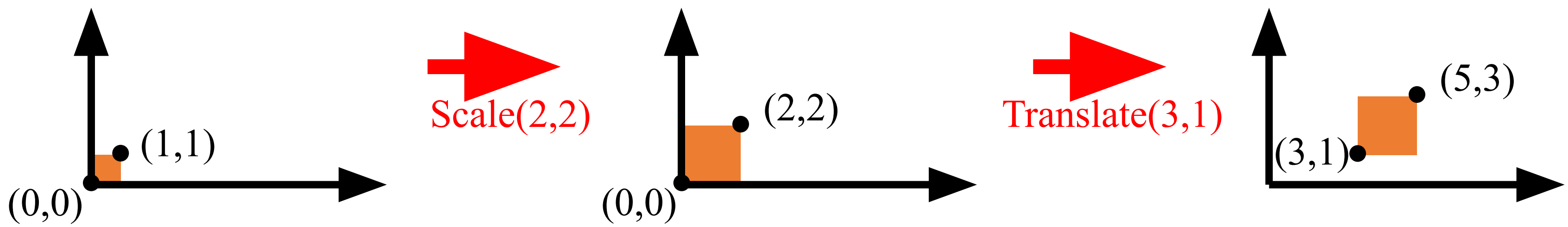
$$\text{Mirror Ref}: \quad M^{-1}_{x} = M_{x}$$

$$M^{-1}_{y} = M_{y}$$

Composing Transformations

How are transforms combined?

Scale then Translate



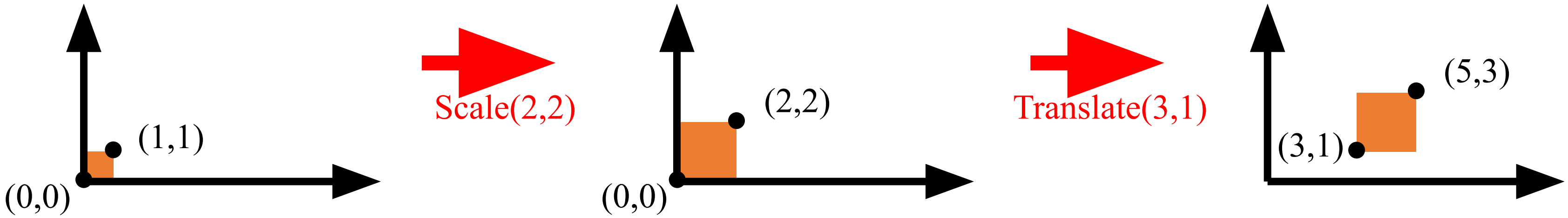
Use matrix multiplication: p' = T (S p) = TS p

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

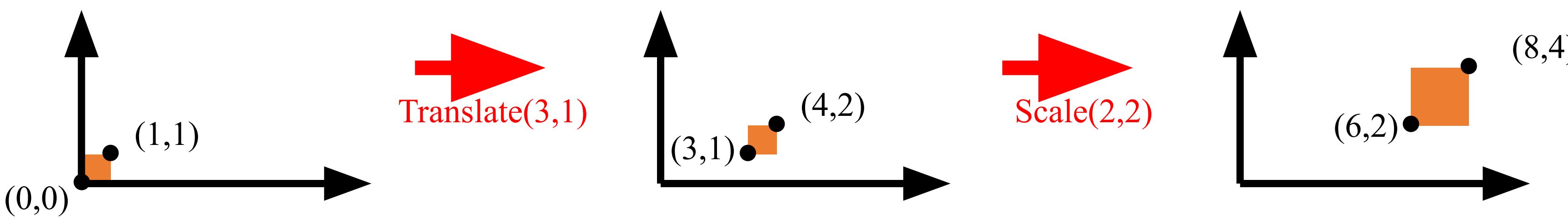
Caution: matrix multiplication is NOT commutative!

Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$



Translate then Scale: $p' = S(T\,p) = ST\,p$



Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$

$$TS = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Translate then Scale: $p' = S(T\,p) = ST\,p$

$$ST = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Combining Translations, Rotations

- Order matters!! TR is not the same as RT (demo)
- General form for rigid body transforms
- We show rotation first, then translation (commonly used to position objects) on next slide. Slide after that works it out the other way

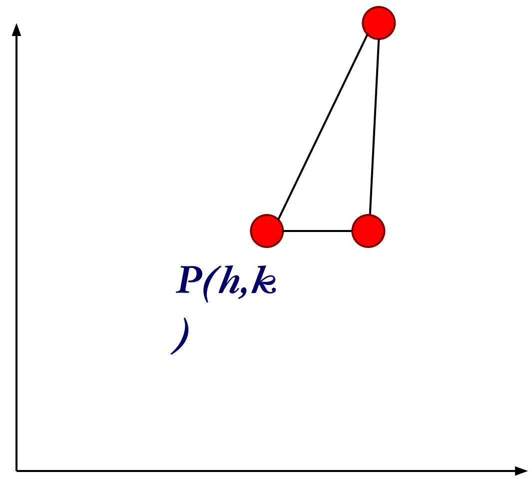
Rotate then Translate

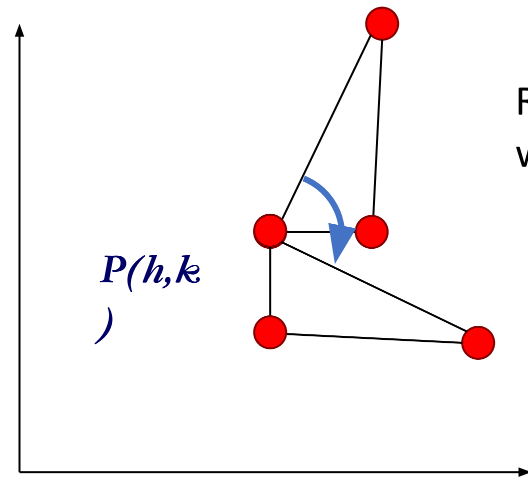
$$P' = (TR)P = MP = RP + T$$

$$M = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{ccc|c} R_{11} & R_{12} & R_{13} & T_x \\ R_{21} & R_{22} & R_{23} & T_y \\ R_{31} & R_{32} & R_{33} & T_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$$

Example Composite Transforms

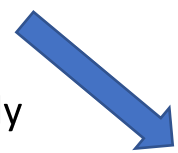
2D Case

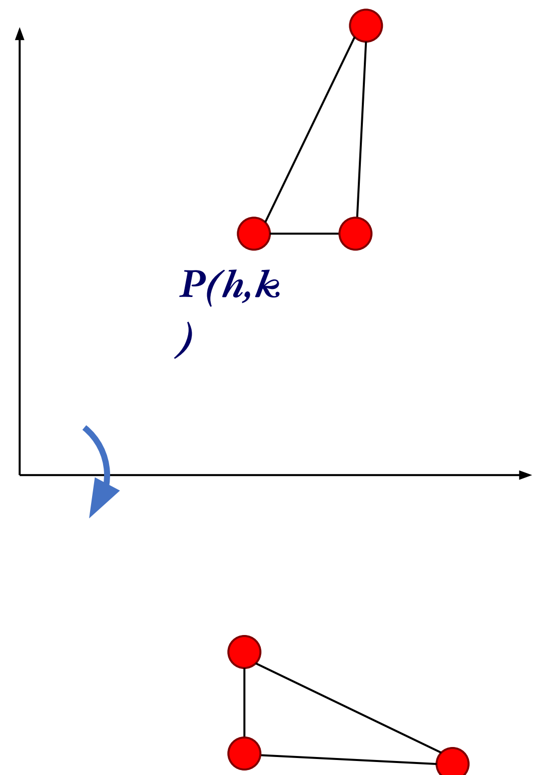


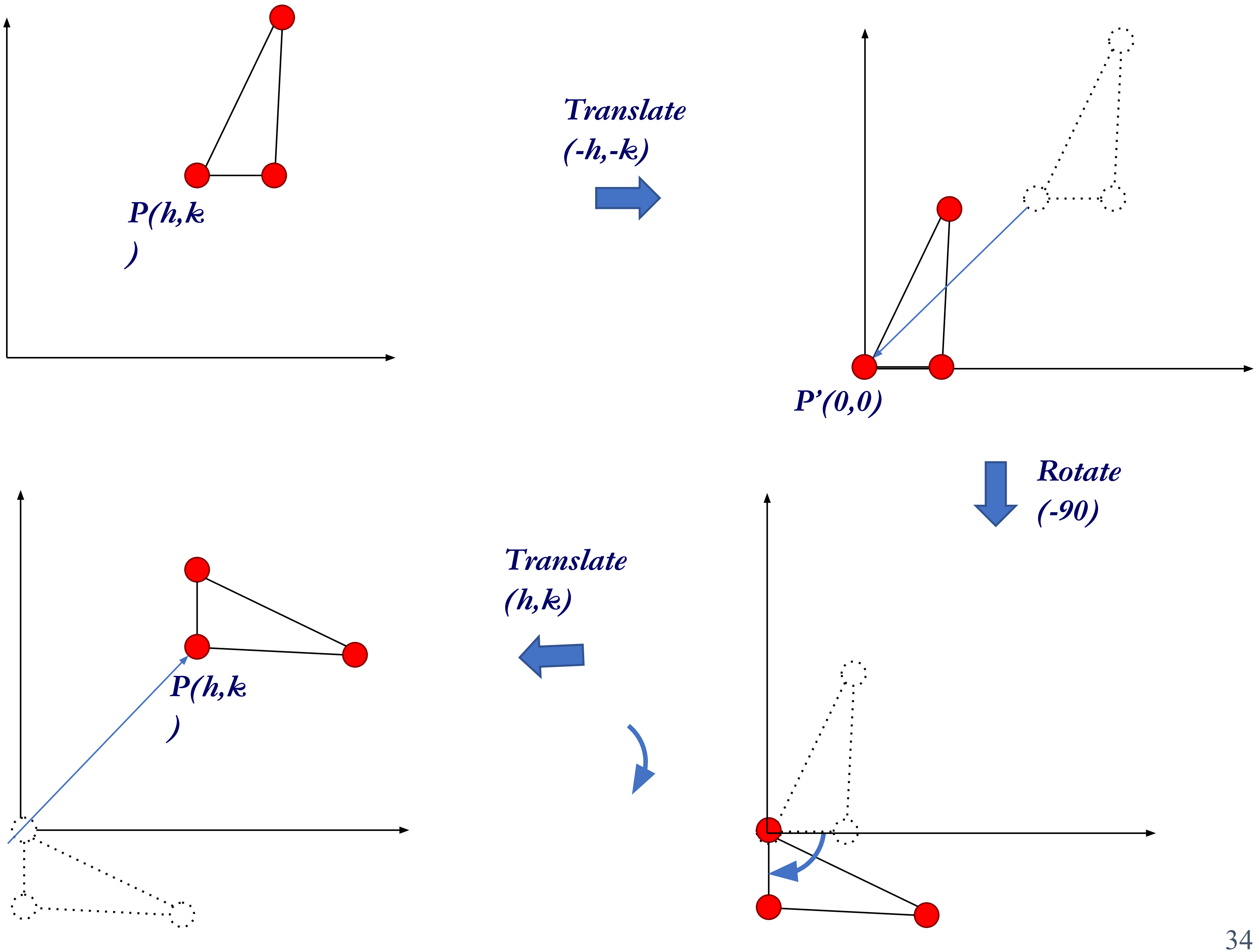


Rotate clockwise 90
with respect to P

If we only apply
Rotate(-90)






Translate
(-h,-k)
P(h,k
)
P'(0,0)
Rotate
(-90)
Translate
(h,k)
P(h,k
)

$$P' = T_{(h,k)} \times R_{(-90)} \times T_{(-h,-k)} \times P$$

$$Composite\ matrix, M = T_{(h,k)} \times R_{(-90)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90) & -\sin(-90) & 0 \\ \sin(-90) & \cos(-90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = T_{(h,k)} \times R_{(\theta)} \times T_{(-h,-k)} \times P$$

$$Composite\ matrix, M = T_{(h,k)} \times R_{(\theta)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose, a composite transformation is defined as rotating 90 degree counterclockwise with respect to point (5, 5). Calculate the composite transformation matrix in homogenous form. Then, find the new coordinates of the point (10, 10) after transformation.

$$Composite\ matrix, M = T_{(5,5)} \times R_{(90)} \times T_{(-5,-5)}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

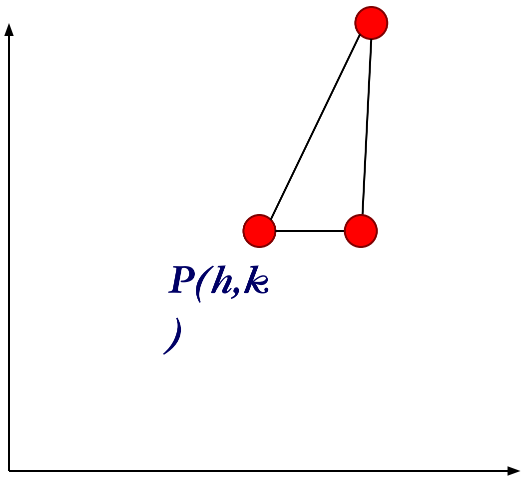
Suppose, a composite transformation is defined as rotating 90 degree counterclockwise with respect to point (5, 5). Calculate the composite transformation matrix in homogenous form. Then, find the new coordinates of the point (10, 10) after transformation.

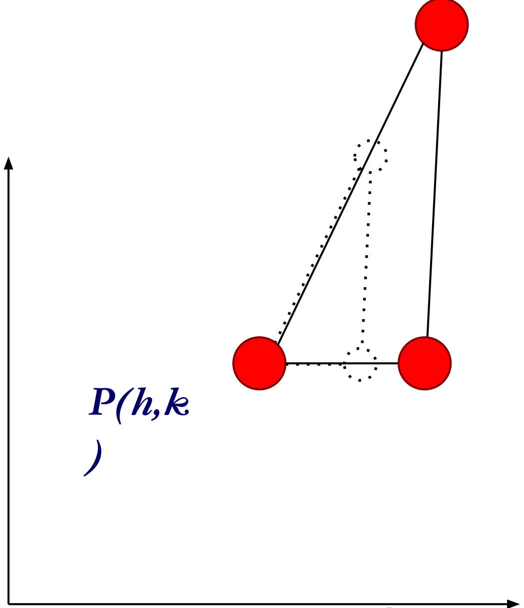
$$P' = M \times P$$

$$= \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix}$$

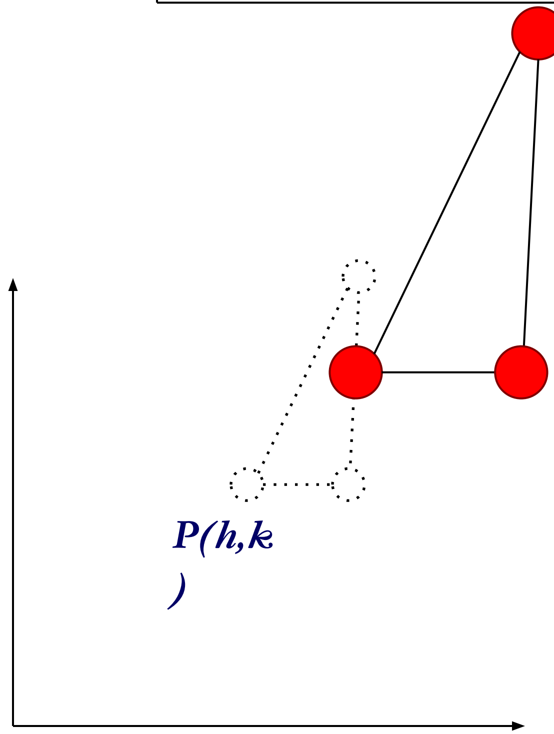
So, the new coordinate of P is (0, 10)

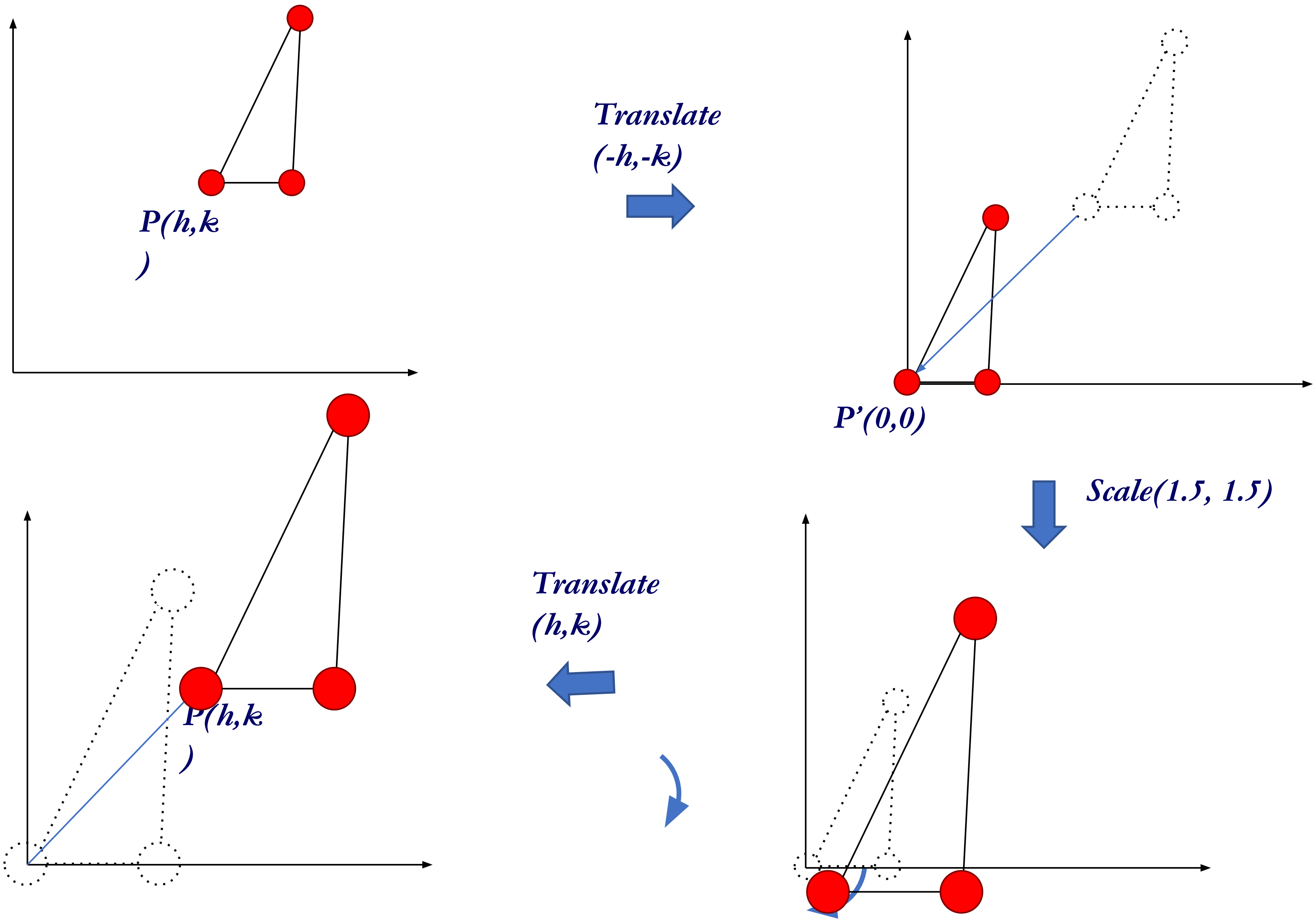




Isotropic scale by 1.5 with respect to P(h,k)

If we only apply Scale(1.5, 1.5)




Translate
(-h,-k)
P(h,k
)
P'(0,0)
Scale(1.5, 1.5)
Translate
(h,k)
P(h,k
)

$$P' = T_{(h,k)} \times S_{(a,b)} \times T_{(-h,-k)} \times P$$

$$Composite\ matrix, M = T_{(h,k)} \times S_{(a,b)} \times T_{(-h,-k)}$$

$$= \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose, a composite transformation is defined as scaling 2 times in both axis with respect to point (5, 5). Calculate the composite transformation matrix in homogenous form. Then, find the new coordinates of the point (10, 10) after transformation.

$$Composite\ matrix, M = T_{(5,5)} \times S_{(2,2)} \times T_{(-5,-5)}$$

$$= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose, a composite transformation is defined as scaling 2 times in both axis with respect to point (5, 5). Calculate the composite transformation matrix in homogenous form. Then, find the new coordinates of the point (10, 10) after transformation.

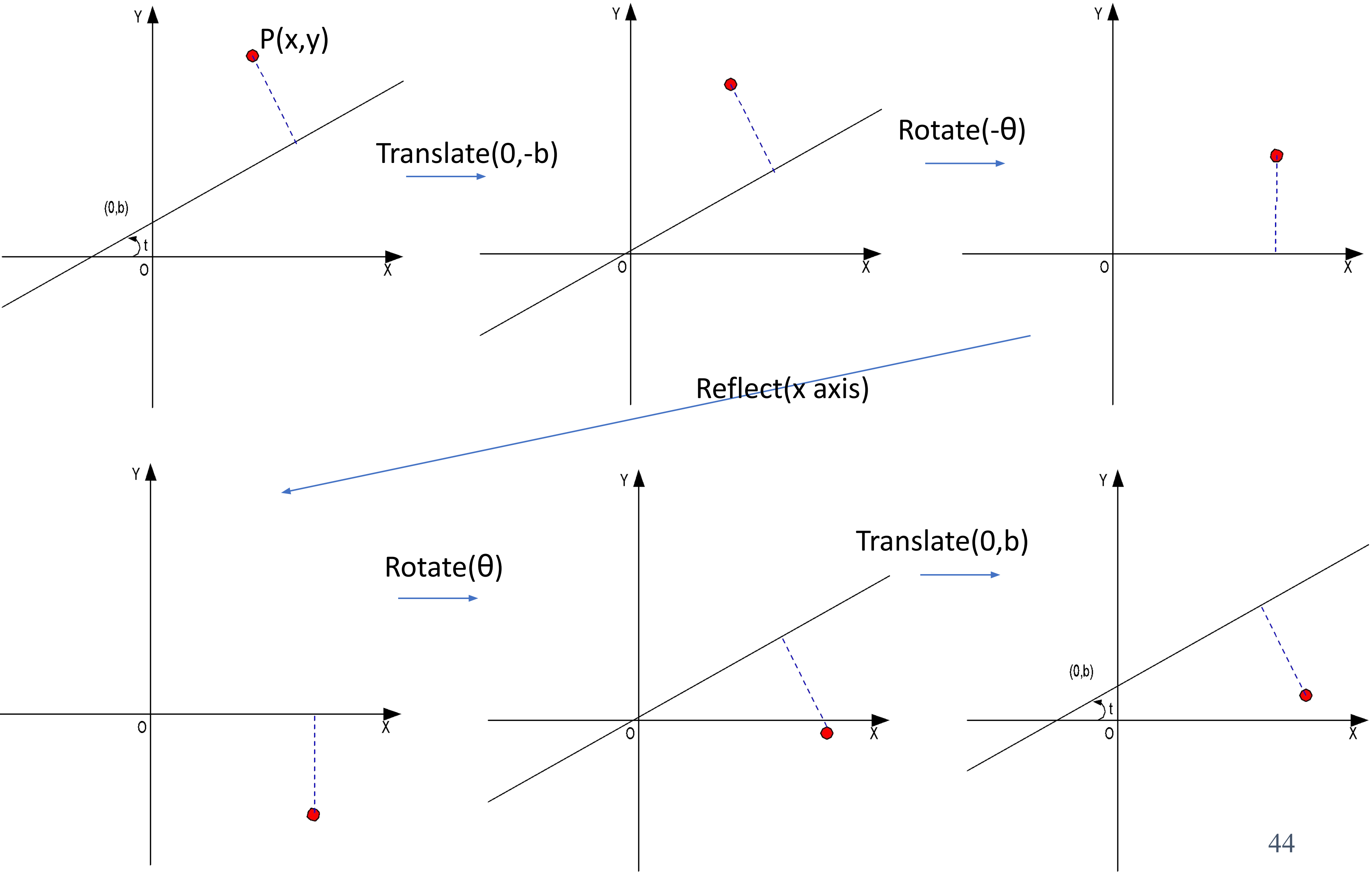
$$P' = M \times P$$

$$= \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 15 \\ 1 \end{bmatrix}$$

So, the new coordinate of P is (15, 15)

Reflection about line L, M_L



Reflection about line L, M_L

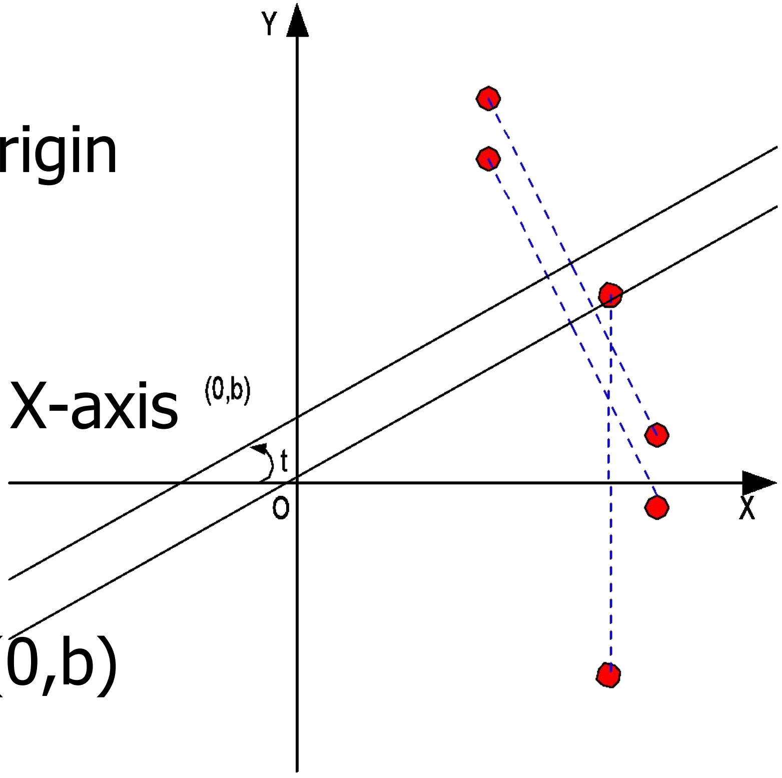
Step 1: Translate (0,b) to origin

Step 2: Rotate -θ degrees

Step 3: Mirror reflect about X-axis

Step 4: Rotate θ degrees

Step 5: Translate origin to (0,b)



$$M_L = T(0,b) * R(\theta) * M_x * R(-\theta) * T(0,-b)$$

Reflect the point (10, 5) with respect to the line y=x+2.

$$b = 2, m = 1, \theta = \tan^{-1} 1 = 45$$

$$M_{composite} = T_{(0,2)} \times R_{(45)} \times Refl_{(x\ axis)} \times R_{(-45)} \times T_{(0,-2)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 1 \end{bmatrix}$$

New coordinate (3, 12)