



Introduction to Robotics

CSE 461

Chapter 2: Lecture 6 (Forward Kinematics)

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Last Class

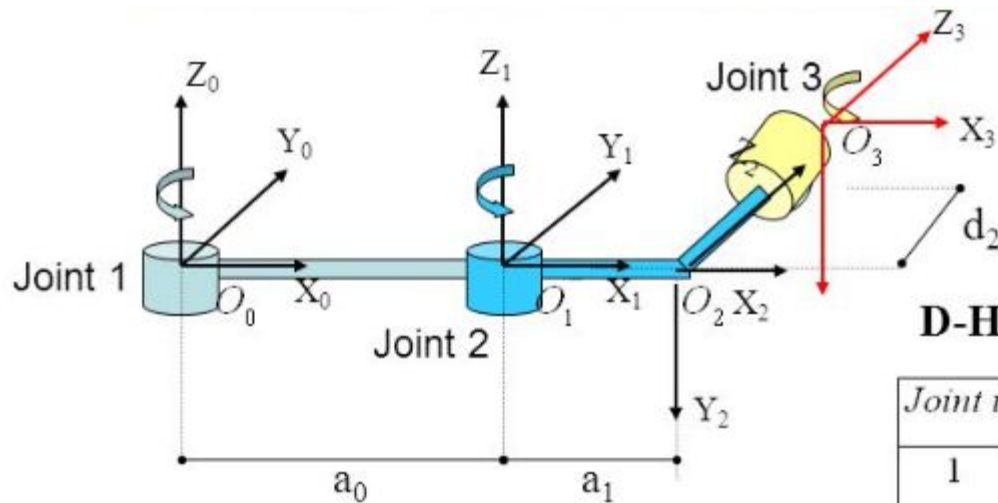
Kinematics

Degrees of Freedom (DOF)

D-H Parameters

Recall D-H Parameters

D-H Parameters



D-H Link Parameter Table

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_1
2	-90	a_1	0	θ_2
3	0	0	d_2	θ_3

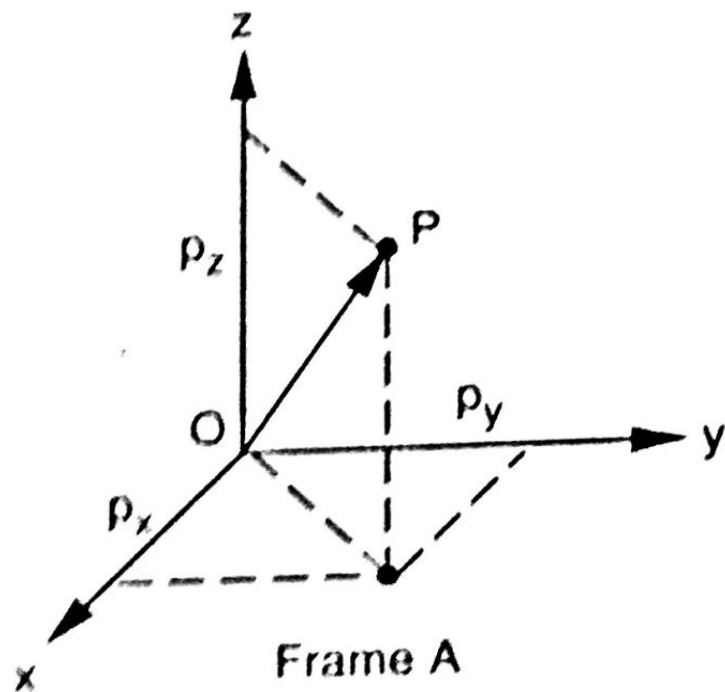
α_i : rotation angle from Z_{i-1} to Z_i about X_i

a_i : distance from intersection of Z_{i-1} & X_i to origin of i coordinate along X_i

d_i : distance from origin of $(i-1)$ coordinate to intersection of Z_{i-1} & X_i along Z_{i-1}

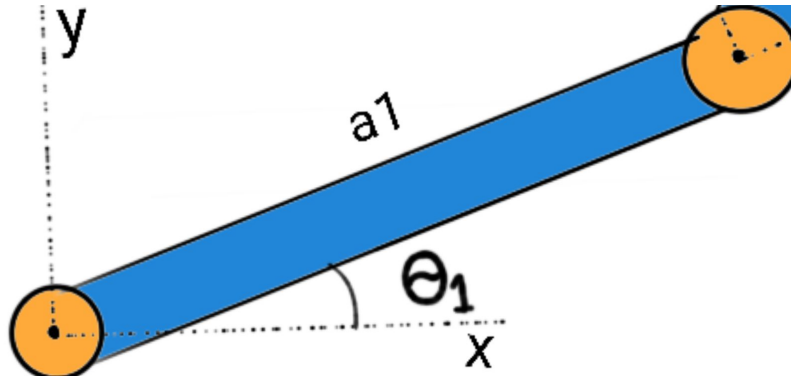
θ_i : rotation angle from X_{i-1} to X_i about Z_{i-1}

Position



$${}^A\mathbf{p} = \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}$$

Rotation

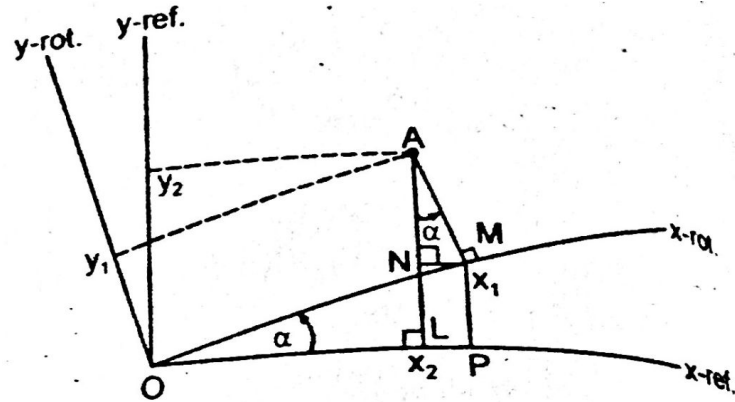


$$\text{Z-axis: } \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$\text{X-axis: } \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$\text{Y-axis: } \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Rotation



$$\begin{aligned}x_2 &= |OL| = |OP| - |LP| \\y_2 &= |LA| = |LN| + |NA| \\|OP| &= |OM| \cos \alpha \\&= x_1 \cos \alpha \\|LN| &= |PM| = |OM| \sin \alpha \\&= x_1 \sin \alpha \\|LP| &= |NM| = |MA| \sin \alpha \\&= y_1 \sin \alpha \\|NA| &= |MA| \cos \alpha \\&= y_1 \cos \alpha\end{aligned}$$

Combining these eqns., we get

$$\left. \begin{aligned}x_2 &= x_1 \cos \alpha - y_1 \sin \alpha \\y_2 &= x_1 \sin \alpha + y_1 \cos \alpha\end{aligned} \right\}$$

Homogeneous Transformation Matrix

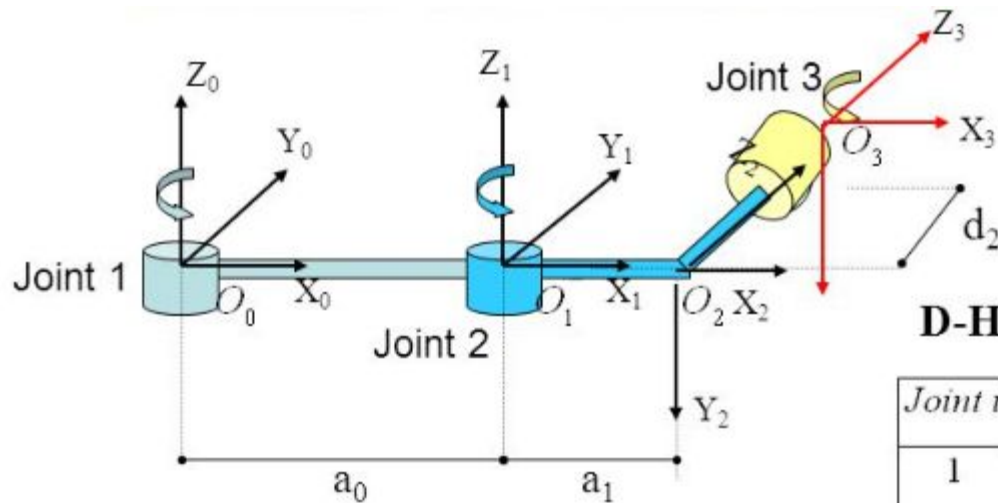
Homogeneous Transformation Matrices are commonly used to represent linkages (for robot arm) in a systematic way for representing *Position Vector* and *Rotation Matrices* {4X4 Matrix}.

$$T = \left[\begin{array}{c|c} R_{3 \times 3} & P_{3 \times 1} \\ \hline - & - \\ \hline f_{1 \times 3} & 1 \times 1 \end{array} \right] = \left[\begin{array}{c|c} \text{Rotation} & \text{Position} \\ \text{Matrix} & \text{Matrix} \\ \hline \text{-----} & \text{-----} \\ \hline \text{Perspective} & \\ \text{Transformation} & \text{Scaling} \end{array} \right]$$

Denavit-Hartenberg (D-H) Homogeneous Transformation matrices

$$\begin{aligned}
 A_i &= R_{z,\theta_i} \text{Trans}_{z,d_i} \text{Trans}_{x,a_i} R_{x,\alpha_i} \\
 &= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

D-H Parameters



D-H Link Parameter Table

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1	0	a_0	0	θ_1
2	-90	a_1	0	θ_2
3	0	0	d_2	θ_3

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a_i : distance from intersection of Z_{i-1} & X_i to origin of i coordinate along X_i

d_i : distance from origin of $(i-1)$ coordinate to intersection of Z_{i-1} & X_i along Z_{i-1}

θ_i : rotation angle from X_{i-1} to X_i about Z_{i-1}

Transformation Matrices

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_1
2	-90	a_1	0	θ_2
3	0	0	d_2	θ_3

$$T_{i-1}^i = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = (T_0^1)(T_1^2)(T_2^3)$$

$$T_0^1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & a_0 \cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & a_0 \sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

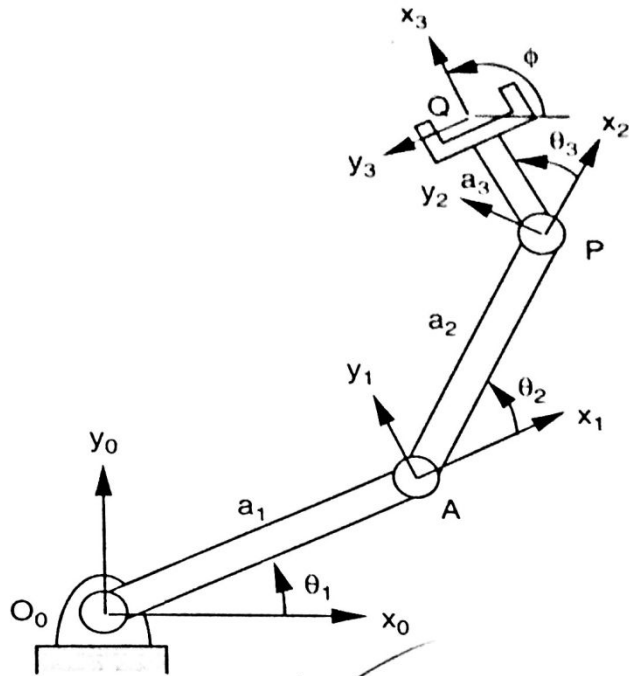
$$T_1^2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & a_1 \cos\theta_2 \\ \sin\theta_2 & 0 & \cos\theta_2 & a_1 \sin\theta_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3



$${}^0A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

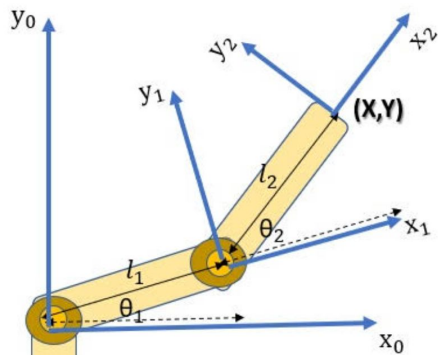
$${}^2A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Overall transformation matrix?

$${}^0A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1c\theta_1 + a_2c\theta_{12} + a_3c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1s\theta_1 + a_2s\theta_{12} + a_3s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

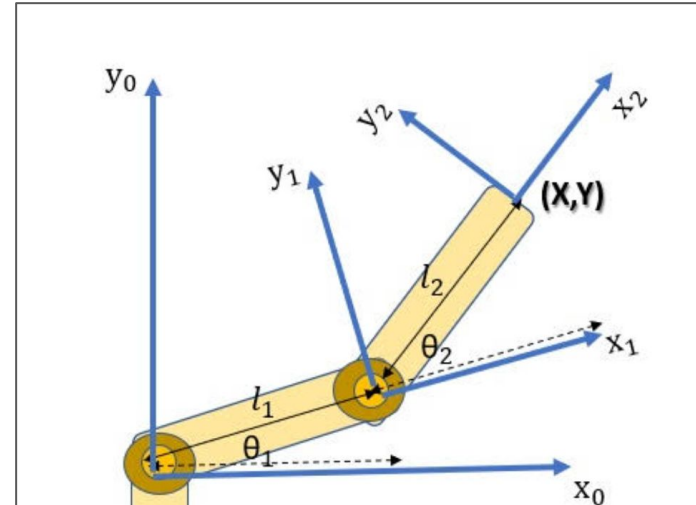
Overall transformation matrix?

$${}^0A_3 = \begin{bmatrix} \begin{matrix} \text{Orientation} \\ c\theta_{123} & -s\theta_{123} & 0 \\ s\theta_{123} & c\theta_{123} & 0 \\ 0 & 0 & 1 \end{matrix} & \begin{matrix} \text{Position} \\ a_1c\theta_1 + a_2c\theta_{12} + a_3c\theta_{123} \\ a_1s\theta_1 + a_2s\theta_{12} + a_3s\theta_{123} \\ 0 \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$



Steps

1. Define z axis
2. Define x, y using R.H rule

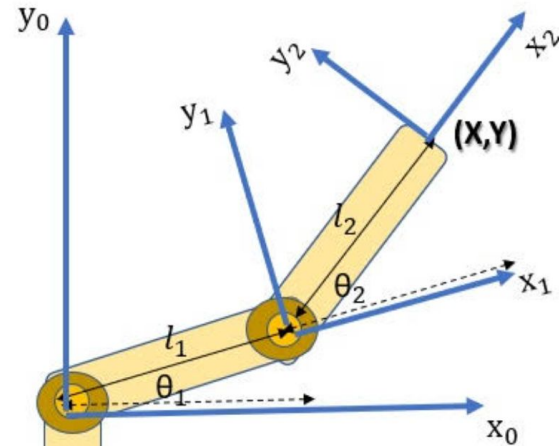


Steps

1. Define z axis
2. Define x,y using R.H rule
3. Calculate D-H parameters

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
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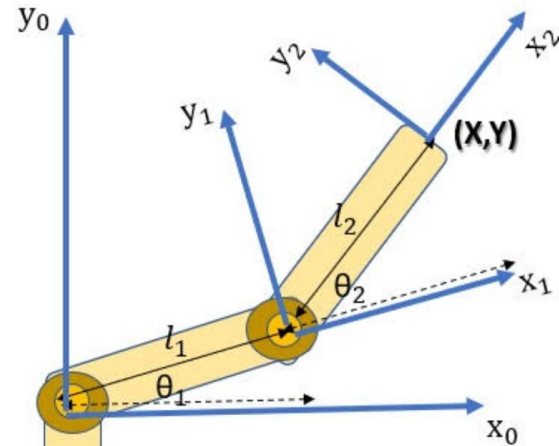
Steps

1. Define z axis
2. Define x,y using R.H rule
3. Calculate D-H parameters
4. Put the parameters to the homogeneous transformation matrices

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Steps

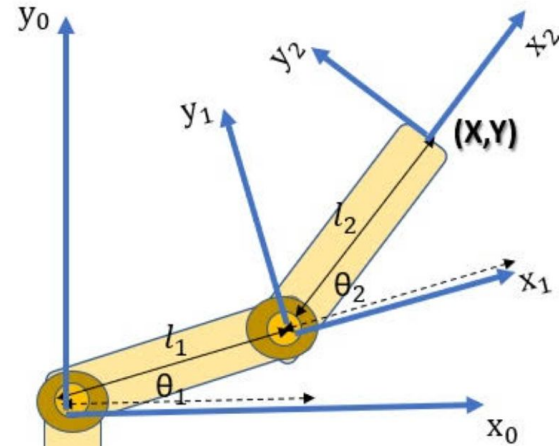
1. Define z axis
2. Define x,y using R.H rule
3. Calculate D-H parameters
4. Put the parameters to the homogeneous transformation matrices
5. Multiply for each joints

$$T_0^3 = (T_0^1)(T_1^2)(T_2^3)$$

TABLE 2.1. D-H Parameters of a 3-DOF Manipulator

Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2

$$\begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



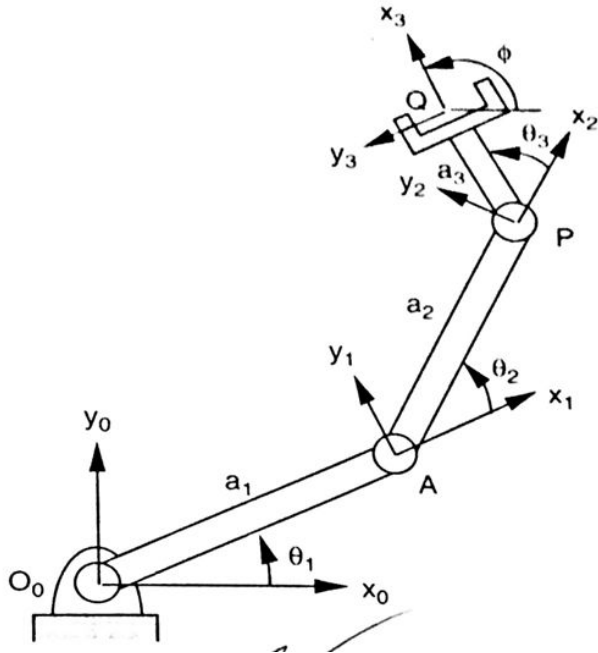
Forward Kinematics

Initial position vector : ${}^3\mathbf{q} = [0, 0, 0, 1]^T$

Final position vector?

$$\begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = {}^0A_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \\ 1 \end{bmatrix}$$

Assignment: Calculate position of end effector of this 3 DOF Manipulator



Angles of three joints are **30 degree, 45 degree and 10 degree.**

$$a_1 = 10$$

$$a_2 = 8$$

$$a_3 = 3$$

Next class

Inverse Kinematics

Thanks