Close-Proximity Maneuver with Obstacle Avoidance: In this project, you are asked to revisit the same problem of close-proximity maneuver in Project 1. However, the chaser has to avoid an obstacle, which has a cubic shape and is placed at a fixed particular location in the LVLH frame. These problems are solved by a numerical technique that involves introducing additional *binary* variables, which converts the original *Linear Programming* (LP) problem into a *Mixed-Integer Linear Programming* (MILP) problem. In addition, MILPs are not convex problems since the set of integer variables is non-convex (i.e., you can connect a line between integer values and the intermediate values lie outside the set).

Linearized Equations of Motion: Let $\boldsymbol{x} = [x, y, z, v_x, v_y, v_z]^\top \in \mathbb{R}^6$ and $\boldsymbol{u} = [T_x, T_y, T_z]^\top \in \mathbb{R}^3$ denote the state and control vectors of a spacecraft. For the problem of interest in this project, a spacecraft (chaser) is assumed to be in close proximity to a main spacecraft (chief). The most common approximation for spacecraft dynamics are the linearized Hill's equations

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x} + \mathbb{B}\boldsymbol{u}, \tag{1}$$

where $\mathbb{A}$ and $\mathbb{B}$ matrices are defined as

$$\mathbb{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbb{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix}. \tag{2}$$

Here, m denotes the spacecraft mass, which is assumed to be constant during the maneuver since the low-thrust propulsion system has a high specific impulse value. Note that the x coordinate is in the radial direction, the y coordinate is in the in-track direction, and the z component is in the out-of-plane direction (along the osculating angular momentum vector). Figure 1 depicts the scenario and the definition of the chaser and target spacecraft along with the co-moving LVLH frame. If the orbit of target vehicle is a circle, then the LVLH frame is called a Clohessy-Wiltshire (CW) frame. The relation for ω can be found in the same chapter. Under this assumption, the equations of motion of the chaser in the CW frame are given in Eq. (1).

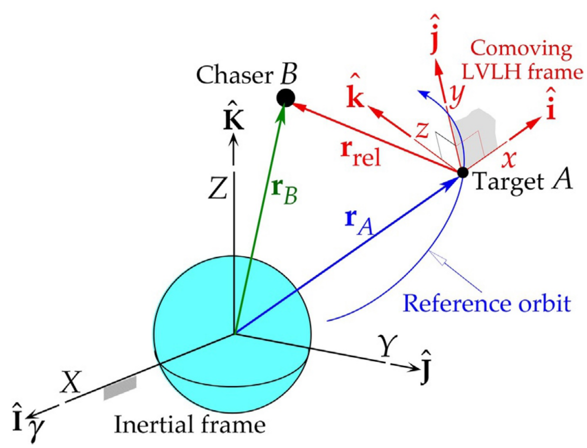

Figure 1: Position of chaser relative to the chief/target spacecraft and the definition of the Local-Vertical Local-Horizontal (LVLH) frame.

The gravitational parameter of the Earth is $\mu = 398600$ km^3/s^2 and the radius of the Earth is assumed to be $Re = 6378$ km. The chief spacecraft is located at the origin of the CW frame, i.e.,

$\boldsymbol{x}_{\text{chief}} = [0, 0, 0, 0, 0, 0]^\top$ with its circular orbit at an altitude of 550 km. The chaser with a mass of $m = 50$ kg is at a stationary position on the negative x axis of the CW frame with the given states, $\boldsymbol{x}_{\text{chaser}} = [-1 \text{ km}, 0, 0, 0, 0, 0]^\top$. The chaser has thrusters on all sides of its body that allows it to thrust along the x, y and z directions of the CW frame at a maximum value of $T_{\max} = 200$ miliNetwon. In other words, we are ignoring the rotational dynamics of the chaser, but allow independent thursting along x, y, and z coordinates of the CW frame as follows

$$\begin{aligned}
-T_{\max} &\leq T_x \leq T_{\max}, \\
-T_{\max} &\leq T_y \leq T_{\max}, \\
-T_{\max} &\leq T_z \leq T_{\max}.
\end{aligned}$$

The optimization problem that is considered for a minimum-fuel maneuver that chaser executes to rendezvous with the chief spacecraft is written as

$$\begin{aligned}
J = \text{minimize} \sum_{i=1}^{N-1} \{|T_{x,i}| + |T_{y,i}| + |T_{z,i}|\} \\
s.t.: \\
Eq.\ (1),\ \boldsymbol{x}_{\text{chief}} = \boldsymbol{0},\ \boldsymbol{x}_{\text{chaser}} = [-1 \text{ km}, 0, 0, 0, 0, 0, 0]^\top.
\end{aligned}$$

where N is the number of points used to discretize the maneuver time, t_f. One interpretation of the considered cost is to minimize the sum of absolute thrust values in each direct at each instant of the discretized points. Note that the thrust value will be fixed until the next discrete point. This is referred to as a *zero-order hold* control. In this problem, the time of fight is fixed at $t_f = 30$ minutes and $N = 100$.

Obstacle Avoidance Constraint: In order to avoid static obstacles (and also for collision avoidance of multiple spacecraft), additional constraints have to be introduced to the linear program, which makes the problem an MILP. As the name implies, part of the decision variables will be integer, and in the case of obstacle avoidance, we deal with binary variables, i.e., variables each taking only either zero or one values, i.e., $\eta \in \{0, 1\}$. In fact, the resulting optimization problems are sometimes referred to as "mixed binary integer programs".

The formulation of the problem and introduction of the new variables is explained through an example. Remark: In general and for simplicity, obstacles can be modeled as convex polygons of any number of sides, but, to even further simplify the problem, we use cubes for 3D problems (or rectangles for 2D problems). Here, "obstacle" refers to any region that the chaser must avoid. Collisions are prevented by ensuring that the trajectories lie outside the obstacles at each of the discrete time points. For a 2D motion, the constraint that the spacecraft does not hit an obstacle can be represented as

$$\begin{aligned}
x(t) &\leq x_{\min}, \\
or\ \ x(t) &\geq x_{\max}, \\
or\ \ y(t) &\geq y_{\max}, \\
or\ \ y(t) &\leq y_{\min}.
\end{aligned}$$

The above four constraints define a region outside the obstacle. At each point along the solution, these four constraints have to be satisfied in order to have a feasible solution. The questions is: How can we represent the above constraints using binary variables? Note that implementation of the above constraints using "or" is not possible. We need to convert the above set of constraints into "and" constraints. The representation is achieved through the following method.

The "Big-M" Method: In this method, we first introduce a constant large value, M, which is greater than the largest value that any of the states can take. Then, for the k-th point along the trajectory ($k = 1, \cdots, N$) the above constraints are written as

$$\begin{aligned} x(t_k) &\le x_{\min} + \eta_{1,k} M, \\ \textit{and} \quad -x(t_k) &\le -x_{\max} + \eta_{2,k} M, \\ \textit{and} \quad -y(t_k) &\le -y_{\max} + \eta_{3,k} M, \\ \textit{and} \quad y(t_k) &\le y_{\min} + \eta_{4,k} M, \\ \sum_{i=1}^{4} \eta_{i,k} \le 3 \rightarrow \eta_{1,k} + \eta_{2,k} + \eta_{3,k} &+ \eta_{4,k} \le 3, \end{aligned}$$

where a set of binary variables, $\eta_{i,k}$ (for $i = 1, \cdots, 4$ and $k = 1, \cdots, N$), is added to the problem. For instance, at $t = t_k$, if $\eta_{1,k} = 1$, it means that the first inequality will always be satisfied since M is selected to be large such that the first equation becomes $x(t_k) \le M$, which is always true by construction. Since M is a large value, it dominates the other term in the right-hand side of the inequality. However, if $\eta_{1,k} = 0$, it means that the first inequality will become active. Thus, the value of the binary variable determines the activation/relaxation of the respective constraint. However, we should ensure that not all of the constraints are inactive (or relaxed); otherwise, we have not solved the problem! Thus, the last equation on the sum of the binary values (at each time step) guarantees that, at least, one constraint is *always* active. Put it another way, no more than three constraints are relaxed. The same technique can be used for the spacecraft problem with its trajectory in the 3D space.

The MILP problems are typically solved by a combination of *Branch and Bound* techniques in which the integer-valued variables are relaxed into continuous variables (just as they are in the LP problems). In other words, initially, we ignore the fact that they can only take integer values. Watch minute 23 of this video from MATHWORKS, where a high-level discussion of the solution methodology is presented:
`https://www.mathworks.com/videos/mixed-integer-linear-programming-in-matlab-91541.html`.

Tasks: Figure 2 shows the optimal trajectory of the Project 1, which passes through an obstacle. Use the "Big-M" method and solve the minimum-fuel problem while avoiding an obstacle with the feasible region defined as

$$\begin{aligned} x(t) &\leq x_{\text{min}} = -0.6 \text{ km}, \\ \textit{or} \quad x(t) &\geq x_{\text{max}} = -0.5 \text{ km}, \\ \textit{or} \quad y(t) &\geq y_{\text{max}} = 0.55 \text{ km}, \\ \textit{or} \quad y(t) &\leq y_{\text{min}} = 0.45 \text{ km}, \\ \textit{or} \quad z(t) &\geq z_{\text{max}} = 0.01 \text{ km}, \\ \textit{or} \quad z(t) &\leq z_{\text{min}} = -0.01 \text{ km}. \end{aligned}$$

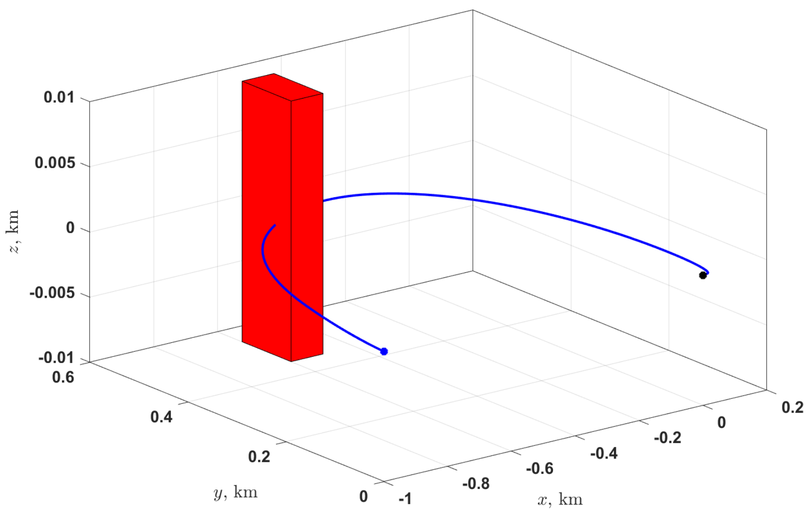

Figure 2: Optimal trajectory of Project 1 that leads to an infeasible solution since it collides with the obstacle.

What to Report: Your report should be typeset in Overleaf and your final report has to be in a pdf format. In addition, you should upload your code as a zip file with its main function named as 'MainObsAvoidMILP-YourName.m'.

Item 1: Summarize the key steps in formulating the MILP problem. Solve the resulting MILP problem using the *problem-based* approach, i.e., using the `solve` command with the optimization algorithm set to `intlinprog` with N = 100.

Item 2: Is it possible for the trajectory to cut into the obstacle even if the solver returns a feasible solution? If yes, elaborate about it.

Item 3: Include a figure that overlays the $x - y$ view of the trajectory from project 1 (w/o the obstacle) and the new solution (in the presence of the obstacle). Include a figure with 3 sub-plots that overlay the time histories of thrust components, i.e., T_x, T_y and T_z. Use a different line style or color to distinguish the solutions.

Item 4: What happens if you set $y_{\text{min}} = 0.35$ km instead of $y_{\text{min}} = 0.45$ km?

Item 5: Conclusion/Summary: Summarize the main challenges you faced in this project and what you learned from this assignment. Report the total number of hours spent on this project.