

AVL Trees

COMP 282

Mahdi Ebrahimi

Slide Courtesy : ilhelm HarDouglas Wder, MMath, UWaterloo

`dwharder@alumni.uwaterloo.ca`

Back to Binary Search Trees

Building BST

Let's consider building a BST (insert values starting from an empty tree)

Insert values 1, 2, 3, 4, 5, 6, 7, 8, 9 into an empty BST

- If inserted in given order, what is the tree?
- What big-O runtime for building the BST on this sorted input?
- Is inserting in the reverse order any better?

Building BST

Insert values 1, 2, 3, 4, 5, 6, 7, 8, 9 into an empty BST

What we if could somehow re-arrange them

- median first, then left median, right median, etc.
5, 3, 7, 2, 1, 4, 8, 6, 9
- What tree does that give us?
- What big-O runtime?

Balancing Binary Search Trees

BST: Efficiency of Operations?

Problem:

Worst-case running time:

- find, insert, delete

How can we make a BST efficient?

Observation

Solution: Require a **Balance Condition** that

- When we **build** the tree, make sure it's balanced.
- **BUT**...Balancing a tree **only** at build time is insufficient.
- We also need to also **keep** the tree balanced as we perform operations.

- Binary search trees store linearly ordered data
- Best case height: $O(\log(n))$
- Worst case height: $O(n)$

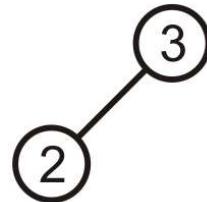
Requirement:

- Define and maintain a *balance* to ensure $O(\log(n))$ height

https://en.wikipedia.org/wiki/Self-balancing_binary_search_tree

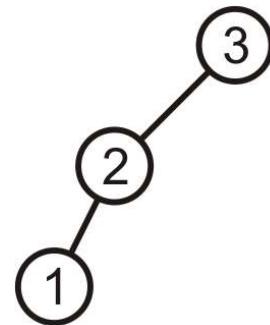
Prototypical Examples

These two examples demonstrate how we can correct for imbalances: starting with this tree, add 1:



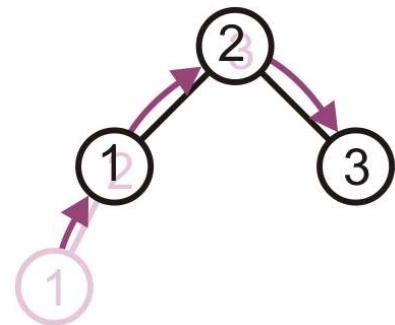
Prototypical Examples

This is more like a linked list; however, we can fix this...



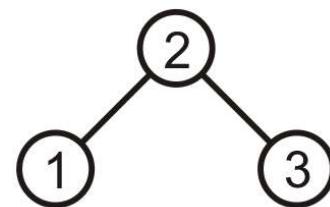
Prototypical Examples

Promote 2 to the root, demote 3 to be 2's right child, and 1 remains the left child of 2



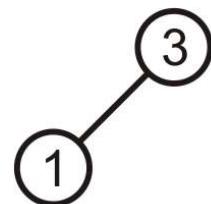
Prototypical Examples

The result is a perfect, though trivial tree



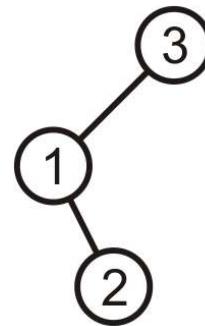
Prototypical Examples

Alternatively, given this tree, insert 2



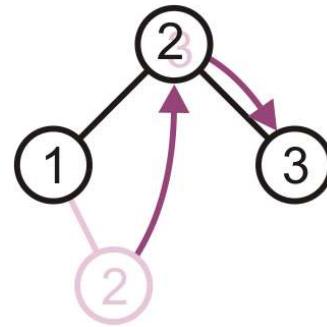
Prototypical Examples

Again, the product is a linked list; however, we can fix this, too



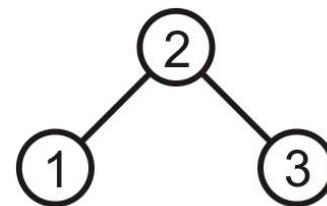
Prototypical Examples

Promote 2 to the root, and assign 1 and 3 to be its children



Prototypical Examples

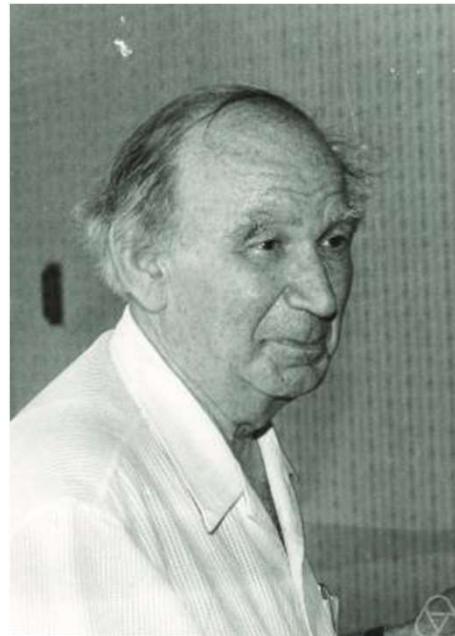
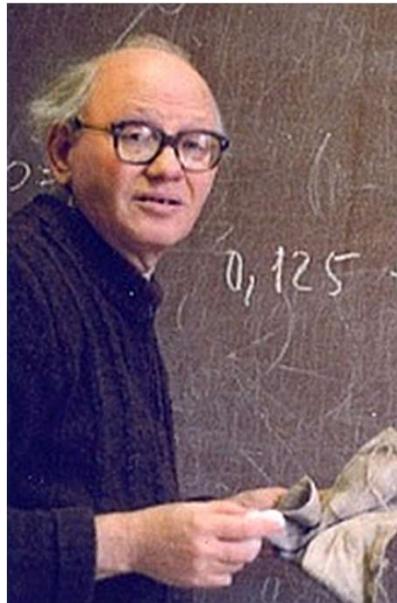
The result is, again, a perfect tree



These examples may seem trivial, but they are the basis for the corrections in the next data structure we will see: AVL trees

AVL Tree (Bonus material: etymology)

Invented by Georgy **A**delson-**V**elsky and Evgenii **L**andis in 1962



AVL Trees

Balance is defined by comparing the **height** of the two sub-trees

Recall:

- An empty tree has height -1
- A tree with a single node has height 0

AVL Trees

A binary search tree is said to be AVL balanced if:

- The difference in the heights between the left and right sub-trees is at most 1, and
- Both sub-trees are themselves AVL trees

Definition for Balance

We must develop a quantitative definition of *balance* which can be applied

Balanced may be defined by:

- *Height balancing*: comparing the heights of the two sub trees
- *Null-path-length balancing*: comparing the null-path-length of each of the two sub-trees (the length to the closest null sub-tree/empty node)
- *Weight balancing*: comparing the number of null sub-trees in each of the two sub trees

Mathematically it can proved that if a tree satisfies the definition of balance, its height is $O(\ln(n))$

The AVL Tree Data Structure

An **AVL tree** is a *self-balancing* binary search tree.

Structural properties

1. **Binary tree** property (same as BST)
2. **Order** property (same as for BST)
3. **Balance condition:**
balance of every node is between -1 and 1 (-1, 0 , 1)

where **balance(node/factor)** = height(*node.left*) – height(*node.right*)

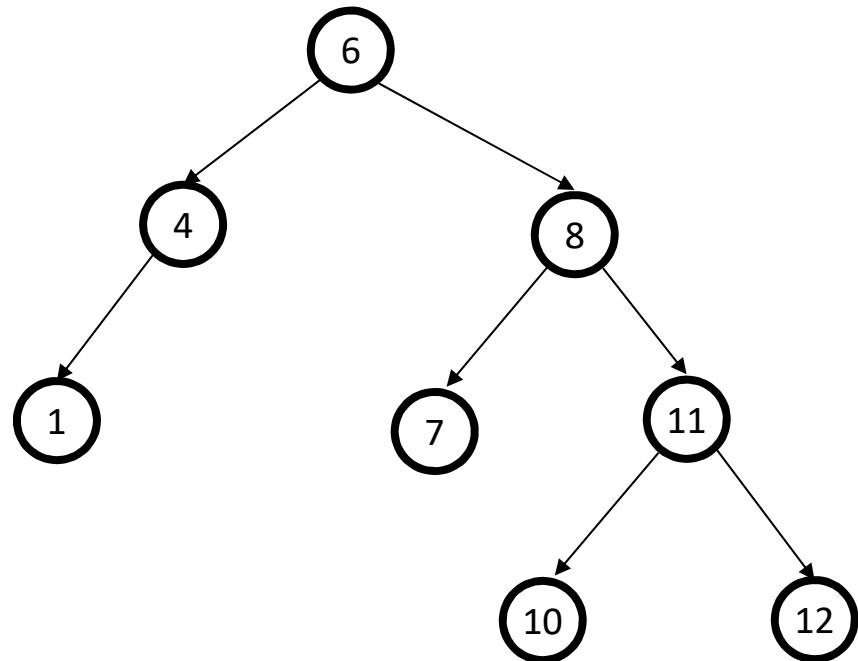
https://en.wikipedia.org/wiki/Self-balancing_binary_search_tree

Example #1: Is this an AVL Tree?

Balance Condition:

balance of every node is between -1 and 1

where $\text{balance}(\text{node}) = \text{height}(\text{node.left}) - \text{height}(\text{node.right})$

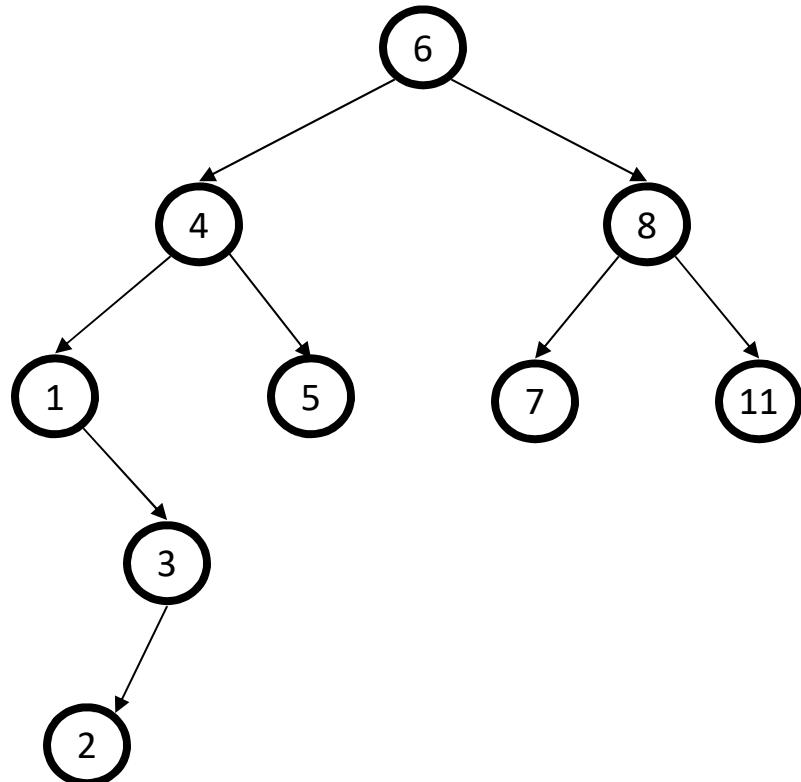


Example #2: Is this an AVL Tree?

Balance Condition:

balance of every node is between -1 and 1

where $\text{balance}(\text{node}) = \text{height}(\text{node.left}) - \text{height}(\text{node.right})$



AVL Trees

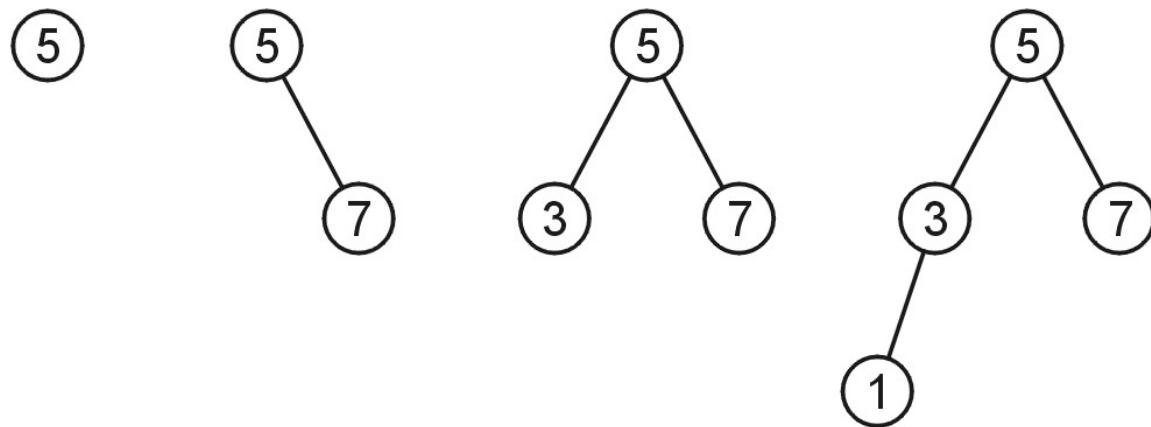
Good News:

Because height of AVL tree is $O(\log(n))$, then find

But as we insert and delete elements, we need to:

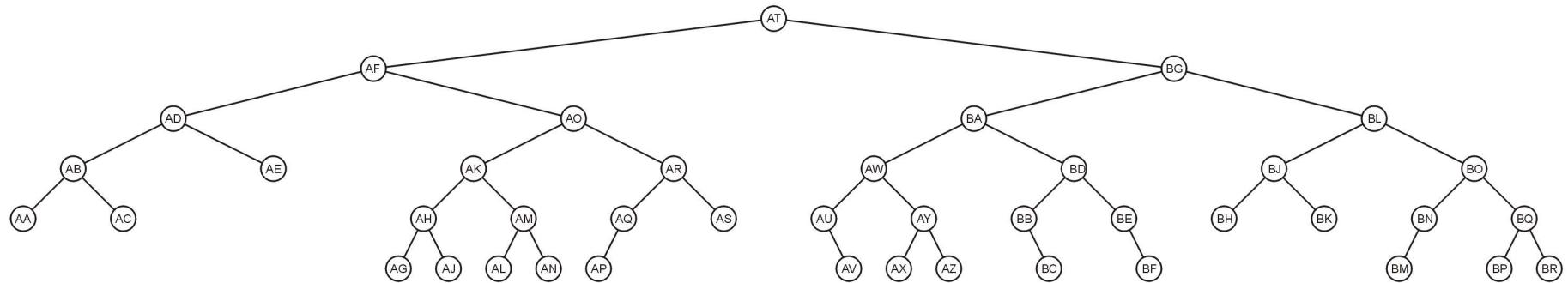
AVL Trees

AVL trees with 1, 2, 3, and 4 nodes:



AVL Trees

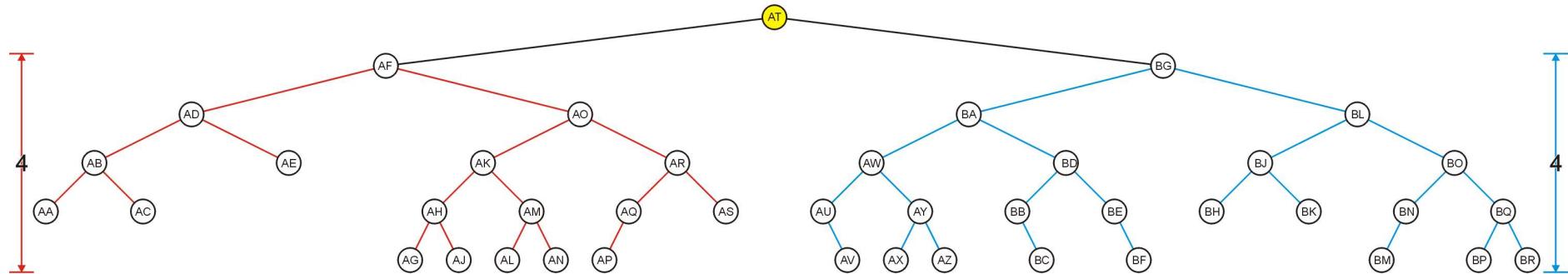
Here is a larger AVL tree (42 nodes):



AVL Trees

The root node is AVL-balanced:

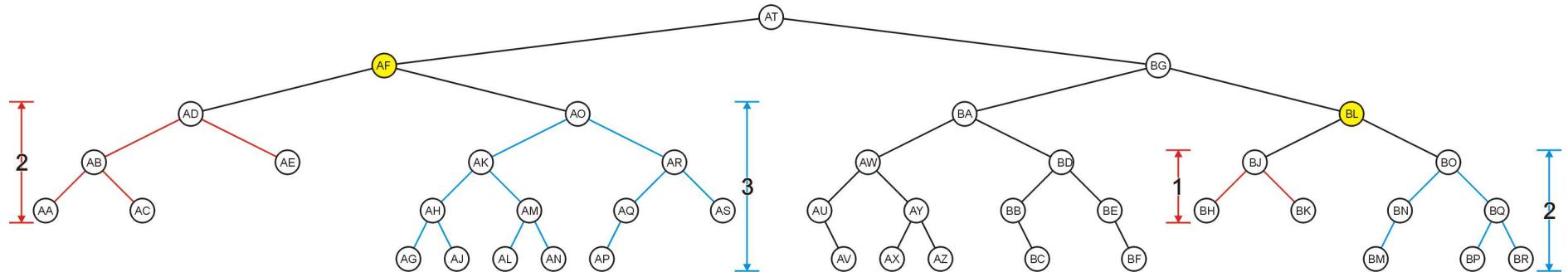
- Both sub-trees are of height 4:



AVL Trees

All other nodes are AVL balanced

- The sub-trees differ in height by at most one



Height of an AVL Tree

By the definition of complete trees, any complete binary search tree is an AVL tree

Thus an upper bound on the number of nodes in an AVL tree of height h a perfect binary tree with $2^{h+1} - 1$ nodes

– What is an lower bound?

Height of an AVL Tree

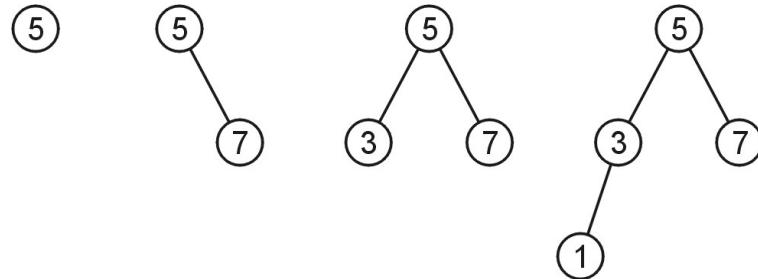
Let $F(h)$ be the fewest number of nodes in a tree of height h

From a previous slide:

$$F(0) = 1$$

$$F(1) = 2$$

$$F(2) = 4$$



Can we find $F(h)$?

Height of an AVL Tree

The worst-case AVL tree of height h would have:

- A worst-case AVL tree of height $h - 1$ on one side,
- A worst-case AVL tree of height $h - 2$ on the other, and
- The **root** node

We get: $F(h) = F(h - 1) + 1 + F(h - 2)$

Height of an AVL Tree

This is a recurrence relation:

$$F(h) = \begin{cases} 1 & h = 0 \\ 2 & h = 1 \\ F(h - 1) + F(h - 2) + 1 & h > 1 \end{cases}$$

The solution?

Fibonacci Numbers

$$F_0 = 0$$

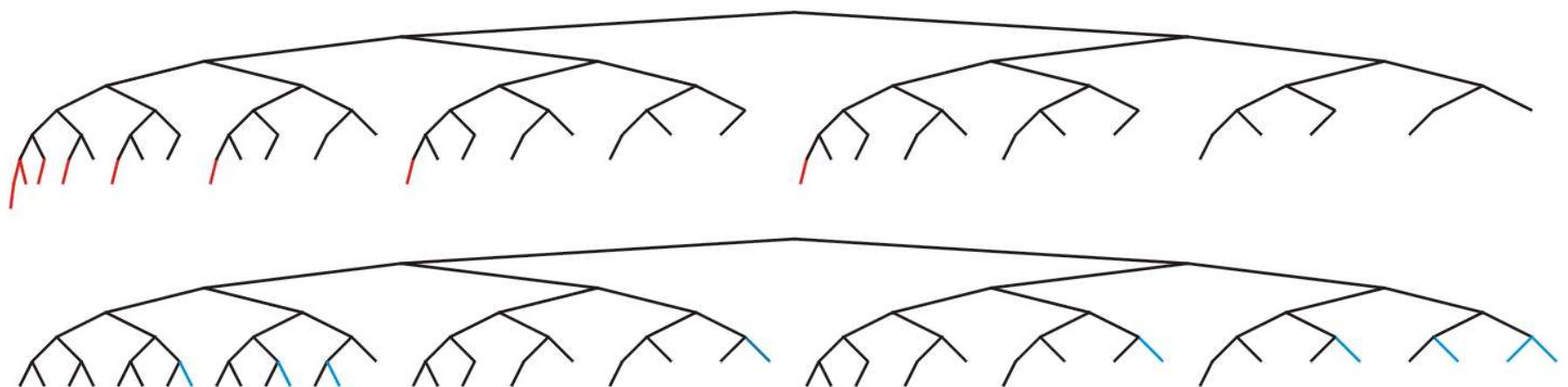
$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}, n > 1$$

$$F(n) = \frac{\frac{1+\sqrt{5}}{2}^n - \frac{1-\sqrt{5}}{2}^n}{\sqrt{5}}$$

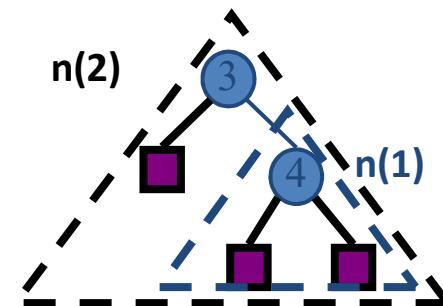
Height of an AVL Tree

In this example, $n = 88$, the worst- and best-case scenarios differ in height by only 2



Height of an AVL Tree

- **Fact:** The *height* of an AVL tree storing n keys is $O(\log n)$.
- **Proof:** Let us bound $n(h)$: the minimum number of internal nodes of an AVL tree of height h .
- We easily see that $n(1) = 1$ and $n(2) = 2$
- For $n > 2$, an AVL tree of height h contains the root node, one AVL subtree of height $h-1$ and another of height $h-2$.
- That is, $n(h) = 1 + n(h-1) + n(h-2)$
- Knowing $n(h-1) > n(h-2)$, we get $n(h) > 2n(h-2)$. So
 - $n(h) > 2n(h-2), n(h) > 4n(h-4), n(h) > 8n(h-6), \dots$ (by induction),
 - $n(h) > 2^i n(h-2i)$
- Solving the base case we get: $n(h) > 2^{h/2-1}$
- Taking logarithms: $h < 2\log n(h) + 2$
- Thus the height of an AVL tree is $O(\log n)$



AVL tree operations

- **AVL find:**
 - Same as usual BST find
- **AVL insert:**
- **AVL delete:**
 - The “easy way” is lazy deletion
 - Otherwise, do the deletion and then check for several imbalance cases (we will skip this)

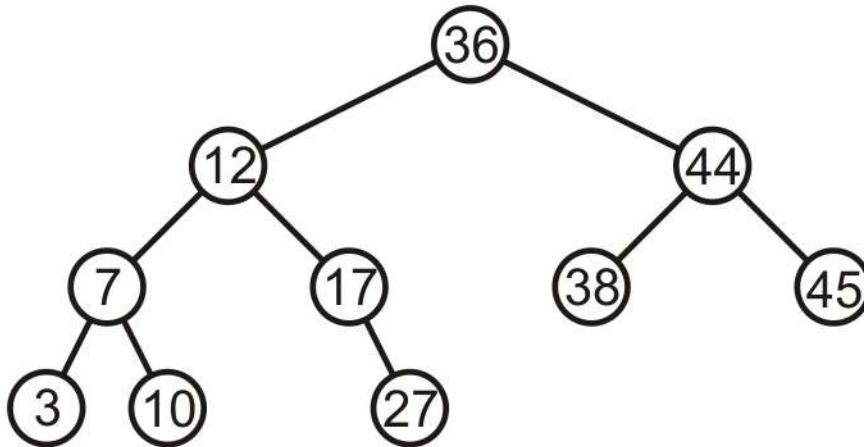
Maintaining Balance

To maintain AVL balance, observe that:

- Inserting a node can increase the height of a tree by at most 1
- Removing a node can decrease the height of a tree by at most 1

Maintaining Balance

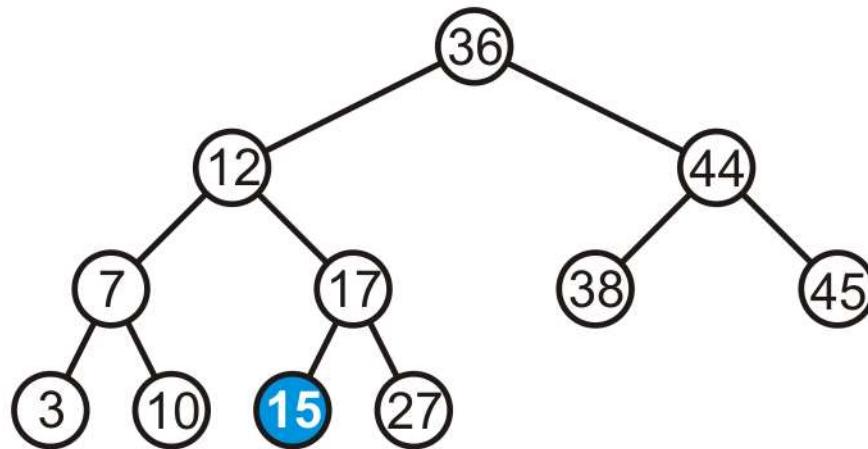
Consider this AVL tree



Maintaining Balance

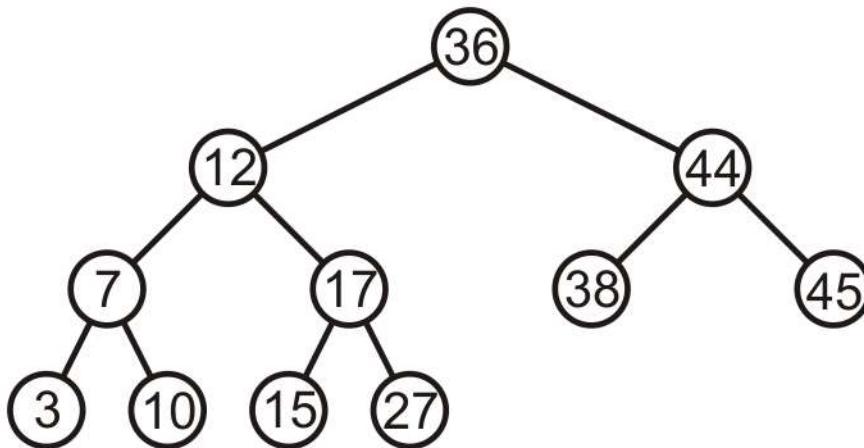
Consider inserting 15 into this tree

- In this case, the heights of none of the trees change



Maintaining Balance

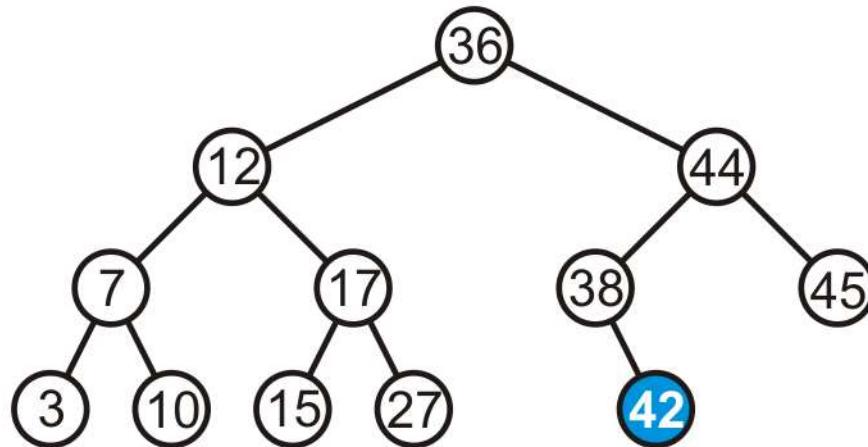
The tree remains balanced



Maintaining Balance

Consider inserting 42 into this tree

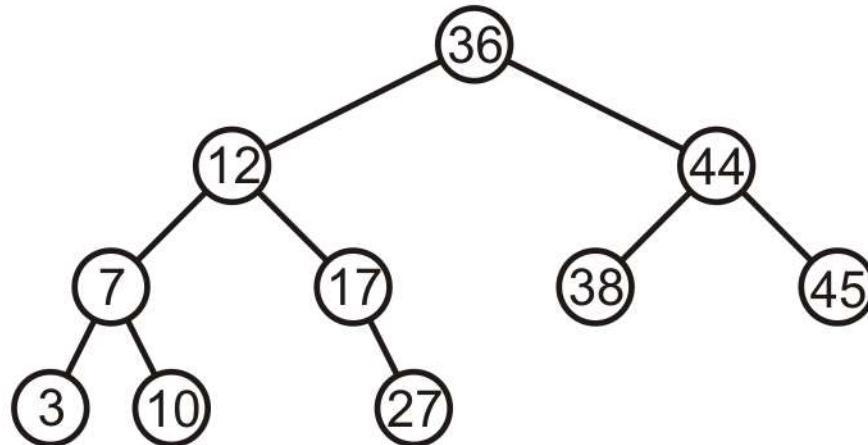
- In this case, the heights of none of the trees change



Maintaining Balance

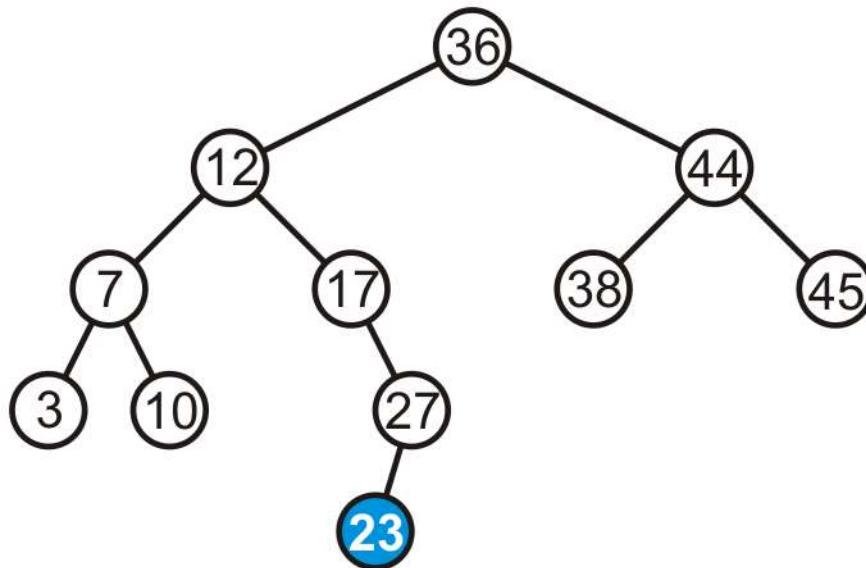
If a tree is AVL balanced, for an insertion to cause an imbalance:

- The heights of the sub-trees must differ by 1
- The insertion must increase the height of the deeper sub-tree by 1



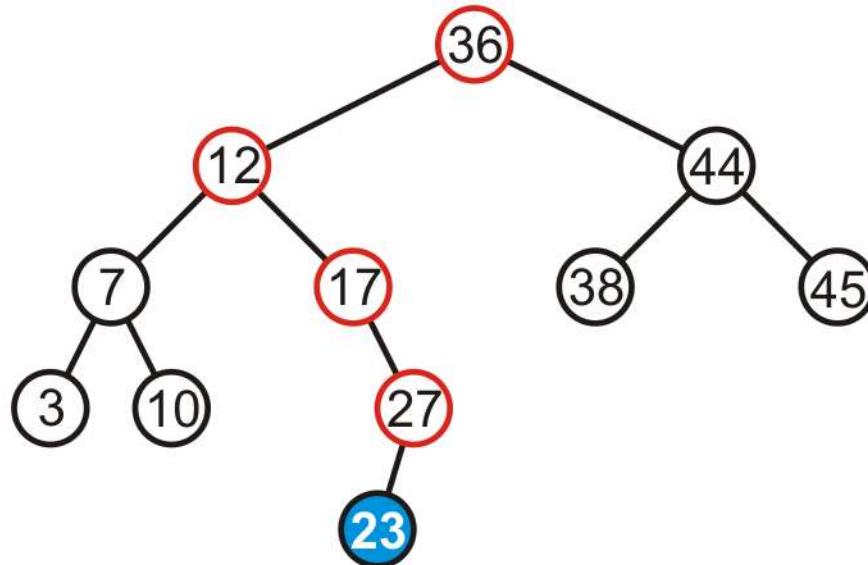
Maintaining Balance

Suppose we insert 23 into our initial tree



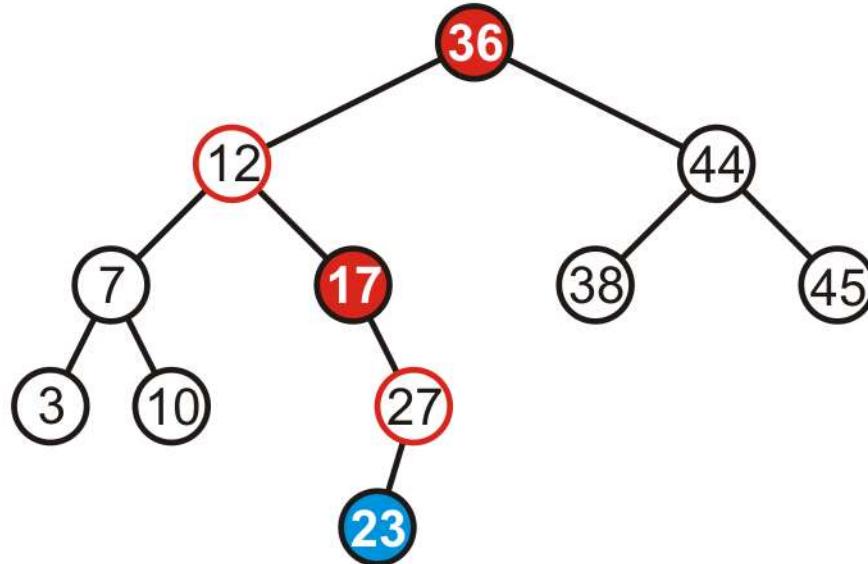
Maintaining Balance

The heights of each of the sub-trees from here to the root are increased by one



Maintaining Balance

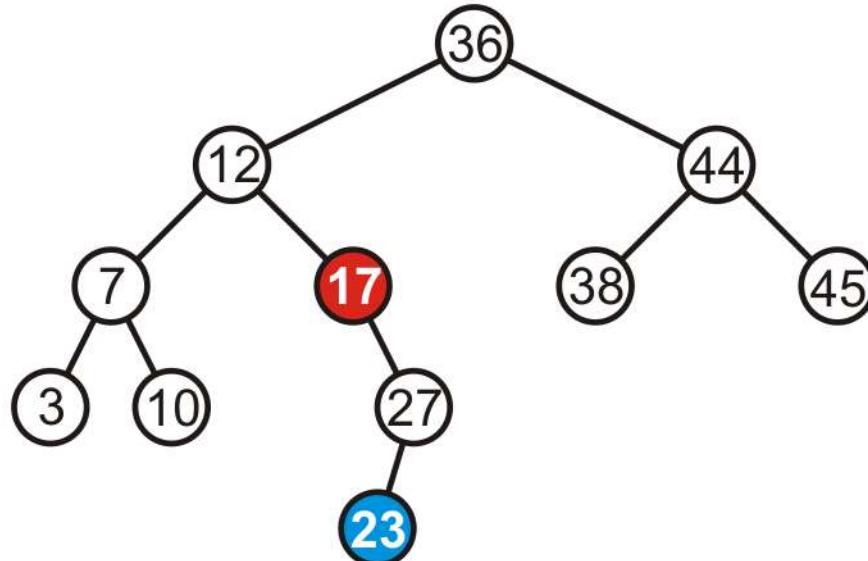
However, only two of the nodes are unbalanced: 17 and 36



Maintaining Balance

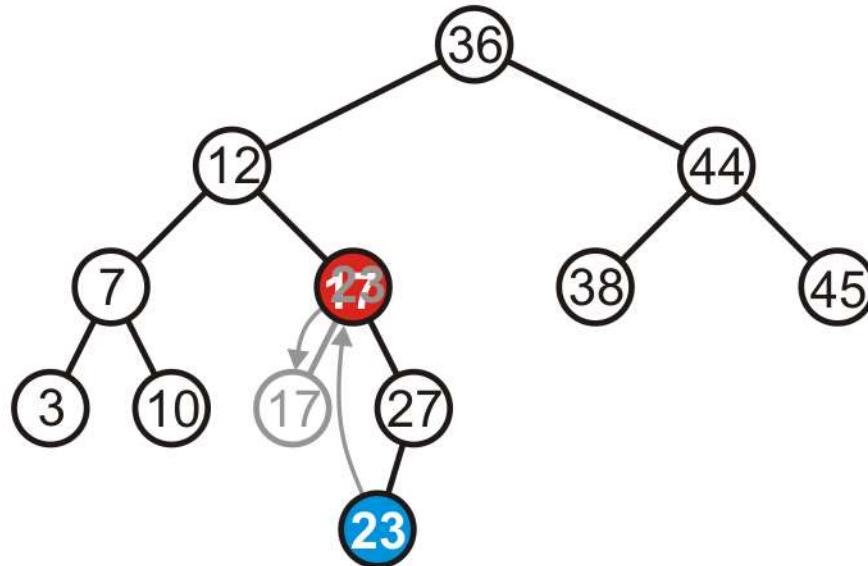
However, only two of the nodes are unbalanced: 17 and 36

- We only have to fix the imbalance at the lowest node



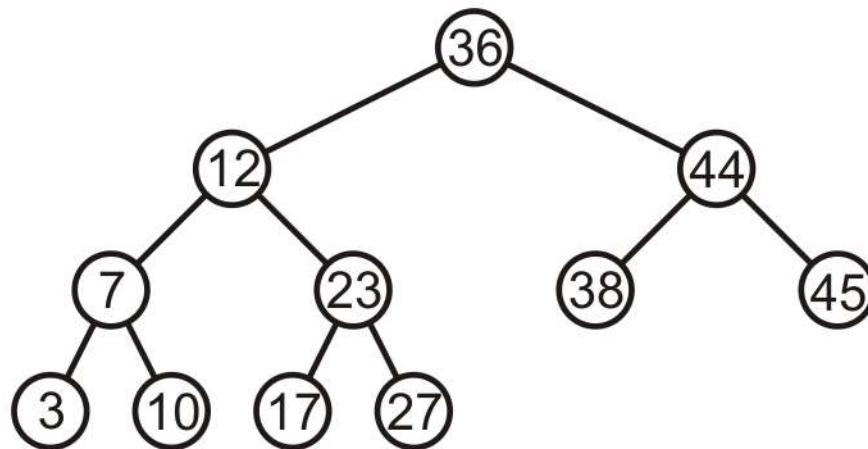
Maintaining Balance

We can promote 23 to where 17 is, and make 17 the left child of 23



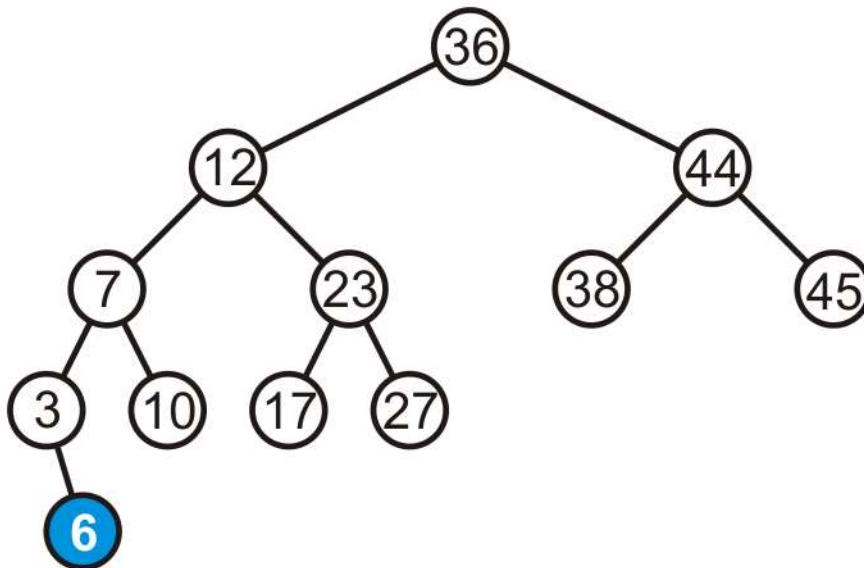
Maintaining Balance

Thus, that node is no longer unbalanced
– Incidentally, neither is the root now balanced again, too



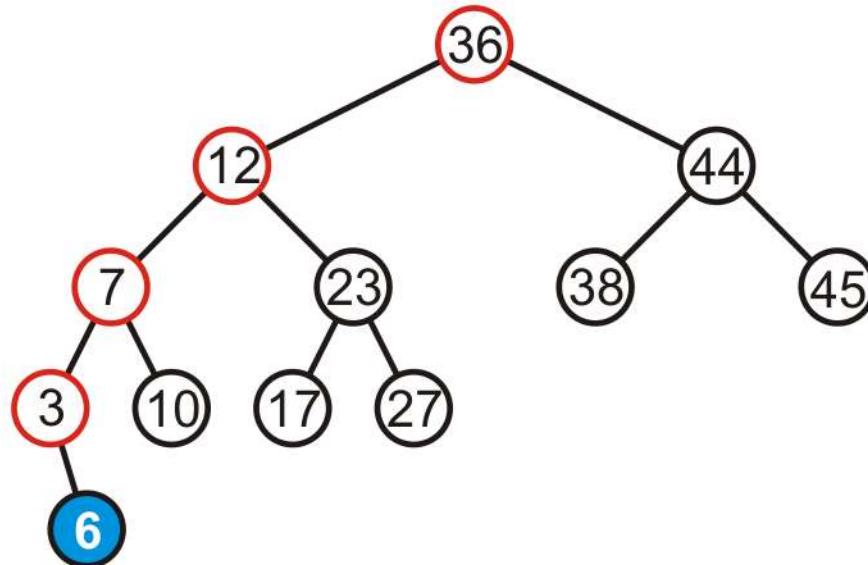
Maintaining Balance

Consider adding 6:



Maintaining Balance

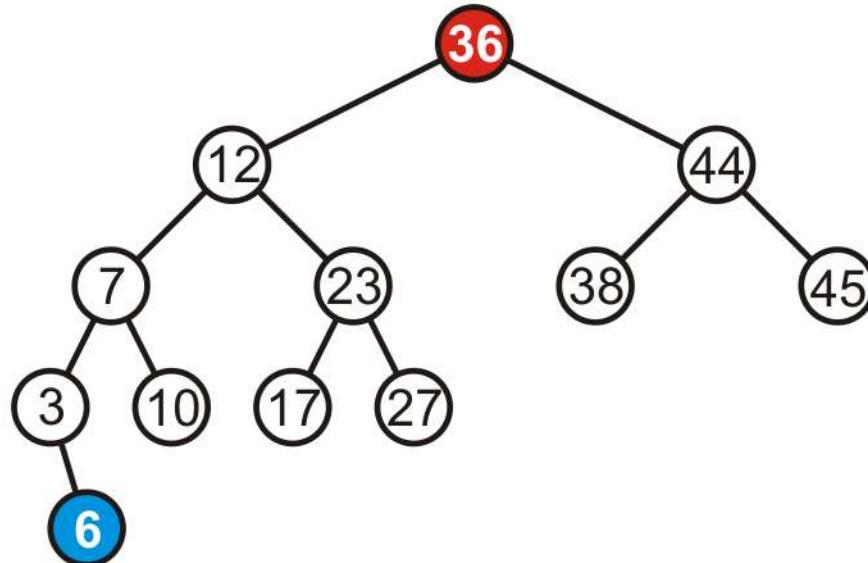
The height of each of the trees in the path back to the root are increased by one



Maintaining Balance

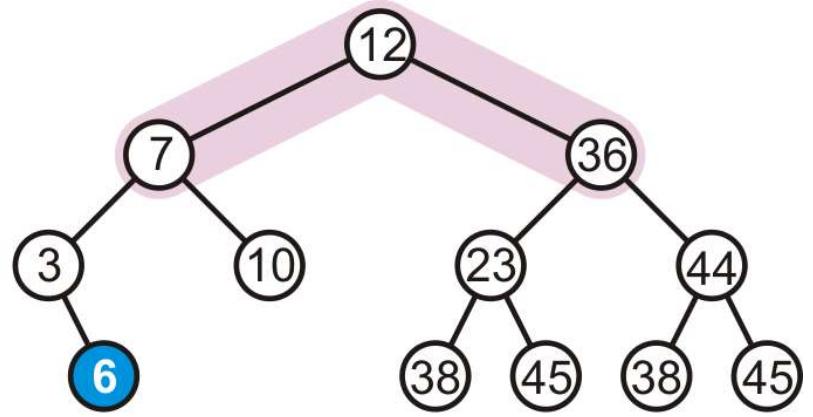
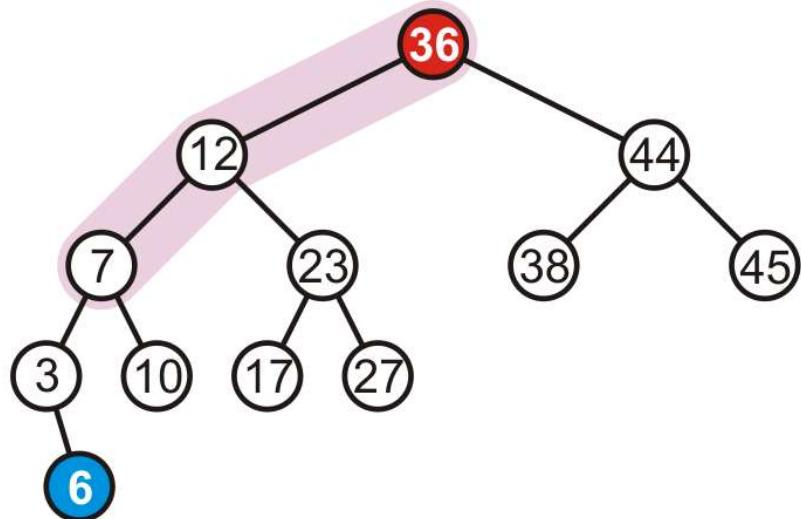
The height of each of the trees in the path back to the root are increased by one

- However, only the root node is now unbalanced



Maintaining Balance

We may fix this by rotating the root to the right

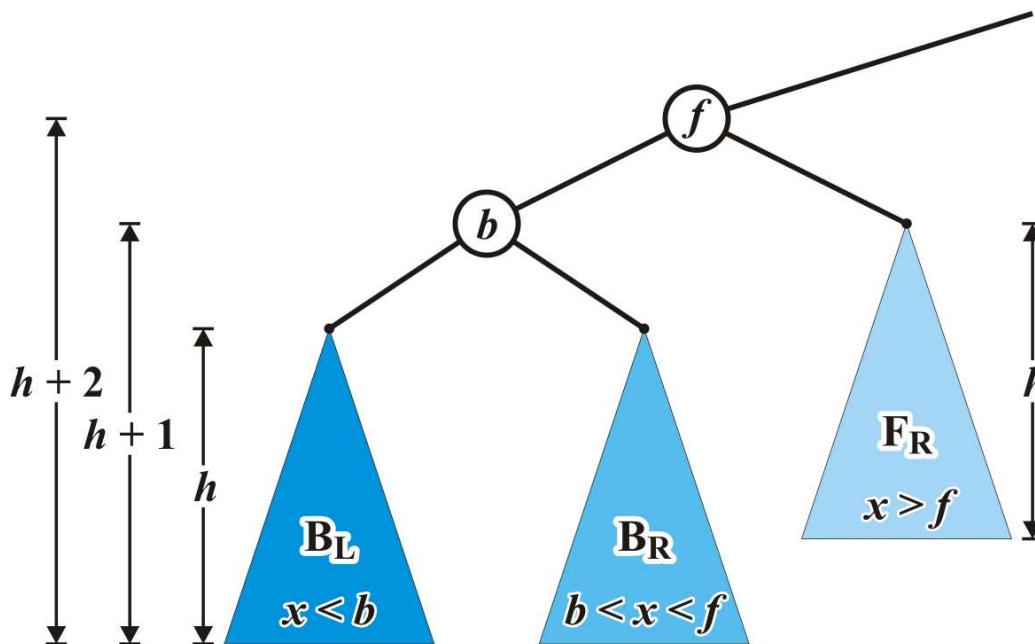


Note: the right subtree of 12 became the left subtree of 36

Case 1 setup

Consider the following setup

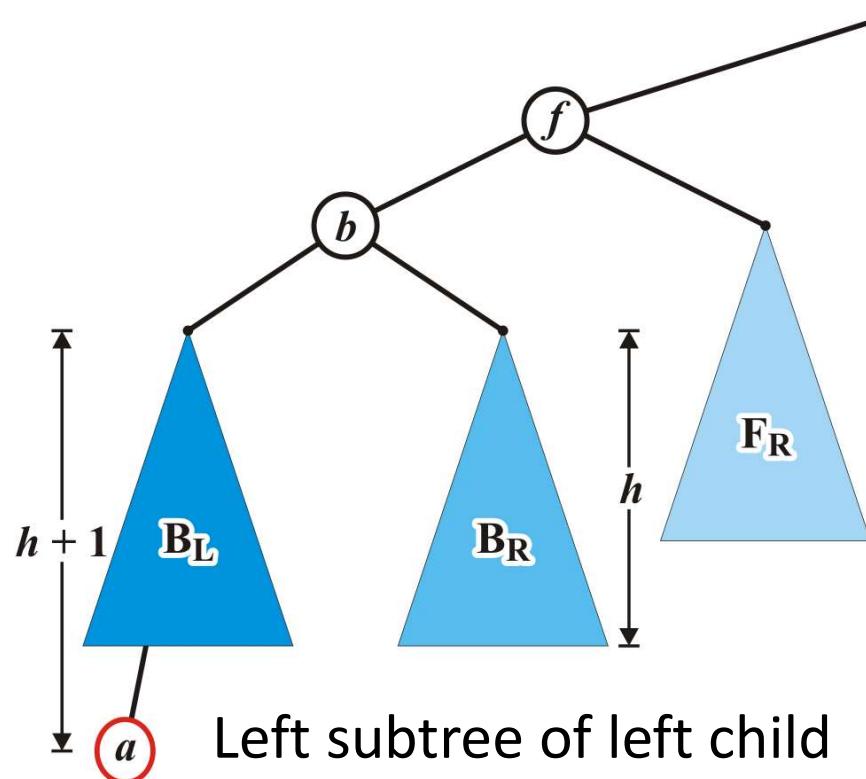
- Each blue triangle represents a tree of height h



Maintaining Balance: Case 1

Insert a into this tree: it falls into the **left subtree B_L of b**

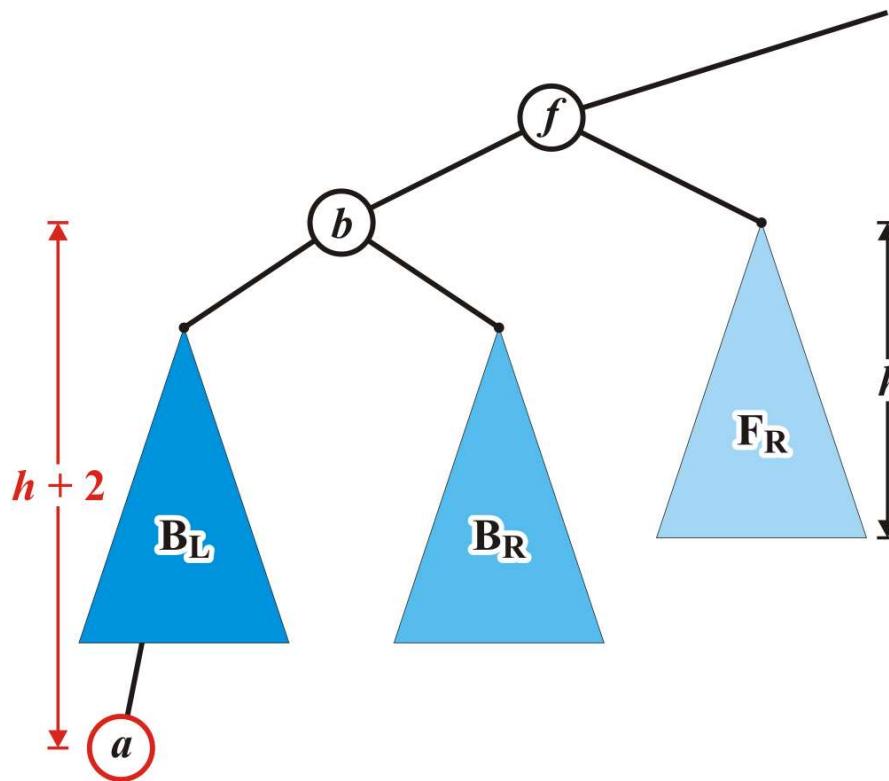
- Assume B_L remains balanced
- Thus, the tree rooted at b is also balanced



Maintaining Balance: Case 1

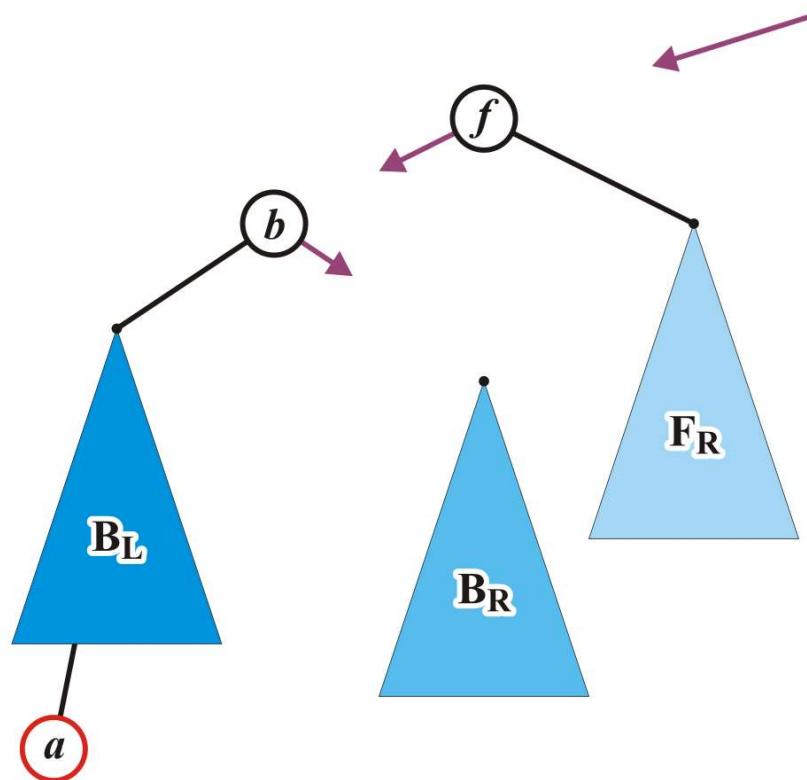
The tree rooted at node f is now unbalanced

- We will correct the imbalance at this node



Maintaining Balance: Case 1

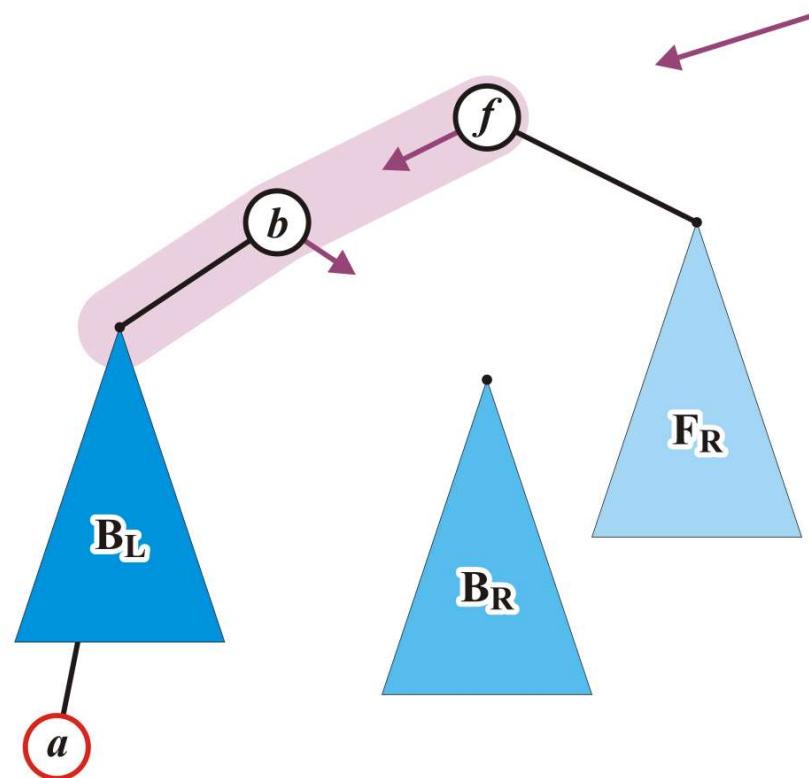
We will modify these three pointers:



Maintaining Balance: Case 1

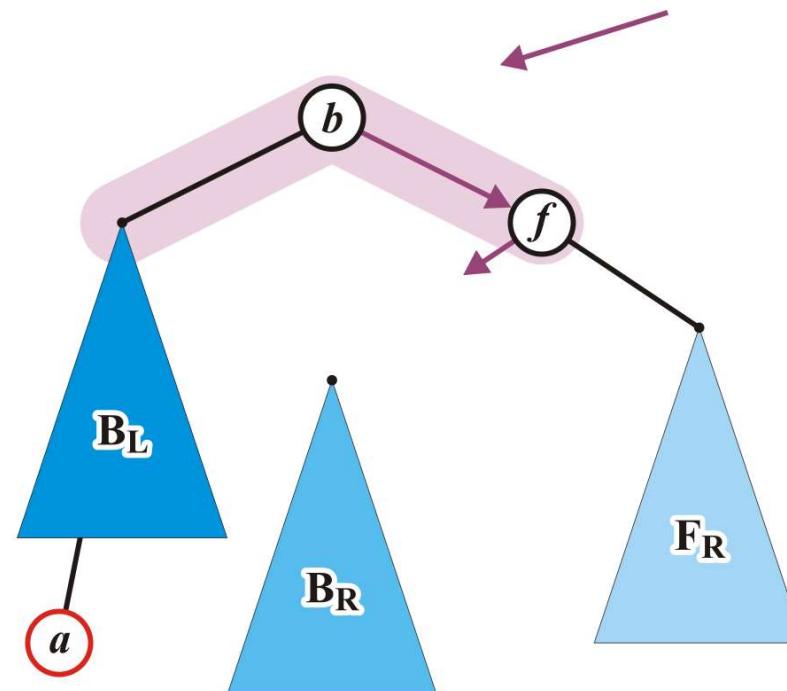
Specifically, we will rotate these two nodes around the root:

- Recall the first prototypical example
- Promote node b to the root and demote node f to be the right child of b



Maintaining Balance: Case 1

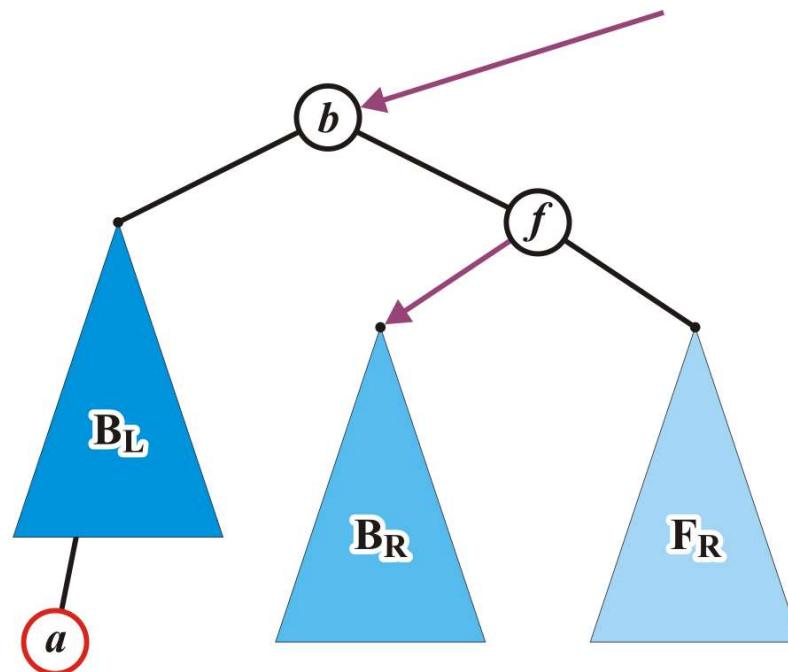
Make f the right child of b



Maintaining Balance: Case 1

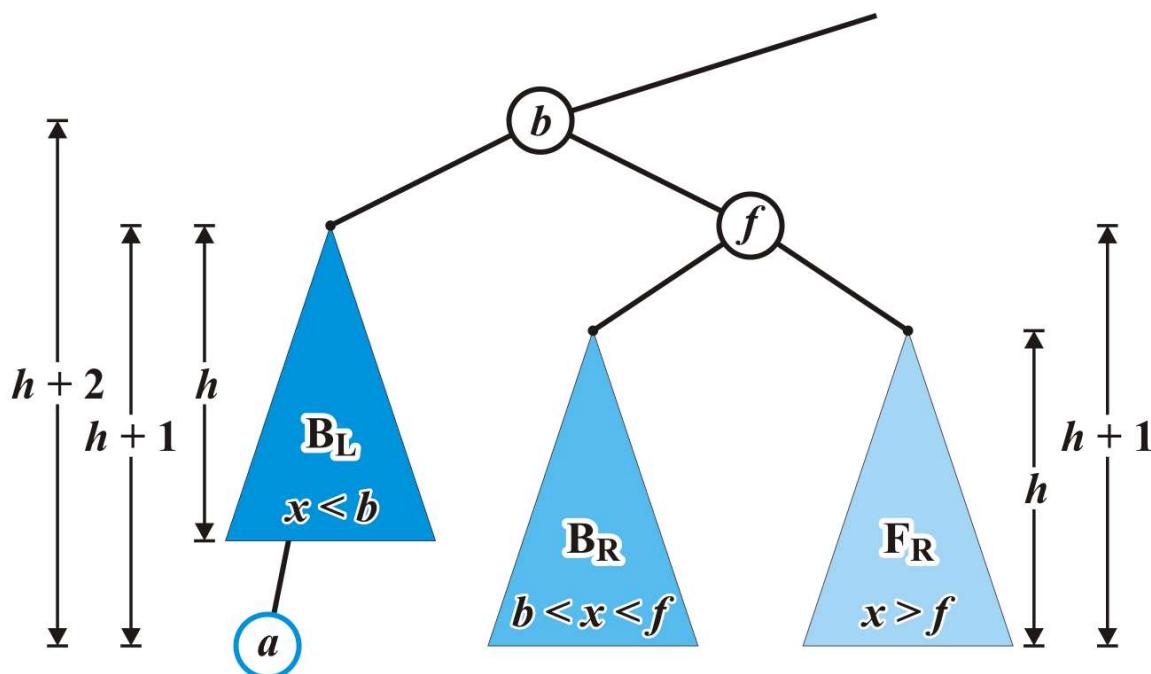
Assign former parent of node f to point to node b

Make B_R left child of node f



Maintaining Balance: Case 1

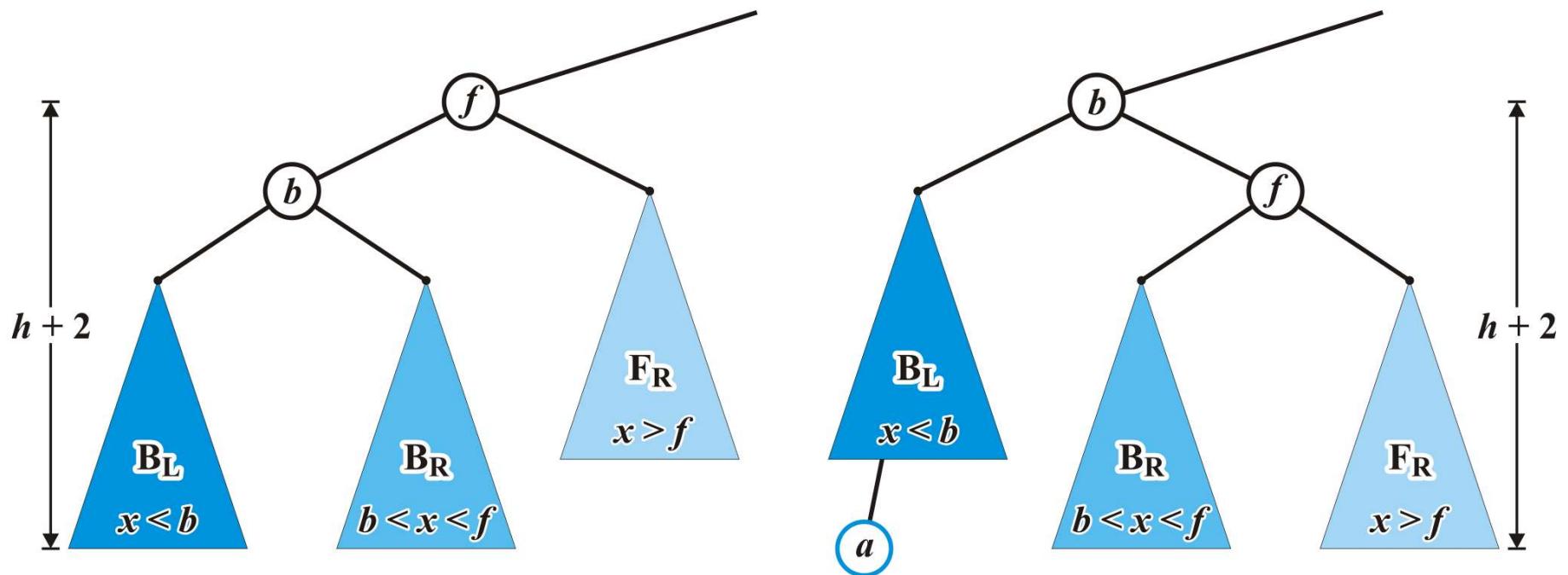
The nodes b and f are now balanced and all remaining nodes of the subtrees are in their correct positions



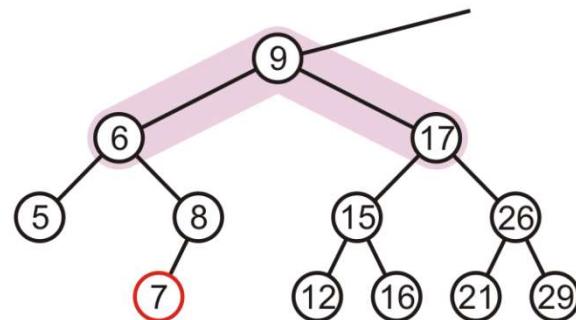
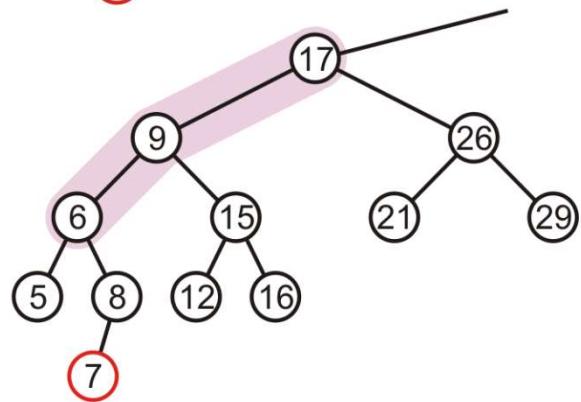
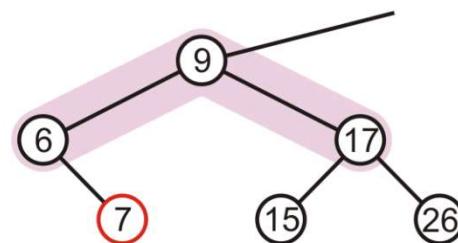
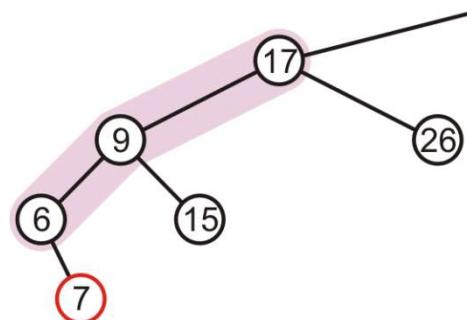
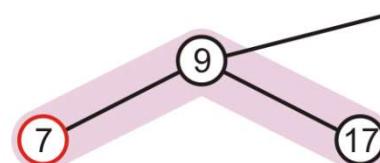
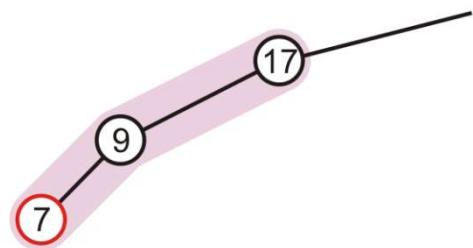
Maintaining Balance: Case 1

Additionally, height of the tree rooted at b equals the original height of the tree rooted at f

- Thus, this insertion will no longer affect the balance of any ancestors all the way back to the root

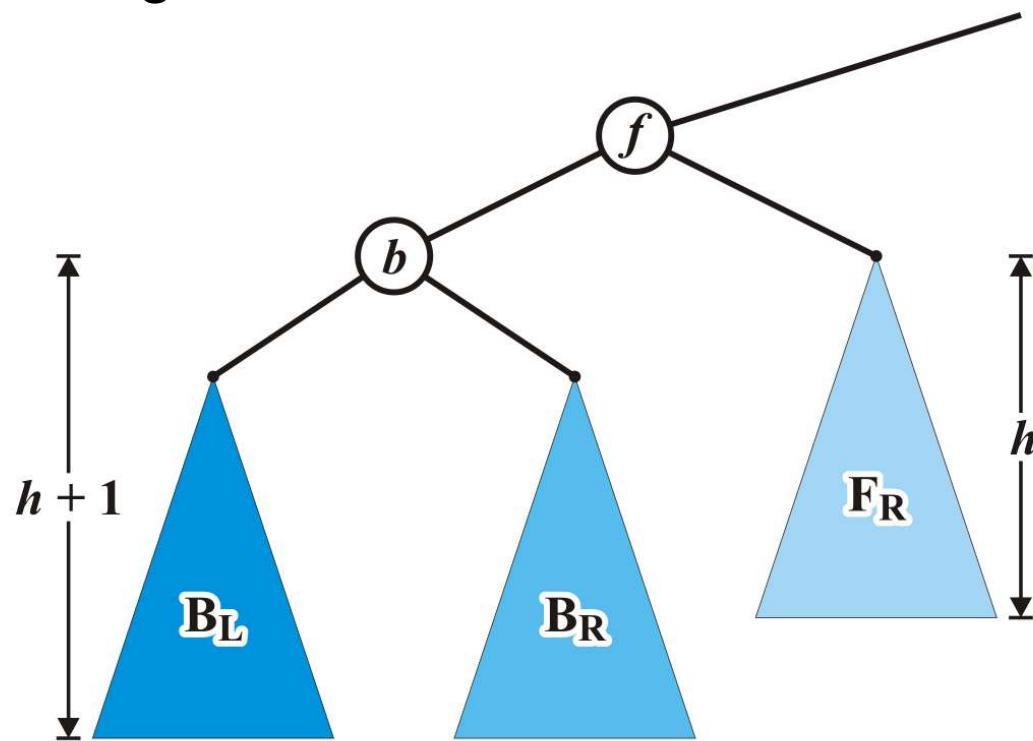


More Examples



Maintaining Balance: Case 2

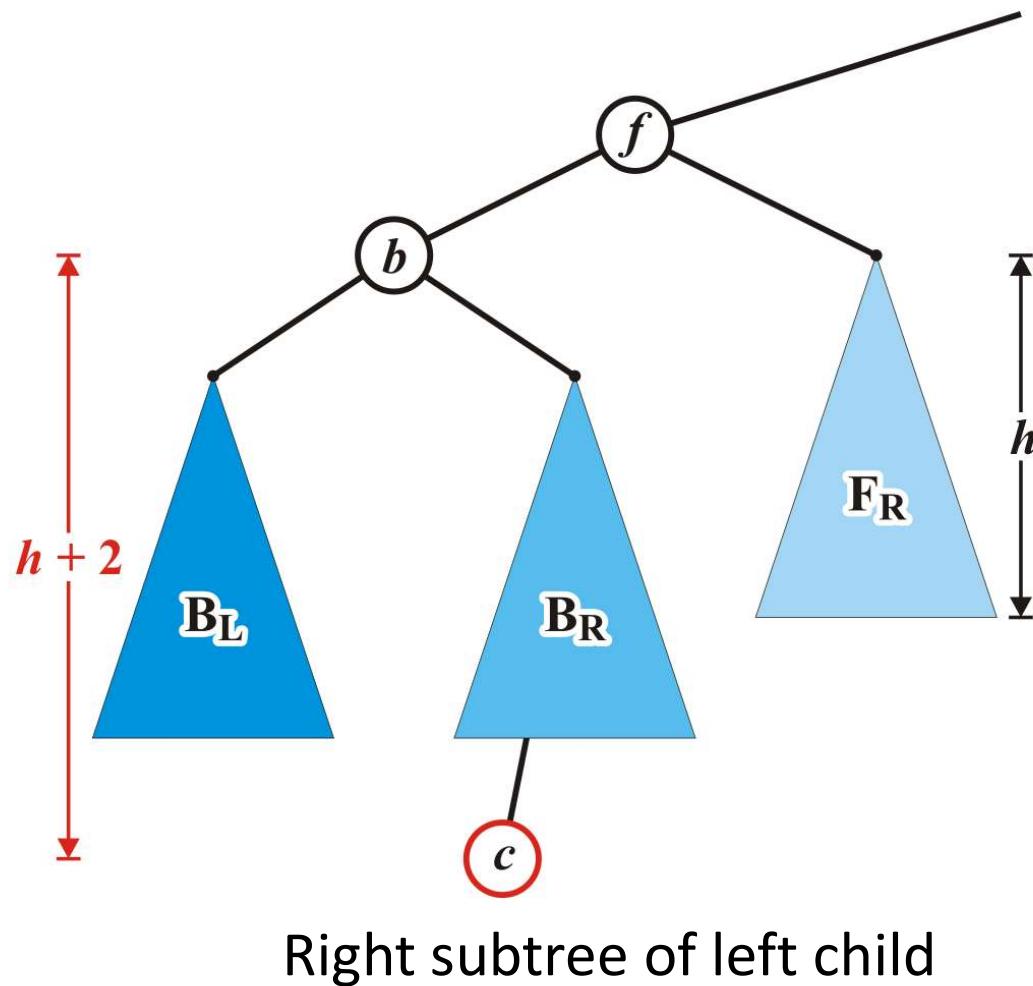
Alternatively, consider the insertion of c where $b < c < f$ into our original tree



Maintaining Balance: Case 2

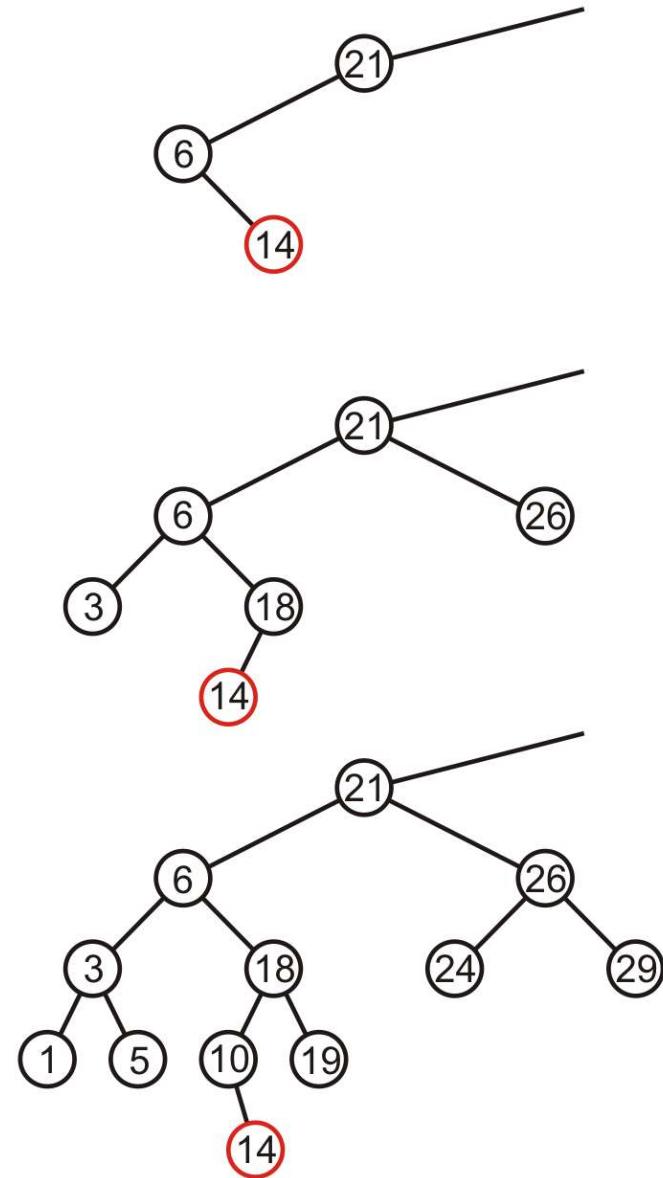
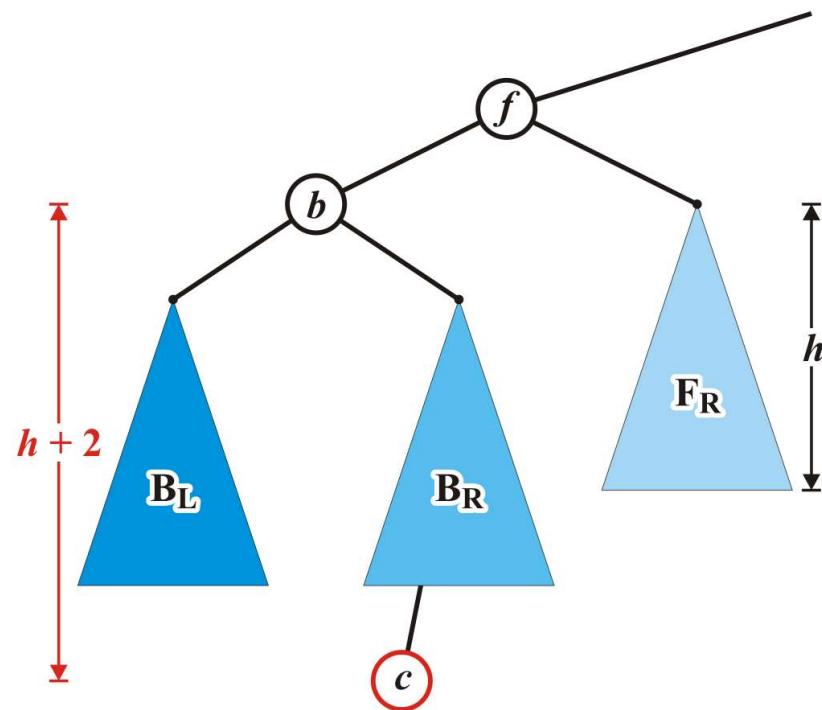
Assume that the insertion of c increases the height of B_R

- Once again, f becomes unbalanced



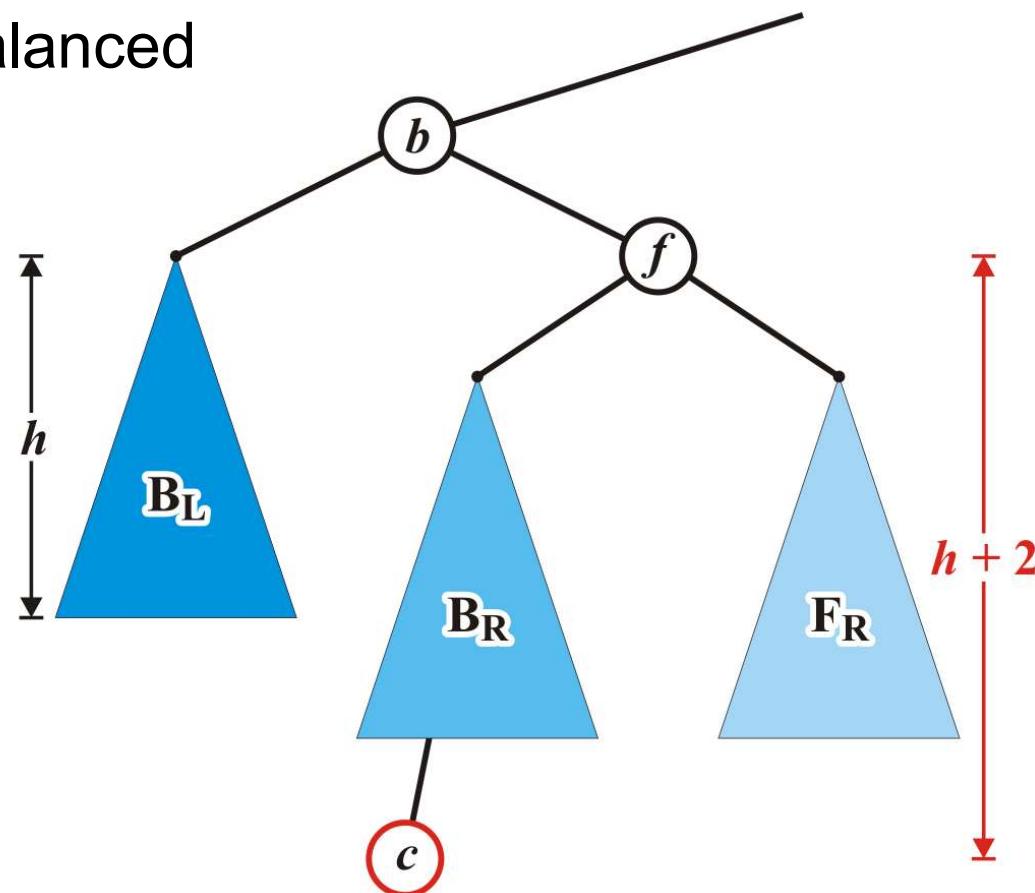
Maintaining Balance: Case 2

Here are examples of when the insertion of 14 may cause this situation when $h = -1, 0, \text{ and } 1$



Maintaining Balance: Case 2

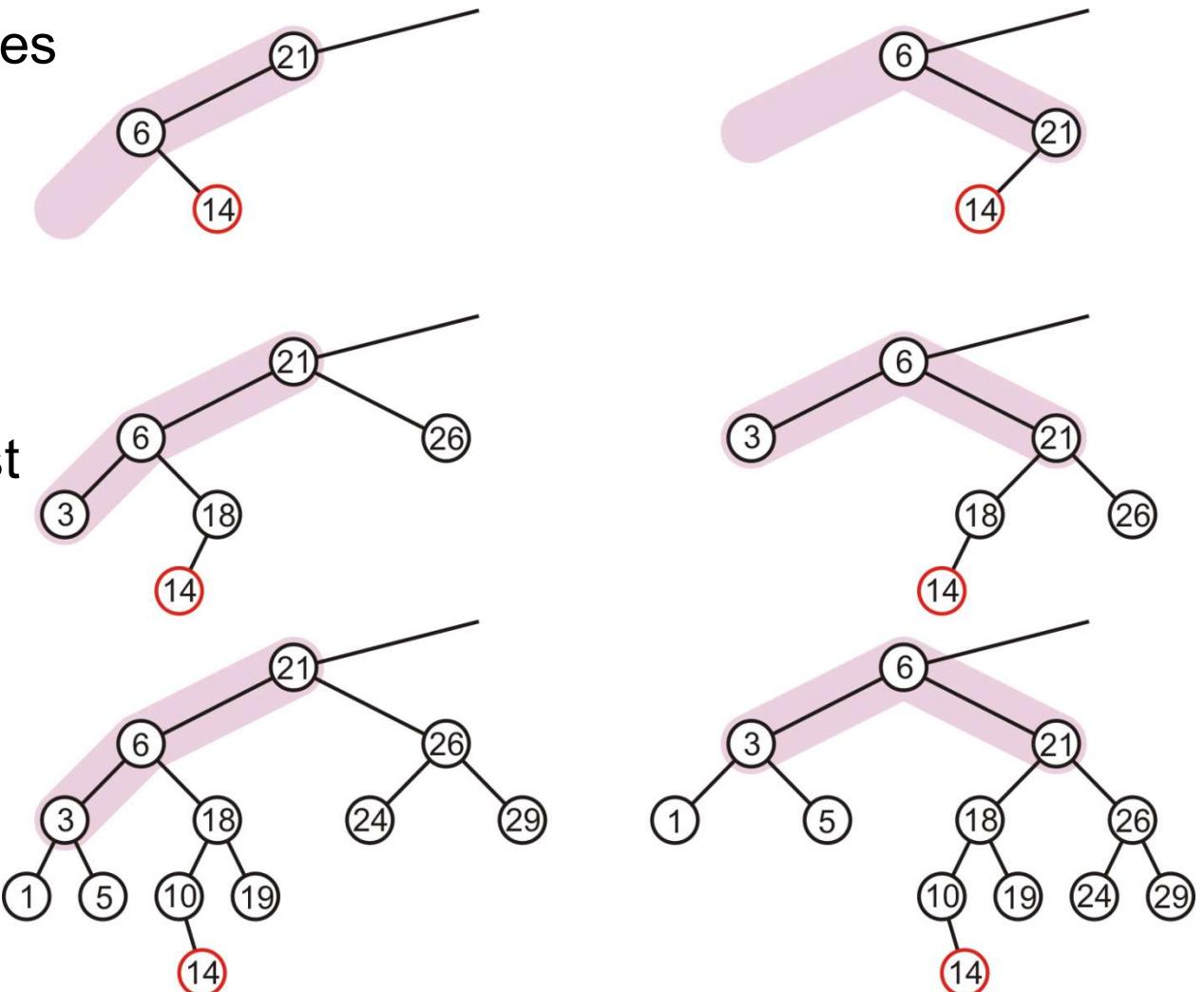
Unfortunately, the previous correction does not fix the imbalance at the root of this sub-tree: the new root, b , remains unbalanced



Maintaining Balance: Case 2

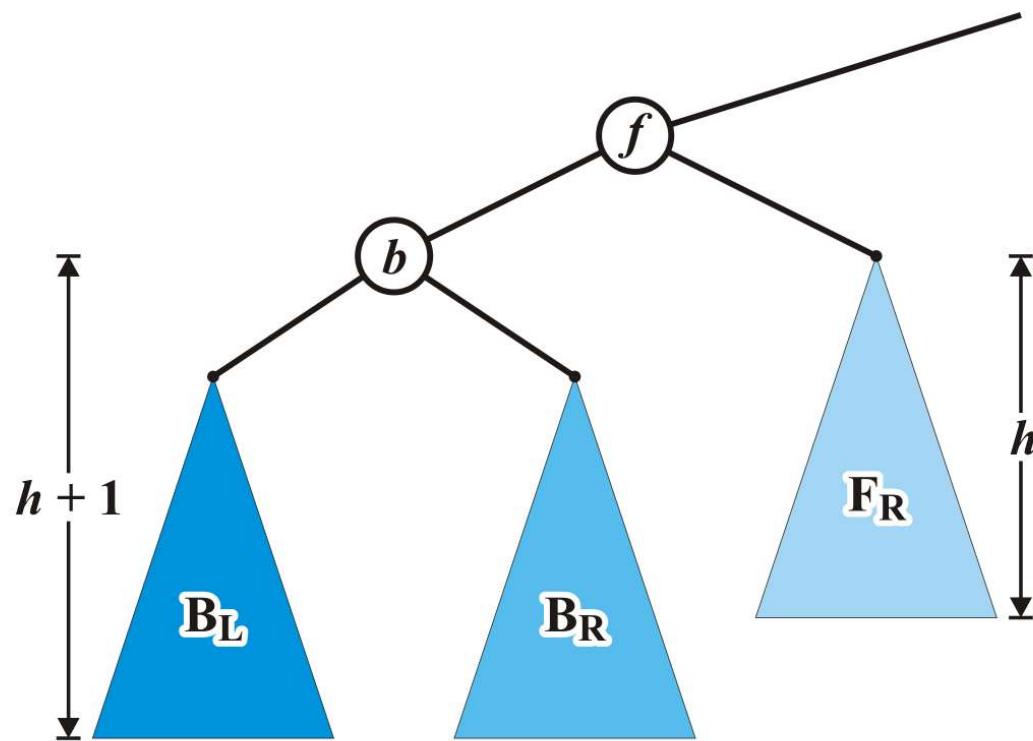
In our three sample cases with $h = -1, 0, \text{ and } 1$, doing the same thing as before results in a tree that is still unbalanced...

- The imbalance is just shifted to the other side



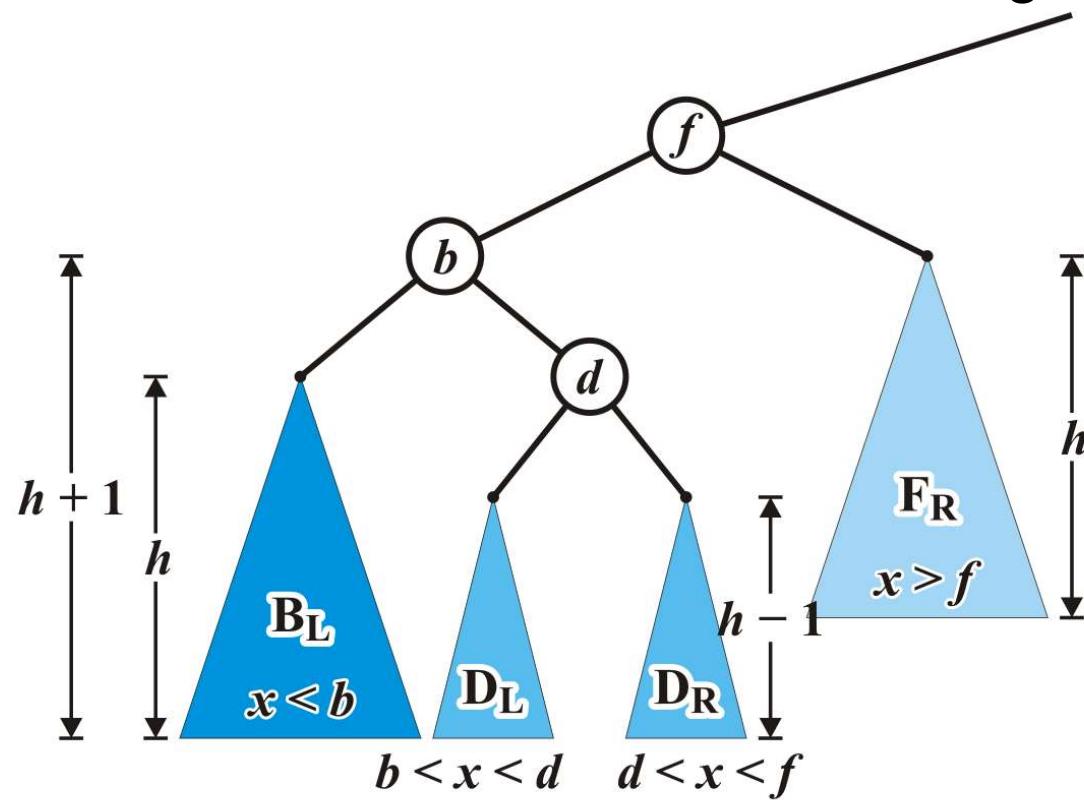
Maintaining Balance: Case 2

Lets start over ...



Maintaining Balance: Case 2

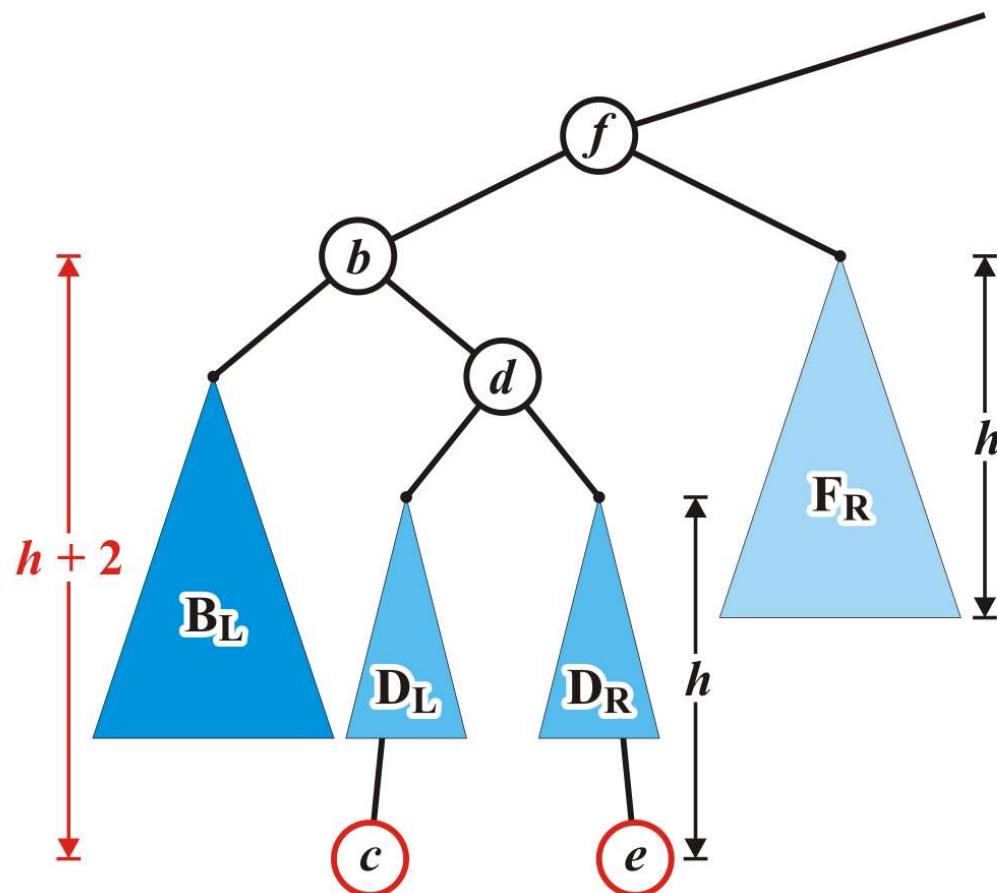
Re-label the tree by dividing the left subtree of f into a tree rooted at d with two subtrees of height $h - 1$



Maintaining Balance: Case 2

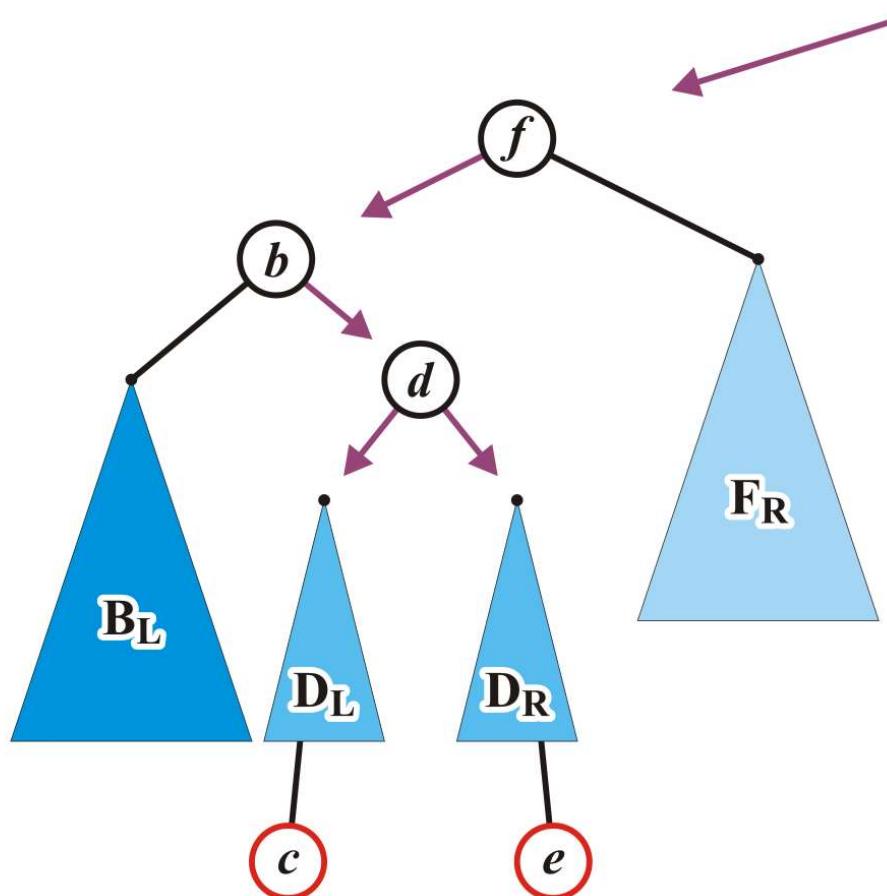
Now an insertion causes an imbalance at f

- The addition of either c or e will cause this



Maintaining Balance: Case 2

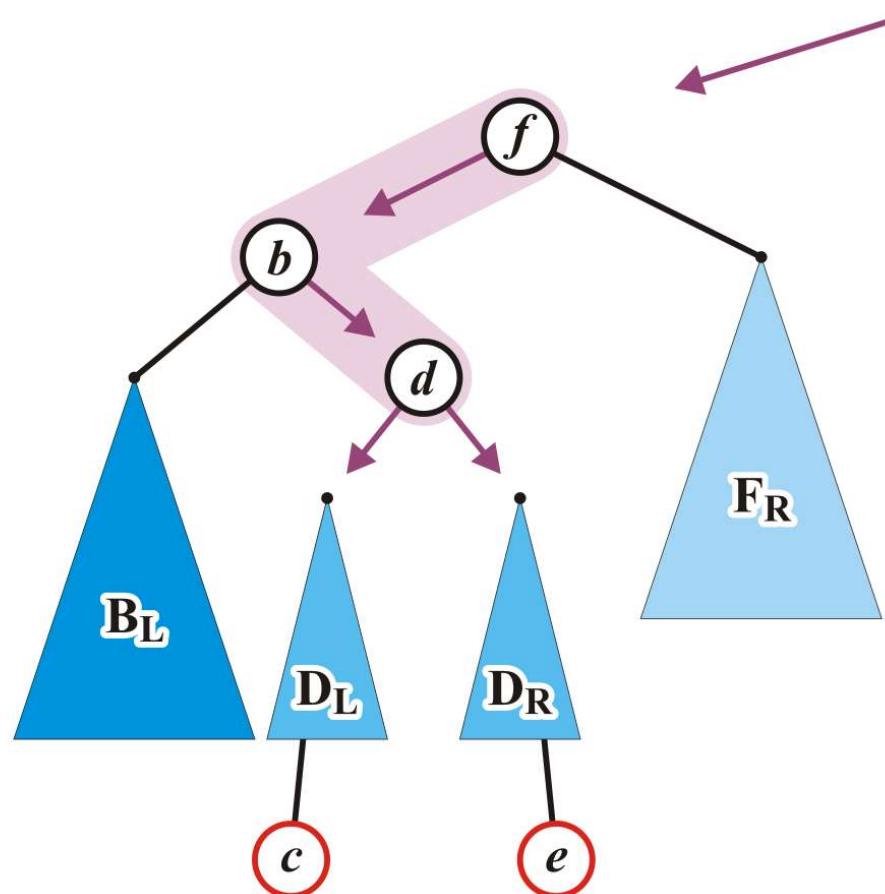
We will reassign the following pointers



Maintaining Balance: Case 2

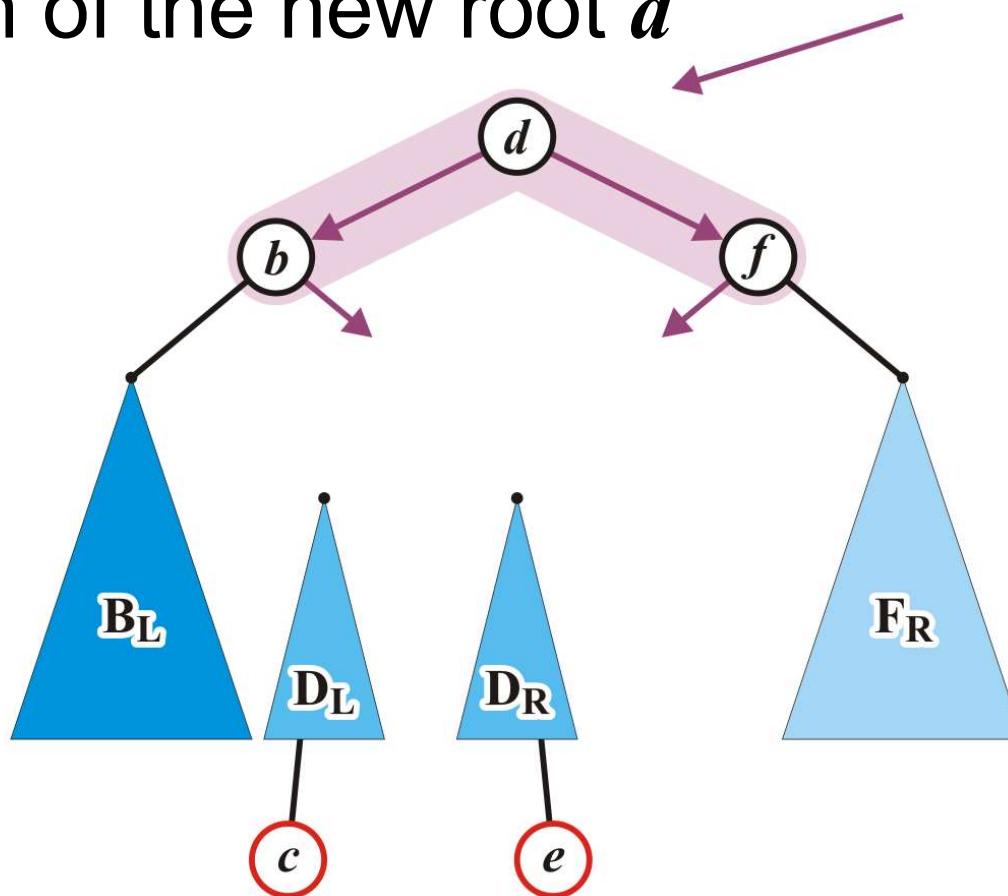
Specifically, we will order these three nodes as a perfect tree

- Recall the second prototypical example



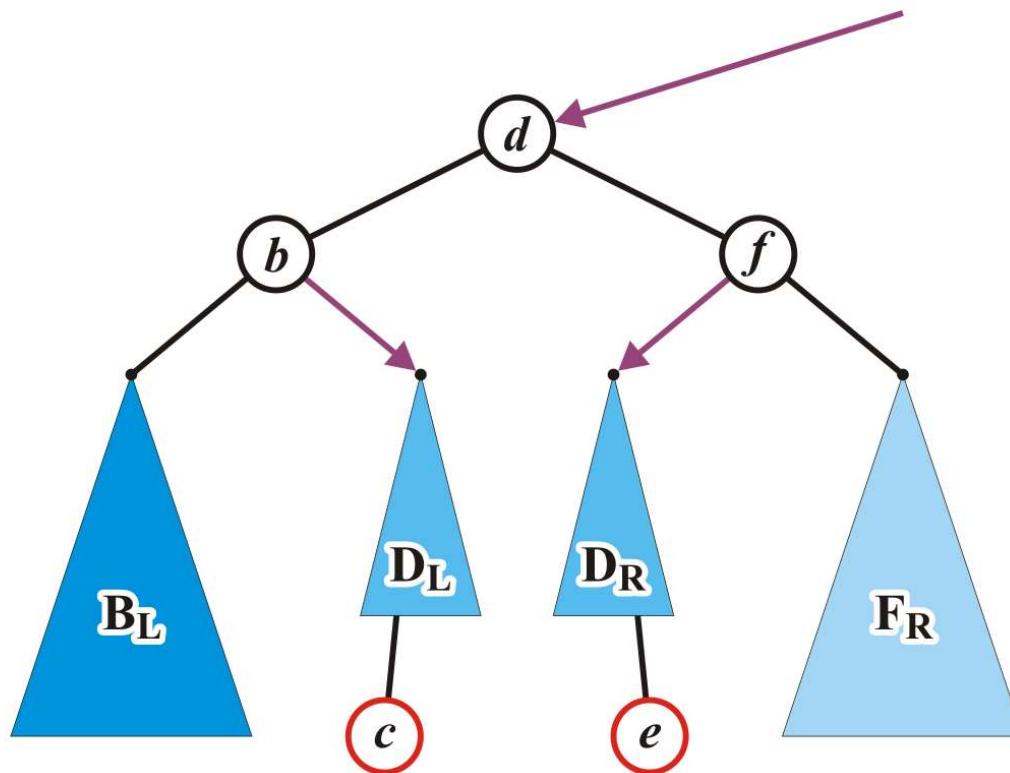
Maintaining Balance: Case 2

To achieve this, b and f will be assigned as children of the new root d



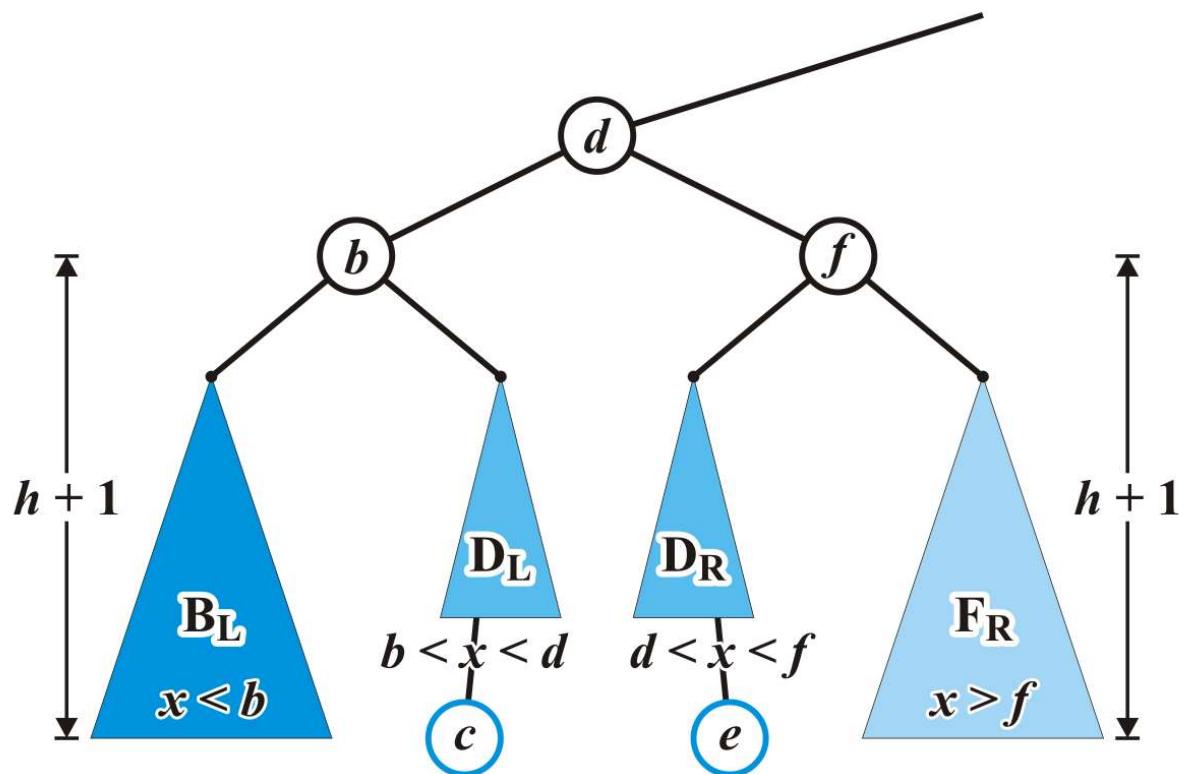
Maintaining Balance: Case 2

We also have to connect the two subtrees and original parent of f



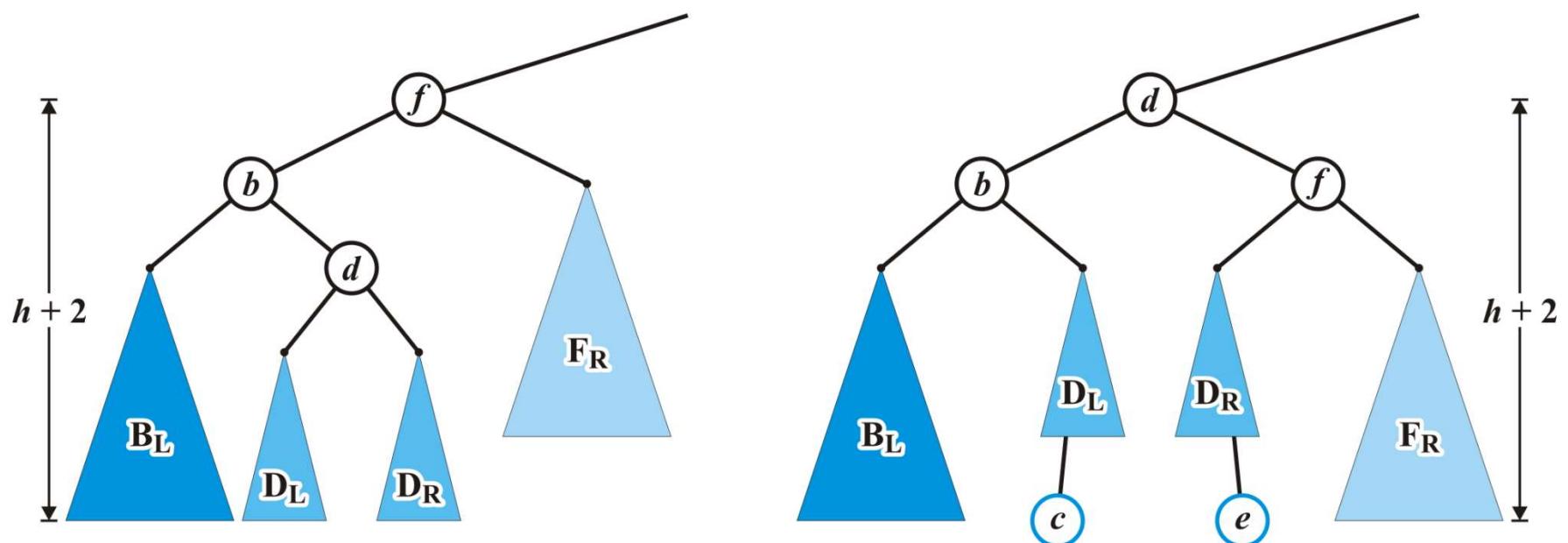
Maintaining Balance: Case 2

Now the tree rooted at d is balanced



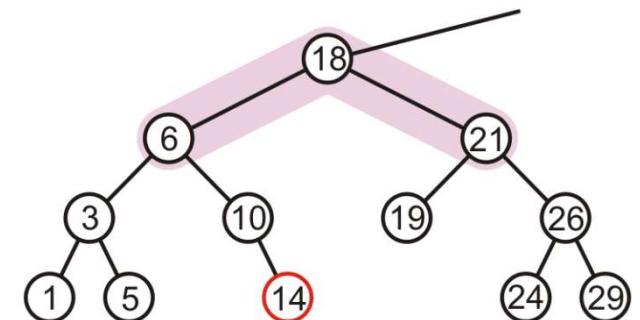
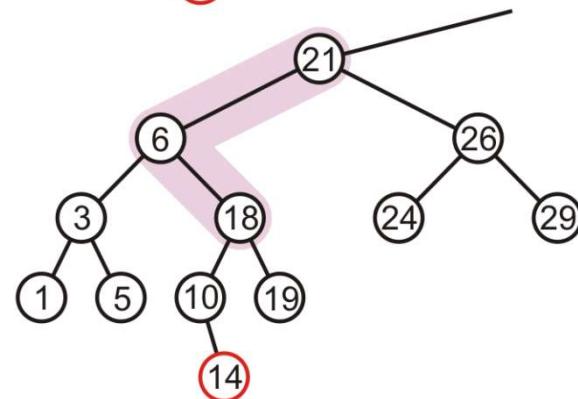
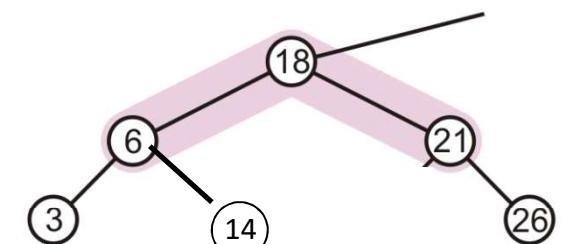
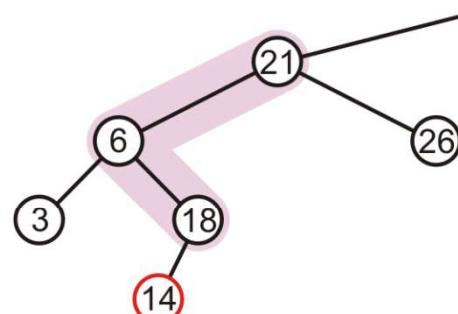
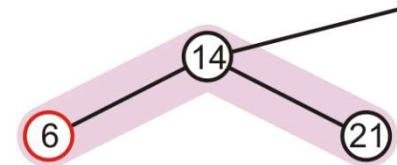
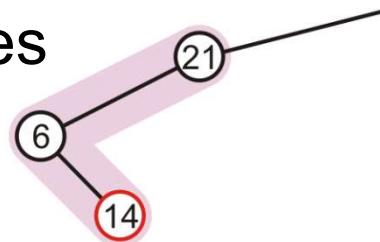
Maintaining Balance: Case 2

Again, the height of the root did not change



Maintaining Balance: Case 2

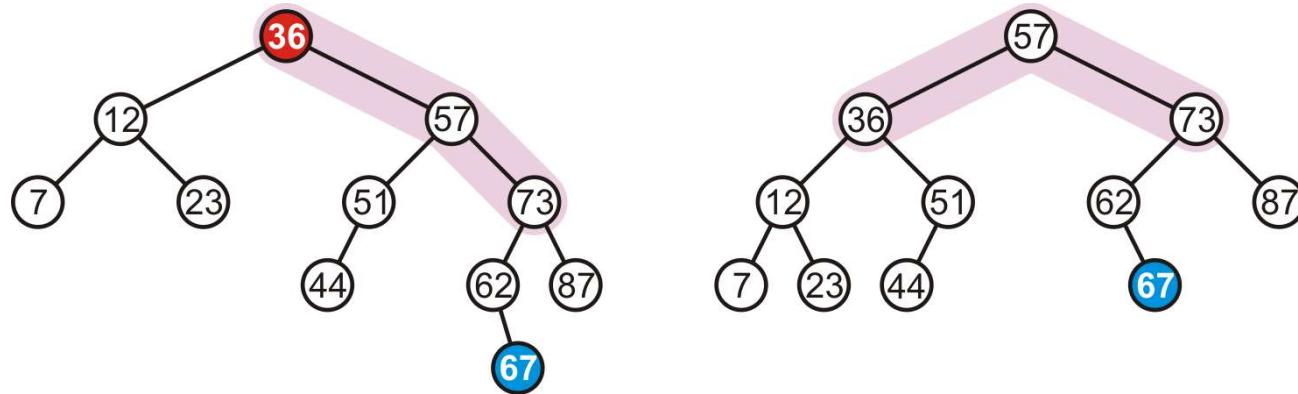
In our three sample cases with $h = -1, 0, \text{ and } 1$, the node is now balanced and the same height as the tree before the insertion



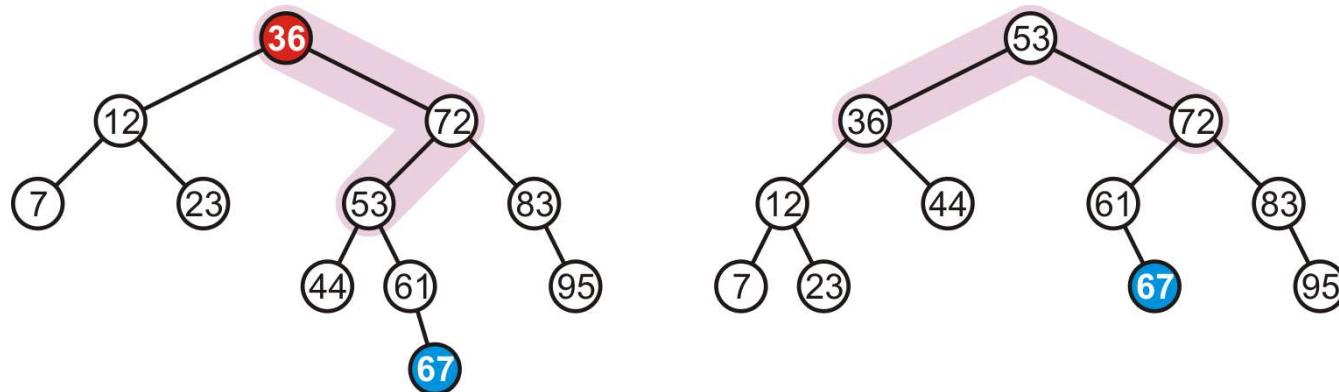
Maintaining balance: Summary

There are two symmetric cases to those we have examined:

- Insertions into the right-right sub-tree

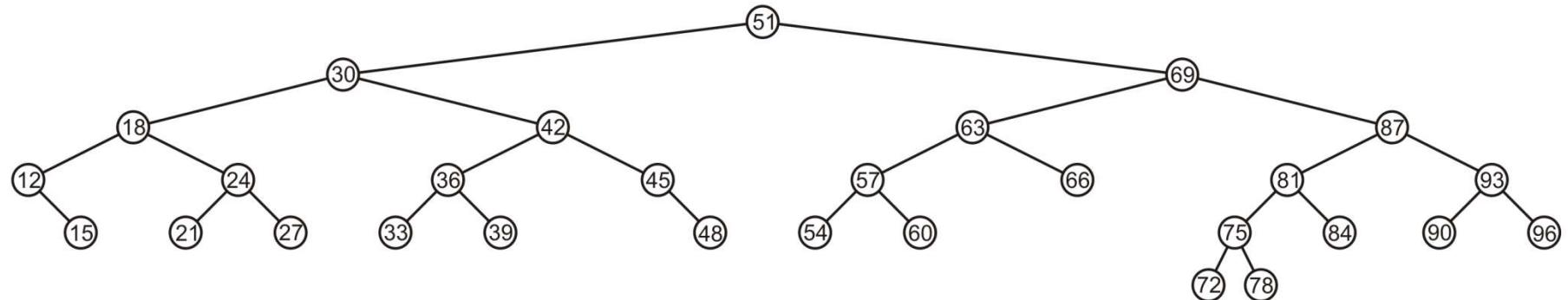


- Insertions into either the right-left sub-tree



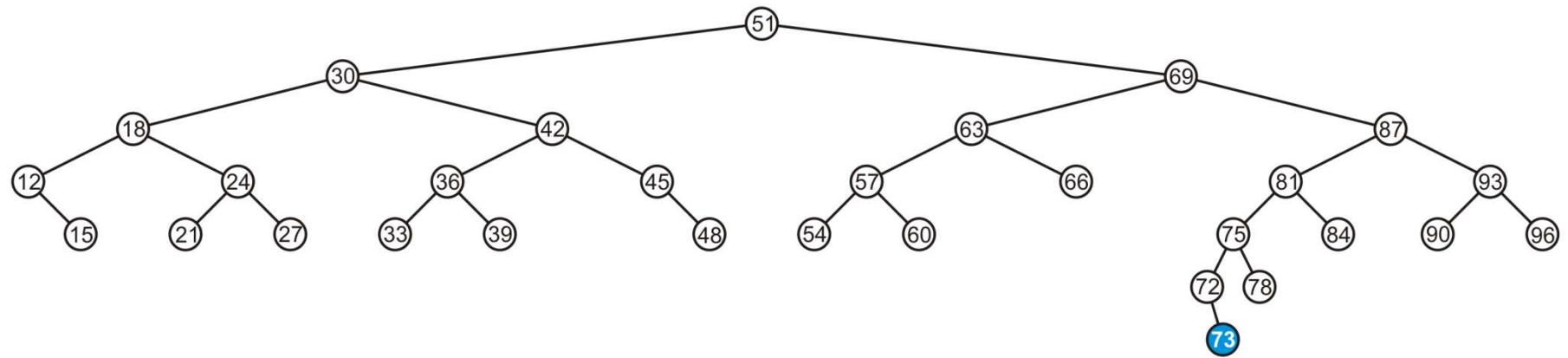
More examples : Insertion

Consider this AVL tree



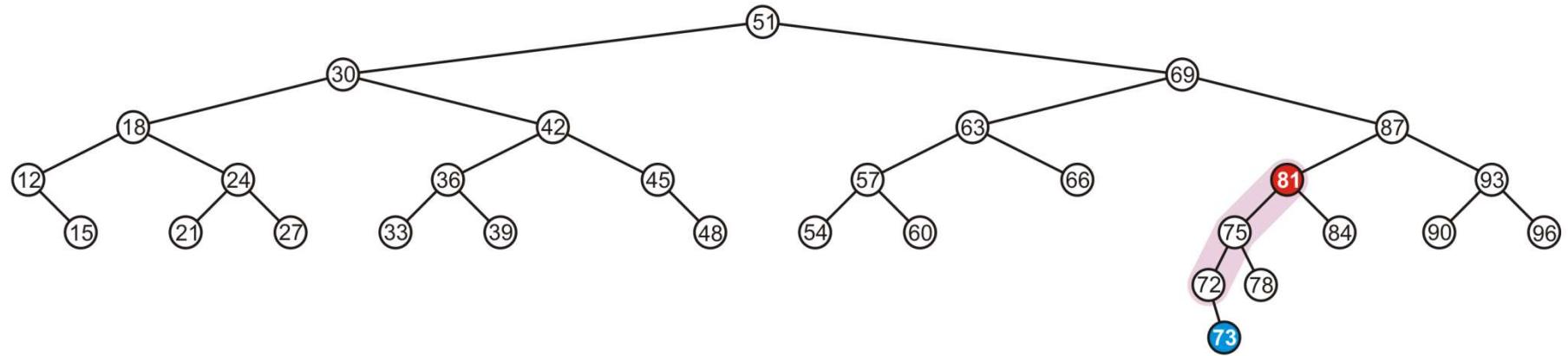
Insertion

Insert 73



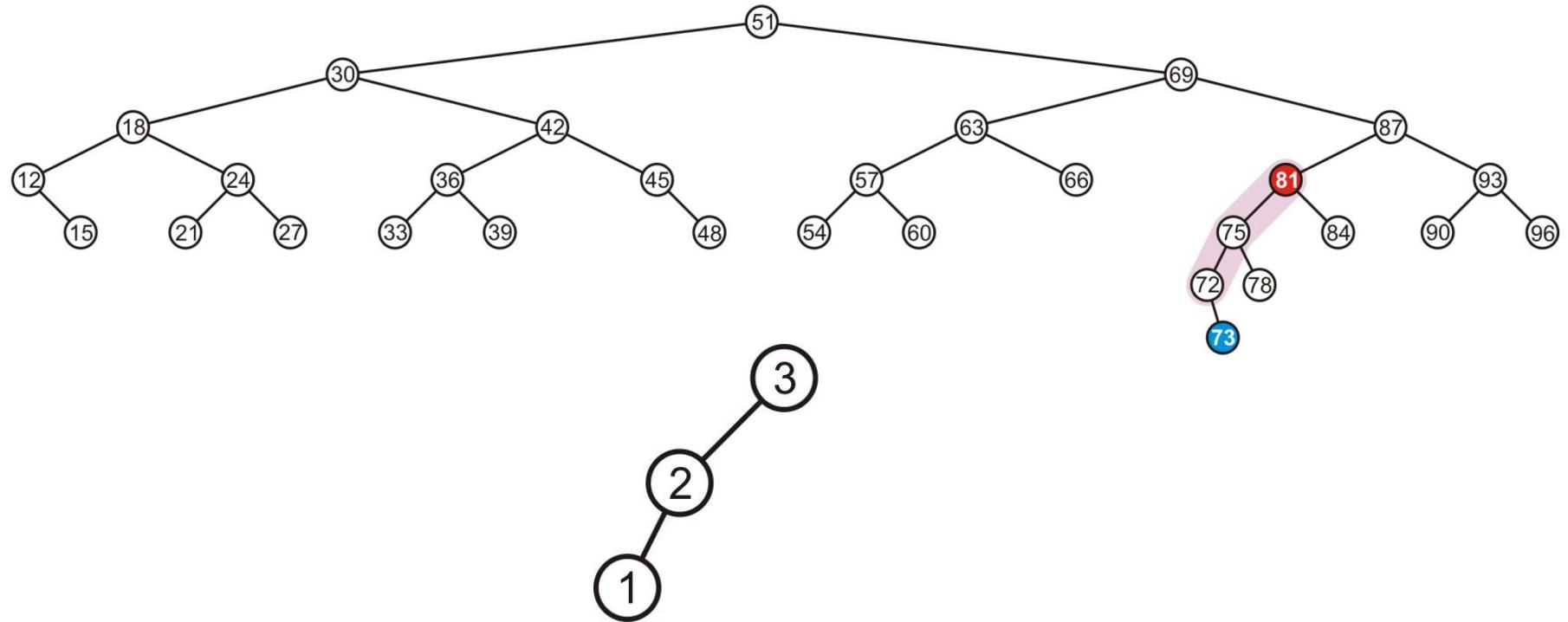
Insertion

The node 81 is unbalanced
– A left-left imbalance



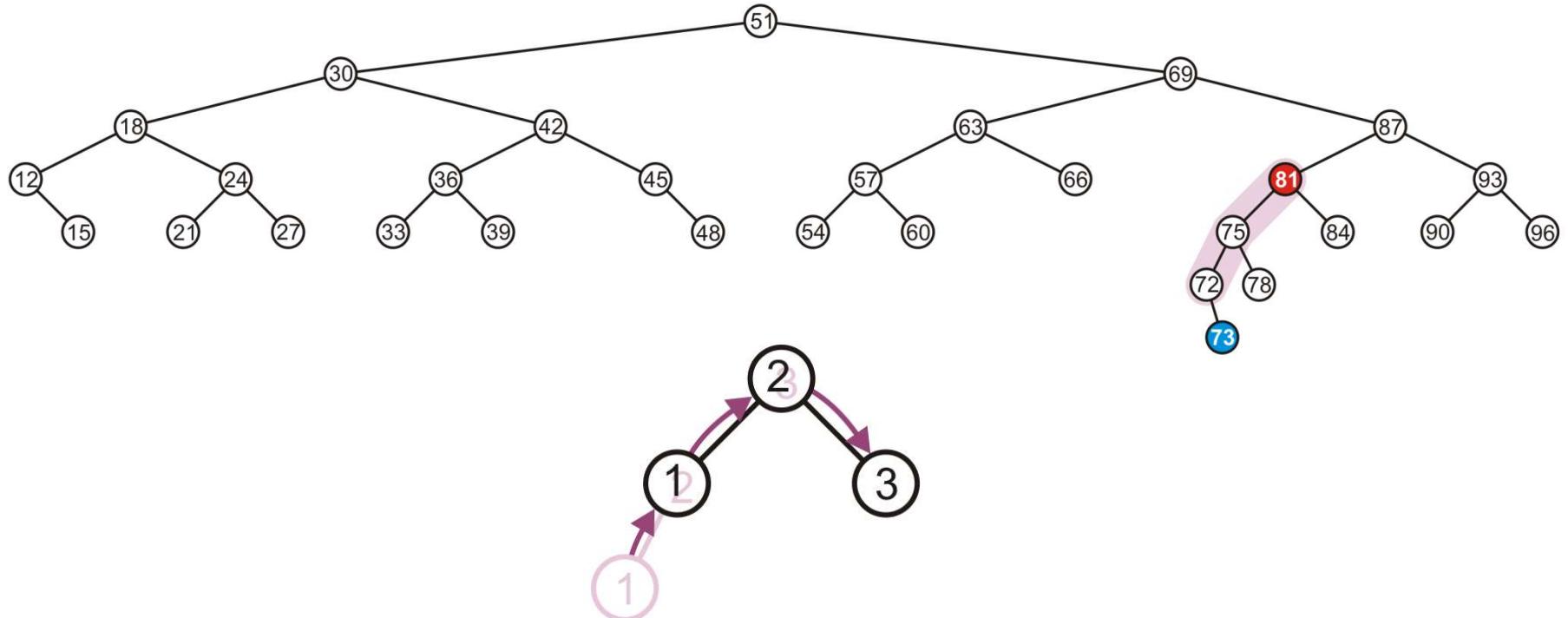
Insertion

The node 81 is unbalanced
– A left-left imbalance



Insertion

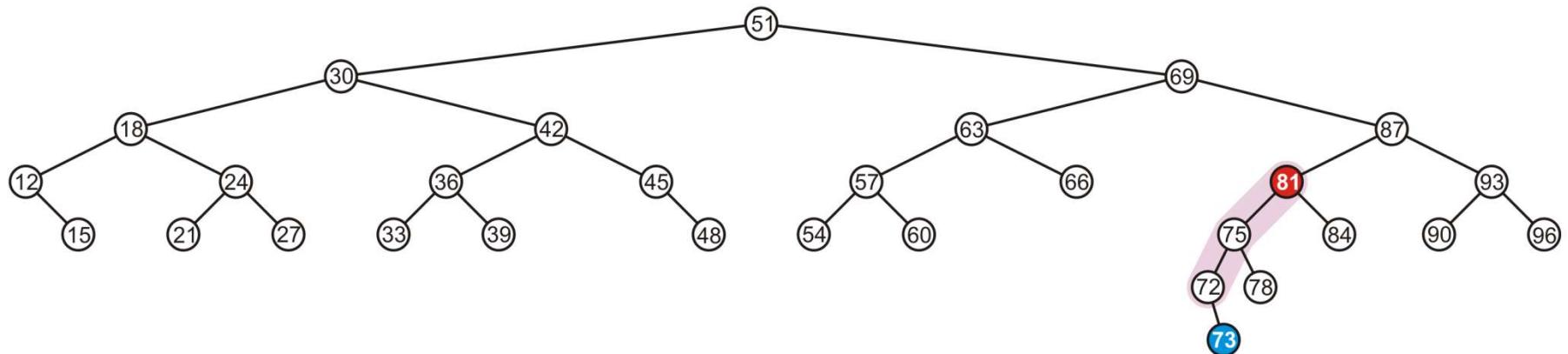
The node 81 is unbalanced
– A left-left imbalance



Insertion

The node 81 is unbalanced

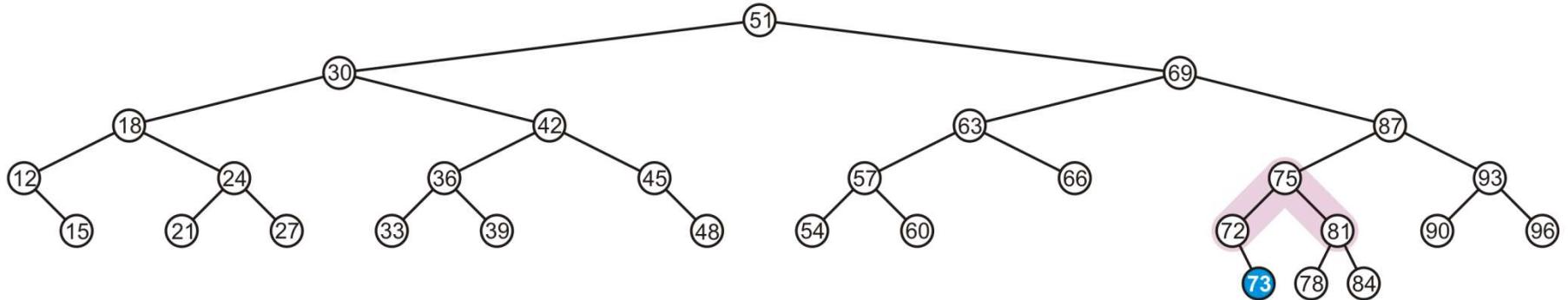
- A left-left imbalance
- Promote the intermediate node to the imbalanced node
- 75 is that node



Insertion

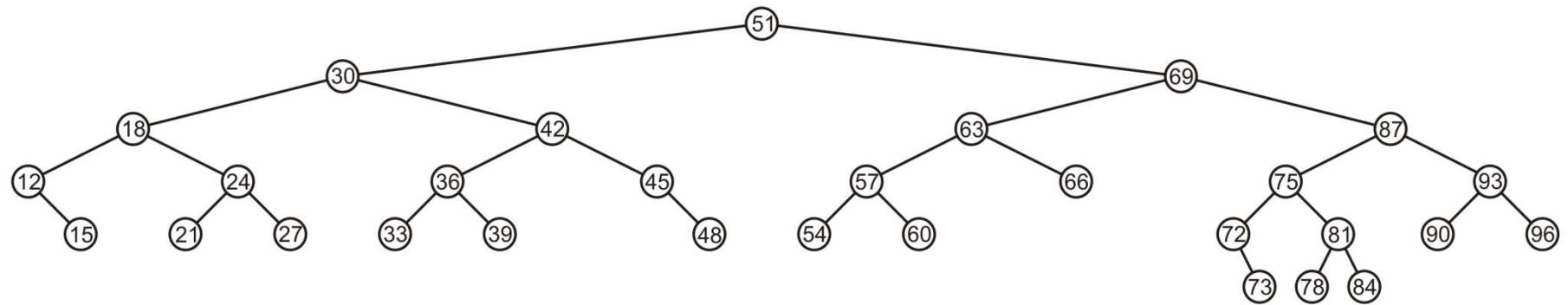
The node 81 is unbalanced

- A left-left imbalance
- Promote the intermediate node to the imbalanced node
- 75 is that node



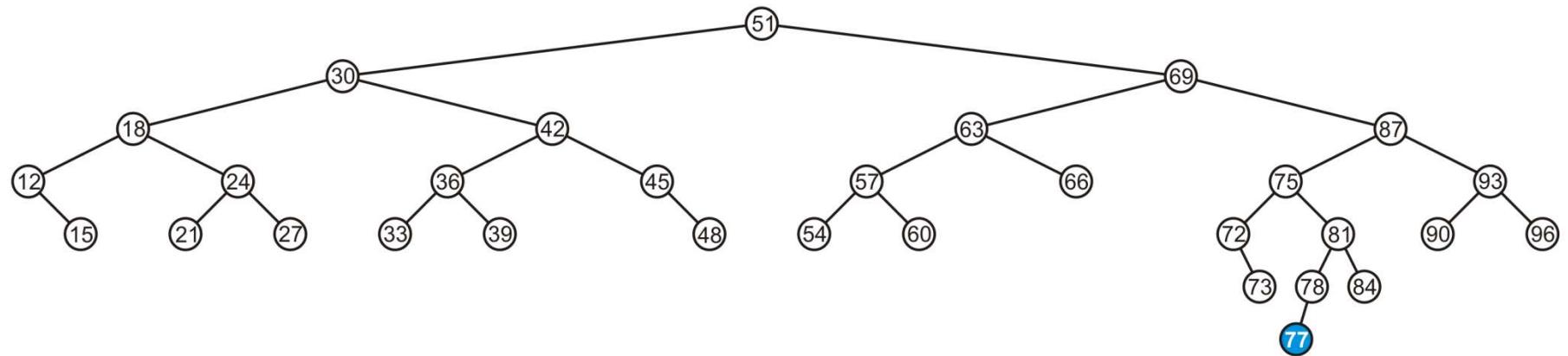
Insertion

The tree is AVL balanced



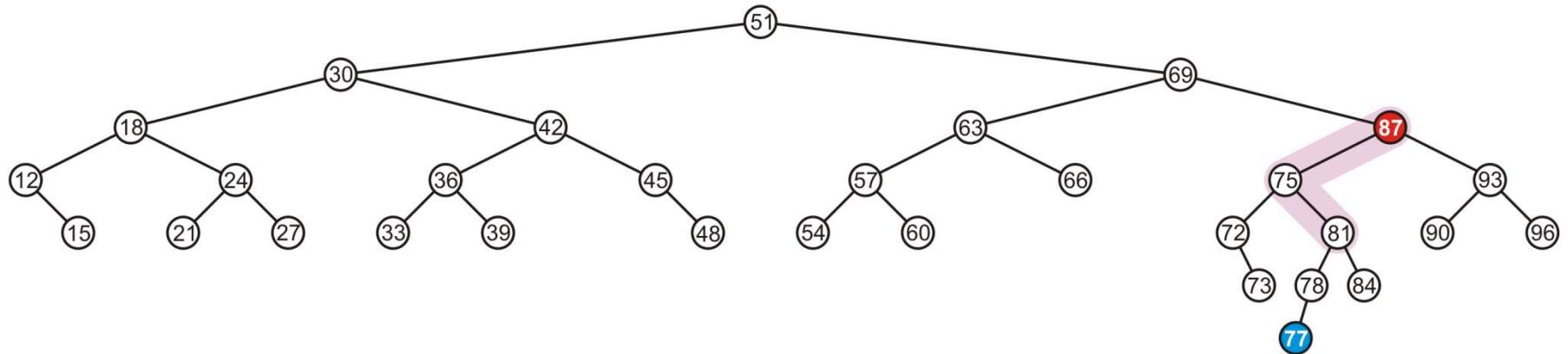
Insertion

Insert 77



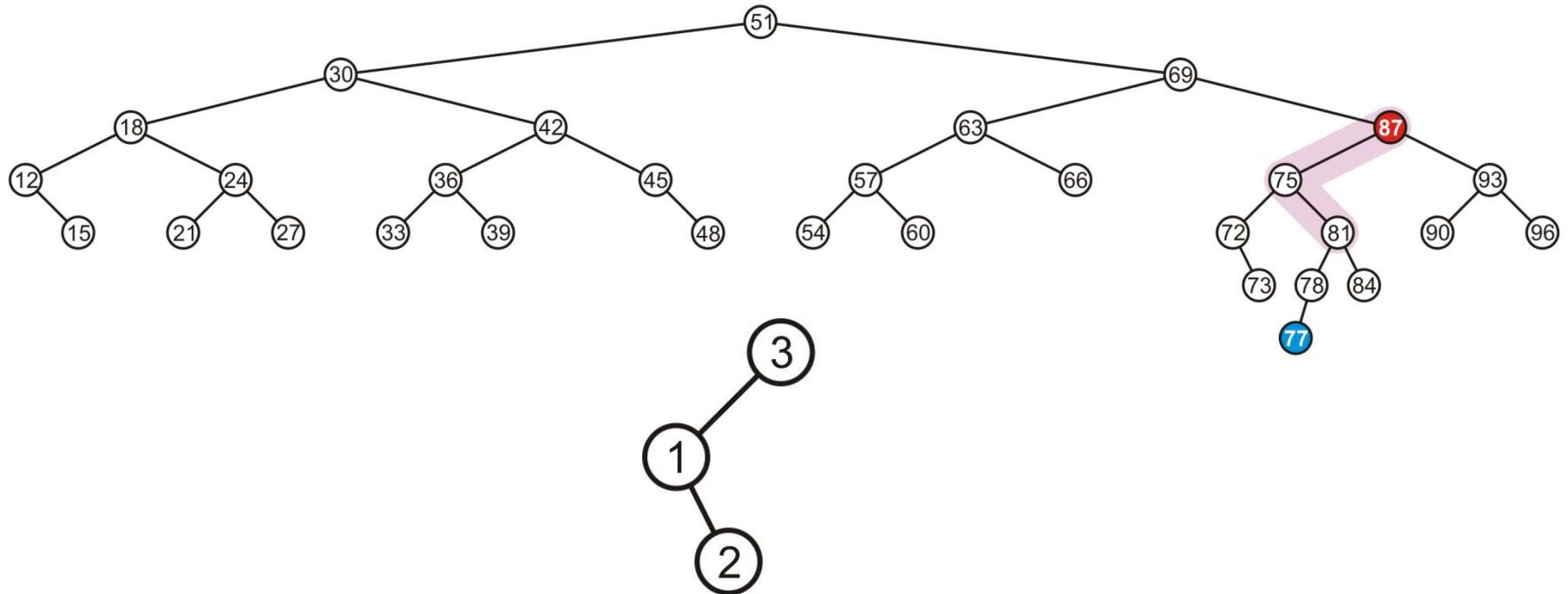
Insertion

The node 87 is unbalanced
– A left-right imbalance



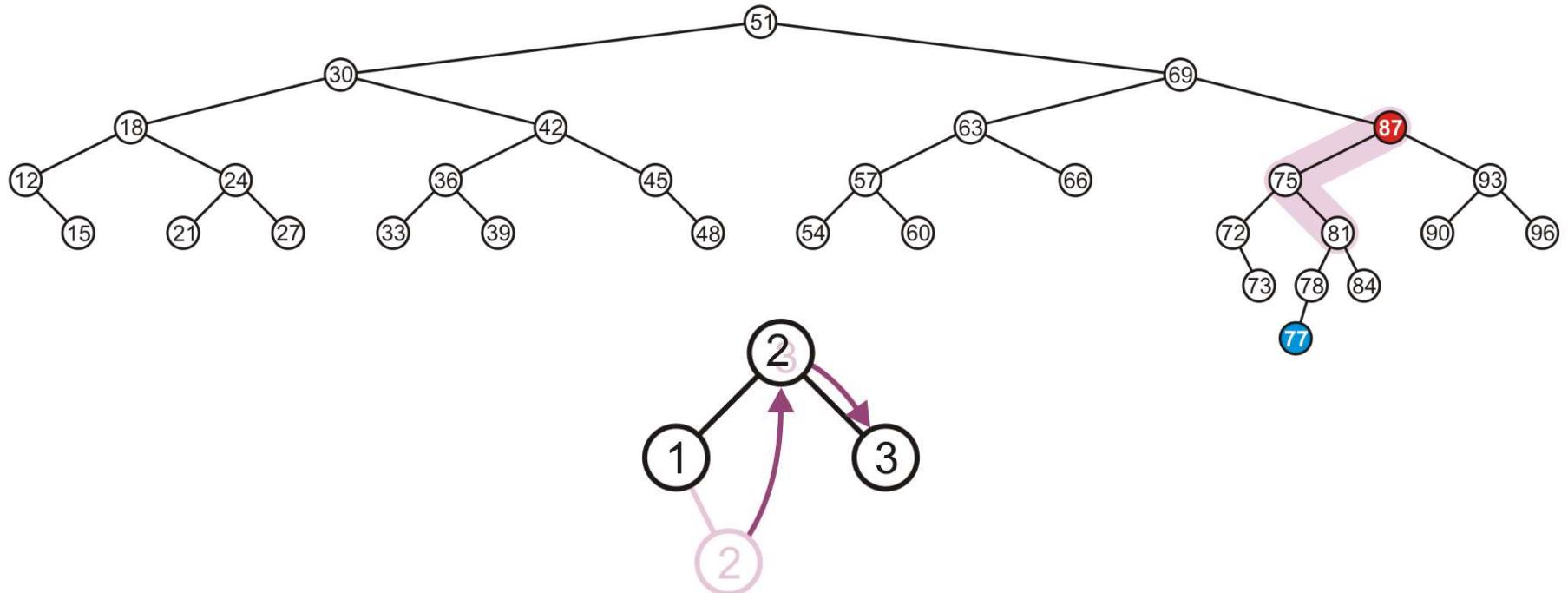
Insertion

The node 87 is unbalanced
– A left-right imbalance



Insertion

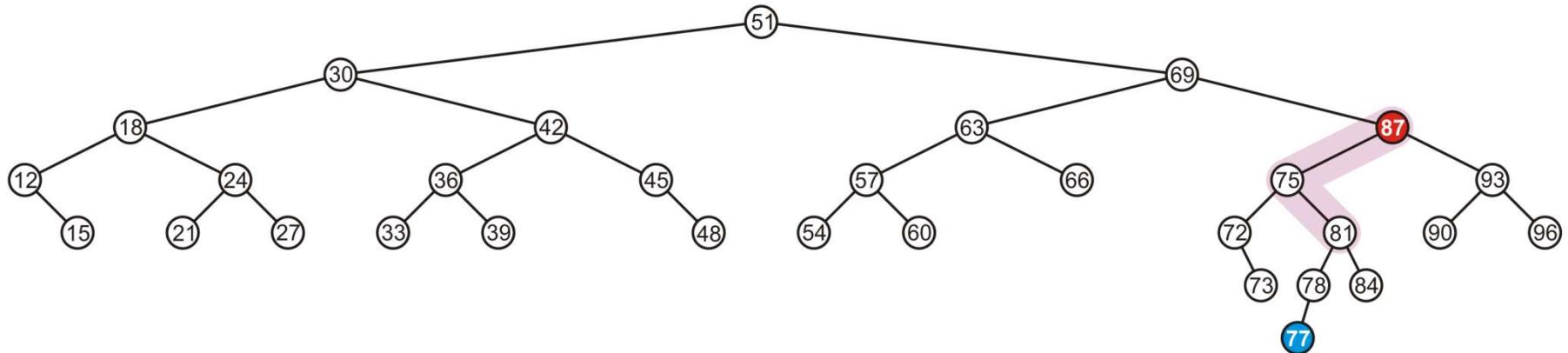
The node 87 is unbalanced
– A left-right imbalance



Insertion

The node 87 is unbalanced

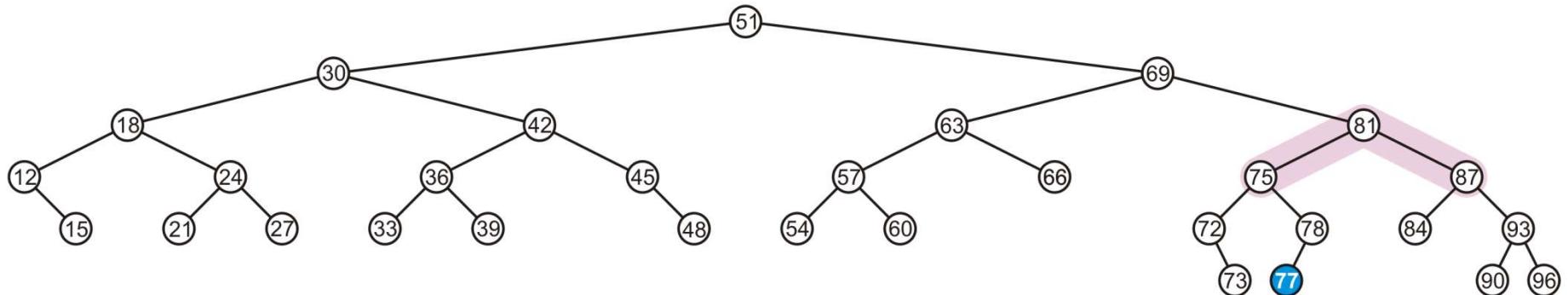
- A left-right imbalance
- Promote the intermediate node to the imbalanced node
- 81 is that value



Insertion

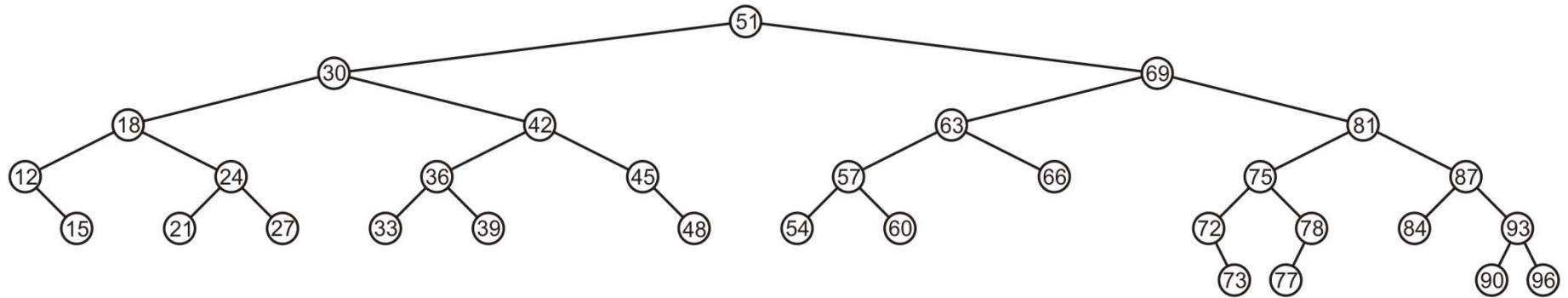
The node 87 is unbalanced

- A left-right imbalance
- Promote the intermediate node to the imbalanced node
- 81 is that value



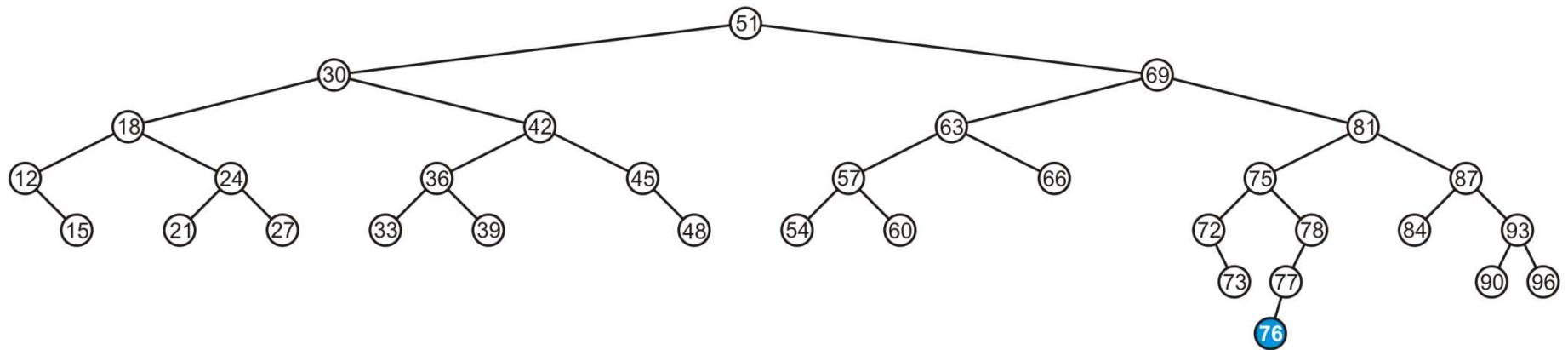
Insertion

The tree is balanced



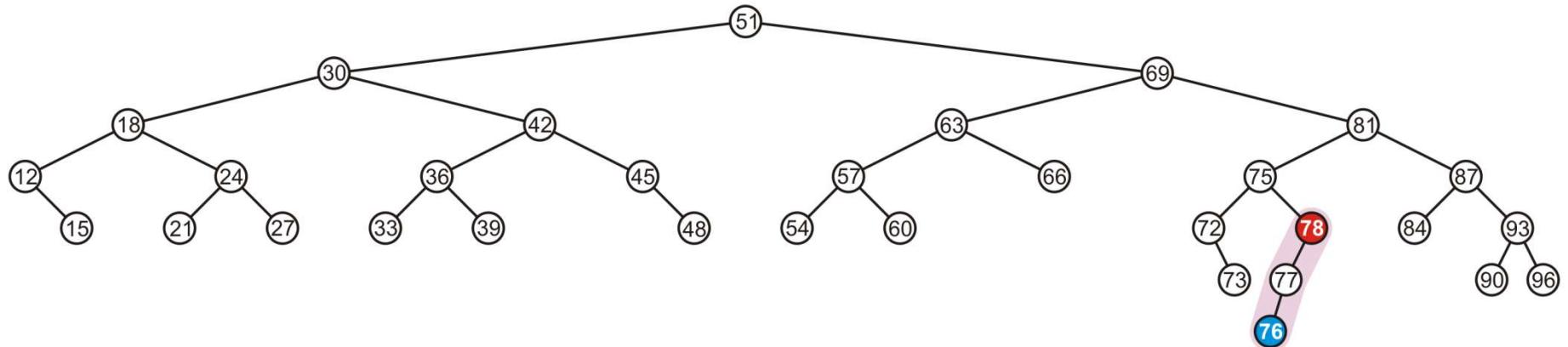
Insertion

Insert 76



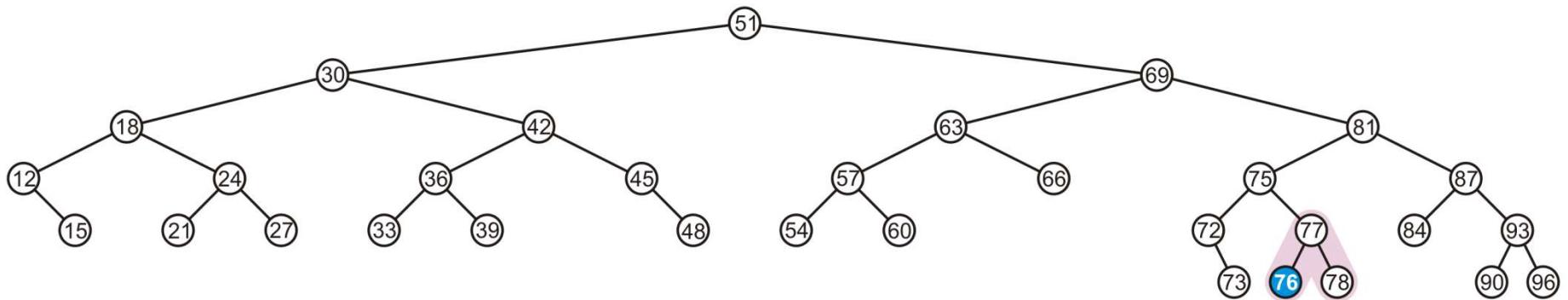
Insertion

The node 78 is unbalanced
– A left-left imbalance



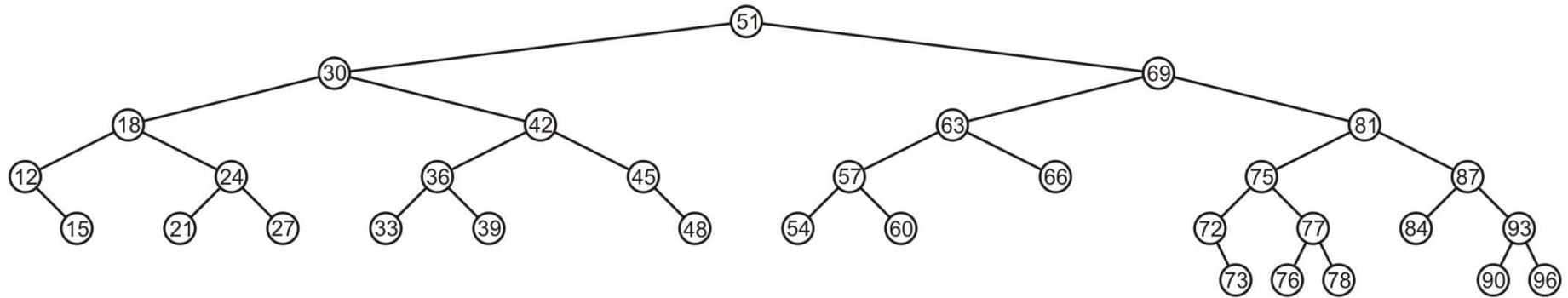
Insertion

The node 78 is unbalanced
– Promote 77



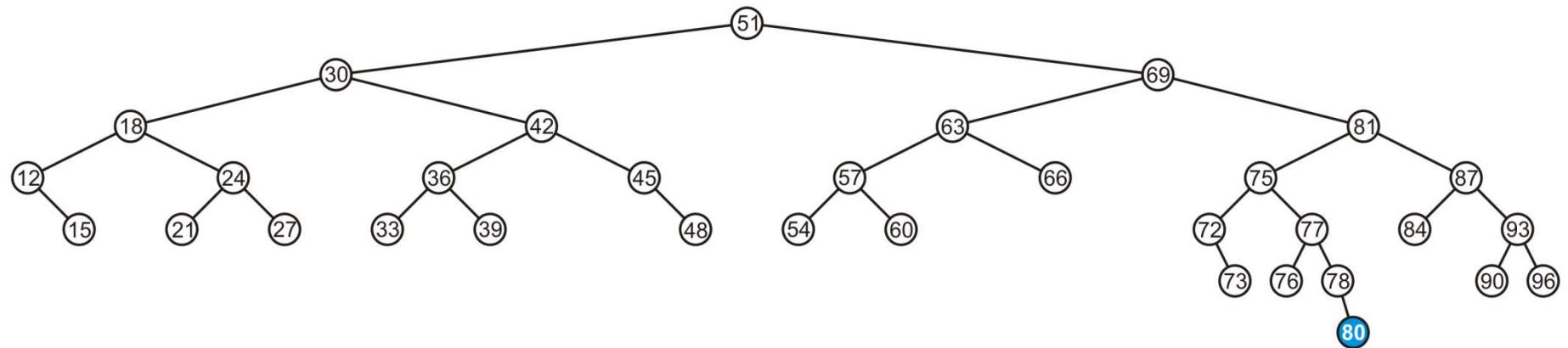
Insertion

Again, balanced



Insertion

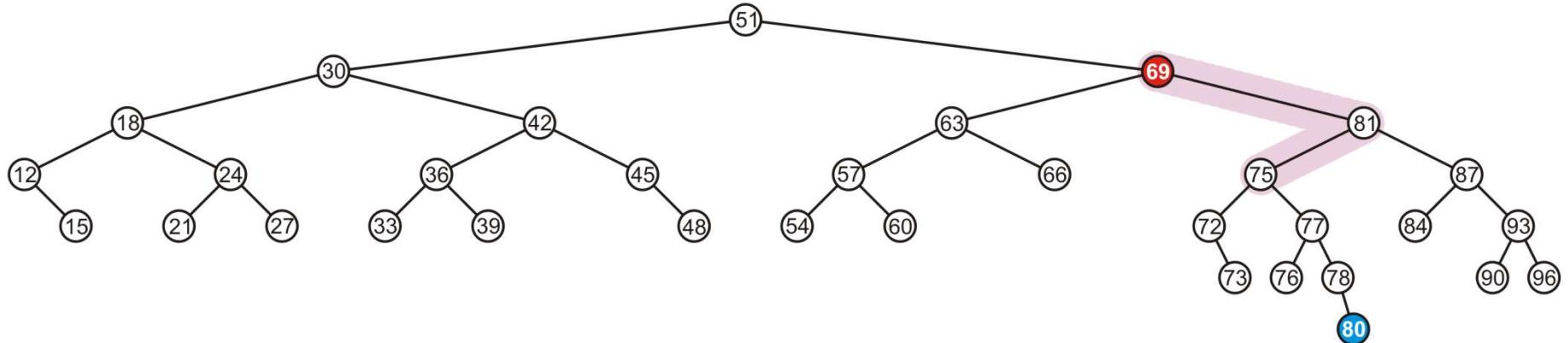
Insert 80



Insertion

The node 69 is unbalanced

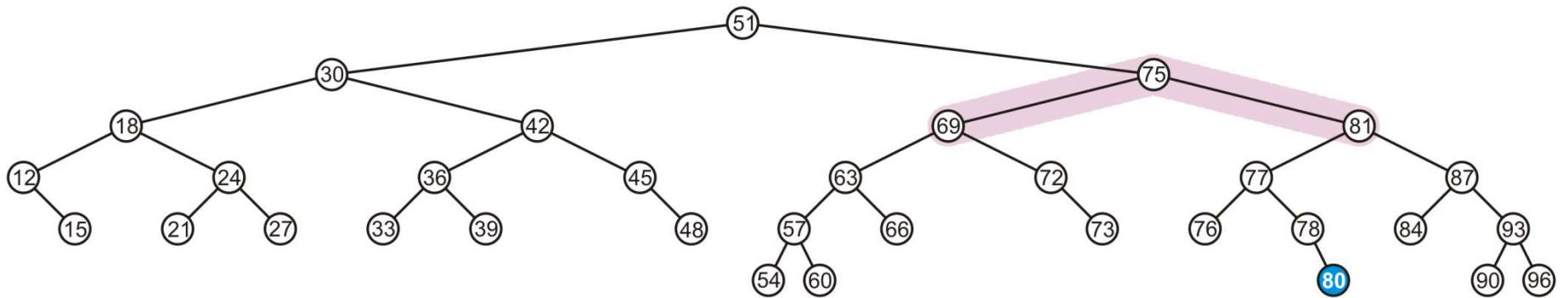
- A right-left imbalance
 - Promote the intermediate node to the imbalanced node



Insertion

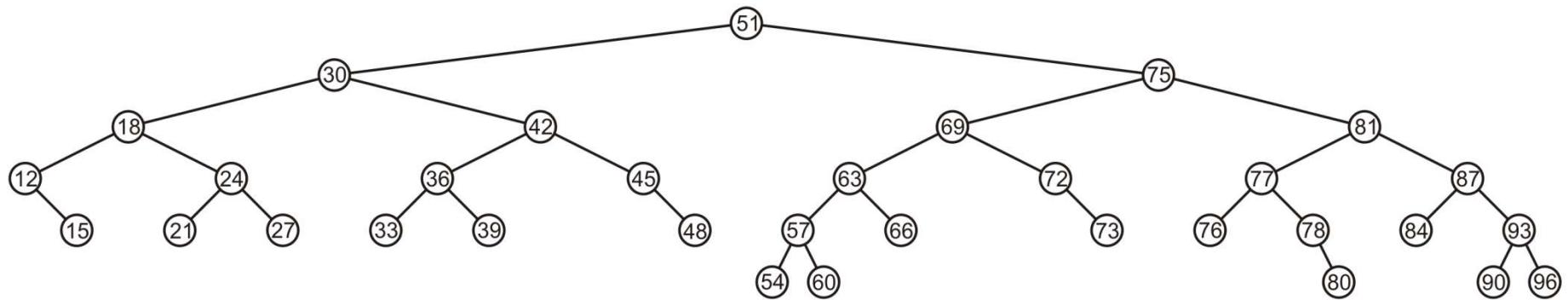
The node 69 is unbalanced

- A left-right imbalance
- Promote the intermediate node to the imbalanced node
- 75 is that value



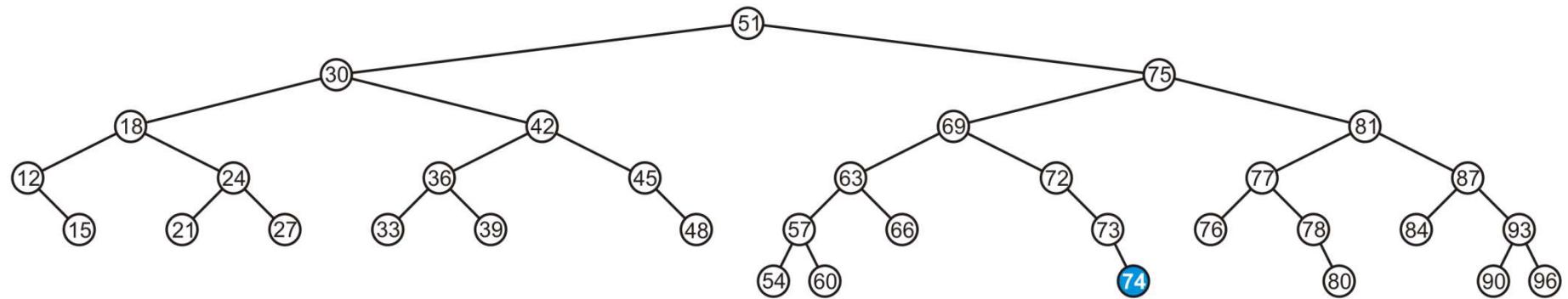
Insertion

Again, balanced



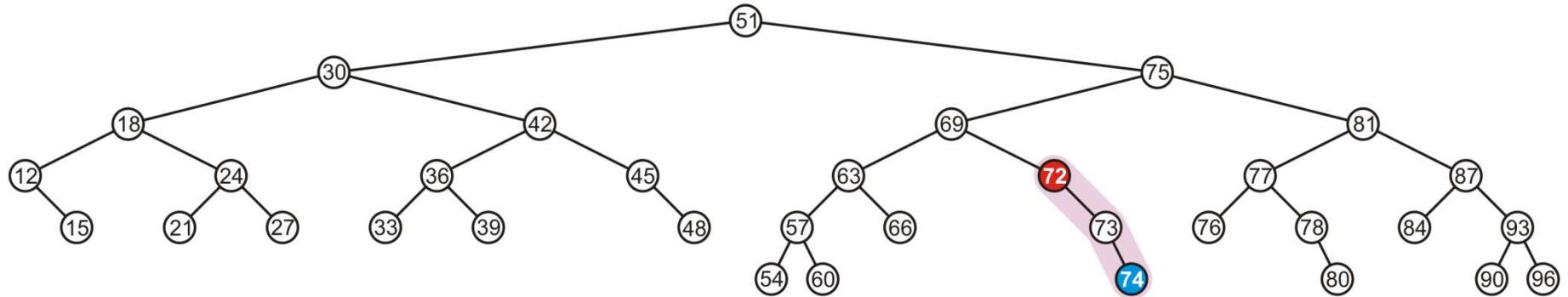
Insertion

Insert 74



Insertion

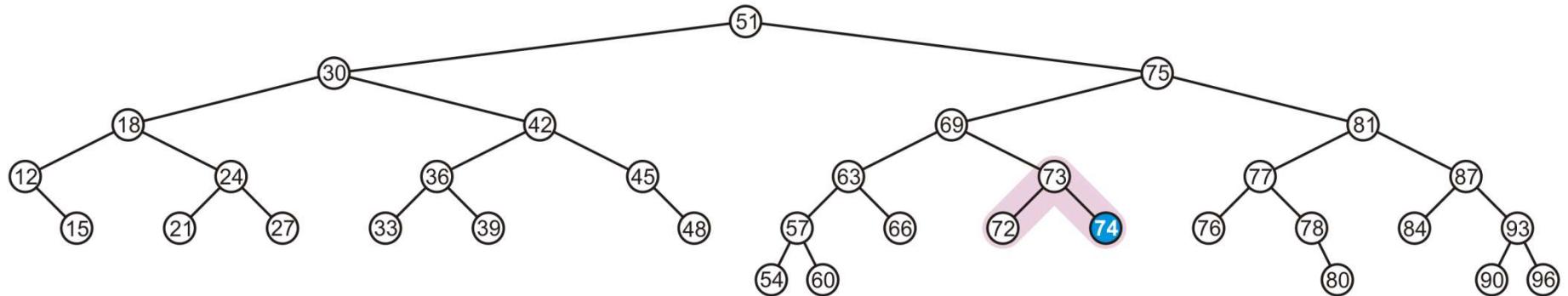
The node 72 is unbalanced
– A right-right imbalance



Insertion

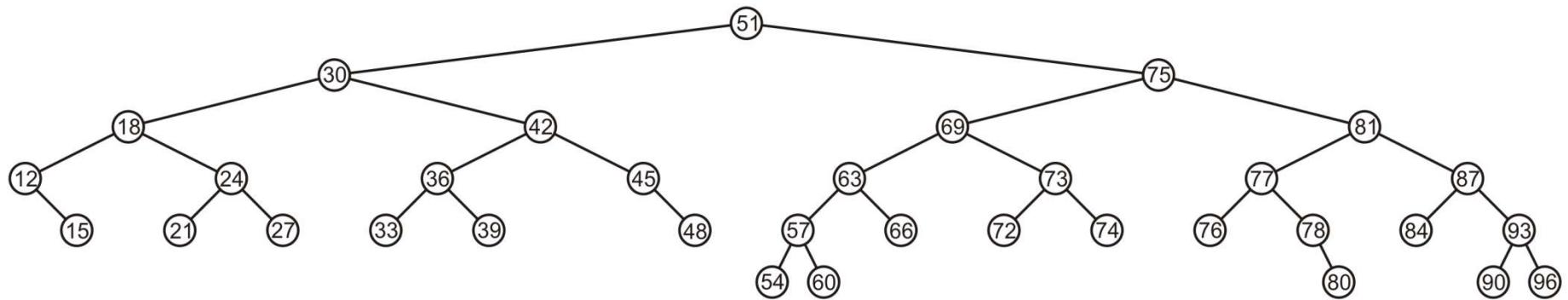
The node 72 is unbalanced

- A right-right imbalance
- Promote the intermediate node to the imbalanced node



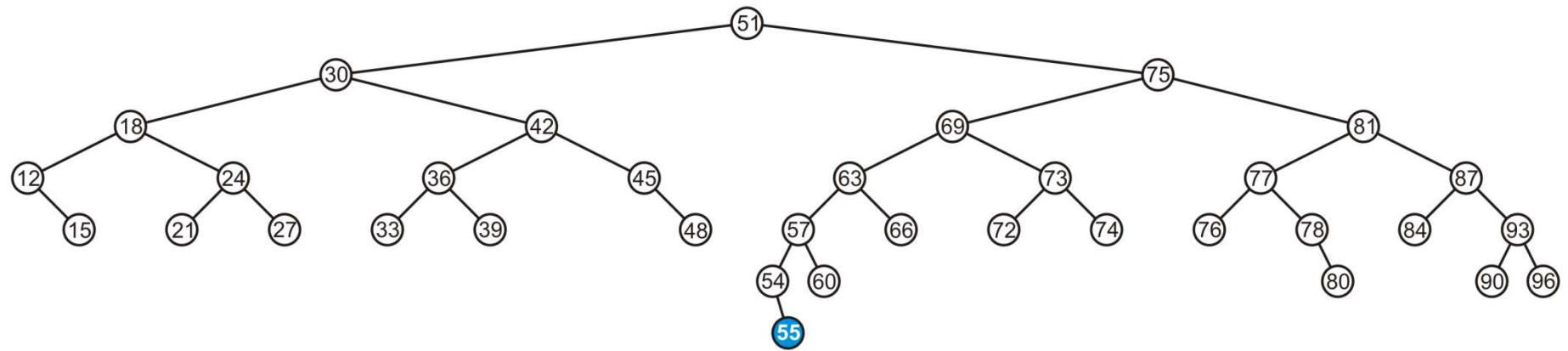
Insertion

Again, balanced



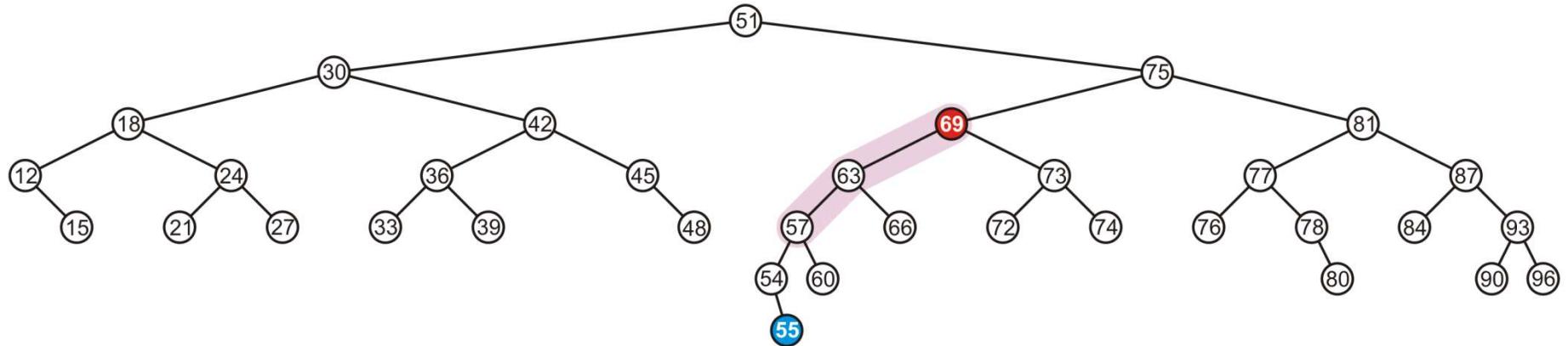
Insertion

Insert 55



Insertion

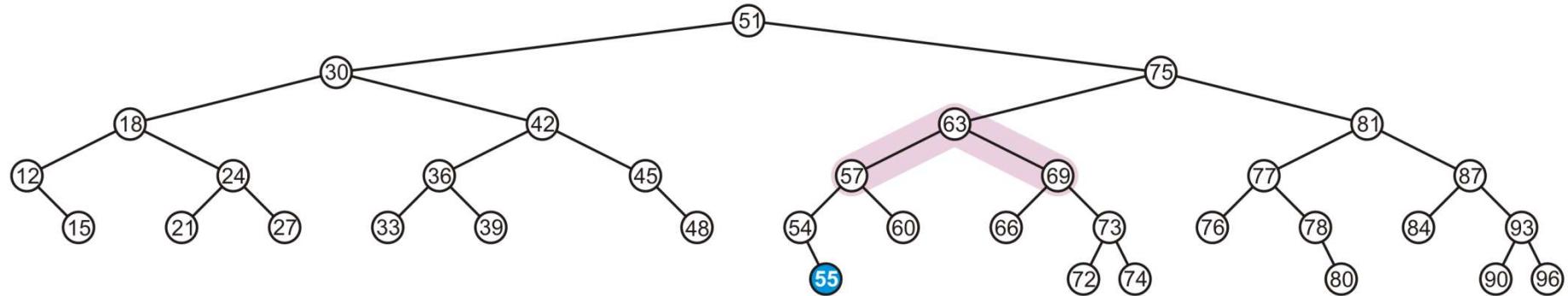
The node 69 is imbalanced
– A left-left imbalance



Insertion

The node 69 is imbalanced

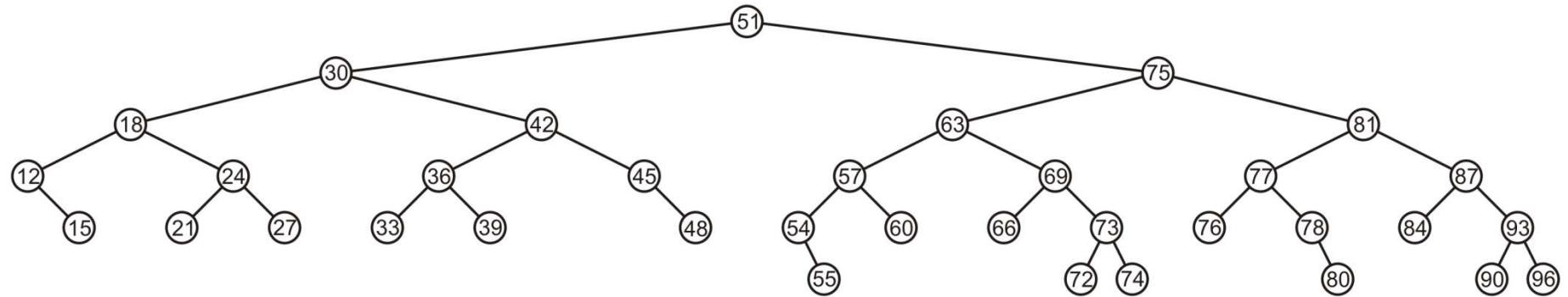
- A left-left imbalance
- Promote the intermediate node to the imbalanced node
- 63 is that value



Insertion

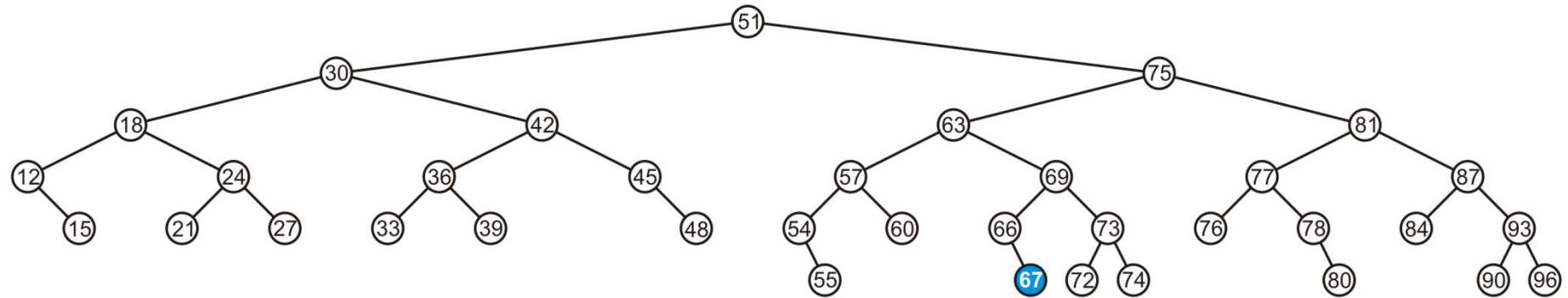
Insert 55

– No imbalances



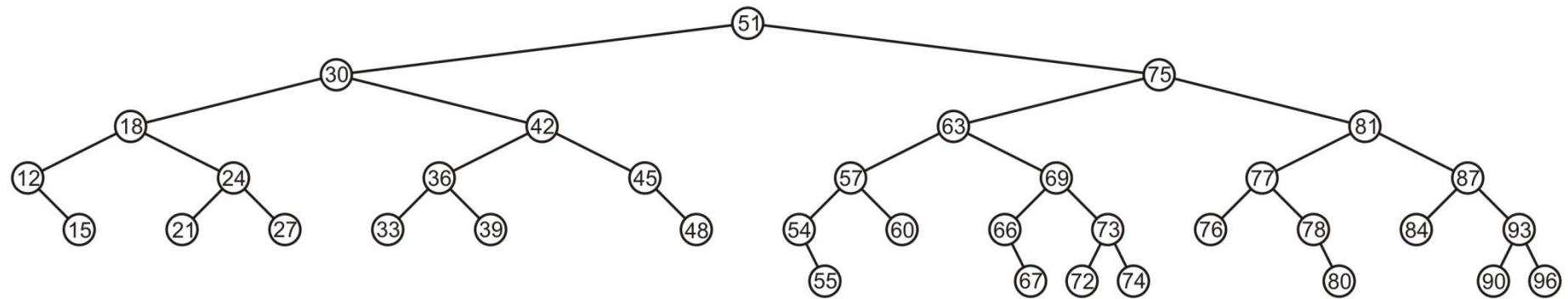
Insertion

Again, balanced



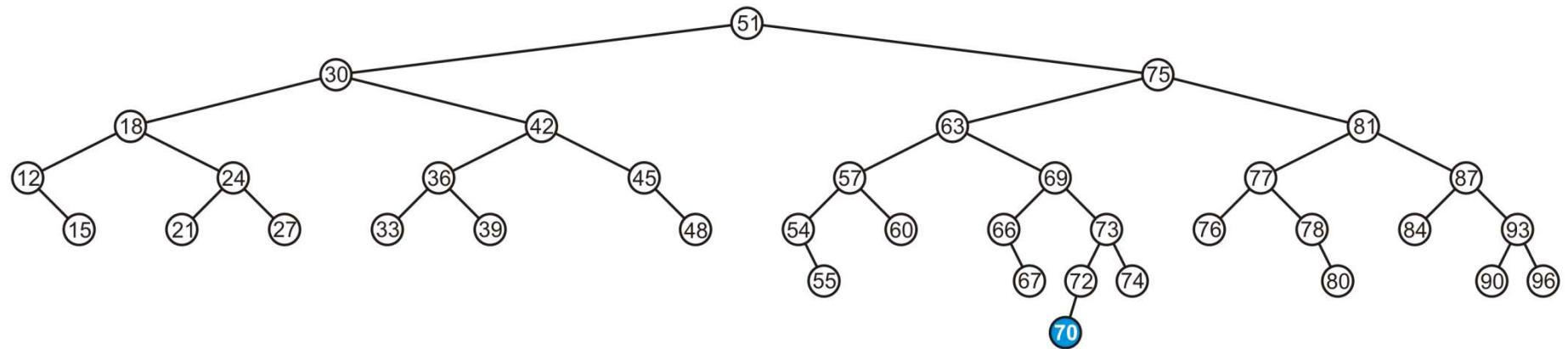
Insertion

Again, balanced



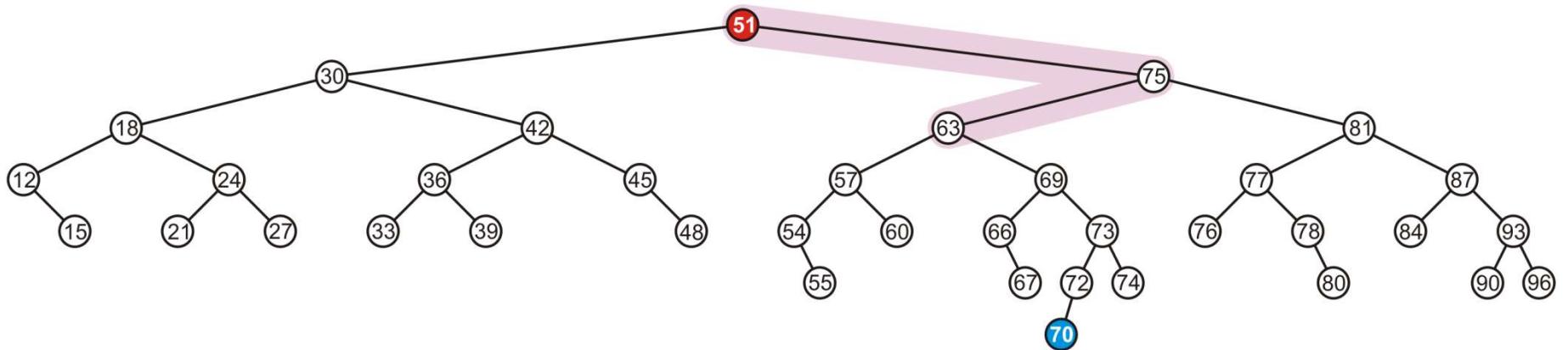
Insertion

Insert 70



Insertion

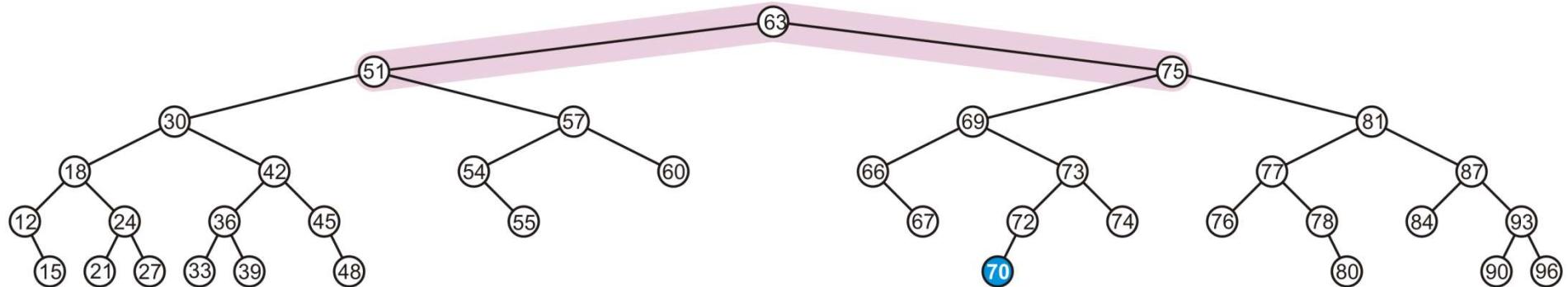
The root node is now imbalanced
– A right-left imbalance



Insertion

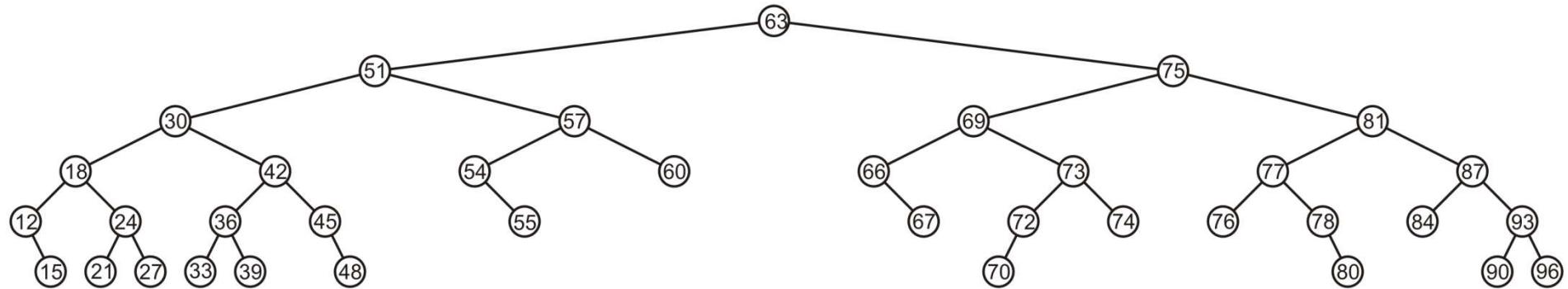
The root node is imbalanced

- A right-left imbalance
- Promote the intermediate node to the root
- 63 is that node



Insertion

The result is balanced



Summary : Insertions

Let the node that needs rebalancing be **j**.

There are 4 cases:

Outside Cases (require single rotation) :

1. Insertion into **left** subtree **of left** child of **j**.
2. Insertion into **right** subtree **of right** child of **j**.

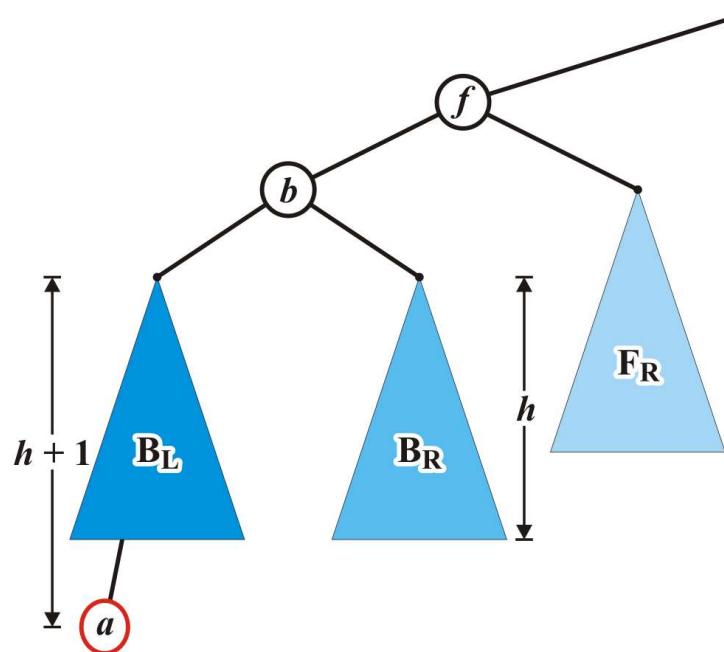
Inside Cases (require double rotation) :

3. Insertion into **right** subtree **of left** child of **j**.
4. Insertion into **left** subtree **of right** child of **j**.

The rebalancing is performed through four separate rotation algorithms.

Outside Case

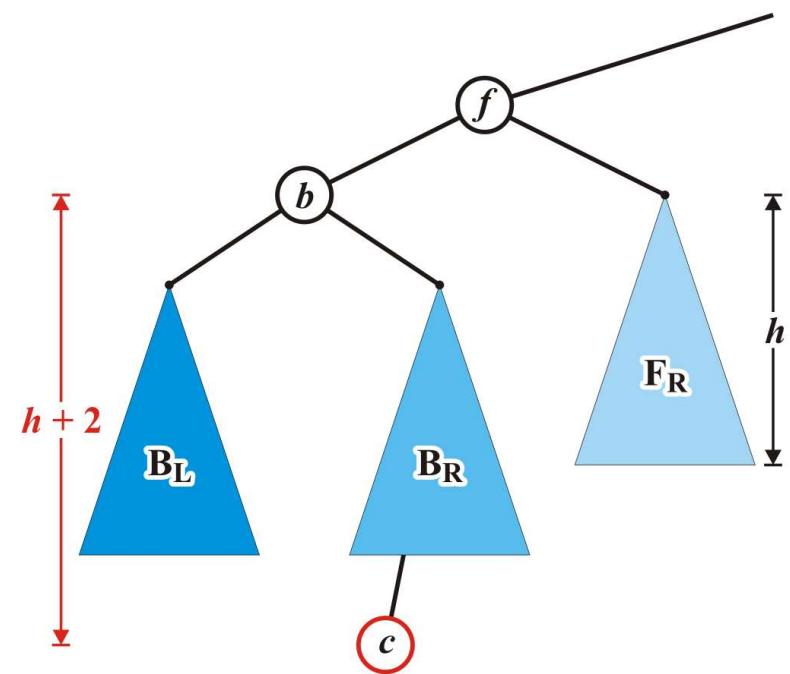
Left subtree of left child



Single “right” Rotation

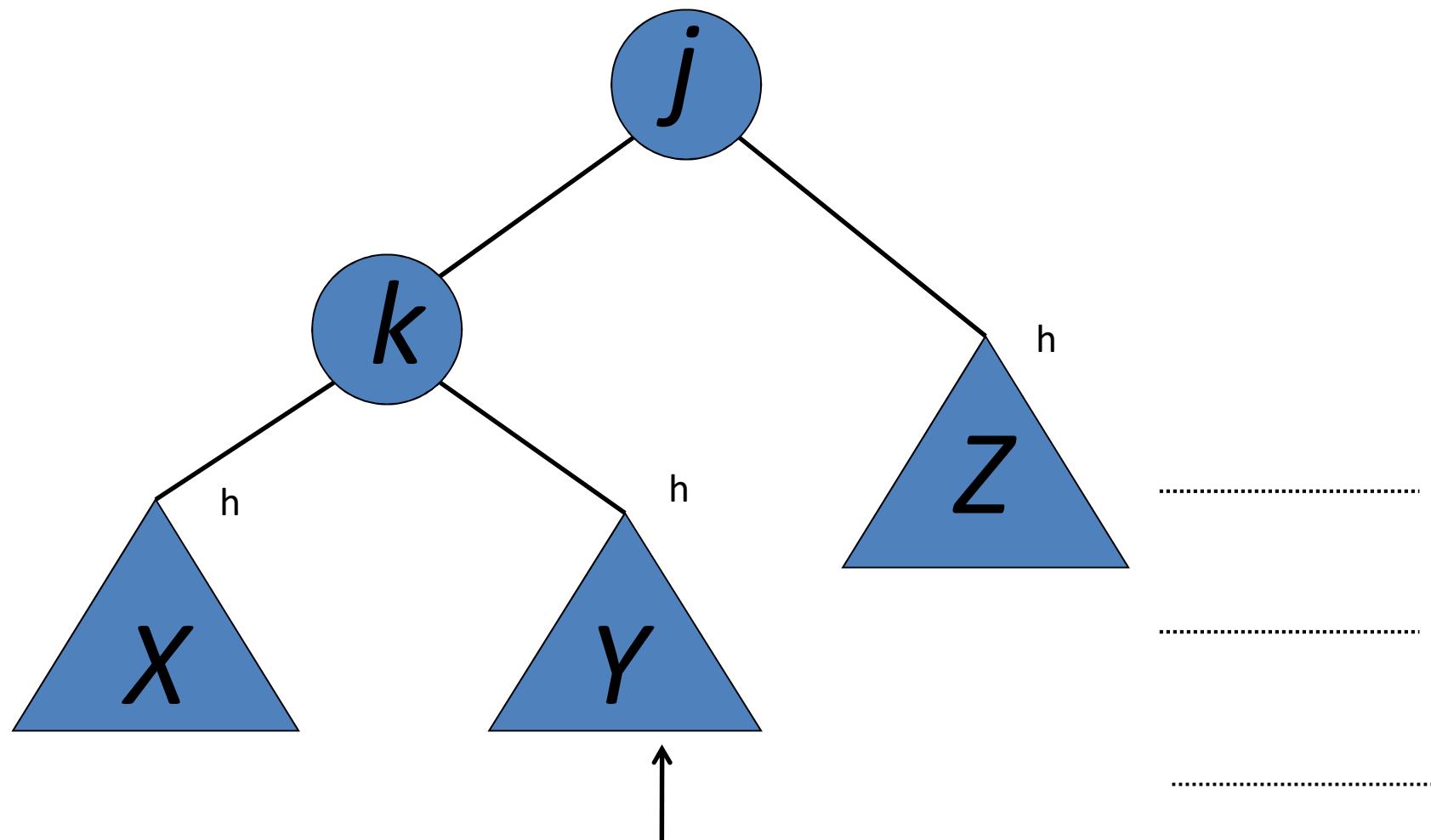
Inside Case

Right subtree of left child



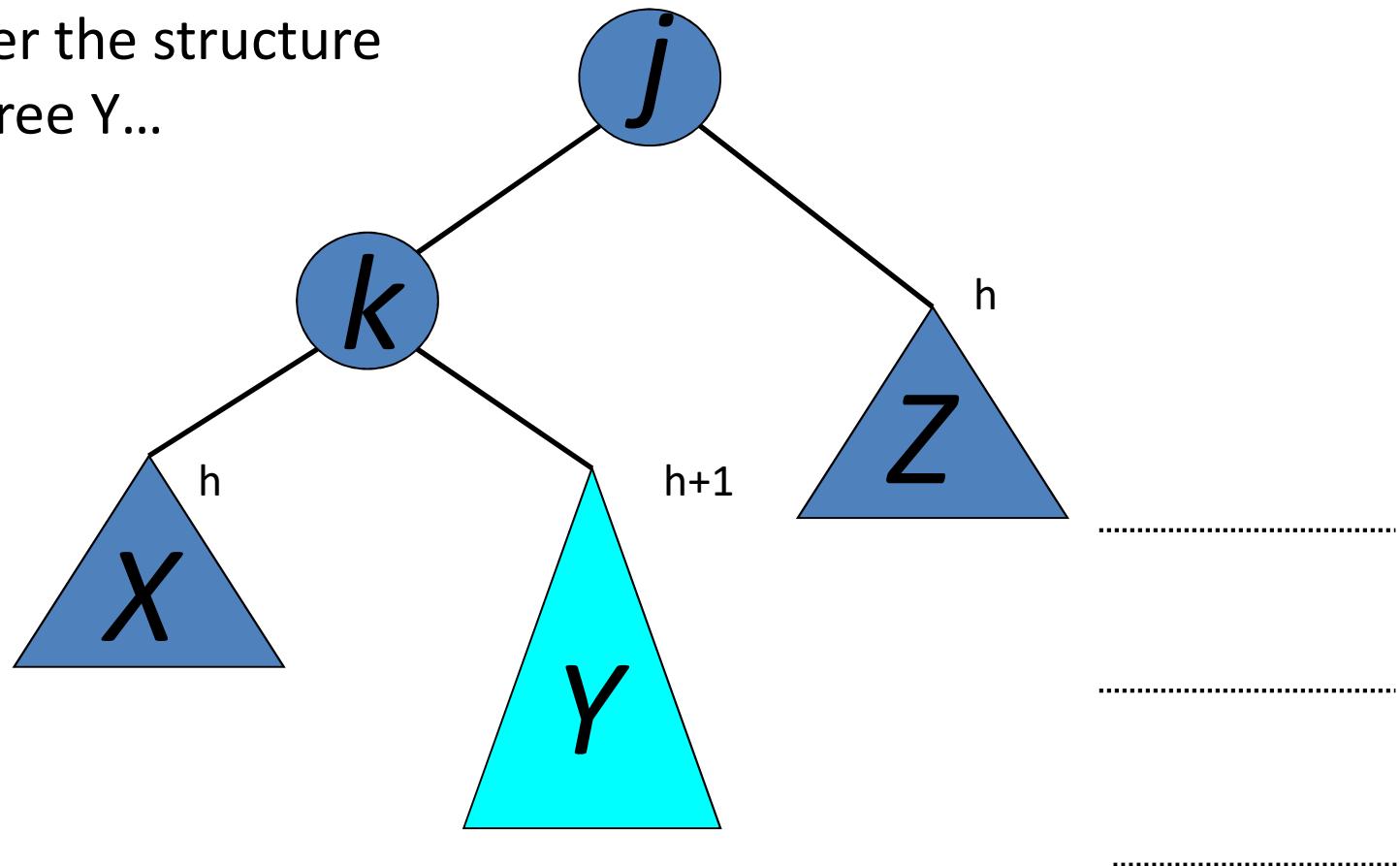
“left-right” Double Rotation

Inside Case Recap

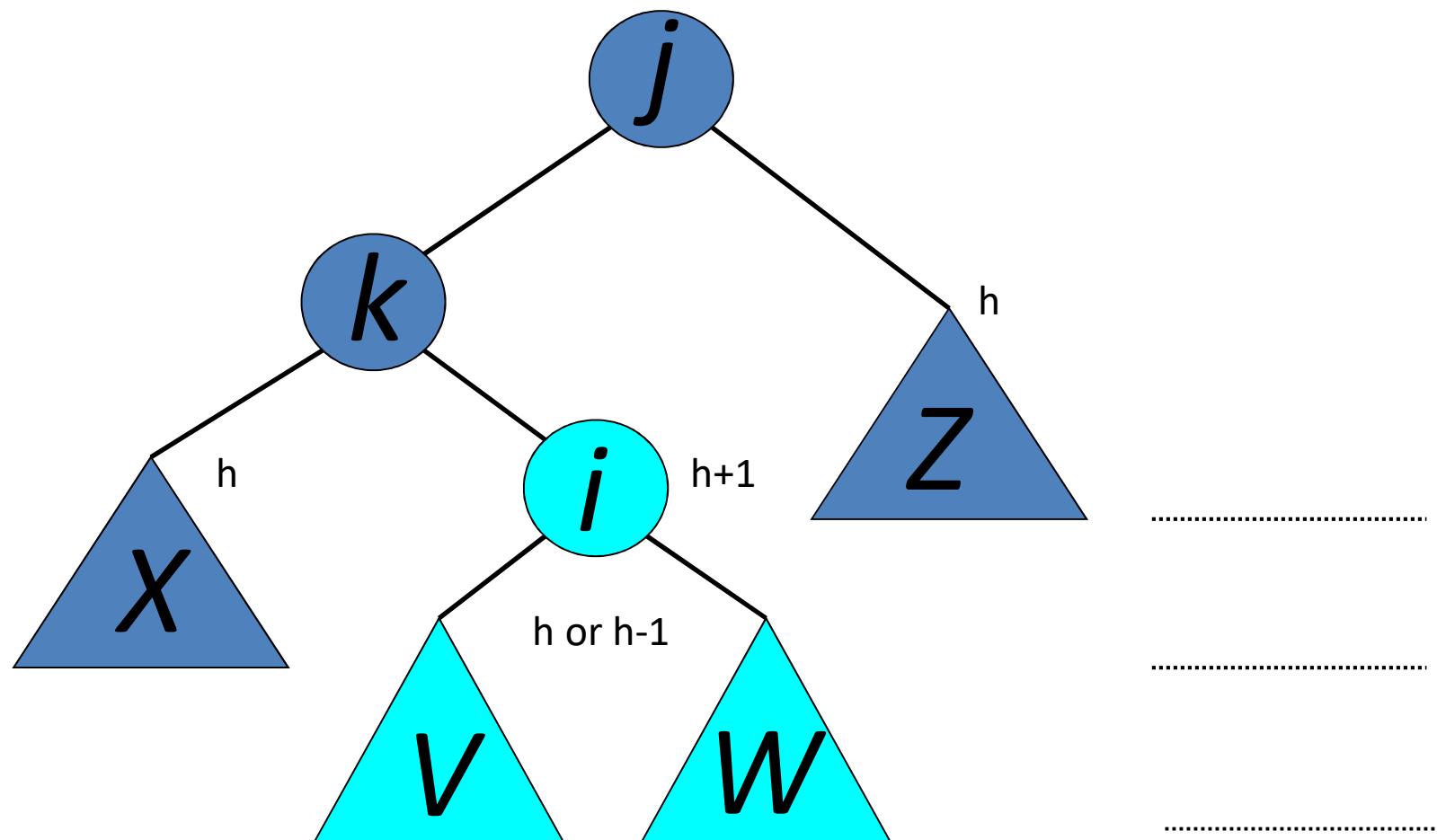


AVL Insertion: Inside Case

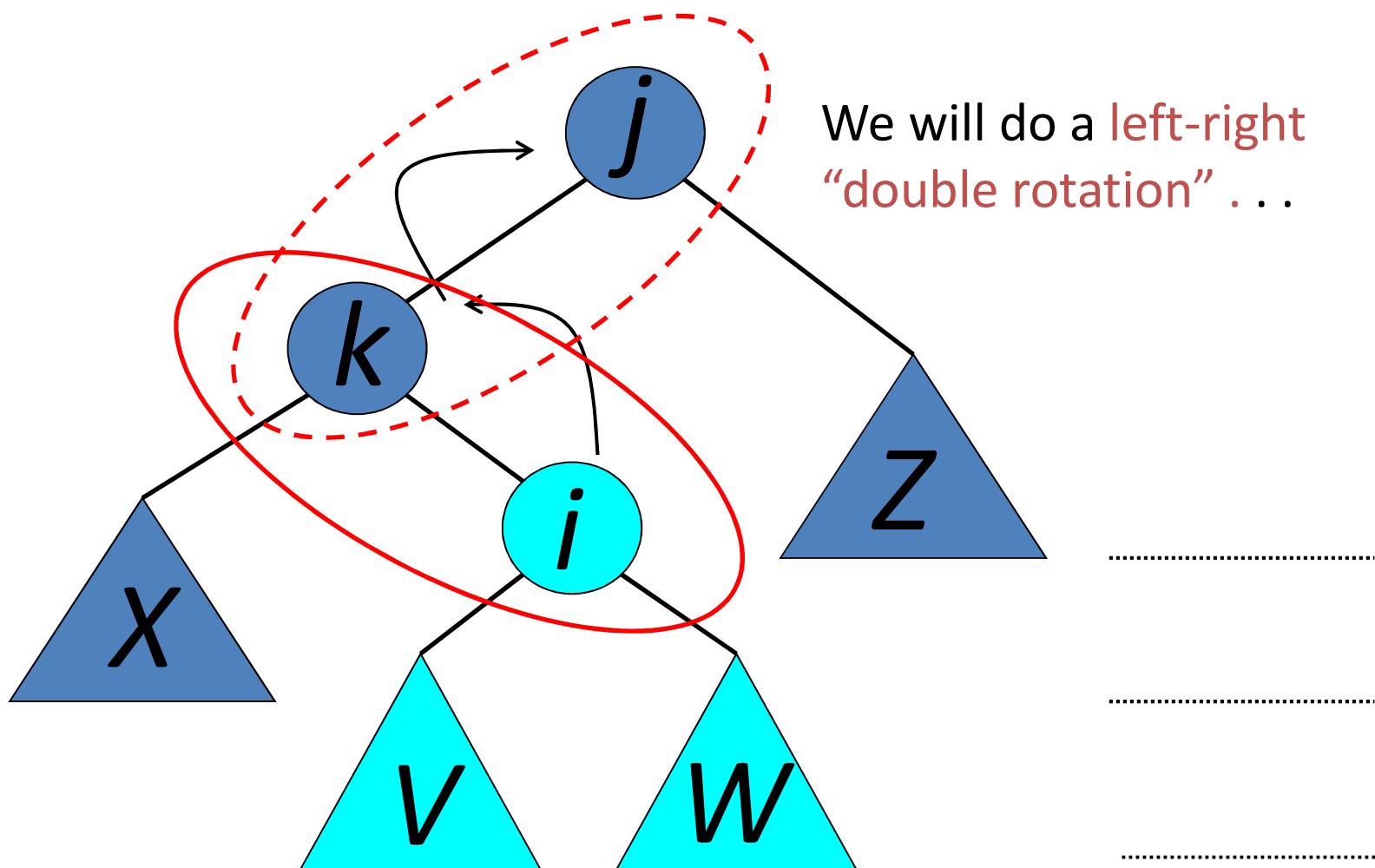
Consider the structure
of subtree Y...



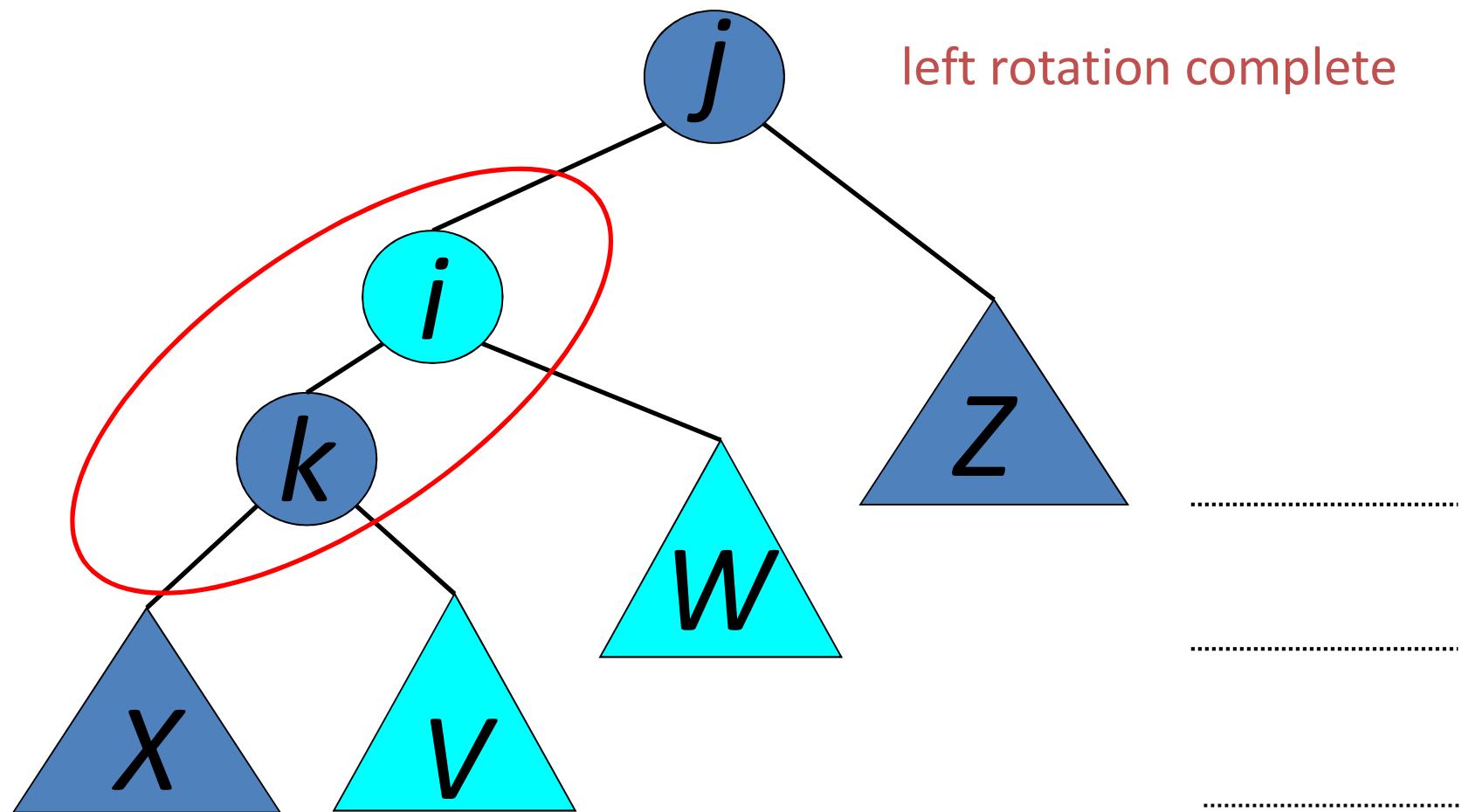
AVL Insertion: Inside Case



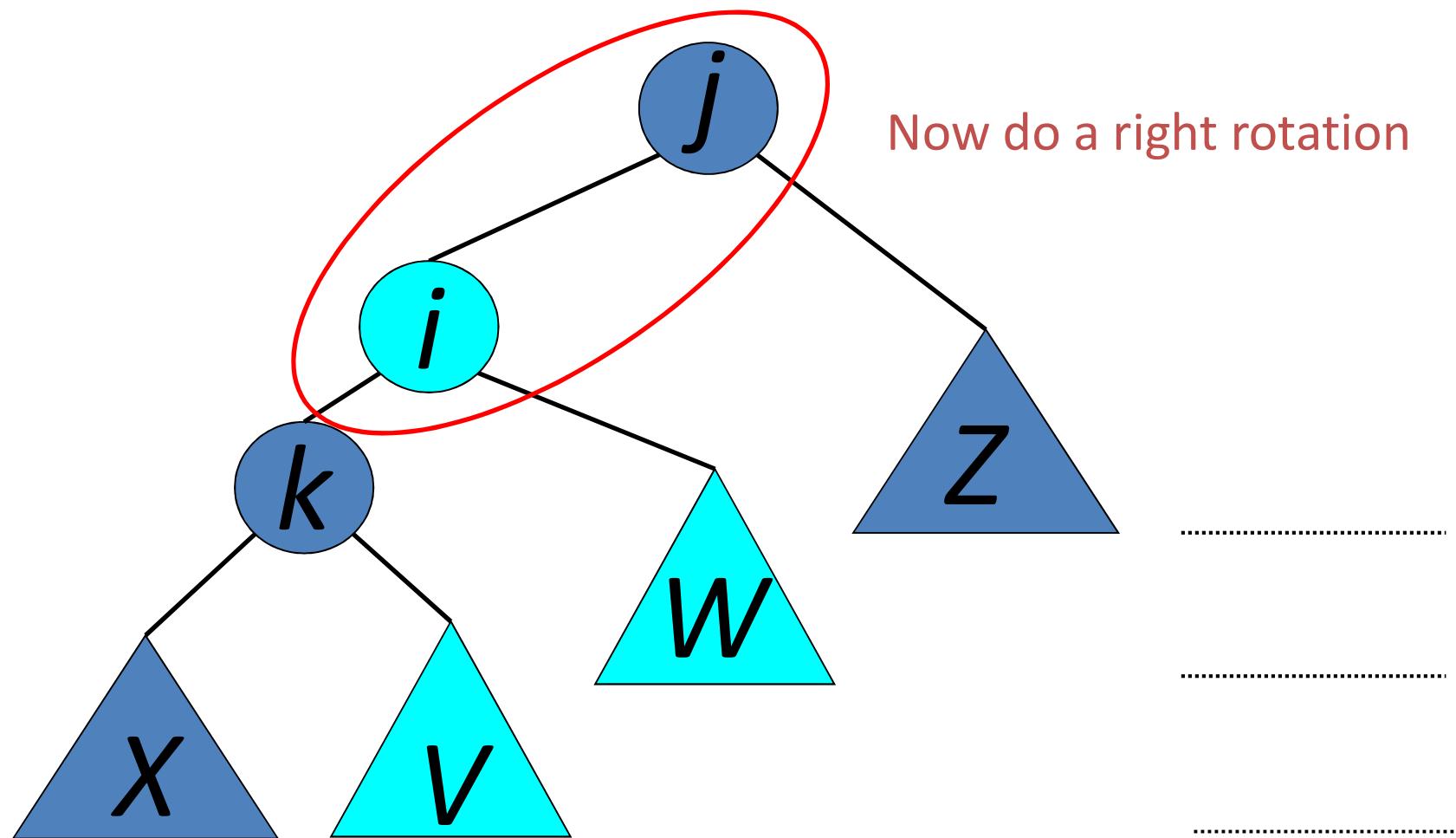
AVL Insertion: Inside Case



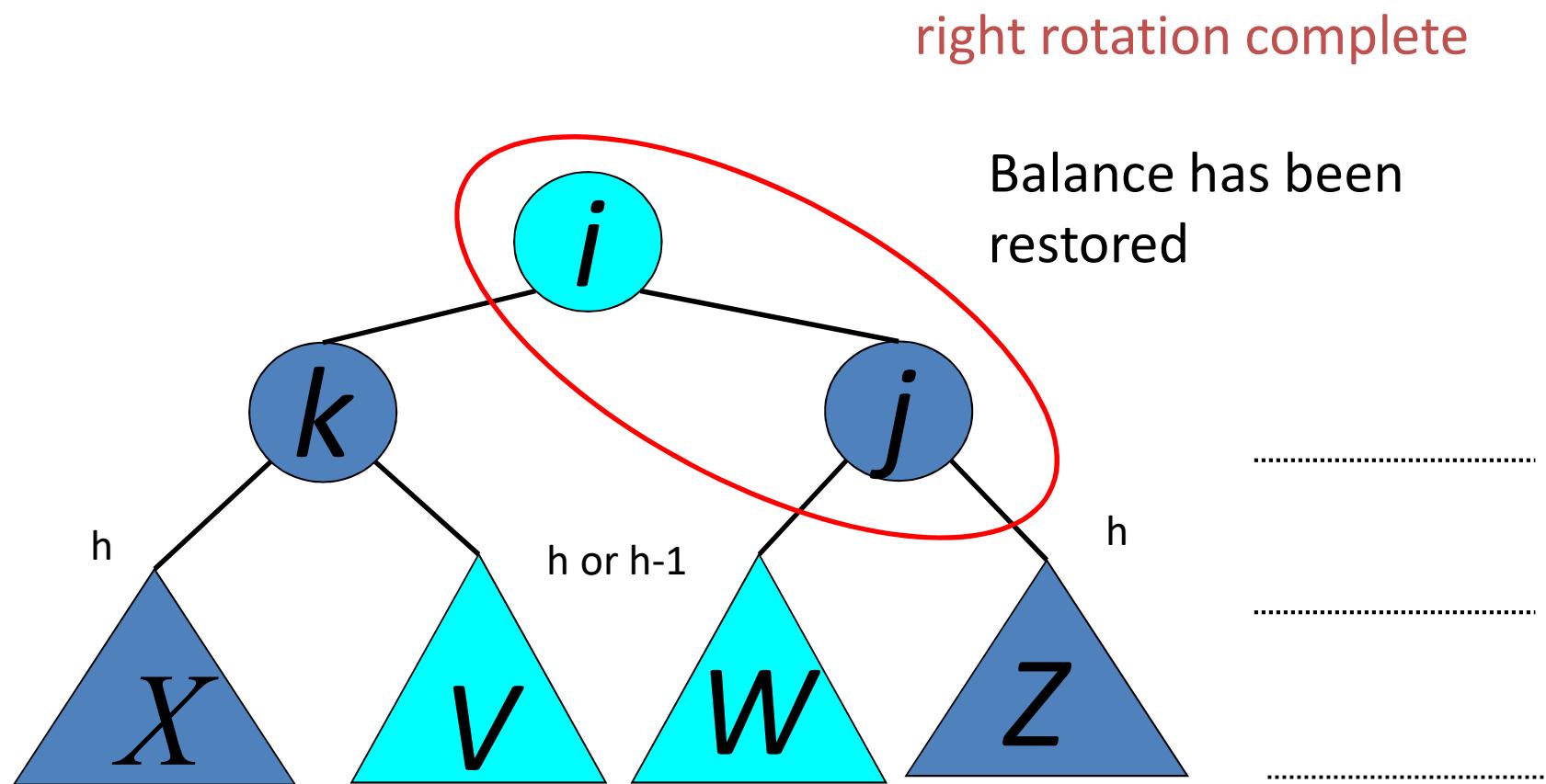
Double rotation : first rotation



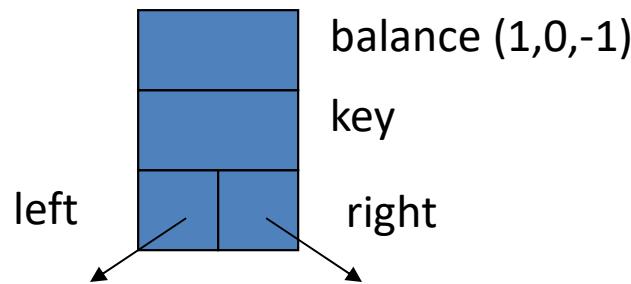
Double rotation : second rotation



Double rotation : second rotation



Implementation



No need to keep the height; just the difference in height, i.e. the **balance** factor; this has to be modified on the path of insertion even if you don't perform rotations

Once you have performed a rotation (single or double) you won't need to go back up the tree

Insertion in AVL Trees

- Insert at the leaf (as for all BST)
 - only nodes on the path from insertion point to root node have possibly changed in height
 - So after the Insert, go back up to the root node by node, updating heights
 - If a new balance factor (the difference $h_{\text{left}} - h_{\text{right}}$) is 2 or -2, adjust tree by *rotation* around the node

Correctness: Rotations preserve inorder traversal ordering

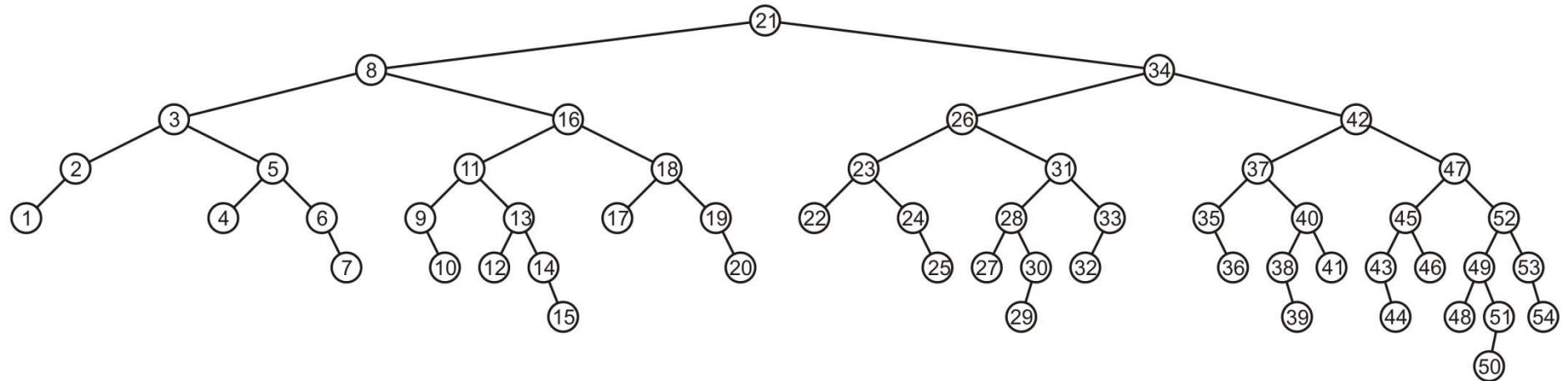
Erase

Removing a node from an AVL tree may cause more than one AVL imbalance

- Like insert, erase must check after it has been successfully called on a child to see if it caused an imbalance
- Unfortunately, it may cause multiple imbalances that must be corrected
 - Insertions will only cause one imbalance that must be fixed

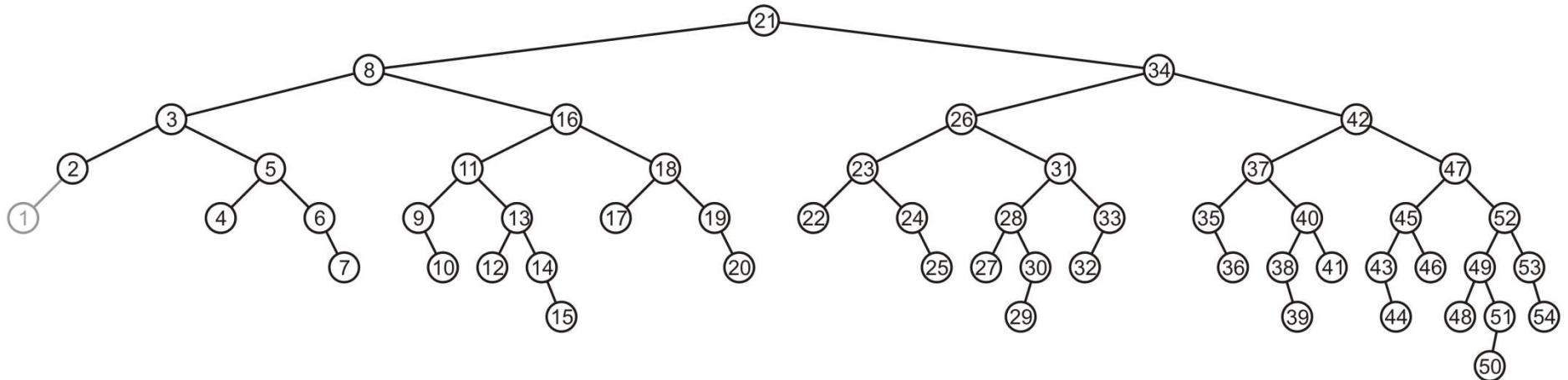
Erase

Consider the following AVL tree



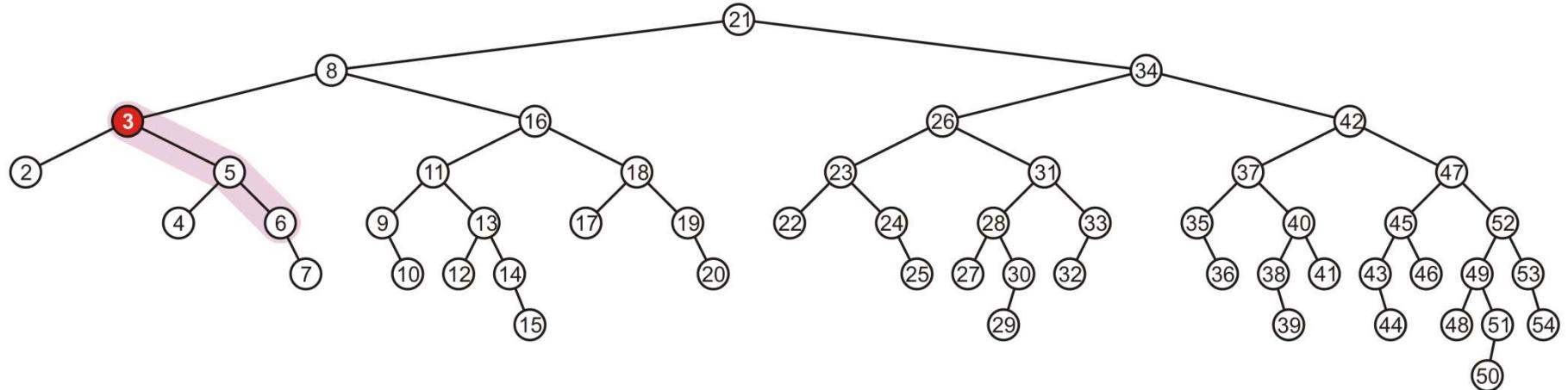
Erase

Suppose we erase the front node: 1



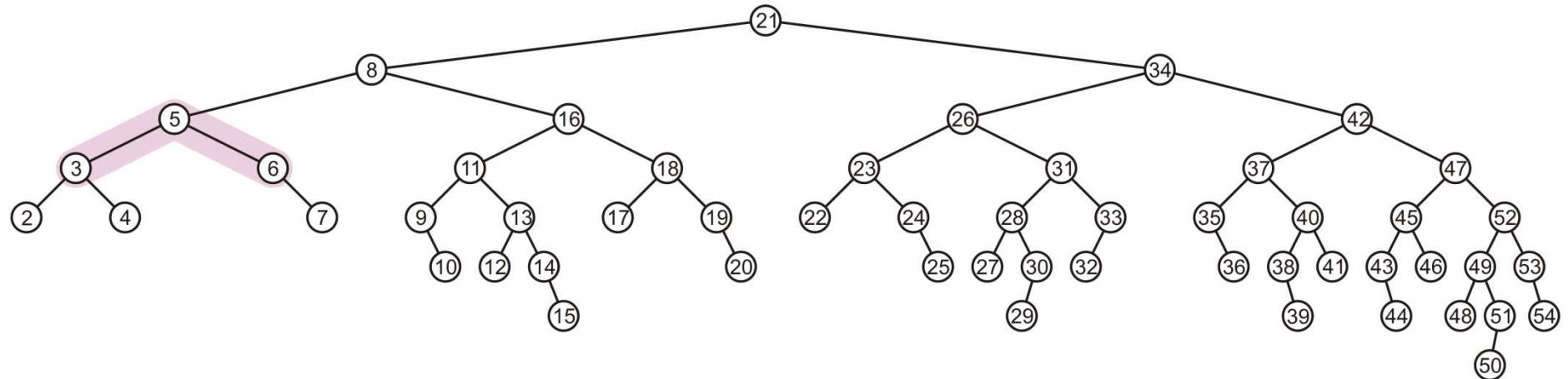
Erase

While its previous parent, 2, is not unbalanced, its grandparent 3 is



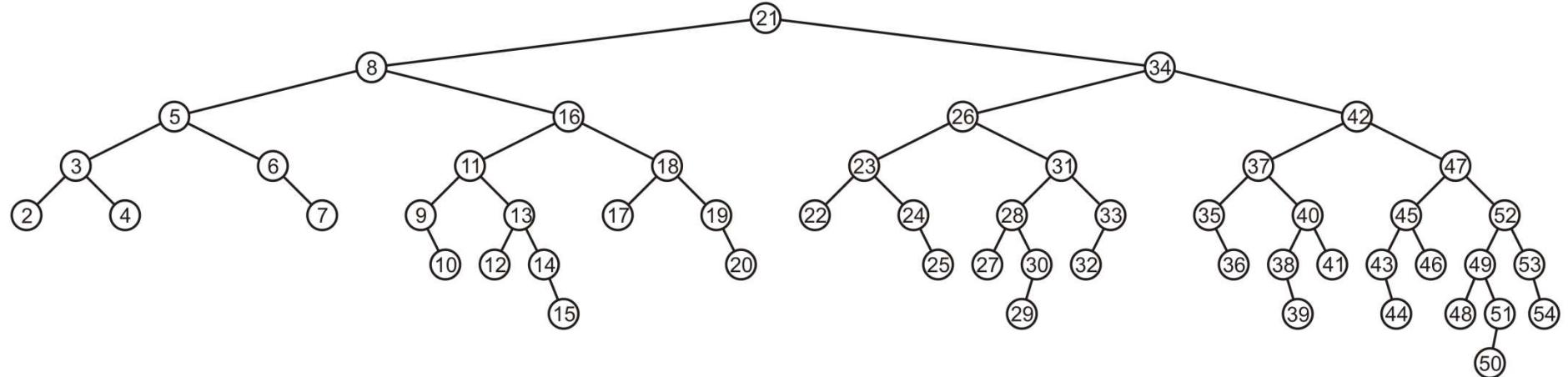
Erase

We can correct this with a simple balance



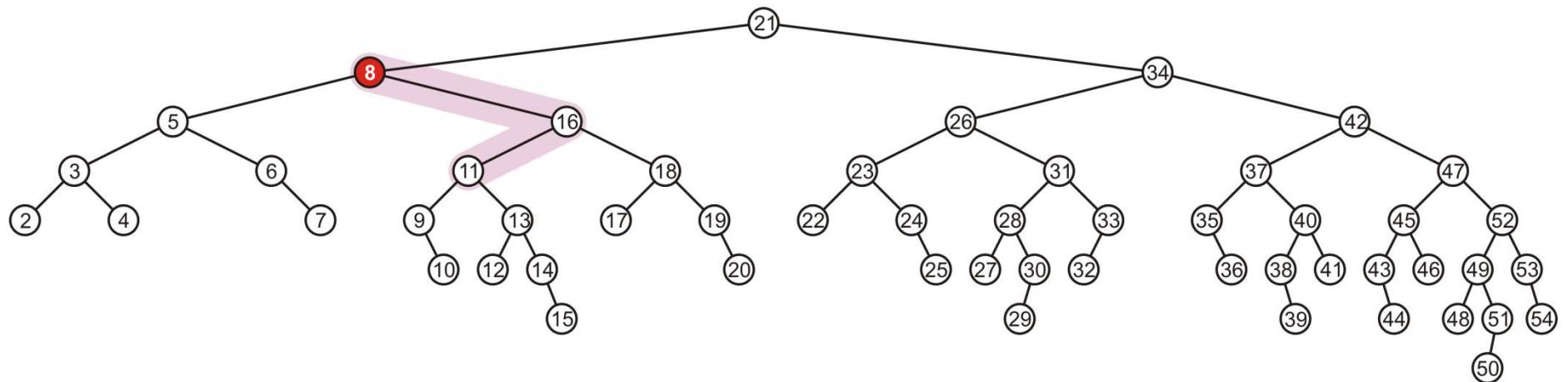
Erase

The node of that subtree, 5, is now balanced



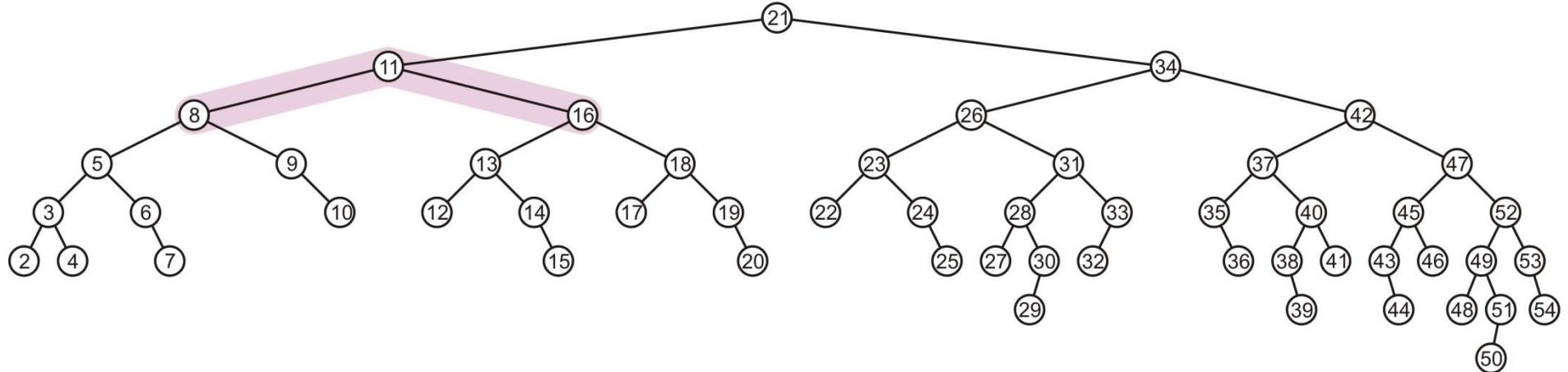
Erase

Recurse to the root, however, 8 is also unbalanced
– This is a right-left imbalance



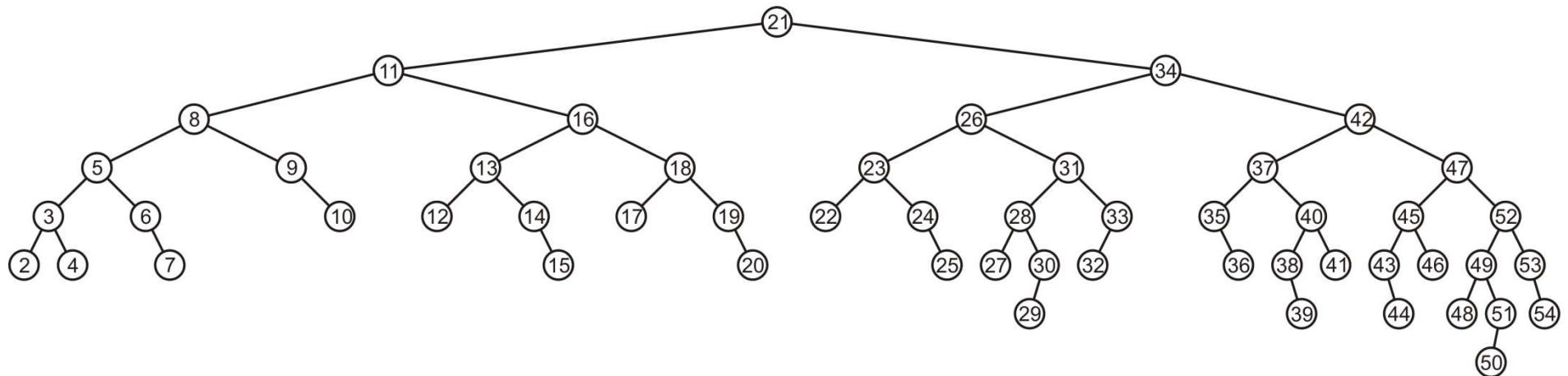
Erase

Promoting 11 to the root corrects the imbalance



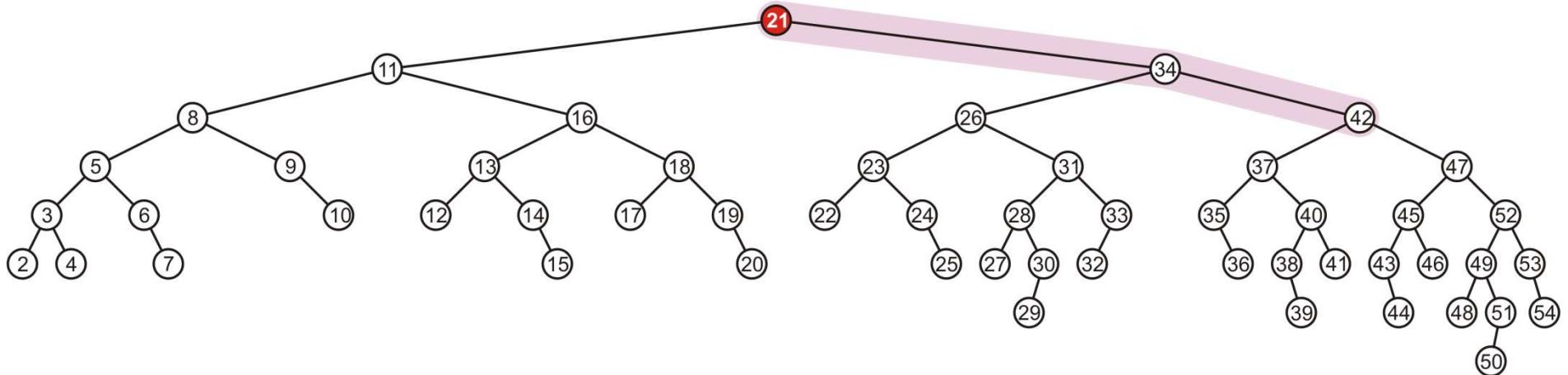
Erase

At this point, the node 11 is balanced



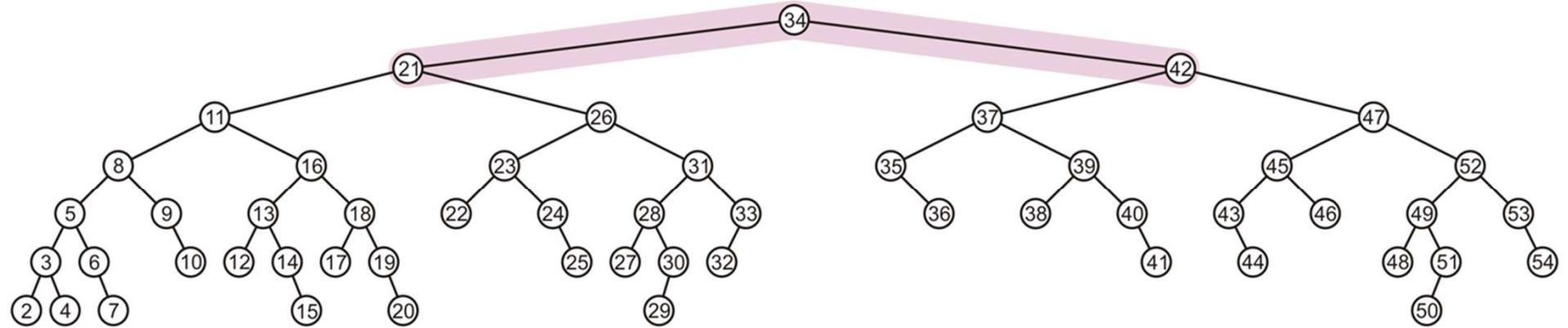
Erase

Still, the root node is unbalanced
– This is a right-right imbalance



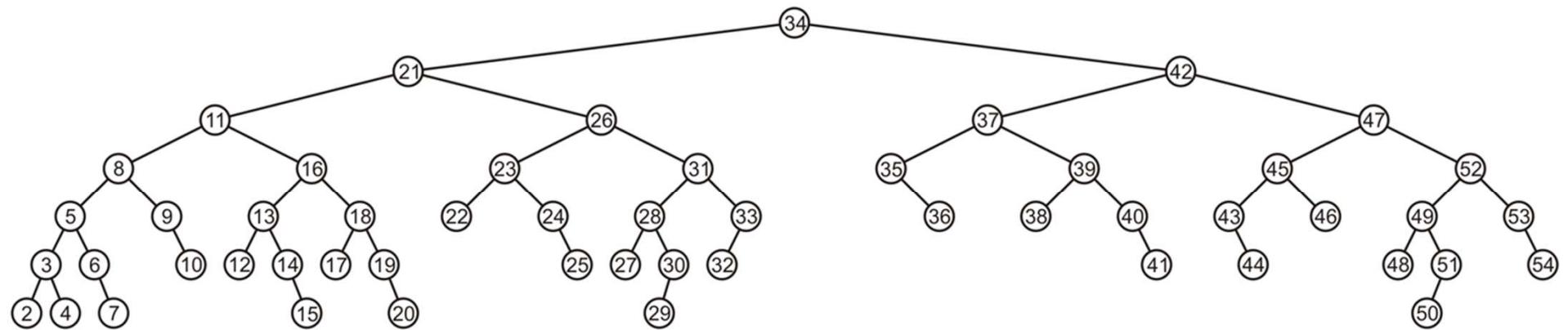
Erase

Again, a simple balance fixes the imbalance



Erase

The resulting tree is now AVL balanced



Pros and Cons of AVL Trees

Arguments for AVL trees:

1. Search is $O(\log N)$ since AVL trees are **always balanced**.
2. Insertion and deletions are also $O(\log n)$
3. The height balancing adds no more than a constant factor to the speed of insertion.

Arguments against using AVL trees:

1. Difficult to program & debug; more space for balance factor.
2. Asymptotically faster but rebalancing costs time.
3. Most large searches are done in database systems on disk and use other structures (e.g. B-trees).
4. May be OK to have $O(N)$ for a single operation if total run time for many consecutive operations is fast (e.g. Splay trees).