Lecture 3: Model-Free Policy Evaluation: Policy Evaluation Without Knowing How the World Works[1]

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CS234 Reinforcement Learning

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[1]Material builds on structure from David SIlver's Lecture 4: Model-Free Prediction. Other resources: Sutton and Barto Jan 1 2018 draft Chapter/Sections: 5.1; 5.5; 6.1-6.3

Today's Plan

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- **Today**
  - **Policy evaluation without known dynamics & reward models**
- Next Time:
  - Control when don't have a model of how the world works

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
    - Policy evaluation when don't have a model of how the world work
        - Given on-policy samples
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

Recall

- Definition of Return, G_t (for a MRP)
  - Discounted sum of rewards from time step t to horizon

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$$

- Definition of State Value Function, $V^\pi(s)$
  - Expected return from starting in state s under policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s] = \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s]$$

- Definition of State-Action Value Function, $Q^\pi(s, a)$
  - Expected return from starting in state s, taking action a and then following policy π

$$\begin{aligned} Q^\pi(s, a) &= \mathbb{E}_\pi[G_t | s_t = s, a_t = a] \\ &= \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s, a_t = a] \end{aligned}$$

Dynamic Programming for Policy Evaluation

- Initialize $V_0^\pi(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

Dynamic Programming for Policy π, Value Evaluation

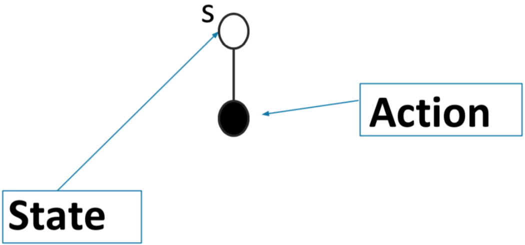
- Initialize $V_0^\pi(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

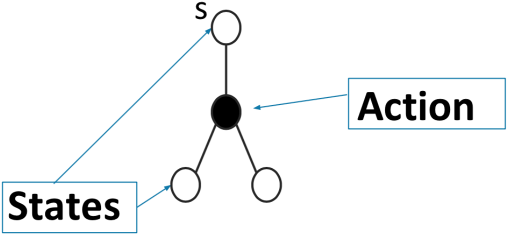
- $V_k^\pi(s)$ is exact value of k-horizon value of state s under policy π
- $V_k^\pi(s)$ is an estimate of infinite horizon value of state s under policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t|s_t = s] \approx \mathbb{E}_\pi[r_t + \gamma V_{k-1}|s_t = s]$$

Dynamic Programming Policy Evaluation $V^\pi(s) \leftarrow \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$



Dynamic Programming Policy Evaluation $V^\pi(s) \leftarrow \mathbb{E}_\pi[r_t + \gamma V_{k-1}|s_t = s]$



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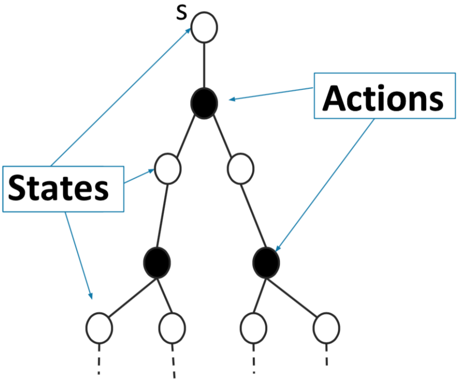

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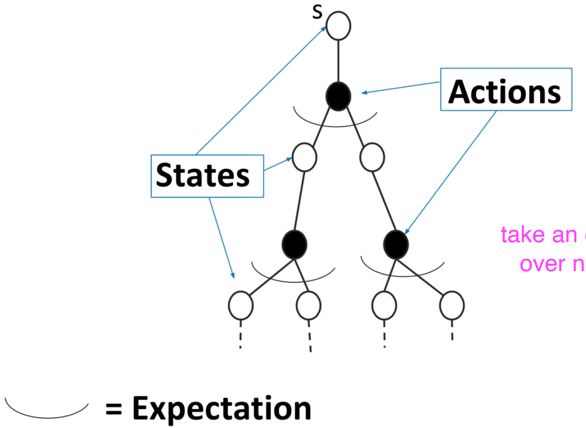
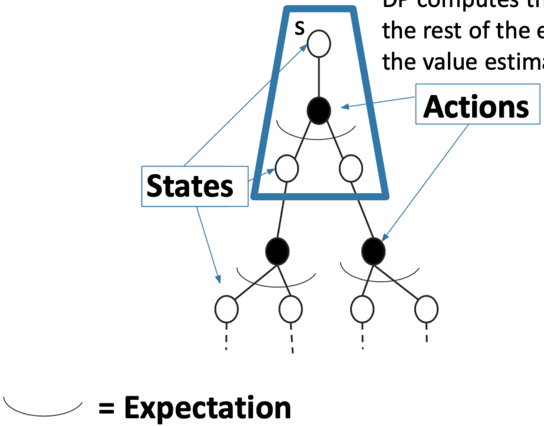


take an expectation over next states

Dynamic Programming Policy Evaluation $V^\pi(s) \leftarrow \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$

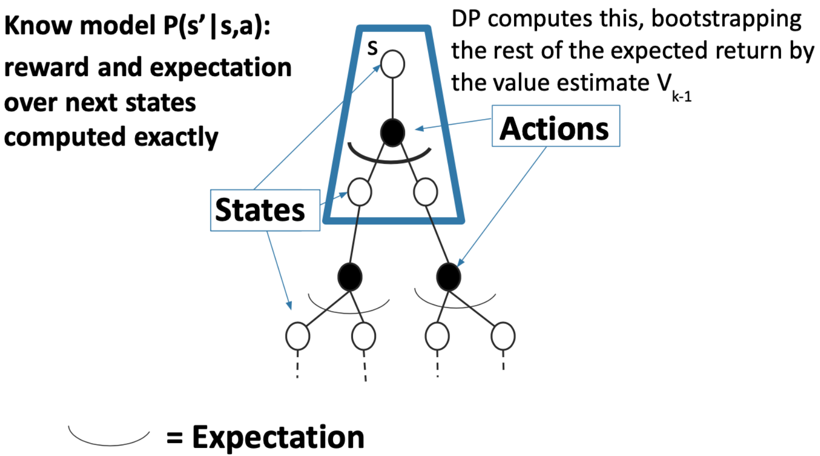
compute a one timestep expectation exactly

DP computes this, **bootstrapping** the rest of the expected return by the value estimate V_{k-1}



- Bootstrapping: Update for V uses an estimate

Dynamic Programming Policy Evaluation $V^\pi(s) \leftarrow \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$



- Bootstrapping: Update for V uses an estimate

Policy Evaluation: $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- Dynamic Programming
  - $V^\pi(s) \approx \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$
  - **Requires model of MDP** M
  - Bootstraps future return using value estimate
  - Requires Markov assumption: bootstrapping regardless of history
- What if don't know dynamics model P and/ or reward model R?
- **Today: Policy evaluation without a model**
  - Given data and/or ability to interact in the environment
  - Efficiently compute a good estimate of a policy π

This Lecture Overview: Policy Evaluation

- Dynamic Programming
- **Evaluating the quality of an estimator**
- **Monte Carlo policy evaluation**
    - Policy evaluation when don't know dynamics and/or reward model
        - Given on policy samples
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

Monte Carlo (MC) Policy Evaluation

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_{T\sim\pi}[G_t | s_t = s]$ now we have no transition(P)
  - Expectation over trajectories T generated by following π
- Simple idea: Value = mean return
- If trajectories are all finite, sample set of trajectories & average returns

Monte Carlo (MC) Policy Evaluation

- If trajectories are all finite, sample set of trajectories & average returns
- Does not require MDP dynamics/rewards
- No bootstrapping

  It doesn't try to maintain V_{k-1}, it's simply sums up all the rewards from each of your trajectories and then averages across those.
- Does not assume state is Markov
- Can **only** be applied to episodic MDPs
  - Averaging over returns from a complete episode
  - Requires each episode to terminate

Monte Carlo (MC) On Policy Evaluation

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- MC computes empirical mean return
- Often do this in an incremental fashion
  - After each episode, update estimate of V^π

First-Visit Monte Carlo (MC) On Policy Evaluation

#times visited a state

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$

Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
  - For **first** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Bias, Variance and MSE

- Consider a statistical model that is parameterized by θ and that determines a probability distribution over observed data $P(x|\theta)$
- Consider a statistic $\hat{\theta}$ that provides an estimate of θ and is a function of observed data x
  - E.g. for a Gaussian distribution with known variance, the average of a set of i.i.d data points is an estimate of the mean of the Gaussian
- Definition: the bias of an estimator $\hat{\theta}$ is:

$$Bias_\theta(\hat{\theta}) = \mathbb{E}_{x|\theta}[\hat{\theta}] - \theta$$

tips: we know
Var[X]=E[X²] - E[X]² ,
now expand the idea to
X=θ̂-θ

- Definition: the variance of an estimator $\hat{\theta}$ is:

https://www.youtube.com/watch?v=KtNwjbWbnh8

$$Var(\hat{\theta}) = \mathbb{E}_{x|\theta}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]$$

- Definition: mean squared error (MSE) of an estimator $\hat{\theta}$ is:

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias_\theta(\hat{\theta})^2$$

First-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0$ $\forall s \in S$

Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
    - For **first** time t that state s is visited in episode i
        - Increment counter of total first visits: $N(s) = N(s) + 1$
        - Increment total return $G(s) = G(s) + G_{i,t}$
        - Update estimate $V^\pi(s) = G(s)/N(s)$

Properties:

- V^π estimator is an $\boxed{\text{unbiased}}$ estimator of true $\mathbb{E}_\pi[G_t | s_t = s]$
- By law of large numbers, as $N(s) \to \infty$, $V^\pi(s) \to \mathbb{E}_\pi[G_t | s_t = s]$

This is reasonable, but it might not be very efficient. You might be at same state for many many steps, and you're only going to use the first state in an episode to update.

Every-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$

Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
  - For **every** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Every-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \dots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each state s visited in episode i
  - For **every** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Properties:

- V^π every-vist MC estimator is an $\boxed{\textbf{biased}}$ estimator of V^π
- But consistent estimator and often has better MSE

Incremental Monte Carlo (MC) On Policy Evaluation

In practice often you want do this incrementally. You may just want to kind of keep track of a running mean and update your count incrementally.

After each episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots$

- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots$ as return from time step t onwards in ith episode
- For state s visited at time step t in episode i
  - Increment counter of total first visits: $N(s) = N(s) + 1$
  - Update estimate

$$V^\pi(s) = V^\pi(s)\frac{N(s)-1}{N(s)} + \frac{G_{i,t}}{N(s)} = V^\pi(s) + \frac{1}{N(s)}(G_{i,t} - V^\pi(s))$$

Incremental Monte Carlo (MC) On Policy Evaluation, Running Mean

Initialize $N(s) = 0$, $G(s) = 0 \ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For state s visited at time step t in episode i
  - Increment counter of total first visits: $N(s) = N(s) + 1$
  - Update estimate

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

- $\alpha = \frac{1}{N(s)}$: identical to every visit MC
- $\alpha > \frac{1}{N(s)}$: forget older data, helpful for non-stationary domains

Check Your Understanding: MC On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0 \; \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each state s visited in episode i
  - For **first or every** time t that state s is visited in episode i
    - $N(s) = N(s) + 1$, $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Example:

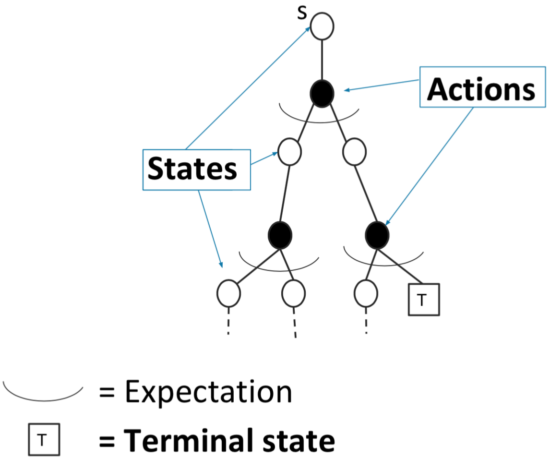
PS: gamma = 1 in this question

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \; \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of V of each state?
  V = [1 1 1 0 0 0 0]
- Every visit MC estimate of s_2?
  $V(s_2) = 1$

$N(s_2)$=2, $G(s_2)$=2 ; $V(s_2)$ =2/2 = 1
just occasionally same, normally different because of gamma setting and immediate reward.

MC Policy Evaluation

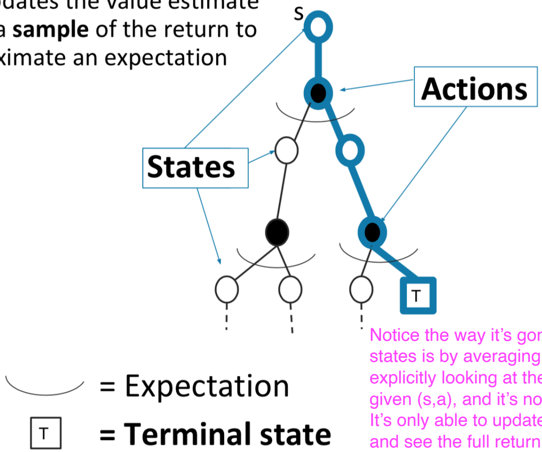
$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$



MC Policy Evaluation

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

MC updates the value estimate using a **sample** of the return to approximate an expectation



Notice the way it's gonna get the expectation over states is by averaging across trajectories. It's not explicitly looking at the probability of next state given (s,a), and it's not bootstrapping.
It's only able to update, when you get all the way out and see the full return.

Monte Carlo (MC) Policy Evaluation Key Limitations

- Generally high variance estimator
  - Reducing variance can require a lot of data
- Requires episodic settings
  - Episode must end before data from that episode can be used to update the value function

e.g. you can NOT do Monte Carlo if there's loop inside your process.

Monte Carlo (MC) Policy Evaluation Summary

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Simple: Estimates expectation by empirical average (given episodes sampled from policy of interest) or reweighted empirical average (importance sampling)
- Updates value estimate by using a **sample** of return to approximate the expectation
- No bootstrapping
- Converges to true value under some (generally mild) assumptions

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- **Temporal Difference (TD)**
- Metrics to evaluate and compare algorithms

Temporal Difference Learning

- "If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be temporal-difference (TD) learning." – Sutton and Barto 2017
- Combination of Monte Carlo & dynamic programming methods
- Model-free
- **Bootstraps and samples**
- Can be used in episodic or infinite-horizon non-episodic settings
- Immediately updates estimate of V after each (s, a, r, s') tuple

Temporal Difference Learning for Estimating V

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Recall Bellman operator (if know MDP models)

$$B^\pi V(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V(s')$$

- In incremental every-visit MC, update estimate using 1 sample of return (for the current ith episode)

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

so that you don't need to wait till the end of episode

- Insight: have an estimate of V^π, use to estimate expected return

$$V^\pi(s) = V^\pi(s) + \alpha([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s))$$

Temporal Difference $[TD(0)]$ Learning

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- Simplest TD learning: update value towards estimated value

$$V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$$

- TD error:

$$\delta_t = r_t + \gamma V^\pi(s_{t+1}) - V^\pi(s_t)$$

- Can immediately update value estimate after (s, a, r, s') tuple
- Don't need episodic setting

you're doing this for every single state compared to DP where you do this for all states.

Temporal Difference [$TD(0)$] Learning Algorithm

Input: α

Initialize $V^\pi(s) = 0, \forall s \in S$

Loop

$\pi(s_t)$

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Check Your Understanding: TD Learning

Input: α
Initialize $V^\pi(s) = 0$, $\forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Example:

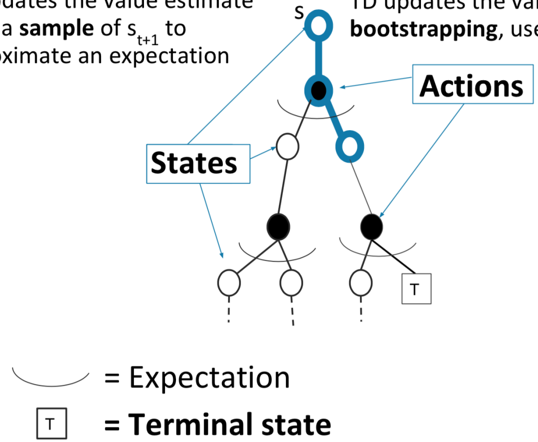
- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \; \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]
- Every visit MC estimate of V of $s2$? 1
- TD estimate of all states (init at 0) with $\alpha = 1$?
  V = [1 0 0 0 0 0 0 0]   this value will be propagate back to s2,s3 slowly, while MC propagate immediately.

Temporal Difference Policy Evaluation

$$V^\pi(s_t) = V^\pi(s_t) + \alpha([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s_t))$$

TD updates the value estimate using a **sample** of s_{t+1} to approximate an expectation

TD updates the value estimate by **bootstrapping**, uses estimate of $V(s_{t+1})$



This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
    - Given off-policy samples
- Temporal Difference (TD)
- **Metrics to evaluate and compare algorithms**

Check Your Understanding: For Dynamic Programming, MC and TD Methods, Which Properties Hold?

- Usable when no models of current domain
- Handles continuing (non-episodic) domains
- Handles Non-Markovian domains
- Converges to true value in limit [1]
- Unbiased estimate of value

[1] For tabular representations of value function. More on this in later lectures

Some Important Properties to Evaluate Policy Evaluation Algorithms

- Usable when no models of current domain
  - DP: No   MC: Yes   TD: Yes
- Handles continuing (non-episodic) domains
  - DP: Yes   MC: No   TD: Yes
- Handles Non-Markovian domains
  - DP: No   MC: Yes   TD: No
- Converges to true value in limit [2]
  - DP: Yes   MC: Yes   TD: Yes
- Unbiased estimate of value
  - DP: NA   MC: Yes   TD: No

[2] For tabular representations of value function. More on this in later lectures

Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Bias/variance characteristics
- Data efficiency
- Computational efficiency

Bias/Variance of Model-free Policy Evaluation Algorithms

- Return G_t is an unbiased estimate of $V^\pi(s_t)$
- TD target $[r_t + \gamma V^\pi(s_{t+1})]$ is a biased estimate of $V^\pi(s_t)$
- But often much lower variance than a single return G_t
- Return function of multi-step sequence of random actions, states & rewards
- TD target only has one random action, reward and next state
- MC
  - Unbiased
  - High variance
  - Consistent (converges to true) even with function approximation
- TD
  - Some bias
  - Lower variance
  - TD(0) converges to true value with tabular representation
  - TD(0) does not always converge with function approximation

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ *Okay Field Site*	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$ 	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ *Fantastic Field Site*

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1\ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]
- Every visit MC estimate of V of s_2? 1
- TD estimate of all states (init at 0) with $\alpha = 1$ is [1 0 0 0 0 0 0]
- TD(0) only uses a data point (s, a, r, s') once
- Monte Carlo takes entire return from s to end of episode

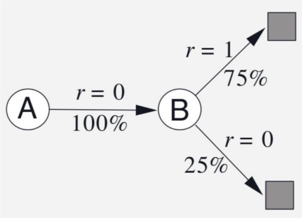
Batch MC and TD

- Batch (Offline) solution for finite dataset
  - Given set of K episodes
  - Repeatedly sample an episode from K
  - Apply MC or TD(0) to the sampled episode
- What do MC and TD(0) converge to?

AB Example: (Ex. 6.4, Sutton & Barto, 2018

- Two states A, B with $\gamma = 1$
- Given 8 episodes of experience:
  - $A, 0, B, 0$
  - $B, 1$ (observed 6 times)
  - $B, 0$
- What are $V(A), V(B)$?

AB Example: (Ex. 6.4, Sutton & Barto, 2018)



- Two states A, B with $\gamma = 1$
- Given 8 episodes of experience:
  - $A, 0, B, 0$
  - $B, 1$ (observed 6 times)
  - $B, 0$
- $V(B) = 0.75$ by TD or MC
- What about $V(A)$?

Batch MC and TD: Converges

- Monte Carlo in batch setting converges to min MSE (mean squared error)
  - Minimize loss with respect to observed returns
  - In AB example, $V(A) = 0$
- TD(0) converges to DP policy V^π for the MDP with the maximum likelihood model estimates
  - Maximum likelihood Markov decision process model

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k-1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k-1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using this model
  - In AB example, $V(A) = 0.75$

Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Data efficiency & Computational efficiency
- In simplest TD, use (s, a, r, s') once to update $V(s)$
  - $O(1)$ operation per update
  - In an episode of length L, $O(L)$
- In MC have to wait till episode finishes, then also $O(L)$
- MC can be more data efficient than simple TD
- But TD exploits Markov structure
  - If in Markov domain, leveraging this is helpful

Alternative: Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s, a, r, s') tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using MLE MDP [3] (e.g. see method from lecture 2)

[3] Requires initializing for all (s, a) pairs

Alternative: Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s, a, r, s') tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} 1(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} 1(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using MLE MDP [4] (e.g. see method from lecture 2)
- Cost: Updating MLE model and MDP planning at each update ($O(|S|^3)$ for analytic matrix solution, $O(|S|^2|A|)$ for iterative methods)
- Very data efficient and very computationally expensive
- Consistent
- Can also easily be used for off-policy evaluation

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ *Okay Field Site*	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$ 	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ *Fantastic Field Site*

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \; \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]
- Every visit MC estimate of V of s_2? 1
- TD estimate of all states (init at 0) with $\alpha = 1$ is [1 0 0 0 0 0 0]
- What is the certainty equivalent estimate?
  $\hat{r} = [1\ 0\ 0\ 0\ 0\ 0\ 0]$,
  $\hat{p}(terminate|s_1, a_1) = \hat{p}(s_1|s_2, a_1) = \hat{p}(s_2|s_3, a_1) = 1$,
  V = [1 1 1 0 0 0 0]

Some Important Properties to Evaluate Policy Evaluation Algorithms

- Robustness to Markov assumption
- Bias/variance characteristics
- Data efficiency
- Computational efficiency

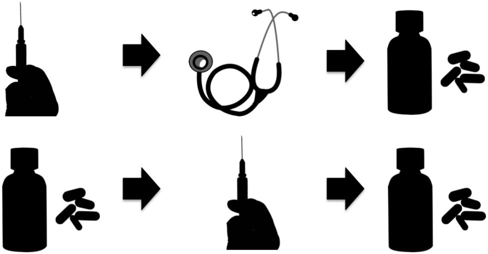
Summary: Policy Evaluation

- Dynamic Programming
- Monte Carlo policy evaluation
  - Policy evaluation when we don't have a model of how the world works
    - Given on policy samples
    - Given off policy samples
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Dynamic programming
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
    - **Given off-policy samples**
- Temporal Difference (TD)
- Metrics to evaluate and compare algorithms

MC Off Policy Evaluation



- Sometimes trying actions out is costly or high stakes
- Would like to use old data about policy decisions and their outcomes to estimate the potential value of an alternate policy

Monte Carlo (MC) Off Policy Evaluation

- Aim: estimate value of policy π_1, $V^{\pi_1}(s)$, given episodes generated under behavior policy π_2
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π_2
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Have data from a different policy, behavior policy π_2
- If π_2 is stochastic, can often use it to estimate the value of an alternate policy (formal conditions to follow)
- Again, no requirement that have a model nor that state is Markov

Monte Carlo (MC) Off Policy Evaluation: Distribution Mismatch

- Distribution of episodes & resulting returns differs between policies

Importance Sampling

- Goal: estimate the expected value of a function $f(x)$ under some probability distribution $p(x)$, $\mathbb{E}_{x \sim p}[f(x)]$
- Have data $x_1, x_2, \ldots, x_n$ sampled from distribution $q(s)$
- Under a few assumptions, we can use samples to obtain an unbiased estimate of $\mathbb{E}_{x \sim q}[f(x)]$

$$\mathbb{E}_{x \sim q}[f(x)] = \int_x q(x) f(x)$$

Importance Sampling (IS) for Policy Evaluation

- Let h_j be episode j (history) of states, actions and rewards

$$h_j = (s_{j,1}, a_{j,1}, r_{j,1}, s_{j,2}, a_{j,2}, r_{j,2}, \ldots, s_{j,L_j(terminal)})$$

Importance Sampling (IS) for Policy Evaluation

- Let h_j be episode j (history) of states, actions and rewards

$$h_j = (s_{j,1}, a_{j,1}, r_{j,1}, s_{j,2}, a_{j,2}, r_{j,2}, \ldots, s_{j,L_j(terminal)})$$

$$\begin{aligned}
p(h_j|\pi, s = s_{j,1}) =& p(a_{j,1}|s_{j,1})p(r_{j,1}|s_{j,1}, a_{j,1})p(s_{j,2}|s_{j,1}, a_{j,1}) \\
& p(a_{j,2}|s_{j,2})p(r_{j,2}|s_{j,2}, a_{j,2})p(s_{j,3}|s_{j,2}, a_{j,2}) \ldots \\
=& \prod_{t=1}^{L_j-1} p(a_{j,t}|s_{j,t})p(r_{j,t}|s_{j,t}, a_{j,t})p(s_{j,t+1}|s_{j,t}, a_{j,t}) \\
=& \prod_{t=1}^{L_j-1} \pi(a_{j,t}|s_{j,t})p(r_{j,t}|s_{j,t}, a_{j,t})p(s_{j,t+1}|s_{j,t}, a_{j,t})
\end{aligned}$$

Importance Sampling (IS) for Policy Evaluation

- Let h_j be episode j (history) of states, actions and rewards, where the actions are sampled from π_2

$$h_j = (s_{j,1}, a_{j,1}, r_{j,1}, s_{j,2}, a_{j,2}, r_{j,2}, \ldots, s_{j,L_j(terminal)})$$

$$V^{\pi_1}(s) \approx \sum_{j=1}^{n} \frac{p(h_j|\pi_1, s)}{p(h_j|\pi_2, s)} G(h_j)$$

Importance Sampling for Policy Evaluation

- Aim: estimate $V^{\pi_1}(s)$ given episodes generated under policy π_2
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π_2
- Have access to $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π_2
- Want $V^{\pi_1}(s) = \mathbb{E}_{\pi_1}[G_t | s_t = s]$
- IS = Monte Carlo estimate given off policy data
- Model-free method
- Does not require Markov assumption
- Under some assumptions, unbiased & consistent estimator of V^{π_1}
- Can be used when agent is interacting with environment to estimate value of policies different than agent's control policy
- More later this quarter about batch learning

Today's Plan

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- Today
  - Policy evaluation when don't have a model of how the world works
- **Next Time**:
  - **Control when don't have a model of how the world works**