Lecture 2: Making Sequences of Good Decisions Given a Model of the World

Emma Brunskill

CS234 Reinforcement Learning

Winter 2019

Today's Plan

- Last Time:
  - Introduction
  - Components of an agent: model, value, policy
- This Time:
  - Making good decisions given a Markov decision process
- Next Time:
  - Policy evaluation when don't have a model of how the world works

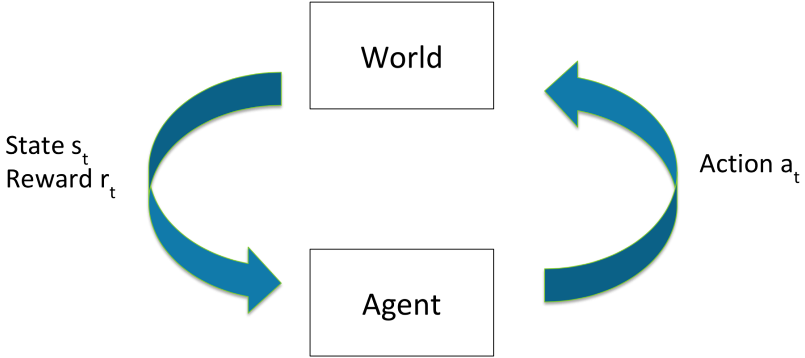
Models, Policies, Values

- **Model**: Mathematical models of dynamics and reward
- **Policy**: Function mapping agent's states to actions
- **Value function**: future rewards from being in a state and/or action when following a particular policy

Today: Given a model of the world

- Markov Processes
- Markov Reward Processes (MRPs)
- Markov Decision Processes (MDPs)
- Evaluation and Control in MDPs

Full Observability: Markov Decision Process (MDP)



- MDPs can model a huge number of interesting problems and settings
  - Bandits: single state MDP
  - Optimal control mostly about continuous-state MDPs
  - Partially observable MDPs = MDP where state is history

Recall: Markov Property

- Information state: sufficient statistic of history
- State s_t is Markov if and only if:

$$p(s_{t+1}|s_t, a_t) = p(s_{t+1}|h_t, a_t)$$

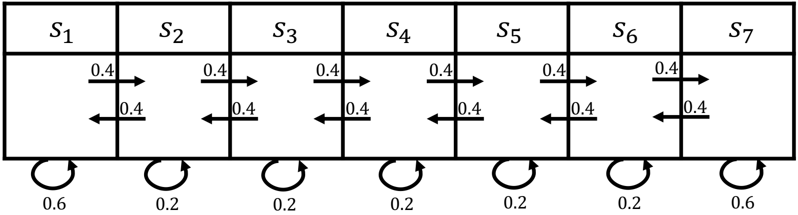
- Future is independent of past given present

Markov Process or Markov Chain

- Memoryless random process
  - Sequence of random states with Markov property
- Definition of Markov Process
  - S is a (finite) set of states ($s \in S$)
  - P is dynamics/transition model that specifices $p(s_{t+1} = s' | s_t = s)$
- Note: no rewards, no actions
- If finite number (N) of states, can express P as a matrix

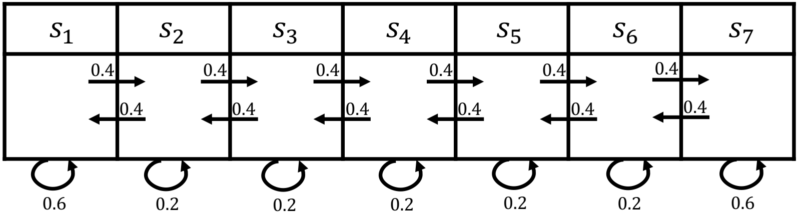
$$P = \begin{pmatrix} P(s_1|s_1) & P(s_2|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & P(s_2|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \vdots & \ddots & \vdots \\ P(s_1|s_N) & P(s_2|s_N) & \cdots & P(s_N|s_N) \end{pmatrix}$$

Example: Mars Rover Markov Chain Transition Matrix, P



$$P = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.2 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{pmatrix}$$

Example: Mars Rover Markov Chain Episodes



Example: Sample episodes starting from S4

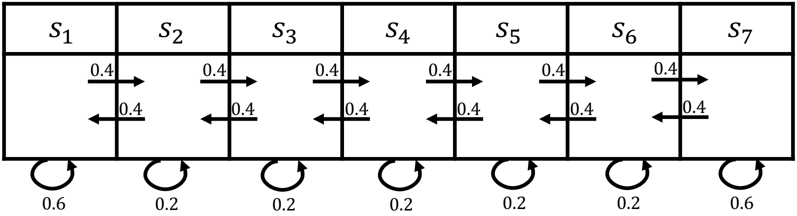
- $s_4, s_5, s_6, s_7, s_7, s_7, \ldots$
- $s_4, s_4, s_5, s_4, s_5, s_6, \ldots$
- $s_4, s_3, s_2, s_1, \ldots$

Markov Reward Process (MRP)

- Markov Reward Process is a Markov Chain + rewards
- Definition of Markov Reward Process (MRP)
  - S is a (finite) set of states ($s \in S$)
  - P is dynamics/transition model that specifices $P(s_{t+1} = s' | s_t = s)$
  - R is a reward function $R(s_t = s) = \mathbb{E}[r_t | s_t = s]$
  - Discount factor $\gamma \in [0, 1]$
- Note: no actions
- If finite number (N) of states, can express R as a vector

Right now we're still in MRP, there's no action. So in this case, the ways you could define rewards would either be over the immediate state, or state and next state.

Example: Mars Rover MRP



- Reward: +1 in s_1, +10 in s_7, 0 in all other states

Return & Value Function

- Definition of Horizon
  - Number of time steps in each episode
  - Can be infinite
  - Otherwise called **finite** Markov reward process
- Definition of Return, G_t (for a MRP)
  - Discounted sum of rewards from time step t to horizon

$$G_t = r_t + \boxed{\gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots}$$

future rewards

- Definition of State Value Function, $V(s)$ (for a MRP)
  - Expected return from starting in state s

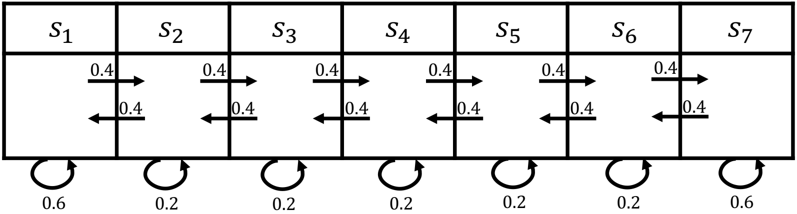
$$V(s) = \mathbb{E}[G_t | s_t = s] = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s]$$

Discount Factor

the value function (discounted sum of returns), is bounded
as long as the reward function is bounded

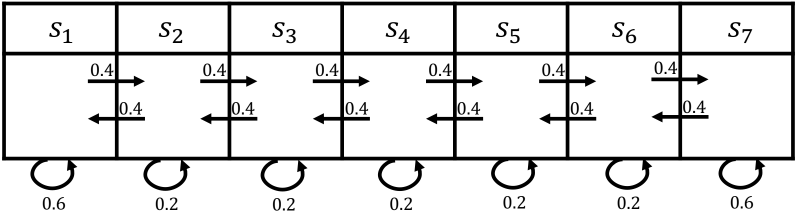
- Mathematically convenient (avoid infinite returns and values)
- Humans often act as if there's a discount factor < 1
- $\gamma = 0$: Only care about immediate reward
- $\gamma = 1$: Future reward is as beneficial as immediate reward
- If episode lengths are always finite, can use $\gamma = 1$

Example: Mars Rover MRP



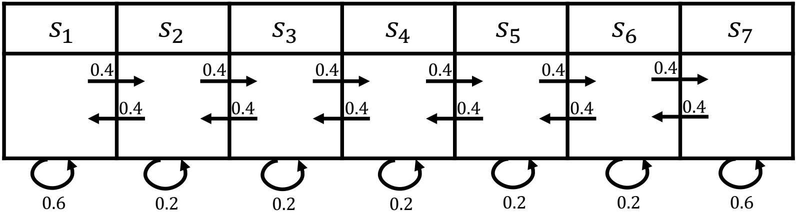
- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states
- Sample returns for sample 4-step episodes, $\gamma = 1/2$
  - s_4, s_5, s_6, s_7: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$

Example: Mars Rover MRP



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states
- Sample returns for sample 4-step episodes, $\gamma = 1/2$
  - s_4, s_5, s_6, s_7: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$
  - s_4, s_4, s_5, s_4: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 0 = 0$
  - s_4, s_3, s_2, s_1: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 1 = 0.125$

Example: Mars Rover MRP



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states
- Value function: expected return from starting in state s

$$V(s) = \mathbb{E}[G_t | s_t = s] = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s]$$

- Sample returns for sample 4-step episodes, $\gamma = 1/2$
  - s_4, s_5, s_6, s_7: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$
  - s_4, s_4, s_5, s_4: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 0 = 0$
  - s_4, s_3, s_2, s_1: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 1 = 0.125$
- $V = [1.53\ 0.37\ 0.13\ 0.22\ 0.85\ 3.59\ 15.31]$

Computing the Value of a Markov Reward Process

- Could estimate by simulation
  - Generate a large number of episodes
  - Average returns
  - Concentration inequalities bound how quickly average concentrates to expected value
  - Requires **no assumption** of Markov structure

Computing the Value of a Markov Reward Process

- Could estimate by simulation
- Markov property yields additional structure
- MRP value function satisfies

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s)V(s')}_{\text{Discounted sum of future rewards}}$$

Matrix Form of Bellman Equation for MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$V = R + \gamma P V$$

R and P are known.

Analytic Solution for Value of MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$\begin{aligned} V &= R + \gamma P V \\ V - \gamma P V &= R \\ (I - \gamma P)V &= R \\ \boxed{V = (I - \gamma P)^{-1} R} \end{aligned}$$

- Solving directly requires taking a matrix inverse $\sim O(N^3)$

Iterative Algorithm for Computing Value of a MRP

- Dynamic programming
- Initialize $V_0(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S
$$V_k(s) = R(s) + \gamma \sum_{s' \in S} P(s'|s) V_{k-1}(s')$$
- Computational complexity: $O(|S|^2)$ for each iteration ($|S| = N$)

Markov Decision Process (MDP)

- Markov Decision Process is Markov Reward Process + actions
- Definition of MDP
  - S is a (finite) set of Markov states $s \in S$
  - A is a (finite) set of actions $a \in A$
  - P is dynamics/transition model for $\boxed{\textbf{each action}}$, that specifies $P(s_{t+1} = s' | s_t = s, a_t = a)$
  - R is a reward function[1]

$$R(s_t = s, a_t = a) = \mathbb{E}[r_t | s_t = s, a_t = a]$$

  - Discount factor $\gamma \in [0, 1]$
- MDP is a tuple: (S, A, P, R, γ)

[1] Reward is sometimes defined as a function of the current state, or as a function of the (state, action, next state) tuple. Most frequently in this class, we will assume reward is a function of state and action

Example: Mars Rover MDP

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

$$P(s'|s,a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad P(s'|s,a_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- 2 deterministic actions

MDP Policies

- Policy specifies what action to take in each state
  - Can be deterministic or stochastic
- For generality, consider as a conditional distribution
  - Given a state, specifies a distribution over actions
- Policy: $\pi(a|s) = P(a_t = a|s_t = s)$

MDP + Policy

- $\boxed{\text{MDP} + \pi(a|s) = \text{Markov Reward Process}}$
- Precisely, it is the MRP $(S, R^{\pi}, P^{\pi}, \gamma)$, where

$$R^{\pi}(s) = \sum_{a \in A} \pi(a|s) R(s,a)$$

$$P^{\pi}(s'|s) = \sum_{a \in A} \pi(a|s) P(s'|s,a)$$

- Implies we can use same techniques to evaluate the value of a policy for a MDP as we could to compute the value of a MRP, by defining a MRP with R^{π} and P^{π}

MDP Policy Evaluation, Iterative Algorithm

- Initialize $V_0(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

- This is a **Bellman backup** for a particular policy

instantiate the reward function by always picking the action that the policy would take.
So in this case, I'm doing it for simplicity for deterministic policy.

Policy Evaluation: Example & Check Your Understanding

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

- Two actions
- Reward: for all actions, $+1$ in state s_1, $+10$ in state s_7, 0 otherwise
- Let $\pi(s) = a_1 \; \forall s$. $\gamma = 0$.
- What is the value of this policy?
- Recall iterativ e

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

$$V^\pi = [1\ 0\ 0\ 0\ 0\ 0\ 10]$$

Practice: MDP 1 Iteration of Policy Evaluation, Mars Rover Example

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, $\ldots$
- Reward: for all actions, $+1$ in state s_1, $+10$ in state s_7, 0 otherwise
- Let $\pi(s) = a_1 \; \forall s$, assume V_k^{π} =[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
  - For all s in S

$$V_k^{\pi}(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^{\pi}(s')$$

$$V_{k+1}(s_6) = r(s_6, a_1) + \gamma * 0.5 * V_k(s_6) + \gamma * 0.5 * V_k(s_7)$$

$$V_{k+1}(s_6) = 0 + 0.5 * 0.5 * 0 + .5 * 0.5 * 10$$

$$V_{k+1}(s_6) = 2.5$$

MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There **exists a unique optimal value function**
- Optimal policy for a MDP in an infinite horizon problem is deterministic

Here, "unique optimal value function" means the optimal value of the state. There may be more than 1 optimal policy, but there exists at least one optimal policy which leads to the maximum value of that state. And there's a single value of that.

Check Your Understanding

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

- 7 discrete states (location of rover)
- 2 actions: Left or Right
- How many deterministic policies are there?
  2^7
- Is the optimal policy for a MDP always unique?
  No, there may be two actions that have the same optimal value function

MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There exists a unique optimal value function
- Optimal policy for a MDP in an infinite horizon problem (agent acts forever is
  - Deterministic
  - Stationary (does not depend on time step)
  - Unique? Not necessarily, may have state-actions with identical optimal values

Policy Search

- One option is searching to compute best policy
- Number of deterministic policies is $|A|^{|S|}$
- Policy iteration is generally more efficient than enumeration

MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

New Definition: State-Action Value Q

- State-action value of a policy

$$Q^{\pi}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi}(s')$$

- Take action a, then follow the policy π

Policy Improvement

- Compute state-action value of a policy π_i
  - For s in S and a in A:

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

- Compute new policy π_{i+1}, for all $s \in S$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a) \;\forall s \in S$$

MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

Delving Deeper Into Policy Improvement Step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

Delving Deeper Into Policy Improvement Step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

$$\max_a Q^{\pi_i}(s,a) \geq R(s,\pi_i(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_i(s)) V^{\pi_i}(s') = V^{\pi_i}(s)$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a)$$

- Suppose we take $\pi_{i+1}(s)$ for one action, then follow π_i forever
    - Our expected sum of rewards is at least as good as if we had always followed π_i
- But new proposed policy is to always follow π_{i+1} ...

Monotonic Improvement in Policy

Monotonic means the new policy is greater than or equal to the old policy for all states.

- Definition
$$V^{\pi_1} \geq V^{\pi_2} : V^{\pi_1}(s) \geq V^{\pi_2}(s), \forall s \in S$$
- Proposition: $V^{\pi_{i+1}} \geq V^{\pi_i}$ with strict inequality if π_i is suboptimal, where π_{i+1} is the new policy we get from policy improvement on π_i

Proof: Monotonic Improvement in Policy

$$
\begin{aligned}
V^{\pi_i}(s) &\leq \max_a Q^{\pi_i}(s,a) \\
&= \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')
\end{aligned}
$$

Proof: Monotonic Improvement in Policy

$$
\begin{aligned}
V^{\pi_i}(s) &\leq \max_a Q^{\pi_i}(s,a) \\
&= \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s') \\
&= R(s, \pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s, \pi_{i+1}(s)) V^{\pi_i}(s') \text{ //by the definition of } \pi_{i+1} \\
&\leq R(s, \pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s, \pi_{i+1}(s)) \left(\max_{a'} Q^{\pi_i}(s', a') \right) \\
&= R(s, \pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s, \pi_{i+1}(s)) \\
&\quad \left(R(s', \pi_{i+1}(s')) + \gamma \sum_{s'' \in S} P(s''|s', \pi_{i+1}(s')) V^{\pi_i}(s'') \right) \\
&\vdots \\
&= V^{\pi_{i+1}}(s)
\end{aligned}
$$

Policy Iteration (PI): Check Your Understanding

- Note: all the below is for finite state-action spaces
- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
    - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
    - $\pi_{i+1} \leftarrow$ Policy **improvement**
    - $i = i + 1$
- **If policy doesn't change, can it ever change again?**
  No
  "policy doesn't change" means $\pi_{i+1} = \pi_i$ for all states
- **Is there a maximum number of iterations of policy iteration?**
  $|A|^{|S|}$ since that is the maximum number of policies, and as the policy improvement step is monotonically improving, each policy can only appear in one round of policy iteration unless it is an optimal policy.

Policy Iteration (PI): Check Your Understanding

- Suppose for all $s \in S$, $\pi_{i+1}(s) = \pi_i(s)$
- Then for all $s \in S$, $Q^{\pi_{i+1}}(s, a) = Q^{\pi_i}(s, a)$
- Recall policy improvement step

$$Q^{\pi_i}(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V^{\pi_i}(s')$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s, a)$$

$$\pi_{i+2}(s) = \arg\max_a Q^{\pi_{i+1}}(s, a) = \arg\max_a Q^{\pi_i}(s, a)$$

- Therefore policy cannot ever change again

MDP: Computing Optimal Policy and Optimal Value

- Policy iteration computes optimal value and policy
- Value iteration is another technique
  - Idea: Maintain optimal value of starting in a state s if have a finite number of steps k left in the episode
  - Iterate to consider longer and longer episodes

Bellman Equation and Bellman Backup Operators

- Value function of a policy must satisfy the Bellman equation

$$V^\pi(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s)V^\pi(s')$$

- Bellman backup operator
  - Applied to a value function
  - Returns a new value function
  - Improves the value if possible

$$BV(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a)V(s')$$

  - BV yields a value function over all states s

Value Iteration (VI)

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until [finite horizon, convergence]:
    - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

    - View as Bellman backup on value function

$$V_{k+1} = BV_k$$

output policy π

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Policy Iteration as Bellman Operations

- Bellman backup operator B^π for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s)V(s)$$

- Policy evaluation amounts to computing the fixed point of B^π
- To do policy evaluation, repeatedly apply operator until V stops changing

$$V^\pi = B^\pi B^\pi \cdots B^\pi V$$

Policy Iteration as Bellman Operations

- Bellman backup operator B^π for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s)V(s)$$

- To do policy improvement

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a)V^{\pi_k}(s')$$

Going Back to Value Iteration (VI)

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until [finite horizon, convergence]:
  - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

  - Equivalently, in Bellman backup notation

$$V_{k+1} = BV_k$$

- To extract optimal policy if can act for $k+1$ more steps,

$$\pi(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_{k+1}(s')$$

Contraction Operator

- Let O be an operator,and $|x|$ denote (any) norm of x
- If $|OV - OV'| \leq |V - V'|$, then O is a contraction operator

Will Value Iteration Converge?

- Yes, if discount factor $\gamma < 1$, or end up in a terminal state with probability 1
- Bellman backup is a contraction if discount factor, $\gamma < 1$
- If apply it to two different value functions, distance between value functions shrinks after applying Bellman equation to each

Proof: Bellman Backup is a Contraction on V for $\gamma < 1$

- Let $\|V - V'\| = \max_s |V(s) - V'(s)|$ be the infinity norm

$$\|BV_k - BV_j\| = \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\|$$

Proof: Bellman Backup is a Contraction on V for $\gamma < 1$

- Let $\|V - V'\| = \max_s |V(s) - V'(s)|$ be the infinity norm

$$
\begin{aligned}
\|BV_k - BV_j\| &= \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\| \\
&\leq \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') - R(s,a) - \gamma \sum_{s' \in S} P(s'|s,a) V_j(s') \right) \right\| \\
&= \left\| \max_a \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \left\| \max_a \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \left\| \gamma \|V_k - V_j\| \max_a \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

- Note: Even if all inequalities are equalities, this is still a contraction if $\gamma < 1$

Check Your Understanding

- Prove value iteration converges to a unique solution for discrete state and action spaces with $\gamma < 1$
- Does the initialization of values in value iteration impact anything?

Value Iteration for Finite Horizon H

V_k = optimal value if making k more decisions
π_k = optimal policy if making k more decisions

- Initialize $V_0(s) = 0$ for all states s
- For $k = 1 : H$
  - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Is optimal policy stationary (independent of time step) in finite horizon tasks? ? In general, no

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until $k == H$:
  - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Value vs Policy Iteration

- Value iteration:
  - Compute optimal value for horizon $= k$
    - Note this can be used to compute optimal policy if horizon $= k$
  - Increment k
- Policy iteration
  - Compute infinite horizon value of a policy
  - Use to select another (better) policy
  - Closely related to a very popular method in RL: policy gradient

What You Should Know

- Define MP, MRP, MDP, Bellman operator, contraction, model, Q-value, policy
- Be able to implement
  - Value Iteration
  - Policy Iteration
- Give pros and cons of different policy evaluation approaches
- Be able to prove contraction properties
- Limitations of presented approaches and Markov assumptions
  - Which policy evaluation methods require the Markov assumption?

Policy Improvement

- Compute state-action value of a policy π_i

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

- Note

$$\begin{aligned}
\max_a Q^{\pi_i}(s,a) &= \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi}(s') \\
&\geq R(s,\pi_i(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_i(s)) V^{\pi}(s') \\
&= V^{\pi_i}(s)
\end{aligned}$$

- Define new policy, for all $s \in S$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a) \; \forall s \in S$$

Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm):
  - Policy **evaluation** of π_i
  - $i = i + 1$
  - Policy **improvement**:

$$Q^{\pi_i}(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V^{\pi_i}(s')$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s, a)$$