6.172 Performance Engineering of Software Systems

LECTURE 3
Bit Hacks

Julian Shun

Binary Representation

Let $x = \langle x_{w-1}x_{w-2}\ldots x_0\rangle$ be a w–bit computer word. The unsigned integer value stored in x is

$$x = \sum_{k=0}^{w-1} x_k 2^k .$$

The prefix 0b designates a Boolean constant.

For example, the 8–bit word 0b10010110 represents the unsigned value $150 = 2 + 4 + 16 + 128$.

The signed integer (two's complement) value stored in x is

$$x = \left(\sum_{k=0}^{w-2} x_k 2^k\right) - \underbrace{x_{w-1}}_{\text{sign bit}} 2^{w-1} .$$

For example, the 8–bit word 0b10010110 represents the signed value $-106 = 2 + 4 + 16 - 128$.

Two's Complement

We have `0b00…0` $= 0$.

What is the value of $x =$ `0b11…1`?

$$
\begin{aligned}
x &= \left(\sum_{k=0}^{w-2} x_k 2^k\right) - x_{w-1} 2^{w-1} \\
&= \left(\sum_{k=0}^{w-2} 2^k\right) - 2^{w-1} \\
&= \left(2^{w-1} - 1\right) - 2^{w-1} \\
&= -1 .
\end{aligned}
$$

Complementary Relationship

Important identity

Since we have $x + \sim x = -1$, it follows that

$$-x = \sim x + 1.$$

Example

```
 x = 0b011011000
~x = 0b100100111
-x = 0b100101000
```

Binary and Hexadecimal

Decimal	Hex	Binary	Decimal	Hex	Binary
0	0	0000	8	8	1000
1	1	0001	9	9	1001
2	2	0010	10	A	1010
3	3	0011	11	B	1011
4	4	0100	12	C	1100
	5	0101	13	D	1101
	6	0110	14	E	1110
	7	0111	15	F	1111

The prefix 0x designates a hex constant.

To transla from hex to binary, translate each hex digit to its b nary equivalent, and concatenate the bits.

Example: 0xDEC1DE2CODE4F00D is

$$\underbrace{1101}_{D}\underbrace{1110}_{E}\underbrace{1100}_{C}\underbrace{0001}_{1}\underbrace{1101}_{D}\underbrace{1110}_{E}\underbrace{0010}_{2}\underbrace{1100}_{C}\underbrace{0000}_{0}\underbrace{1101}_{D}\underbrace{1110}_{E}\underbrace{0100}_{4}\underbrace{1111}_{F}\underbrace{0000}_{0}\underbrace{0000}_{0}\underbrace{1101}_{D}$$

C Bitwise Operators

Operator	Description
&	AND
|	OR
^	XOR (exclusive OR)
~	NOT (one's complement)
<<	shift left
>>	shift right

Examples (8-bit word)

```
A = 0b10110011
B = 0b01101001
```

```
A&B = 0b00100001          ~A = 0b01001100
A|B = 0b11111011      A >> 3 = 0b00010110
A^B = 0b11011010      A << 2 = 0b11001100
```

Set the kth Bit

Problem

Set kth bit in a word x to 1.

Idea

Shift and OR.

```
y = x | (1 << k);
```

Example

`k = 7`

`x`	1011110101101101
`1 << k`	0000000010000000
`x | (1 << k)`	1011110111101101

Clear the kth Bit

Problem

Clear the k th bit in a word x.

Idea

Shift, complement, and AND.

```
y = x & ~(1 << k);
```

Example

k = 7

x	1011110111101101
1 << k	0000000010000000
~(1 << k)	1111111101111111
x & ~(1 << k)	1011110101101101

Toggle the kth Bit

Problem

Flip the kth bit in a word x.

Idea

Shift and XOR.

```
y = x ^ (1 << k);
```

Example (0 → 1)

k = 7

x	1011110101101101
1 << k	0000000010000000
x ^ (1 << k)	1011110111101101

Toggle the kth Bit

Problem

Flip the kth bit in a word x.

Idea

Shift and XOR.

```
y = x ^ (1 << k);
```

Example (1 → 0)

`k = 7`

`x`	1011110111101101
`1 << k`	0000000010000000
`x ^ (1 << k)`	1011110101101101

Extract a Bit Field

Problem

Extract a bit field from a word `x`.

Idea

Mask and shift.

```
(x & mask) >> shift;
```

Example

`shift = 7`

x	1011110101101101
mask	0000011110000000
x & mask	0000010100000000
x & mask >> shift	0000000000001010

Set a Bit Field

Problem

Set a bit field in a word `x` to a value `y`.

Idea

Invert mask to clear, and OR the shifted value.

```
x = (x & ~mask) | (y << shift);
```

Example

`shift = 7`

x	1011110101101101
y	0000000000000011
mask	0000011110000000
x & ~mask	1011100001101101
x = (x & ~mask) | (y << shift);	1011100111101101

Set a Bit Field

Problem

Set a bit field in a word `x` to a val

For safety's sake:
`((y << shift) & mask)`

Idea

Invert mask to clear, and OR the shi value.

```
x = (x & ~mask) | (y << shift);
```

Example

`shift = 7`

`x`	`1011110101101101`
`y`	`0000000000000011`
`mask`	`0000011110000000`
`x & ~mask`	`1011100001101101`
`x = (x & ~mask) | (y << shift);`	`1011100111101101`

Ordinary Swap

Problem

Swap two integers x and y.

```
t = x;
x = y;
y = t;
```

No-Temp Swap

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101			
y	00101110			

No-Temp Swap

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101	10010011		
y	00101110	00101110		

No-Temp Swap

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101	10010011	10010011	
y	00101110	00101110	10111101	

No-Temp Swap

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101	10010011	10010011	00101110
y	00101110	00101110	10111101	10111101

No-Temp Swap

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101	10010011	10010011	00101110
y	00101110	00101110	10111101	10111101

Why it works

XOR is its own inverse:

(x ^ y) ^ y ⇒ x

x	y	x ^ y	(x ^ y) ^ y
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

No-Temp Swap (Instant Replay)

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101			
y	00101110			

Why it works

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0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

No-Temp Swap (Instant Replay)

Problem

Swap x and y without using a temporary.

Mask with 1's where bits differ.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101	10010011		
y	00101110	00101110		

Why it works

XOR is its own inverse:

(x ^ y) ^ y ⇒ x

x	y	x ^ y	(x ^ y) ^ y
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

No-Temp Swap (Instant Replay)

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Flip bits in y that differ from x.

Example

x	10111101	10010011	10010011	
y	00101110	00101110	10111101	

Why it works

XOR is its own inverse:

(x ^ y) ^ y ⇒ x

x	y	x ^ y	(x ^ y) ^ y
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

No-Temp Swap (Instant Replay)

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Flip bits in x that differ from y.

Example

x	10111101	10010011	10010011	00101110
y	00101110	00101110	10111101	10111101

Why it works

XOR is its own inverse:

$(x \wedge y) \wedge y \Rightarrow x$

x	y	x ^ y	(x ^ y) ^ y
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

No-Temp Swap (Instant Replay)

Problem

Swap x and y without using a temporary.

```
x = x ^ y;
y = x ^ y;
x = x ^ y;
```

Example

x	10111101	10010011	10010011	00101110
y	00101110	00101110	10111101	10111101

Why it works

XOR is its own inverse: `(x ^ y) ^ y ⇒ x`

Performance 👎

Poor at exploiting *instruction-level parallelism (ILP)*.

Naive temp variable version WIN !

Minimum of Two Integers

Problem

Find the minimum r of two integers x and y.

```
if (x < y)
  r = x;
else
  r = y;
```

or

```
r = (x < y) ? x : y;
```

Performance

A mispredicted branch empties the processor pipeline.

Caveat

The compiler is usually smart enough to optimize away the unpredictable branch, but maybe not.

No-Branch Minimum

Problem

Find the minimum `r` of two integers `x` and `y` without using a branch.

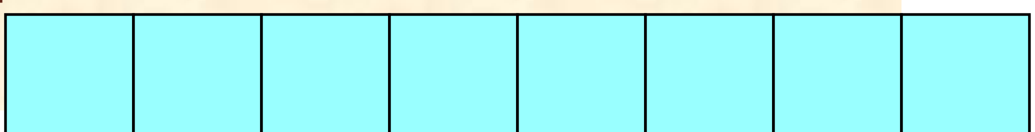
```
r = y ^ ((x ^ y) & -(x < y));
```

Why it works

- The C language represents the Booleans TRUE and FALSE with the integers 1 and 0, respectively.
- If `x < y`, then `-(x < y)` ⇒ `-1`, which is all 1's in two's complement representation. Therefore, we have `y ^ (x ^ y)` ⇒ `x`.
- If $x \geq y$, then `-(x < y)` ⇒ `0`. Therefore, we have `y ^ 0` ⇒ `y`.

Merging Two Sorted Arrays

```
static void merge(long * __restrict C,
                  long * __restrict A,
                  long * __restrict B,
                  size_t na,
                  size_t nb) {
  while (na > 0 && nb > 0) {
    if (*A <= *B) {
      *C++ = *A++; na--;
    } else {
      *C++ = *B++; nb--;
    }
  }
  while (na > 0) {
    *C++ = *A++;
    na--;
  }
  while (nb > 0)  {
    *C++ = *B++;
    nb--;
  }
}
```



Branching

```
  static void merge(long * __restrict C,
                    long * __restrict A,
                    long * __restrict B,
                    size_t na,
                    size_t nb) {
4   while (na > 0 && nb > 0) {
3     if (*A <= *B) {
        *C++ = *A++; na--;
      } else {
        *C++ = *B++; nb--;
      }
    }
2   while (na > 0) {
      *C++ = *A++;
      na--;
    }
1   while (nb > 0)  {
      *C++ = *B++;
      nb--;
    }
  }
```

Branch	Predictable?
1	Yes
2	Yes
3	No
4	Yes

Branchless

```c
static void merge(long * __restrict C,
                  long * __restrict A,
                  long * __restrict B,
                  size_t na,
                  size_t nb) {
  while (na > 0 && nb > 0) {
    long cmp = (*A <= *B);
    long min = *B ^ ((*B ^ *A) & (-cmp));
    *C++ = min;
    A +=  cmp; na -=  cmp;
    B += !cmp; nb -= !cmp;
  }
  while (na > 0) {
    *C++ = *A++;
    na--;
  }
  while (nb > 0)  {
    *C++ = *B++;
    nb--;
  }
}
```

This optimization works well on some machines, but on modern machines using `clang -O3`, the branchless version is usually slower than the branching version. 👎 Modern compilers can perform this optimization better than you can!

Why Learn Bit Hacks?

Why learn bit hacks if they don't even work?

- Because the compiler does them, and it will help to understand what the compiler is doing when you look at the assembly code.
- Because sometimes the compiler doesn't optimize, and you have to do it yourself by hand.
- Because many bit hacks for words extend naturally to bit and word hacks for vectors.
- Because these tricks arise in other domains, and so it pays to be educated about them.
- Because they're fun!

Modular Addition

Problem

Compute $(x + y) \bmod n$, assuming that $0 \le x < n$ and $0 \le y < n$.

```
r = (x + y) % n;
```

Division is expensive, unless by a power of 2.

```
z = x + y;
r = (z < n) ? z : z-n;
```

Unpredictable branch is expensive.

```
z = x + y;
r = z - (n & -(z >= n));
```

Same trick as minimum.

Round up to a Power of 2

Problem

Compute $2^{\lceil \lg n \rceil}$.

Notation

$\lg n = \log_2 n$

Round up to a Power of 2

Problem

Compute $2^{\lceil \lg n \rceil}$.

```
uint64_t n;
 ⋮
--n;
n |= n >> 1;
n |= n >> 2;
n |= n >> 4;
n |= n >> 8;
n |= n >> 16;
n |= n >> 32;
++n;
```

Example

0010000001010000

Round up to a Power of 2

Problem

Compute $2^{\lceil \lg n \rceil}$.

```
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 ⋮
--n;
n |= n >> 1;
n |= n >> 2;
n |= n >> 4;
n |= n >> 8;
n |= n >> 16;
n |= n >> 32;
++n;
```

Example

0010000001010000
0010000001001111

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++n;
```

Example

0010000001010000
0010000001001111
0011000001101111

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n |= n >> 8;
n |= n >> 16;
n |= n >> 32;
++n;
```

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111

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Compute $2^{\lceil \lg n \rceil}$.

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n |= n >> 8;
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n |= n >> 32;
++n;
```

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111
0011111111111111

Round up to a Power of 2

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++n;
```

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111
0011111111111111

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Compute $2^{\lceil \lg n \rceil}$.

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n |= n >> 1;
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n |= n >> 4;
n |= n >> 8;
n |= n >> 16;
n |= n >> 32;
++n;
```

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111
0011111111111111

Round up to a Power of 2

Problem

Compute $2^{\lceil \lg n \rceil}$.

```
uint64_t n;
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n |= n >> 16;
n |= n >> 32;
++n;
```

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111
0011111111111111

Round up to a Power of 2

Problem

Compute $2^{\lceil \lg n \rceil}$.

```
uint64_t n;
 ⋮
--n;
n |= n >> 1;
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n |= n >> 4;
n |= n >> 8;
n |= n >> 16;
n |= n >> 32;
++n;
```

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111
0011111111111111
0100000000000000

Round up to a Power of 2

Problem

pow(2,ceil(lgn))

Compute $2^{\lceil \lg n \rceil}$.

```
uint64_t n;
 ⋮
--n;
n |= n >> 1;
n |= n >> 2;
n |= n >> 4;
n |= n >> 8;
n |= n >> 16;
n |= n >> 32;
++n;
```

Bit $\lceil \lg n \rceil - 1$ must be set

Populate all bits to the right with 1

Set bit $\lceil \lg n \rceil$

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111
0011111111111111
0100000000000000

Round up to a Power of 2

Problem

Compute $2^{\lceil \lg n \rceil}$.

```
uint64_t n;
 ⋮
--n;
n |= n >> 1;
n |= n >> 2;
n |= n >> 4;
n |= n >> 8;
n |= n >> 16;
n |= n >> 32;
++n;
```

Example

0010000001010000
0010000001001111
0011000001101111
0011110001111111
0011111111111111
0100000000000000

Why decrement?

To handle the boundary case when n is a power of 2.

Least-Significant 1

Problem

Compute the mask of the least-significant 1 in word x.

```
r = x & (-x);
```

Example

x	0010000001010000
-x	1101111110110000
x & (-x)	0000000000010000

Why it works

The binary representation of -x is (~x)+1.

Question

How do you find the index of the bit, i.e., lg r?

Log Base 2 of a Power of 2

Problem

Compute lg x, where x is a power of 2.

```
const uint64_t deBruijn = 0x022fdd63cc95386d;
const int convert[64] = {
   0,  1,  2, 53,  3,  7, 54, 27,
   4, 38, 41,  8, 34, 55, 48, 28,
  62,  5, 39, 46, 44, 42, 22,  9,
  24, 35, 59, 56, 49, 18, 29, 11,
  63, 52,  6, 26, 37, 40, 33, 47,
  61, 45, 43, 21, 23, 58, 17, 10,
  51, 25, 36, 32, 60, 20, 57, 16,
  50, 31, 19, 15, 30, 14, 13, 12
};
r = convert[(x * deBruijn) >> 58];
```

Mathemagic Trick

5 volunteers who can follow directions

Introducing Jess Ray,
"The Golden Raytio"

Log Base 2 of a Power of 2

Why it works

A *deBruijn sequence* s of length 2^k is a cyclic 0–1 sequence such that each of the 2^k 0–1 strings of length k occurs exactly once as a substring of s.

0b00011101*2^4 ⇒ 0b11010000
0b11010000 >> 5 ⇒ 6
convert[6] ⇒ 4

start with all 0's

Performance

Limited by multiply and table look-up.

Example: k=3

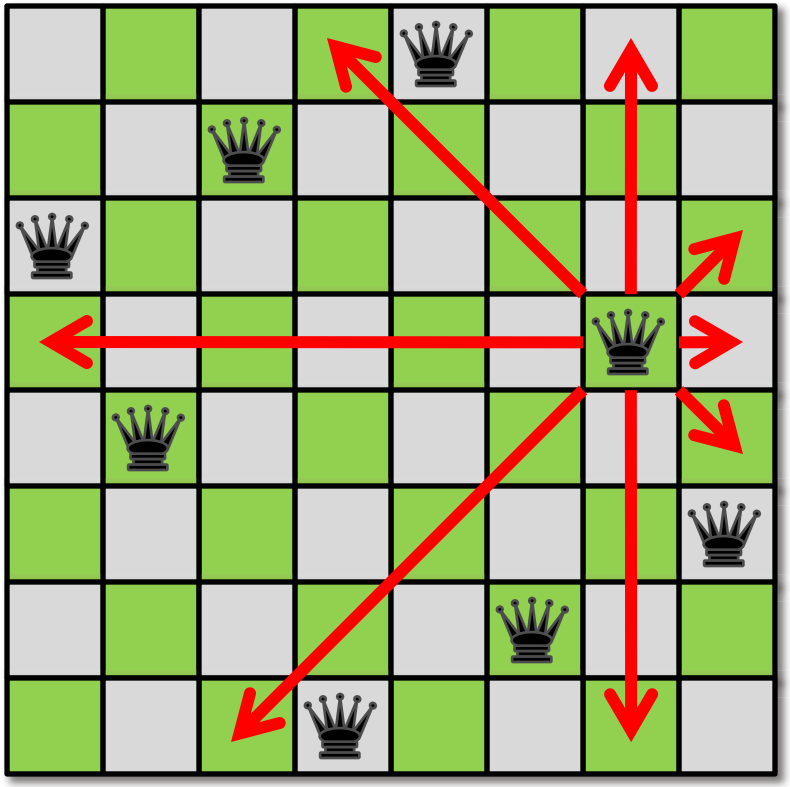
	00011101
0	000
1	001
2	011
3	111
4	110
5	101
6	010
7	100

```
const int convert[8]
= {0,1,6,2,7,5,4,3};
```

Queens Problem

Problem

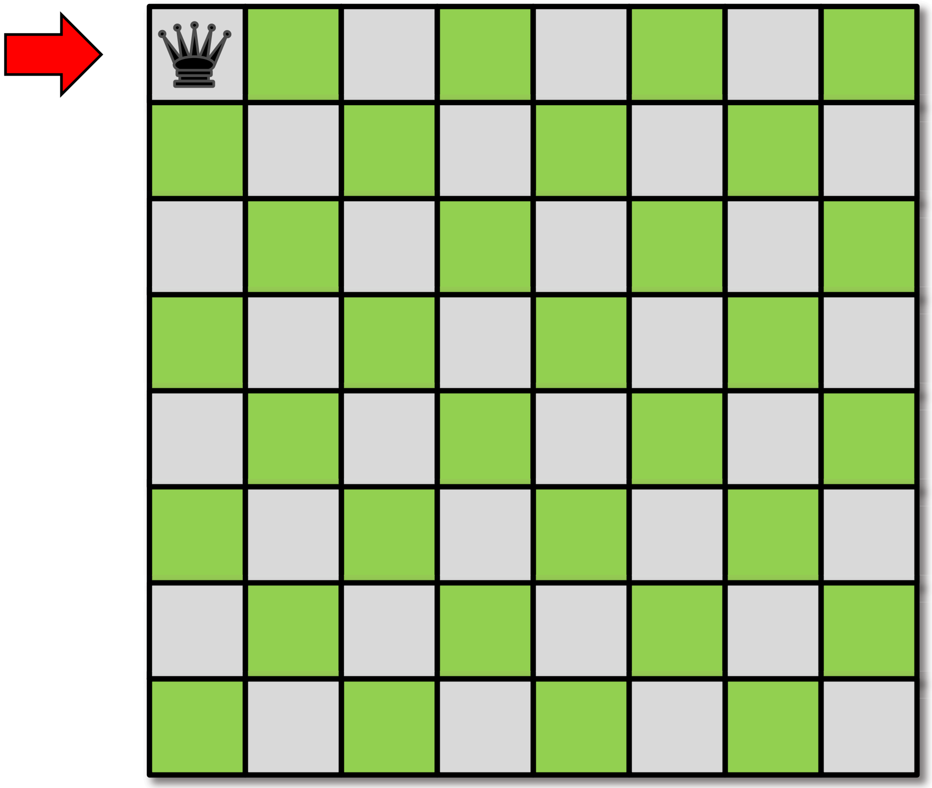
Place n queens on an n × n chessboard so that no queen attacks another, i.e., no two queens in any row, column, or diagonal. Count the number of possible solutions.



Backtracking Search

Strategy

Try placing queens row by row. If you can't place a queen in a row, backtrack.



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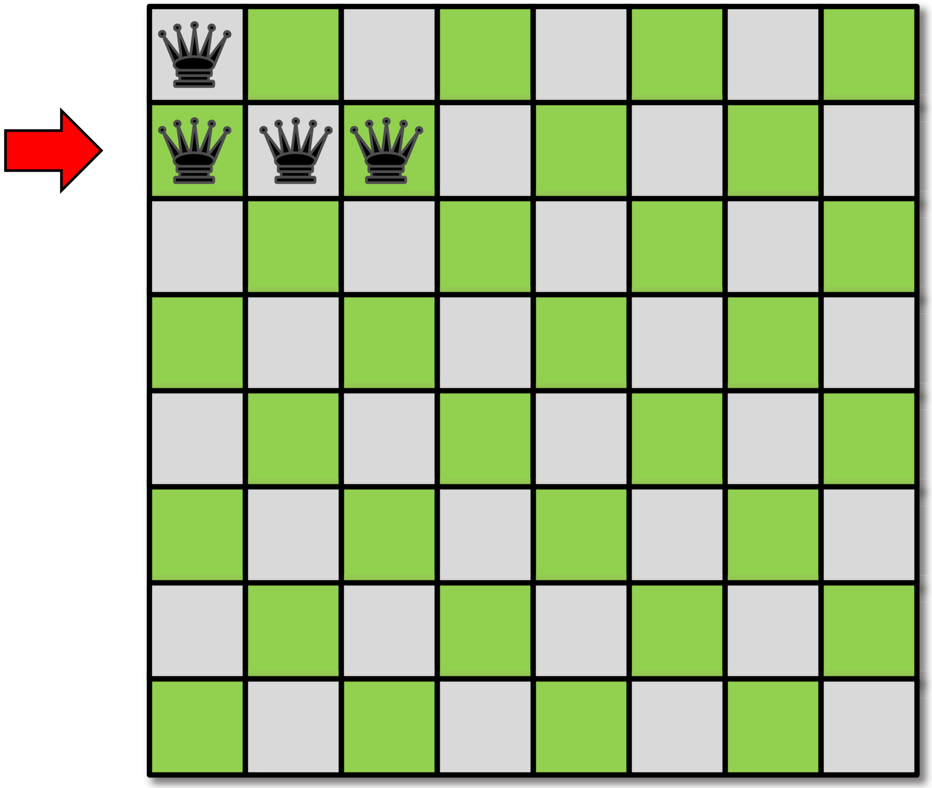

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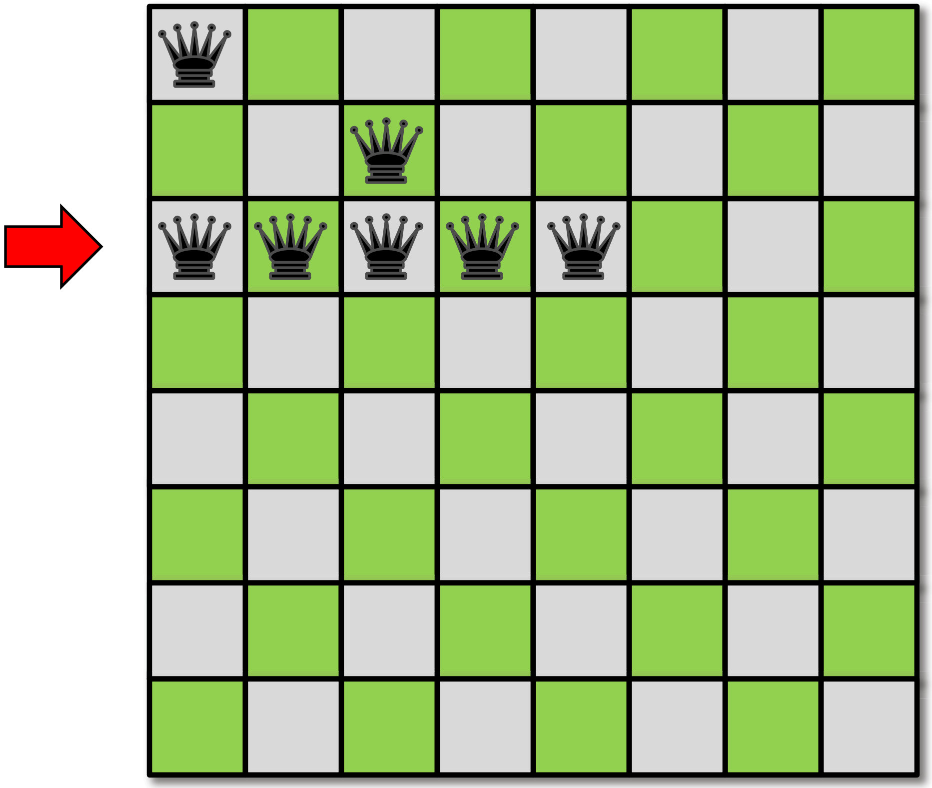

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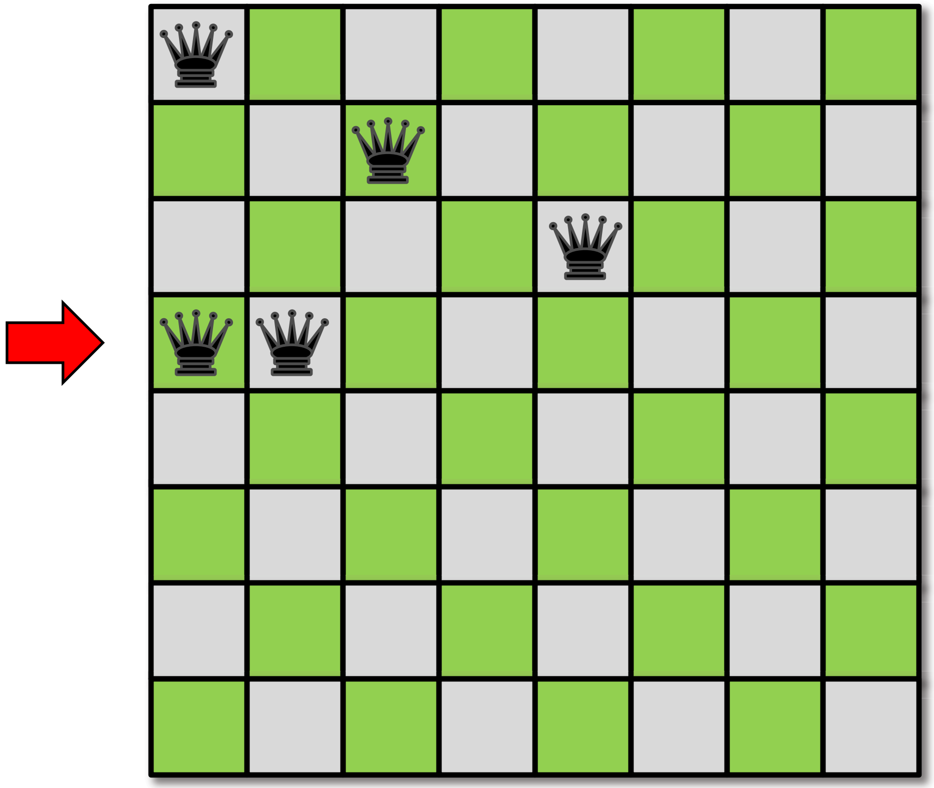

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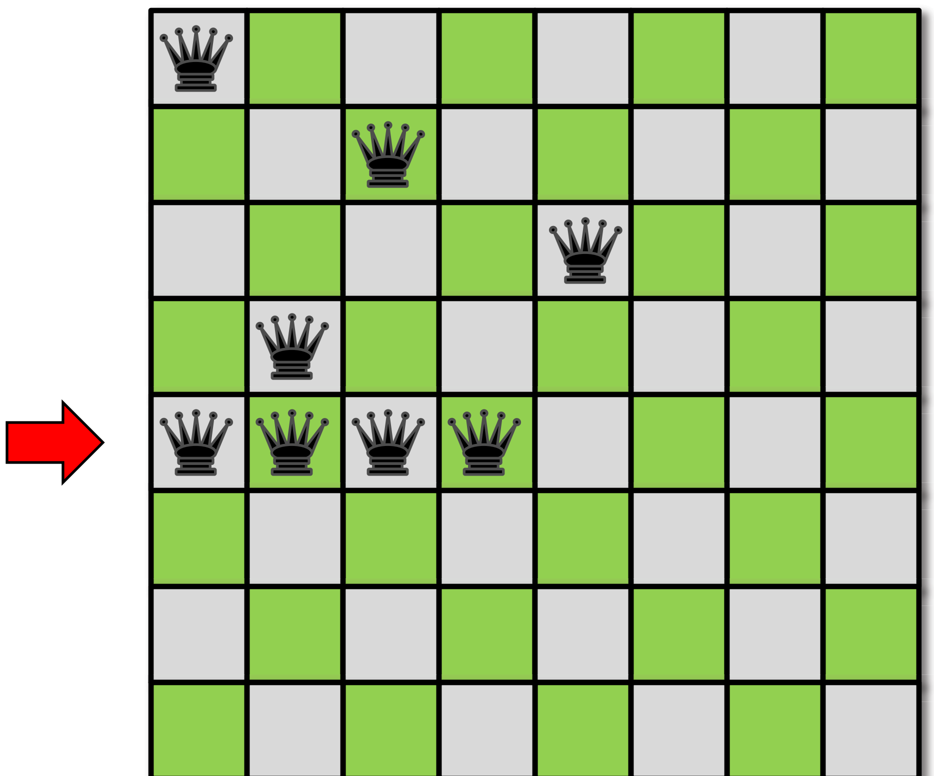
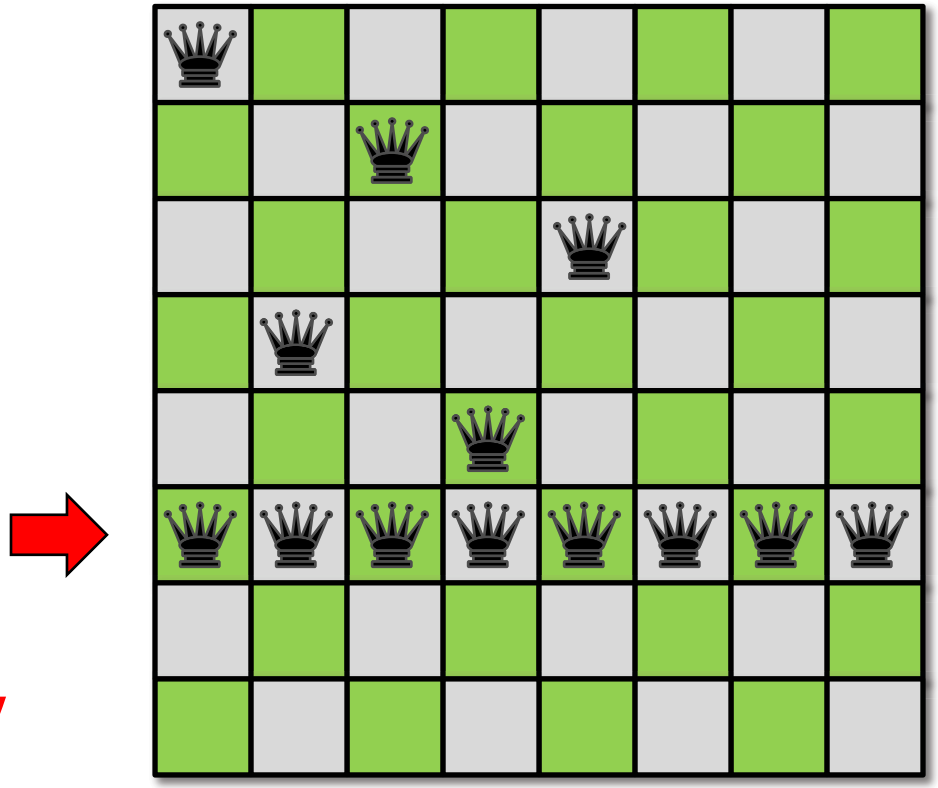


Backtracking Search

Strategy

Try placing queens row by row. If you can't place a queen in a row, backtrack.

Backtrack!



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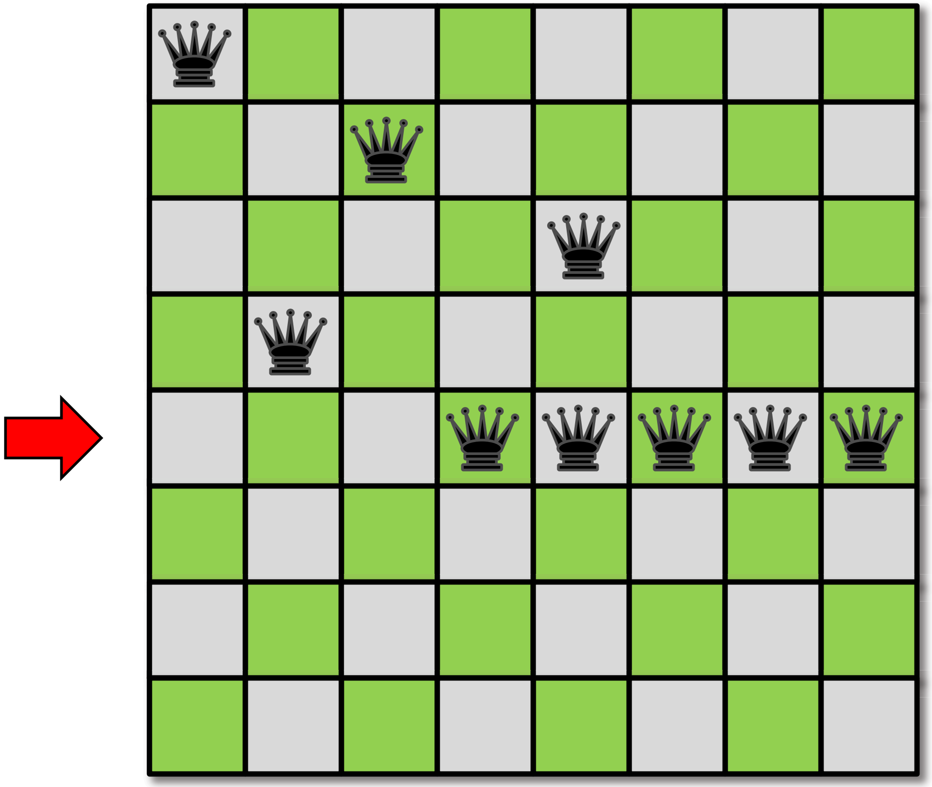

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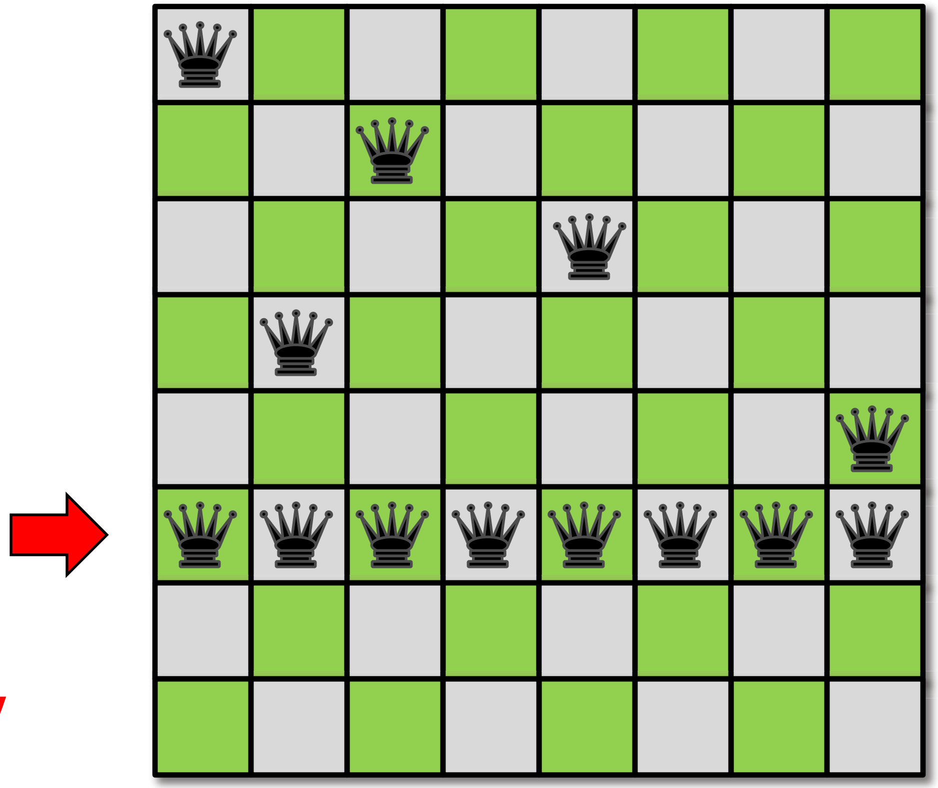

Backtrack!

Backtracking Search

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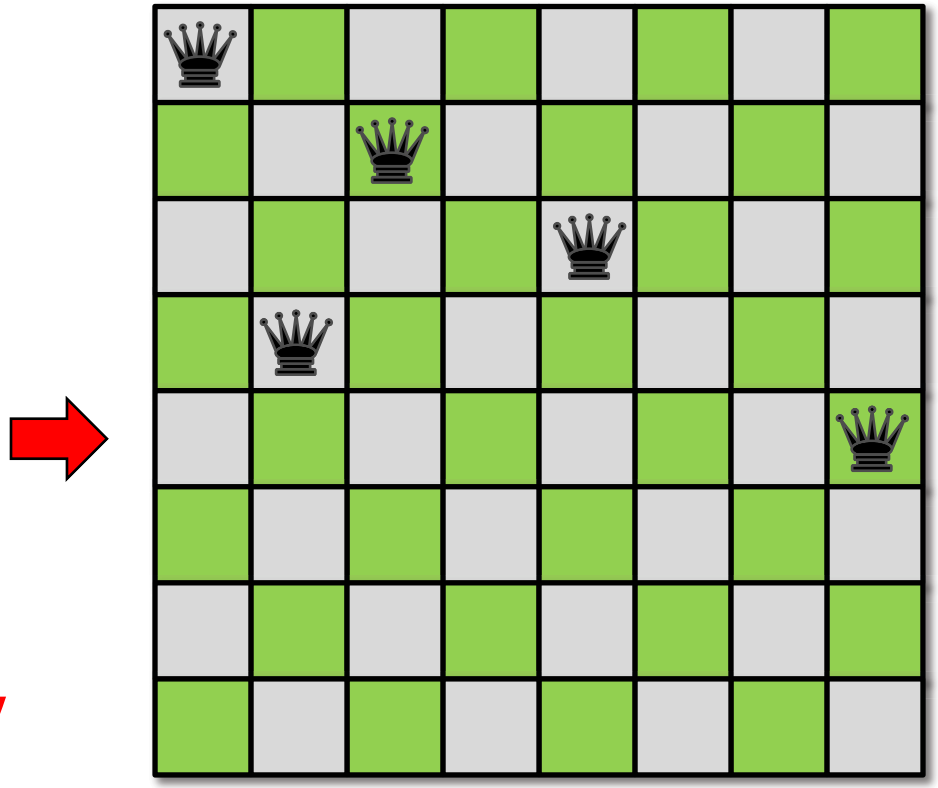
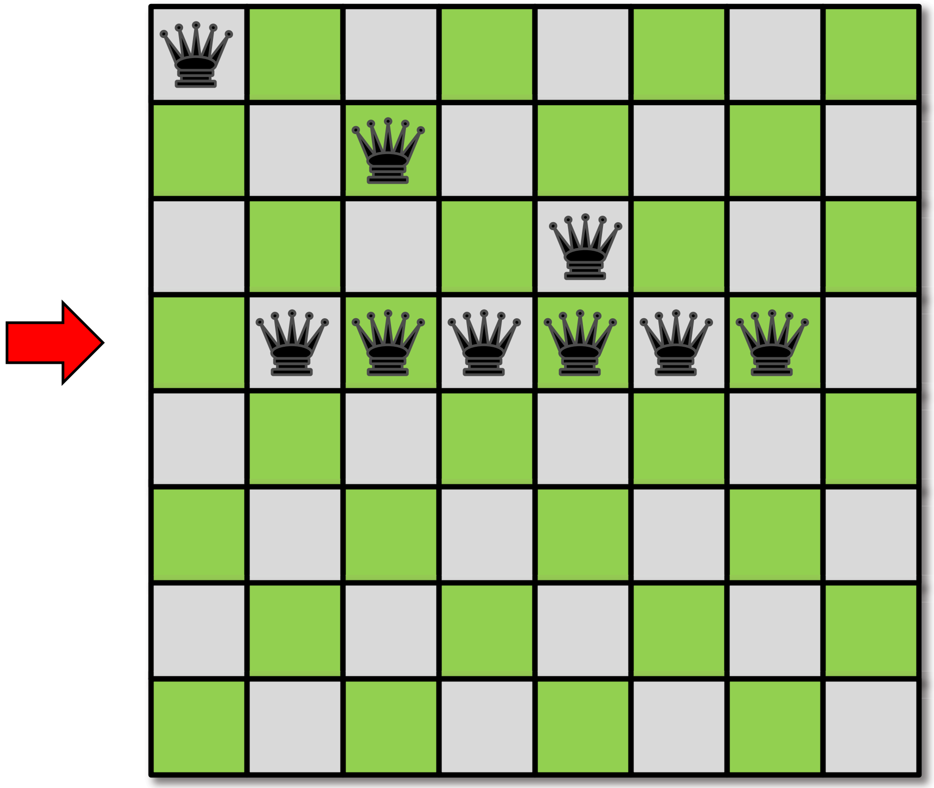


Backtrack!

Backtracking Search

Strategy

Try placing queens row by row. If you can't place a queen in a row, backtrack.

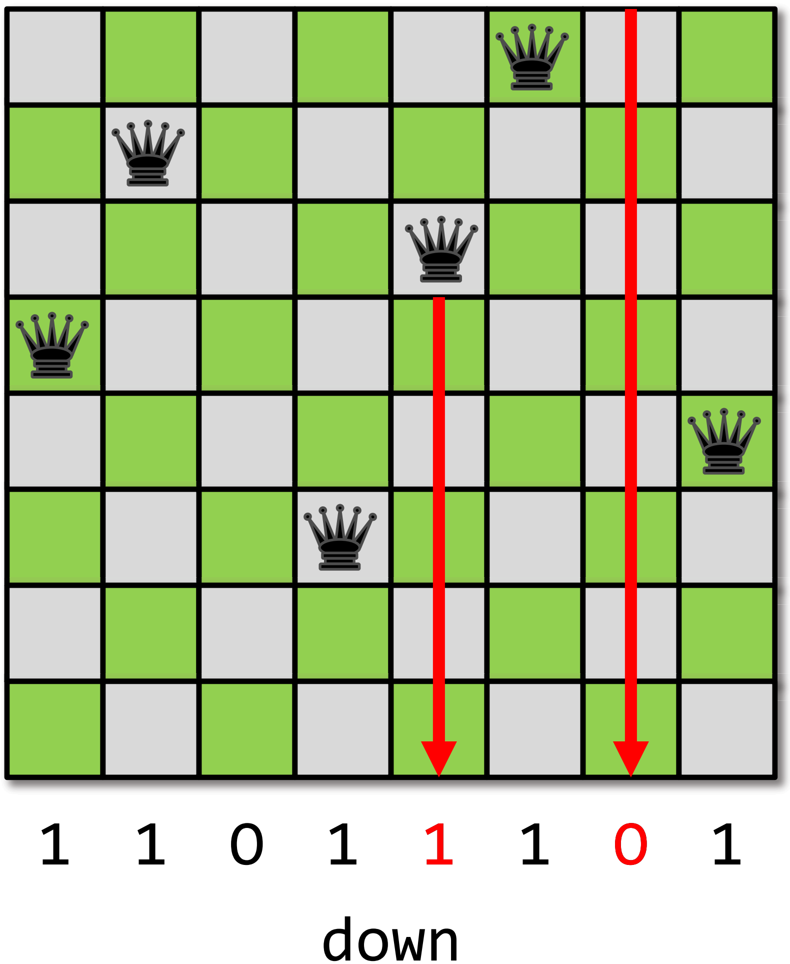


Board Representation

The backtrack search can be implemented as a simple recursive procedure, but how should the board be represented to facilitate queen placement?

- array of n^2 bytes?
- array of n^2 bits?
- array of n bytes?
- 3 bitvectors of size n, $2n-1$, and $2n-1$.

Bitvector Representation

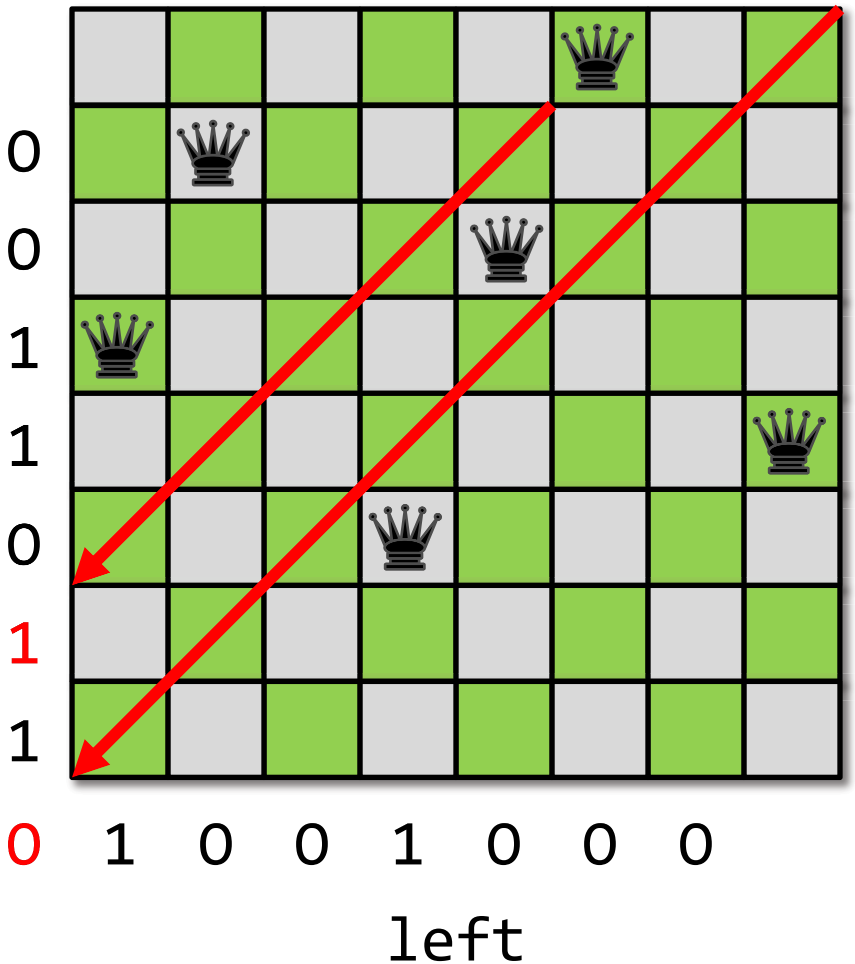


Placing a queen in column `c` is not safe if

```
down & (1 << c);
```

is nonzero.

Bitvector Representation

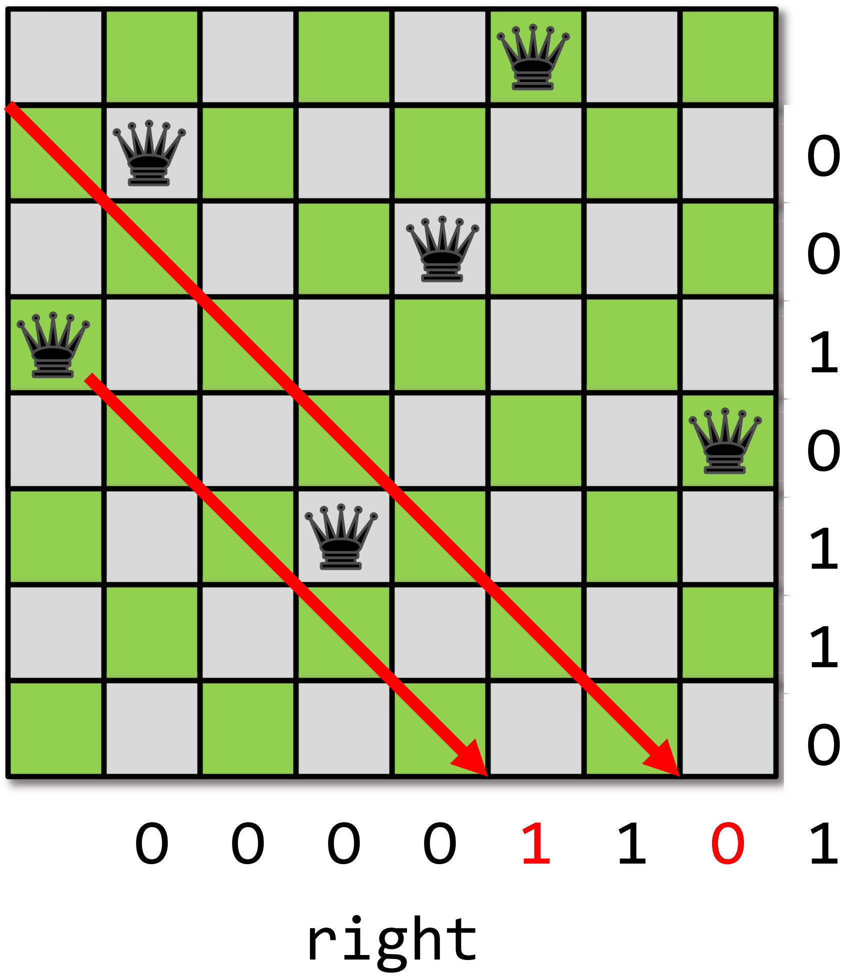


Placing a queen in row `r` and column `c` is not safe if

```
left & (1 << (r+c))
```

is nonzero.

Bitvector Representation



Placing a queen in row r and column c is not safe if

```
right & (1 << (n-1-r+c))
```

is nonzero.

Population Count I

Problem

Count the number of 1 bits in a word x.

```
for (r=0; x!=0; ++r)
  x &= x - 1;
```

Repeatedly eliminate the least-significant 1.

Example

x	0010110111010000
x - 1	0010110111001111
x & (x - 1);	0010110111000000

Issue

Fast if the popcount is small, but in the worst case, the running time is proportional to the number of bits in the word.

Population Count II

Table look-up

```
static const int count[256] =
{ 0, 1, 1, 2, 1, 2, 2, 3, 1, …, 8 };

for (int r = 0; x != 0; x >>= 8)
  r += count[x & 0xFF];
```

Performance depends on the size of x. The cost of memory operations is a major bottleneck. Typical costs:

- register: 1 cycle,
- L1-cache: 4 cycles,
- L2-cache: 10 cycles,
- L3-cache: 50 cycles,
- DRAM: 150 cycles.

per 64-byte cache line

Population Count III

Parallel divide-and-conquer

```
// Create masks
M5 = ~((-1) << 32);   // 0³²1³²
M4 = M5 ^ (M5 << 16); // (0¹⁶1¹⁶)²
M3 = M4 ^ (M4 << 8);  // (0⁸1⁸)⁴
M2 = M3 ^ (M3 << 4);  // (0⁴1⁴)⁸
M1 = M2 ^ (M2 << 2);  // (0²1²)¹⁶
M0 = M1 ^ (M1 << 1);  // (01)³²
// Compute popcount
x = ((x >> 1) & M0) + (x & M0);
x = ((x >> 2) & M1) + (x & M1);
x = ((x >> 4) + x) & M2;
x = ((x >> 8) + x) & M3;
x = ((x >> 16) + x) & M4;
x = ((x >> 32) + x) & M5;
```

Notation:

$x^k = \underbrace{xx\cdots x}_{k \text{ times}}$

Population Count III

```
  1 1 0 0 0 0 1 0 0 1 0 1 1 0 1 1 1 1 1 1 0 1 0 0 0 1 1 1 1 0 0 0

    1   0   0   0   1   1   0   1   1   1   1   0   1   1   0   0      x&M0
+   1   0   0   1   0   0   1   1   1   1   0   0   0   1   1   0      (x>>1)&M0
-----------------------------------------------------------------
      0 0     0 1     0 1     1 0     1 0     0 0     1 0     0 0      x&M1
+     1 0     0 0     0 1     0 1     1 0     0 1     0 1     0 1      (x>>2)&M1
-----------------------------------------------------------------
        0 0 0 1         0 0 1 1         0 0 0 1         0 0 0 1      x&M2
+       0 0 1 0         0 0 1 0         0 1 0 0         0 0 1 1      (x>>4)&M2
-----------------------------------------------------------------
                0 1 0 1 1 0 1 1                 0 0 0 0 0 1 0 0      x&M3
+               1 1 0 0 0 0 1 0                 0 0 0 0 0 1 0 1      (x>>8)&M3
-----------------------------------------------------------------
                                0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1      x&M4
+                               0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0      (x>>16)&M4
-----------------------------------------------------------------
  0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1
                                                         \_______/
                                                            17
```

Population Count III

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M1 = M2 ^ (M2 << 2);  // (0²1²)¹⁶
M0 = M1 ^ (M1 << 1);  // (01)³²
// Compute popcount
x = ((x >> 1) & M0) + (x & M0);
x = ((x >> 2) & M1) + (x & M1);
x = ((x >> 4) + x) & M2;
x = ((x >> 8) + x) & M3;
x = ((x >> 16) + x) & M4;
x = ((x >> 32) + x) & M5;
```

Performance

$\Theta(\lg w)$ time, where w = word length.

Avoid overflow

No worry about overflow.

Popcount Instructions

Most modern machines provide popcount instructions, which operate much faster than anything you can code yourself. You can access them via compiler intrinsics, e.g., in GCC:

```
int __builtin_popcount (unsigned int x);
```

Warning: You may need to enable certain compiler switches to access built-in functions, and your code may be less portable.

Exercise

Compute the log base 2 of a power of 2 quickly using a popcount instruction.

Further Reading

Sean Eron Anderson, "Bit twiddling hacks," http://graphics.stanford.edu/~seander/bithacks.html, 2009.

Donald E. Knuth, *The Art of Computer Programming*, Volume 4A, *Combinatorial Algorithms, Part 1*, Addison-Wesley, 2011, Section 7.1.3.

Henry S. Warren, *Hacker's Delight*, Addison-Wesley, 2003.

Happy Bit-Hacking!