MAP562 Optimal design of structures

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Homework Sheet 3, Jan 22nd, 2020

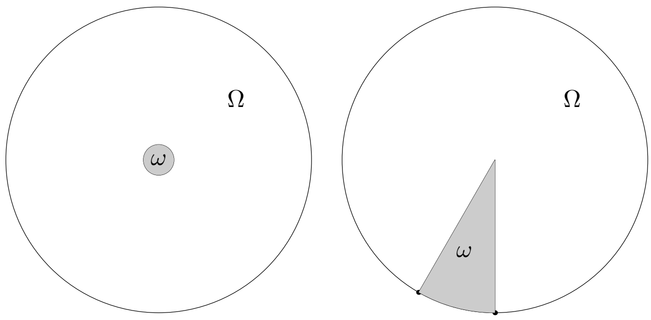
Instructions Choose ONE of the exercises below. The chosen exercise is due on January 28th, 2020. Upload your solutions as separate `FreeFEM` files, including detailed comments, to the course Moodle.

Important. Comments regarding your numerical experiments should be present either in a separate pdf file either in the `FreeFEM` codes. For students who submit codes without pertinent comments the maximal grade is B.

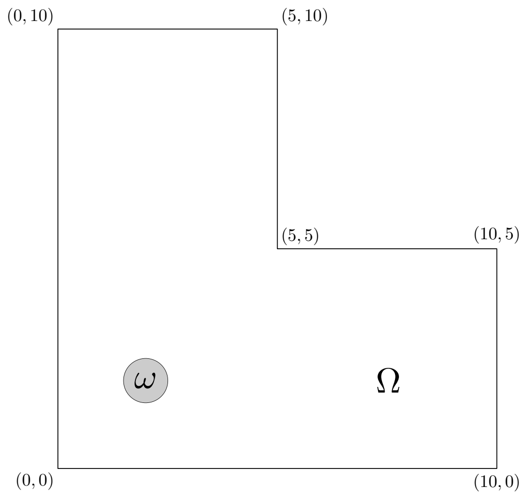
Exercise 1

We denote by (x_1, x_2) the coordinates of a point $x \in \mathbb{R}^2$ and by $B(c,r)$ the ball centered at $c \in \mathbb{R}^2$ of radius $r \in \mathbb{R}$.

1. For a **constant heat source** $v \in \mathbb{R}$, using a gradient method, implement in `FreeFem++` the minimization of the functional:
$$J(v) = \int_\Omega |T - T_0|^2 \, dx$$
where
$$\begin{cases} -\Delta T + u \cdot \nabla T &= 1_\omega v \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega. \end{cases}$$
in which the desired temperature is the constant function $T_0 = 10$.
The following geometries and velocity fields will be considered:
   - $\Omega = B((0,0),1)$, $u(x) = (-x_2, x_1)^\top$
     - $\omega = B((0,0), 0.1)$,
     - ω is a *slice* of Ω with angle $\pi/6$ (see the drawing below).



   - Ω is the L shaped domain (see the drawing below), $u = \nabla\Phi$ where Φ is a solution of
$$\Delta\Phi = 0 \text{ in } \Omega, \text{ and } \frac{\partial\Phi}{\partial n} = \begin{cases} -1 & \text{if } x_1 = 0 \text{ and } 0.5 < x_2 < 1; \\ 1 & \text{if } x_1 = 10; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$
     - $\omega = B((2,2),1)$
     - vary the position of ω and observe the behavior of the solution.



2. Compare your result to the case without convection ($u = [0, 0]$): run the simulation and write a short comment on the influence of the velocity field.

Exercise 2 Optimization of the position of a heater

For a given smooth bounded domain $\Omega \subset \mathbb{R}^d$ and for any $z \in \mathbb{R}^d$, we investigate the minimization of the cost functional:

$$J(z) = \int_\Omega |T - T_0|^2 \, dx$$

where $x \mapsto T_0(x)$ is a given smooth temperature field and $T \equiv T(x, z)$ is the solution of the following boundary value problem for the x variable

$$\begin{aligned} -\Delta T &= f_z \quad \text{in } \Omega \\ T &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f_z(x) = f(x - z)$ and $f(x)$ is a smooth non-negative function with compact support. In the numerical applications, take $d = 2$, $T_0(x) \equiv 0.1$, $\Omega = (0, 10)^2$ and

$$f(x) = \begin{cases} 1 - |x|^2 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In a second application use the L-beam geometry described above.

This problem corresponds to an exercise on Exercise Sheet 3. Solving the theoretical questions should help you with the numerical application.

1. Implement a gradient algorithm for this problem in `FreeFem++`.
2. Implement a gradient algorithm for this problem in `FreeFem++` with the additional constraint that f_z be supported in Ω.
3. **(Verification)** Implement an algorithm which computes $J(z)$ for every point z in the mesh and plot the result for one of the given geometries (as a $\mathbb{P}_1$ function, for example). (Since one PDE should be solved for each z, use a small mesh for this part: less than 1000 nodes). Once the result is plotted, verify that the result produced by the gradient algorithm coincides with a local minimum.
   Indication: The number of vertices in a mesh `Th` can be found with the command `Th.nv`. The coordinates of the node `i` of the mesh can be found with `Th(i).x` and `Th(i).y`. If `g` is a $\mathbb{P}_1$ function defined on `Th` then you can set the value of `g` at node `i` with the command `g[][i]=...`. A similar example can be found among the codes on Moodle.