OPTIMAL DESIGN OF STRUCTURES (MAP 562)

CHAPTER II

PREREQUISITES OF NUMERICAL ANALYSIS

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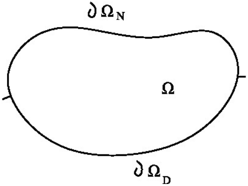
Boundary value problems (B.V.P)



Membrane model: f = bulk force, g = surface load,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega_N \end{cases}$$

$n =$ unit normal vector,
notation: $\frac{\partial u}{\partial n} = \nabla u \cdot n$.

$\partial\Omega_D$: Dirichlet boundary
$\partial\Omega_N$: Neumann boundary

Key idea which must be mastered: the variational approach

- Boundary value problem = p.d.e. + boundary conditions
- It is proved that a boundary value problem is equivalent to its variational formulation.
- From a mechanical point of view, the variational formulation is just the principle of virtual work.
- Any variational formulation can be written as

$$\text{find } u \in V \text{ such that } a(u, v) = L(v) \quad \forall\, v \in V.$$

- This approach gives an existence theory for solutions and yields numerical methods such as finite elements for computing them.
- It is also a key tool for shape optimization.

Technical ingredients

Green's formula:

$$\int_{\Omega} \Delta u(x) v(x)\, dx = -\int_{\Omega} \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(x) v(x)\, ds$$

Sobolev spaces (functions with finite energy):

$$u \in H^1(\Omega) \Leftrightarrow \int_{\Omega} \left(|\nabla u(x)|^2 + |u(x)|^2 \right) dx < +\infty$$

$$u \in H^1_0(\Omega) \Leftrightarrow u \in H^1(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega$$

- The Hilbert space V is usually a Sobolev space.
- To find a and L, the p.d.e. is multiplied by a test function.
- Integrate by parts using Green's formula.
- Use the boundary conditions for simplifying the boundary integrals.

Recipe

How to remember Green's formula ? It is enough to know the simple formula

$$\int_\Omega \frac{\partial w}{\partial x_i}(x)\,dx = \int_{\partial\Omega} w(x) n_i\,ds$$

with $n_i(x)$, the i-th component of the exterior unit normal vector to $\partial\Omega$ (to remember that it is the **exterior** normal, think about the 1-d formula !). All type of Green's formulas are deduced from this one.

As an example, take $w = v\frac{\partial u}{\partial x_i}$ and sum w.r.t. i to get

$$\int_\Omega \Delta u(x) v(x)\,dx = -\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(x) v(x)\,ds.$$

Variational formulation (V.F.)

Integration by parts yields

$$\int_\Omega f\, v\, dx = -\int_\Omega \Delta u\, v\, dx = \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds.$$

- The Dirichlet B.C. is imposed to the test functions.
- The Neumann B.C. is plugged into the variational formulation.

Adequate choice of the Sobolev space:

$$V = \left\{ v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \partial\Omega_D \right\}$$

After simplification we get: Find $u \in V$ such that

$$\boxed{\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f\, v\, dx + \int_{\partial\Omega_N} g\, v\, ds \quad \forall\, v \in V.}$$

Lax-Milgram Theorem $\Rightarrow$ existence and uniqueness of a solution $u \in V$ of the V.F.

Checking the equivalence V.F $\Leftrightarrow$ B.V.P.

We already saw that u *solution of B.V.P.* $\Rightarrow$ u *solution of V.F.*
Let us check that u solution of V.F. $\Rightarrow$ u solution of B.V.P.
Let $u \in V = \left\{v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \partial\Omega_D\right\}$ satisfy

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx + \int_{\partial\Omega_N} g \, v \, ds \quad \forall \, v \in V.$$

Integrating by parts (backwards) yields under sufficient regularity

$$-\int_\Omega \Delta u \, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds = \int_\Omega f \, v \, dx + \int_{\partial\Omega_N} g \, v \, ds \quad \forall \, v \in V.$$

Taking first v with compact support in Ω leads to

$$-\Delta u = f \;\; \text{in} \;\; \Omega.$$

Taking into account this first equality, the V.F. becomes

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds = \int_{\partial\Omega_N} g \, v \, ds \quad \forall \, v \in V.$$

In a second step, v is any function with a trace on $\partial\Omega_N$. Thus

$$\frac{\partial u}{\partial n} = g \ \text{ on } \ \partial\Omega_N.$$

The Dirichlet B.C. $u = 0$ on $\partial\Omega_D$ is guaranteed because $u \in V$. Eventually, u is a (weak) solution of the B.V.P.

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega_N. \end{cases}$$

Remark: if $\partial\Omega_D = \emptyset$ (no clamping), then a necessary and sufficient condition of existence of a solution is the force equilibrium:

$$\int_\Omega f \, dx + \int_{\partial\Omega} g \, ds = 0.$$

Furthermore, uniqueness is obtained up to an additive constant, i.e., up to a rigid displacement.

Linearized elasticity system

$$\begin{cases} -\operatorname{div}\sigma = f & \text{in } \Omega \\ \text{with} \quad \sigma = 2\mu e(u) + \lambda \operatorname{tr}(e(u)) \operatorname{Id} & \\ u = 0 & \text{on } \partial\Omega_D \\ \sigma n = g & \text{on } \partial\Omega_N, \end{cases}$$

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)_{1\le i,j\le N}$$

Variational formulation: find $u \in V$ such that

$$\int_\Omega 2\mu e(u)\cdot e(v)\, dx + \int_\Omega \lambda \operatorname{div} u \operatorname{div} v\, dx = \int_\Omega f\cdot v\, dx + \int_{\partial\Omega_N} g\cdot v\, ds\ \forall\, v \in V,$$

with

$$V = \left\{ v \in H^1(\Omega)^N \text{ such that } v = 0 \text{ on } \partial\Omega_D \right\}$$

The finite element method (F.E.M.)

Variational approximation

Exact variational formulation:

$$\text{Find } u \in V \text{ such that } a(u, v) = L(v) \quad \forall\, v \in V.$$

Approximate variational formulation (Galerkin):

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = L(v_h) \quad \forall\, v_h \in V_h$$

where $V_h \subset V$ is a finite-dimensional subspace.
The finite element method amounts to properly define simple subspaces V_h, linked to the notion of mesh of the domain Ω.

Introducing a basis $(\phi_j)_{1\leq j\leq N_h}$ of V_h, we define

$$u_h = \sum_{j=1}^{N_h} u_j\phi_j \quad \text{with} \quad U_h = (u_1, ..., u_{N_h}) \in \mathbb{R}^{N_h}.$$

The approximate V.F. is equivalent to

$$\text{Find } U_h \in \mathbb{R}^{N_h} \text{ such that } a\left(\sum_{j=1}^{N_h} u_j\phi_j, \phi_i\right) = L(\phi_i) \quad \forall\, 1 \leq i \leq N_h,$$

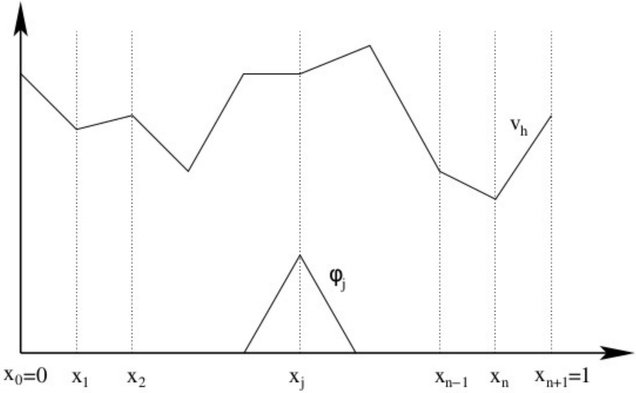
which is nothing but a linear system

$$\mathcal{K}_h U_h = b_h \quad \text{with} \quad (\mathcal{K}_h)_{ij} = a(\phi_j, \phi_i), \quad (b_h)_i = L(\phi_i).$$

Remark: the coercivity of $a(u, v)$ implies that the stiffness matrix $\mathcal{K}_h$ is positive definite. On the same token, the symmetry of $a(u, v)$ implies that of $\mathcal{K}_h$.

Lagrange $\mathbb{P}_1$ finite elements in $N = 1$ dimension

Uniform mesh with nodes (or vertices) $(x_j = jh)_{0\le j\le n+1}$ where $h = \frac{1}{n+1}$.



V_h = space of piecewise affine and globally continuous functions

$$\phi_j(x) = \phi\left(\frac{x - x_j}{h}\right) \quad \text{with} \quad \phi(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Resulting linear system

We have to solve the linear system $\mathcal{K}_h U_h = b_h$ where $\mathcal{K}_h$ is the stiffness matrix. For the membrane problem:

$$\mathcal{K}_h = \left(\int_0^1 \phi_j'(x) \phi_i'(x)\, dx \right)_{1 \leq i,j \leq n}, b_h = \left(\int_0^1 f(x) \phi_i(x)\, dx \right)_{1 \leq i \leq n},$$

$$u_h(x) = \sum_{j=1}^{N_h} u_j \phi_j(x) \quad \text{with} \quad U_h = (u_1, ..., u_{N_h}) \in \mathbb{R}^{N_h}.$$

A straightforward calculation shows that $\mathcal{K}_h$ is tridiagonal, in fact

$$\mathcal{K}_h = h^{-1} \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}.$$

Resulting linear system (ctd.)

To obtain explicitly the right hand side b_h we have to compute the integrals

$$(b_h)_i = \int_{x_{i-1}}^{x_{i+1}} f(x)\phi_i(x)\,dx \quad \text{for} \quad 1 \le i \le n.$$

For that purpose one uses quadrature formulas (or numerical integration). For example, the "trapezoidal rule"

$$\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \psi(x)\,dx \approx \frac{1}{2}\left(\psi(x_{i+1})+\psi(x_i)\right),$$

Remark. In most cases, Gauss quadrature is employed yielding optimal order.

Convergence of the F.E.M.

Theorem. Let $u \in H_0^1(0,1)$ and $u_h \in V_{0h}$ be the exact and approximate solutions, respectively. The $\mathbb{P}_1$ finite element method converges in the sense that

$$\lim_{h \to 0} \|u - u_h\|_{H^1(0,1)} = 0.$$

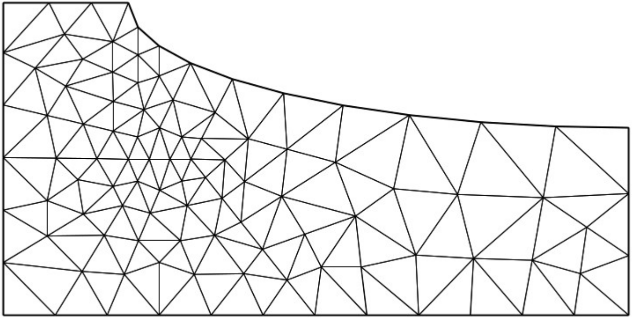
Furthermore, if $u \in H^2(0,1)$ (which is true as soon as $f \in L^2(0,1)$), then there exists a constant C, which does not depend on h, such that

$$\|u - u_h\|_{H^1(0,1)} \leq Ch\|u''\|_{L^2(0,1)} = Ch\|f\|_{L^2(0,1)}.$$

Remark. One advantage of the V.F. (in comparison to the strong form) is that the F.E. basis functions need not to be twice differentiable but merely once.

F.E.M. in higher dimensions $N \geq 2$

Lagrange $\mathbb{P}_1$ finite elements



The domain is meshed by triangles in dimension $N = 2$ or tetrahedra in dimension $N = 3$ with vertices denoted by $(a_j)_{1 \leq j \leq N+1}$ in $\mathbb{R}^N$.

We shall use FreeFem++ `http://www.freefem.org`

Lemma Let K be a triangle or a tetrahedron with vertices $(a_j)_{1 \le j \le N+1}$. Any affine function (polynomial of degree ≤ 1) $p \in \mathbb{P}_1$ can be written as

$$p(x) = \sum_{j=1}^{N+1} p(a_j)\lambda_j(x),$$

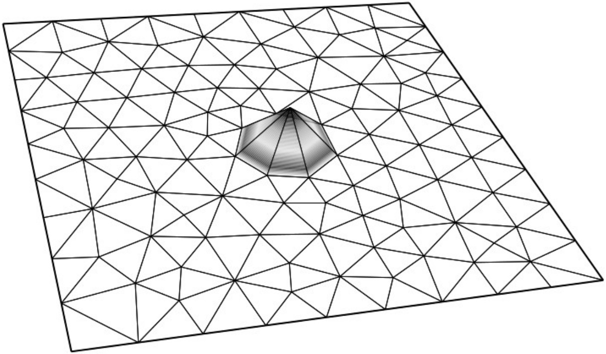
where $(\lambda_j(x))_{1 \le j \le N+1}$ are the barycentric coordinates of $x \in \mathbb{R}^N$ defined by

$$\begin{cases} \sum_{j=1}^{N+1} a_{i,j}\lambda_j = x_i & \text{for } 1 \le i \le N \\ \sum_{j=1}^{N+1} \lambda_j = 1 \end{cases}$$

In other words, any $\mathbb{P}_1$ function is uniquely characterized by its (nodal) values at the vertices or nodes of the mesh.

The Lagrange $\mathbb{P}_1$ finite element method (**triangular F.E. of order** 1) associated to a mesh $\mathcal{T}_h$ is defined by

$$V_h = \left\{ v \in \mathcal{C}(\overline{\Omega}) \text{ such that } v\,|_{K_i} \in \mathbb{P}_1 \text{ for any } K_i \in \mathcal{T}_h \right\}.$$



Basis function of V_h associated to one node or vertex of the mesh.

Resulting linear system

We have to solve the linear system $\mathcal{K}_h U_h = b_h$ where $\mathcal{K}_h$ is the stiffness matrix. For the membrane problem:

$$\mathcal{K}_h = \left(\int_\Omega \nabla \phi_j \cdot \nabla \phi_i \, dx \right)_{1 \le i,j \le n_{dl}}, b_h = \left(\int_\Omega f \phi_i \, dx \right)_{1 \le i \le n_{dl}},$$

$$u_h(x) = \sum_{j=1}^{N_h} u_j \phi_j(x) \quad \text{with} \quad U_h = (u_h(\hat{a}_j))_{1 \le j \le n_{dl}} \in \mathbb{R}^{n_{dl}}$$

Example of quadrature formula for an approximate computation of integrals:

$$\int_K \psi(x) \, dx \approx \frac{\mathrm{Volume}(K)}{N+1} \sum_{i=1}^{N+1} \psi(a_i)$$