OPTIMAL DESIGN OF STRUCTURES (MAP 562)

CHAPTER I

INTRODUCTION TO OPTIMAL DESIGN

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Basic concepts

A problem of optimal design (or shape optimization) of structures is defined by three ingredients:

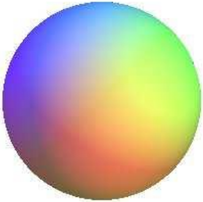
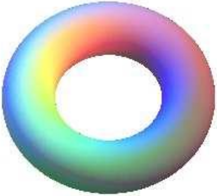
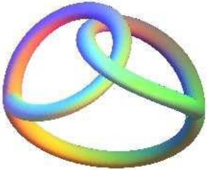
- a **model** (typically a system of partial differential equations) to assess the mechanical behavior of a structure,
- an **objective function** which has to be minimized or maximized, or sometimes several objectives (also called cost functions or criteria),
- a **set of admissible designs** which precisely defines the optimization variables, including possible constraints.

Optimal design problems can roughly be classified in three categories from the "easiest" ones to the "most difficult" ones:

- **parametric or sizing** optimization for which designs are parametrized by a vector variable (for example, thickness or member sizes), implying that the set of admissible designs is considerably simplified,
- **geometric or shape** optimization for which all designs are obtained from an initial guess by moving its boundary (without changing its topology, i.e., its number of holes in 2-d),
- **topology** optimization where both the shape and the topology of the admissible designs can vary without any explicit or implicit restrictions.

Notion of topology

Two shapes have the same topology if there exists a continuous deformation from one to the other.
In dimension 2 topology is characterized by the number of holes or of connected components of the boundary.
In dimension 3 it is quite more complicated ! Not only the hole's number matters but also the number and intricacy of "handles" or "loops".
(a ball $\neq$ a ball with a hole inside $\neq$ a torus $\neq$ a bretzel)

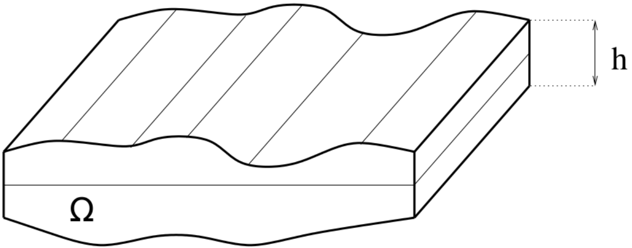
Goals of the course

1. To introduce numerical algorithms for computing optimal designs in a **"systematic" way** and not **by "trials and errors"**.
2. To obtain optimality conditions (necessary and/or sufficient) which are crucial both for the theory (characterization of optimal shapes) and for the numerics (they are the basis for gradient-type **algorithms**).
3. A (very) brief survey of theoretical results on existence, uniqueness, and qualitative properties of optimal solutions ; such issues will be discussed only when they matter for numerical purposes.

A **continuous** approach of shape optimization is preferred to a **discrete** one.

Example of sizing or parametric optimization

Thickness optimization of a membrane



- $\Omega =$ mean surface of a (plane) membrane
- $h =$ thickness in the normal direction to the mean surface

The membrane deformation is modeled by its vertical displacement $u(x) : \Omega \to \mathbb{R}$, solution of the following partial differential equation (p.d.e.), the so-called **membrane model**,

$$\begin{cases} -\operatorname{div}(h\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the thickness h, bounded by minimum and maximum values

$$0 < h_{min} \le h(x) \le h_{max} < +\infty.$$

The thickness h is the optimization variable.
It is a **sizing or parametric** optimal design problem because the optimization variable is a function defined over a fixed computational domain Ω.

The set of admissible thickness is

$$\mathcal{U}_{ad} = \Big\{ h(x) : \Omega \to \mathbb{R} \text{ s. t. } h_{min} \leq h(x) \leq h_{max} \\ \text{and } \int_\Omega h(x)\,dx = h_0|\Omega| \Big\},$$

where h_0 is an imposed average thickness and $0 < h_{min} < h_{max}$.

Possible additional "feasibility" constraints: according to the production process of membranes, the thickness $h(x)$ can be discontinuous, or on the contrary continuous. A uniform bound can be imposed on its first derivative $h'(x)$ (molding-type constraint) or on its second-order derivative $h''(x)$, linked to the curvature radius (milling-type constraint).

The **optimization criterion** is related to the mechanical behaviour of the membrane, evaluated through its displacement u by

$$J(h) = \int_\Omega j(u)\, dx,$$

where, of course, u depends on h. For example, the global stiffness of a structure is often measured by its **compliance**, i.e. the work done by the load: the smaller is the work, the larger is the stiffness (be careful ! compliance = - stiffness). In such a case,

$$j(u) = fu.$$

Another example is to reach (at least approximately) a **target displacement** $u_0(x)$, which can be achieved by

$$j(u) = |u - u_0|^2.$$

Those two criteria are the typical examples studied in this course.

Other examples of objective functions

- Introducing the stress vector $\sigma(x) = h(x)\nabla u(x)$, we can minimize the maximum stress norm

$$J(h) = \sup_{x\in\Omega} |\sigma(x)|$$

or more generally, for any $p \geq 1$,

$$J(h) = \left(\int_\Omega |\sigma|^p dx \right)^{1/p}.$$

- For a vibrating structure, introducing the first eigenfrequency ω, defined by

$$\begin{cases} -\operatorname{div}(h\nabla u) = \omega^2 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we consider $J(h) = -\omega$, in order to maximize ω.

Other examples of criteria (ctd.)

- Multiple loads optimization: for n given loads $(f_i)_{1\leq i\leq n}$ the independent displacements u_i are solutions of

$$\begin{cases} -\operatorname{div}(h\nabla u_i) = f_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

and we introduce an aggregated criteria

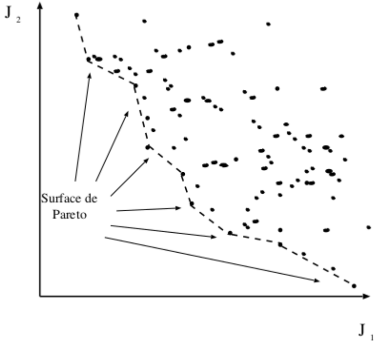
$$J(h) = \sum_{i=1}^{n} c_i \int_\Omega j(u_i)\,dx,$$

with given coefficients c_i, or

$$J(h) = \max_{1\leq i\leq n} \left(\int_\Omega j(u_i)\,dx \right).$$

- Multi-criteria optimization: notion of Pareto front (see next slide).

Multi-criteria optimization: Pareto front



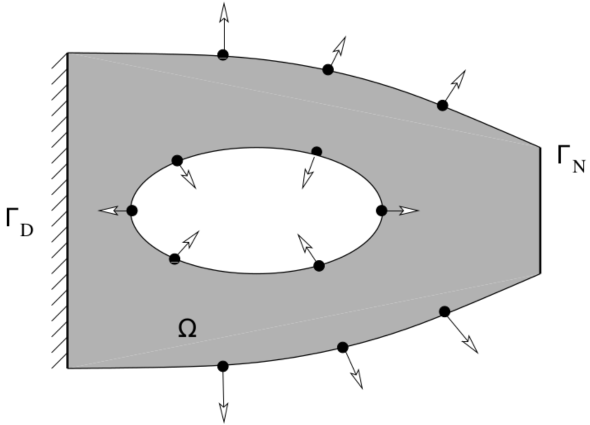
Assume we have n objective functions $J_i(h)$.
A design h is said to dominate another design $\tilde{h}$ if

$$J_i(h) \leq J_i(\tilde{h}) \quad \forall\, i \text{ and } \exists i \text{ s.t. } J_i(h) < J_i(\tilde{h}).$$

The Pareto front is the set of designs which are not dominated by any other.

Example of geometric optimization

Optimization of a membrane's shape



A reference domain for the membrane is denoted by Ω, with a boundary made of three disjoint parts

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

where Γ is the variable part, Γ_D is the Dirichlet (clamped) part and Γ_N is the Neumann part (loaded by g).
The vertical displacement u is the solution of the **membrane model**

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

From now on the membrane thickness is fixed, equal to 1.

The set of admissible shapes is thus

$$\mathcal{U}_{ad} = \left\{ \Omega \subset \mathbb{R}^N \text{ such that } \Gamma_D \bigcup \Gamma_N \subset \partial\Omega \text{ and } \int_\Omega dx = V_0 \right\},$$

where V_0 is a given volume. The **geometric** shape optimization problem reads

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega),$$

with, as a **criteria**, the compliance

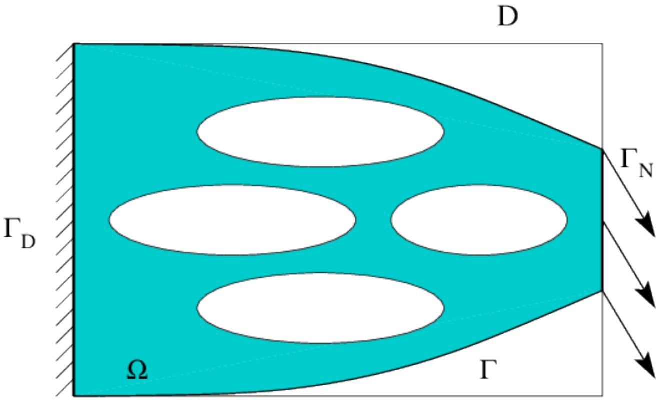
$$J(\Omega) = \int_{\Gamma_N} g u \, dx,$$

or a least square functional to achieve a target displacement $u_0(x)$

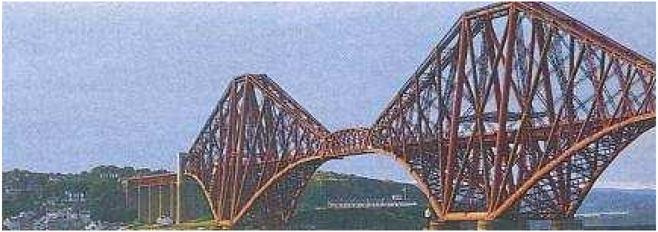
$$J(\Omega) = \int_\Omega |u - u_0|^2 dx.$$

The true optimization variable is the free boundary Γ.

Example of topology optimization



Not only the shape boundaries Γ are allowed to move but new connected components (holes in 2-d) of Γ can appear or disappear.
Topology is now optimized too.



Shape optimization in the elasticity setting

The **model of linearized elasticity** gives the displacement vector field $u(x) : \Omega \to \mathbb{R}^N$ as the solution of the system of equations

$$\begin{cases} -\operatorname{div}(A\, e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\, e(u)\big) n = g & \text{on } \Gamma_N \\ \big(A\, e(u)\big) n = 0 & \text{on } \Gamma \end{cases}$$

with $e(u) = \big(\nabla u + (\nabla u)^t\big)/2$, and $A\xi = 2\mu\xi + \lambda(tr\xi)\,\mathrm{Id}$, where μ and λ are the Lamé coefficients.
The domain boundary is again made of three disjoint parts

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

where Γ is the free boundary, the true optimization variable.

The set of admissible shapes is again

$$\mathcal{U}_{ad} = \left\{ \Omega \subset \mathbb{R}^N \text{ such that } \Gamma_D \bigcup \Gamma_N \subset \partial\Omega \text{ and } \int_\Omega dx = V_0 \right\},$$

where V_0 is a given imposed volume. The **criteria** is either the compliance

$$J(\Omega) = \int_{\Gamma_N} g \cdot u \, dx,$$

or a least-square criteria for the target displacement $u_0(x)$

$$J(\Omega) = \int_\Omega |u - u_0|^2 dx.$$

As before, the shape optimization problem reads

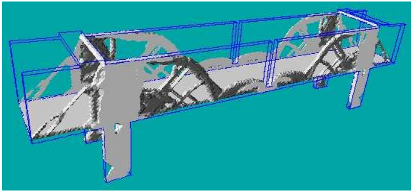
$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega).$$

Three possible approaches: **parametric**, **geometric**, **topology**.

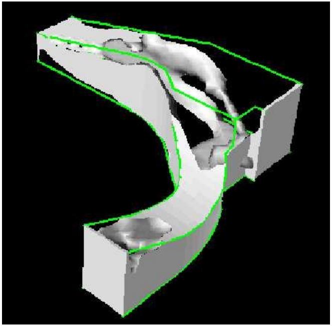
Applications

See the web site `http://www.cmap.polytechnique.fr/~optopo` (and links therein).

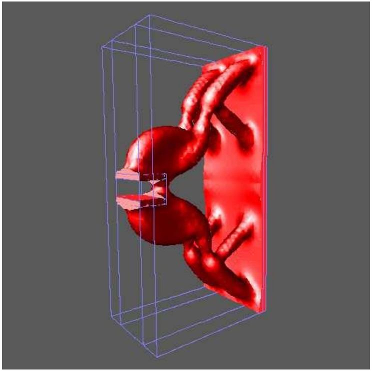
Civil engineering



Mechanical engineering



Micromechanics (MEMS)



Aeronautics

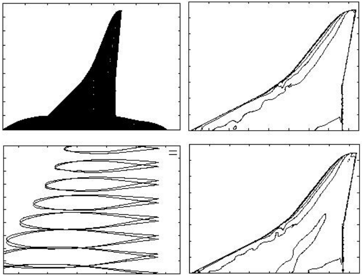
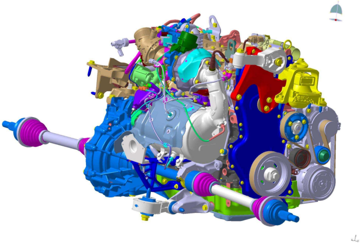


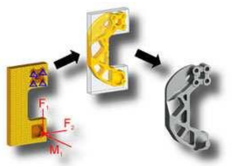
Figure 1: 3D optimization for a supersonic civil transport: top left: a view from above of the airplane with the trace of the mesh. Top right: Mach lines after optimization (minimal drag) and bottom right shows the same before optimization. Finally the bottom left figure shows cross sections of the initial (dotted lines) and optimized wings.

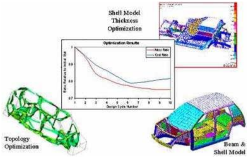
Industrial examples at Airbus, Renault, Safran...



Commercial softwares

Optistruct, Ansys DesignSpace, Genesis, MSC-Nastran, Tosca, devDept...

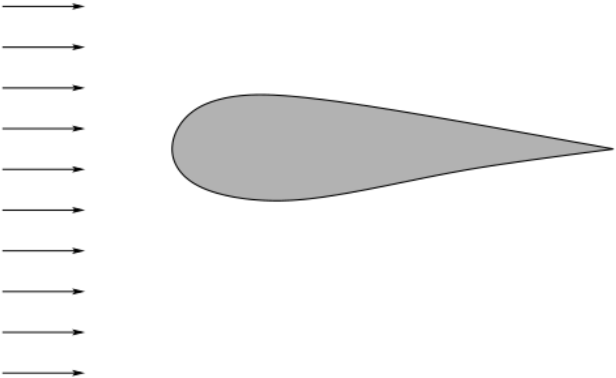




Example in fluid mechanics

Optimization of a wing profile

Drag minimization and lift maximization.



Constant velocity at infinity U_0.

Potential flow: simplification of Navier-Stokes equations for a perfect incompressible and irrotational fluid in a steady state regime. The velocity U derives from a scalar potential ϕ

$$U = \nabla \phi.$$

Bernoulli's law for the pressure:

$$p = p_0 - \frac{1}{2}|\nabla \phi|^2.$$

Governing equations for the potential:

$$\begin{cases} -\Delta \phi = 0 & \text{in } \Omega \\ \lim\limits_{|x| \to +\infty} (\phi(x) - U_0 \cdot x) = 0 & \text{at infinity} \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \partial P. \end{cases}$$

D'Alembert paradox: zero drag, zero lift !
We choose a criteria on the pressure

$$J(P) = \int_{\partial P} j(p)\, ds\,,$$

where the function j is typically a least square criteria for a target pressure

$$j(p) = |p - p_{target}|^2.$$

The **geometric shape optimization** problem reads

$$\inf_{P \in \mathcal{U}_{ad}} J(P).$$

A priori, there is no need of topology optimization for a wing profile...

Parametric optimization of a thin profile (in 2-d)

Example on how to reduce a geometric optimization problem into a parametric one.

Thin profile P with upper and lower boundaries (extrados and intrados) defined by

$$y = f^+(x) \quad \text{for} \quad 0 \le x \le L, \qquad y = f^-(x) \quad \text{for} \quad 0 \le x \le L,$$

where L is the length of the profile chord. We assume that the velocity at infinity U_0 is aligned with the x-axis. The Neumann boundary condition for the potential is

$$\frac{\partial \phi}{\partial y} - \frac{df^\pm}{dx}\frac{\partial \phi}{\partial x} = 0 \text{ on } \partial P,$$

which, at first order, becomes

$$\frac{\partial \phi}{\partial y} = U_0 \frac{df^\pm}{dx} \text{ on the chord } [0, L].$$

Simplified model with $\Sigma = [0, L]$

$$\begin{cases} -\Delta\phi = 0 & \text{in } \Omega \setminus \Sigma \\ \lim_{|x|\to+\infty} (\phi(x) - U_0 \cdot x) = 0 & \text{at infinity} \\ \frac{\partial\phi}{\partial y} = U_0 \frac{df^+}{dx} & \text{on } \Sigma^+ \\ \frac{\partial\phi}{\partial y} = U_0 \frac{df^-}{dx} & \text{on } \Sigma^-. \end{cases}$$

Parametric optimization problem

$$\inf_{f^\pm \in \mathcal{U}_{ad}} J(f^\pm),$$

with

$$\mathcal{U}_{ad} = \left\{ \begin{array}{l} f^+ : [0, L] \to \mathbb{R}^+ \\ f^- : [0, L] \to \mathbb{R}^- \end{array} \text{ s. t. } f^+(0) = f^-(0) = f^+(L) = f^-(L) = 0 \right\}$$

The main advantage is that the domain Ω is now **fixed**.

Modeling choices

Modeling is typically an engineering issue.

- Choice of the model: a **compromise** between accuracy and the CPU cost (optimization requires many successive analyses of the model).
- Choice of the criterion: difficulty of **measuring** a qualitative property, of **combining** several criteria.
- Choice of the admissible set: **selecting** the most appropriate constraints from the point of view of the applications but also of the numerical algorithms.

We shall not discuss this issue during the course. It is however an important aspect of the personal projects (EA).

Other fields related to shape optimization

The technical tools in this course are also useful for the following areas:

- Optimal control.
- Inverse problems.
- Sensitivity analysis to parameters.