MAP562 Optimal design of structures (École Polytechnique)
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Session 6: Feb 12th, 2020 – Geometric Optimization

Exercise 1

We consider a structure $\Omega \in \mathbb{R}^d, d = 2, 3$ (open, bounded, and sufficiently regular). Let x_Ω be its gravitational center and $V(\Omega)$ its volume. We define the admissible set as

$$\mathcal{U}_{ad} = \left\{ \Omega \subset \mathbb{R}^N \text{ such that } V(\Omega) = V_0 \right\}.$$

The goal is the minimization of the trace of the inertia tensor:

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega); \quad J(\Omega) = \frac{1}{2} \int_\Omega |x - x_\Omega|^2 dx.$$

We suppose that a minimiser exists and is regular.

Question: What can we say about possible minima of the above problem?

Exercise 2

Let $V(x)$ be a vector field, i.e., a smooth function from $\mathbb{R}^d$ into $\mathbb{R}^d$. We define the functional

$$J(\Omega) = \int_{\partial\Omega} V \cdot n \, ds,$$

where n is the unit exterior normal to the domain Ω.

Question: Compute the shape derivative of $J(\Omega)$.

Exercise 3

We consider the optimization of a membrane with a constant thickness in a domain $\Omega \subset \mathbb{R}^2$ (open, bounded, and sufficiently regular). The boundary is split into three parts: $\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$. The parts Γ_D and Γ_N are fixed and only Γ can vary. We furthermore introduce the admissible set of shapes:

$$\mathcal{U}_{ad} = \left\{ \Omega \subset \mathbb{R}^2 \text{ such that } (\Gamma_D \cup \Gamma_N) \subset \partial\Omega \right\}.$$

The displacement $u(x)$ of the membrane is the solution of the PDE:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{1}$$

where $f \in L^2(\mathbb{R}^2)$ is a volume force and $g \in L^2(\Gamma_N)$ is a surface traction force. Let $u_0(x) \in L^2(\mathbb{R}^2)$ be a given displacement that we want to match. The objective functional can then be formulated as:

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega); \quad J(\Omega) = \int_\Gamma |u - u_0|^2 ds,$$

where u is the solution of ((1)).

1. Formulate the Lagrangian $\mathcal{L}(\Omega, v, q)$ and deduce the adjoint state.
2. Calculate (formally) the shape derivative of the objective function.
3. If we find $u = u_0$ on Γ for some shape Ω, what is the value of the shape derivative?

Exercise 4   Optimize an inclusion

In this exercise we consider an elastic body Ω which is clamped on Γ_D and is subjected to some loadings on part Γ_N. We assume that $\partial\Omega = \Gamma_D \cup \Gamma_N$. We suppose that there is a hole $\omega \subset \Omega$ which is traction-free and will be subject to optimization. As usual, the displacements u are computed using the linearized elasticity equation

$$\begin{cases} -\operatorname{div}\sigma & = & 0 & \text{in } \Omega \setminus \omega \\ u & = & 0 & \text{on } \Gamma_D \\ \sigma n & = & \sigma_0 n & \text{on } \Gamma_N \\ \sigma n & = & 0 & \text{on } \partial\omega \end{cases}.$$

The objective function is the compliance

$$J(\omega) = \int_{\Gamma_N} \sigma_0 n \cdot u.$$

1. Show that an alternate formula for the compliance is

$$J(\omega) = \int_{\Omega\setminus\omega} \sigma \cdot e(u) dx$$

where σ is the stress tensor $2\mu e(u) + \lambda \operatorname{div} u$.

2. Compute the shape derivative with respect to ω, when ω is a general inclusion in the direction given by a general vector field θ. (Note that the outer boundary $\partial\Omega$ does not move.)

In the rest of the exercise we suppose that the inclusion ω is an ellipse, which will simplify the numerical aspects. The ellipse is characterized by its center $\mathbf{c} = (x_0, y_0)$ its principal axes a, b and its orientation α.

3. What are the vector fields θ which correspond to the following transformations T of $\partial\omega$:
   - Translations $T\mathbf{x} = \mathbf{x} + \mathbf{v}$ with $\mathbf{v} = (v_x, v_y)$
   - Scalings $T\mathbf{x} = (\alpha, \beta) \cdot (\mathbf{x} - \mathbf{c}) + \mathbf{c}$, with $\alpha, \beta \in \mathbb{R}_+$
   - **(Optional)** Rotations $T\mathbf{x} = \mathbf{A}(\mathbf{x} - \mathbf{c}) + \mathbf{c}$ where $\mathbf{A} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ for some angle α.

   Deduce the partial derivatives of $J(\omega)$ with respect to the parameters defining the ellipse: x_0, y_0, a, b (**(Optional)** α).

4. Implement a `FreeFem++` algorithm which performs the optimization of $J(\omega)$, when $|\omega|$ is fixed, in the following cases:
   - $\partial\omega$ is a circle ($a = b = r$) and the center (x_0, y_0) is variable.
   - $\partial\omega$ is an ellipse with fixed center and variable axes
   - $\partial\omega$ is a general ellipse with axes oriented parallel to the coordinate axes
   - **(Optional)** $\partial\omega$ is a general ellipse (including variable orientation)

5. **(Homework)** Consider the following types of parametric curves:
   1. **Stadiums:** rectangles of dimensions $a \times b$ with a half disk of diameter b glued to the two opposite sides of length b. Optimization variables: a, b and the center.
   2. **Superellipse:** parametric curve defined by $\frac{|x|^p}{a^p} + \frac{|y|^p}{b^p} = 1$ where $p = 4$. Optimization variables: a, b and the center.
   3. **Generic parametric curve:** curve with given radial parametrization $\rho(\beta) = 1 + a\cos(2\beta) + b\cos(3\beta)$. Recall that given a radial function $[0, 2\pi] \ni \beta \mapsto \rho(\beta) \in (0, \infty)$ the corresponding parametric curve is defined by $(x(\beta), y(\beta)) = (\rho(\beta)\cos\beta, \rho(\beta)\sin\beta)$. (notice that this is one of the standard ways to define a curve in `FreeFem++`) You may suppose that $|a| \leq 0.5, |b| \leq 0.5$ and the bounding box is the square $(-2.5, 2.5)^2$. For this type of curve you may assume that the center for the radial parametrization is fixed. The optimization variables are: a, b. Note that in the computation of the derivative you may need the value of the angle β on the boundary of the curve $\partial\omega$. You may use the `FreeFem++` function `atan2` to do this (an example will be posted on Moodle).

   Choose one of the categories of curves defined above and compute the partial derivatives of $J(\omega)$ with respect to the relevant parameters. As above, it is useful to find what is the corresponding vector field θ when varying the desired parameter and then use the general shape derivative formula found at Question 1.