MAP562 Optimal design of structures (École Polytechnique)

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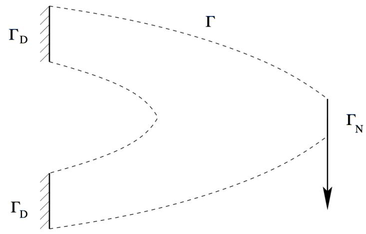
Session 7: Feb 26th, 2020 – Geometric Optimization
Exercise and Homework Sheet

Instructions The homework is Exercise 3 (note that the theoretical part is treated in class in Exercise 2). Upload your solutions to the course Moodle by March 4th.

Exercise 1

We now turn to a practical exercise to study in more detail geometric optimization. To this end, download the `FreeFem++` script `Cantilever_Geometric_Opt.edp`, which can be found on the course Moodle. The goal is compliance minimization in order to obtain optimal stiffness of a cantilever. The cantilever is given in a domain $\mathbb{R}^2$ and clamped on Γ_D. On Γ_N the solid is subject to a traction force. The space for the displacements $u(\Omega)$ is defined as

$$u(\Omega) \in V := \{v \in H^1(\Omega)^2 : v = 0 \text{ on } \Gamma_D\}.$$



The state equation in variational form is given by:

$$\int_\Omega (2\mu e(u) \cdot e(v) + \lambda \operatorname{div}(u) \operatorname{div}(v))\, dx = \int_{\Gamma_N} g \cdot v \, ds \quad \forall v \in V, \tag{1}$$

where μ and λ are the Lamé coefficients, and $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. The objective functional is given by:

$$J_c(\Omega) = \int_{\Gamma_N} g \cdot u(\Omega)\, ds.$$

The admissible set is determined by the fact that Γ_D and Γ_N are fixed and a prescribed volume V_0:

$$\mathcal{U}_{ad} := \{\Omega \text{ open in } \mathbb{R}^2 \text{ such that } (\Gamma_D \cup \Gamma_N) \subset \partial\Omega \text{ and } \int_\Omega dx = V_0\}.$$

Algorithm. The code for this problem is given in the file `Cantilever_Geometric_Opt.edp`. In order to simplify the computation we model the volume constraint using a fixed Lagrange multiplier $\ell \in \mathbb{R}$. In practice we minimize the problem

$$J_c(\Omega) + \ell|\Omega|$$

with ℓ fixed.

The algorithm has the following steps:

(i) At each iteration we solve the elasticity problem and we compute the corresponding shape derivative. Recall that in the case of the compliance the shape derivative can be computed directly from the state equation, since the problem is auto-adjoint.

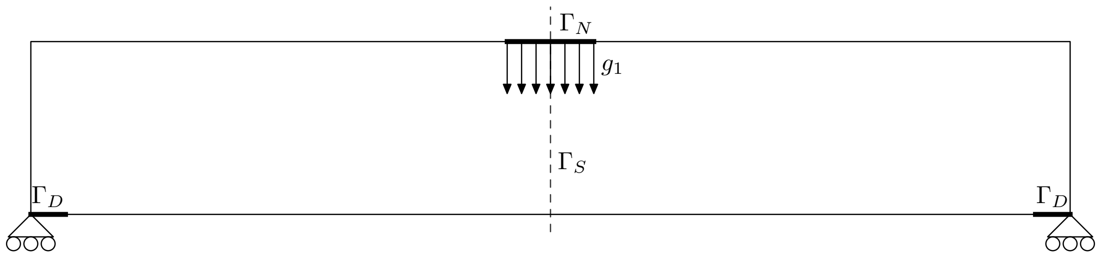
(ii) We wish to perturb the free part of the boundary $\partial\Omega$ using mesh deformations in `FreeFem++`. In order to define the mesh we need to extend the vector field given by the shape derivative to all the nodes of the mesh. This is done with an extension/regularization procedure. We also define some regions close to the parts of the boundary which are fixed (Γ_D and Γ_N) where the shape is not optimizable.

(iii) We use the command `movemesh` in order to change the mesh of Ω. Before updating the mesh we check that the procedure does not invert triangles using the command `checkmovemesh` and we also check that the objective function has decreased.

(iv) In the end we use the command `adaptmesh` in order to improve the quality of the new mesh and we go to the next iteration.

Tasks.

1. Read the `FreeFem++` script and become familiar with its implementation. Identify for yourself the state equation, cost functional, and optimization loop.
2. Test different initial domains Ω_0 (different exterior boundaries, different number of holes) and observe the final optimal shapes.
3. Comment the results: local minima vs global minima.
4. Implement a similar algorithm for the case of a MBB-beam represented in the Figure below: a structure is included in a rectangle of size 6×1 with prescribed zero vertical displacements in the lower corners and a vertical applied load on the middle of the top part. In order to simplify the computations you could suppose that the structure is symmetric with respect to Γ_S, as shown in the Figure.



Exercise 2

We consider again a heat optimization problem, which we addressed in the optimal control chapter some weeks ago. Let Ω be a domain representing a room. In this room there is a heat source whose shape is given by $\omega \subset \Omega$ and a convection flux whose velocity is given by u with the condition that $\nabla \cdot u = 0$. We assume that the temperature outside the room is zero, and that the heater is at a given constant temperature $T_1 > 0$. The state equation that describes the temperature distribution is given by:

$$\begin{aligned} -\Delta T + u \cdot \nabla T &= 0 \quad \text{in } \Omega \setminus \omega, \\ T &= 0 \quad \text{on } \partial\Omega, \\ T &= T_1 \quad \text{on } \partial\omega. \end{aligned}$$

The goal is to achieve a constant temperature T_0 in the entire room Ω. To this end, we introduce the cost functional:

$$J(\omega) = \int_{\Omega\setminus\omega} |T(\omega) - T_0|^2 \, dx,$$

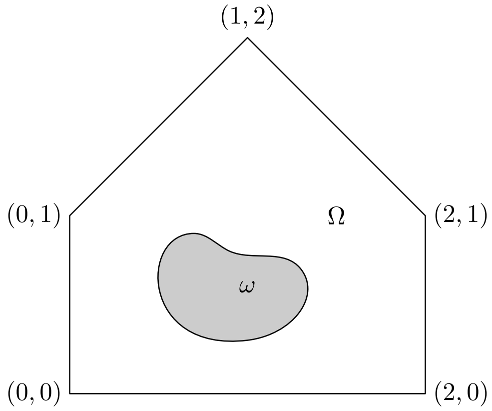
where $T(\omega)$ is the solution of the state equation. Note that ω is the variable of optimization.

1. Determine the standard variational formulation of the state equation.
2. Recast the variational problem by expressing the constraint $T(\omega) = T_1$ on $\partial\omega$ (namely a Dirichlet boundary condition on the inner boundary) with a Lagrange multiplier λ.
3. Determine the Lagrangian associated to the minimization problem where we do not use the standard variational formulation of the state equation.
   Hint : write both the strong forms of the equation and of the boundary condition on ω as contraints.
4. We assume that both $T(\omega)$ and $\lambda(\omega)$ are differentiable with respect to ω. Compute the shape derivative $\langle \frac{\partial J}{\partial \omega}, \theta \rangle$ for all vector fields θ such that $\theta \cdot n = 0$ on $\partial\Omega$.
5. We change the boundary condition on the heater to
$$\frac{\partial T}{\partial n} - u \cdot nT = \Phi_1, \text{ on } \partial\omega. \tag{2}$$
   Compute the shape derivative of J with respect to ω.

Exercise 3 Numerical Application: Homework

We consider a numerical application of the previous exercise. Each group should choose to work on **one of the tasks described below**. When necessary, multiple files can be submitted.

This exercise is a numerical application for the Exercise 2 shown previously. The outer geometry Ω is described in the following figure:



Inside the above polygon we consider the heater ω. In each case write a `Freefem++` code that minimizes J with respect to the shape of ω. For simplicity, a volume constraint on ω may be implemented using a fixed Lagrange multiplier.

1. Implement an optimization algorithm and show the result obtained when the initial hole is a disk and the Dirichlet boundary condition $T = T_1$ is imposed on $\partial\omega$. What happens when considering multiple inclusions? (starting from two disks, for example) You may use the following values for the constants in the problem:

$$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad T_1 = 100; \quad T_0 = 20.$$

2. Do the same task as above in the case of the boundary condition (2) for the particular case $u = 0$, $\Phi_1 = 0$.
3. Consider the case of the boundary condition (2) in the particular case where $\partial\omega$ is a disk $u = (1, 1)$ and $\Phi_1 = 1$. (note that in this case the curvature is known explicitly)
   Hints: 1. Use the script given as an example for Exercise 1, with the difference that only the inner part is allowed to move. In this case, **no cutoff function is necessary**.
   2. In order to see if the derivative is coded well, always check that the value of the objective function decreases when the volume Lagrange multiplier ℓ is zero.

Note: It may be difficult to observe the convergence to an optimal shape if the volume constraint is not handled properly. In any case, in a correctly implemented code the value of the objective function should decrease at every iteration.