MAP562 Optimal design of structures (École Polytechnique)
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For details concerning the Homework questions and the associated deadlines consult the Homework Sheet.

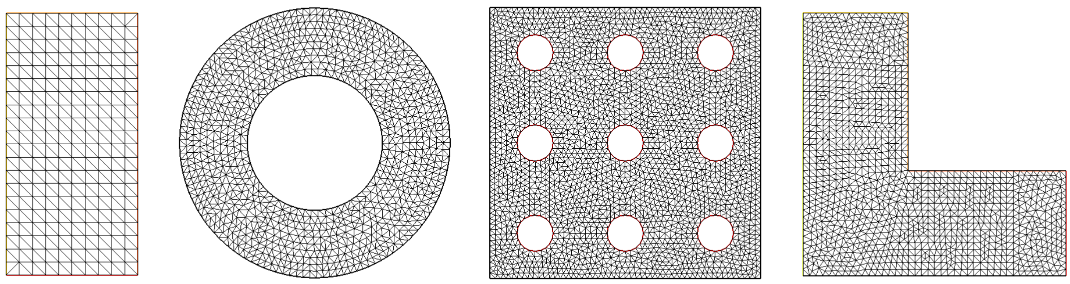
Exercise 1

Let Ω be a smooth bounded and connected open set of $\mathbb{R}^n$. Let Γ_F and Γ_D be boundary parts such that $\partial\Omega = \Gamma_F \cup \Gamma_D$ of negligible intersection , each of them being of non zero surface measure. Let $g \in L^2(\Gamma_F)$, $f \in L^2(\Omega)$ and $\alpha \geq 0$. We consider the Partial Differential Equation

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} + \alpha u = g \text{ on } \Gamma_F. \end{cases} \tag{1}$$

1. Derive the variational formulation associated to (1).
2. Assume that the solution u of the variational formulation is smooth. Then prove that it is also a solution of the initial PDE (1).
3. Suppose that $\Gamma_D = \emptyset$ and $\alpha = 0$ such that a pure Neumann problem is obtained. Show that a solution of the variational formulation exists only if the data satisfies a compatibility condition.

   Remark 1. It is possible to prove that the variational formulation is well posed in the space $V = \{v \in H^1(\Omega) \text{ such that } \int_\Omega v(x)dx = 0\}$. In order to do this a Poincaré type inequality can be established.
4. Find numerically the solution of problem (1) on some of the domains illustrated below. Consult the file `geometry.edp` in order to see how to define such geometries in `FreeFem++`. Choose various parts of the boundaries as Γ_D and Γ_F and observe the behavior of the solutions.



1. 2. 3. 4.

For example:

- Geometry 1 (Rectangle): Consider as Γ_D the left boundary of the rectangle and Γ_F the right boundary. You may also try other combinations.
- Geometry 2 (Ring): Consider as Γ_D the boundary of the inner circle and Γ_N the boundary of the outer circle.
- Geometry 3 (Square with holes): Consider as Γ_D the boundaries of the small disks and let Γ_F be the boundary of the square.
- Geometry 4 (L-Beam): Consider as Γ_D the top left boundary and Γ_F the right lower boundary.

Exercise 2 Linearized Elasticity

In the following we consider the elasticity problem with surface loadings replaced by loadings proportional to the displacements (analogue of Robin or Fourier conditions for the scalar case). Denote by Γ_D the Dirichlet part and Γ_N the part where the Fourier conditions are applied. The associated linearized elasticity system is given by

$$\begin{cases} -\operatorname{div} Ae(u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_D \\ Ae(u)n + g \odot u &= 0 \quad \text{on } \Gamma_N \\ Ae(u)n &= 0 \quad \text{on } \partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \end{cases} \tag{2}$$

where g is a vector containing the coefficients of the proportional deformations for each component of u and $\odot$ is the pointwise product of vectors (the Hadamard product). As recalled in the course A is given by Hooke's law: $Ae(u) = 2\mu e(u) + \lambda \operatorname{div} u \operatorname{Id}$, where λ and ν are the Lamé coefficients, $e(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is the symmetrized gradient and Id is the identity matrix.

1. Derive the variational formulation associated to the problem (2) on the appropriate space.
2. As in Exercise 1, study the case where $g = 0$ and $\Gamma_D = \emptyset$ and deduce the necessary conditions that need to be satisfied by the force f. In particular, you may study the set of displacements $v \in H^1(\Omega)^n$ for which $e(v) = 0$.

   Remark 2. It is possible to fix just one of the displacements equal to zero on various regions defined by Γ_D and still have a well posed problem.
3. Use one of the given geometries in the file `geometry.edp` (illustrated in the previous figure) and solve problem (2) in `FreeFem++` in the case where $f = (0, -1)$ and $g \in \{(1, 1), (10^{-6}, 1), (10^{15}, 1)\}$. Give an interpretation of the results obtained.
4. **(Homework)** Study the case of the L-beam for different mesh sizes. Trace the displacements and the norm of the constraints tensor. You should observe that the displacements converge, but the norm of the tensor of constraints diverges. Use the command `adaptmesh` in order to improve the results.

Exercise 3 Numerics in Freefem++

1. Implement the Poisson problem with Fourier boundary conditions. This is the problem from Exercise 1 where $\Gamma_F = \partial\Omega$. Use the following parameters:
   - $\Omega = \mathcal{C}((0, 0), 100)$ where $\mathcal{C}(M, r)$ denotes the circle centered at M with radius equal to r,
   - $f = xy + 1$, $g = x$, $\alpha = 2$.
2. Solve the scalar elliptic problem (k is a varying coefficient):

$$\begin{cases} -\nabla \cdot (k(x, y)\nabla u(x, y)) = f(x, y) \text{ in } \Omega, \\ \quad \alpha u(x, y) + \partial_n u(x, y) = g(x, y) \text{ on } \partial\Omega, \end{cases}$$

   where
   - $k(x, y) = 1$ everywhere except for $k(x, y) = k_1$ in $\Omega_1 = \mathcal{C}((0, 0), 20)$.
   - $f(x, y) = 0$ in Ω_1.
3. **(Homework)** Try different values of the constant k_1. What happens for $k_1 \gg 1$ and $0 < k_1 \ll 1$?
4. **(Homework)** Solve the same problem by constructing the matrix and right hand side explicitly in Freefem++ instead of using the keyword Problem. (consult the file `FreeFem_matrices.edp` for an example)

Exercise 4 Non-Local Boundary Conditions

In this exercise we consider **non local boundary conditions**. Let Ω_0 and Ω_1, be open bounded sets of $\mathbb{R}^2$ such that $\overline{\Omega}_1$ is included in Ω_0 which is assumed to be connected. We denote by Ω the open set $\Omega_0 \setminus \overline{\Omega}_1$. Let f be a map in $L^2(\Omega)$ and $q \in L^\infty(\Omega)$. We would like to establish the existence of a solution u to the following problem

$$\begin{cases} -\Delta u(x) + q(x)u(x) = f(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n}(x) = -\alpha \displaystyle\int_{\Gamma_1} u(s)ds & \text{on } \Gamma_1 \\ u(x) = 0 & \text{on } \Gamma_0 \end{cases} \tag{3}$$

where α is a non negative real number, $\Gamma_0 = \partial\Omega_0$ and $\Gamma_1 = \partial\Omega_1$.

Remark 3. The above situation is a simplification of a heat transfer problem. Therefore, the boundary conditions are also known as *nonlocal radiation boundary conditions.*

1. *Variational Formulation.* Prove that all smooth solutions of the PDE are solutions of a variational problem to determine. Conversely, prove that all smooth solutions of the variational problem are solutions of the PDE.
2. **(Optional)** *Existence.* We assume that there exists a positive real number q_0 such that for almost every $x \in \Omega$,
$$q(x) \geq q_0.$$
Prove that there exists a unique solution u to the variational problem and that this solution depends continuously on the data.
3. **(Optional)** *Poincaré like inequality.* Prove by contradiction (reductio ad absurdum), that there exists a constant C such that for all u satisfying $u(x) = 0$ on Γ_0:
$$\|u\|_{H^1(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$
Can the conditions imposed on α and q be relaxed while still preserving the existence result ?
4. **(Homework)** Write a `FreeFem++` code which solves the problem (3). (Hints about the implementation are given in the Homework Sheet)