HW2_EX5.1-2

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Ex 1:

Consider the following non-linear problem modeling the radiative heat transfer:

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_0 \\ \dfrac{\partial u}{\partial n} + k|u|^3 u = 0 \text{ on } \Gamma_N \end{cases}$$

With the help of the Green's formula:

$$\int_\Omega \Delta u \cdot v = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \frac{\partial u}{\partial n} v$$

So, for all $v \in V = \{u \in H^1 : u = 0 \text{ on } \Gamma_0\}$, we can have the variational formulation of this system:

$$\int_\Omega -\Delta u \cdot v = \int_\Omega fv$$

$$\Leftrightarrow \quad \int_\Omega \nabla u \cdot \nabla v - \int_{\Gamma_0} \frac{\partial u}{\partial n} \cdot v - \int_{\Gamma_N} \frac{\partial u}{\partial n} \cdot v = \int_\Omega fv$$

$$\Leftrightarrow \quad \int_\Omega \nabla u \cdot \nabla v - \int_{\Gamma_N} k|u|^3 u \cdot v = \int_\Omega fv$$

Ex 2:

Consider the functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{k}{5}\int_{\Gamma_N} |u|^5 \, d\sigma - \int_\Omega fudx = J_1(u) + J_2(u) - J_3(u)$$

$$J_1(u+h) = \frac{1}{2}\int_\Omega |\nabla(u+h)|^2 = \frac{1}{2}\int_\Omega (|\nabla u|^2 + 2\nabla u \cdot \nabla h + |\nabla h|^2)$$

$$= J_1(u) + \int_\Omega \nabla u \cdot \nabla h + \frac{1}{2}\int_\Omega |\nabla h|^2$$

$$J_2(u+h) = \frac{k}{5}\int_{\Gamma_N} |u+h|^5 = \frac{k}{5}\int_{\Gamma_N} |u|^5 + 5|u^4 h| + |u^3 h^2| + O(h^3)$$

$$= J_2(u) + k\int_{\Gamma_N} |u^4 h| + 2k\int_{\Gamma_N} |u^3 h^2| + O(h^3)$$

$$J_3(u+h) = \int_\Omega fu + \int_\Omega fh = J_3(u) + \int_\Omega fh$$

Then we can have the derivative of J:

$$< J'(u), h >_{H_1} = \int_\Omega \nabla u \cdot \nabla v + k\int_{\Gamma_N} |u^4 h| - \int_\Omega fh$$

Which is the same as the variational formulation found at the previous question

Then we have:

$$J(u+h) = J(u) + J'(u)(h) + \frac{1}{2}\int_\Omega |\nabla h|^2 + k\int_{\Gamma_N} 2|u^3 h^2| + 2|u^2 h^3| + |u\, h^4| + \frac{1}{5}|h^5|$$

where the rest of the term is positive, so the functional J is strictly convex.

And when u tends to infinity, J also tends to infinity, so if a minimizer exists it is unique.