Particle Kinematics - II

Sections: 2.4, 2.6

Overview

- Time derivative of a Vector
- Polar coordinates

Velocity: Derivative of position vector

Consider a vector represented as $\vec{r} = r\hat{u}_r$. The velocity of the particle is given by $\dot{\vec{r}}$.

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{dt}$$

A change in the vector $\vec{r}$ can be due to change in the magnitude r or $\hat{u}_r$.

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{dt} = \dot{r}\hat{u}_r + y\dot{\hat{u}}_r$$

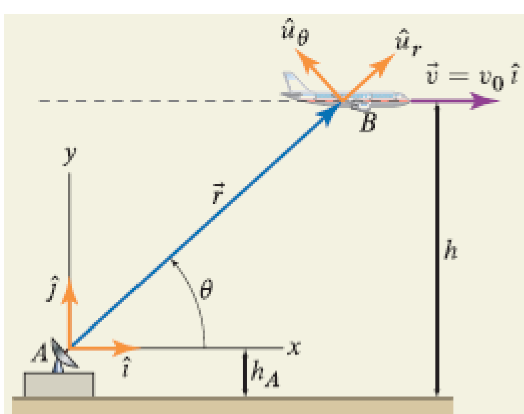
Two ways $\dot{\vec{r}}$ can change,

1. change in $\dot{r}$, refers to a change in the length of the vector
2. change in $\dot{\hat{u}}_r$, refers to a change in direction of the unit vector

Derivative of a unit vector

$\hat{u}_r$ can be written as,

$$\hat{u}_r = cos(\theta)\hat{i} + sin(\theta)\hat{j}$$



Derivative of $\hat{u}_r$ is,

$$\dot{\hat{u}}_r = -sin(\theta)\dot{\theta}\hat{i} + cos(\theta)\dot{\theta}\hat{j}$$
$$\dot{\hat{u}}_r = \dot{\theta}\left(-sin(\theta)\hat{i} + cos(\theta)\hat{j}\right)$$

$$\dot{\hat{u}}_r = \dot{\theta}\left(sin(\theta)\hat{k} \times \hat{j} + cos(\theta)\hat{k} \times \hat{i}\right)$$
$$\dot{\hat{u}}_r = \dot{\theta}\hat{k} \times \left(cos(\theta)\hat{i} + sin(\theta)\hat{j}\right)$$

Derivative of $\hat{u}_r$ is,

$$\dot{\hat{u}}_r = \dot{\theta}\hat{k} \times \left(cos(\theta)\hat{i} + sin(\theta)\hat{j}\right) = \dot{\theta}\hat{k} \times \hat{u}_r = \vec{\omega} \times \hat{u}_r$$

Therefore, derivative of vector $\vec{r}$ is given by,

$$\dot{\vec{r}} = \dot{r}\hat{u}_r + r\dot{\hat{u}}_r$$

$$= \dot{r}\hat{u}_r + r\vec{\omega} \times \hat{u}_r$$

Therefore, velocity is given by

$$\vec{v} = \dot{\vec{r}} = \underbrace{\dot{r}\hat{u}_r}_{change\ in\ length} + \underbrace{\vec{\omega} \times \vec{r}}_{change\ in\ direction}$$

Acceleration: Second derivative of position vector

Given, $\vec{r} = r\hat{u}_r$ compute acceleration.

From before, velocity is given by,

$$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{u}_r + \vec{\omega} \times \vec{r}$$

Acceleration is derivative of velocity,

$$\vec{a} = \ddot{\vec{r}} = \dot{\vec{v}} = \ddot{r}\hat{u}_r + \dot{r}\dot{\hat{u}}_r + +\dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}$$

$$= \ddot{r}\hat{u}_r + \dot{r}\dot{\hat{u}}_r + +\dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}$$

From before,

$$\dot{\hat{u}}_r = \vec{\omega} \times \hat{u}_r$$

and

$$\dot{\vec{r}} = \dot{r}\hat{u}_r + \vec{\omega} \times \vec{r}$$

Substituting gives

$$\vec{a} = \ddot{\vec{r}} = \dot{\vec{v}} = \ddot{r}\hat{u}_r + \dot{r}\dot{\hat{u}}_r + +\dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}$$

$$= \ddot{r}\hat{u}_r + \dot{r}(\vec{\omega} \times \hat{u}_r) + +\dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\dot{r}\hat{u}_r + \vec{\omega} \times \vec{r})$$

$$= \ddot{r}\hat{u}_r + \dot{r}(\vec{\omega} \times \hat{u}_r) + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\dot{r}\hat{u}_r + \vec{\omega} \times \vec{r})$$

Rearranging terms gives,

$$= \ddot{r}\hat{u}_r + 2\dot{r}(\vec{\omega} \times \hat{u}_r) + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Position, Velocity and Acceleration of a vector

Position

$$\vec{r} = r\hat{u}_r$$

Velocity

$$\vec{v} = \dot{r}\hat{u}_r + \vec{\omega} \times \vec{r}$$

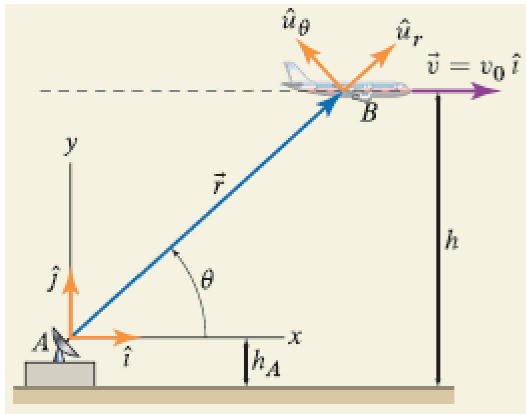
Acceleration

$$\vec{a} = \underbrace{\ddot{r}\hat{u}_r}_{linear\ acceleration} + \underbrace{2\dot{r}(\vec{\omega} \times \hat{u}_r)}_{coriolis} + \underbrace{\vec{\alpha} \times \vec{r}}_{rotational\ acceleration} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{rotational\ acceleration}$$

These relations are always true, we will next apply them under specific coordinate frame assumptions

Polar coordinates

In polar coordinates, we define $\hat{u}_r$ and $\hat{u}_\theta$, and simplify previous equations.



Choose coordinate frames such that,

$$\hat{u}_r, \hat{u}_\theta, \; and \; \hat{k}$$

are orthogonal, and

$$\hat{u}_r \times \hat{u}_\theta = \hat{k}$$

$$\hat{u}_\theta \times \hat{k} = \hat{u}_r$$

$$\hat{k} \times \hat{u}_r = \hat{u}_\theta$$

Under this notation, $\vec{\omega} = \omega\hat{k}$

Velocity

$$\vec{v} = \dot{r}\hat{u}_r + \vec{\omega} \times \vec{r}$$

$$= \dot{r}\hat{u}_r + \omega\hat{k} \times r\hat{u}_r = \dot{r}\hat{u}_r + r\omega\hat{k} \times \hat{u}_r$$

$$= \dot{r}\hat{u}_r + r\omega\hat{u}_\theta$$

Acceleration

$$\vec{a} = \ddot{r}\hat{u}_r + 2\dot{r}(\vec{\omega} \times \hat{u}_r) + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$\vec{a} = \ddot{r}\hat{u}_r + 2\dot{r}(\omega\hat{k} \times \hat{u}_r) + \alpha\hat{k} \times r\hat{u}_r + \omega\hat{k} \times (\omega\hat{k} \times r\hat{u}_r)$$

$$= \ddot{r}\hat{u}_r + 2\dot{r}\omega(\hat{k} \times \hat{u}_r) + r\alpha\hat{k} \times \hat{u}_r + \omega\hat{k} \times (\omega\hat{k} \times r\hat{u}_r)$$
$$= \ddot{r}\hat{u}_r + 2\dot{r}\omega(\hat{k} \times \hat{u}_r) + r\alpha\hat{k} \times \hat{u}_r + r\omega^2\hat{k} \times (\hat{k} \times \hat{u}_r)$$
$$\vec{a} = (\ddot{r} - r\omega^2)\hat{u}_r + (\alpha r + 2\dot{r}\omega)\hat{u}_\theta$$

Position

$$\vec{r} = r\hat{u}_r$$

Velocity

$$\vec{v} = \dot{r}\hat{u}_r + r\omega\hat{u}_\theta$$

Acceleration

$$\vec{a} = (\ddot{r} - r\omega^2)\hat{u}_r + (\alpha r + 2\dot{r}\omega)\hat{u}_\theta$$

EXAMPLE