Operador Variacional

$\delta(F_1 \pm F_2) = \delta F_1 + \delta F_2$ si $G = G(u, v, w)$ $\delta \frac{du}{dx} = \frac{d}{dx}(\delta u)$

$\delta(F_1 \cdot F_2) = \delta F_1 \cdot F_2 + F_1 \delta F_2$ $\delta G = \delta_u G + \delta_v G + \delta_w G$ $\delta \int_\Omega u \, d\Omega = \int_\Omega \delta u \, d\Omega$

$\delta(F_1/F_2) = \dfrac{\delta F_1 \cdot F_2 - F_1 \delta F_2}{\delta F_2^{\,2}}$

Para un funcional $I(v) = \int_a^b F(x, v, v')\,dx$ $v = v(x)$

$\delta(F_1)^n = n\,F_1^{n-1}\,\delta F_1$ su primer variación es $\delta I(v) = \int_a^b \left(\frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta v' \right) dx$

El funcional $I(v)$ tiene un extremo (max o min) o ensilladura si $\delta I = 0$

Demuestre las ecuaciones de Euler-Lagrange

$\delta v' = \frac{d(\delta v)}{dx}$

$\Rightarrow \delta I(v) = \int_a^b \left(\frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\frac{d\delta v}{dx} \right) dx$ (Integral por partes) $\rightarrow \int \frac{\partial F}{\partial v'} \frac{d\,\delta v}{\cancel{dx}}\,\cancel{dx} =$

$\int \underbrace{\frac{\partial F}{\partial v'}}_{u} \cdot \underbrace{d\delta v}_{v} = u.v - \int v.du$

$\delta I(v) = \int_a^b \left(\frac{\partial F}{\partial v}\delta v - \frac{d}{dx}\frac{\partial F}{\partial v'}\delta v \right) dx + \underbrace{\frac{\partial F}{\partial v'}\delta v \Big]_a^b}_{0}$

$\frac{\partial F}{\partial v'} \cdot \delta v \Big|_a^b - \int \delta v \cdot \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) dx$

$\delta I(v) = \int_a^b \left(\frac{\partial F}{\partial v} - \frac{d}{dx}\frac{\partial F}{\partial v'} \right) \delta v \, dx$ $\delta v_a = \delta v_b = 0$

δv puede tomar cualquier valor arbitrario $\Rightarrow \delta I(v) = 0 \Leftrightarrow \left(\frac{\partial F}{\partial v} - \frac{d}{dx}\frac{\partial F}{\partial v'} \right) = 0$ ec Euler-Lagrange

Demuestre que para el funcional $I(y) = \int_{x_1}^{x_2} f(y)\sqrt{1+y'^2}\,dx$ la función $y(x)$ que encuentra su extremo satisface $1 + y'^2 = B\,f(y)^2$

$\mathcal{L} = f(y) \cdot \sqrt{1+y'^2} \Rightarrow \frac{d}{dx}\left[\frac{\partial \mathcal{L}}{\partial y'}\right] = \frac{\partial \mathcal{L}}{\partial y} \Rightarrow \frac{d}{dx}\left[f(y) \cdot \frac{1}{2}(1+y'^2)^{-1/2} \cdot 2 \cdot y' \right] = f' \cdot \sqrt{1+y'^2}$

$\frac{f'(y)\,y' \cdot y'}{\sqrt{1+y'^2}} + \frac{f \cdot y''}{\sqrt{1+y'^2}} + f \cdot y'\left(-\frac{1}{2}\right)(1+y'^2)^{-3/2} \cdot 2 \cdot y' \cdot y'' = f' \cdot \sqrt{1+y'^2}$ M.M.A.M. $(1+y'^2)^{3/2}$

$() = (1+y'^2)$

$f' \cdot y'^2() + f \cdot y''() - f y'^2 y'' = f'()^2 \rightarrow f y'' = f'(1+y'^2)$

$f y''[() - y'^2] = f'[()^2 - y'^2()]$

$f y'' = f'() \cdot \frac{\cancel{[() - y'^2]}}{\cancel{[() - y'^2]}}$

$\frac{y''}{1+y'^2}\frac{dy}{\cancel{dx}} = \frac{f'}{f}\frac{dy}{\cancel{dx}}$ $\cdot\frac{dy}{\cancel{dx}}$ MAM

$\int \frac{y''}{1+y'^2}\,dy = \int \frac{f'}{f}\,dy$

$\frac{1}{2}\ln(1+y'^2) = \ln f + C \Rightarrow 1 + y'^2 = f^2 e^{2C} \rightarrow \boxed{1+y'^2 = B\,f(y)^2}$

Encuentre la expresión para la distancia más corta entre dos puntos de un plano con una trayectoria $y = f(x)$

x_1 x_2

Long arco $= \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$ $\mathcal{L} = \sqrt{1+y'^2}$

$\frac{d}{dx}\left[\frac{\partial \mathcal{L}}{\partial y'}\right] = \frac{\partial \mathcal{L}}{\partial y} \Rightarrow \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = cte \Rightarrow y' = cte \Rightarrow \frac{dy}{dx} = b \Rightarrow \int dy = \int b\,dx$ $\boxed{y = bx + c}$

$\frac{y'^2}{1+y'^2} = c^2$

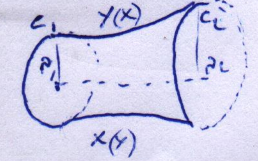
$y'^2 = c^2 + c^2 \cdot y'^2$

$y'^2(1-c^2) = c^2$

$y'^2 = \frac{c^2}{1-c^2} = ct$

$y' = cte$

Mínima superficie de revolución entre dos anillos de radio c_1 y c_2



Recap área cilindro $= \underbrace{2\pi r}_{\text{perímetro}} \cdot h$

$$\boxed{\text{Tip: } \int \frac{1}{\sqrt{y^2-1}}\,dy = \cosh^{-1} y + d}$$

$$A=\int_{c_1}^{c_2} 2\pi\, y(x)\cdot\sqrt{1+x'^2}\,dy = 2\pi\int_{c_1}^{c_2} y\cdot\sqrt{1+x'^2}\,dy \qquad \mathcal{L} \propto y\cdot\sqrt{1+x'^2}$$

$$\frac{d}{dy}\left(\frac{\partial\mathcal{L}}{\partial x'}\right)=\frac{\partial \mathcal{L}}{\partial x} \qquad \frac{d}{dy}\Big[\underbrace{y\,\tfrac12(1+x'^2)^{-1/2}\cdot 2x'}_{cte}\Big]=0 \Rightarrow \frac{yx'}{\sqrt{1+x'^2}}=b \Rightarrow \frac{y^2x'^2}{1+x'^2}=b^2$$

$$\frac{y^2}{b^2}x'^2 = 1+x'^2 \Rightarrow \left(\frac{y^2}{b^2}-1\right)\cdot x'^2=1 \Rightarrow dx = \frac{dy}{\sqrt{y^2/b^2-1}} \xRightarrow{\text{TIP}} x = b\cosh^{-1}\left(\frac{y}{b}\right)+d \Rightarrow \boxed{y=b\cosh\frac1b(x+d)}$$

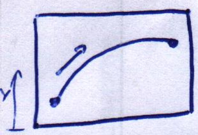
$$A=\int_{c_1}^{c_2} 2\pi\, y(x)\sqrt{1+y'^2}\,dx \Rightarrow \mathcal{L}\propto y\sqrt{1+y'^2} \Rightarrow \quad 1+y'^2 = By^2 \qquad \text{sea } \frac{1}{b^2}=B$$

$$\frac{dy}{dx}=\pm\sqrt{\frac{y^2}{b^2}-1} \qquad dx=\frac{dy}{\sqrt{\left(\frac{y}{b}\right)^2-1}}$$

sigue igual que el caso anterior

Condiciones de contorno

$$\begin{cases} c_1 = b\cosh 1/b\,(\partial_1+d) \\ c_2 = b\cosh 1/b\,(\partial_2+d)\end{cases}$$



supongo que la velocidad de la luz en un material es proporcional a la altura desde la base del material

Muestre que en ese caso la luz se mueve en trayectorias circulares

Recuerde que la luz sigue la trayectoria que minimiza el tiempo de recorrido entre dos puntos (Principio de Fermat)

sea la trayectoria $y(x)$ y la velocidad $v\propto y$ el tiempo para ir desde (x_1,y_1) a (x_2,y_2) es:

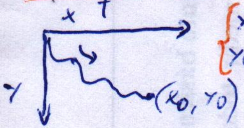
$$T=\int_{x_1}^{x_2}\frac{ds}{v}\propto\int_{x_1}^{x_2}\frac{\sqrt{1+y'^2}}{y}\,dx \Rightarrow L\propto \frac{\sqrt{1+y'^2}}{y}$$

$$\Rightarrow 1+y'^2=\frac{B}{y^2}\Rightarrow y^2+y^2\cdot y'^2=B\Rightarrow y^2y'^2=B-y^2 \qquad \frac{y^2y'^2}{B-y^2}=1\Rightarrow \int\frac{y}{\sqrt{B-y^2}}\,dy=\mp\int dx$$

$$x+A=\mp\sqrt{B-y^2}$$

$$(x+A)^2+y^2=B$$

Braquistócrono



$$\begin{cases} y(0)=0 \\ y(x_0)=y_0 \end{cases}$$

$$\frac{m}{2} v^2 = m g y \qquad vel = \sqrt{2gy}$$

$$T = \int_0^x \frac{ds}{vel} = \int_0^x \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx = \frac{1}{\sqrt{2g}} \int_0^x \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$

$$\mathcal{L} \propto \frac{\sqrt{1+y'^2}}{\sqrt{y}} \Rightarrow \mathcal{L} \propto (y^{-1/2}) \cdot \sqrt{1+y'^2} \Rightarrow 1+y'^2 = B\,(y^{-1/2})^2 \Rightarrow 1+y'^2 = \frac{B}{y} \Rightarrow y + y \cdot y'^2 = B$$

$$\frac{dy}{dx} = \pm\sqrt{\frac{B-y}{y}}$$

$$\Rightarrow \frac{\sqrt{y}}{\sqrt{B-y}}\, dy = \pm dx$$

Propongo un conveniente cambio de variable $y = B \sin^2\phi$

$$dy = 2B \sin\phi \cos\phi \, d\phi$$

$$\frac{\sqrt{B\sin^2\phi}}{\sqrt{B - B\sin^2\theta}} \cdot 2B \sin\phi\cos\phi\, d\phi = \pm dx \Rightarrow \frac{\sqrt{B}\cdot \sin\phi \cdot 2B\sin\theta\cos\theta\, d\theta}{\sqrt{B(1-\sin^2\phi)}} = \pm dx$$

$$\boxed{\cancel{\begin{array}{l} \int \sin^2(\theta)\, d\theta \\ \frac{1}{2}\theta - \frac{1}{2}\sin(2\theta) + C \end{array}}}$$

$$\boxed{\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)}$$

$$\frac{\cancel{\sqrt{B}}}{\cancel{\sqrt{B}}} \frac{\sin^2\phi \cdot 2B \cancel{\cos\phi}\, d\phi}{\cancel{\sqrt{\cos^2\phi}}} = \pm dx \Rightarrow 2B \sin^2\phi\, d\phi = \pm dx$$

$$2B\left(\frac{1}{2} - \frac{1}{2}\cos(2\phi)\right) d\phi = \pm dx \qquad \cdot 2\text{ M.A.M}$$

$$B \int 2 - 2\cos(2\phi)\, d\theta = \int \pm 2\, dx$$

$$B\,(2\phi - \sin(2\phi)) = 2x + C$$

$\theta = 2\phi$

$$x = a(\theta - \sin\theta)$$

$$y = B\sin^2\phi = B(1-\cos\theta)$$

$$y = a(1-\cos\theta)$$

la cicloide