In this assignment you will use quadratic Lyapunov functions to estimate the region of attraction of an asymptotically stable equilibrium. Consider the system

$$\dot{x} = f(x)$$

for a C^1 vector field f, and suppose x_e is an equilibrium, that is, suppose $f(x_e) = 0$. Next, suppose the Jacobian matrix $A = Df(x_e)$ is Hurwitz (which implies that x_e is asymptotically stable). Pick a positive definite matrix Q (the same size as A), and let P be the unique positive definite solution to the Lyapunov matrix equation

$$A^T P + PA + Q = 0\,. \tag{1}$$

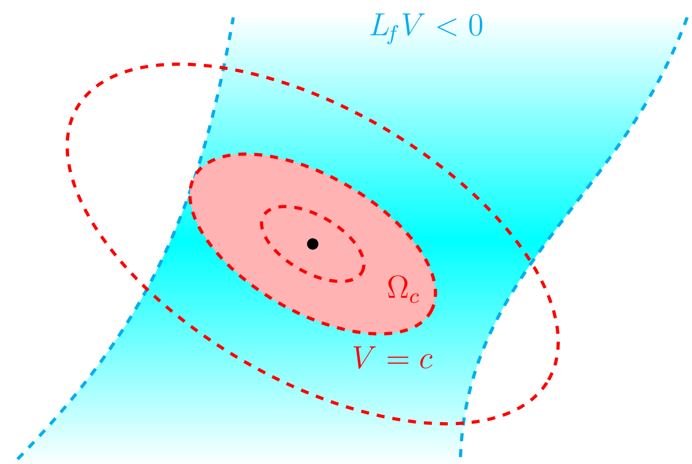
Then the quadratic function

$$V(x) = (x - x_e)^T P (x - x_e) \tag{2}$$

is a strict Lyapunov function for this equilibrium x_e. To estimate the stable set $W^s(x_e)$, which is the region of attraction for the equilibrium x_e, we define the following sublevel set of V for each constant $c > 0$:

$$\Omega_c = \big\{ x \in X \,:\, V(x) < c \big\}\,.$$

If we choose c such that $L_f V(x) < 0$ for all $x \in \Omega_c$ (except at $x = x_e$ where $L_f V$ is zero), then Ω_c will be a subset of $W^s(x_e)$. Thus to obtain an estimate of $W^s(x_e)$, we choose c as large as possible while keeping Ω_c within the region where $L_f V$ is negative, as shown in the following picture:



The black dot represents x_e, the red dashed lines are lines of constant V, and c is chosen so that Ω_c just touches the boundary of the region where $L_f V$ is negative. Repeating this procedure with a different choice for the matrix Q will result in a different Lyapunov function V (with a different region where $L_f V$ is negative) and thus a different estimate Ω_c. The union of all such estimates is again a subset of $W^s(x_e)$.

Consider the following system on the state space $X = \mathbb{R}^2$:

$$\dot{x}_1 = x_1 - x_1^3 + x_2$$
$$\dot{x}_2 = 3x_1 - x_2\,.$$

This system is forward complete: for every initial state $x_0 \in \mathbb{R}^2$, the flow $\varphi(t, x_0)$ exists for all forward time $t \geqslant 0$.

1. Show that this system has a symmetric flow in the sense that $\varphi(t, x_0) = -\varphi(t, -x_0)$ for all $t \geqslant 0$ and $x_0 \in \mathbb{R}^2$. In other words, show that if $x(t)$ is a solution then $-x(t)$ is also a solution.

2. Find all equilibria of the system. Perform the linearization stability test at each equilibrium, and if the test is conclusive, state whether the equilibrium is unstable or asymptotically stable.

3. Choose Q to be the 2×2 identity matrix, and use the above procedure to estimate the region of attraction of one of the asymptotically stable equilibria. Calculate the area of the resulting sublevel set Ω_c using the formula

$$\text{Area}(\Omega_c) = \frac{\pi c}{\sqrt{\det(P)}}\,.$$

   Note that Matlab, GNU Octave, and Julia all have a function `lyap` which will compute the solution P to the Lyapunov equation (1), but if you read the documentation of this function carefully, you will see that you need to send this function the *transpose* of A as an input argument, rather than A itself. Also, there are different ways to find the boundary of the region where $L_f V$ is negative. You can use a contour plot, or, because $L_f V$ is quadratic in x_2, you can solve the boundary equation $L_f V(x) = 0$ for x_2 as a function of x_1. Hand in a plot similar to the picture on the previous page which shows the boundary of your estimate Ω_c together with the boundary of the region where $L_f V$ is negative.

4. Repeat the previous problem with different choices for the matrix Q. How large can you make the area of the resulting estimate Ω_c by adjusting Q? Report the value of this largest area you found. Note that simply multiplying Q by a positive scalar will not change Ω_c at all, because both P and c will end up just being multiplied by the same scalar. Thus the search for a matrix Q that maximizes the area is a search over a two-dimensional parameter space. You can either search this space manually or use an optimization algorithm to search it for you. Hand in a plot which shows the boundary of your largest-area estimate Ω_c together with the boundary of the region where $L_f V$ is negative.

5. Suppose there is a scalar control u in the second system equation, so that the new system is

$$\dot{x}_1 = x_1 - x_1^3 + x_2$$
$$\dot{x}_2 = 3x_1 - x_2 + u\,.$$

   Suppose you have a clean measurement of x_2, and consider a static feedback of the form $u = \kappa(x_2)$ for some feedback function $\kappa(\cdot)$. Your goal in choosing this feedback function is to keep the same asymptotically stable equilibrium you analyzed above but to enlarge its region of attraction. How large can you make this region of attraction by using feedback? Hint: try using V in (2) with $P = I$.