ĐỔI BIẾN TRONG TÍCH PHÂN KÉP

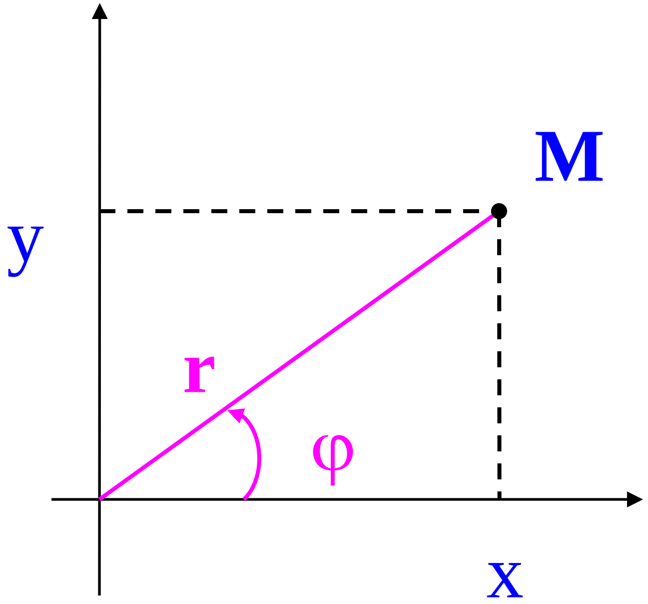
Tích phân cần tính $I = \iint\limits_D f(x,y)\,dxdy$

Tọa độ Đề - Các Oxy: Cận theo x và y (Livestream)

Tọa độ cực: Cận theo r và φ $\rightarrow$ Đường tròn

$\rightarrow$ Elip

TỌA ĐỘ CỰC



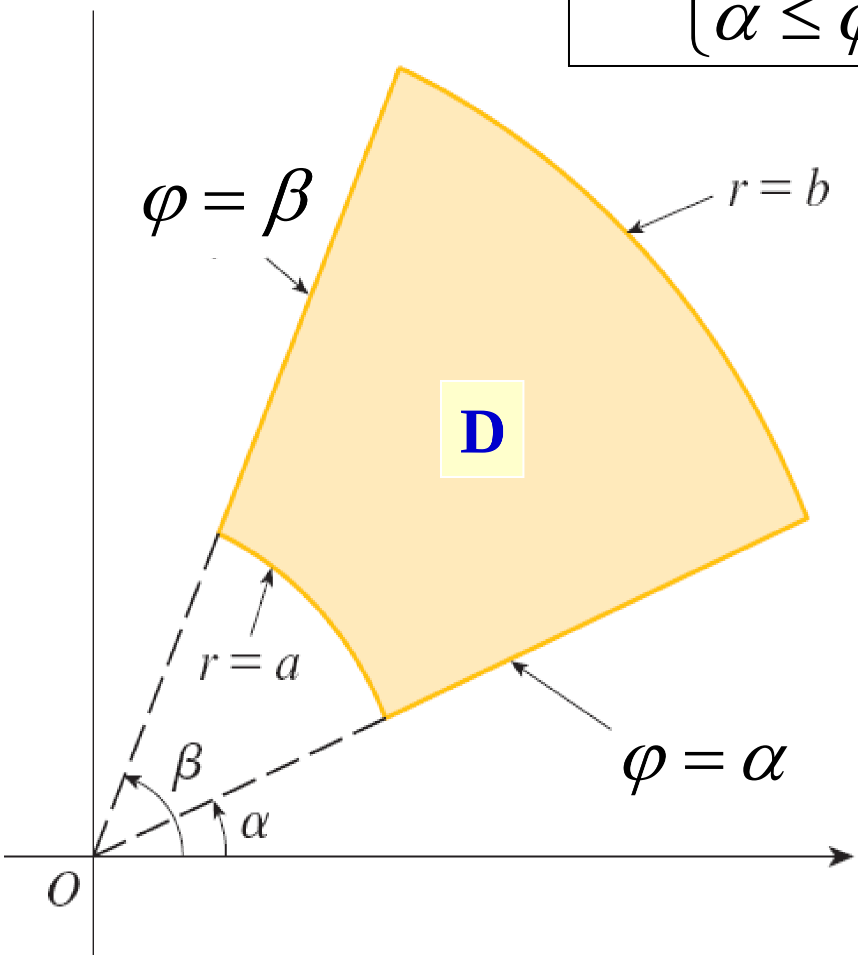
$$r = \sqrt{x^2 + y^2} \geq 0$$

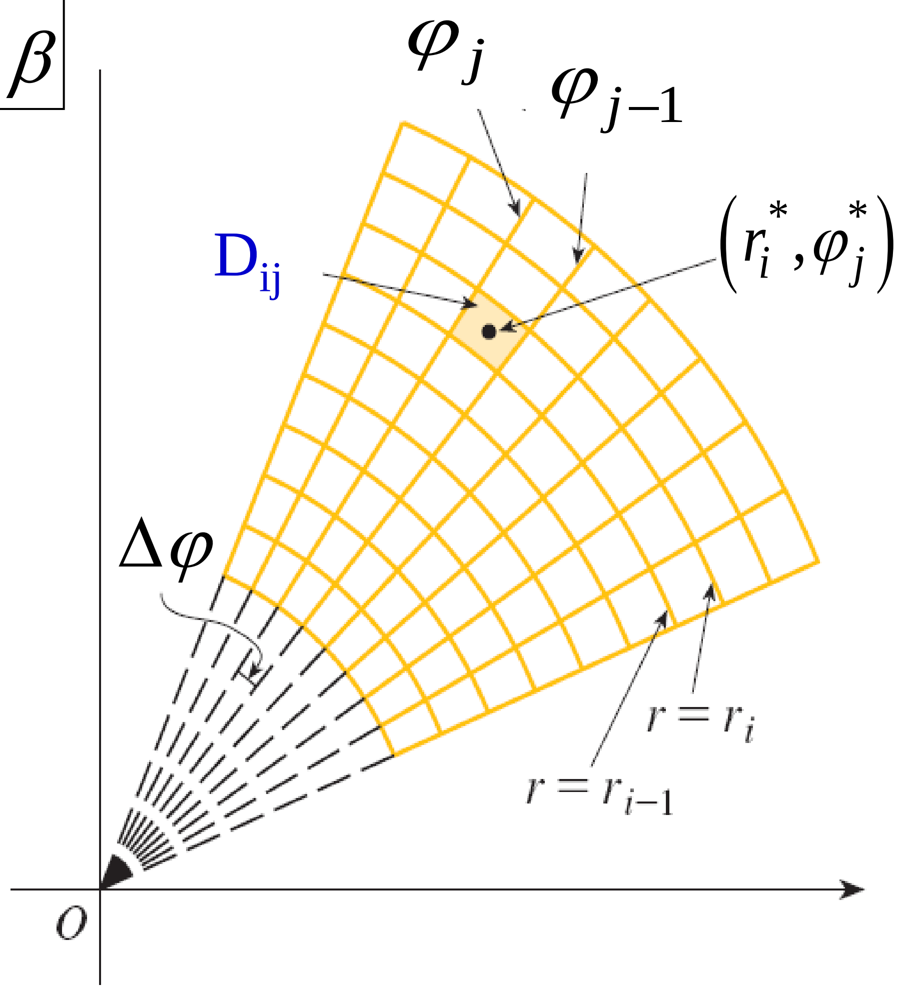
$$x = r\cos\varphi,\ y = r\sin\varphi$$

$$\varphi \in [0, 2\pi] \text{ hay } \varphi \in [-\pi, \pi]$$

TÍCH PHÂN KÉP TRONG TỌA ĐỘ CỰC

$$D:\begin{cases} a \le r \le b \\ \alpha \le \varphi \le \beta \end{cases}$$





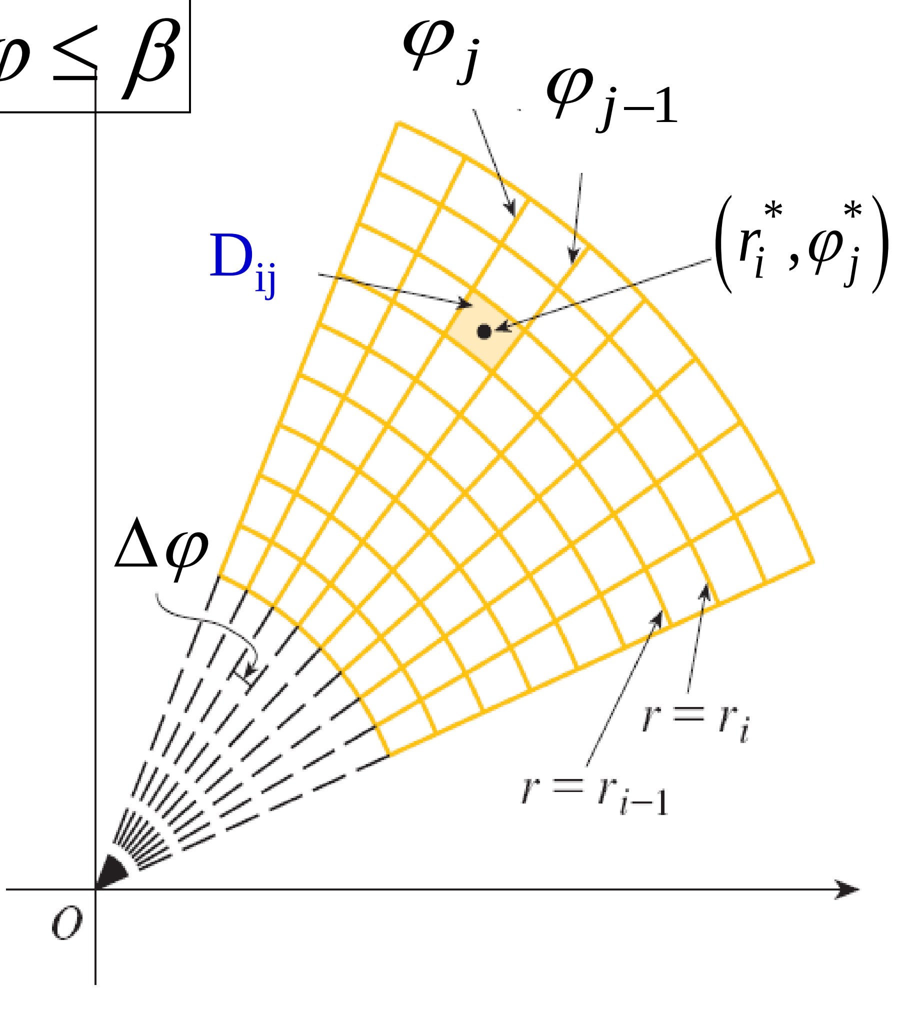
TÍCH PHÂN KÉP TRONG TỌA ĐỘ CỰC

$$\boxed{D:\begin{cases} a \le r \le b \\ \alpha \le \varphi \le \beta \end{cases}}$$

$$S\left(D_{ij}\right) = \frac{1}{2}r_i^2\Delta\varphi - \frac{1}{2}r_{i-1}^2\Delta\varphi$$

$$= \frac{1}{2}\left(r_i + r_{i-1}\right)\left(r_i - r_{i-1}\right)\Delta\varphi$$

$$\approx r_i^*\Delta r\Delta\varphi$$



Tổng tích phân

$$S_n = \sum_{i,j} f(r_i^* \cos\varphi_j^*, r_i^* \sin\varphi_j^*) r_i^* \Delta r \Delta\varphi$$

$$\iint_D f(x,y)dxdy = \lim_{d\to 0} S_n$$

$$\lim_{d\to 0} S_n = \iint_D f(r\cos\varphi, r\sin\varphi) r dr d\varphi$$

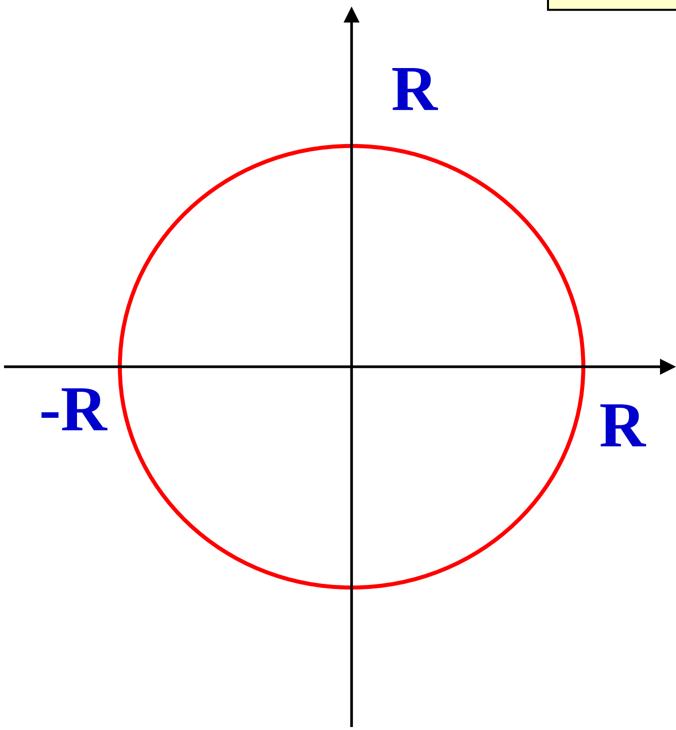
Công thức đổi biến sang tọa độ cực

$$x = r\cos\varphi,\ y = \sin\varphi$$

$$\iint\limits_D f(x,y)\,dxdy = \iint\limits_D f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi$$

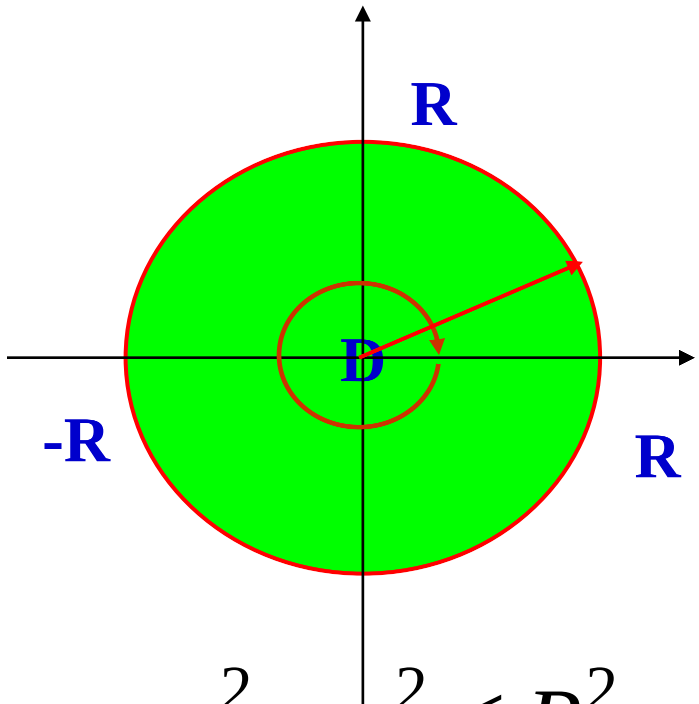
Một số đường cong và miền D trong tọa độ cực

$$x = r\cos\varphi,\ y = r\sin\varphi$$



$$x^2 + y^2 = R^2$$

$$\Leftrightarrow r = R$$



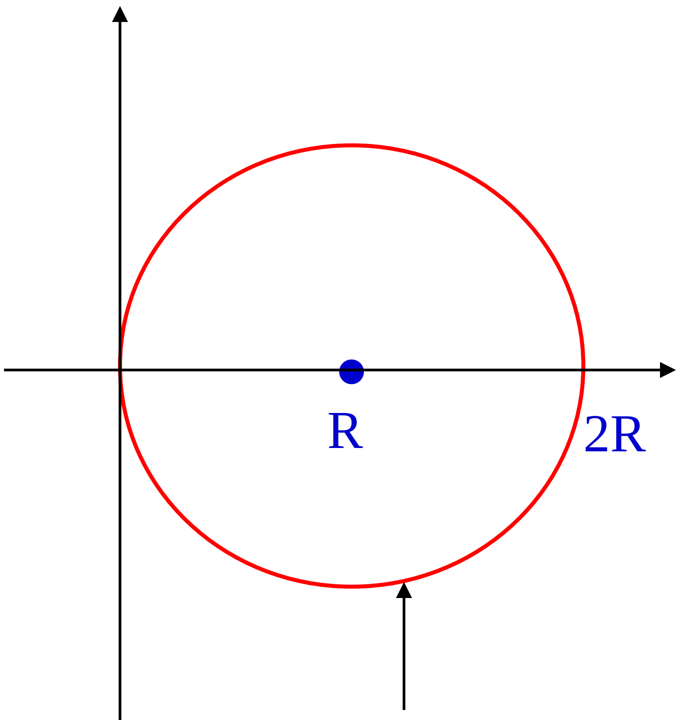
$$x^2 + y^2 \le R^2$$

$$\Leftrightarrow \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$x^2 + y^2 = 2Rx$$

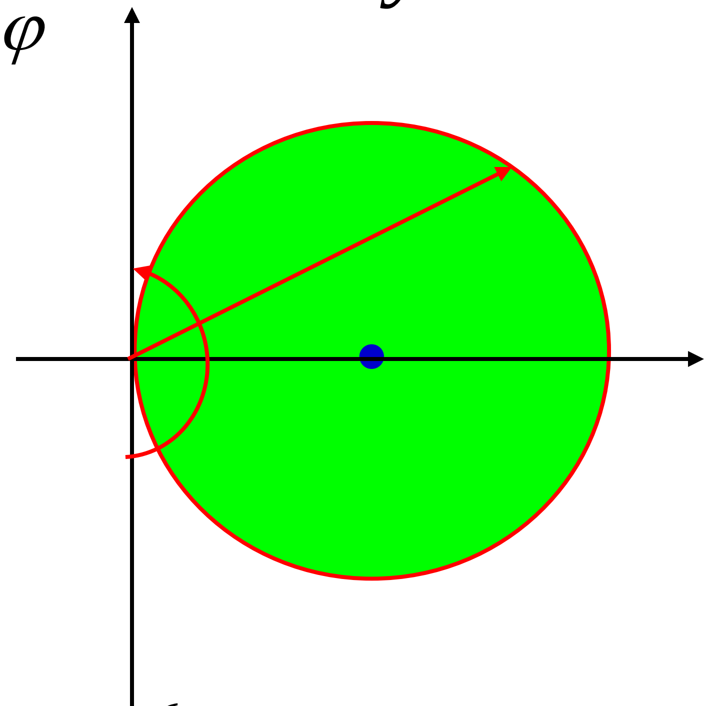
$$\Leftrightarrow (r\cos\varphi)^2 + (r\sin\varphi)^2 = 2Rr\cos\varphi$$

$$\Leftrightarrow r^2 = 2Rr\cos\varphi \Leftrightarrow r = 2R\cos\varphi$$



$$r = 2R\cos\varphi$$

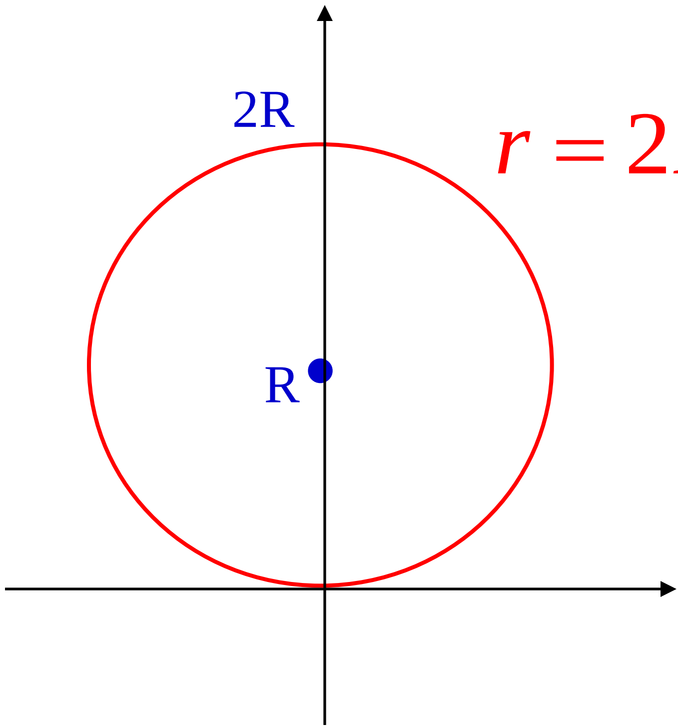
$$x^2 + y^2 \le 2Rx$$



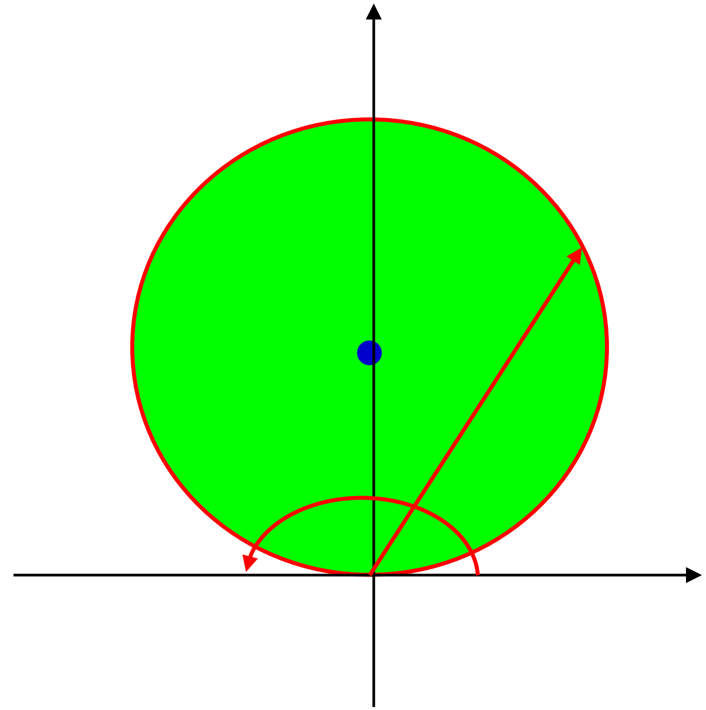
$$\begin{cases} 0 \le r \le 2R\cos\varphi \\ -\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2} \end{cases}$$

$$x^2 + y^2 = 2Ry$$

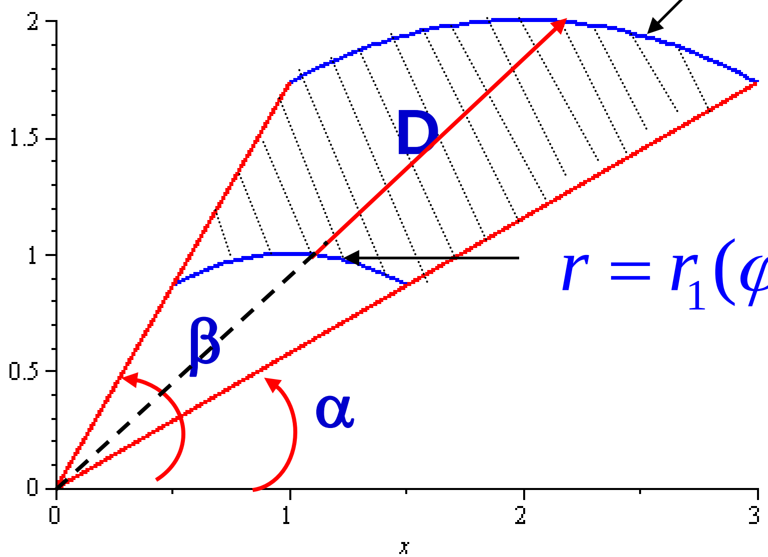
$$x^2 + y^2 \le 2Ry$$



$$r = 2R\sin\varphi$$



$$\begin{cases} 0 \le r \le 2R\sin\varphi \\ 0 \le \varphi \le \pi \end{cases}$$



$$D:\begin{cases} r_1(\varphi) \le r \le r_2(\varphi) \\ \alpha \le \varphi \le \beta \end{cases}$$

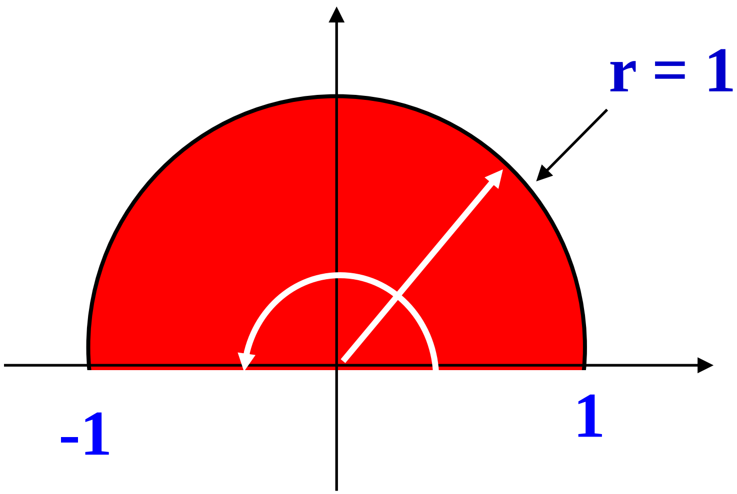
$$(0 < \beta - \alpha \le 2\pi)$$

$$\iint\limits_D f(r\cos\varphi, r\sin\varphi) r\,dr\,d\varphi$$

$$= \int\limits_{\alpha}^{\beta} d\varphi \int\limits_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi, r\sin\varphi) r\,dr$$

VÍ DỤ

1/ Tính: $I = \iint\limits_D \sqrt{x^2 + y^2}\,dxdy$ với $D: \begin{cases} x^2 + y^2 \le 1 \\ y \ge 0 \end{cases}$



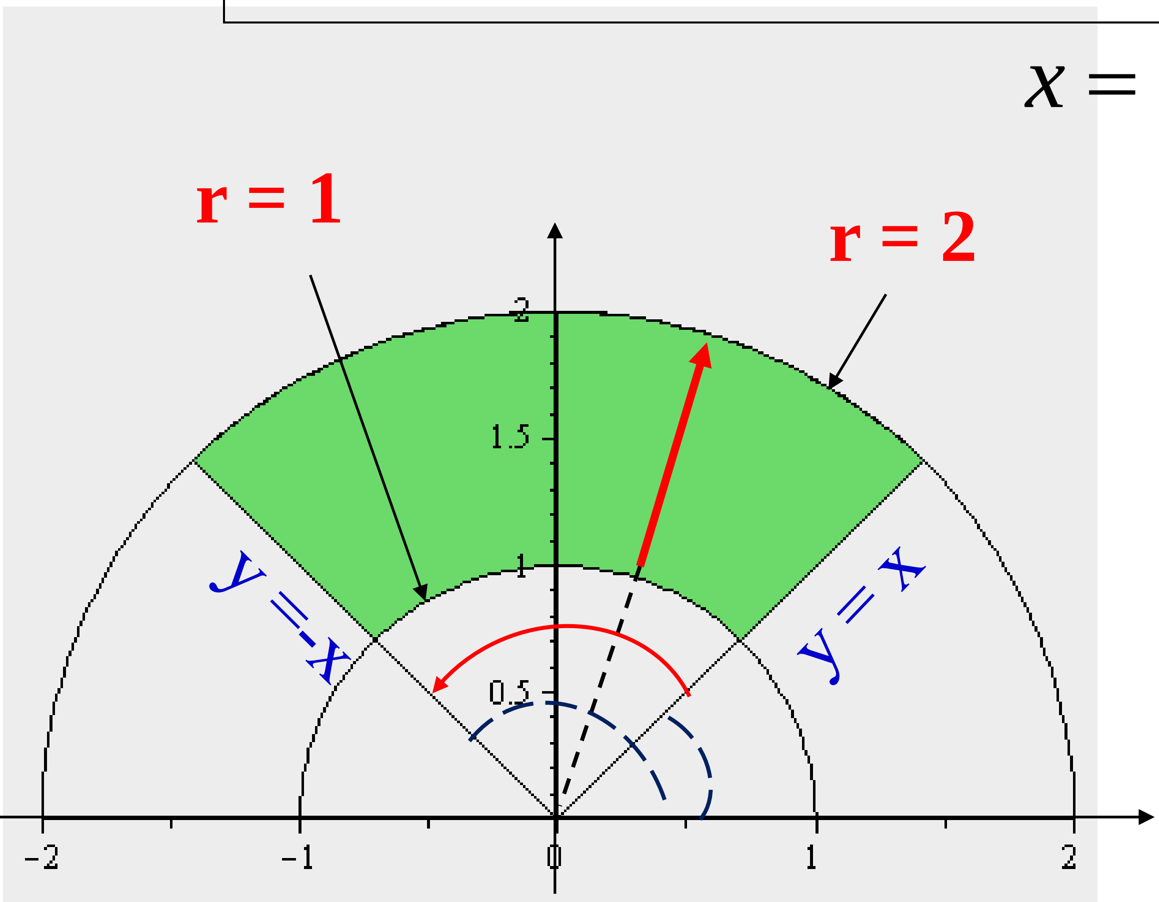
$$x = r\cos\varphi,\ y = r\sin\varphi$$

$$D: \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \end{cases}$$

$$I = \iint\limits_D r.rdrd\varphi = \int_0^{\pi} d\varphi \int_0^1 r^2 dr = \frac{1}{3}\int_0^{\pi} d\varphi = \frac{\pi}{3}$$

2/ Tính: $$I = \iint_D (x-y)\,dxdy$$

$$D:\begin{cases} 1 \le x^2 + y^2 \le 4 \\ y \ge x, y \ge -x \end{cases}$$



$$x = r\cos\varphi,\ y = r\sin\varphi$$

$$D:\begin{cases} 1 \le r \le 2 \\ \dfrac{\pi}{4} \le \varphi \le \dfrac{3\pi}{4} \end{cases}$$

$$I=\iint_D (x-y)\,dxdy \qquad D:\begin{cases}1\le r\le 2\\ \dfrac{\pi}{4}\le\varphi\le\dfrac{3\pi}{4}\end{cases}$$

$$=\iint_D (r\cos\varphi - r\sin\varphi).rdrd\varphi$$

$$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi\int_1^2 r^2(\cos\varphi-\sin\varphi)dr$$

$$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos\varphi-\sin\varphi)\left(\frac{8}{3}-\frac{1}{3}\right)d\varphi \quad =-\frac{7}{3}\sqrt{2}$$

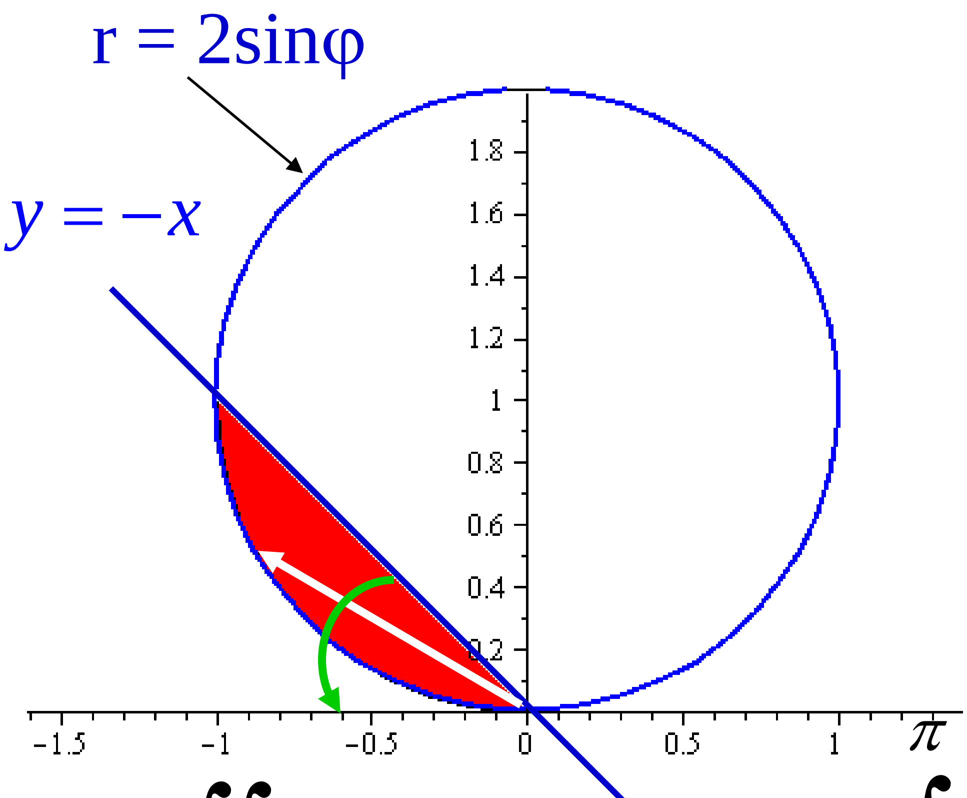
$$f(x,y)=g(x).h(y)$$

$$\iint_D f(x,y)dxdy=\int_a^b g(x)dx.\int_c^d h(y)dy$$

$$I=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi\int_1^2 r^2(\cos\varphi-\sin\varphi)dr$$

$$=\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(\cos\varphi-\sin\varphi)d\varphi\int_1^2 r^2dr=\frac{-7}{3}\sqrt{2}$$

3/ Tính: $I = \iint\limits_D x\,dx\,dy$ với $D: \begin{cases} x^2 + y^2 \le 2y \\ y \le -x \end{cases}$

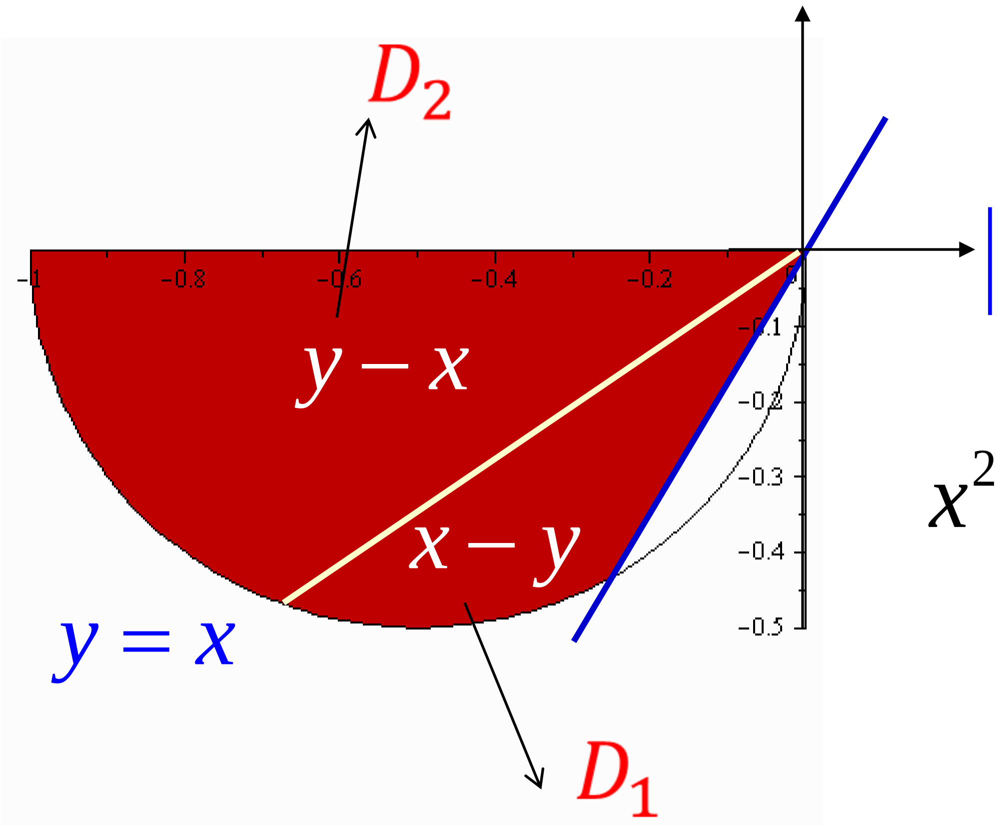


$$x = r\cos\varphi,\ y = r\sin\varphi$$

$$D: \begin{cases} 0 \le r \le 2\sin\varphi \\ \dfrac{3\pi}{4} \le \varphi \le \pi \end{cases}$$

$$I = \iint\limits_D r\cos\varphi\, r\,dr\,d\varphi = \int_{\frac{3\pi}{4}}^{\pi} d\varphi \int_0^{2\sin\varphi} r^2\cos\varphi\,dr = -\frac{1}{6}$$

Tính: $I = \iint\limits_D |x - y|\,dxdy \quad D: \begin{cases} x^2 + y^2 \le -x \\ \sqrt{3}x \le y \le 0 \end{cases}$



$$|x - y| = \begin{cases} x - y,\ x - y \ge 0\ (D_1) \\ y - x,\ x - y < 0\ (D_2) \end{cases}$$

$$x^2 + y^2 = -x \Leftrightarrow r = -\cos\varphi$$

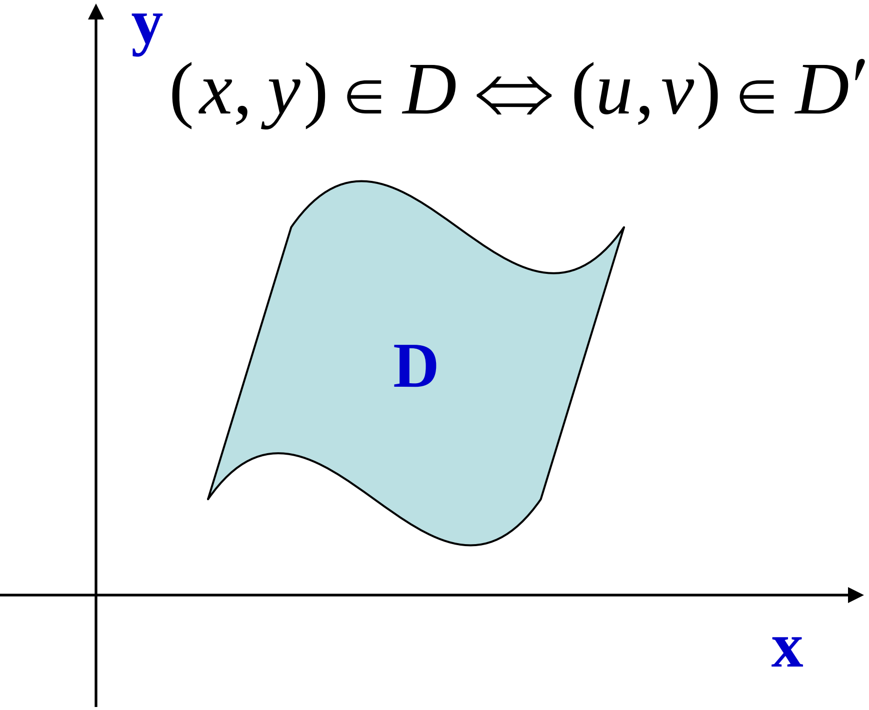
$$I=\iint\limits_{D}\left|x-y\right|dxdy=\iint\limits_{D_1}\left(x-y\right)dxdy+\iint\limits_{D_2}\left(y-x\right)dxdy$$

$$=\int\limits_{\frac{5\pi}{4}}^{\frac{4\pi}{3}}d\varphi\int\limits_{0}^{-\cos\varphi}\left(r\cos\varphi-r\sin\varphi\right)rdr+\int\limits_{\pi}^{\frac{5\pi}{4}}d\varphi\int\limits_{0}^{-\cos\varphi}\left(r\sin\varphi-r\cos\varphi\right)rdr$$

$$=\int\limits_{\frac{5\pi}{4}}^{\frac{4\pi}{3}}\left(\cos\varphi-\sin\varphi\right)d\varphi\int\limits_{0}^{-\cos\varphi}r^2dr+\int\limits_{\pi}^{\frac{5\pi}{4}}\left(\sin\varphi-\cos\varphi\right)d\varphi\int\limits_{0}^{-\cos\varphi}r^2dr$$

$$=\int\limits_{\frac{5\pi}{4}}^{\frac{4\pi}{3}}\left(\cos\varphi-\sin\varphi\right)\frac{\left(-\cos\varphi\right)^3}{3}d\varphi+\int\limits_{\pi}^{\frac{5\pi}{4}}\left(\sin\varphi-\cos\varphi\right)\frac{\left(-\cos\varphi\right)^3}{3}d\varphi$$

ĐỔI BIẾN TỔNG QUÁT



$$(x, y) \in D \Leftrightarrow (u, v) \in D'$$

x = x(u,v), y= y(u,v)

$$J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}$$

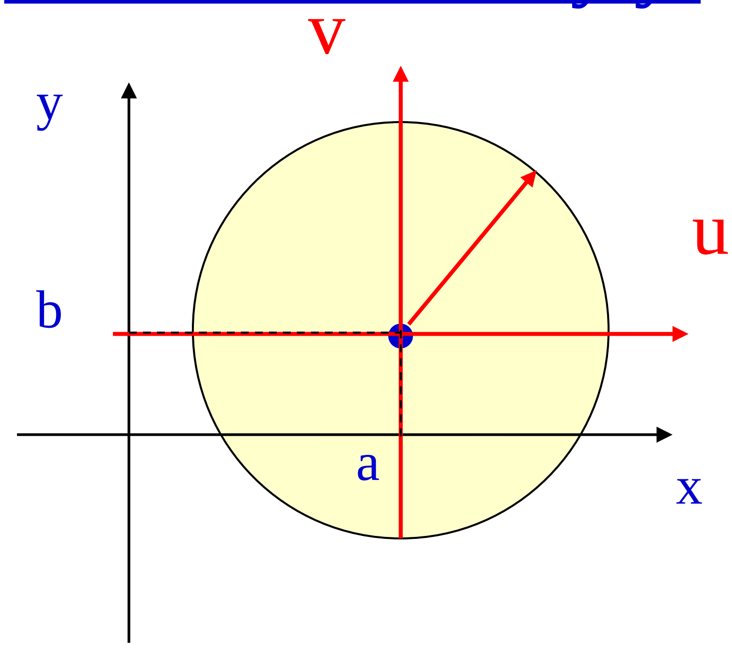
$$J = \frac{1}{\dfrac{D(u,v)}{D(x,y)}}$$

Công thức đổi biến

$$\iint\limits_D f(x,y)dxdy = \iint\limits_{D'} f(x(u,v), y(u,v))\,|J|\,dudv$$

Hình tròn tâm tùy ý:

$D: (x-a)^2 + (y-b)^2 \le R^2$



Dời gốc tọa độ đến tâm

$x = u + a,\ y = v + b$

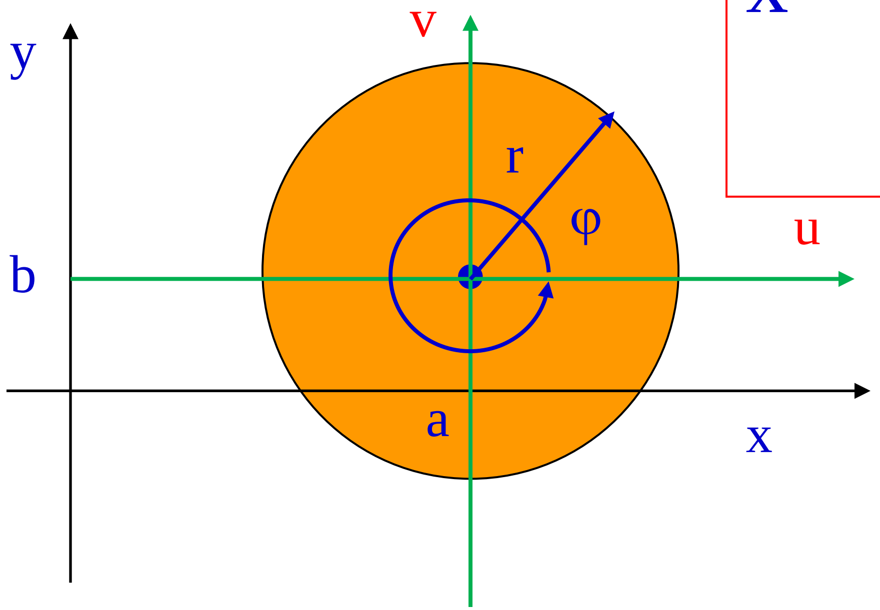
$$J = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\iint_D f(x,y)\,dxdy = \iint_{u^2+v^2\le R^2} g(u,v).1\,dudv$$

Đổi tiếp sang tọa độ cực: $u = r\cos\varphi, v = r\sin\varphi$

Tóm tắt:

$$D: (x - a)^2 + (y - b)^2 \le R^2$$



$$x = a + r\cos\varphi,\ y = b + r\sin\varphi$$

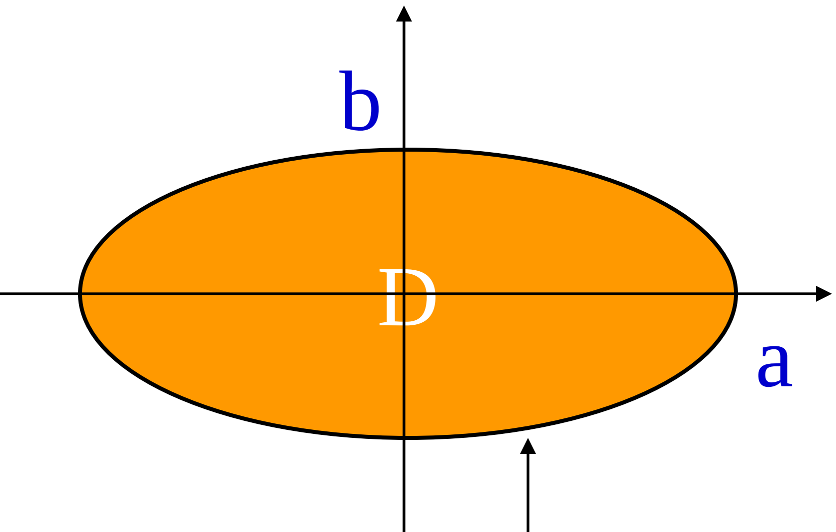
$$J = r$$

$$D': \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\iint_D f(x,y)\,dxdy = \iint_{D'} f(a + r\cos\varphi, b + r\sin\varphi)\,r\,dr\,d\varphi$$

Đổi biến trong ellipse

$$D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

x = arcosφ, y = brsinφ

J = abr

$$D': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\iint_D f(x,y)dxdy = \iint_{D'} f(ar\cos\varphi, br\sin\varphi)abr\,drd\varphi$$

1/ Tính: $I = \iint\limits_D \frac{y}{x} dxdy$ với D giới hạn bởi

$$xy = 1, xy = 2, y = x, y = 3x (x, y \geq 0)$$

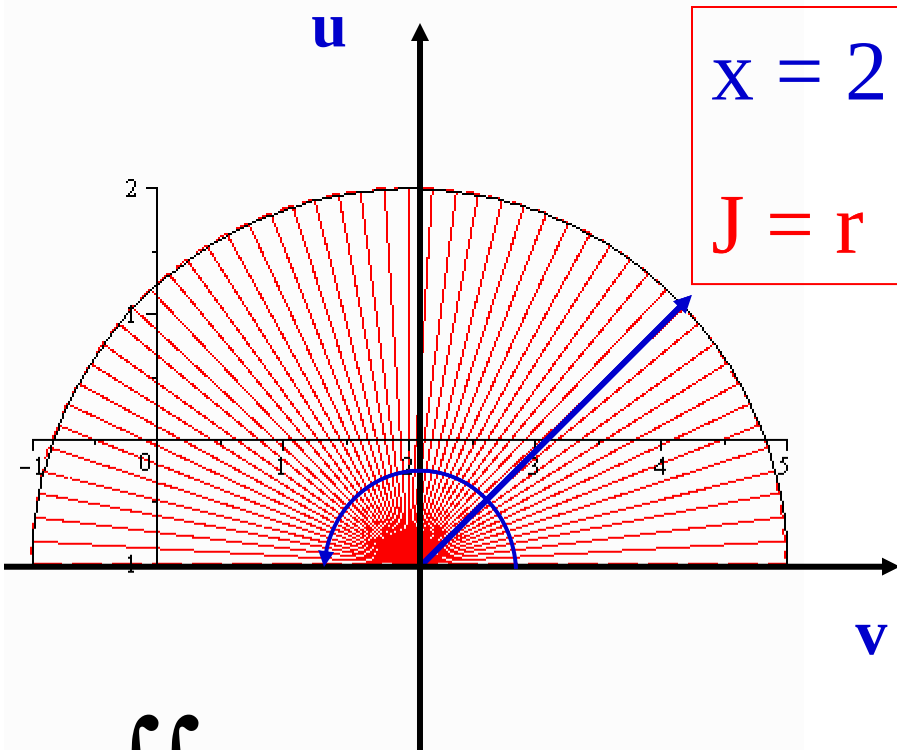
Đổi biến: $u = xy, v = \frac{y}{x}$ Khi đó: $D': \begin{cases} 1 \leq u \leq 2 \\ 1 \leq v \leq 3 \end{cases}$

$$J = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = 2v$$

Vậy:

$$I = \int_1^2 du \int_1^3 2v^3 dv$$

2/ Tính: $I = \iint\limits_D xy\,dxdy$ với D là nửa trên của hình tròn: $(x-2)^2 + (y+1)^2 \leq 9$



$x = 2 + r\cos\varphi$, $y = -1 + r\sin\varphi$

$J = r$

$$D: \begin{cases} 0 \leq r \leq 3 \\ 0 \leq \varphi \leq \pi \end{cases}$$

$$I = \iint\limits_D (2 + r\cos\varphi)(-1 + r\sin\varphi)\,r\,drd\varphi$$

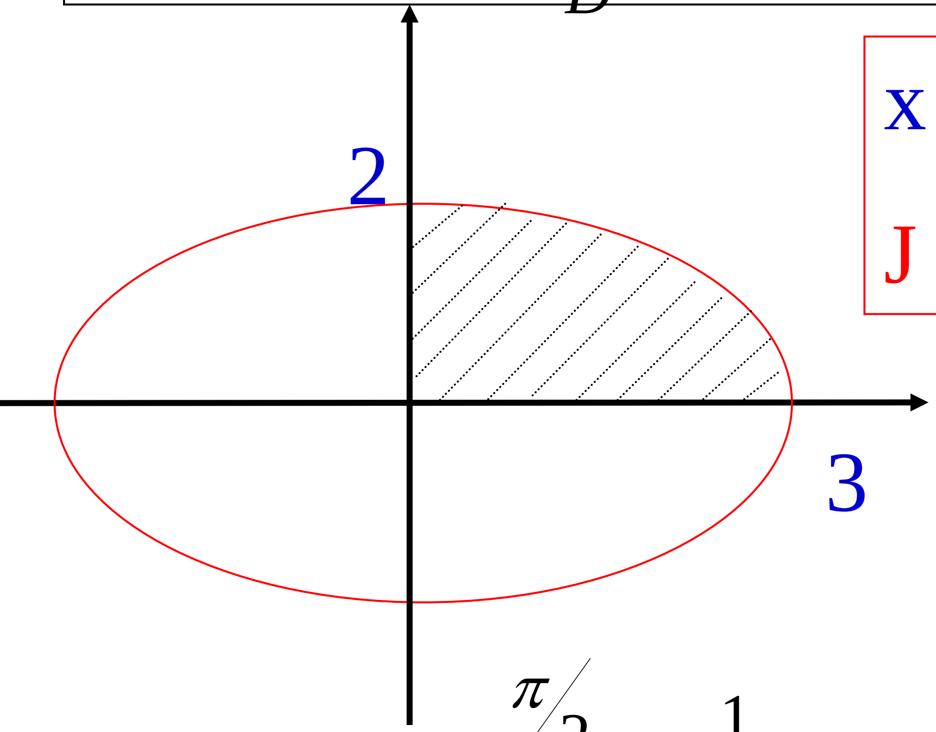
$$I=\iint\limits_{D'}(2+r\cos\varphi)(-1+r\sin\varphi)rdrd\varphi$$

$$=\int\limits_0^{\pi}d\varphi\int\limits_0^{3}(-2-r\cos\varphi+2r\sin\varphi+r^2\sin\varphi\cos\varphi)rdr$$

$$=-9\pi+18$$

Ví dụ

3/ Tính: $I=\iint\limits_D xy\,dxdy,\ D:\dfrac{x^2}{9}+\dfrac{y^2}{4}\le 1;\ y\ge 0;\ x\ge 0$



x = 3rcosφ, y = 2rsinφ

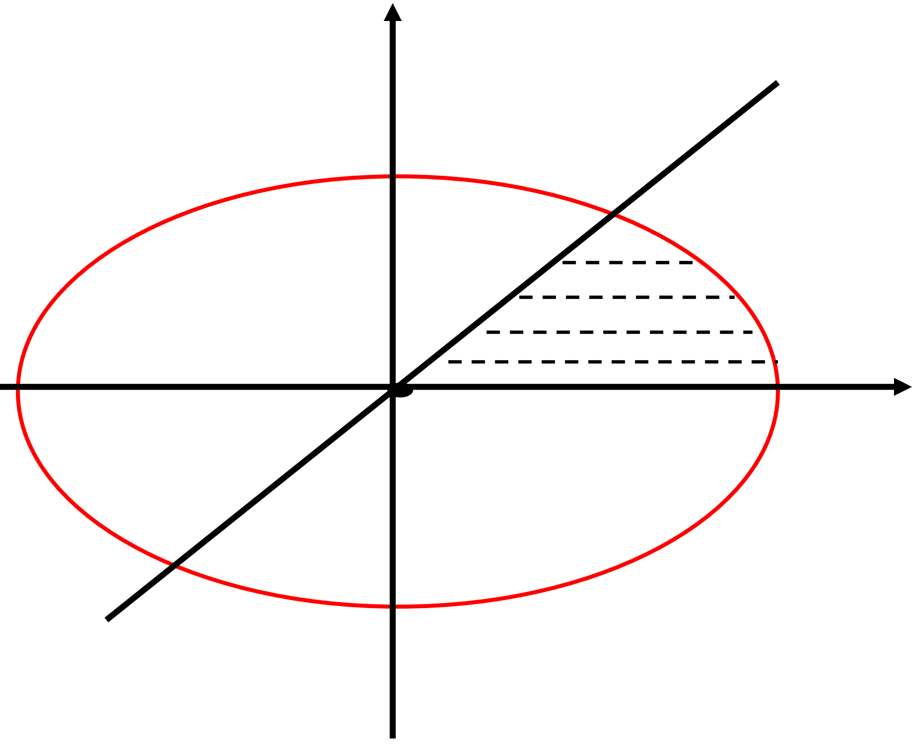
J = 3.2.r = 6r

$$D:\begin{cases}0\le r\le 1\\ 0\le\varphi\le \pi/2\end{cases}$$

$$\iint\limits_D xy\,dxdy=\int_0^{\pi/2}d\varphi\int_0^1 3r\cos\varphi.2r\sin\varphi.6r\,dr=\frac{9}{2}$$

4/ Tính diện tích miền giới hạn bởi

$$ellipse \quad \frac{x^2}{3} + y^2 = 1,\ y = 0,\ y = x,\ x \geq 0$$



$$x = \sqrt{3} r\cos\varphi, y = r\sin\varphi$$

$$J = \sqrt{3} r$$

Miền D được viết lại:

$$\frac{x^2}{3} + y^2 \leq 1,\ 0 \leq y \leq x$$

$$\Leftrightarrow \begin{cases} 0 \leq r \leq 1, \\ 0 \leq r\sin\varphi \leq \sqrt{3} r\cos\varphi \end{cases}$$

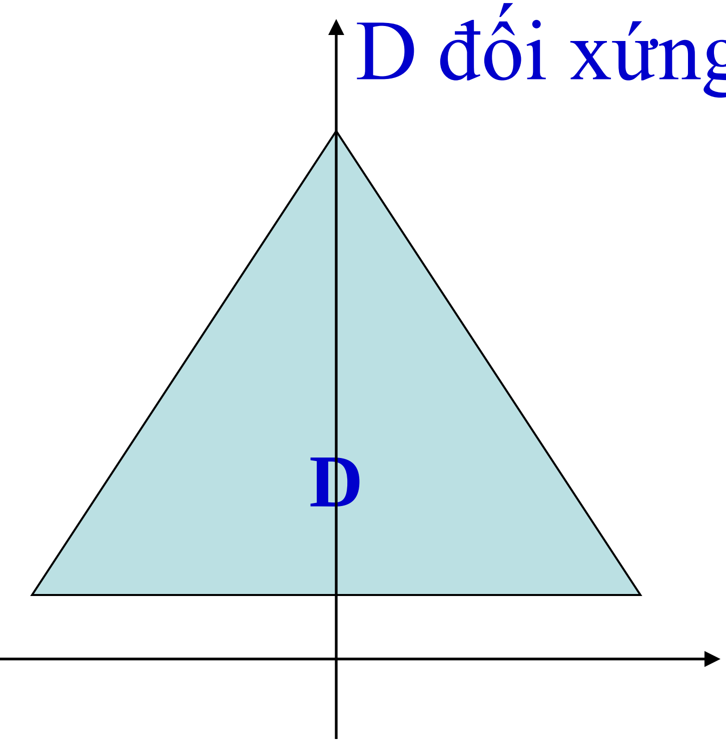
$$\begin{cases} 0 \le r \le 1, \\ 0 \le r\sin\varphi \le \sqrt{3}r\cos\varphi \end{cases}$$

$$\Leftrightarrow \begin{cases} 0 \le r \le 1, \\ 0 \le \tan\varphi = \dfrac{\sin\varphi}{\cos\varphi} \le \sqrt{3} \end{cases} \Leftrightarrow \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \dfrac{\pi}{3} \end{cases}$$

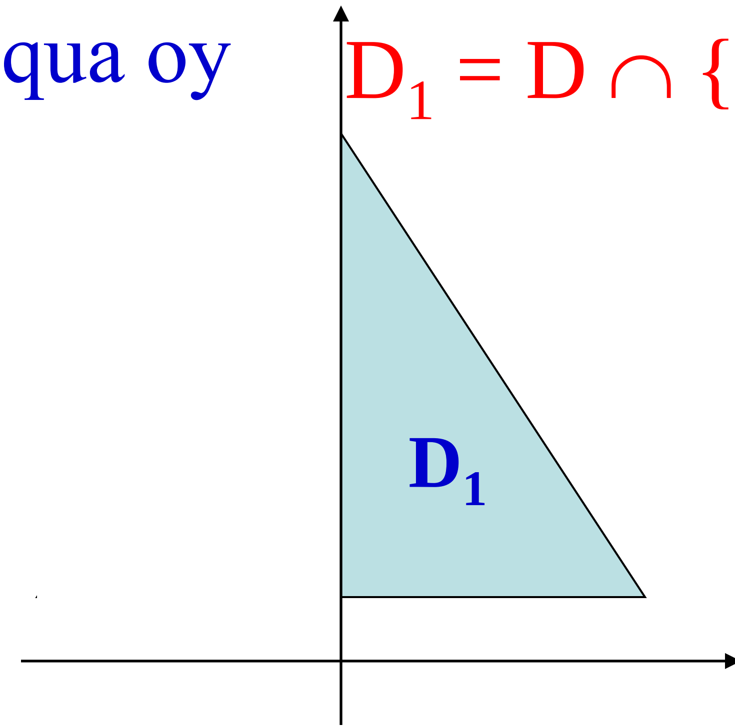
$$S(D) = \iint\limits_D dxdy = \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 \sqrt{3}rdr$$

Tính đối xứng của miền D trong tính tp kép

D đối xứng qua oy



$D_1 = D \cap \{(x,y) / x \geq 0\}$



f(x,y) chẵn theo x: $\iint\limits_D f(x,y)dxdy = 2\iint\limits_{D_1} f(x,y)dxdy$

f(x,y) lẻ theo x: $\iint\limits_D f(x,y)dxdy = 0$

Ví dụ

3/ Tính: $I = \iint\limits_D |x| + |y| dxdy, D : |x| + |y| \le 1$