6

Product of Vectors

Product (multiplication operation) of two vectors can be performed in the following two ways:

i) Scalar product or Dot product

ii) Vector product or Cross product

In this section we discuss only about the scalar product or dot product of two vectors.

<u>Scalar product or dot product</u>

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then, the dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and is defined as $|\vec{a}|\,|\vec{b}|\cos\theta$.

Thus, $\mathbf{\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta}$.

Clearly, the dot a product of two vectors is a scalar quantity due to which this product is also called a dot or scalar product. Since, we are putting a dot between $\vec{a}$ and $\vec{b}$, it is called dot product.

Also, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\vec{a}.\vec{b}}{a\,b}$

<u>Geometrical interpretation of scalar product</u>

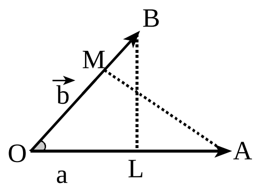
Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw BL ⊥ OA at L and AM ⊥ OB at M.

From Δ OBL and Δ OAM,

OL = OB cosθ and OM = OA cosθ

Here, OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.

Now, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$



$= |\vec{a}|$ (OB $\cos\theta$)

$= |\vec{a}|$ (OL)

$\Rightarrow \vec{a}.\vec{b}$ = (Magnitude of $\vec{a}$) (projection of $\vec{b}$ on $\vec{a}$) ......(i)

Again, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}|(|\vec{a}|\cos\theta)$

$= |\vec{b}|$ (OM)

$\Rightarrow \vec{a}.\vec{b}$ = (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$) ……….(ii)

Thus, the scalar product of two vectors is the product of modulus of either vector and the projection of the other vector in its direction.

Scalar Projection:

From above equation (i),

Scalar projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}.\vec{b} = \hat{a}.\vec{b}$,

where, $\hat{a}$ is the unit vector along a.

Similarly, from equation (ii),

Scalar projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \vec{a}.\dfrac{\vec{b}}{|\vec{b}|} = \vec{a}.\hat{b}$.

Thus, the scalar projection of $\vec{a}$ on $\vec{b}$ is the dot product of $\vec{a}$ with the unit vector along $\vec{b}$ and the scalar projection of $\vec{b}$ on $\vec{a}$ is the dot product of $\vec{b}$ with the unit vector along $\vec{a}$.

Notes:- (i) Dot product of $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ is $\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2$

ii) Dot product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is

$\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

i) $\cos\theta = \dfrac{\vec{a}.\vec{b}}{ab} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{a_1b_1 + a_2b_2}{|(a_1, a_2)|(b_1, b_2)|} = \dfrac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}$

ii) $\cos\theta = \dfrac{\vec{a}.\vec{b}}{ab} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{\vec{a}.\vec{b}}{|(a_1,a_2,a_3)|(b_1,b_2,b_3)|} = \dfrac{a_1b_1+a_2b_2+a_3 b_3}{\sqrt{a_1^2 + a_2^2+a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

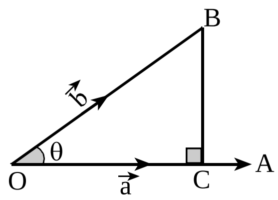
Vector projection of one vector on another vector.

Let, $\overrightarrow{OA} = \vec{a}$, and $\overrightarrow{OB} = \vec{b}$. Let BC ⊥ OA at C. Then,

$\overrightarrow{OC} = OC\ (\hat{a})$

$= (OB \cos\theta)\ \hat{a}$

$= |\vec{b}| \cos\theta \dfrac{\vec{a}}{|\vec{a}|}$



$= \dfrac{|\vec{a}||\vec{b}| \cos\theta}{|\vec{a}||\vec{a}|}\vec{a}$

$= \dfrac{(\vec{a}.\vec{b})}{|\vec{a}|^2}\vec{a}$

Hence, **vector projection of $\vec{b}$ on $\vec{a}$** $= \dfrac{(\vec{a}.\vec{b})}{|\vec{a}|^2}\vec{a}$

Similarly, **vector projection of $\vec{a}$ on $\vec{b}$** $= \dfrac{(\vec{a}.\vec{b})}{|\vec{b}|^2}\vec{a}$

Note: 1

Vector projection of $\vec{b}$ on $\vec{a}$

= (**Scalar projection of $\vec{b}$ on $\vec{a}$) (unit vectore along $\vec{a}$)**

$$= \frac{\vec{a}.\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{(\vec{a}.\vec{b})}{|\vec{a}|^2}\vec{a}$$

Similarly for **vector projection of $\vec{a}$ on $\vec{b}$.**

Note:2

Two vecetors $\vec{a}$ and $\vec{b}$ are said to be perpendicular or **orthogonal**

if $\vec{a}.\vec{b} = 0$.

Proof:

We have,

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta.$

If, $\theta = 90^0$ then

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos 90^0 = |\vec{a}|\,|\vec{b}|\,.\,0 = 0$

So, $\vec{a}.\vec{b} = 0$.

(ii) $\vec{a}.\vec{a} = |\vec{a}|^2 = a^2 = \vec{a}^2$, Also $a = \sqrt{\vec{a}.\vec{a}}$.

Note:2

$(\vec{a} \pm \vec{b})^2 = (\vec{a} \pm \vec{b})\,.\,(\vec{a} \pm \vec{b}) = a^2 \pm 2\vec{a}.\vec{b} + b^2$

$(\vec{a} + \vec{b}).\,(\vec{a} - \vec{b}) = a^2 - b^2$

$(\vec{a} + \vec{b} + \vec{c})^2 = a^2 + b^2 + c^2 + 2\,(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

Note: 3

If $\hat{i}$, $\hat{j}$ and $\hat{k}$ are three mutually perpendicular unit vectors along the co-ordinate axes then $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$ and $\hat{i}.\,\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

.	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	1	0	0
$\hat{j}$	0	1	0
$\hat{k}$	0	0	1

we have, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$.

For $\hat{i}$ and $\hat{i}$, $\theta = 0$

So, $\hat{i}\,.\,\hat{i} = 1.\,1.\cos 0 = 1.\,1 = 1$

Again, for $\hat{i}$ and $\hat{j}$, $\theta = 90$

$\hat{i}.\hat{j} = 1.\,1.\cos 90 = 1.\,0 = 0$

<u>Worked out examples</u>

1. If $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 3\vec{k} + 2\vec{j} + 3\vec{k}$, find (a) $\vec{a}.\,\vec{b}$ (b) angle between $\vec{a}$ and $\vec{b}$ (c) scalar projection of $\vec{a}$ on $\vec{b}$.

Solution:

a) $\vec{a}.\vec{b} = (2\vec{i} - \vec{j} + 2\vec{k})\,.\,(3\vec{i} + 2\vec{j} + 3\vec{k})$

$= 2.3 + (-1).\,2 + 2.3 = 10.$

b) $|\vec{a}| = \sqrt{(2)^2 + (-1)^2 + (2)^2} = 3$, $|\vec{b}| = \sqrt{(3)^2 + (2)^2 + (3)^2} = \sqrt{22}$.

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \cos\theta = \dfrac{10}{3\sqrt{22}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{10}{3\sqrt{22}}\right)$.

c) Scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \dfrac{10}{\sqrt{22}}$

2) Find the value of 'n' for which the vectors $\vec{a} = 3\vec{i} + 2\vec{j} + 9\vec{k}$ and $\vec{b} = \vec{i} + n\vec{j} + 3\vec{k}$ are perpendicular.

Solution:

i) If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}.\vec{b} = 0$.

$\Rightarrow (3\vec{i} + 2\vec{j} + 9\vec{k}) . (\vec{i} + n\vec{j} + 3\vec{k}) = 0$

$\Rightarrow 3 + 2n + 27 = 0 \Rightarrow n = -15.$

3) For any two vectors $\vec{a}$ and $\vec{b}$

(a) $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ if and only if $\vec{a}$ is perpendicular to $\vec{b}$. Interpret it geometrically.

Proof: Given, $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

$\Leftrightarrow |\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$

$\Leftrightarrow |\vec{a}|^2 + 2(\vec{a}.\vec{b}) + |\vec{b}|^2 = |\vec{a}|^2 - 2(\vec{a}.\vec{b}) + |\vec{b}|^2$

$\Leftrightarrow 2(\vec{a}.\vec{b}) = -2(\vec{a}.\vec{b})$

$\Leftrightarrow 4(\vec{a}.\vec{b}) = 0$

$\Leftrightarrow \vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}.$

Geometrical Interpretation:-

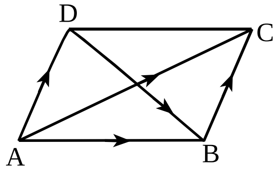
Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$

In ΔABC,

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\Rightarrow \vec{a} + \vec{b} = \overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC} = \vec{a} + \vec{b}$



In ΔABD

$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$

$\Rightarrow \overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{DB}$

$\Rightarrow \vec{a} - \vec{b} = \overrightarrow{DB}$

$\Rightarrow \overrightarrow{DB} = \vec{a} - \vec{b}$

Hence, the diagonals of a parallelogram are equal if it is a rectangle.

b) If $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$, then prove that $|\vec{a}| = |\vec{b}|$.

Solution: Given, $(\vec{a} + \vec{b}).\,(\vec{a} - \vec{b}) = 0$

$\Rightarrow \vec{a}\,.\,\vec{a} - \vec{a}\,.\,\vec{b} + \vec{b}.\,\vec{a} - \vec{b}.\,\vec{b} = 0$

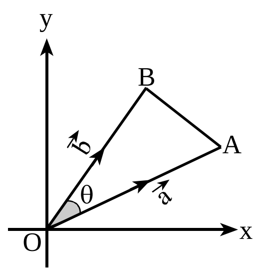
$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 0\ (\because\ \vec{a}.\vec{b} = \vec{b}.\vec{a})$

$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow |\vec{a}| = |\vec{b}|$.

4) Prove that if θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\,.\,\vec{b}$ = ab $\cos\theta$.

Proof:- Let $O\vec{A} = \vec{a}$, $O\vec{B} = \vec{b}$ and $\rightarrow AOB = \theta$.



Then, $|O\vec{A}| = a$, $|O\vec{B}| = b$, $|A\vec{B}| = |O\vec{B} - O\vec{A}| = |\vec{b} - \vec{a}|$

Using cosine law in ΔOAB,

$AB^2 = OA^2 + OB^2 - 2OA.OB.\cos\theta$

$\Rightarrow |\vec{b} - \vec{a}|^2 = a^2 + b^2 - 2ab\cos\theta$

$\Rightarrow |\vec{b}|^2 - 2\,\vec{b}\,.\,\vec{a} + |\vec{a}|^2 = -2ab\cos\theta\ (\because\ \vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a})$

$\Rightarrow \vec{a}\,.\,\vec{b} = ab\cos\theta$.

5) If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle θ, then prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a} - \vec{b}|$

Proof:

Since, $\vec{a}$ and $\vec{b}$ are two unit vectors, $|\vec{a}| = a = 1$ and $|\vec{b}| = b = 1$

Now, $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2(\vec{a}\,.\,\vec{b}) + |\vec{b}|^2$

$= 1^2 - 2\,a\,b\cos\theta + 1^2 = 1 - 2.1.1\cos\theta + 1 = 2 - 2\cos\theta$

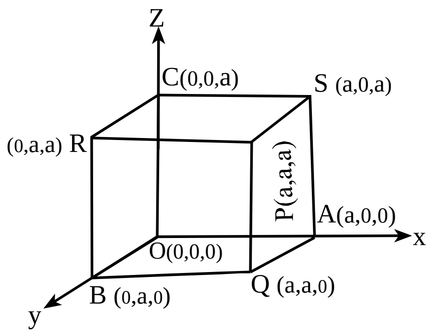
$= 2(1 - \cos\theta) = 2.2\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2} = (2\sin\frac{\theta}{2})^2$

$\Rightarrow |\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2} \quad \Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}|\vec{a} - \vec{b}|$

6) Show that the angle between two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

Solution:-

Let, a be the length of an edge of the cube and let one of the corners be at the origin as shown as in the figure. Clearly, OP, AR, BS and CQ are the diagonals of the cube. Consider the diagonals OP and AR.



We have, $\overrightarrow{OP} = a\hat{i} + a\hat{j} + a\hat{k}$

And, $\overrightarrow{AR} = (a\hat{i} + a\hat{j} + a\hat{k}) - (a\hat{i} + o\hat{j} + a\hat{k}) = -a\hat{i} + a\hat{j} + a\hat{k}$

Let θ be the angle between $\overrightarrow{OP}$ and $\overrightarrow{AR}$. Then

$$\cos\theta = \frac{\overrightarrow{OP}.\overrightarrow{AR}}{|\overrightarrow{OP}||\overrightarrow{AR}|} = \frac{(a\hat{i} + a\hat{j} + a\hat{k}).(-a\hat{i} + a\hat{j} + a\hat{k})}{\sqrt{a^2 + a^2 + a^2}\sqrt{(-a)^2 + a^2 + a^2}}$$

$$= \frac{-a^2 + a^2 + a^2}{\sqrt{3a^2}\sqrt{3a^2}} = \frac{a^2}{3a^2} = \frac{1}{3}.$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right).$$

Application of vector in Trigonometry

1. **Cosine formula(law)**:

If a, b, c are length of the sides opposite respectively to the angle A, B, C of a triangle ABC, show that $a^2 = b^2 + c^2 - 2bc \cos A$.

Furthermore, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

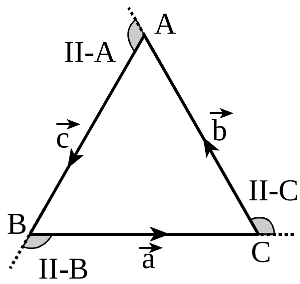
Solution:-

Let, $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$ in Δ ABC. Then,

$|\vec{a}| = |\overrightarrow{BC}| = a$, $|\vec{b}| = |\overrightarrow{AB}| = b$

And $|\vec{c}| = |\overrightarrow{AB}| = c$

We have from triangle law,



$\Rightarrow \overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$\Rightarrow \overrightarrow{BC} = -\overrightarrow{AB} - \overrightarrow{CA}$

$\Rightarrow \vec{a} = -\vec{c} - \vec{b}$

$\Rightarrow \vec{a} = -(\vec{b} + \vec{c})$

squaring,

$\Rightarrow \vec{a}^2 = (\vec{b} + \vec{c})^2$

$\Rightarrow a^2 = \vec{b}^2 + 2\,\vec{b}.\vec{c} + \vec{c}^2$

$\Rightarrow a^2 = b^2 + 2|\vec{b}|\,|\vec{c}|\,.\cos(\pi - A) + c^2$

$\Rightarrow a^2 = b^2 - 2bc \cos A + c^2$

$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$

Also,

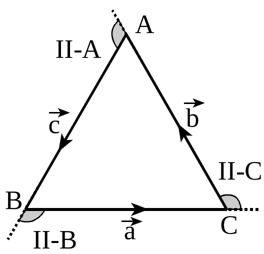
$2bc \cos A = b^2 + c^2 - a^2$

$\Rightarrow \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

2. Projection formula:-

If a, b, c are the lengths of the sides opposite respectively to the angle A, B, C of a triangle ABC, show that a = b cosC + c cosB.

Solution:-



Let, $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$ in ΔABC, then

$|\overrightarrow{BC}| = |\vec{a}| = a$, $|\overrightarrow{CA}| = |\vec{b}| = b$

And $|\overrightarrow{AB}| = |\vec{c}| = c$

By the triangle law of addition of vectors,

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$\Rightarrow \overrightarrow{BC} = -\overrightarrow{AB} - \overrightarrow{CA}$

$\Rightarrow \vec{a} = -\vec{c} - \vec{b}$

$\Rightarrow \vec{a} = -(\vec{b} + \vec{c})$

Taking dot product with $\vec{a}$ on both sides

$\vec{a} \cdot \vec{a} = -\vec{a} \cdot (\vec{b} + \vec{c})$

$\Rightarrow |\vec{a}|^2 = -\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$

$\Rightarrow a^2 = -ab\cos(\pi - C) - ac\cos(\pi - B)$

$\Rightarrow a^2 = ab\cos C + ac\cos B$

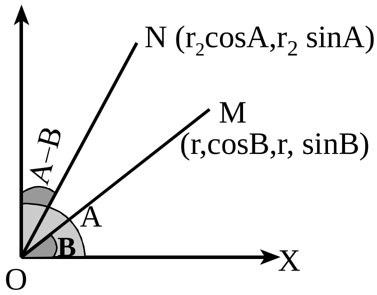
$\Rightarrow a = b\cos C + c\cos B$

3. Prove that cos (A – B) = cosA cosB + sinA sinB

Proof:- Let O be origin, OX be x-axis and OY be y-axis. Let $\overrightarrow{OM}$ and $\overrightarrow{ON}$ be two vectors with magnitude r_1 and r_2 respectively.

Let $\rightarrow$MOX = $\rightarrow$B, NOX = A Then

$\rightarrow$MON = A – B



$\overrightarrow{OM}$ = (r, cosB, r, sinB) and $\overrightarrow{ON}$ = (r_2 cosA, r_2 sinA)

$\therefore$ $\overrightarrow{OM}.\ \overrightarrow{ON} = r_1\, r_2$ (cosA cosB + sinA sinB)

We have,

$$\text{Cos}(A-B) = \frac{\overrightarrow{OM}.\overrightarrow{ON}}{|\overrightarrow{OM}|.|\overrightarrow{ON}|}$$

$$= \frac{r_1 r_2(\cos A \cos B + \sin A \sin B)}{r_1 r_2}$$

$\therefore$ cos (A – B) = CosA cosB + sinA sinB

4. Prove that cos (A + B) = cosA cosB –sinA sinB

Proof:- Let O be origin, XX' be x-axis

YY' be y-axis. Let $\overrightarrow{OM}$ and $\overrightarrow{ON}$ be two vectors such that $|\overrightarrow{OM}| = r_1$ and $|\overrightarrow{ON}| = r_2$.

Let $\rightarrow$MOX = A and $\rightarrow$NOX' = B. Then $\rightarrow$MON = π – (A + B). $\overrightarrow{OM}$ = (r_1 cosA, r_1, sinA) and $\overrightarrow{ON}$ = (– r_2 cosB, r_2, sinB).

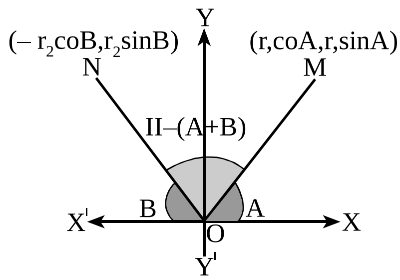
So, $\overrightarrow{OM}.\ \overrightarrow{ON} = -\,r_1\, r_2$ (cosA cosB – SinA sinB)

We have,

$$\text{Cos}\{\pi - (A+B)\} = \frac{\overrightarrow{OM}.\overrightarrow{ON}}{|\overrightarrow{OM}|.|\overrightarrow{ON}|}$$

$$\Rightarrow -\cos(A+B) = \frac{-\,r_1\, r_2\,(\cos A\cos B - \sin A\sin B)}{r_1 r_2}$$

$\therefore$ cos (A + B) = cosA cosB – sinA sinB.



Worked out example

7. For what value of m is the pair of vectors $\vec{i} - 2\vec{j} + 4\vec{j}$ and $2\vec{i} + 7\vec{j} + m\vec{k}$ orthogonal or perpendicular?

Solution:

Let $\vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}$ and $\vec{b} = 2\vec{i} + 7\vec{j} + m\vec{k}$

The vectors $\vec{a}$ and $\vec{b}$ will be orthogonal if $\vec{a}.\vec{b} = 0$

$\Rightarrow (\vec{i} - 2\vec{j} + 4\vec{k}).(2\vec{i} + 7\vec{j} + m\vec{k}) = 0$

$\Rightarrow 2 - 14 + 4m = 0$

$\Rightarrow m = -3$

8. If $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + \vec{j} + \vec{k}$, find the scalar projection of $\vec{b}$ on $\vec{a}$. Also find the scalar projection of $\vec{a}$ on $\vec{b}$.

Solution:– Given $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - \vec{j} + \vec{k}$, Then, $\vec{a}.\vec{b} = 1 - 2 - 1 = -2$

Also, $|\vec{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

and $|\vec{b}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|} = \dfrac{-2}{\sqrt{6}}$

and projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \dfrac{-2}{\sqrt{3}}$

9. Show that the three points whose position vectors and $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ and $3\vec{j} - 4\vec{k}$ from the sides of a right angled triangle. Also find the remaining two angles.

Solution: Let $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ and $3\vec{i} - 4\vec{j} - 4 - 4\vec{k}$ respectively be the position vectors of A, B and C of ΔABC. Let O be the origin. Then,

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} - 2\vec{j} - 6\vec{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\vec{i} - \vec{j} - \vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{i} - 3\vec{j} - 5\vec{k}$

$|\overrightarrow{AB}| = AB = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{41}$

$|\overrightarrow{BC}| = BC = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

$|\overrightarrow{AC}| = AC = \sqrt{2^2 + (-3)^2 + (-5)^2} = \sqrt{35}$

Here, $BC^2 + AC^2 = 6 + 35 = 41 = AB^2 \therefore \to C = 90^o$

Now, $\cos A = \dfrac{\overrightarrow{AB}.\overrightarrow{AC}}{AB.AC} = \dfrac{-1+6+30}{\sqrt{41}\sqrt{35}} = \sqrt{\dfrac{35}{41}}$

and $\cos B = \dfrac{\overrightarrow{BC}.\overrightarrow{BA}}{BC.BA} = \dfrac{2-2+6}{\sqrt{6}\sqrt{41}} = \sqrt{\dfrac{6}{41}}$

$\Rightarrow B = \cos^{-1}\left(\sqrt{\dfrac{6}{41}}\right)$

Therefore, the remaining two angles are

$\cos^{-1}\left(\sqrt{\dfrac{35}{41}}\right)$ and $\cos^{-1}\left(\sqrt{\dfrac{6}{41}}\right)$.

10. Find a unit vector which is perpendicular to the vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $3\hat{i} - \hat{j} + 2\hat{k}$.

Solution:- Let $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$ be a unit vector perpendicular to the vectors

$\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$.

If $\vec{a}.\vec{b} = 0$,

$(x\vec{i} + y\vec{j} + z\vec{k}).(\vec{i} + 2\vec{j} - \vec{k}) = 0$

$\Rightarrow x + 2y - z = 0$ ……….(i)

Similarly, if $\vec{a}$ and $\vec{c}$ are perpendicular, then $\vec{a}.\vec{c} = 0$

$\Rightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(3\vec{i} - \vec{j} + 2\vec{k}) = 0$

$\Rightarrow 3x - y + 2z = 0$ ……….. (ii)

Solving (i) & (ii) by cross-multiplication method.

$\dfrac{x}{4-1} = \dfrac{y}{-3-2} = \dfrac{z}{-1-6}$

$\Rightarrow \frac{x}{3} = \frac{y}{-5} = \frac{z}{-7} = \lambda$ (say)

$\Rightarrow \quad x = 3\lambda, y = -5\lambda, z = -7\lambda$

Since, $\vec{a}$ is a unit vector,

$|\vec{a}| = 1$

$\Rightarrow \sqrt{x^2 + y^2 + z^2} = 1$

$\Rightarrow x^2 + y^2 + z^2 = 1$

$\Rightarrow 9\lambda^2 + 25\lambda^2 + 49\lambda^2 = 1$ [Using (iii)]

$\Rightarrow \lambda = \pm \frac{1}{\sqrt{83}}$

From (iii)

$x = \pm \frac{3}{\sqrt{83}}, y = \mp \frac{5}{\sqrt{83}}$ and $z = \mp \frac{7}{\sqrt{83}}$

Putting the values of x, y and z, $\vec{a} = \pm \frac{3}{\sqrt{83}} \vec{i} \pm \frac{7}{\sqrt{83}} \vec{k}$ is the required unit vector.

11. If $\vec{a} = (-2, 3, 1)$ and $\vec{b} = (1, 1, 2)$, find the

(a) vector projection of $\vec{a}$ on $\vec{b}$ and

(b) vector projection of $\vec{b}$ on $\vec{a}$.

Solution:– $\vec{a}.\vec{b} = -2 + 3 + 2 = 3$

$|\vec{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}, |\vec{b}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

(a) Vector projection of $\vec{a}$ on $\vec{b} = \frac{(\vec{a}.\vec{b})}{|\vec{b}|^2} \vec{b} = \frac{3}{6}(1,1,2) = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$

b) Vector projection of $\vec{b}$ on $\vec{a} = \frac{(\vec{a}.\vec{b})}{|\vec{a}|^2} \vec{a} = \frac{3}{14}$ (– 2, 3, 1) = (–3/7, 9/14. 3/14)

Exercise

1. Define a dot product. Interpret it geometrically.

2. If $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$ are any two vectors, find the cosine of the angle between the two vectors. ***Ans:*** $\boldsymbol{\frac{1}{2}}$

3. For what value of *x* are the pair of vectors $x\vec{i} - 2\vec{j} + 4\vec{k}$ and $2\vec{i} + 7\vec{j} + \vec{k}$ orthogonal? ***Ans: 5***

4. If $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, find the scalar projection of $\vec{a}$ an $\vec{b}$ and the scalar projection of $\vec{b}$ on $\vec{a}$. Ans: $\frac{20}{\sqrt{29}}$ and $\frac{20}{\sqrt{14}}$

5. Find the vector projection of $\vec{b}$ on $\vec{a}$ if $\vec{a} = (3, 6, -2)$ and $\vec{b} = (1, 2, 3)$.

   Ans: $\boldsymbol{\left(\frac{27}{49}, \frac{54}{49}, -\frac{18}{49}\right)}$

6. If a, b, c are the length of the sides opposite respectively to the angles A, B, C of a triangle ABC, prove that

   i) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$ ii) $\cos c = \dfrac{a^2 + b^2 - c^2}{2ab}$

   iii) $b = c \cos A + a \cos c$ (iv) $c = a \cos B + b \cos A$

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