Unit 4 Probability

4.1 Concept of Probability

Probability is one of the important branches of Statistics which is concerned with random phenomena. It is the synonym of the words most likely, chance, probably etc. The theory of probability was developed in the middle of the seventeenth century. It was originated from the problem related to gambling such as tossing a coin, throwing a die, drawing cards from a pack of cards etc. But nowadays, there is hardly any discipline (field) where "probability" has not been used. The theory of probability is used in almost all discipline like physical science, natural science, biological science, medical science, engineering, social sciences, business, industry etc. It is extensively used in quantitative analysis of business and economics problems. It is an essential tool in statistical inference and backbone of probability sampling and forms the basis of the "Decision Theory", viz. decision making in the case of uncertainty with calculated risks.

Business firms, factory managers, stock market policy makers often have been facing the problems regarding the chance of selling the goods, chance of receiving better demand, chance of getting defective or non-defective product and chance of deterioration and prosperity of stock market etc. The word probability is a chance or possibility which is widely used in daily life also.

e.g. What is the chance of heavy rain in this morning?

What is the chance of passing BCA student in a final examination?

Probability is a numerical measure (with a value lying between 0 and 1) of the likelihood of chance that a particular event will occur or not. Probability is simply a number lies between zero and one. That is; $0 \leq P \leq 1$ or $0\% \leq P\% \leq 100\%$. It is generally denoted by 'P'. If there is absolute possibility of occurrence of an event, then its probability is equal to one. This probability is also called the probability of certainty. If there is complete impossibility that an event will occur, then its probability is zero and is called the probability of uncertainty. For example, probability of rising the sun form east is 1 and probability of rising the sun from west is 0. This shows that the range of probability is in between zero and one. This range of probability is shown as follows.

0	1/2	1
Probability of uncertainty	Probability of event as likely to occur or not	Probability of certainty

Basic Terminologies in Probability

1. **Experiment:** The process (phenomena) performed which result different possible outcomes is known as experiment. For example, tossing a coin, attempting in an examination, throwing a dice etc.
2. **Random Experiment:** An experiment in which all the possible outcomes are known in advance and no personal bias are expected is called 'Random experiment'. In other words, an experiment is called

a random experiment if it is performed a large number of times under essentially homogeneous conditions; the result is not unique but may be any one of the various possible outcomes. Tossing an unbiased (fair) coin and rolling a uniform die are some examples of random experiment.

3. **Trial and Event:** Performing a random experiment is called a trial and outcome or combination of outcomes (i.e. results) of random experiments called an event. For example; flipping (tossing) an unbiased coin is a trial and getting either head or tail is an event.
4. **Sample Space:** The **set of all possible outcomes** of a random experiment is called sample space. Each outcome is thus considered as a sample point in the sample space. It is usually denoted by S. For example, in tossing an unbiased coin once, there are two possible outcomes head (H) and tail (T). So, the sample space is given by $S = \{H, T\}$
5. **Exhaustive Cases:** The total number of possible outcomes of a random experiment or the **cardinality** of a sample space is called the exhaustive cases for the experiment. In a toss of single coin, we can get head (H) or tail (T). Hence, exhaustive number of cases is 2. Similarly, in rolling a six faced die, the exhaustive number of cases = 6.
6. **Favourable Cases or Events:** The number of outcomes of a random experiment which result in the happening of an event or which are desired to occur are termed as the cases favourable to the event. In other words, the numbers of outcomes which are in favour of happening of an event are called favourable cases. For example; in drawing a card from a pack of 52 playing cards, the number of cases favourable to drawing a queen is 4. Similarly, in a toss of two coins, the number of cases favourable to the event 'exactly one head ' is 2, viz., *HT* , *TH* and for 'two heads' is one viz., *HH*.
7. **Equally Likely Events:** Events or cases are said to be equally likely or equally probable if none of them is expected to occur in preference to other. In other words, two or more events are said to be equally likely if all of them have equal chance of occurrence. In tossing an unbiased coin, head and tail are equally likely, throwing a unbiased die (the faces 1,2,3,4,5, 6) are equally likely.
8. **Mutually Exclusive Events:** Two or more events are said to be mutually exclusive if the happening of any one of them excludes the happening of all others in the same experiments i.e. if two or more than two events cannot occur simultaneously at the same time in the same trial. For example in a single tossing of a coin, we may get either head or tail but not both. Thus, the events head and tail in a single tossing of a coin are mutually exclusive. Similarly, in the throw of a die, the six faces numbered 1, 2, 3, 4, 5 & 6 are mutually exclusive. Thus, no two or more of them can happen simultaneously.
9. **Independent Events:** Events are said to be independent if the occurrence of one event does not affect the occurrence of the other events and vice versa. In tossing of a coin, the occurrence of head in first tossing is independent of the occurrence of head in the second tossing. Similarly, drawing of balls from an urn gives independent events if the draws are made **with replacement**.
10. **Dependent Events:** Events are said to be dependent if the occurrence of one event affects the occurrence of the other events and vice versa. For example, if a pack of cards is playing **without replacement**, the occurrence of a king in first draw affects the occurrence of other cards in the second draw.

4.2 Fundamental(Basic) Principle of Counting

It is also known as basic principle of counting. The counting rules facilitate to calculate all the possible outcomes in an experiment.

The fundamental principle of counting states as 'If one operation can be performed in n_1 different ways and another operation can be performed in n_2 different ways, then the two operations when associated together can be performed in $n_1 \times n_2$ ways.'

That is, an event A_1 can occur in n_1 ways and after its occurrence, event A_2 can occur in n_2 ways, then both the events can occur in a total of $n_1 \times n_2$ different ways in a given order of occurrences.

The result can be generalized to more than two operations:

If there are '*k*' separate parts of an experiment, and the first part can be done in n_1 ways, second successive part in n_2 ways ... and k^{th} successive part in n_k ways, then the total number of possible outcomes is given by $n_1 \times n_2 \times \cdots \times n_k$.

For example, tossing an unbiased coin thrice. There are two possible outcomes in each toss namely head '*H*' and tail '*T*'. Hence, the total number of possible outcomes = $2 \times 2 \times 2 = 2^3 = 8$.

This can be listed as follows:

HHH, *HHT*, *HTH*, *HTT*, *TTT*, *THH*, *THT*, *TTH.*

Permutation and Combination

1. **Permutation:** The literal meaning of the word permutation is "Arrangement". Therefore, permutation is the arrangement of objects taken some or all at time in some order. If there are '*n*' objects and they are to be placed in any definite arrangement or order. The number of permutations of 'n' different objects taken 'r' objectives at a time is denoted by ${}^np_r$ or P (n, r) and defined as

$$n_{p_r} = \frac{n!}{(n-r)!} \text{ where, } r \leq n,\ n! = n \times (n-1) \times (n-2)!$$

$$= n(n-1)\,.\,(n-2) \cdots 3 \cdot 2 \cdot 1$$

Example 4.1 How many numbers of 4 different digits can be formed with digits 1, 2, 3, 4,5 ?

Solution: Here $n = 5$, $r = 4$

Then, the required permutation is:

$$n_{p_r} = \frac{n!}{(n-r)!}$$

$${}^5P_4 = \frac{5!}{(5-4)!} = \frac{5!}{0!} = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Hence, 120 different numbers of 5 digits can be formed.

Permutation of Objects not at all Different

The number of permutations of '*n*' objects taken all at a time, when '*p*' objects are of one kind '*q*', objects are of second kind, '*r*' objects are of third kind, '*s*' objects are of fourth kind and so on is given by,

$$\frac{n!}{p!q!r!s! \cdots}$$

Note that: 0! = 1 and 1! = 1

Example 4.2 Find the total number of arrangements of the letters of the word 'STATISTICS' taken all at a time.

Solution: Here, the required arrangement is given by = $\frac{10!}{3!\,3!\,2!\,1!\,1!} = \frac{10.9.8.7.6.5.4}{3.2.1.2.1} = 504000$

2. **Combination:** The literal meaning of the word combination is "Selection". Therefore, combination is the selection of objects taken some or all at a time without specific order. Note that the position of the object is meaningless in combination.

A combination of 'n' different objects taken 'r' objects at a time, is denoted by ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$ and defined as

$$ {}^nC_r = \frac{n!}{(n-r)!\, r!} \text{ for } r \le n $$

Example 4.3 In how many ways can a committee of 4 persons be chosen out of 8 persons?

Solution: Here, $n = 8$, $r = 4$

$$\text{Required number of ways} = {}^8C_4 = \frac{8!}{(8-4)!4!} = \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)\,(4 \times 3 \times 2 \times 1)} = 70 \text{ ways}$$

OR,

$$ {}^8C_4 = \frac{8!}{(8-4)!4!} = \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5 \times 4!}{(4 \times 3 \times 2 \times 1)(4!)} = 70 $$

OR

Therefore, the committee can be formed in 70 ways.

Approaches of Probability

The chance of occurrence or non-occurrence of any event in a random experiment is called probability. There are four approaches to define the probability.

1. Mathematical or classical or priori approach
2. Statistical or empirical or relative frequency approach
3. Subjective approach
4. Axiomatic approach

1. Mathematical or Classical or Priori Approach of Probability

Let n be the exhaustive, mutually exclusive and equally likely cases (or outcomes) out of which 'm' are favourable cases to the happening of an event A. Then the probability of happening (occurrence) of an event A is given by

$$P(A) = \frac{\text{Favourable number of cases to } A}{\text{Exhaustive number of cases}} = \frac{m}{n}$$

$$\text{or,} \quad P(A) = \frac{m}{n} \qquad \text{... (1)}$$

If $(n - m)$ cases are favourable to the non-occurrence of an event A, then its probability is given by $P(A^c)$ or $P(\bar{A})$.

$$\text{And,} \quad P(\bar{A}) = q = \frac{\text{Favourable number of cases for not happening of event } A}{\text{Exhaustive number of cases}} = 1 - \frac{m}{n} = \frac{n-m}{n}$$

$$\Rightarrow \quad P(\bar{A}) = q = 1 - \frac{m}{n}$$

$$\Rightarrow \quad P(\bar{A}) = q = 1 - p(A)$$

$$\text{Hence,} \quad P(A) + P(\bar{A}) = 1 \qquad \text{... (2)}$$

$$\Rightarrow \quad p + q = 1,\ q = 1 - p$$

This shows that total probability is equal to 1.

Hence, probability of occurrence of event A + Probability of non-occurrence of event A = 1.

i.e., 100%

Note that probability lies between zero and 1 i.e. $0 \le p \le 1$.

Limitations of Classical Approach of Probability

This approach breaks down in the following cases

a. If *n*, the exhaustive number of cases (outcomes) of the random experiment is infinite.
b. If the various outcomes of the random experiment are not equally likely.
c. If the actual value of n is not known.

2. Statistical or Empirical or Relative Frequency Approach

This approach of probability is based on statistical data. If an experiment is performed repeatedly a large number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the total number of trials of the experiments the number trials becomes indefinitely large, is called the probability of happening of the event. It is being assumed that the limit is finite and unique.

Suppose an event *A* occurs m times in n repetitions of a random experiment. Then the ratio $\frac{m}{n}$ gives the relative frequency of the event *A*. In the limiting case when n becomes sufficiently large, then the number is called the probability of *A* and symbolically denoted by

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

Limitations of statistical or empirical approach of probability

a. The experimental conditions may not remain essentially homogeneous and identical in a large number of repetitions of the experiment.
b. The relative frequency $\frac{m}{n}$, may not attain a unique value, no matter however large *n* may be.

3. Subjective Approach

The subjective probability approach is purely individualistic in nature. Therefore, this approach of probability is completely based on the personal beliefs, feelings, experience, judgment, personal discretion of a person. Since, different persons may assign different probabilities one cannot arrive at objective conclusions using probabilities assigned by this subjective method. This method of assigning probability is generally used by top level authorities on the basis of their discretion.

Note: When probability is not given directly, then it can be categorized into two cases:

Case I: When one item is selected at a time.

Case II: When more than one item is selected at a time.

For example,

Example 4 A card is drawn at random from a well-shuffled pack of 52 cards. What is the probability of getting?

(a) a red card, (b) a black king

Solution: There are 52 cards in a pack.

Total number of cases (n) = 52

(a) P(*a* red card) = ?

Since, there are 26 red cards.

Favourable number of cases (m) = 26

Required probability of getting a red card is

$$P(a \text{ red card}) = \frac{m}{n} = \frac{26}{52} = \frac{1}{2}$$

(a) $P(a \text{ black king})$ =?

There are 2 black kings

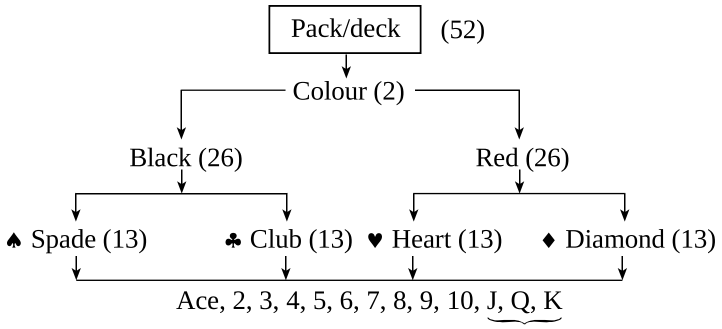
Favourable number of cases (m) = 2

Required probability of getting a black king is

$$P(a \text{ black king}) = \frac{2}{52} = \frac{1}{26}$$

Example 5 A card is drawn from a pack of 52 cards at random what is the probability that it is

(i) Red, (ii) Spade, (iii) Face card, (iv) an ace, (v) red king, (vi) Knave of heart, (vii) King or queen, (viii) Heart or club, (ix) a red 2 or black 8 or a queen, (x) Spade or ace, (xi) Heart or face card, (xii) Red or face card**.**



Solution:

Since, there are 52 cards in a pack.

n = Total number of cases (or outcomes) = 52

Since, one card is drawn at random

(i) $P(a \text{ red card})$ =?
Let 'E' denotes the event of drawing red cards
Favourable number of case (m) = 26

$$\therefore \; P(a \text{ red}) = P(E) = \frac{m}{n} = \frac{26}{52} = \frac{1}{2}$$

(ii) $P(a \text{ spade})$ = ?
Let E denotes the event of drawing spade cards
Favourable number of cases (m) = 13

$$\therefore \; P(a \text{ spade}) = P(E) = \frac{m}{n} = \frac{13}{54} = \frac{1}{4}$$

Similarly,

(iii) $P(a \text{ face card})$ = ?
Let 'E' denotes the event of drawing a face card
Favourable number of cases (m) = 12
(3 face cards in each suit)

$$P\ (a \text{ face card}) = P\ (E) = \frac{m}{n} = \frac{12}{52} = \frac{3}{13}$$

(iv) $P(\text{an ace})$ = ?
Let 'E' denotes the event of drawing an ace
Favourable number of outcomes (m) = 4

$$P(\text{an ace}) = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$$

(v) P(a red king) =?

Favourable number of cases (m) = number of red kings = 2

$$P(\text{a red king}) = \frac{2}{52} = \frac{1}{26}$$

(vi) P(a knave of heart) = ?

Favourable number of cases (m) = number of knave of heart = 1

$$P(\text{a knave of heart}) = \frac{1}{52}$$

(vii) P(a king or queen) = ?

Favourable number of cases (m) = number of kings or queen = 4 + 4 = 8

$$P(\text{king or queen}) = \frac{8}{52} = \frac{2}{13}$$

(viii) P(Heart or club) = ?

$m = 13 + 13 = 26$

$$P(\text{Heart or club}) = \frac{26}{52} = \frac{1}{2}$$

(ix) P(a red 2 or black 8 or a queen) = ?

m = number of red 2 or black 8 or a queen

= 2 + 2 + 4 = 8

$$P(\text{a red 2 or black 8 or a queen}) = \frac{8}{52} = \frac{2}{13}$$

(x) P(a spade or ace) = ?

m = number of spade or ace

= 13 + 4 – 1 ($\because$ spade ace is common in both)

= 16

$$P(\text{a spade or ace}) = \frac{16}{52} = \frac{4}{13}$$

(xi) P(a heart or a face card) = ?

Favourable number of cases (m) = Drawing a heart or face card = 13 + 12 – 3

= 22 ($\because$3 face cards of heart are common)

$$P(\text{a heart or face card}) = \frac{22}{52} = \frac{11}{26}$$

(xii) P(a red or face card)

Favourable number of cases (m) = Drawing a red or face card = 26 + 12 – 6 ($\because$ 6 red cards are common)

$$P(\text{a red or face card}) = \frac{32}{52} = \frac{8}{13}$$

Example 6 Twenty balls are numbered from 1 to 20. If one ball is drawn at random, what is the probability that the ball drawn is multiple of 4 or 7?

Solution: Total number of cases (n) = 20

Favourable number of cases (m) = The number of cases which are multiple of 4 or 7

i.e. {4, 7, 8, 12, 14, 16, and 20}

i.e. $m = 5 + 2 = 7$

$\therefore$ Required probability that the ball drawn is multiple of 4 or 7 = $\frac{m}{n} = \frac{7}{20}$

Example 7 Two fair dice are thrown at random. What is the probability that the face turn up show (i) a sum 7 (ii) a sum of 8 or 9 (iii) sum less than 5 (iv) number 6 in the first die (v) odd number in the second die (vi) same faces (vii) different faces.

Solution: Since, two dice are thrown

$$n = \text{Total number of cases} = 6 \times 6 = 36$$

The sample spaces (total outcomes) are presented below:

		Faces in second die					
		1	2	3	4	5	6
Faces in first die	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

(i) $P(\text{a sum } 7) = ?$

m = Favourable number of outcomes

= getting a sum 7 in both dice

i.e. (1, 6) (2, 5) (3, 4) (4, 3) (5, 2) (6, 1)

$m = 6$

$$P(\text{a sum } 7) = \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

(ii) $P(\text{a sum of 8 or 9}) = ?$

m = Favourable number of outcomes

= number of cases getting a sum of 8 or 9

i.e. (2, 6) (3, 5) (4, 4) (5, 3) (6, 2) (3, 6) (4, 5) (5, 4) (6, 3)

$m = 9$

$$P(\text{a sum is 8 or 9}) = \frac{9}{36} = \frac{1}{4}$$

(iii) $P(\text{sum less than 5}) = ?$

m = Favourable number of outcomes

= Number of cases getting a sum of less than 5

i.e., (1, 1) (1, 2), (2, 1), (1, 3), (2, 2) (3, 1)

$m = 6$

$$P(\text{sum less than 5}) = \frac{6}{36} = \frac{1}{6}$$

(iv) $P(\text{number 6 in the first die}) = ?$

m = Favourable number of outcomes

= number of cases getting a 6 in 1st die i.e. (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

$m = 6$

$\therefore \quad P(6 \text{ in first die}) = \dfrac{6}{36} = \dfrac{1}{6}$

(v) P(odd number in the second die)

m = Favourable number of outcomes

= number of cases of odd number in the second die

i.e., (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (1, 3), (2, 3)
(3, 3), (4, 3), (5, 3), (6, 3), (1, 5), (2, 5), (3, 5), (4, 5) (5, 5), (6, 5)

$m = 18$

$P(\text{Odd number in the second die}) = \dfrac{18}{36} = \dfrac{1}{2}$

(vi) P(same faces)

m = Favourable number of outcomes (cases)

= number of cases of getting same faces in both dice

i.e. (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6)

$m = 6$

$P(\text{same faces in both dice}) = \dfrac{6}{36} = \dfrac{1}{6}$

(vii) P(different faces) = ?

m = Favourable number of outcomes

= Number of cases of different faces in both dice

i.e., (1, 2) (2, 1) (1, 3) (3, 1) .... , (1, 6) (6, 1)

$m = 30$

$P(\text{different faces}) = \dfrac{m}{n} = \dfrac{30}{36} = \dfrac{5}{6}$

Since, P (different faces) + P (same faces) = 1

$\therefore \quad P$ (different faces) = 1 – P(same faces)

$$= 1 - \frac{6}{36} = \frac{30}{36} = \frac{5}{6}$$

Example 8 What is the chance that a leap year selected randomly consists of 53 Sundays?

Solution: In a leap year, there are 366 days i.e. 52 complete weeks and 2 days over. These 2 days may be either (i) Sunday and Monday or (ii) Monday and Tuesday or (iii) Tuesday and Wednesday or (iv) Wednesday and Thursday or (v) Thursday and Friday or (vi) Friday and Saturday or (vii) Saturday and Sunday.

Total number of cases (n) = 7

P(53 Sundays) = ?

Favourable number of cases (m) = Number of cases consisting Sunday = 2

$\therefore$ Required probability that a leap year selected randomly consists of 53 Sundays i.e.

$$P(A) = \frac{m}{n} = \frac{2}{7}$$

Example 9 What is the chance that a non-leap year selected randomly consists of 53 Sundays?

Solution: In a non leap year, there are 365 days i.e. 52 complete weeks and 1 day over. The 1 day may be either (i) Sunday or (ii) Monday or (iii) Tuesday or (iv) Wednesday or (v) Thursday or (vi) Friday or (vii) Saturday.

Total number of cases (n) = 7

P(53 Sundays) = ?

Favourable number of cases (m) = Number of cases consisting Sunday = 1

∴ Required probability that a non-leap year selected randomly consists of 53 Sundays i.e.

$$P(A) = \frac{m}{n} = \frac{1}{7}$$

Example 10 Annual sales commission from a survey of 300 media sales persons were observed as follows:

Annual sales commission ('000' Rs.)	0-5	5-10	10-15	15-20	20-25	25 and above
No. of sales persons	15	25	35	125	70	30

What is the probability that a media sales person makes a commission (a) between Rs. 5000 to 10,000 (b) less than Rs. 15,000?

Solution: Since, total number of sales persons is 300

∴ Total number of cases (n) = 300

(a) P(a media sales person makes a commission between Rs. 5,000 to 10,000) = ?

Favourable number of cases (m) = number of media sales persons make a commission between Rs. 5000 to 10,000

∴ $m = 25$

P(a media sales person makes a commission between Rs. 5,000 to 10,000) $= \frac{m}{n} = \frac{25}{300} = \frac{1}{12}$

(b) P(a media sales person makes a commission less than Rs. 15000) = ?

Favourable number of cases (m) = number of media sales persons make a commission bless than Rs. 15000

∴ $m = 75$

P(a media sales person makes a commission less than Rs. 15000) $= \frac{m}{n} = \frac{35 + 25 + 15}{300} = \frac{75}{300} = \frac{1}{4}$

Example 11 A bag contains 8 red, 4 white and 5 black coloured balls. Three balls are drawn randomly from a bag. Find the probability that (i) all are red (ii) 2 is red and 1 white (iii) 2 are red and 1 other (iv) all colour balls.

Solution: Exhaustive (total) number of cases (n) = number of cases of selection of 3 balls out of 17 balls

$$= {}^{17}C_3 = \frac{17 \times 16 \times 15}{1 \times 2 \times 3} = 680.$$

(i) Favourable number of cases of drawing all 3 are red balls $(m) = {}^8C_3 = \frac{1 \times 7 \times 6}{1 \times 2 \times 3} = 56$

$$P(\text{all are red}) = \frac{m}{n} = \frac{56}{680} = 0.082$$

(ii) Favourable cases for 2 red and 1 white $= m = {}^8C_2 \times {}^4C_1 = \frac{8 \times 7}{1 \times 2} \times \frac{4}{1} = 112$

$$\therefore \quad P(2 \text{ is red and 1 white}) = \frac{m}{n} = \frac{112}{680} = 0.1647$$

(iii) Favourable cases for 2 are red out of three drawn balls i.e. 2 are red and 1 other

$$(m) = {}^8C_2 \times {}^9C_1 = \frac{8 \times 7}{1 \times 2} \times \frac{9}{1} = 252$$

$$\therefore \quad P\ (2 \text{ are red and 1 other}) = \frac{m}{n} = \frac{252}{680} = 0.37058$$

(iv) Favourable number of cases for all coloured balls $(m) = {}^8C_1 \times {}^4C_1 \times {}^5C_1 = 8 \times 4 \times 5 = 160$

Hence, $P(\text{all colour balls}) = \frac{m}{n} = \frac{160}{680} = \frac{4}{17} = 0.235$

Example 12 Five men in a group of 20 are graduates. If 3 are chosen out of 20 at random, what is the probability that

a) all are graduates b) none of them is graduate

c) at least one of them being graduate

Solution: Here, Total number of men = 20

Number of graduates = 5

Number of non-graduates = 20 – 5 = 15

If three are choosen at random

total number of possible outcomes $(n) = {}^{20}C_3$

$$= \frac{20!}{(20-3)!\,3!} = \frac{20!}{17!\,3!} = \frac{20 \times 19 \times 18 \times 17!}{17!\,3!} = 1140$$

(a) $P(\text{all are graduates}) = ?$

Favourable number of cases $(m) = {}^5C_3 = \frac{5!}{(5-3)!\,3!} = 10$

$$\therefore \quad P(\text{all are graduate}) = \frac{{}^5C_3}{{}^{20}C_3} = \frac{10}{1140} = \frac{1}{114}$$

(b) $P(\text{none of them are graduate}) = ?$

$\therefore$ Favourable number of cases for none of them are graduate $(m) = {}^{15}C_3 = \frac{15!}{(15-3)!\,3!} = 455$

$$\therefore \quad P(\text{none of them are graduate}) = \frac{{}^{15}C_3}{{}^{20}C_3} = \frac{455}{1140} = \frac{91}{228}$$

(ii) $P(\text{at least one of them being graduate}) = 1 - P(\text{none them are graduate})$

$$= 1 - \frac{91}{228} = \frac{228 - 91}{228} = \frac{137}{228}$$

Example 13 A class consists of 40 boys and 60 girls. If two students are chosen at random, what will be the probability that

(a) both are boys (b) both are girls (c) one boy and one girl

Solution: Number of boys in a class = 40

Number of girls in a class = 60

Total number of student = 40 + 60 = 100

If two students are choosen at random, then total number of possible outcomes (n) = ${}^{100}C_2$ = 4950

(i) P(both are boys) = ?

Favourable case for both are boys (m) = ${}^{40}C_2 = 780$

$$\therefore \quad P(\text{both are boys}) = \frac{{}^{40}C_2}{{}^{100}C_2} = \frac{780}{4950} = \frac{26}{165}$$

(ii) P(both are girls) = ?

Favourable case for both are girls (m) = ${}^{60}C_2 = 1770$

$$\therefore \quad P(\text{both are girls}) = \frac{{}^{60}C_2}{{}^{100}C_2} = \frac{1770}{4950} = \frac{177}{495} = \frac{59}{165}$$

(iii) P(one boy and one girl) = ?

Favourable number of cases for one boy and one girl (m) = ${}^{40}C_1 \times {}^{60}C_1 = 40 \times 60 = 2400$

$$\therefore \quad P(\text{one boy and one girl}) = \frac{{}^{40}C_1 \times {}^{60}C_1}{{}^{100}C_2} = \frac{2400}{4950} = \frac{48}{99} = \frac{16}{33}$$

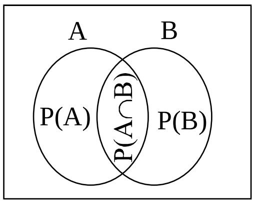
Laws of Probability

There are two important theorems on probabilitiy. They are termed as laws of probability and following are the laws of probability.

1. Additive law of probability
2. Multiplicative law of probability

Additive Law of Probability (Addition Theorem of Probability/Law of total probability)

Case I: when events are not mutually exclusive



If A and B are not mutually exclusive events, then the probability of occurrence of at least one of them is given by

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Also, we have $P(A \cup B) + P(\overline{A \cup B}) = 1$

$$P(A \cup B) = 1 - P(\overline{A \cup B})$$

By De-Morgan's Law

$$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B}) \text{ and } P(\overline{A \cup B}) = P(\bar{A} \cup \bar{B})$$

If A, B and C are three not mutually exclusive events then the occurrence of at least one of them is given by

$$\begin{aligned} P(A \text{ or } B \text{ or } C) &= P(A \cup B \cup C) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \end{aligned}$$

Also, we have

$P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C})$

Also, $P(A \cup B \cup C) + P(\overline{A \cup B \cup C}) = 1$

De-Morgan's law

$P(\overline{A \cup B \cup C}) = P(\bar{A} \cap \bar{B} \cap \bar{C})$ and $P(\overline{A \cap B \cap C}) = P(\bar{A} \cup \bar{B} \cup \bar{C})$

Again,

$P(A \cup B \cup C) \ = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

In general,

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots \cup A_n) = 1 - P(\overline{A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n})$$
$$= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cdots \cap \bar{A}_n)$$

Example 14 The probability that a new airport will get an award for its design is 0.16, the probability that it will get an award for the efficient use of materials is 0.24, and the probability that it will get both awards is 0.11.

a. What is the probability that it will get at least one of the two awards?

b. What is the probability that it will get only one of two awards?

Solution: Probability that a new airport will get award for its design, $P(A) = 0.16$

Probability that a new airport will get award for efficient, $P(B) = 0.24$

Probability that a new airport will get award for both, $P(A \cap B) = 0.11$

a. Probability that it will get at least one of the two awards,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.16 + 0.24 - 0.11 = 0.29$$

b. Probability that it will get only one of two awards.

$$= [P(A) - P(A \cap B)] \text{ or } [P(B) - P(A \cap B)]$$
$$= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)$$
$$= (0.16 - 0.11) + (0.24 - 0.11)$$
$$= 0.05 + 0.13 = 0.18$$

Example 15 A construction company is bidding for two contracts A and B. The probability that the company will get contract A is $\frac{3}{5}$, the probability that the company will get contract B is $\frac{1}{3}$ and the probability that the company will get both the contracts is $\frac{1}{8}$. What is the probability that the company will get contract A or B?

Solution: We have, $P(A) = \frac{3}{5}$, $P(B) = \frac{1}{3}$, $P(A \cap B) = \frac{1}{8}$, $P(A \text{ or } B) = ?$

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{5} + \frac{1}{3} - \frac{1}{8} = \frac{97}{120}$$

Example 16 The probability that a boy will get a scholarship is 0.9 and that a girl will get is 0.8. What is the probability that at least one of them will get the scholarship?

Solution: Let B = event of a boy getting scholarship

G = event of a girl getting scholarship

Then,

$P(B) = 0.9, \quad P(\bar{B}) = 0.1 \quad \text{and} \quad P(G) = 0.8, \quad P(\bar{G}) = 0.2$

The probability that at least one of them will get scholarship is given by

$$\begin{aligned} P(B \cup G) &= 1 - P(\overline{B \cup G}) \\ &= 1 - P(\bar{B} \cap \bar{G}) \text{ (Using De-Morgan's law)} \\ &= 1 - P(\bar{B}) \times P(\bar{G}) \text{ (Since, a boy and a girl getting scholarship are independent)} \\ &= 1 - 0.1 \times 0.2 = 1 - 0.02 = 0.98 \end{aligned}$$

Alternative method (i)

The probability that at least one of them will get scholarship is given by $P(B \cup G) = P$ (Boy get the scholarship and girl not or girl get the scholarship and boy not or both get the scholarships)

$$\begin{aligned} &= P(B \cap \bar{G} \text{ or } G \cap \bar{B} \text{ or } B \cap G) \\ &= P(B \cap \bar{G}) + P(G \cap \bar{B}) + P(B \cap G) \\ &= P(B) \times P(\bar{G}) + P(G) \times P(\bar{B}) + P(B) \times P(G) \\ &= 0.9 \times 0.2 + 0.8 \times 0.1 + 0.9 \times 0.8 \\ &= 0.18 + 0.08 + 0.72 = 0.98 \end{aligned}$$

Alternative method (ii)

The probability that at least one of them will get scholarship is given by

$$\begin{aligned} P(B \cup G) &= P(B) + P(G) - P(B \cap G) \\ &= P(B) + P(G) - P(B) \times P(G) \\ &= 0.9 + 0.8 - 0.9 \times 0.8 = 1.7 - 0.72 = 0.98 \end{aligned}$$

Example 17 A bag contains 24 balls numbered from 1 to 24. One ball is drawn at random. Find the probability that the ball drawn has a number which is multiple of 3 or 4.

Solution: Let A and B be the events of drawing a number which is multiple of 3 and 4 respectively.

Then, $A = \{3, 6, 9, 12, 15, 18, 21, 24\}$, $B = \{4, 8, 12, 16, 20, 24\}$

$$P(A) = \frac{m}{n} = \frac{8}{24}, \; P(B) = \frac{m}{n} = \frac{6}{24}$$

But 12 and 24 repeated. So, $P(A \cap B) = \dfrac{m}{n} = \dfrac{2}{24}$

$$\therefore \quad P(3 \text{ or } 4) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{8}{24} + \frac{6}{24} - \frac{2}{24} = \frac{12}{24} = \frac{1}{2}$$

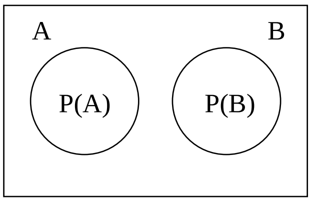
Alternatively,

Total number of cases $(n) = 24$

Favourable number of cases (m) = Number of cases of multiple of 3 or 4 = 8 + 6 – 2 = 12

$$\therefore \quad P(3 \text{ or } 4) = \frac{m}{n} = \frac{12}{24} = \frac{1}{2}$$

Case II: When the events are mutually exclusive



Let A and B be two mutually exclusive events. Then the probability of the occurrence of either event A or event B is the sum of their individual probabilities. Hence, the probability of occurrence of either event A or event B is given by

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

If A, B and C are three mutually exclusive events, then the probability of occurrence of either events A or B or C is given by

$$P(A \text{ or } B \text{ or } C) = P\,(A \cup B \cup C) = P(A) + P(B) + P(C)$$

In general, if events $A_1, A_2, \cdots, A_n$ are n mutually exclusive then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_n) + P(A_2) + P(A_3) + \cdots + P(A_n)$$

Example 18 The probability that a company execute will travel by plane is $\frac{2}{3}$ and that he will travel by train is $\frac{1}{5}$. Find the probability of travelling by plane or train.

Solution: Let A and B be the events that a company execute will travel by plane and train respectively.

$$P(A) = \frac{2}{3},\ P(B) = \frac{1}{5},\ P(\text{Plane or train}) = P\,(A \text{ or } B) = ?$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{2}{3} + \frac{1}{5} = \frac{10+3}{15} = \frac{13}{15} \end{aligned}$$

Example 19 If an experiment has the three possible and mutually exclusive outcomes A, B and C, check in each case whether the assignment of probabilities is permissible.

(a) $P(A) = 1/3,\ P(B) = 1/3,\ P(C) = 1/3$

(b) $P(A) = 0.35,\ P(B) = 0.52,\ P(C) = 0.26$

Solution: (a) $P(A) + P(B) + P(C) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

Hence, this probability is permissible since total probability = 1.

(b) $P(A) + P(B) + P(C) = 0.35 + 0.52 + 0.26 = 1.13 > 1$

Which is impossible? Hence, this probability is not permissible.

Example 20 Three events A, B and C are mutually exclusive events and their respective probabilities are as follows.

$$P(A) = 2/3;\ P(B) = 1/4;\ P(C) = 1/6.$$

Comment on the result.

Solution: If A, B and C are mutually exclusive events then

$$\begin{aligned} P(A \text{ or } B \text{ or } C) &= P\,(A \cup B \cup C) = P(A) + P(B) + P(C) \\ &= \frac{2}{3} + \frac{1}{4} + \frac{1}{6} \\ &= \frac{8+3+2}{12} = \frac{13}{12}. \end{aligned}$$

$\therefore \quad P(A \text{ or } B \text{ or } C) = \frac{13}{12} = 1.08 > 1$, which is not possible.

Hence, the given information is not correct.

Example 21 Suppose that a manager of a large apartment complex provides the following subjective probability estimates about the number of vacancies that will exist next month.

Vacancies	0	1	2	3	4	5
Probability	0.05	0.15	0.35	0.25	0.10	0.10

List the sample points in each of the following events and provide the probability of the event.

(a) no vacancies (b) at least four vacancies
(c) two or fewer vacancies

Solution: Suppose the number of vacancies from 0 to 5 is denoted by A_0 to A_5 respectively.

(a) Probability of selecting no vacancies, $P(A_0) = 0.05$

(b) Probability of selecting at least four vacancies,

$$P(A_4 \cup A_5) = P(A_4) + P(A_5) = 0.10 + 0.10 = 0.2$$

(c) Probability of selecting two or fewer vacancies,

$$P(A_2 \cup A_1 \cup A_0) = P(A_2) + P(A_1) + P(A_0) = 0.35 + 0.15 + 0.05 = 0.55$$

Example 22 A card is drawn at random from a pack of 52 cards. Find the probability of drawing (i) a jack, a queen or a king (ii) a card which is neither a jack , a queen nor a king.

Solution: Let A, B & C be the events of drawing a jack, a queen and a king respectively. Then

$$P(A) = \frac{4}{52}, P(B) = \frac{4}{52}, P(C) = \frac{4}{52}$$

$\therefore$ Probability of drawing a jack, a queen or a king

$$= P(\text{a jack, a queen or a king}) = P(A \text{ or } B \text{ or } C)$$

$$= P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{12}{52} = \frac{3}{13}$$

P(Neither a jack, a queen nor a king) = 1 – P (a jack, a queen or a king)

$$= 1 - \frac{3}{13} = \frac{10}{13}$$

Multiplicative Law of Probability (or Multiplication Theorem of Probability)

Case I: For independent events

Let A and B are two independent events, then the probability of occurrence of both the events is the product of their individual probabilities

i.e. $\quad P(A \text{ and } B) = P(A \cap B) = P(A)\, P(B)$

If A, B and C are three independent events, then

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

In general, if $A_1, A_2, \cdots, A_n$ are independent events.

Then

$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n)$

Where, $P(A \cap B)$ = Joint probability of events A and B

$P(A \cap B \cap C)$ = Joint probability of events A, B and C

$P(A_1 \cap A_2 \cap \cdots \cap A_n)$ = Joint probability of events $A_1, A_2, \cdots, A_n$

and $P(A), P(B), P(C), P(A_1), P(A_2)$ etc. are the marginal probabilities of occurrence of events A, B, C, A_1, A_2 etc. respectively.

Example 23 Two brothers: Mr. X and Mr. Y appear in an interview for getting the scholarship. The scholarship can be provided for two persons. The probability of getting scholarship by Mr. X is $\frac{1}{7}$ and getting by Mr. Y is $\frac{1}{5}$.

What is the probability that,

(a) Both of them will get scholarship.

(b) Only one of them will get scholarship.

(c) None of them will get scholarship**.**

Solution: Let $P(A)$ = Probability that Mr. X will get scholarship

$P(\bar{A})$ = Probability that Mr. X will not get scholarship

$P(B)$ = Probability that Mr. Y will get scholarship

$P(\bar{A})$ = Probability that Mr. X will not get scholarship

Given that,

$$P(A) = \frac{1}{7} \text{ and } P(\bar{A}) = 1 - \frac{1}{7} = \frac{6}{7}$$

$$P(B) = \frac{1}{5} \text{ and } P(\bar{B}) = 1 - \frac{1}{5} = \frac{4}{5}$$

(a) Probability that both of them will get scholarship,

$P(A \cap B) = P(A) \cdot P(B)$

 A and B events A and B are independent

$$= \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$$

(b) Probability that only one of them will get scholarship

$$= \text{P } (\text{A} \cap \bar{B} \text{ } or \text{ } \bar{A} \cap \text{B})$$

$$= \text{P } (\text{A} \cap \bar{B}) + \text{P } (\bar{A} \cap \text{B})$$

$$= P(A) \cdot P(\bar{B}) + \text{P}(\bar{A}) \cdot P(B)$$

$$= \frac{1}{7} \times \frac{4}{5} + \frac{6}{7} \times \frac{1}{5} = \frac{4}{35} + \frac{6}{35} = \frac{10}{35}$$

(c) Probability that non of them will get scholarship

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$$

$$= \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$$

Example 24 Probability that a man will be alive 25 years hence is 0.3 and the probability that his wife will be alive 25 years hence is 0.4. Find the probability that 25 years hence (i) both will be alive (ii) only the man will be alive (iii) only the woman will be alive(iv) none will be alive (v) at least one of them will be alive.

Solution: Given,

Probability that a man will be alive 25 years, hence i.e.

$$P(A) = 0.3 \text{ and then, } P(\bar{A}) = 1 - 0.3 = 0.7$$

Similarly, probability that his wife will be alive 25 years, hence, i.e.

$$P(B) = 0.4, \ P(\bar{B}) = 1 - 0.4 = 0.6$$

(i) Required probability that both will be alive 25 years

$$P(A \cap B) = P(A) \times P(B) = 0.3 \times 0.4 = 0.12$$

(ii) Probability that only the man will be alive

$$= P(A \cap \bar{B}) = \mathrm{P(A)} \times P(\bar{B}) = 0.3 \times 0.6 = 0.18$$

(iii) Probability that the only woman will be alive

$$= P(B \cap \bar{A}) = P(B) \times (P\bar{A}) = 0.4 \times 0.7 = 0.28$$

The probability that non of them will alive $= P(\overline{A \cup B}) = P(\bar{A}) \cdot P(\bar{B}) = 0.7 \times 0.6 = 0.42$

Required probability that at least one of them will be alive

$$= P(A \cap \bar{B} \text{ or } \bar{A} \cap B \text{ or } A \cap B)$$

$$= [P(A) \times P(\bar{B})] + [P(\bar{A}) \times P(B)] + [P(A) \times P(B)]$$

$$= 0.3 \times 0.6 + 0.7 \times 0.4 + 0.3 \times 0.4 = 0.18 + 0.28 + 0.12 = 0.58$$

Alternatively, (iv) can be calculated as follows.

$$P(A \cup B) = 1 - \mathrm{P}\,(\bar{A})\ \mathrm{P}(\bar{B}) = 1 - 0.7 \times 0.6$$

$$= 1 - 0.42 = 0.58$$

Example 25 A problem in statistics is given to three students *A*, *B*, and *C* whose chances of solving it are $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ respectively. Find the probability that

(a) The problem will be solved.

(b) Only one of them can solve the problem.

(c) None of them will solve the problem

(d) *A* solves it but *B* and *C* cannot.

(e) All three students *A*, *B* and *C* can solve the problem**.**

Solution: Given,

Probability that *A* solves a problem i.e. $P\,(A) = \frac{1}{3}$ and $P(\bar{A}) = 1 - \frac{1}{3} = \frac{2}{3}$

Probability that *B* solves a problem i.e. $P(B) = \frac{1}{4}$ and $P(\bar{B}) = 1 - \frac{1}{4} = \frac{3}{4}$

Probability that *C* solves a problem i.e. $P(C) = \frac{1}{5}$ and $P(\bar{C}) = 1 - \frac{1}{5} = \frac{4}{5}$

Required probability that the problem will be solved is given by

$$\mathrm{P}(A \cup B \cup C) = P(A) + P\,(B) + P(C) - P\,(A \cap B) - P\,(B \cap C) - P\,(C \cap A) + \mathrm{P}\,(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})$$

$$= P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) - P(C) \cdot P(A) + P(A) \cdot P(B) \cdot P(C)$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{3} \times \frac{1}{4} - \frac{1}{4} \times \frac{1}{5} - \frac{1}{5} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} = \frac{3}{5}$$

[Since, A, B and C events A and are independent]

Alternatively,

Required probability that the problem will be solved

$$= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$

$$= 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}$$

$$= 1 - \frac{2}{5} = \frac{3}{5}$$

(b) Probability that only one of them can solve the problem.

$$= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{C} \cap C)$$

$$= P(A) \cdot P(\bar{B}) \cdot P(\bar{C})\ P(\bar{A}) \cdot P(B) \cdot P(\bar{C}) + P(\bar{A}) \cdot P(\bar{B}) \cdot P(C)$$

$$= \frac{1}{3} \times \frac{3}{4} \times \frac{4}{5} + \frac{2}{3} \times \frac{1}{4} \times \frac{4}{5} + \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5}$$

$$= \frac{12}{60} + \frac{8}{60} + \frac{6}{60} = \frac{26}{60} = 0.433$$

(c) Probability that non of them will solve the problem,

$$P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}) = \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{2}{5} = 0.4$$

(d) Probability that A solves it but B and C cannot,

$$P(A \cap \bar{B} \cap \bar{C}) = \mathrm{P(A)} \times \mathrm{P}(\bar{B}) \times \mathrm{P}(\bar{C}) = \frac{1}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5} = 0.2$$

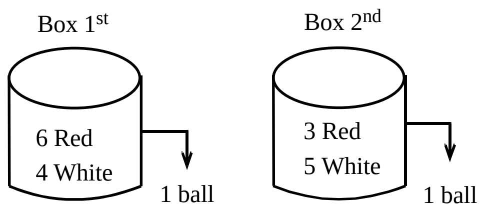
(e) Probability that all three students can solve the problem,

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C) = \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} = \frac{1}{60} = 0.017.$$

Example 26 A box contains 6 red and 4 white balls. Another box contains 3 red and 5 white balls. One ball is drawn at random from each bag, find the probability that (i) both balls are red; (ii) one is red and one is white; (iii) same colors.

Solution: Total number of balls in 1st box = 6 + 4 = 10

Total number of balls in 2nd box = 3 + 5 = 8



Let,

$$P(R_1) = \text{Probability of getting a red ball from 1}^{\text{st}} \text{ box} = \frac{6}{10}$$

$P(W_1)$ = Probability of getting a white ball from 1st box = $\frac{4}{10}$

$P(R_2)$ = Probability of getting a red ball from 2nd box = $\frac{3}{8}$

$P(W_2)$ = Probability of getting a white ball from 2nd box = $\frac{5}{8}$

Now,

(i) The probability of getting both red balls is given by,

$$P(\text{both red balls}) = P(R_1 \cap R_2) = P(R_1) \times P(R_2) = \frac{6}{10} \times \frac{3}{8} = \frac{9}{40}$$

(ii) The probability of one is red and other is white is given by,

P(one red and one white) = p(red ball from 1st box and white from 2nd box or white ball from 1st box and red from 2nd box)

$$= P(R_1 \cap W_2 \quad \text{or,} \quad W_1 \cap R_2)$$
$$= P(R_1 \cap W_2) + P(W_1 \cap R_2)$$
$$= P(R_1) \times P(W_2) + P(W_1) \times P(R_2)$$
$$= \frac{6}{10} \times \frac{5}{8} \times \frac{4}{10} \times \frac{3}{8} = \frac{21}{40}$$

The probability of different colors means probability of getting ball of each color. In above example probability of different colors is given by,

P(different colors) = P(One red and one white)

(iii) The probability of getting balls of same color is given by,

$$P(\text{same colors}) = P(\text{both red or both white})$$
$$= P(\text{both red}) + P\ (\text{both white})$$
$$= P(R_1 \cap R_2) + P(W_1 \cap W_2)$$
$$= P(R_1) \times P(R_2) + P(W_1) \times P(W_2)$$
$$= \frac{6}{10} \times \frac{3}{8} + \frac{3}{8} + \frac{4}{10} \times \frac{5}{8} = \frac{19}{40}$$

Example 27 The odds in favour that A speaks the truth are 3: 2 and the odds in favour that B speaks the truth are 5:3. In what percentage of cases are they likely to contradict and do not contradict each other on an identical point?

Solution: Given,

Probability that A speaks the truth = $P(A) = \frac{3}{3+2} = \frac{3}{5}$

Probability that A does not speak truth = $P(\bar{A}) = \frac{2}{3+2} = \frac{2}{5}$

Probability that B speaks the truth = $P(B) = \frac{5}{5+3} = \frac{5}{8}$

Probability that B does not speak truth = $P(\bar{B}) = \frac{3}{5+3} = \frac{3}{8}$

Required probability that they are likely to contradict each other on an identical point

$$= P(A \cap \bar{B} \text{ or } \bar{A} \cap \bar{B})$$

$$= P(A) \times P(\bar{B}) + P(\bar{A}) \times P(B)$$

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40}$$

Required percentage of contradiction $= \frac{19}{40} \times 100 = 47.5\%$

Required probability that they do not contradict each other on an identical point

$$= P(A \cap B \text{ or } \bar{A} \cap \bar{B})$$

$$= \mathrm{P(A)\,.\,P(B) + P(\bar{A})\,.\,P(\bar{B})} \qquad [\because A \text{ and } B \text{ are independent}]$$

$$= \frac{3}{5} \times \frac{5}{8} + \frac{2}{5} \times \frac{3}{8} = \frac{21}{40} = \frac{21}{40} \times 100 = 52.5\%$$

Alternatively,

Required probability that they do not contradict each other on an identical point

$$= 1 - \frac{19}{40} = \frac{21}{40} = \frac{21}{40} \times 100 = 52.5\%.$$

Case II: For dependent events

If A and B are two dependent events, then the probability of simultaneous happening of two events A and B is given by

$$P(A \cap B) = P(A)\, P\left(\frac{B}{A}\right)$$

Similarly, $P(A \cap B) = P(B)\, P\left(\frac{A}{B}\right)$

Where,

$P(B/A)$ is the conditional probability of the occurrence of event B given that (if) event A has already occurred & $P(A/B)$ is the conditional Probability of occurrence (happening) of event A given that event B has already occurred (happened).

If A, B and C are three dependent events, then

$$P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B)$$

Where, $P(C/A \cap B)$ is the conditional probability of occurrence (happening) of event C given that (if) both events A and B have already occurred (happened).

In general, for n events $A_1, A_2, A_3, A_4, \cdots, A_n$

We have,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \cdots P(A_n/A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{n-1})$$

Note: (i) If events A and B are independent, then $P(A/B) = P(A)$ and $P(B/A) = P(B)$

(ii) $P(A \cap B)$ can also be denoted by $P(AB)$

Conditional Probability

Conditional probability is the probability that an event will occur given that another event has already occurred. If A and B are two dependent events, then the conditional probability of event A given that (if) event B has already occurred is given by,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) > 0$$

Similarly, the conditional probability of event B given that (if) event A has already occurred is given by

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0$$

Example 28 The probability that a manufacturer will produce 'brand X' product is 0.13, the probability that he will produce 'brand Y' product is 0.28 and the probability that he will produce both brand is 0.06. What is the probability that the manufacturer who has produced 'brand Y' will also have produced 'brand X'?

Solution: Let X and Y be the events that a manufacturer will produce brand X and brand Y respectively.

$$P(X) = 0.13, P(Y) = 0.28, P(X \cap Y) = 0.06, \text{P}\left(\frac{X}{Y}\right)$$

$$\text{P}\left(\frac{\text{X}}{\text{Y}}\right) = \frac{P(X \cap Y)}{P(Y)} = \frac{0.06}{0.28} = 0.214$$

∴ The probability that the manufacturer who has produced 'brand Y' will also have produced 'brand X' is 0.214

Example 29 In a certain school, 20% students failed in English, 15% students failed in Mathematics and 10% of students failed in both English and Mathematics. A student is selected at random. If he failed in English, what is the probability that he also failed in Mathematics.

Solution: Let E and M be the events that denote the students failed in English and failed in Mathematics respectively. Then

$$P(E) = 20\% = 0.2, \quad P(M) = 15\% = 0.15$$

$$P(E \cap M) = 10\% = 0.10$$

If one student is selected at random, probability that if he failed in English then he also failed in Mathematics is given by

$$P(M/E) = \frac{P(M \cap E)}{P(E)} = \frac{0.10}{0.2} = \frac{1}{2}$$

Example 30 What is the probability that a couple's second child will be

(a) a boy, given that their first child was a girl. (b) a girl, given that their first child was a girl.

Solution: Let B_1 and B_2 be the events that denote the first child is boy and second child is also boy respectively.

Similarly, G_1 and G_2 be the events that denote that first child is girl and second child is also girl respectively. There are only two possibilities; either boy child or girl child. Then

Probability of first boy child, $P(B_1) = 1/2$

Probability of first girl child, $P(G_1) = 1/2$

(a) Probability of being second child a boy given that their first child was a girl

$$P(B_2/G_1) = \frac{P(B_1 \cap B_2)}{P(G_1)}$$

$$= \frac{P(B_2)\,P(G_1)}{P(G_1)} \quad [\because B_2 \text{ and } G_1 \text{ are independent}]$$

$$= \frac{1}{2}$$

(b) Similarly, the probability of being second child a girl given that their first child was a girl,

8R
6W

$$P(G_2/G_1) = \frac{P(G_2 \cap G_1)}{P(G_1)} = \frac{P(G_2)\,P(G_1)}{P(G_1)} = \frac{1}{2}$$

[∵ First birth and second birth are independent.]

Example 31 A bag contains 8 red and 6 white balls. Two balls are drawn randomly from the bag one after other without replacement. Find the probability that both balls are white.

Solution: Probability of drawing a white ball in the first draw = $P(W_1) = \frac{6}{14}$

Probability of drawing a white ball in the second draw given that first drawn ball is white = $P\left(\frac{W_2}{W_1}\right) = \frac{5}{13}$.

Since, drawn is made without replacement. (i.e. dependent case)

∴ Required probability of drawing both white balls,

$$P(W_1 \cap W_2) = P(W_1) \cdot P\left(\frac{W_2}{W_1}\right) = \frac{6}{14} \times \frac{5}{13} = 0.1648$$

Example 32 Find the probability of drawing an ace, a king and a queen in that order from a pack of cards in three consecutive draws, cards drawing not being replaced.

Solution: Let A, K and Q denote the event of drawing an ace, a king and a queen respectively.

The probability of drawing an ace, a king and a queen in that order in three consecutive draws, when the drawn cards is not replaced (without replacement) in the pack of cards is given by

$$P(A \cap K \cap Q) = P(A) \times P(K/A) \times P(Q/A \cap K)$$

Where, $P(A)$ = Probability of drawing an ace card = $\frac{4}{52}$

$P(K/A)$ = Probability of drawing a king card given that an ace has been already drawn = $\frac{4}{51}$

$P(Q/A \cap K)$ = Probability of drawing a queen card given that an ace and a king have been already drawn = $\frac{4}{50}$

$$\therefore \quad P(A \cap K \cap Q) = P(A) \times P(K/A) \times P(Q/A \cap K) = \frac{4}{52} \times \frac{5}{51} \times \frac{4}{50} = 5/8788$$

Example 33 In an opinion poll about the economic program of the political parties following results were found.

Total respondents	1000
In favour of party *A*	445
In favour of party *B*	455
In favour of both *A and B*	50
Not in favour either	50

In the light of above facts, which party do you thing have highest chance to win the incoming election on the basis of economic agenda.

Solution: Calculation of probabilities for the given result.

	No. of polls	**Probability**
In favour of party A	445	445/1000 = 0.445
In favour of party B	455	455/1000 = 0.455
In favour of both AB	50	50/1000 = 0.05
Not in favour of either	50	50/1000 = 0.05
Total responds	1000	1

Here, it is seen that the highest probability of getting in favour for party *B* which is slightly higher than that for party *A*. Hence, party *B* would have the highest chance to win incoming election on the basis of economic agenda.

Example 34 If *A* and *B* are two events such that $P(A) = 0.4$, $P(B) = 0.6$ and $P(B/A) = 0.5$, find $P(A/B)$ and $P(A \cup B)$.

Solution:

$$P(A) = 0.4,\ P(B) = 0.6$$

$$P(B/A) = 0.5$$

We have,
$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

or,
$$0.5 = \frac{P(A \cap B)}{0.4}$$

$$\therefore \quad P(A \cap B) = 0.5 \times 0.4 = 0.2$$

Now,
$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.6} = 0.334$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.6 - 0.2 = 0.8.$$

Example 35 A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are (i) white, (ii) black, (iii) different colors (or one white and one black) (iv) same colors.

Solution:

(i) Let

W_1 = Event of drawing white ball in the first draw

W_2 = Event of drawing white ball in the second draw

5 White
3 Black

Then,

The probability of drawing a white ball in the first draw is

$$P(W_1) = \frac{5}{8}$$

The probability of drawing a white ball in the second draw given that the first ball drawn is white is

$$P(W_2/W_1) = \frac{4}{7}$$

Thus, the probability that both balls drawn are white is given by

$$P(W_1 \cap W_2) = P(W_1) \times P(W_2/W_1) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$$

(Since, drawn is made without replacement i.e. W_1 & W_2 are dependent events)

(ii) Let

B_1 = Event of drawing black ball in the first draw

B_2 = Event of drawing black ball in the second draw

Then,

The probability of drawing a black ball in the first draw is

$$P(B_1) = \frac{3}{8}$$

The probability of drawing a black ball in the second draw given that the first ball drawn is black is

$$P(B_2/B_1) = \frac{2}{7}$$

The probability that both balls drawn are black is given by

$$P(B_1 \cap B_2) = P(B_1) \times P(B_2/B_1) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$$

(Since, drawn is made without replacement i.e. W_1 & W_2 are dependent events)

(iii) The probability of different colors is given by

P(different colors) $= P$ (one white and one black)

$= P$ (first drawn ball is white and second drawn ball is black or first drawn ball is black and second drawn ball is white)

$$= P(W_1 \cap B_2 \text{ or, } B_1 \cap W_2)$$

$$= P(W_1 \cap B_2) + P(B_1 \cap W_2)$$

$$= P(W_1) \times P(B_2/W_1) + P(B_1) \times P(W_2/B_1)$$

$$= \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7} = \frac{30}{56}$$

The probability of same colors is given by

P(same colors)

$= P$ (both are white or both are black)

$= P$ (both are white) $+ P$ (both are black)

$= P\ (W_1 \cap W_2) + P(B_1 \cap B_2)$

$= P(W_1) \times P(W_2/W_1) + P(B_1) \times P(B_2/B_1)$

$$= \frac{5}{8} \times \frac{4}{7} + \frac{3}{8} \times \frac{2}{7} = \frac{26}{56}$$

Example 36 A bag contains 5 white and 8 red balls. Two drawing of 3 balls are made such that (i) the balls are replaced before the second trial and (ii) the balls are not replaced before the second trial. Find the probability that the first drawing will give 3 white and the second 3 red balls in each case.

Solution: Total number of balls = 5 + 8 = 13 balls

Let A and B denote the events of drawing 3 white balls is the first draw and 3 red balls in the second draw respectively.

(i) **With replacement:** If the balls drawn in the first draw are replaced back in the bag before the second draw, then events A and B are independent. Then the probability of drawing 3 white balls in the first draw and 3 red balls in the second draw is given by

$$P(A \cap B) = P(A) \times P(B) = \frac{^5C_3}{^{13}C_3} \times \frac{^8C_3}{^{13}C_3} = \frac{10}{286} \times \frac{56}{286} = 0.0068$$

(ii) **Without replacement :** If the balls drawn in the first draw are not replaced back in the bag before the second draw, then events A and B are dependent. The probability of drawing 3 white balls in the first draw and 3 red balls in the second draw is given by

$$P(A \cap B) = P(A) \times P(B/A) \qquad \text{... (I)}$$

Where,

$P(A)$ = Probability of drawing 3 white balls in the first draw $= \frac{^5C_3}{^{13}C_3}$

Now, if 3 white balls drawn in the first draw are not replaced back in the bag, there are 13 – 3 = 10 balls left in which 5 – 3 = 2 are white and 8 are red balls.

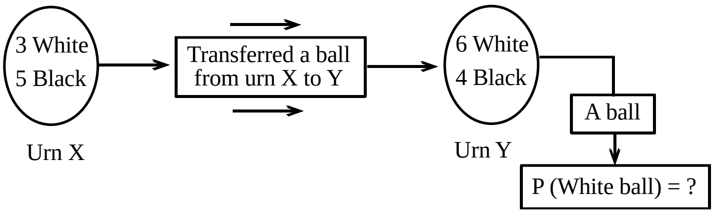
$P(B/A)$ = Probability of drawing 3 red balls from the bag containing 2 white and 8 red balls

$$= \frac{^8C_3}{^{10}C_3}$$

$$\therefore \quad P(A \cap B) = P(A)\ P(B/A) = \frac{^5C_3}{^{13}C_3} \times \frac{^8C_3}{^{10}C_3} = \frac{10}{286} \times \frac{56}{120} = 0.0163$$

Example 37 An urn X contains 3 white balls and 5 black balls. Another urn Y contains 6 white balls and 4 black balls. A ball is transferred from the urn X to urn Y and then a ball is taken from urn Y. Find the probability that it will be a white ball.

Solution:



Let W_1 and B_1 be the events drawing a white ball and a black ball from urn X respectively.

Then,

$$P(W_1) = \frac{3}{3+5} = \frac{3}{8},\ P(B_1) = \frac{5}{3+5} = \frac{5}{8}$$

Let, W be the event of drawing a white ball from the urn Y after transferring a ball from urn X to urn Y.

Probability of drawing white ball from urn Y given that a white ball is transferred from urn X to urn Y is

$$P(W/W_1) = \frac{6+1}{(6+1)+4} = \frac{7}{11}$$

Probability of drawing white ball from urn Y given that a black ball is transferred from urn X to urn Y is

$$P(W/B_1) = \frac{6}{6+(4+1)} = \frac{6}{11}$$

Hence, probability of drawing a white ball from urn Y after transferring a ball from urn X to urn Y is given by

$$\begin{aligned} P(W) &= P(W_1 \cap W) + P(B_1 \cap W) \\ &= P(W_1) \times P(W/W_1) + P(B_1) \times P(W/B_1) = \frac{3}{8} \times \frac{7}{11} + \frac{5}{8} \times \frac{6}{11} = \frac{51}{88} \end{aligned}$$

Probability Distribution

Probability distribution is the distribution of probability with certain condition, mathematical expression and logical consideration. There are two types of probability distribution; discrete probability distribution and continuous probability distribution.

Binomial Distribution

Binomial distribution is widely used probability distribution of discrete random variable. It was introduced in 1705 by James Bernoulli. The binomial distribution describes the possible number of time that a particular event will occur in a sequence of distribution which have four conditions:

Condition of Binomial Distribution:

1. The number of trial should be fixed and finite; given and number of expression of random variable greater than the trial.
2. There is dichotomous case (yes, no) (passed, failed), defective and non-defective, good and bad) $[p + q = 1]$
3. Probability is fixed, finite i.e., probability of success in each trial constant and does n ot change from to trial.
4. Random variables are independent to each other.

Definition of Binomial Function

A discrete random distribution takes the probability mass function;

$$\text{Probability } [P(r)] = {}^nC_r\, p^r\, q^{n-r} = \frac{n!}{(n-r)!r!}\, p^r\, q^{n-r}$$

Example 38 (i) Four unbiased coins are tossed simultaneously what is the probability of getting (i) 2 heads and 2 tails (ii) At least two heads and (iii) at least one heads.

Solution: Number of coins = n = 10 (Unbiased coins)

Probability of getting = $(P) = \frac{1}{2}$

Probability of getting tail in each coin = $(q) = \frac{1}{2}$

Let r be the number of heads

Then probability of getting r heads is

$$P(X = 2) = {}^nC_r\, p^r\, q^{n-r} = {}^4C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{4-r}$$

If $r = 2$ then Head = 2, tail = 2. For four

$$\Rightarrow \quad P(X = 2) = {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$$

$$\Rightarrow \quad P(X = 2) = \frac{4!}{(4-2)!2!} \left(\frac{1}{2}\right)^4$$

$$\Rightarrow \quad P(X = 2) = \frac{4 \times 3 \times 2 \times 1}{2 \times 1\ \ 2 \times 1} \left(\frac{1}{2}\right)^4 = \frac{16}{16} = \frac{3}{8} = 0.375$$

ii) $P(X \geq 2)$ = At least two head

$$= 1 - [P(0) + P(1)]$$

$$= 1 - \left[{}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0} + {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1} \right]$$

$$= 1 - \left(\frac{1}{2}\right)^4 (1 + 4)$$

$$= 1 - \frac{5}{16} = \frac{11}{16} = 0$$

iii) $P(X \geq 1) = 1 - P(0) = 1 - {}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0}$

$$= 1 - \left[\frac{4!}{4\,!0!}\left(\frac{1}{2}\right)^4\right] = 1 - \frac{1}{16} = \frac{15}{16}$$

Characteristics of Binomial Distribution

1. Binomial distribution have two parameter n and p. If we have n and p then we can introduce probabilities of the random variable.
2. In Binomial distribution, mean is always greater than variance.
3. Binomial distribution is symmetrical if $p = 1/2$.
4. Binomial distribution have positive skewness if $p < 1/2$. Binomial distribution is negatively skewed if $p > 1/2$.
5. In Binomial distribution, mean = np, variance = npq.
6. Expected frequency of binomial distribution

$$E = NP(r) = N.\ {}^nC_r\, p^r\, q^{n-r}$$

Fitting of Binomial Distribution

The process of finding the expected or theoretical frequencies on the basis of available information supplied by observed or experimental frequency is called fitting of a distribution.

Thus, the process of finding the expected or theoretical frequencies for a binomial distribution is called fitting of binomial distribution.

Suppose a random experiment consists of 'n' trails satisfying the conditions of binomial distribution repeated N times, then the expected frequency of getting exactly 'r' successes is given by

$$f(r) = N \times P(X = r) = N \times {}^nC_r\, p^r\, q^{n-r}$$

The expected frequencies for different values of $r = 0, 1, 2, \cdots, n$ can be calculated as,

No. of success ($X = r$)	$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$	$f(r) = N \times P(X = r)$
0	$P(X = 0) = {}^0C_0\, p^0\, q^{n-0}$	$f(0) = N \times P(X = 0)$
1	$P(X = 1) = {}^nC_1\, p^1\, q^{n-1}$	$f(1) = N \times P(X = 1)$
2	$P(X = 2) = {}^nC_2\, p^2\, q^{n-2}$	$f(2) = N \times P(X = 2)$
⋮	⋮	⋮
n	$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$	$f(\mathrm{r}) = N \times P(X = r)$

If the probability of success 'p' is not known, then first of all we calculate the mean of the given frequency distribution as $\bar{X} = \frac{\Sigma fx}{N}$ and equating it to the mean of the Binomial distribution np. That is, $np = \bar{X}$. Thus, the probability of success 'p' is estimated as $p = \frac{\bar{X}}{n}$ and then probability of failure $(q) = 1 - p$.

Example 39 Five coins are tossed 3200 times, find the expected frequencies of the distribution of heads and tails and tabulate the results. Calculate the mean number of success and standard deviation.

Solution: Given $(n) = 5$

Total number of trials $(N) = 3200$

Probability of getting or success $(p) = 0.5$

Probability of getting tail or failure $(q) = 1 - 0.5 = 0.5$

The expected or theoretical of getting 'r' heads is denoted by $f(r)$ and is given by

$$\begin{aligned} f(r) &= N \times P(X = r) \\ &= N \times {}^nC_r\, p^r\, q^{n-r};\ r = 0, 1, 2, \cdots, n. \\ &= 3200 \times {}^5C_r \times (0.5)^r \times (0.5)^{5-r};\ r = 0, 1, 2, \cdots, 5. \end{aligned}$$

Now,

When $r = 0$, $f(0) = 3200 \times {}^5C_0 \times (0.5)^0 \times (0.5)^{5-0} = 100$

When $r = 1$, $f(1) = 3200 \times {}^5C_1 \times (0.5)^1 \times (0.5)^{5-1} = 500$

When $r = 2$, $f(2) = 3200 \times {}^5C_2 \times (0.5)^2 \times (0.5)^{5-2} = 1000$

When $r = 3$, $f(3) = 3200 \times {}^5C_3 \times (0.5)^3 \times (0.5)^{5-3} = 1000$

When $r = 4$, $f(4) = 3200 \times {}^5C_4 \times (0.5)^4 \times (0.5)^{5-4} = 500$

When $r = 5$, $f(5) = 3200 \times {}^5C_5 \times (0.5)^5 \times (0.5)^{5-5} = 100$

Hence, the expected frequency distribution of heads and tails is

No. of heads (X)	0	1	2	3	4	5
Expected frequency	100	500	1000	1000	500	100

Again,

Mean number of success = $np = 5 \times 0.5 = 2.5$

Standard deviation = $\sqrt{npq} = \sqrt{5 \times 0.5 \times 0.5} = 1.118$

Example 40 Seven coins are tossed and the number of heads noted. The experiment is repeated 128 times and the following distribution is obtained:

No. of heads	0	1	2	3	4	5	6	7
Frequencies	7	6	19	35	30	23	7	1

Fit a Binomial distribution assuming

i) The coins is unbiased

ii) The nature of the coin is not known

Solution:

i) When the coin is unbiased:

Here,

Number of coins (n) = 7

Probability of getting head or success (p) = 0.5

Probability of getting tail or failure (q) = 1 – 0 .5 = 0.5

The expected or theoretical frequency of getting 'r' heads is denoted by

$$\begin{aligned} f(r) &= N \times P(X = r) \\ &= N \times {}^{n}C_{r}\, p^{r}\, q^{n-r};\ r = 0, 1, 2, \cdots, n. \\ &= 128 \times {}^{7}C_{r}\,(0.5)^{r} \times (0.5)^{7-r}\ ;\ r = 0, 1, 2, \cdots, 7. \end{aligned}$$

Now,

When $r = 0$, $f(0) = 128 \times {}^{7}C_{0} \times (0.5)^{0} \times (0.5)^{7-0} = 1$

When $r = 1$, $f(1) = 128 \times {}^{7}C_{1} \times (0.5)^{1} \times (0.5)^{7-1} = 7$

When $r = 2$, $f(2) = 128 \times {}^{7}C_{2} \times (0.5)^{2} \times (0.5)^{7-2} = 21$

When $r = 3$, $f(3) = 128 \times {}^{7}C_{3} \times (0.5)^{3} \times (0.5)^{7-3} = 35$

When $r = 4$, $f(4) = 128 \times {}^{7}C_{4} \times (0.5)^{4} \times (0.5)^{7-4} = 35$

When $r = 5$, $f(5) = 128 \times {}^{7}C_{5} \times (0.5)^{5} \times (0.5)^{7-5} = 21$

When $r = 6$, $f(6) = 128 \times {}^{7}C_{6} \times (0.5)^{6} \times (0.5)^{7-6} = 7$

When $r = 7$, $f(7) = 128 \times {}^{7}C_{7} \times (0.5)^{7} \times (0.5)^{7-7} = 1$

Hence, the expected frequency distribution of heads and tails is

No. of heads (X)	0	1	2	3	4	5	6	7
Expected frequencies	1	7	21	35	35	21	7	1

ii) When the nature of the coin is not known:

Here,

Number of trails (N) = 128

Calculate of mean of the distribution

No. of heads (X)	Frequency (f)	fX
0	7	0
1	6	6
2	19	38
3	35	105
4	30	120
5	23	115
6	7	42
7	1	7
	$N = 128$	$\Sigma fx = 433$

We have,

$$\text{Mean } (\bar{X}) = \frac{\Sigma fX}{N} = \frac{433}{128} = 3.3828$$

$$\text{Now, } np = \bar{X} = \frac{\bar{X}}{n} = \frac{3.3828}{7} = 0.48$$

$\therefore$ Probability of getting head or success (p) = 0.48

Probability of getting tail or failure (q) = 1 – 0.48 = 0.52

The expected or theoretical frequency of getting 'r' heads is denoted by $f(r)$ and is given by

$$\begin{aligned} f(r) &= N \times P(X = r) \\ &= N \times {}^nC_r\, p^r\, q^{n-r}\ ;\ r = 0, 1, 2, \cdots, n. \\ &= 128 \times {}^7C_r\, (0.48)^r \times (0.52)^{7-r};\ r = 0, 1, 2, \cdots, 7. \end{aligned}$$

Now,

When $r = 0$, $f(0) = 128 \times {}^7C_0 \times (0.52)^0 \times (0.5)^{7-0} = 1$

When $r = 1$, $f(1) = 128 \times {}^7C_1 \times (0.52)^1 \times (0.5)^{7-1} = 7$

When $r = 2$, $f(2) = 128 \times {}^7C_2 \times (0.52)^2 \times (0.5)^{7-2} = 21$

When $r = 3$, $f(3) = 128 \times {}^7C_3 \times (0.52)^3 \times (0.5)^{7-3} = 35$

When $r = 4$, $f(4) = 128 \times {}^7C_4 \times (0.52)^4 \times (0.5)^{7-4} = 35$

When $r = 5$, $f(5) = 128 \times {}^7C_5 \times (0.52)^5 \times (0.5)^{7-5} = 21$

When $r = 6$, $f(6) = 128 \times {}^7C_6 \times (0.52)^6 \times (0.5)^{7-6} = 7$

When $r = 7$, $f(7) = 128 \times {}^7C_7 \times (0.52)^7 \times (0.5)^{7-7} = 1$

Hence, the expected frequency distribution of heads and tails is

No. of defective (X)	0	1	2	3	4	5	6	7
Observed Frequency	7	6	19	35	30	23	7	1
Expected Frequency	1	7	21	35	35	21	7	1

Again,

Mean and variance of the number of defective in the sample is

Mean = $n \times p = 7 \times 0.5 = 3.5$

Variance = $n \times p \times q = 7 \times 0.5 \times 0.5 = 1.75$

Example 41 If hens of a certain breed normally lay eggs on 5 days a week in an average, find how many days during a season of 100 days a poultry keeper with 5 hens of this breed will expect to receive (a) no eggs (b) exactly 2 eggs (c) at least 4 eggs (d) at most 1 eggs?

Solution: Here,

Number of hens (n) = 5

Total number of days (N) = 100

Probability of laying eggs or failure (q) = $1 - \frac{5}{7} = \frac{2}{7}$

Let, X denote the number eggs, then using binomial distribution, the probability of exactly 'r' eggs is given by

$$\begin{aligned} P(X = r) &= {}^nC_r\, p^r\, q^{n-r}\ ;\ r = 0, 1, 2, 3, \cdots, n \\ &= {}^5C_r \left(\frac{5}{7}\right)^r \left(\frac{2}{7}\right)^{5-r};\ r = 0, 1, 2, \cdots, 5. \end{aligned}$$

i) The probability of receiving no eggs is

$$= {}^5C_0 \left(\frac{5}{7}\right)^0 \left(\frac{2}{7}\right)^{5-0} = 0.0019$$

The expected number of days for no eggs out of 100 days is

$$f(0) = N \times P(X = 0) = 100 \times 0.0019 = 0.19 \approx 0$$

ii) The probability of receiving exactly 2 eggs is

$$P(X = 2) = {}^5C_2 \left(\frac{5}{7}\right)^2 \left(\frac{2}{7}\right)^{5-2} = 0.1190$$

The expected number of days for exactly 2 eggs out of 100 days is

$$f(2) = N \times P(X = 2) = 100 \times 0.1190 = 11.9 \approx 12$$

iii) The probability of receiving at least 4 eggs is

$$f(X \geq 4) = P(X = 4) + P(X = 5)$$

$$= {}^5C_4 \left(\frac{5}{7}\right)^4 \left(\frac{2}{7}\right)^{5-4} + {}^5C_5 \left(\frac{5}{7}\right)^5 \left(\frac{2}{7}\right)^{5-5}$$

$$= 0.3719 + 0.1859 = 0.5578$$

The expected number of days for at most 1 egg out of 1000 days is

$$f(X \leq 1) = N \times \mathrm{P(X \leq 1)} = 100 \times 0.0257 = 2.57 \approx 3.$$

Example 42 The probability of a bomb hitting a target is 1/5. Two bombs are enough to destroy a bridge. If 6 bombs are aimed to the bridge. Find the probability 6 that the bridge is destroyed.

Solution:

$$P = \frac{1}{5}, q = \frac{4}{5} \quad (\text{ since, } p + q = 1)$$

$$n = 6$$

$$P(X = x) = P(x) = {}^nC_x\, p^x\, q^{n-x}$$

$$= {}^6C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}$$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X = 0) + P(x = 1)]$$

$$= 1 - (0.262144 - 0.3932116)$$

$$= 1 - 0.65536 = 0.345$$

Poisson Distribution

Introduction

A Poisson distribution is a tool that helps to predict the probability of certain events from happening when you know how often the event has occurred. The Poisson distribution gives us the probability of a given number of events happening in a fixed interval of time. It was introduced by French Mathematician **Simeon Denis Poisson**.

Unlike binomial distribution, Poisson distribution cannot be deduced on purely theoretical grounds based on the conditions of the experiment. In fact, it must be based on experience, i.e., on the empirical results of past experiments relating to the problem under study. The Poisson distribution is a good approximation of the binomial distribution when n is greater than or equal to 20 and the probability of success for each trial is less than or equal to 0.05.

This distribution deals with the evaluation of probabilities of rare events such as " no. of car accidents on road", "no. of earthquakes in a year", "no. of printing mistakes per page", etc.

Attributes of a Poisson Experiment

A Poisson experiment is a statistical experiment that has the following properties:

i. The experiment results in outcomes that can be classified as successes or failures.

ii. The average number of successes (λ) that occurs in a specified region is known.

iii. The probability that a success will occur is proportional to the size of the region.

iv. The probability that a success will occur in an extremely small region is virtually zero.

Notation

The following notation is helpful, when we talk about the Poisson distribution.

i) *e*: A constant equal to approximately 2.71828. (Actually, *e* is the base of the natural logarithm system.)

ii) λ: The mean number of successes that occur in a specified region (parametric of poison distribution)

iii) *n*: The total number of trials

iv) *p*: The probability of success for each trial

v) *r*: no. of occurrence of a given event

Formula

Suppose we conduct a Poisson experiment, in which the average number of successes within a given region is λ. Then, the Poisson probability is:

$$f(r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

Where, r = The actual number of successes that result from the experiment,

e = Approximately equal to 2.71828 (exponential constant).

λ = Mean of Poison distribution

In Poison distribution mean is equal to the variance of distribution. It is also limiting case of Binomial distribution in which $n \to \infty;\ p \to 0$ (rare cases).

Real World Application of Poisson Distribution

Whether one observes patients arriving at an emergency room, cars driving up to a gas station, bank customers coming to their bank or shoppers (client) being served at a cash register, the streams of such events typically follow the Poisson process. The underlying assumption is that the events are statistically independent and the rate, λ, of these events (the expected number of the events per time unit) is constant. The list of applications of the Poisson distribution is very long. To name just a few more:

1. The number of mutations on a given strand of DNA per time unit.
2. The number of bankruptcies that are recorded in a month.
3. The number of arrivals at a car wash in one hour.
4. The number of network failures per day.
5. The number of file server virus infection at a data center during a 24-hour period.
6. The number of Airbus 330 aircraft engine shutdowns per 100,000 flight hours.
7. The number of asthma patient arrivals in a given hour at a walk-in clinic.
8. The number of hungry persons entering Mc Donald's restaurant.
9. The number of work- related accidents over a given production time.
10. The number of birth, deaths, marriages, divorces, suicides, and homicides over a given period of time.

Example 43 The number of calls coming per minute into a hotels reservation center is Poisson random variable with mean 3.

(a) Find the probability that no calls come in a given 1 minute period.

(b) Find the probability that at least 2 calls come in a given 1 minute period.

Solution: Given Information:

Mean (λ) = 3

a) Find the probability that no calls come in a given 1 minute period.

Let "r" denotes the number of calls coming in that given 1 minute period. Then,

$$r \sim \text{Poisson}(3)$$

$$P(r = 0) = ?$$

We Know,

$$P(r; \lambda) = \frac{e^{-\lambda}\lambda^{r}}{r!}$$

or, $P(r = 0) = \left(e^{-3}\right) \cdot (3°)/0!$

or, $P(r = 0) = \left(e^{-3}\right)$

or, $P(r = 0) = (2.71828) - 3$

or, $P(r = 0) = 0.0498$

or, $P(r = 0) = 4.98\%$

Therefore, the probability that no calls come in a given 1 minute period is 4.98%.

b) Find the probability that at least 2 calls come in a given 1 minute period.

Let "r" denotes the number of calls coming in that given 1 minute period. Then,

$$r \sim \text{Poisson}(3)$$

$$P(r \geq 2) = ?$$

We know,

$$\begin{aligned} P(r \geq 2) &= 1 - P(r < 2) \\ &= 1 - [P(0) + P(1)] \\ &= 1 - [(e^{-3}) \cdot (3^0)/0! + (e^{-3}) \cdot (3^1)/1!] \\ &= 1 - [0.0498 + 0.1494] \\ &= 0.8008 \end{aligned}$$

Therefore, the probability that at least 2 calls will come in a given 1 minute period is 80.08%

Normal Distribution

The Binomial and Poisson distribution so far are discrete probability distributions because of the discrete random phenomenon. Normal probability distribution simply called normal distribution is one of the most important continuous theoretical distributions in statistics. Because of its characteristics, most of the data relating to economics, business or even in social and physical science confirm to this distribution. It assumes all the values within infinite interval or range $-\infty$, to ∞. Actually, the normal curve represents a family of curves. The normal distribution was first propounded by English Mathematician **De-Moivre** (1667 – 1754) in 1733 obtained the mathematical equation for this distribution while dealing with problems arising in the game of chance. It is also called Gaussian distribution (Gaussian law of errors) after Karl Friedrich Gauss (1777 – 1855) who used this distribution to describe the theory of accidental errors of measurements involved in the calculation of orbits of heavenly bodies. At present, normal probability model is one of the most important probability models in statistical analysis.

It is a very important and useful theoretical continuous probability distribution. Most of statistical data in our society relating to economic and business, rural development even in social and physical science confirm to continuous probability distribution. It is most important probability models in statistical analysis of data.

Definition of Normal Distribution

If X is a continuous random variable ranging between $(-\infty, \infty)$ and follows normal distribution with mean μ, and standard deviation σ (or variance (σ^2), then the probability density function (*pdf*) of X is given by

$$p(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2} \left(\frac{X-\mu}{\sigma}\right), -\infty < X \infty$$

where X = values of continuous random variable.

$\pi = \frac{22}{7}$, $e = 2.7183$ are constants

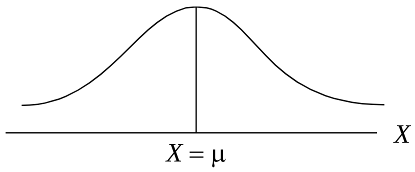
μ = Mean of normal distribution

σ = Standard deviation of normal distribution

The mean μ and standard deviation σ are called parameters of the normal distribution, Symbolically, X follows normal variable with mean μ and variance σ^2 is written as $X \sim N(\mu, \sigma^2)$.

The graph of normal curve is symmetrical bell shaped that extends indefinitely in both directions approaching closer to the horizontal axis without touching it, as shown below:

Left and right hand tail extended without touching horizontal axis and the curve is symmetrical about a vertical line erected at the mean.



Mean = Median = Mode

Figure: *Frequency curve of the Normal Probability Distribution*

Properties of Normal Distribution

The normal probability curve with parameters μ and σ has the following properties:

1. The curve is bell shaped and symmetrical about the line $X = \mu$ (i.e. $z = 0$) so that $\beta_1 = \gamma$, $g_1 = 0$, $\beta_2 = 3$ and $\gamma_2 = 0$.

2. Mean, Median and Mode of the distribution coincide

3. Area under the normal curve is unity and the mean divides it into two equal parts.

4. The curve has a single peak, thus it is unimodal.

5. Since is the probability can never be negative, no portion of the curve lies below the x-axis.

6. The two tails of the normal curve extended indefinitely and never touch the horizontal axis (i.e. the curve tapers off x-axis)

7. Points of inflexion (points at which the curve changes its deviation) of the normal curve are at $X = \mu \pm \sigma$, i.e. they are equidistant from mean at a distance of σ.

8. A linear combination of independent normal variates is also normal variate i.e. if $X_1, X_2, \cdots, X_n$ are independent normal variates with means $\mu_1, \mu_2, \cdots, \mu_n$ and standard deviation $\sigma_1, \sigma_2, \cdots, \sigma_n$ respectively, then their linear combination $a_1 X_1 + a_2 X_2 \cdots + a_n X_n$ where $a_1 a_2, \cdots, a_n$ are constants is also a normal variate with:

$$\text{Mean } (\mu) = a_1 \mu_1 + a_2 \mu_2 + \cdots a_n \mu_n$$

and
$$\text{Variate } (\sigma_2) = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \cdots + a_n^2 \sigma_n^2$$

Thus, the sum (or difference) of independent normal variates is a normal variate.

9. Normal distribution is a limiting case of binomial and Poisson distribution.

10. In a normal distribution the quartiles (first and third) are equidistant from median i.e.

$$Q_3 + Q_1 = 2Md = 2\mu$$

11. In a normal distribution, quartile deviation is approximately equal to $\frac{2}{3}$ times its standard deviation i.e. Q.D. $= \frac{2\sigma}{3}$ and mean deviation about mean is approximately equal to 4/5 times its standard deviation. i.e. $M.D. = \frac{4\sigma}{5}$

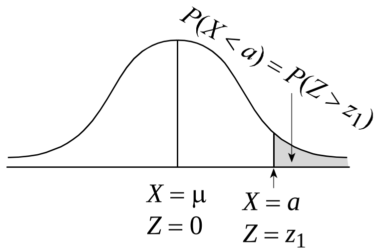
12. Area property: The most fundamental property of normal probability curve is its area property. The area under the normal probability curves between the ordinates at $X = \mu - \sigma$ and $X = \mu + \sigma$ is 0.6826. In other words, the range $\mu \pm \sigma$ covers 68.26% of the observations or

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$$

Similarly $$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9974 \text{ which is almost unity.}$$

Standard Normal Distribution



Standard normal distribution is a special case of the normal distribution. If a random variable X follows normal distribution with mean μ and standard deviation σ, define a random variable Z as,

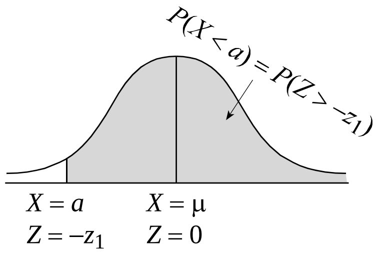
$$Z = \frac{X - \mu}{\sigma}$$ is called standard normal variate

The expected value and variance of the standard normal variate Z is given by,

$$E(Z) = 0, V(Z) = 1$$

The probability density function of the standard normal variate Z is given by,

$$p(Z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty < X \infty$$



Symbolically, we write $Z \sim N(0, 1)$ to denote standard normal variate follows normal distribution with mean 0 and variance 1.

How to compute areas under normal probability curve?

Mathematically, the area bounded by their curve $f(x)$, X-axis and the ordinates $X = a$ and $X = b$ is given by integrating $f(x)$ over the limits $[a, b]$ and is denoted as $P(a \le X \le b)$

In order to calculate the area $P(a \le X \le b)$ under the normal probability curve first convert the normal variate X into standard normal variate Z by using the relation $Z = \dfrac{X - \mu}{\sigma}$.

Computation of area to the Right of the ordinate at $X = A$, i.e., to find $P(X > a)$

Case (i): $a > \mu$, i.e., a is to the right of the mean ordinate at $X = \mu$

When, $X = a$, $Z = \dfrac{X - \mu}{\sigma} = \dfrac{a - \mu}{\sigma} = z_1$ (say)

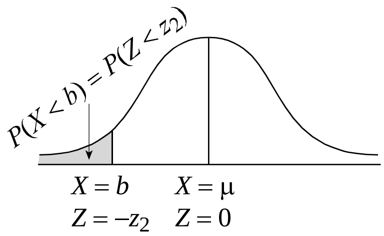
Now,

$$\begin{aligned} P(X > a) &= P(Z > z_1) \\ &= P(0 < Z < \infty) - P(0 < Z < z_1) \\ &= 0.5 - P(0 < Z < z_1) \end{aligned}$$

The value of probability $P(0 < Z < z_1)$ is obtained from the area under standard normal curve.

Case (ii): $a < \mu$, i.e., a is to the left of the mean ordinate at $X = \mu$.

When, $X = a$, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{a-\mu}{\sigma} = z_1$ (say)



Now,

$$\begin{aligned} P(X > a) &= P(Z > -z_1) \\ &= P(-z_1 < Z < 0) + P(0 < Z < \infty) \\ &= P(0 < Z < z_1) + 0.5 \end{aligned}$$

The value of probability $P(0 < z < Z_1)$ is obtained from the area under standard normal curve.

Computation of Area to the left of the ordinate at $X = b$, i.e., to find $P(X < b)$

Case (i): $b > \mu$, i.e., b is to the right of the mean ordinate at $X = \mu$

When, $X = b$, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{b-\mu}{\sigma} = Z_2$ (say)

Now,

$$\begin{aligned} P(X < a) &= P(Z < z_2) \\ &= P(-\infty < Z < 0) + P(0 < Z < z_2) \\ &= 0.5 + P(0 < Z < z_2) \end{aligned}$$

The value of probability, $P(0 < Z < z_2)$ is obtained from the area under standard normal curve.

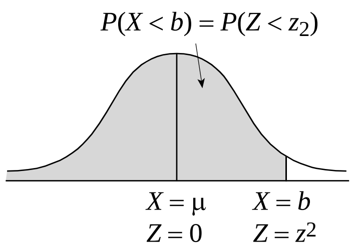
Case (ii): $b < \mu$, i.e., b is to the left of the mean ordinate at $X = \mu$.

When, $X = b$, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{b-\mu}{\sigma} = z_2$ (say)

Now,

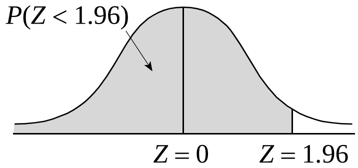
$$\begin{aligned} P(X < b) &= P(Z < -z_2) \\ &= P(-\infty < Z < 0) - P(-z_2 < Z < 0) \\ &= 0.5 - P(0 < Z < z_2) \end{aligned}$$

The value of probability $P(0 < Z < z_2)$ is obtained from the area under standard normal curve.



Example 44 Given a standardized normal distribution, determine the following probabilities: (i) $P(Z < 1.96)$, (ii) $P(Z > 1.64)$, (iii) $P(0 < Z < 2.34)$, (iv) $P(Z < -1.64)$, (v) $P(Z > -0.34)$, (vi) $P(-1.25 < Z < 0)$, (vii) $P(-1.64 < Z < +1.64)$, (viii) $P(0.17 < Z < 1.64)$, (ix) Z is less than -0.84 or greater than $+2.08$, and (x) what is the value of Z if only 31.87% of all possible Z values are smaller?

Solution: Here, Z is a standard normal variate



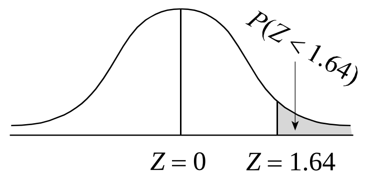
i) $P(Z < 1.96) = P(-\infty < Z \leq 0) + P(0 < Z < 1.96)$

$= 0.50 + 0.475 = 0.975$

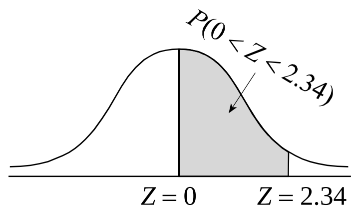
ii) $P(Z > 1.64) = P(1.64 < Z < \infty)$

$= P(0 < Z < \infty) - P(0 < Z \le 1.64)$

$= 0.50 - 0.4495 = 0.0505$

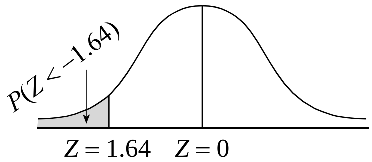


iii) $P(0 < Z < 2.34) = 0.4904$



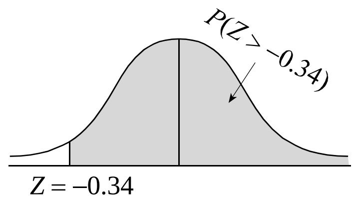
iv) $P(Z < -1.64) = P(-\infty < Z < 0) - P(-1.64 < Z < 0)$

$= 0.50 - 0.4495 = 0.050$



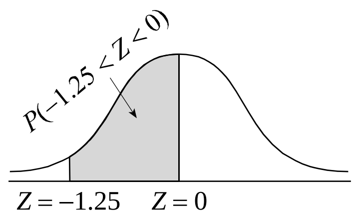
v) $P(Z > -0.34) = P(-0.34 < Z < 0) + P(0 \le Z < \infty)$

$= 0.1331 + 0.50 = 0.6331$



vi) $P(-1.25 < Z < 0) = P(0 < Z < 1.25)$ [By Symmetry]

$= 0.3944$

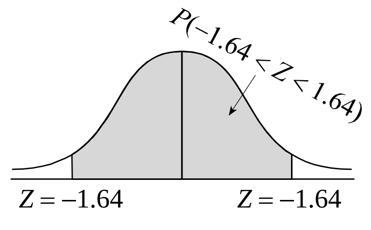


vii) $P(-1.64 < Z < +1.64) = P(-1.64 < Z < 0) + P(0 < Z < 1.64)$

$= P(0 < Z < 1.64) + P(0 < Z < 1.64)$

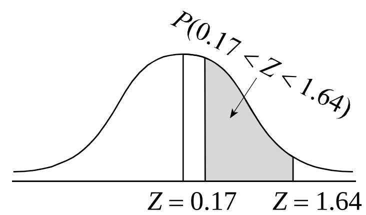
$= 2 \times P(0 < Z < 1.64)$

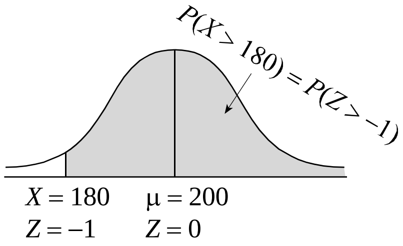
$= 2 \times 0.4495 = 0.899$



viii) $P(0.17 < Z < 1.64) = P(0 < Z < 1.64) - P(0 < Z < 0.17)$

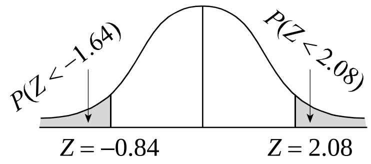
$= 0.4495 - 0.0675 = 0.382$





ix) The probability of Z is less than –0.84 or greater than +2.08 is

$$= P(Z < -0.84) + P(Z > 2.08)$$
$$= [P(-\infty < Z < 0) - P(-0.84 < Z < 0)]$$
$$+ [P(0 < Z < \infty) - P(0 < Z < 2.08)]$$
$$= [0.50 - P(0 < Z < 0.48)] + [0.50 - 0.4812]$$
$$= 0.50 - 0.2995 + 0.50 - 0.4812 = 0.2193$$



x) Left value of z be $-z_1$ such that only 31.87% of all values of z are smaller than $-z_1$, i.e.,

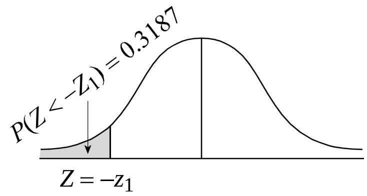
$$P(Z < -z_1) = 0.3187$$

or, $P(-\infty < Z < 0) - P(-z_1 < Z < 0) = 0.3187$

or, $0.50 - P(0 < Z < z_1) = 0.3187$

or, $P(0 < Z < z_1) = 0.50 - 0.3187$

or, $P(0 < Z < z_1) = 0.1813$



The value of probability closer to 0.1813 in Z-table at $Z = 0.47$.

$$z_1 = 0.47$$

Hence, the required value of Z is –0.47.

Example 45 Given a normal distribution with mean $\mu = 200$ and $\sigma = 20$, find the probability that (i) $P(X > 180)$, (ii) $P(X < 220)$, (iii) $P(160 < X < 240)$, (iv) $P(X > 220)$, (v) $P(X < 180 \text{ or } X > 220)$, (vi) 10% of the values are less than the value of X?

Solution: Here, let X be the normal variate that follows normal distribution with mean $(\mu) = 2000$ and standard deviation $(\sigma) = 20$.

i) For $P(X > 180)$: When $X = 180$

$$Z = \frac{X - \mu}{\sigma} = \frac{180 - 200}{20}$$

$$\therefore \quad P(X > 180) = P(Z > -1)$$
$$= P(-1 < Z < 0) + P(0 < Z < \infty)$$
$$= P(0 < Z < 1) + 0.50$$
$$= 0.3413 + 0.50 = 0.8413$$

ii) For $P(X < 220)$: When $X = 220$

$$Z = \frac{X - \mu}{\sigma} = \frac{220 - 200}{20} = 1$$

$$\therefore \quad P(X < 220) = P(Z < 1)$$
$$= P(-\infty < Z < 0) + P(0 < Z < 1)$$
$$= 0.50 + 0.3413 = 0.8413$$

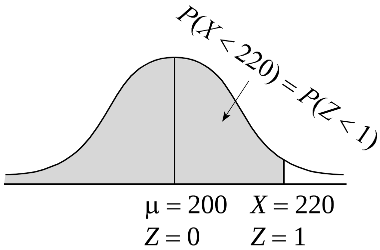
iii) For $P(160 < X < 240)$: When $X = 160$

$$Z = \frac{X - \mu}{\sigma} = \frac{160 - 200}{20} = -2$$

When $X = 240$

$$Z = \frac{X - \mu}{\sigma} = \frac{160 - 200}{20} = 2$$

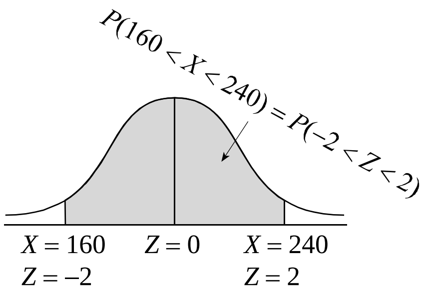
$$\therefore \quad P(160 < X < 240) = P(-2 < Z < 2)$$
$$= P(-2 < Z < 0) + P(0 < Z < 2)$$
$$= P(0 < Z < 2) + P(0 < Z < 2)$$
$$= 2 \times 0.4772 = 0.9544$$



iv) For $P(X > 220)$: When $X = 220$

$$Z = \frac{X - \mu}{\sigma} = \frac{220 - 200}{20} = 1$$

$$\therefore \quad P(X > 220) = P(Z > 1)$$
$$= P(0 < Z < \infty) - P(0 < Z < 1)$$
$$= 0.50 - 0.3413$$
$$= 0.1587$$



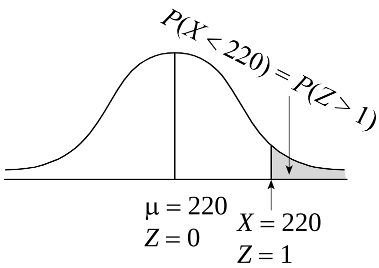
v) For $P(X < 180 \text{ or } X > 220)$: When $X = 180$

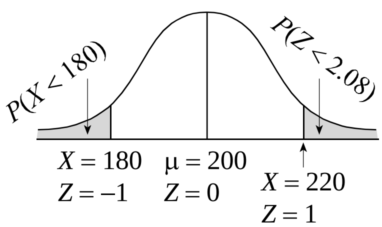
$$Z = \frac{X - \mu}{\sigma} = \frac{180 - 200}{20} = -1$$

When $X = 220$

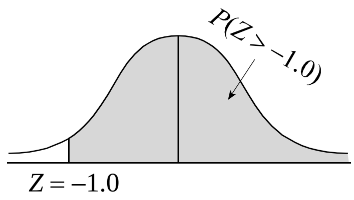
$$Z = \frac{X - \mu}{\sigma} = \frac{220 - 200}{20} = 1$$

$$\therefore \quad P(X < 180 \text{ or } X > 220)$$
$$= P(X < 180) + P(X > 220)$$
$$= P(Z < -1) + P(Z > 1)$$
$$= [P(-\infty < Z < 0) - P(-1 < Z < 0)] + [P(0 < Z < \infty) - P(0 < Z < 1)]$$
$$= [0.50 - P(0 < Z < 1)] - [0.50 - P(0 < Z < 1)]$$
$$= 1 - 2 \times P(0 < Z < 1)$$





$= 1 - 2 \times 0.3413 = 0.3174$



vi) Let x_1 be the value of X for which 10% of the values are less than x_1, i.e.,

$$P(X < x_1) = 0.10$$

When $X = x_1$,

$$Z = \frac{X - \mu}{\sigma} = \frac{x_1 - 200}{20} = z_1 \text{ ... (i)}$$

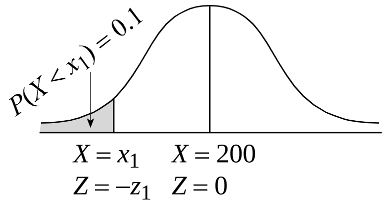
$\therefore$ $P(X - x_1) = 0.10$

$$= P(Z < -z_1) = 0.10$$

$$= P(-\infty < Z < 0) - P(-z_1 < Z < 0) = 0.10$$

$$= 0.50 - P(0 < Z < z_1) = 0.10$$

$$= P(0 < Z < z_1) = 0.40$$



The value of probability closer to 0.40 in Z-table is 0.3997 at $Z = 1.28$

$\therefore$ $z_1 = 1.28$

From equation (i)

$$\frac{x_1 - 200}{20} = -1.28$$

or, $x_1 = 200 - 1.28 \times 20$

or, $x_1 = 174.4$

Hence, the required value of X is 174.4

Example 46 A Kathmandu municipality putts 10,0000 light bulbs on the streets of a city. If lives of bulbs follow a normal distribution with a mean of 60 days and a standard deviation of 20 days, how many bulbs will have to be replaced after (i) 40 days? and (ii) 80 days?

Solution: Let X be the normal variated denoting the life of the electric bulbs that follows normal distribution with mean life of bulbs (μ) = 60 days.

Standard deviation of life of bulbs (σ) = 20 days

Here, total number of light bulbs (N) = 1000

i) The probability that the bulbs will have to be replaced after 40 days is $P(X > 40)$.

For, $X = 40$

$$Z = \frac{X - \mu}{\sigma} = \frac{40 - 60}{20} = -1$$

$\therefore$ $P(X > 40) = P(Z > -1)$

$$= P(-1 < Z < 0) + P(0 < Z < \infty)$$

$$= P(0 < Z < 1) + 0.5$$

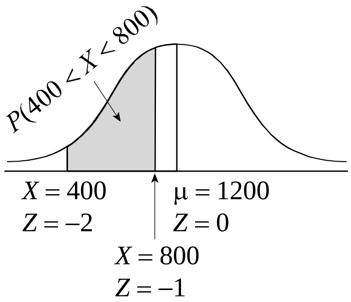
$$= 0.3413 + 0.5 = 0.8413$$

Hence, the expected number of bulbs that will have to be replaced after 40 days = $N \times P(X > 40) = 10000 \times 0.8413 = 8413$

ii) The probability that the bulbs will have to be replaced after 80 days is $P(X > 80)$.

For, $X = 80$,

$$Z = \frac{X - \mu}{\sigma} = \frac{80 - 60}{20} = 1$$



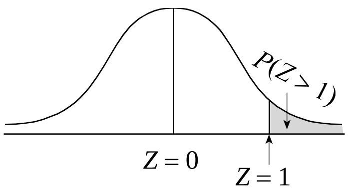
$\therefore \quad P(X > 80) = P(Z > 1)$

$= P(0 < Z < \infty) - P(0 < Z < 1)$

$= 0.5 - 0.3413 = 0.1587$

Hence, the expected number of bulbs that will have to be replaced after 80 days = $N \times P(X > 80) = 10000 \times 0.1587 = 1587$

Example 47 Then mean and a standard deviation of the wages of 6,000 workers engaged in a factory are Rs. 1,200 and Rs. 400 respectively. Assuming that the distribution to be normal estimate: i) the percentage of workers getting wages above Rs. 1600; ii) number of workers getting wages between Rs. 400 and Rs. 800; iii) number of workers getting wages between Rs. 1000 and Rs. 1400.



Solution: Let X be the normal variate denoting the wage of the workers engaged in a factory that follows normal distribution with mean wages (μ) = Rs. 1200

Standard deviation of workers (σ) = Rs. 400

Here, total number of workers (N) = 6000

i) The probability of workers wages above Rs. 1600 is $P(X > 1600)$.

For, $X = 1600$.

$$Z = \frac{X - \mu}{\sigma} = \frac{1600 - 1200}{400} = 1$$

$\therefore \quad P(X > 1600) = P(Z > 1)$

$= P(0 < Z < \infty) - P(0 < Z < 1)$

$= 0.5 - 0.3413$

$= 0.1587$

$= 15.87\%$

Hence, the percentage of workers getting wages above Rs. 1600 is 15.87%.

ii) The probability of workers getting wages between Rs. 400 to Rs. 800 is $P(400 < X < 800)$.

For, $X = 400$

$$Z = \frac{X - \mu}{\sigma} = \frac{400 - 1200}{400} = -2$$

For, $X = 800$

$\therefore \quad P(400 < X < 800) = P(-2 < Z < -1)$

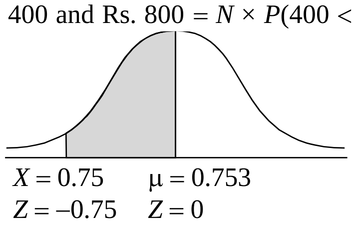
$= P(-2 < Z < 0) - P(-1 < Z < 0)$

$= P(0 < Z < 2) - P(0 < Z < 1)$

$= 0.4772 - 0.3413$

$= 0.1359$

Hence, the number of worker getting wages between Rs. 400 and Rs. 800 = $N \times P(400 < X < 800) = 6000 \times 0.1359 = 815.4 = 815$



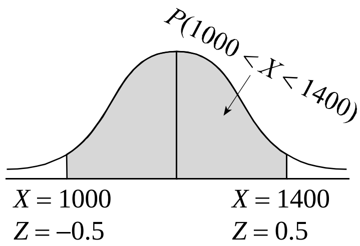
iii) The probability of workers getting wages between Rs. 1000 to Rs. 1400 is $P(10000 < X < 1400)$

For, $X = 1000$

$$Z = \frac{X - \mu}{\sigma} = \frac{1000 - 1200}{400} = 0.5$$

For, $X = 1400$

$$Z = \frac{X - \mu}{\sigma} = \frac{1400 - 1200}{400} = 0.5$$



$\therefore \quad P(1000 < X < 1400) = P(-0.5 < Z < 0.5)$

$= P(-0.5 < Z < 0) + P(0 < Z < 0.5)$

$= P(0 < Z < 0.5) + P(0 < Z < 0.5)$

$= 2 \times P(0 < Z < 0.5)$

$= 2 \times 0.1915$

$= 0.3830$

Hence, the number of worker getting wages between Rs. 1000 and Rs. 1400 = $N \times P(1000 < X < 1400) = 6000 \times 0.3830 = 2298$

Example 48 An industrial sewing machine uses ball bearings that are targeted to have a diameter of 0.75 inch. The lower and upper specification limits under which the 4 ball bearing can operate are 0.74 and 0.76 inch, respectively. Experience has indicated that the actual diameter of the ball bearings is approximately normally distributed with a mean of 0.753 inch and a standard deviation of 0.004 inch. What is the probability that a ball bearing will be (i) between the target and the actual mean, (ii)between the lower specification limit and the actual mean, (iii) above the upper specification limit, (iv) below the lower specification limit, (v) above which value in diameter will 93% of the ball bearing be?

Solution: Let X be the normal variate denoting the diameters of the ball bearings that follows normal distribution with

Mean diameter of ball the bearings (μ) = 0.753 inch.

Standard deviation of the ball bearings (σ) = 0.004 inch.

Here, targeted value = 0.75

Lower specification limit = 0.74

Upper specification limit = 0.76

i) The probability that a ball bearing will be between the targeted value and the actual mean is $P(0.75 < X < 0.753)$.

For, $X = 0.75$

$$Z = \frac{X - \mu}{\sigma} = \frac{0.75 - 0.753}{0.004} = 0.75$$

For, $X = 0.753$

$$Z = \frac{X - \mu}{\sigma} = \frac{0.753 - 0.753}{0.004} = 0$$

$$\therefore\ P(0.75 < X < 0.753) = P(-0.75 < Z < 0)$$
$$= P(0 < Z < 0.75) = 0.2734$$

ii) The probability that a ball bearing will be between the lower specification limit and the actual mean is $P(0.74 < X < 0.753)$.

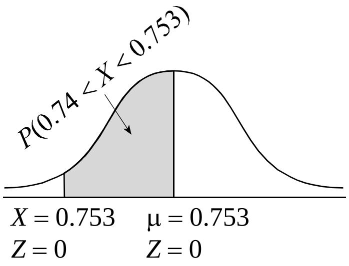
For, $X = 0.74$

$$Z = \frac{X - \mu}{\sigma} = \frac{0.74 - 0.753}{0.004} = -3.25$$

For, $X = 0.753$

$$Z = \frac{X - \mu}{\sigma} = \frac{0.75 - 0.753}{0.004} = 0$$

$$\therefore\ P(0.74 < X < 0.753) = P(-3.25 < Z < 0)$$
$$= P(0 < Z < 3.25) = 0.4994$$

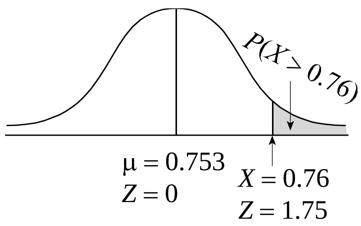


iii) The probability that a ball bearing will be above the upper specification limit is $P(X > 0.76)$.

For, $X = 0.76$

$$Z = \frac{X - \mu}{\sigma} = \frac{0.76 - 0.753}{0.004} = 1.75$$

$$\therefore\ P(X < 0.76) = P(Z > 1.75)$$
$$= P(0 < Z < \infty) - P(0 < Z < 1.75)$$
$$= 0.5 - 0.4599$$
$$= 0.0401$$

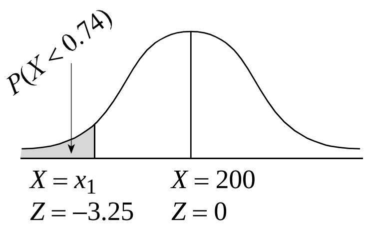


iv) The probability that a ball bearing will be below the lower specification limit is $P(X > 0.74)$.

For, $X = 0.74$

$$Z = \frac{X - \mu}{\sigma} = \frac{0.74 - 0.753}{0.004} = 3.25$$

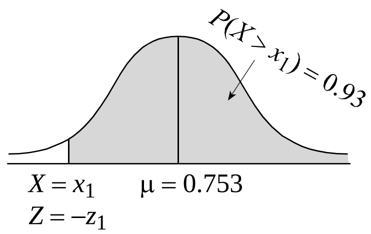
$$\therefore\ P(X < 0.76) = P(Z > 1.75)$$
$$= P(-\infty < Z < 0) - P(-3.25 < Z < 0)$$
$$= 0.5 - P(0 < Z < 3.25)$$
$$= 0.5 - 0.4994 = 0.0006$$



v) Let x_1 be the value above which there are 93% of the ball bearings, i.e., $P(X > x_1) = 0.93$

For, $X = x_1$

$$Z = \frac{X - \mu}{\sigma} = \frac{x_1 - 0.753}{0.004} = -z_1 \quad \text{... (i)}$$



$$\therefore\ P(X < x_1) = P(Z > -z_1) = 0.93$$

$$= P(-z_1 < Z < 0) + P(0 < Z < \infty)$$

$$= P(0 < Z < z_1) + 0.5 = 0.93$$

$$= P(0 < Z < z_1) = 0.43$$

The value of probability closer to 0.43 in the normal table is 0.4306 at $Z = 1.48$.

$$\therefore\ z_1 = 1.48$$

From equation (i),

$$\frac{x_1 - 0.753}{0.004} = -1.48$$

or, $x_1 = 0.753 - 1.48 \times 0.004$

or, $x_1 = 0.7471$

Hence, the required value is 0.7471.

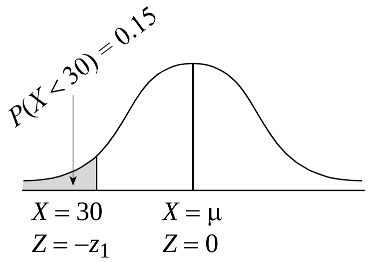
Example 49 The marks obtained by 1000 students in an examination are known to be normally distributed. If 15% of the students got less than 30 marks and 10% of the students got over 90, find the mean and standard deviation of the distribution.

Solution: Let X be the normal variate denoting the marks of the students that follows normal distribution with mean μ and standard deviation σ.

According to question,

$$P(X < 30) = 0.05 \quad \text{... (i)}$$

$$P(X > 90) = 0.10 \quad \text{... (ii)}$$



From equation (i),

For, $X = 30$

$$Z = \frac{X - \mu}{\sigma} = \frac{30 - \mu}{\sigma} = -z_1 \quad \text{... (i)}$$

$$\therefore\ P(X < 30) = 0.15$$

or, $P(Z < -z_1) = 0.15$

or, $P(-\infty\ Z < 0) - P(z_1 < Z < 0) = 0.15$

or, $0.5 - P(0 < Z < z_1) = 0.15$ [By symmetry]

or, $P(0 < Z < z_1) = 0.35$

The value of probability closer to 0.35 in the normal table is 0.3508 at $Z = 1.04$. Therefore $z_1 = 1.04$

From equation (iii), we get

$$\frac{30 - \mu}{\sigma} = -1.04$$

or, $\mu - 1.04 \times \sigma = 13 \qquad \text{(iv)}$

Again, from equation (ii)

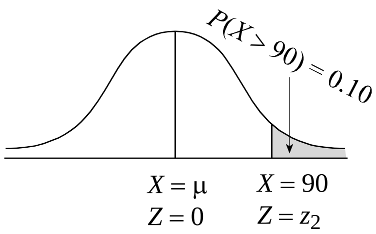
For, $X = 90$

$$Z = \frac{X - \mu}{\sigma} = \frac{90 - \mu}{\sigma} = z_2 \quad \text{... (v)}$$

or, $P(Z > z_2) = 0.10$

or, $P(0 < Z < \infty) - P(0 < Z < z_2) = 0.10$

or, $P(0 < Z < z_2) = 0.40$



The value of probability closer to 0.40 in the normal table is 0.3997 at $Z = 1.28$. Therefore, $z_2 = 1.28$

From equation (v), we get

or, $\frac{90 - \mu}{\sigma} = 1.28$

or, $\mu + 1.28 \times \sigma = 90$ (vi)

Subtracting equation (iv) from equation (vi), we get

$2.32\,\sigma = 60$

$\Rightarrow \quad \sigma = 25.86$

Now, substituting the value of $\sigma = 25.86$ in equation (iv), we get

$\mu - 1.04 \times 25.86 = 30$

or, $\mu = 30 + 1.04 \times 25.86$

or, $\mu = 56.89$

Hence, the mean and standard deviation of the distribution are 56.89 and 25.86 respectively.

Exercise 4.1

A. Theoretical questions:

1. Define probability and explain the importance of the probability in decision making.

2. What do you understand by (i) equally likely (ii) mutually exclusive and (iii) independent events?

3. Explain the concept of probability from the following:

(a) Mathematical or a prior approach, (b) relative frequency or empirical approach

4. State the addition theorem of probability and illustrate it with suitable examples.

5. State the addition theorem of probability and illustrate it with suitable examples.

6. Give the classical definition of probability and state its limitations.

7. Explain with examples the concept of independent and mutually exclusive events in probability.

8. When are two events said to be independent in the probability sense? Give examples of dependent and independent events.

9. Define conditional probability with examples.

10. State Baye's theorem. Explain it with suitable examples.

11. Explain the different approaches of probability.

12. What are the conditions for binomial distribution?

13. Define binomial distribution. Also explain the properties of binomial distribution.

14. What are the condition of Poisson distribution?

15. What are the important uses of Poisson distribution?

16. Compare difference between binomial and Poisson distribution.

17. What is normal distribution? Define in term of standard normal variate.

18. What are characteristics of normal distribution?

19. Write the important application of normal distribution.

B. Numerical and practical problems

1. Two coins are tossed simultaneously. What is the probability that (i) both are heads (ii) both are tails (iii) one head and one tail (iv) at least one tail.

2. A die is thrown twice. Determine the probability of getting

(i) the sum of two faces is 6 (ii) sum of two faces is 12.

3. A bag contains 9 red, 7 white and4 black balls. A ball is drawn at random. Find the probability of drawing

(i) a white ball (ii) not a black ball

(iii) a red ball (iv) a white ball or a red ball or a black ball.

4. A card is drawn at random from a pack of 52 cards. Find the probability of drawing (i) a black card (ii) not black (iii) a king (iv) either a card 2 or a card 3 (v) a red or a black card (vi) a red card or an ace (vii) a queen of spade (viii) a spade.

5. What is the probability that an English alphabet selected at random is (i) a vowel (ii) a consonant

6. A bag contains 20 tickets marked numbers 1 to 20. One ticket is drawn at random. Find the probability the number on the ticket selected is a multiple of (a) 2 or 5 (b) 10 or 11 (c) it is an even number (d) it is prime number.

7. What is the probability that a leap year selected at random, will contain 53 Sundays?

8. (a) The following table represents the sales of Maruti car (800 cc.) for a period of 200 days.

Sales	0−100	100−200	200−300	300−400	400−500
No. of days	10	40	60	70	20

Find the probability of selling

(i) 150 or more cars (ii) between 200 and 300 cars

(iii) less than 300 cars (iv) between 100 and 400 cars

(b) Income of employees in an industrial concern are given below:

Income (Rs.)	0-50	50-100	100-150	150-200	200-250	250 and above
No. of employees	90	150	100	80	70	10

Find the probability that an employee selected at random has

(i) Income below Rs.100 (ii) Income above Rs.200

(iii) Income between Rs. 100 and Rs. 200

9. (a) You are given below the income distribution of 1000 persons

Income in Rs.	0–500	500–1000	1000–1500	1500–2000	2000–2500	2500–3000	3000–3500
No.of days	150	250	300	100	80	70	50

Find the probability that a person selected at random has

(i) income below Rs. 2000 (ii) income Rs. 2500 or more

(iii) income more than Rs.2250 (iv) income less than Rs. 2750

(v) income between Rs. 750 and Rs. 2650.

(b) The distribution of 500 workers of a factory according to the sex and nature of work is as follows:

	Skilled	Unskilled	Total
Male	250	50	300
Female	150	50	200
Total	400	100	500

If a worker is chosen at random, what is the probability that the worker is (i) Male and skilled (ii) Unskilled

10. Two cards are drawn at random from a deck of well shuffled 52 cards. Find the probability that the two cards drawn are

(a) both red (b) both club (c) one red and one black

11. From a pack of 52 cards, three cards are drawn at random. Find the chance that

(a) They are a king , a knave and an ace.

(b) Two are from black and one is from red cards.

(c) All are red cards.

12. A bag contains 4 red balls and 5 green balls. 3 balls are drawn at random. What is the probability that.

(a) All of them are green. (b) Two of them are red

(c) all of them are of same colour.

13. Five men in a group of 20 are graduates. If 3 are chosen out of 20 at random, what is the probability that

(a) all are graduates (b) none of them is graduate

(c) at least one of them being graduate.

14. A class consists of 40 boys and 60 girls. If two students are chosen at random, what will be the probability that

(a) both are boys (b) both are girls (c) one boy and one girl

15. (a) If A and B are two independent events with P (A) = $\frac{2}{3}$ and P (B) =$\frac{3}{5}$, find P $\overline{(A \cup B)}$.

(b) For two mutually exclusive events A and B if P(A) = 0.31 and P (B) = 0.42. Find P.

(c) For two independent events A and B if P (A) = 0.25 and P (B) = 0.35, find P (A∩B), P (A∪B) and P(A∩$\bar{B}$)

16. If the probability of happening two mutually exclusive events A and B are respectively $\frac{2}{5}$ and $\frac{1}{5}$, find the probability of happening one of these events.

17. A husband and his wife appear in an interview for the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that (i) both of them will be selected (ii) none of them will be selected (iii) at least one of them will be selected (iv) only one of them selected.

18. The probability that a boy passes the examination is $\frac{3}{4}$ and that of a girl is $\frac{2}{5}$. Find the probability that (i) both will pass the examination (ii) at least one of them will pass the examination.

19. A person is applying for the post of manager in bank X and bank Y. The probability of his selection in bank X is 0.20 and in bank Y is 0.30. He has also chance of selection in both banks at the same time with probability is 0.085. What is the probability that he will be selected in at least one of the banks?

20. One shot is fired from each of the two guns G_1 and G_2 denote the events that the target will hit by the first and second guns respectively. If P (G_1) = 0.6 and P (G_2) = 0.7 and G_1 and G_2 are independent events, find the probability that

(i) Exactly one hit is registered (ii) the target will be hit?

21. A problem in Mathematics is given to three students A, B and C where the chances of solving it by them are $\frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ respectively.

Find the probability that,

(a) All of them can solve the problem.
(b) None of them will solve the problem.
(c) The problem will be solved.

22. The odds against A solving a problem as 8:6 and odds in favour of B solving the same problem are 14:10. What is the probability that (i) both A and B will solve it? (ii) A solves it but B fails to solve it? (iii) At least one of them will solve the problem.

23. (a) If A and B are two events such that P(A) = 0.5, P(B) = 0.6 and $P(A \cup B)$ = 0.58, find $P(A/B)$ and $P(B/A)$

(b) If $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, find $P(A/B)$ and $P(B/A)$. Are A and B are independent?

(c) Given that P (A) = 0.4, P (B) = 0.7, P (A/B) = 0.2 find
(i) $P(A/B)$ (ii) $P(A'/B)$ (iii) $P(A/B')$ (iv) $P(A'/B')$

(d) Gavin P(A) = $\frac{3}{14}$, P(B) = $\frac{1}{6}$, P(C) = $\frac{1}{3}$, P(AC) = $\frac{1}{7}$, find the probabilities P(A/C) and P(C/A).

(e) Two events A and B are statistically dependent. If $P(A) = 0.35, P(B) = 0.20, P(A \text{ or } B) = 0.45$, Find the probability (i) neither A nor B will occur (ii) B will occur, given that A has occurred.

24. In a certain school, 20% students failed in English, 15% students failed in Mathematics and 10% students failed in both English and Mathematics. A student is selected at random. If he failed in English, what is the probability that he also failed in Mathematics.

25. The probability that a manufacturer will produce 'brand X ' product is 0.13, the probability that he will produce 'brand Y' product is 0.28 and the probability that he will produce both brand is 0.06. What is the probability that the manufacturer who has produced 'brand Y' will also have produced 'brand X'?

26. The following information was obtained concerning 1000 employees of an industrial concern.

Department	**Sex**		
	Male	**Female**	**Total**
Manufacturing	280	220	500
Production control	175	125	300
Quality control	115	85	200
Total	570	430	1000

If an employee is chosen at random, what is the probability that

(a) Employee chosen is male given that he belongs to production control department.
(b) Employee chosen is female from manufacturing department.

27. A bag contains 5 white and 3 black balls. Two balls are drawn a random one after the other without replacement. Find the probability that (i) both balls are white (ii) both are black, (iii) different colors, (iv) same colors.

28. Two cards drawn successively one after another from a well-shuffled pack of 52 cards. If the cards are not replaced, find the probability that all of them are queens.

29. A lot contains 10 items of which 3 are defective. Three items are chosen from the lot at random one after another without replacement. Find the probability that all three are defective.

30. An urn contains 5 red, 7 white, and 8 black balls. Three balls are drawn one after another without replacement, find the probability that they are in order of red, white and balls.

31. A box contains 4 red and 6 white balls. Two balls are drawn one after another with replacing (i.e. with replacement) first ball before drawing second ball. Find the probability of getting (i) both red (ii) different colour (iii) same colour (iv) first red and second white in order

32. Bag A contains 5 white balls and 3 black balls. Another bag B contains 4 white and 5 black balls. A ball is transferred from bag A to the bag B. Then a ball is drawn from the urn B. Find the probability that it will be white ball.

33. In a group of equal number of men and women, 60% man and 80% women are employed in a certain town. A person is selected at random and found to be employed. What is the probability that the person is (i) a man (ii) a woman?

34. In a certain locality 80% of the people read the Kantipur, 75% of the people read the Gorkhapatra and 60% of people read both Kantipur and Gorkhapatra. A people is selected at random

(i) If he reads Gorkhapatra, what is the probability that he will also read Kantipur?

(ii) If he reads Kantipur, what is the probability that he will read Gorkhapatra?

(iii) What is the probability that he read Kantipur or Gorkhapatra?

35. The chances of X, Y and Z becoming manager will be of a certain company are 4:2:3. The probabilities that bonus scheme will be introduced if X, Y and Z become manager are 0.3, 0.5 and 0.8 respectively. If the bonus scheme has been introduced, what is the probability that X is appointed as the manager?

36. In a factory producing portable radio, there are three machines producing 1000, 2000 and 3000 radio per hour respectively. These machines produce 1% ,2% and 1% defectives respectively. One radio is selected at random from a hour production of the three machines and found to be defective. What is the probability that this radio is produced from first machine?

37. An insurance company insured 2,000 scooter drivers, 4,000 car drivers and 6,000 truck drivers. The probability of an accident involving a scooter, a car and a truck are respectively 0.01, 0.03, and 0.15 respectively. One of the insured drivers meets with an accident. What is the probability that s/he is (i) scooter driver (ii) car driver (iii) truck driver.

38. In a college, there will be three candidates for the position of Principal. Mr. Chhetri, Mr. Thapa and Mr. Shrestha whose chances of getting the appointment are in the proportion 4:2:3 respectively. The probability that Mr. Chhetri would start electronics department in the college is 0.3. The probabilities of Mr. Thapa and Mr. Shrestha doing the same are 0.5 and 0.8 respectively. Find

(i) What is the probability that there will be electronics department in the college?

(ii) If there is electronics department in the next year, what is the probability that Mr. Shrestha is the principal in the college?

39. Seven fair coins are tossed simultaneously. Find the probability of getting (a) no head (b) one head (c) exactly two heads (d) less than two heads (e) more than two heads (f) at least two heads (g) at most two heads (h) between 2 and 5 heads.' [**Ans.** a) 0.0078; b) 0.0547; c) 0.1641; d) 0.0625; e) 0.7734; f) 0.9375; g) 0.2266; h) 0.5469]

40. A lot contains 500 items of which 50 are selected with replacement at random from the box, find the probability that (i) exactly 2 are defectives, (ii) 2 or less are defectives, more than two are defectives.

[**Ans.** i) 0.1937; ii) 0.9298; iii) 0.2639; iv) 0.7361 v) 0.0702]

41. An accountant is to audit 24 accounts of a firm. Sixteen of these are if highly valued customers. If the accountant selects 4 of accounts at random with replacement, what is the probability that the chooses at least one highly valued customer? [**Ans.** 0.9881]

42. The average number of defective items, in the manufacturing of an article, is 1 in 10. Find the probability of getting exactly 3 defectives in a packet of 10 articles selected at random.[**Ans:** 0.0574]

43. On a very long mathematics test, Ram got 70% of the items right. For a 10 item quiz calculate the probability that Ram will get (i) at least 8 items right, (ii) less than 3 items right.[**Ans.** i) 0.3828 ii) 0.0015]

44. 10% of the DVDs manufactured by a large electronics company are defectives. A quality control injector selects ten DVDs from the production line. Find the probability that (i) exactly two are defective; (ii) at most two are defectives; (iii) at least two are defectives; 9iv) more than two are defectives; (v) less than two are defectives.[**Ans:** i) 0.1937; ii) 0.9298; iii) 0.2639; iv) 0.0702; v) 0.7361]

45. Determine the number trial of binomial distribution fro which the mean is 4 and variance is?

[**Ans.** 16]

46. Fit the binomial distribution for the following data.

Success	5	4	3	2	1	0
Frequency	190	500	900	960	500	150

Assume the probability of success in each case as 0.5.

Ans.

X	0	1	2	3	4	5
$f(x)$	100	500	1000	1000	500	100

47. Fit the binomial distribution for the following data. Assuming the nature of the coin is unknown.

Number of heads	0	1	2	3	4
No. of experiment	28	62	46	20	4

Ans.

X	0	1	2	3	4
$f(r)$	27	60	51	19	3

48. Out of 1,000 families of 3 children each how many families would you expect to have two boys and one girls assuming that boys and girls are equally likely? [**Ans.** 375]

49. An unbiasd coin is tossed six times. Find the probability obtaining (a) exactly 4 heads (b) no heads.

[**Ans.** (a) 0.2344 and (b) 0.08562]

50. The customer accounts of a certain departmental store have an average balance of Rs.120 and a standard deviation of Rs. 40. Assume that the account balances are normally distributed.

(i) What proportion of the accounts is over Rs. 150?

(ii) What proportion of the accounts is between Rs. 100 and Rs. 150?

(iii) What proportion of the accounts is between Rs. 60 and Rs. 90?

[**Ans.** (i) 22.66% (ii) 46.49% (iii) 15.98%]

51. Ram and Co. manufactures chrome and glass lamp manually. It requires 40 labour hours to complete on with a standard deviation of 10 hours.

(i) What is the probability that it will take between 35 and 42 labour hours.

(ii) What is the probability that it will take more than 48 hours?

(Given: $Z_{0.2} = 0.0793$, $Z_{0.5} = 0.1915$, $Z_{0.8} = 0.2881$) [**Ans.** (i) 0.2708; (ii) 0.2119]

52. A banker claims that the life of a regular saving account opened in his bank averages 18 months with a standard deviation of 6.45 months. What is the probability that:

(i) there will still be money in a savings account between 20 to 22 months by a depositor.

(ii) the bank will be closed (no money in the deposit) after two years ?

(Given: $Z_{0.31} = 0.\ 1217$, $Z_{0.62} = 0.2324$, $Z_{0.93} = 0.3238$) [**Ans.** (i) 0.1107; (ii) 0.1762]

53. The mean weight of products is 68.22 grams with a variance of 10.8 grams. How many products in a batch of 1000 would you expect (i) to be over 72.0 grams (ii) between 70 and 72 grams.

(Given $Z_{1.15} = 0.3749$, $Z_{0.59} = 0.2054$) [**Ans.** (i) 125; (ii) 170]

54. The number of a group of 10000 persons was found to be normally distributed with mean Rs. 750 per month and standard deviation Rs.50. Find (i) the number of persons with income less than Rs. 700 p.m. (ii) the number of persons with income between Rs. 700 and Rs. 800 p.m.

[**Ans.** (i) 1587; (ii) 6826]

55. Incomes of a group of 10000 persons were found to be normally distributed with mean Rs. 520 and standard deviation Rs. 60. Find

(i) the number of persons having income between Rs. 400 and Rs. 550.

(ii) the lowest income of richest 1000 persons. [**Ans.** (i) 6687; (ii) Rs. 596.80]

56. The heights of 1000 students follow normal distribution with $\mu = 66"$ and $\sigma = 2"$.

(i) How many observations may be expected to lie between 63" and 69".

(ii) Find the height in inches beyond which 10% of the students would lie.

(Given: $Z_{1.5} = 0.4332$; $Z_{0.1} = 0.0398$; $Z_{0.255} = 0.1000$, $Z_{1.28} = 0.3997$) [**Ans.** (i) 866; (ii) 68.56]

57. The weekly wages of 1000 workmen are normally distributed around a mean of Rs. 70 and with a standard deviation of Rs. 58. Estimate the number of workers whose weekly wages will be

i. Between Rs. 70 and 72. ii. Between Rs. 69 and 72.

iii. More than Rs. 75 iv. Less than 63.

v. More than Rs. 80

Also, estimate the lowest weekly wages of the 100 highest paid workers. [**Ans.** (i) 155; (ii) 235]

58. In an aptitude test administrated to 900 college students, the mean was found to be 50 and standard deviation 20.

i. Find the number of students securing between 30 and 70.

ii. Find the value of the score exceeded by the top 90 students.

Area under the normal curve	0.3413	0.3997	0.4015	0.4332
Value of *Z*	190	500	900	960

59. In an intelligence test administered to 1,000 students the average score was 42 and standard deviation 24. Find (i) the number of students exceeding a score 50, (ii) the number of students lying between 30 and 54, (iii) the value of score exceeding by the top 100 students. [**Ans.** (i) 371; (ii) 383; (iii) 73]

60. a. A set of examination marks is approximately normally distributed with a mean of 75 and standard deviation of 5. If the top 5% of the students get grade A and the bottom 25% get grade F, what mark is the lowest A and what mark is the highest F?

b. Income of a group of 20000 persons were found to be normally distributed with mean Rs. 5000 and standard deviation Rs. 500, find (i) lowest income of richest 2000 persons (ii) highest income of poorest 2000 persons. [**Ans.** (a) 83.72 (b) (i) Rs. 5640 (ii) Rs. 4360]

61. The highest of 1000 cakes baked with a certain mix have a normal distribution with a mean of 5. 75 cm, and a standard deviation of 0.75. Find

i. the number of cakes having heights between 5 cm and 6.25 cm and

ii. the maximum height of the flattest 200 cakes. [**Ans.** (i) 589 (ii) 5.12 cm]

62. Given a normal distribution with $\mu = 50$ and $\sigma = 10$, find the value of X that maximum height of the flattest 200 cakes. [**Ans.** (i) 38.7; (ii) 67.5]

63. Assume that the marks in M.B.A. examination are normally distributed with mean $\mu = 400$ and $\sigma = 100$ of 600 students taking this examination, it is desired to pass 500 of them. What should be the lowest marks permitted for passing? [**Ans.** 303]

64. The mean I.Q. intelligence quotients of a large number of children of age 14 was 100 and s.d. 16; assuming that the distribution was normal. Find between what limits the I.Q.'s of the middle 40% of the children lie? [**Ans.** 91.52 and 108.48]

65. In an examination, average marks secured by 400 students is 45 with s.d. of 10. Assuming the distribution to be normal, find (i) the number of students securing marks between 50 and 60 (ii) the range of marks within which middle 50% of students would lie. [**Ans.** (i) 97 (ii) 38.3; 51.7; 314]

66. In an examination 15% of the candidates got first class (60% marks or above), while 40% failed (securing below 40%). Assuming the marks to be normally distributed, estimate the mean and standard deviation.

(Given $Z_{0.1} = 0.25$, $Z_{0.4} = 1.28$, $Z_{0.35} = 1.04$, $Z_{0.15} = 0.385$) [**Ans.** 43.88, 15.50]

67. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and the standard deviation of the distribution.

(Given: $Z_{0.42} = 1.4$, $Z_{0.19} = 0.5$) [**Ans.** 18. 50, 10]

68. Of a large group of men, 5 percent are under 60 inches in height and 40 percent are between 60 and 65 inches. Assuming a normal distribution, find the mean height and standard deviation. [**Ans.** 65.42, 3.29]

69. At a certain examination, 10% of the students who appeared for the paper in statistics get less than 30 marks and 97% of the students got less than 62 marks. Assuming the distribution to be normal, find the mean and the standard deviation of the distribution. [**Ans.** 20. $\mu = 42.7$, $\sigma = 10.13$]

70. 5000 candidates appeared in a certain examination paper carrying a maximum of 100 marks. It was found that the marks were normally distributed with mean 39.5 and standard deviation 12.5. Determine the approximate number of candidates who secured a first class for which a minimum of 60 is necessary. [**Ans.** 21. 253]

71. Find the probability of getting 5 heads in 12 tosses of a fair coin by using

i. Binomial distribution

ii. Normal approximation to Binomial distribution. [**Ans**. (i) 0.1934 ;(ii) 0.1975]

72. Ten percent of the tools produced in a certain manufacturing process turn out to be defective. Find the probability that in a sample of 10 tools chosen at random exactly two will be defective by using (i) the Binomial distribution (ii) Poisson approximation to the Binomial distribution. [**Ans.** (i) 0.194; (ii) 0.184]

Answers

B. Numerical and Practical Problems

1. (i) $\frac{1}{4}$ (ii) $\frac{1}{4}$ (iii) $\frac{1}{2}$ (iv) $\frac{3}{4}$

2. (i) $\frac{5}{36}$ (ii) $\frac{1}{36}$

3. (i) $\frac{7}{20}$ (ii) $\frac{4}{5}$ (iii) $\frac{9}{20}$ (iv) 1

4. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{13}$ (iv) $\frac{2}{13}$

(v) 1 (vi) $\frac{7}{13}$ (vii) $\frac{1}{52}$ (viii) $\frac{1}{4}$

5. (i) $\frac{6}{26}$ (ii) $\frac{21}{26}$

6. (a) $\frac{3}{5}$ (b) $\frac{3}{20}$ (c) $\frac{1}{2}$ (d) $\frac{2}{5}$

7. $\frac{2}{7}$

8. (a) i. 0.85, ii. 0.30, iii. 0.55, iv. 0.85 (b) i. 0.48, ii. 0.16, iii. 0.36

9. (a) i. 0.71, ii. 0.12, iii. 0.16, iv. 0.825, v. 0.536 (b) i. 0.5 and ii. 0.2

10. (a) $\frac{325}{1326}$ (b) $\frac{1}{17}$ (c) $\frac{676}{1326}$

11. (a) 0.0029 (b) 0.3829 (c) 0.1176

12. (a) $\frac{5}{42}$ (b) $\frac{5}{14}$ (c) $\frac{1}{6}$

13. (a) $\frac{1}{114}$ (b) $\frac{91}{228}$ (c) $\frac{137}{228}$

14. (a) $\frac{26}{165}$ (b) $\frac{59}{165}$ (c) $\frac{16}{33}$

15. (a) $\frac{13}{15}$ (b) 0.27 (c) 0.0875, 0.5125, 0.9125

16. $\frac{3}{5}$

17. (i) $\frac{1}{35}$ (ii) $\frac{24}{35}$ (iii) $\frac{11}{35}$ (iv) $\frac{2}{7}$

18. (i) $\frac{3}{10}$ (ii) $\frac{17}{20}$ **19.** 0.415

20. (i) 0.46 (ii) 0.88 **21.** (a) $\frac{1}{60}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$

22. (i) 0.25 (ii) 0.178 (iii) 0.761

23. (a) $\frac{1}{2}, \frac{3}{5}$ (b) $\frac{2}{5}, \frac{2}{3}$ A and B are not independent

(c) (i)) $\frac{2}{7}$, (ii) $\frac{5}{7}$ (iii) $\frac{2}{3}$ (iv) $\frac{1}{3}$

(d) $\frac{3}{7}, \frac{2}{3}$ (e) (i) 0.55 (ii) 0.2857

24. 0.5 **25.** 0.214 **26.** (a) 0.583 (b) 0.727

27. (i) 5/14, (ii) 3.28, (iii) 15/28, (iv) 13/28 **28.** 1/22 **29.** $\frac{1}{120}$

30. $\frac{7}{171}$

31. (i) 0.16 (ii) 0.48 (iii) 0.52 (iv) 0.24

32. 37/80 **33.** (i) 3/7, (ii) 4/7 **34.** (i) 0.8 (ii) 0.75 (iii) 0.95

35. 0.426 **36.** 0.84 **37.** 0.5446 **38.** 0.2588

39. 0.126 **40.** (i) 0.0192 (ii) 0.1154 (iii) 0.8652 **41.** (i) 0.511 (ii) 0.521

Exercise 4.3

Multiple Choice Questions circle (O) the correct answer.

1. The outcome of tossing a coin is a:

(a) simple event (b) mutually exclusive event

(c) complementary event (d) compound event

2. Classical probability is measured in terms of:

(a) an absolute value (b) a ratio

(c) absolute value and ratio both (d) none of the above

3. Probability can take values

(a) $-\infty$ to ∞ (b) $-\infty$ to 1 (c) -1 to 1 (d) 0 to 1

4. Probability is expressed as:

(a) ratio (b) proportion (c) percentage (d) all the above

5. Two events are said to be independent if:

(a) each outcome has equal chance of occurrence

(b) there is no common point in between them

(c) one does not affect the occurrence of the other

(d) both the events have only one point

6. If *A* and *B* are two events which have no point in common, the events *A* and *B* are:

(a) complementary to each other (b) independent

(c) mutually exclusive (d) dependent

7. Classical probability is also known as:

(a) Laplace's probability (b) mathematical probability

(c) a priori probability (d) all the above

8. Each outcome of a random experiment is called:

(a) Primary event (b) compound event

(c) derived event (d) all the above

9. If A and B are two events, the probability of occurrence of either A or B is given as:

(a) $P(A) + P(B)$ (b) $P(A \cup B)$ (c) $P(A \cap B)$ (d) $P(A)\, P(B)$

10. If A and B are two events, the probability of occurrence of A and B simultaneously is given as:

(a) $P(A) + P(B)$ (b) $P(A \cup B)$ (c) $P(A \cap B)$ (d) $P(A)\, P(B)$

11. The limiting relative frequency approach of probability is known as:

(a) statistical probability (b) classical probability (c) mathematical probability

(d) all the above

12. The definition of statistical probability was originally given by:

(a) De Moivre (b) Laplace (c) Von-Mises (d) Pascal

13. The definition of a priori probability was originally given by:

(a) De Moivre (b) Laplace (c) Von-Mises (d) Feller

14. If it is known that an event A has occurred, the probability of an event E given A is called:

(a) empirical probability (b) a priori probability

(c) posteriori probability (d) conditional probability

15. The probability of Mr R living 20years more is1/5 and that of Mr S is 1/7. The probability that at least one of them will survive 20 years hence is:

(a) 12/35 (b) 1/35 (c) 13/35 (d) 11/35

16. Given that $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$ and $P(A \cup B) = \frac{11}{12}$, probability, $P(B/A)$ is:

(a) 1/6 (b) 4/9 (c) 1/2 (d) none of the above

17. What is the minimum value of probability?

(a) 1 (b) 100 (c) 0 (d) $-\infty$

18. In binomial distribution which relation is true,

(a) Mean = Variance (b) Mean < variance (c) Mean $\geq$ variance (d) Mean $\leq$ variance

19. If for a binomial distribution $b(n, p) = 4$ and also $P(X = 2) = 3\,P(X = 3)$ the value of P is

(a) $\frac{9}{11}$ (b) 1 (c) $\frac{1}{3}$ (d) $\frac{5}{13}$

20. In which discrete probability distribution mean is equal to variance,

(a) binomial (b) Normal (c) Poisson (d) Gamma

21. For a normal distribution, the QD, MD and SD are in the ratio

(a) 5 : 6 : 7 (b) 10 : 12 : 15 (c) 2 : 3 : 4 (d) 10 : 13 : 17

22. If $X \sim N(\mu, \sigma^2)$, the points of inflexion of normal distribution curve is

(a) $\pm \mu$ (b) $\mu \pm \sigma$ (c) $\sigma^2 \pm \mu$ (d) $\pm \sigma$

23. X is a binomial variate with parameters n and P if $n = 1$, the distribution of X reduces to

(a) Poisson distribution (b) Binomial distribution

(c) Bernoulli distribution (d) Normal distribution

24. The are under the standard normal curve beyond lines $z = \pm 1.96$ is

(a) 95% (b) 90% (c) 5% (d) 10%

Answer Key

1. (a)	2. (c)	3. (d)	4. (b)	5. (c)	6. (c)	7. (b)	8. (d)	9. (b)	10. (c)	11. (a)	12. (c)	13. (b)
14. (d)	15. (d)	16. (c)	17. (c)	18. (c)	19. (c)	20. (c)	21. (b)	22. (b)	23. (c)	24. (a)		

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