Transformation

A transformation (or mapping or function) T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a rule that called the **range of T.** It is denoted by T : $\mathbb{R}^n \to \mathbb{R}^n$**.**

Example:-

Let A = $\begin{pmatrix} 1 & 2 \end{pmatrix}$, u = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and define a transformation defined by T(x) = Ax, Then find T(u).

Solution:-

Given T(x) = Ax

T(u) = Au = $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (5)$

Linear Transformation:-

A transformation(or mapping or function) T : $\mathbb{R}^n \to \mathbb{R}^n$ is said to be linear if

(i) T(u + v) = T(u) + T(v) for all u, v in the domain of T (i. e. T is **additive**).

(ii) T(ru) = r T(u) for a vector u in the domain of T and scalar r (i.e. T is **homogeneous**)

Equivlently, for all u, v ∈ V, T(ru +sv) = rT(u) + s T(v), where r and s are scalars.

Example:

Let T: $\mathbb{R} \to \mathbb{R}$ be a linear transformation defined by T(x) = mx, where m is a fixed real number(or a scalar). Show that T is a linear Transformation.

Solution:

We must show that T is adidtive and homogeneous.

i. For the additivity, ler u and v be in $\mathbb{R}$.

Given, T(x) = mx then

T(u+v) = m (u+v) = mu + mv = T(u) + T(v)

ii. For homoginity, let x be in $\mathbb{R}$ and r be any scalar then

T(ru) = m(ru) = r(mu) = r T(u).

Hence, T is linear.

Matrix of a Linear Transformation

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then there exists a unique matrix A such that T(x) = A x for all x in $\mathbb{R}^n$, where A = [$T(e_1)$, $T(e_2)$, ..., $T(e_n)$], where e_j is the j^{th} column of the identity matrix in $\mathbb{R}^n$.

Note:- For x in $\mathbb{R}^3$, $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$, ...

Example:-

Find the standard matrix A for the transformation T(x) = 2x for x in $\mathbb{R}^2$.

Solution:

In $\mathbb{R}^2$, we take $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$,

Given, T(x) = 2x then

$T(e_1) = 2e_1 = 2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ and $T(e_2) = 2e_2 = 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$

Hence, the standard matrix is

$A = [T(e_1), T(e_2)] = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$

Orthogonal Matrix

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$

Clearly, we see that $A^T = A^{-1}$.

Note: Determinant of any of any orthogonal matrix is ± 1.

Example:

Prove that the matrix $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ is an orthogonal matrix.

Solution:-

Given, $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

So, $A^T = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

Now,

$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Also,

$A^TA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since, $AA^T = A^TA = I$, A is orthogonal matrix.

Orthogonal Transformation

A linear transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by T(x) = A x is said to be orthogonal transformation if the matrix A is orthogonal.

In another words, A linear transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is called orthogonal if it preserves the length. That is length of T(x) = length of x for all x in R^n.

Example:

The transformation T: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by T(x) = A x where,

$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal transformation since A is an orthogonal matrix.

Rank of Matrices

An m×n matrix is said to have a rank **r** if it has at least one square submatrix of order r which is **non-singular** and all submatrice of order greater than r are singular. It is denoted by ρ(A) and read as rank of A.Note: (i) The rank of an m×n matrix can at most be equal to the smaller of the numbers m and n. i.e. Rank = min{m, n}.

ii. An n×n matrix A has rank iff |A| ≠ 0.

iii. An n×n matrix A has rank less than n iff |A|= 0.

Example:

a. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

So, rank of A is less than 2.

Consider a submatrix of order 1; say [2]

$|2| = 2 \neq 0$

So, rank of A = 1

b. Let $A = \begin{bmatrix} 2 & 3 & 0 \\ -5 & 2 & 1 \end{bmatrix}$

The size of A is 2×3. So, consider a submatrix of order 2 namely $\begin{bmatrix} 2 & 3 \\ -5 & 2 \end{bmatrix}$

$\begin{vmatrix} 2 & 3 \\ -5 & 2 \end{vmatrix} = 4 + 15 = 19 \neq 0$

So, rank of A = 2.

c. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 4 & 8 & 12 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 4 & 8 & 12 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 4 \times 0 \; (\because R_1 = R_3) = 0$

So, rank of A is not 3

Consider a submatrix of order 2 namely $\begin{bmatrix} 1 & 2 \\ -4 & 7 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ -4 & 7 \end{vmatrix} = 7 + 8 = 15 \neq 0$

Rank A = 2.

5. **Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by**

(a) $T(x, y, z) = (x, y, 0)$ is linear. **(b) $T(x, y, z) = (0, y, z)$ is linear.**

Solution

a. Let $u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$ in $\mathbb{R}^3$

Let α and β be the scalars.

$T(\alpha u + \beta v) = T(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2))$

$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, 0)$

$= (\alpha x_1, \alpha y_1, 0) + (\beta x_2, \beta y_2, 0) = \alpha(x_1, y_1, 0) + \beta(x_2, y_2, 0) = \alpha T(u) + \beta T(v)$

$\therefore$ T is linear.

b. Let $u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$ in $\mathbb{R}^3$

Let α and β be the scalars.

$T(\alpha u + \beta v) = T(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2))$

$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$= (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) = (0, \alpha y_1, \alpha z_1) + (0, \beta y_2, \beta z_2) = \alpha(0, y_1, z_1) + \beta(0, y_2, z_2)$

$= \alpha T(u) + \beta T(v)$

$\therefore$ T is linear.

6. Show that the transformation T defined by $T(x_1,x_2) = (4x_1-2x_2, 3|x_2|)$ is not linear.

Solution: Consider, $(1,-1), (0, 1) \in \mathbb{R}^2$.

$T(1,-1) = (4.1-2.(-1), 3|-1|) = (6, 3)$

$T(0, 1) = (4.0-2.1, 3.|1|) = (-2, 3)$

$T(1, -1) + T(0, 1) = (6, 3) + (-2, 3) = (4, 6)$ .....(i)

And $T((1, -1) + (0, 1)) = T(1, 0) = (4.1 – 2.0, 3. |0|) = (2, 0)$ .....(ii)

From (i) and (ii), $T(1, -1) + T(0, 1) \neq T((1, -1)\ \ (0, 1))$

i.e. $T(u) + T(v) \neq T(u, v)$. Hence, T is not linear.

Worked Out Examples

1. If $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1+x_2, x_2, x_1)$ be a the linear transformation the find matrix associatd with this linear map.

Solution: see Asmita's Book (Example-3), Page 195.

Let matrix associated with T be $A= [T(e_1), T(e_2)]$,

where, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Given, $T(x_1, x_2) = (x_1+x_2, x_2, x_1)$. Then

$T(e_1) = T(1,0) = (1+0, 0, 1) = (1, 0, 1)$

$T(e_2) = T(0,1) = (0+1, 1, 0) = (1, 1, 0)$

Now, the matrix associated with the map T is

$$A = [T(e_1), T(e_2)] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2. If $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x+y, y, x)$ then prove that T is a linear transformation.

Solution:

Let, $u = (x_1,y_1)$ and $v = (x_2, y_2)$ in $\mathbb{R}^2$ and r and s be two scalars.

Now, $T(ru + sv) = T(r(x_1,y_1) + s(x_2, y_2)) = T((rx_1,ry_1) + (sx_2, sy_2)) =$

$T((rx_1+ sx_2) , (ry_1 + sy_2))$

$= ((rx_1+ sx_2+ ry_1+sy_2) , (ry_1 + sy_2), (rx_1+ sx_2))$

$= (r(x_1 + y_1)+ s(x_2+sy_2)) , (ry_1 + sy_2), (rx_1+ sx_2))$

$= (r(x_1+ y_1), ry_1, rx_1)+ (s(x_2+ y_2), sy_2, sx_2)$

$= r(x_1+ y_1, y_1, x_1)+ s(x_2+ y_2, y_2, x_2)$

$= r\, T(x_1,y_1) + s\, T(x_2, y_2)$

$= r\, T(u) + s\, T(v)$

Hence, T is linear.

2. Given $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Transform u, v, u + v and u − v by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution

Let, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Here, $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$Au = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 4+0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Scanned by CamS

$$Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-3 \\ 2+0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$A(u+v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4+2 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0-4 \\ 6+0 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

$$A(u-v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4-2 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0+2 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

4. (a) Transform $u = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and check whether th

transformation is linear or not?

(b) Transform $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and show th

this transformation is linear.

Solution

a. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$Au = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Let, $T(x) = Ax$

$$\text{Now, } T(u+v) = A(u+v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2+2 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{... (i)}$$

$$T(u) + T(v) = Au + Av = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{... (ii)}$$

From (i) and (ii),

$T(u+v) = T(u) + T(v)$

Let α be a scalar,

$T(\alpha u) = A(\alpha u)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2\alpha \\ -2\alpha \end{bmatrix} = \begin{bmatrix} 2\alpha+0 \\ 0+2\alpha \end{bmatrix} = \begin{bmatrix} 2\alpha \\ 2\alpha \end{bmatrix} = \alpha\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \alpha \cdot Au = \alpha T($$

And, $T(\alpha v) = A(\alpha v)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 2\alpha \\ -\alpha \end{bmatrix} = \alpha\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \alpha\, Av = \alpha\, T(v)$$

$\therefore$ T is linear.

9. **If a transformation T is defined by $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that T is orthogonal transformation.**

Solution

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore \quad AA^T = A^TA = I.$

So, A is orthogonal.

Hence, T is orthogonal transformation.