BCA

Unit – 1

Set Theory and Real & Complex Number

EXERCISE – 1 A

1. (a) Given U = {1, 2, 3, 4, …, 12}, A = {2, 3, 5, 6, 8, 10}, B = {1, 4, 6, 9, 10} and C = {2, 3, 6, 8, 12}. Find:

(i) $A \cup B$ (ii) $A - B$

(iii) $(A \cup C)^c$ (iv) $(B \cap C)^c$ (v) $A \Delta B$

(b) Given:

U = {1, 2, 3, 4, …, 10}, A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6}. Find:

(i) $(A \cup B) \cup C$ (ii) $(A \cup B) - C$ (iii) $(A - B) \cap C$

(c) Given U = {1, 2, 3, …, 20}, A = $\{x : x \geq 10\}$, B = $\{x : x \leq 14\}$ where, A and B are subsets of the universal set U. Find:

(i) $A - B$ (ii) $A^c \cup B^c$ (iii) $(B - A)^c$

(d) If U = $\{x : 2 \leq x + 1 \leq 11, x$ is an integer$\}$ A = $\{x : x$ is an even number$\}$ and B = $\{x : x$ is a prime number$\}$; find the followings.

(i) $A \cap B$ (ii) $(A \cap B)^c$ (iii) $(A \cup B)^c$

Solution

(a) (i) $A \cup B$ = {2, 3, 5, 6, 8, 10} $\cup$ {1, 4, 6, 9, 10} = {1, 2, 3, 4, 5, 6, 8, 9, 10}

(ii) $A - B$ = {2, 3, 5, 6, 8, 10} – {1, 4, 6, 9, 10} = {2, 3, 5, 8}

(iii) $(A \cup C)$ = {2, 3, 5, 6, 8, 10} $\cup$ {2, 3, 6, 8, 12} = {2, 3, 5, 6, 8, 10, 12}

$\therefore$ $(A \cup C)^c$ = U – $(A \cup C)$ = {1, 2, ..., 12} – {2, 3, 5, 6, 8, 10, 12} = {1, 4, 7, 9, 11}

(iv) $(B \cap C)$ = {1, 4, 6, 9, 10} $\cap$ {2, 3, 6, 8, 12} = {6}

$\therefore$ $(B \cap C)^C$ = U – $(B \cap C)$ = {1, 2, ..., 12} – {6} = {1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12}

(v) $A \Delta B$ = $(A - B) \cup (B - A)$ = {1, 2, 3, 4, 5, 8, 9}

(b) (i) $(A \cup B)$ = {1, 2, 3, 4,} $\cup$ {2, 4, 6, 8,} = {1, 2, 3, 4, 6, 8}

$(A \cup B) \cup C$ = {1, 2, 3, 4, 6, 8} $\cup$ {3, 4, 5, 6} = {1, 2, 3, 4, 5, 6, 8}

(ii) $(A \cup B) - C$ = {1, 2, 3, 4, 6, 8} – {3, 4, 5, 6} = {1, 2, 8}

(iii) $(A - B)$ = {1, 2, 3, 4,} – {2, 4, 6, 8,} = {1, 3}

$(A - B) \cap C$ = {1, 3} $\cap$ {3, 4, 5, 6} = {3}

(c) Given U = {1, 2, 3, ..., 20}

A = $\{x : x \geq 10\}$ = {10, 11, ..., 20} B = $\{x : x \leq 14\}$ = {1, 2, ..., 14}

(i) $A - B$ = {10, 11, ..., 20} – {1, 2, ..., 14} = {15, 16, 17, 18, 19, 20}

(ii) A^c = U – A = {1, 2, 3, ..., 20} – {10, 11, ..., 20} = {1, 2, ..., 9}

B^c = U – B = {1, 2, 3, ..., 20} – {1, 2, ..., 14} = {15, 16, 17, 18, 19, 20}

$\therefore$ $A^c \cup B^c$ = {1, 2, ..., 9} $\cup$ {15, 16, 17, 18, 19, 20}

= {1, 2, 3, 4, 5, 6, 7, 8, 9, 15, 16, 17, 18, 19, 20}

(iii) $B - A$ = {1, 2, ..., 14} – {10, 11, ..., 20} = {1, 2, 3, ..., 9}

$\therefore$ $(B - A)^c$ = U – $(B - A)$ = {1, 2, 3, ..., 20} – {1, 2, 3, ..., 9} = {10, 11, ..., 20}

(d) Here, $U = \{x : 2 \le x + 1 \le 11, x \text{ is an integer}\} = \{x : 1 \le x \le 10, x \text{ is an integer}\} = \{1, 2, ..., 10\}$
$A = \{x : x \text{ is an even number}\} = \{2, 4, 6, 8, 10\}$
$B = \{x : x \text{ is a prime number}\} = \{2, 3, 5, 7\}$

(i) $(A \cap B) = \{2, 4, 6, 8, 10\} \cap \{2, 3, 5, 7\} = \{2\}$

(ii) $(A \cap B)^c = U - (A \cap B) = \{1, 2, ..., 10\} - \{2\} = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$

(iii) $A \cup B = \{2, 4, 6, 8, 10\} \cap \{2, 3, 5, 7\} = \{2, 3, 4, 5, 6, 7, 8, 10\}$

$\therefore \quad (A \cup B)^c = U - (A \cup B) = \{1, 2, ..., 10\} - \{2, 3, 4, 5, 6, 7, 8, 10\} = \{1, 9\}$

2. (a) If $A = \{a, b, x, y\}$ and $B = \{c, d, x, y\}$ then find the following by Venn diagram (i) $B - A$ (ii) $A \cup B$.

(b) Given set $U = \{x : x$ is a positive integer less than $11\}$, $A = \{2, 3, 6, 7, 9\}$, $B = \{2, 4, 6, 8\}$ and $C = \{2, 3, 4, 5, 9\}$. Find by Venn-diagram.

(i) $B \cup C$ (ii) $A - B$

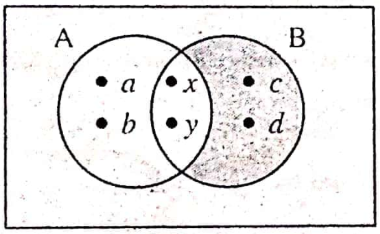
(iii) $A \cup (B \cup C)$ (iv) $\overline{A \cap (B \cap C)}$.

Solution

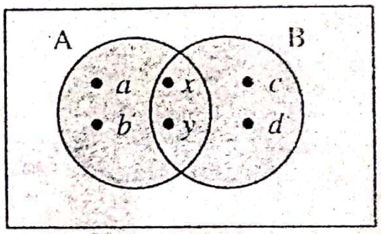
(a) $U = \{a, b, c, d, x, y\}$

(i) $B - A = \{c, d, x, y\} - \{a, b, x, y\}$
$B - A = \{c, d\}$

(ii) $A \cup B = \{a, b, c, d, x, y\}$



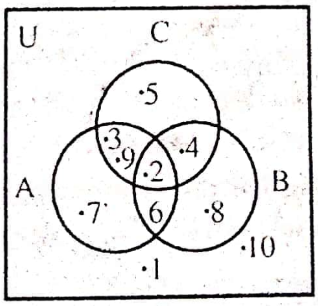
B – A



A ∪ B

(b) $U = \{1, 2, 3, ..., 10\}$
$A = \{2, 3, 6, 9\}$
$B = \{2, 4, 6, 8\}$
$C = \{2, 3, 4, 5, 9\}$

Venn-diagram



From Venn-diagram

(i) $B \cup C = \{2, 3, 4, 5, 6, 8, 9\}$

(ii) $A - B = \{3, 7, 9\}$

(iii) $A \cup (B \cup C) = \{2, 3, 4, 5, 6, 7, 8, 9\}$

(iv) $\overline{A \cap (B \cap C)} = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$

3. Let $U = \{a, b, c, d, e\}$; $A = \{a, b, c\}$; $B = \{c, d, e\}$; $C = \{b, c\}$.

Verify that

(i) $(A \cap B)^c = A^c \cup B^c$ (ii) $(A \cup B)^c = A^c \cap B^c$

(iii) $A - (B \cap C) = (A - B) \cup (A - C)$.

Solution

(i) $A \cap B = \{c\}$

$(A \cap B)^c = U - (A \cap B) = \{a, b, c, d, e\} - \{c\} = \{a, b, d, e\}$

$A^c = U - A = \{a, b, c, d, e - \{a, b, c\} = \{d, e\}$

$B^c = U - B = \{a, b, c, d, e\} - \{c, d, e\} = \{a, b\}$

$A^c \cup B^c = \{d, e\} \cup \{a, b\} = \{a, b, d, e\}$

$\therefore (A \cap B)^c = A^c \cup B^c$ Verified

(ii) $A \cup B = \{a, b, c, d, e\}$

$(A \cup B)^c = U - (A \cup B) = \{a, b, c, d, e\} - \{a, b, c, d, e\} = \phi$

$A^c \cap B^c = \{d, e\} \cap \{a, b\}$ [From (i)]

$= \phi$

$\therefore (A \cup B)^c = A^c \cap B^c$ Verified.

(iii) $B \cap C = \{c, d, e\} \cap \{b, c\} = \{c\}$

$A - (B \cap C) = \{a, b, c\} - \{c\} = \{a, b\}$

$A - B = \{a, b, c\} - \{c\} = \{a, b\}$

$A - C = \{a, b, c\} - \{b, c\} = \{a\}$

$(A - B) \cup (A - C) = \{a, b\} \cup \{a\} = \{a, b\}$

$\therefore A - (B \cap C) = (A - B) \cup (A - C)$ Verified.

4. **If $n(U) = 200$, $n(A) = 150$, $n(B) = 80$, $n(A \cup B) = 160$ find $n(A \cap B)$, $n(A - B)$, $n(\overline{A \cup B})$. Also if $B \subset A$, find $n(A \cup B)$ and $n(A \cap B)$.**

Solution

$n(U) = 200 n(A) = 150$ $n(B) = 80$ $n(A \cup B) = 160$

$n(A \cap B) = ?$ $n(A - B) = ?$ $n(\overline{A \cup B}) = ?$

We have,

$n(A \cap B) = (A) + n(B) - n(A \cup B) = 150 + 80 - 160 = 70$

$n(A - B) = n(A) - n(A \cap B) = 150 - 70 = 80$

$n(\overline{A \cup B}) = n(U) - n(A \cup B) = 200 - 160 = 40$

Next Part

If $B \subset A$, then

$n(A \cup B) = n(A) = 150$ and $n(A \cap B) = n(B) = 80$

5. **If $n(U) = 100$, $n(A) = 70$ and $n(B) = 40$, find:**

(a) maximum value of $n(A \cup B)$ (b) maximum value of $n(A \cap B)$

(c) minimum value of $n(A \cup B)$ (d) minimum value of $n(A \cap B)$.

Solution

(a) $n(A \cup B)$ is maximum when $A \cap B = \phi$.

In this case, $n(A \cup B) = n(A) + n(B) = 70 + 40 = 110 > n(U)$ which is not possible.

Hence, maximum value of $n(A \cup B) = 100$ as $n(U) = 100$.

(b) $n(A \cap B)$ is maximum when $B \subset A$. [$\because n(B) < n(A)$]

Thus, maximum value of $n(A \cap B) = n(B) = 40$.

(c) Minimum value of $n(A \cup B) = n(A) + n(B) -$ max. value of $n(A \cap B) = 70 + 40 - 40 = 70$

Alternatively,

$n(A \cup B)$ is minimum when $B \subset A$ as $n(B) < n(A)$.

$\therefore$ Min. value of $n(A \cup B) = n(A) = 70$.

(d) Minimum value of $n(A \cap B) = n(A) + n(B) -$ max. value of $n(A \cup B) = 70 + 40 - 100 = 10$

6. **20 students play football and 15 students play hockey. It is found that 5 students play both games. Find the number of students playing at least one game.**

Solution

$n(F) = 20$ $\quad$ $n(H) = 15$

$n(F \cap H) = 5$ $\quad$ $n(F \cup H) = ?$

We have,

$n(F \cup H) = n(F) + n(H) - n(F \cap H) = 20 + 15 - 5 = 30.$

7. **In a survey of a city market, it was found that 143 families used Colgate toothpaste. 135 used Everest toothpaste and 70 families used both. Find the number of families using at least one type of toothpaste.**

Solution

$n(C)$ = no. of people who used Colgate toothpaste = 143

$n(E)$ = no. of people who used Everest toothpase = 135

$n(C \cap E)$ = no. of people who used both toothpaste = 70,

$n(C \cup E)$ = number of families using at least one type of toothpaste

$= n(C) + n(E) - n(C \cap E)$

$\therefore$ $n(C \cup E) = 143 + 135 - 70 = 208.$

8. **In a college of 500 students, 400 use Facebook, 300 use Twitter and 50 use neither of them. Find the number of students who use both Facebook and Twitter.**

Solution

Total number of students, $n(U) = 500$

Number of students using Facebook, $n(F) = 400$

Number of students using Twitter, $n(T) = 300$

Number of students who use neither of them, $n(\overline{F \cup T}) = 50$

Number of students who use both, $n(F \cap T) = ?$

We have,

$n(F \cup T) = n(U) - n(\overline{F \cup T}) = 500 - 50 = 450$

Also,

$n(F \cap T) = n(F) + n(T) - n(F \cup T) = 400 + 300 - 450 = 250$

9. **32 students play basketball and 25 students play volleyball. It is found that 20 students play both the games. Find the number of students playing at least one game. Also, find total number of students if 13 students play none of these games.**

Solution

$n(B) = 32$ $\quad$ $n(V) = 25$

$n(B \cap V) = 20$ $\quad$ $n(B \cup V) = ?$

$n(\overline{B \cup V}) = 13$ $\quad$ $n(U) = ?$

We have, $n(B \cup V) = n(B) + n(V) - n(B \cap V) = 32 + 25 - 20 = 37$

Again, we have, $n(U) = n(B \cup V) + n(\overline{B \cup V}) = 37 + 13 = 50.$

10. **In a city of 26000 populations, 5000 read English newspaper, 12000 read Nepali newspaper and 1000 read both. What percentage read neither English nor Nepali newspaper?**

Solution

$n(U)$ = total population in a city = 26,000

$n(A)$ = population who read English local newspaper = 5000

$n(B)$ = population who read Nepali local newspaper = 12000

$n(A\cap B)$ = population who read both newspaper = 1000

$n(A\cup B)$ = population who read at least one newspaper

$= n(A) + n(B) - n(A\cap B) = 5000 + 12000 - 1000 = 16000$

$n(A\cup B)^c$ = population who read neither English nor Nepali newspaper

$= n(U)\ n(A\cup B) = 26000 - 16000 = 10000$

% of population who read neither English nor Nepali newspaper

$$= \frac{10,000}{26,000} \times 100 = 38.46\%$$

11. In a survey of a city market. It was noted that 300 families were randomly selected, out of which 142 used Laptop and 139 used Desktop computers and 70 families used both. Find the number of families who used exactly one of these types of computers.

Solution

$n(U)$ = total number of families = 300

$n(L)$ = number of families who used Laptop = 142

$n(D)$ = number of families who used Desktop = 139

$n(L\cap D)$ = number of families who used both computers = 70

$n_0(L) = n(L) - n(L\cap D) = 142 - 70 = 72$

$n_0(D) = n(D) - n(L\cap D) = 139 - 70 = 69$

Required no. of families who used exactly one of the computers

$= n_0(L) + n_0(D) = 72 + 69 = 141$

12. In a market survey of 1000 consumers of tea it was found that 500 purchased Soktim Tea, 400 purchased Tokla Tea and 150 purchased both brands. How many purchased (a) Soktim only (b) Tokla only (c) exactly one of these brands and (d) neither of them.

Solution

$n(U)$ = total number of consumers = 1000

$n(A)$ = no. of consumers who purchased Soktim Tea = 500

$n(B)$ = no. of consumers who purchased Tokla Tea = 400

$n(A\cap B)$ = no. of consumers who purchased both brands of tea = 150

(a) $n_0(A)$ = no. of consumers who purchased only Soktim Tea

$= n(A) - n(A\cap B) = 500 - 150 = 350.$

(b) $n_0(B)$ = no. of consumers who purchased only Tokla Tea

$= n(B) - n(A\cap B) = 400 - 150 = 250.$

(c) Total number of consumers who purchased exactly one of the brands

$= n_0(A) + n_0(B) = 350 + 250 = 600.$

(d) $n(A\cup B)^c$ = no. of consumers who purchased neither of them

$= n(U) - n(A\cup B) = 1000 - [n(A) + n(B) - n(A\cap B)]$

$= 1000 - [500 + 400 - 150] = 1000 - 750 = 250.$

13. Of the number of three athletic teams, 25 are in the basketball team, 30 in hockey team and 28 in football team, 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. How many members are there in all?

Solution

$n(B) = 25$ $\quad$ $n(H) = 30$

$n(F) = 28$ $\quad$ $n(B\cap H) = 14$

$n(H\cap F) = 15$ $\quad$ $n(F\cap B) = 12$

$n(B\cap H\cap F) = 8$ $\quad$ $n(B\cup H\cup F) = ?$

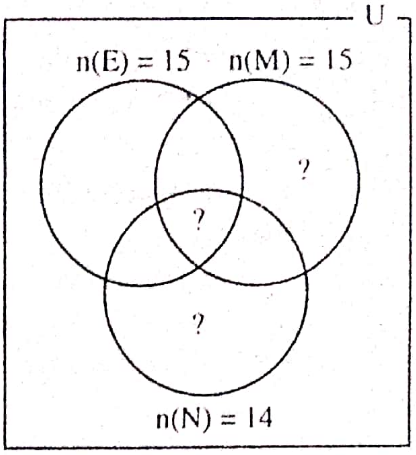
We have,

$n(B\cup H\cup F) = n(B) + n(H) + n(F) - n(B\cap H) - n(H\cap F) - n(F\cap B) + n(B\cap H\cap F)$

$= 25 + 30 + 28 - 14 - 15 - 12 + 8 = 50.$

14. In a group of twenty eight teachers of a school, 15 teach English, 15 teach Maths, 14 teach Nepali, 7 teach English and Maths, 6 teach English and Nepali, 5 teach Maths and Nepali. Find how many teach (a) all three subjects, (b) Maths only (c) Nepali only.

Solution



Let E, M and N be the sets of teachers teaching English, Maths and Nepali respectively. Now,

$n(E \cup M \cup N)$ = no. of teachers of a school = 28

$n(E)$ = no. of teachers teaching English = 15

$n(M)$ = no. of teachers teaching Maths = 15

$n(N)$ = no. of teachers teaching Nepali = 14

$n(E \cap M)$ = no. of teachers teaching English and Maths = 7

$n(E \cap N)$ = no. of teachers teaching English and Nepali = 6

$n(M \cap N)$ = no. of teachers teaching Maths and Nepali = 5

$n(E \cap M \cap N)$ = no. of teachers teaching all 3 subjects = ?

$n_0(M)$ = no. of teachers teaching Maths only = ?

$n_0(N)$ = no. of teachers teaching Nepali only = ?

We know that,

$n(E \cup M \cup N) = n(E) + n(M) + n(N) - n(E \cap M) - n(M \cap N) - n(E \cap N) + n(E \cap M \cap N)$

or, $28 = 15 + 15 + 14 - 7 - 5 - 6 + n(E \cap M \cap N)$

or, $n(E \cap M \cap N) = 28 - 26$

$\therefore$ $n(E \cap M \cap N) = 2$

Again,

$n_0(M) = n(M) - n(E \cap M) - n(M \cap N) + n(E \cap M \cap N)$

$= 15 - 7 - 5 + 2$

$\therefore$ $n_0(M) = 5$

and, $n_0(N) = n(N) - n(M \cap N) - n(E \cap N) + n(E \cap M \cap N)$

$= 14 - 5 - 6 + 2$

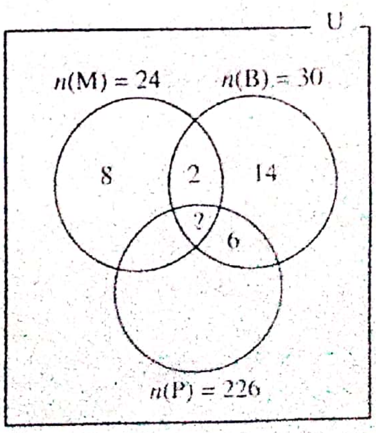
$n_0(N) = 5$

15. In a group of students 24 study Maths, 30 study Biology, 22 study Physics, 8 study Math only, 14 study Biology only, 6 study Biology and physics only and 2 study Maths and Biology only. Find:

(a) How many study all three subjects?

(b) How many students were in the group?

Solution



Let M, B and P be the sets of students studying Mathematics, Biology and Physics respectively. Then,

$n(M)$ = no. of students studying Maths = 24

$n(B)$ = no. of students studying Biology = 30

$n(P)$ = no. of students studying Physics = 22

$n_0(M)$ = no. of students studying Maths only = 8

$n_0(B)$ = no. of students studying Biology only = 14

$n_0(B \cap P)$ = no. of students studying Biology and Physics only = 6

$n_0(M \cap B)$ = no. of students studying Maths and Biology only = 2

$n(M \cap B \cap P)$ = no. of students studying all three subjects = ?

$n(M \cup B \cup P)$ = no. of students in the group = ?

Now,

a. We know that

$$n(B) = n_0(B) + n_0(B \cap P) + n_0(M \cap B) + n(M \cap B \cap P)$$

or, $30 = 14 + 6 + 2 + n(M \cap B \cap P)$

$\therefore$ $n(M \cap B \cap P) = 30 - 22 = 8$

b. We have,

$$n(M) = n_0(M) + n_0(M \cap B) + n_0(M \cap P) + n(M \cap B \cap P)$$

or, $24 = 8 + 2 + n_0(M \cap P) + 8$

$\therefore$ $n_0(M \cap P) = 24 - 18 = 6$

Again, we have,

$$n(P) = n_0(P) + n_0(B \cap P) + n_0(M \cap P) + n(M \cap B \cap P)$$

or, $22 = n_0(P) + 6 + 6 + 8$

or, $n_0(P) = 22 - 20 = 2$

Now, $n(M \cup B \cup P) = n_0(M) + n_0(B) + n_0(P) + n_0(B \cap P) + n_0(M \cap B) + n_0(M \cap P) + n(M \cap B \cap P) = 8 + 14 + 2 + 6 + 2 + 6 + 8 = 46.$

16. In a city of 50,000 population, 20,000 read The Rising Nepal, 25,000 read The Kathmandu Post, 30,000 read The Annapurna, 10,000 read none of these newspapers, 5,000 read the Rising Nepal and The Kathmandu Post 15,000 read The Rising Nepal and The Annapurna and 20,000 read the Kathmandu Post and the Annapurna. Find

(a) The number of readers reading all newspapers.

(b) The number of readers reading the Rising Nepal only.

(c) The number of readers reading the Kathmandu Post only.

(d) The number of readers reading the Kathmandu Post and the Annapurna only.

Solution

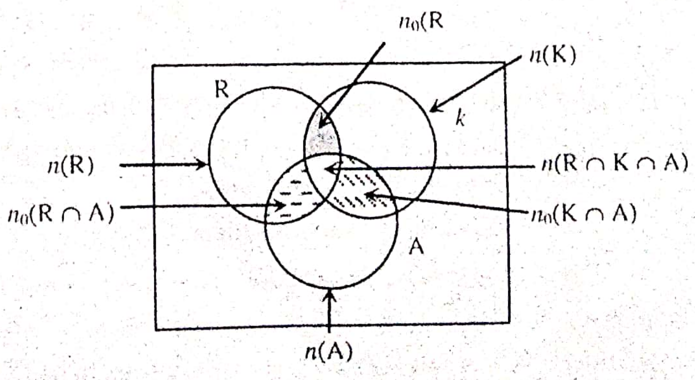
Let U denote the set of total population and R, K and A denote the set of population reading Rising Nepal, Kathmandu Post and Annapurna restpectively. Then,

$n(U) = 50000$

$n(R) = 20000$

$n(K) = 25000$

$n(A) = 30000$



$n(\overline{R \cup K \cup A}) = 10000$

$n(R \cap K) = 5000$

$n(R \cap A) = 15000$

$n(K \cap A) = 20000$

(a) $n(R \cap K \cap A) = ?$

$n(R \cup K \cup A) = n(U) - n(\overline{R \cup K \cup A}) = 50000 - 10000 = 40000$

$n(R \cup K \cup A) = n(R) + n(K) + n(A) - n(R \cap K) - n(K \cap A) - n(A \cap R) + n(R \cap K \cap A)$

or, $40000 = 20000 + 25000 + 30000 - 5000 - 20000 - 15000 + n(R \cap K \cap A)$

or, $n(R \cap K \cap A) = 5000$

(b) $n_0(R) = n(R) - n(R \cap K) - n(R \cap A) + n(R \cap K \cap A)$

$= 20000 - 5000 - 15000 + 5000 = 5000$

(c) $n_0(K) = n(K) - n(R \cap K) - n(K \cap A) + n(R \cap K \cap A)$

$= 25000 - 5.000 - 20000 + 5000 = 5000$

(d) $n_0(K \cap A) = n(K \cap A) - n(R \cap K \cap A) = 20000 - 5000 = 15000.$

Objective Questions

1. Which of the following collection is not a set?

(a) $\{x : x^2 + 1 = 0, x \in \mathbb{R}\}$

(b) collection of smart BCA students of Tribhuvan University.

(c) collection of rivers of Nepal.

(d) collection of prime numbers.

Ans: b

Options a, c and d are well defined but b is not well defined.

2. The set $\{x : x^2 + 4 = 0, x \in \mathbb{R}\}$ is a / an

(a) null set (b) singleton set

(c) infinite set (d) set having two students.

Ans: a

$x^2 + 4 = 0$

$\therefore$ $x^2 = -4$

Since square of any real numbers can't be negative. So this equation has no real solution.

So, $\{x: x^2 + 4 = 0, x \in \mathbb{R}\} = \phi$

3. The number of subsets of set $\{a, b, c\}$ is

(a) 3 (b) 4 (c) 6 (d) 8

Ans: d

$n = 3$

No. of subsets $= 2^n = 2^3 = 8$

4. The number of proper subsets of A = $\{a, b, c, d\}$ is

(a) 15 (b) 16 (c) 31 (d) 32

Ans: a

$n = 4$

No. of proper subsets $= 2^n - 1 = 2^4 - 1 = 15$

5. If A = {1, 2, 3} then $n(P(A)) =$

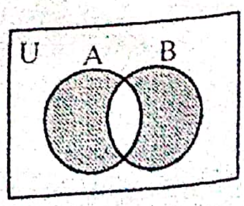
(a) 3 (b) 6 (c) 8 (d) 12

Ans: c

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

$n(P(A)) = 8$

6. The shaded region in the Venn-diagram is



(a) $A \cup B$ (b) $A \cap B$

(c) $A - B$ (d) $A \Delta B$

Ans: d

7. Which of the following is true?

(a) $A - B = B - A$ (b) $A - B \subseteq A \cup B$

(c) $A - B \subseteq A \cap B$ (d) $A \cup B \subseteq A \cap B$

Ans: b

8. If A and B are two subsets of universal set U, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$ is called

(a) Identity laws (b) Associate laws

(c) Distributive laws (d) De-Morgans' law

Ans: d

9. Let $A = \{x : x^2 = 16, x \in \mathbb{R}\}$ and $B = \{x : 2x = 8\}$ then $A \cap B =$

(a) ϕ (b) $\{4\}$ (c) $\{-4, 4\}$ (d) $\{8, 16\}$

Ans: b

$A = \{x : x^2 = 16, x \in \mathbb{R}\} = \{-4, 4\}$

$B = \{x : 2x = 8, x \in \mathbb{R}\} = \{4\}$

$A \cap B = \{-4, 4\} \cap \{4\} = \{4\}$

10. If $A = \{x : x = 2n + 1, n \leq 4, n \in \mathbb{N}\}$ and $B = \{7, 9, 11\}$ then $A - B =$

(a) $\{3, 5\}$ (b) $\{3, 5, 7\}$ (c) $\{7\}$ (d) $\{9, 11\}$

Ans: a

$A = \{x : x = 2n + 1, n \leq 4, n \in \mathbb{N}\} = \{3, 5, 7, 9\}$

$B = \{7, 9, 11\}$

$A - B = \{3, 5, 7, 9\} - \{7, 9, 11\} = \{3, 5\}$

11. If $n(A) = 30$, $n(B) = 32$ and $n(A \cap B) = 12$ then $n(A \cup B) =$

(a) 30 (b) 32 (c) 2 (d) 50

Ans: d

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 32 - 12 = 50$

12. If $n(U) = 150$, $n(A) = 90$, $n(B) = 70$, what is the maximum value of $n(A \cup B)$?

(a) 70 (b) 90 (c) 150 (d) 160

Ans: c

For max. value of $n(A \cup B)$, $A \cap B = \phi$

For this, $n(A \cup B) = n(A) + n(B) = 90 + 70 = 160 > n(U)$

Max. value of $n(A \cup B) = 150$.

13. In a group of 130 people, 80 people like football, 20 like both football and cricket. How many like cricket and not football?

(a) 50 (b) 40 (c) 30 (d) 15

Ans: a

$n(F) = 80$, $n(F \cap C) = 20$,

$n_0(F) = n(F) - n(F \cap C) = 80 - 20 = 60$

$n_0(C) = 130 - 80 = 50$

14. If A and B are two subsets of universal set U such that $n(U) = 350$, $n(A) = 100$, $n(B) = 150$ and $n(A \cap B) = 50$, then $n(\overline{A} \cap \overline{B}) =$

(a) 200 (b) 250 (c) 300 (d) 150

Ans: d

$n(\overline{A} \cap \overline{B}) = n(\overline{A \cup B})$ [$\because$ From De Morgan's law, $\overline{A \cup B} = \overline{A} \cap \overline{B}$]

$= n(U) - [n(A) + n(B) - n(A \cap B)] = 350 - [100 + 150 - 50] = 150$

EXERCISE – 1 B

1. (a) If $-10 < 5x + 10 < 5$, prove that $-4 < x < -1$.
   (b) If $-5 < 7x + 9 < 30$, prove that $-2 < x < 3$.
   (c) If $-8 \le 2x + 2 < -2$ prove that $-5 \le x < -2$.
   (d) If $0 \le 3x + 9 \le 27$, prove that $-3 \le x \le 6$.

Solution

(a) Here, $-10 < 5x + 10 < 5$
Adding -10 to each side
or, $-10 - 10 < 5x + 10 - 10 < 5 - 10$
or, $-20 < 5x < -5$
Dividing each sides by 5, we get
or, $-4 < x < -1$.

(b) Here, $-5 < 7x + 9 < 30$
Adding -9 to each side
or, $-5 - 9 < 7x < 30 - 9$
or, $-14 < 7x < 21$
Dividing each side by 7
or, $-2 < x < 3$.

(c) Here, $-8 \le 2x + 2 < -2$
Adding -2 on each side
or, $-8 - 2 \le 2x < -2 - 2$
or, $-10 \le 2x < -4$
Dividing 2 on each side
or, $-5 \le x < -2$.

(d) Here, $0 \le 3x + 9 \le 27$
Adding -9 to each side
or, $-9 \le 3x \le 27 - 9$
or, $-9 \le 3x \le 18$
Dividing each side by 3
or, $-3 \le x \le 6$
or, $-3 \le x \le 6$.

2. If $x = -3$, $y = 5$, verify that:
   (a) $|x + y| \le |x| + |y|$
   (b) $|x| - |y| \le |x - y|$
   (c) $|xy| = |x||y|$
   (d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

Solution

(a) $|x + y| = |-3 + 5| = |2| = 2$
$|x| + |y| = |-3| + |5| = 3 + 5 = 8$
$\because$ $2 < 8$, so $|x + y| < |x| + |y|$ verified.

(b) $|x| - |y| = |-3| - |5| = 3 - 5 = -2$
$|x - y| = |-3 - 5| = |-8| = 8$
$\therefore$ $|x| - |y| < |x - y|$.

(c) $|xy| = |(-3) \times 5| = |-15| = 15$
$|x||y| = |-3||5| = 3 \times 5 = 15$
$\therefore$ $|xy| = |x||y|$

(d) $\left|\frac{x}{y}\right| = \left|\frac{-3}{5}\right| = \frac{3}{5}$ and $\frac{|x|}{|y|} = \frac{|-3|}{|5|} = \frac{3}{5}$
$\therefore$ $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

3. Express $\frac{1}{9}$ as a repeating decimal, using a bar to indicate the repeating digits. What are the representations of $\frac{2}{9}, \frac{7}{9}$ and $\frac{8}{9}$?

Solution

By division, $\frac{1}{9} = 0.111 \ldots = 0.\overline{1}$

$\frac{2}{9} = 0.222 \ldots = 0.\overline{2}$

$\frac{7}{9} = 0.777 \ldots = 0.\overline{7}$

$\frac{8}{9} = 0.888 \ldots = 0.\overline{8}$

4. If $2 < x < 6$, which of the following statements about x are necessarily true and which are not necessarily true?

(a) $0 < x < 4$ (b) $1 < \frac{x}{2} < 3$

(c) $1 < \frac{6}{x} < 3$ (d) $|x - 4| < 2$.

Solution

(a) not necessarily true

(b) $1 < \frac{x}{2} < 3$

Multiplying each side by 2

$2 < x < 6$. (True)

(c) $1 < \frac{6}{x} < 3$

or, $1 > \frac{x}{6} > \frac{1}{3}$

or, $\frac{1}{3} < \frac{x}{6} < 1$

Multiplying each side by 6

$2 < x < 6$. (True)

(d) $|x - 4| < 2$

$-2 < x - 4 < 2$

or, $-2 + 4 < x - 4 + 4 < 2 + 4$

$\therefore$ $2 < x < 6$ (True)

5. Solve the inequalities.

(a) $-2x > 4$ (b) $5x - 3 \leq 7 - 3x$

(c) $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$ (d) $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$

(e) $-\frac{x+5}{2} \leq \frac{12+3x}{4}$

Solution

(a) $-2x > 4$

Dividing both sides by -2

or, $x < -\frac{4}{2}$

$\therefore$ $x < -2$

(b) $5x - 3 \leq 7 - 3x$

or, $5x + 3x \leq 7 + 3$

or, $8x \leq 10$

or, $x \leq \frac{10}{8}$

$\therefore$ $x \leq \frac{5}{4}$.

(c) $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$

or, $2x - 7x \geq \frac{7}{6} + \frac{1}{2}$

or, $-5x \geq \frac{7+3}{6}$

or, $-5x \geq \frac{10}{6}$

Dividing both sides by -5

or, $\frac{-5x}{-5} \leq \frac{10}{6 \times (-5)}$

$\therefore$ $x \leq -\frac{1}{3}$

(d) $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$

or, $12(x - 2) < 5(x - 6)$

or, $12x - 24 < 5x - 30$

or, $12x - 5x < 24 - 30$

or, $7x < -6$

$\therefore$ $x < -\frac{6}{7}$.

(e) $-\frac{x+5}{2} \leq \frac{12+3x}{4}$

or, $-x + 5 \leq \frac{12+3x}{2}$

or, $-2x + 10 \leq 12 + 3x$

or, $10 - 12 \leq 3x + 2x$

or, $-2 \leq 5x$

or, $5x \geq -2$

$\therefore$ $x \geq -\frac{2}{5}$.

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6. Solve:

(a) $|x| = 3$ (b) $|2x + 5| = 4$ (c) $|8 - 3x| = \frac{9}{2}$

Solution

(a) $|x| = 3$

$\therefore \quad x = \pm 3$

(b) $|2x + 5| = 4$

Let $2x + 5$ be positive. Then,

$2x + 5 = 4$

or, $2x = 4 - 5$

or, $2x = -1$

$\therefore \quad x = -\frac{1}{2}$

Again, let $2x + 5$ be negative. Then,

$-(2x + 5) = 4$

or, $2x + 5 = -4$

or, $2x = -4 - 5$

or, $2x = -9$

$\therefore \quad x = -\frac{9}{2}$

Hence, $x = -\frac{1}{2}, -\frac{9}{2}$

(c) $|8 - 3x| = \frac{9}{2}$

Let $8 - 3x$ be positive. Then,

$8 - 3x = \frac{9}{2}$

or, $16 - 6x = 9$

or, $16 - 9 = 6x$

or, $6x = 7$

$\therefore \quad x = \frac{7}{6}$

Again, let $8 - 3x$ be negative. Then,

$-(8 - 3x) = \frac{9}{2}$

or, $-16 + 6x = 9$

or, $6x = 9 + 16$

or, $6x = 25$

or, $x = \frac{25}{6}$

$\therefore \quad x = \frac{7}{6}, \frac{25}{6}$.

7. (i) **Rewrite the following without using modules sign.**

(a) $|x + 2| \le 1$ (b) $|2x - 3| \le 1$

(ii) **If A = [– 2, 1) and B = (– 1, 3] then find**

(a) $A \cup B$ (b) $A \cap B$

(c) $A - B$ (d) $B - A$

(iii) **Solve the inequalities expressing the solution sets as intervals or unions of intervals. Also, graph each solution set on the real line.**

(a) $|x| < 2$ (b) $|x - 2| < 6$ (c) $|3x - 7| < 4$

(d) $\left|3 - \frac{1}{y}\right| < \frac{1}{2}$ (e) $|3 - 5x| \le 2$ (f) $|1 - x| > 1$

(g) $\left|\frac{x + 1}{2}\right| \ge 1$

Solution

(i)

(a) $|x + 2| \le 1$

$\Rightarrow \quad -1 \le x + 2 \le 1$

$\Rightarrow \quad -1 - 2 \le x + 2 - 2 \le 1 - 2$

$\Rightarrow \quad -3 \le x \le -1$

$\therefore \quad -3 \le x \le -1.$

(b) $|2x - 3| \le 1$

$\Rightarrow \quad -1 \le 2x - 3 \le 1$

$\Rightarrow \quad -1 + 3 \le 2x - 3 + 3 \le 1 + 3$

$\Rightarrow \quad 2 \le 2x \le 4$

$\Rightarrow \quad 1 \le x \le 2$

$\therefore \quad 1 \le x \le 2.$

(ii) (a) $A \cup B = [-2, 1) \cup (-1, 3]$

$= \{x : -2 \le x < 1\} \cup \{x : -1 < x \le 3\} = \{x : -2 \le x \le 3\} = [-2, 3]$

(b) $A \cap B = (-2, 1) \cap (-1, 3]$

$= \{x : -2 \le x < 1\} \cap \{x : -1 < x \le 3\} = \{x : -1 < x < 1\} = (-1, 1)$

(c) $A - B = [-2, 1) - (-1, 3]$

$= \{x : -2 \le x < 1\} - \{-x : -1 < x \le 3\} = \{x : -2 \le x \le -1\} = [-2, -1]$

(d) $B - A = (-1, 3] - [-2, 1)$

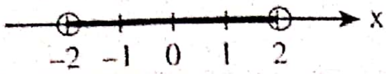
$= \{x : -1 < x \le 3\} - \{x : -2 \le x < 1\} = \{x : 1 \le x \le 3\} = [1, 3]$

(iii)

(a) $|x| < 2$

$\Rightarrow \quad -2 < x < 2$

$\therefore \quad x \in (-2, 2)$



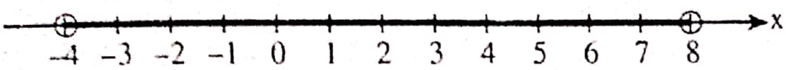
(b) $|x - 2| < 6$

$\Rightarrow \quad -6 < x - 2 < 6$

$\Rightarrow \quad -6 + 2 < x - 2 + 2 < 6 + 2$

$\Rightarrow \quad -4 < x < 8$

$\therefore \quad x \in (-4, 8)$



(c) $|3x - 7| < 4$

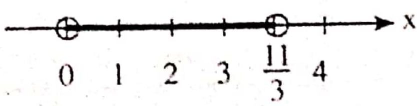
$\Rightarrow \quad -4 < 3x - 7 < 4$

$\Rightarrow \quad -4 + 7 < 3x - 7 + 7 < 4 + 7$

$\Rightarrow \quad 3 < 3x < 11$

$\Rightarrow \quad 1 < x < \frac{11}{3}$

$\Rightarrow \quad x \in \left(1, \frac{11}{3}\right)$



(d) $\left|3 - \frac{1}{y}\right| < \frac{1}{2}$

$\Rightarrow \quad -\frac{1}{2} < 3 - \frac{1}{y} < \frac{1}{2}$

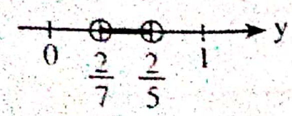
$\Rightarrow \quad -\frac{1}{2} - 3 < 3 - \frac{1}{y} - 3 < \frac{1}{2} - 3$

$\Rightarrow \quad -\frac{7}{2} < -\frac{1}{y} < -\frac{5}{2}$

$\Rightarrow \quad \frac{7}{2} > \frac{1}{y} > \frac{5}{2}$

$\Rightarrow \quad \frac{2}{7} < y < \frac{2}{5}$

$\Rightarrow \quad y \in \left(\frac{2}{7}, \frac{2}{5}\right)$



(e) $|3 - 5x| \le 2$

$\Rightarrow \quad -2 \le 3 - 5x \le 2$

$\Rightarrow \quad -2 - 3 \le 3 - 5x - 3 \le 2 - 3$

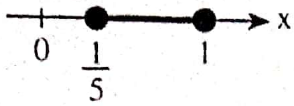
$\Rightarrow \quad -5 \le -5x \le -1$

$\Rightarrow \quad 5 \geq 5x \geq 1$

$\Rightarrow \quad 1 \geq x \geq \frac{1}{5}$

$\Rightarrow \quad \frac{1}{5} \leq x \leq 1$

$\Rightarrow \quad x \in \left[\frac{1}{5}, 1\right]$



(f) $|1 - x| > 1$

Let $(1 - x)$ be positive. Then,

$1 - x > 1$

$\Rightarrow \quad 1 - 1 > x$

$\Rightarrow \quad 0 > x$

$\Rightarrow \quad x < 0$

$\Rightarrow \quad x \in (-\infty, 0)$

Again, let $(1 - x)$ be negative. Then,

$-(1 - x) > 1$

$\Rightarrow \quad 1 - x < -1$

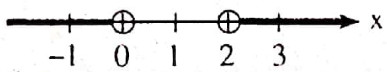
$\Rightarrow \quad 1 + 1 < x$

$\Rightarrow \quad 2 < x$

$\Rightarrow \quad x > 2$

$\Rightarrow \quad x \in (2, \infty)$

Required solution is $x \in (-\infty, 0) \cup x \in (2, \infty)$



(g) $\left|\frac{x+1}{2}\right| \geq 1$

Let $\frac{x+1}{2}$ be positive. Then,

$\frac{x+1}{2} \geq 1$

or, $x + 1 \geq 2$

or, $x \geq 2 - 1$

or, $x \geq 1$

$\therefore \quad x \in [1, \infty)$

Again, let $\frac{x+1}{2}$ be negative. Then,

$-\left(\frac{x+1}{2}\right) \geq 1$

or, $\frac{x+1}{2} \leq -1$

or, $x + 1 \leq -2$

or, $x \leq -3$

$\therefore \quad x \in (-\infty, -3]$

Required solution is $x \in (-\infty, -3] \cup [1, \infty)$.

8. **Rewrite the following by using the modulus sign.**

(a) $-3 < x < 3$ (b) $-3 < x < 9$ (c) $-4 \le x \le 1$

(d) $-3 \le x \le -2$ (e) $-5 \le x \le -2$

Solution

(a) $-3 < x < 3$

$\Rightarrow |x| < 3$

[$\because$ By definition, if $-a < x < a$ then $|x| < a$]

(b) $-3 < x < 9$

$\Rightarrow \quad -3 - 3 < x - 3 < 9 - 3$

$\Rightarrow \quad -6 < x - 3 < 6$

$\Rightarrow \quad |x - 3| < 6$

(c) $-4 \le x \le 1$

$\Rightarrow \quad -8 \le 2x \le 2$

$\Rightarrow \quad -8 + 3 \le 2x + 3 \le 2 + 3$

$\Rightarrow \quad -5 \le 2x + 3 \le 5$

$\Rightarrow \quad |2x + 3| \le 5$

(d) $-3 \le x \le -2$

$\Rightarrow \quad -6 \le 2x \le -4$

$\Rightarrow \quad -6 + 5 \le 2x + 5 \le -4 + 5$

$\Rightarrow \quad -1 \le 2x + 5 \le 1$

$\Rightarrow \quad |2x + 5| \le 1$

(e) $-5 \le x \le -2$

$\Rightarrow \quad 2 \times -5 \le 2x \le -4$

$\Rightarrow \quad -10 \le 2x \le -4$

$\Rightarrow \quad -10 + 7 \le 2x + 7 \le -4 + 7$

$\Rightarrow \quad -3 \le 2x + 7 \le 3$

$\therefore \quad |2x + 7| \le 3$

9. **Solve:**

(a) $x^2 < 2$ (b) $(x - 1)^2 < 4$

(c) $x^2 - x < 0$ (d) $x^2 - x - 2 \ge 0.$

Solution

(a) $x^2 < 4$

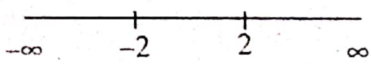
or, $x^2 - 4 < 0$

The corresponding eq^n is

$x^2 - 4 = 0$

or, $(x - 2)(x + 2) = 0$

$\therefore \quad x = -2, 2$



These points divide the real line in three intervals $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

Intervals	**Sign of**		
	$(2 - x)$	**$x + 2$**	**$x^2 - 4$**
$(-\infty, -2)$	$-$	$-$	$+$
$(-2, 2)$	$-$	$+$	$-$
$(2, \infty)$	$+$	$+$	$+$

From above table, required solution is $x \in (-2, 2)$.

(b) $(x - 1)^2 < 4$

or, $x^2 - 2x + 1 - 4 < 0$

or, $x^2 - 2x - 3 < 0$

The corresponding eq^n is

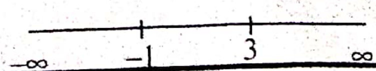
$x^2 - 2x - 3 = 0$

or, $x^2 - 3x + x - 3 = 0$

or, $x(x - 3) + 1(x - 3) = 0$

or, $(x + 1)(x - 3) = 0$

$\therefore \quad x = -1, 3$



Intervals	**Sign of**		
	$x + 1$	**$x - 3$**	**$(x + 1)(x - 3)$**
$(-\infty, -1)$	$-$	$-$	$+$
$(-1, 3)$	$+$	$-$	$-$
$(3, \infty)$	$+$	$+$	$+$

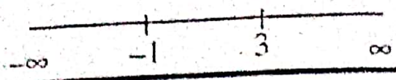
From above table required solution is $x \in (-1, 3)$.

(c) The corresponding eq^n is

$x^2 - x < 0$

or, $x(x-1) = 0$

$\therefore$ $x = 0, 1$



Intervals	Sign of		
	x	$(x-1)$	$x(x-1)$
$(-\infty, 0)$	–	–	+
$(0, 1)$	+	–	–
$(1, \infty)$	+	+	+

From above table, the solution is $x \in (0, 1)$.

(d) The corresponding eq^n is

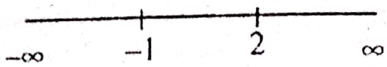
$x^2 - x - 2 = 0$

or, $x^2 - 2x + x - 2 = 0$

or, $x(x-2) + 1(x-2) = 0$

or, $(x-2)(x+1) = 0$

$\therefore$ $x = -1, 2$



Intervals	Sign of		
	$(x-2)$	$(x+1)$	$(x-2)(x+1)$
$(-\infty, -1)$	–	–	+
$(-1, 2)$	–	+	–
$(2, \infty)$	+	+	+

From above table, the solution is $(-\infty, -1) \cup (2, \infty)$

Also, at $x = -1$ and $x = 2$, $x^2 - x - 2 = 0$

Hence, the required solution is $(-\infty, -1] \cup [2, \infty)$.

Objective Questions

1. π is

(a) a natural number. (b) an integer.

(c) a rational number. (d) an irrational number.

Ans: d

π is an irrational number.

2. $0.\overline{45}$ is

(a) a natural number (b) an integer

(c) a rational number (d) an irrational number

Ans: c

Repeating decimal numbers are rational

3. $[2, 3] =$

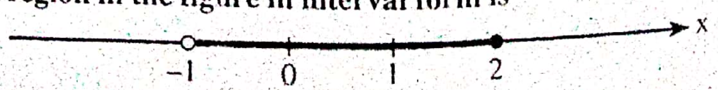
(a) $\{x : 2 < x < 3\}$ (b) $\{x : 2 \le x \le 3\}$

(c) $\{x : 2 < x \le 3\}$ (d) $\{x : 2 \le x < 3\}$

Ans: b

$[2, 3] = \{x : 2 \le x \le 3\}$

4. **The shaded region in the figure in interval form is**



(a) $(-1, 2)$ (b) $(-1, 2]$ (c) $[-1, 2)$ (d) $[-1, 2]$

Ans: b

$(-1, 2]$

5. **If $A = [-3, 1)$ and $B = (0, 3]$ then $A \cup B =$**

(a) $\{-3, -2, -1, 0, 1, 2, 3\}$ (b) $[-3, 3]$

(c) $(0, 1)$ (d) $[0, 1]$

Ans: b

$A \cup B = [-3, 1] \cup (0, 3]$

$= \{x : -3 \le x < 1\} \cup \{x : 0 < x \le 3\} = \{x : -3 \le x \le 3\} = [-3, 3]$

6. **Which of the following is not true?**

(a) $|-a| = |a|$ (b) $|ab| = |a|\,|b|$

(c) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (d) $|a + b| = |a| + |b|$

Ans: d

7. **If $2 < x < 4$ and $3 < y < 7$, the largest integer value of $x + y$ is**

(a) 9 (b) 10 (c) 11 (d) 12

Ans: b

8. **The number of integers that satisfy the inequality $|x| < \pi$ is**

(a) 2 (b) 5 (c) 7 (d) 11

Ans: c

The integers satisfying $|x| < \pi$ are $0, \pm 1, \pm 2, \pm 3$.

9. **The solution set of $|x| = 2$ is**

(a) $\{2\}$ (b) $\{-2\}$ (c) $\{-2, 2\}$ (d) ϕ

Ans: c

$|x| = 2 \Rightarrow x = \pm 2$

Solution set $= \{-2, 2\}$

10. **$-7 < -2x + 3 < 5$ implies**

(a) $-1 < x < 5$ (b) $1 < x < 5$ (c) $-5 < x < 1$ (d) $-5 < x < -1$

Ans: a

$-7 < -2x + 3 < 5$

or, $-7 - 3 < 2x + 3 - 3 < 5 - 3$

or, $-10 < -2x < 2$

or, $\frac{-10}{-2} > \frac{-2x}{-2} > \frac{2}{-2}$

or, $5 > x > -1$

$\therefore$ $-1 < x < 5$

11. **$-1 \le x \le 4$ implies**

(a) $|x| \le 4$ (b) $|x| \le 4$ (c) $|2x - 3| \le 5$ (d) $|2x - 3| \le 2$

Ans: c

$-1 \le x \le 4$

or, $-2 \le 2x \le 8$

or, $-2 - 3 \le 2x - 3 \le 8 - 3$

or, $-5 \le 2x - 3 \le 5$

$\therefore$ $|2x - 3| \le 5.$

12. **$1 \le x \le 2$ implies**

(a) $|2x - 3| \le 1$ (b) $|2x + 3| \le 4$ (c) $|2x - 3| \le 5$ (d) $|2x - 3| \le 2$

Ans: a

$1 \le x \le 2$

or, $2 \le 2x \le 4$

or, $2 - 3 \le 2x - 3 \le 4 - 3$

or, $-1 \le 2x - 3 \le 1$

$\therefore$ $|2x - 3| \le 1$

EXERCISE – 1 C

1. **Find the values of**

(a) i^8 **(b)** i^{15} **(c)** $(1, 0)^{100}$

(d) $(0, 1)^{405}$ **(e)** $i^2 + i^4 + i^6 + i^8$ **(f)** $i^{10} + i^{11} + i^{12} + i^{13}$

Solution

(a) $i^8 = (i^2)^4 = (-1)^4 = 1 = i.$

(b) $i^{15} = i^{14} \cdot i = (i^2)^7 \cdot i = (-1)^7 \cdot i = (-1) \cdot i. = -i.$

(c) $(1, 0)^{100} = (1 + 0 \cdot i)^{100} = 1^{100} = 1.$

(d) $(0, 1)^{405} = (0 + 1 \cdot i)^{405} = i^{405} = i^{404} \cdot i = (i^2)^{202} \cdot i = (-1)^{202} \cdot i = 1 \cdot i = i.$

(e) $i^2 + i^4 + i^6 + i^8 = i^2 + (i^2)^2 + (i^2)^3 + (i^2)^4 = (-1) + (-1)^2 + (-1)^3 + (-1)^4$
$= -1 + 1 - 1 + 1 = 0.$

(f) $i^{10} + i^{11} + i^{12} + i^{13}$
$= i^{10} + i^{10} \cdot i + i^{12} + i^{12} \cdot i = (i^2)^5 + (i^2)^5 \cdot i + (i^2)^6 + (i^2)^6 \cdot i$
$= (-1)^5 + (-1)^5 \cdot i + (-1)^6 + (-)^6 \cdot i = -1 - i + 1 + i = 0.$

2. **Find the values of**

(a) i^{19} **(b)** $(3 + 4i)(7 + i)$ **(c)** $(3 - 2i)(5 - 4i)$

(d) $\dfrac{(1 + i)^2}{(1 - i)^2}$ **(e)** $\dfrac{(1 + i)^2 + (1 - i)^2}{(1 + i)^2 - (1 - i)^2}$ **(f)** $(1 - i)^3 + (1 + i)^3$

(g) $(1 + i)^4 \left(1 + \dfrac{1}{i}\right)^4$

Solution

(a) $i^{19} = (i^2)^9 \cdot i = (-1)^9 \cdot i = -1 \times i = -i.$

(b) $(3 + 4i)(7 + i) = 21 + 3i + 28i - 4 = 17 + 31i.$

(c) $(3 - 2i)(5 - 4i) = 15 - 12i - 10i - 8 = 7 - 22i.$

(d) $\dfrac{(1 + i)^2}{(1 - i)^2} = \dfrac{1^2 + 2i + i^2}{1^2 - 2i + i^2} = \dfrac{1 + 2i - 1}{1 - 2i - 1} = \dfrac{2i}{-2i} = -1.$

(e) $\dfrac{(1 + i)^2 + (1 - i)^2}{(1 + i)^2 - (1 - i)^2}$

$= \dfrac{1^2 + 2.1.i + i^2 + 1^2 - 2.1.i + i^2}{(1^2 + 2.1.i + i^2) - (1^2 - 2.1.i + i^2)} = \dfrac{1 + 2i + i^2 + 1 - 2i + i^2}{1 + 2i + i^2 - (1 - 2i + i^2)}$

$= \dfrac{1 + 2i + (-1) + 1 - 2i + (-1)}{1 + 2i + (-1) - 1 + 2i - (-1)}$ $[\because i^2 = -1]$

$= \dfrac{1 + 2i - 1 + 1 - 2i - 1}{1 + 2i - 1 - 1 + 2i + 1} = \dfrac{0}{4i} = 0$

(f) $1^3 - 3 \cdot 1^2 \cdot i + 3 \cdot 1 \cdot i^2 - i^3 + 1^3 + 3 \cdot 1^2 \cdot i + 3 \cdot 1 \cdot i^2 + i^3 = 1 - 3i + 3i^2 - i^3 + 1 + 3i + 3i^2 + i^3$
$= 2 + 6i^2 = 2 - 6 = -4$

(g) $(1 + i)^4 \left(1 + \dfrac{1}{i}\right)^4$

$= (1 + i)^4 \left(\dfrac{i + 1}{i}\right)^4 = \left\{(1 + i)\left(\dfrac{i + 1}{i}\right)\right\}^4 = \left\{\dfrac{(1 + i)^2}{i}\right\}^4 = \left\{\dfrac{1 + 2i + i^2}{i}\right\}^4 = \left(\dfrac{2i}{2}\right)^4$

$= 2^4 = 16.$

3. **Express each of the following complex number in the from of A + *i*B.**

(a) $(3 + 4i) + (4 + 3i)$ **(b)** $(3 - 2i)(6 - 8i)$ **(c)** $\dfrac{1 + 2i}{1 - i} + \dfrac{1 - 2i}{1 + i}$

(d) $\dfrac{1 - i}{1 + i}$ **(e)** $\left(\dfrac{1 - i}{1 + i}\right)^3$ **(f)** $\sqrt{-1}\sqrt{-4}$ **(g)** $\dfrac{3 - \sqrt{-25}}{2 - \sqrt{-16}}$.

Solution

(a) $(3 + 4i) + (4 + 3i)$

$= 3 + 4i + 4 + 3i$

$= 7 + 7i$ which is in form of A + iB where A = 7 & B = 7

(b) $(3 - 2i)(6 - 8i) = 18 - 24i - 12i + 16i^2$

$= 18 - 36i - 16$

$= 2 + (-36)\,i$ which is in form of A + iB where A = 2 & B = – 36

(c) $\frac{1+2i}{1-i} + \frac{1-2i}{1+i} = \frac{(1+2i)(1+i) + (1-2i)(1-i)}{(1-i)(1+i)} = \frac{1+i+2i+2i^2+1-i-2i+2i^2}{1-i^2}$

$= \frac{2+4i^2}{1-i^2} = \frac{2+4(-1)}{1-(-1)} = \frac{2-4}{1+1}$

$\Rightarrow \frac{-2}{2} = -1 + 0.i$ which is in form of A + iB where A = –1, B = 0

(d) $\frac{1-i}{1+i}$

Given complex number is $\frac{1-i}{1+i}$

$= \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1-i^2} = \frac{1-2i+i^2}{1-(-1)} = \frac{1-2i+(-1)}{1+1} = \frac{-2i}{2} = -i$

$= 0 + (-1)\,i$ which is in form of A + iB where A = 0, B = –1.

(e) $\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^3 = \left(\frac{1^2 - 2.1.i + i^2}{1^2 - i^2}\right)^3 = \left(\frac{1-2i-1}{1-(-1)}\right)^3 = \left(\frac{-2i}{2}\right)^3$

$= -i^3 = -i.\,i^2 = -i(-1) = i$

$= 0 + 1 \cdot i$ which is in the form of A + iB where A = 0, B = 1

(f) $\sqrt{-1}\sqrt{-4} = i \cdot 2i = 2i^2 = -2 = -2 + 0 \cdot i$ which is in the form of A + iB where A = –2, B = 0.

(g) $\frac{3-\sqrt{-25}}{2-\sqrt{-16}} = \frac{3-\sqrt{(-1)\times 25}}{2-\sqrt{(-1)\times 16}} = \frac{3-\sqrt{25\,i^2}}{2-\sqrt{16\,i^2}} = \frac{3-5i}{2-4i} = \frac{3-5i}{2-4i} \times \frac{2+4i}{2+4i}$

$= \frac{6+12i-10\,i-20i^2}{4-16\,i^2} = \frac{6+12i-10\,i-20(-1)}{4-16(-1)} = \frac{6+12i-10\,i+20}{4+16}$

$= \frac{26+2i}{20} = \frac{13}{10} + \frac{1}{10}\,i$ which is in form of A + iB where, A = $\frac{13}{10}$ and B = $\frac{1}{10}$.

4. Find the additive inverse of

(a) $3 + 2i$ **(b) $-1 + 5i$**

Solution

(a) Here $z = 3 + 2i$

Let $w = x + iy$ be the addtive invese of z. Then

$z + w = 0$

or, $3 + 2i + x + iy = 0$

or, $x + 3 + iy + 2i = 0$

or, $(x + 3) + i\,(y + 2) = 0 + 0.i$

Equating real and imaginary parts.

$x + 3 = 0$

$\therefore\ x = -3$

and $y + 2 = 0$

$\therefore\ y = -2$

$\therefore\ w = x + iy$

$= -3 - 2i$

(b) Here, $z = -1 + 5i$

Let $w = x + iy$ be the additive inverse of z. Then

$z + w = 0$

or, $-1 + 5i + x + iy = 0$

or, $x - 1 + iy + 5i = 0$

or, $(x - 1) + i(y + 5) = 0$

Equating real and imaginary parts, we have

$x - 1 = 0$

$\therefore$ $x = 1$

and $y + 5 = 0$

$\therefore$ $y = -5$

$\therefore$ $w = x + iy$

$= 1 - 5i.$

5. Find the multiplicative inverse of

(a) $2 + i$ **(b)** $\dfrac{3+4i}{12-5i}$

Solution

(a) Here $z = 2 + i$

Let W be the multiplicative inverse of z.

Then,

$z \cdot w = 1$

or, $(2 + i)\, w = 1$

or, $w = \dfrac{1}{2+i} \cdot \dfrac{2-i}{2-i}$

or, $w = \dfrac{2-i}{4-i^2}$

or, $w = \dfrac{2-i}{5}$

$\therefore$ $w = \dfrac{2}{5} - \dfrac{1}{5}i$

(b) Here $z = \dfrac{3+4i}{12-5i}$

Let w be the multiplicative inverse of z.

Then,

$z \cdot w = 1$

or, $\dfrac{3+4i}{12-5i}\, w = 1$

or, $w = \dfrac{12-5i}{3+4i} \cdot \dfrac{3-4i}{3-4i}$

or, $w = \dfrac{36 - 48i - 15i + 20i^2}{9 - 16i^2}$

or, $w = \dfrac{36 - 63i - 20}{9 + 16}$

$\therefore$ $w = \dfrac{16}{25} - \dfrac{63}{25}i$

6. (a) If $x = 3 - 2i$, $y = 2 + 3i$, find the values of

(i) $x^2 + y^2$ **(ii)** $x^2 + xy + y^2$

(b) If $x = 4 + 5i$ and $y = 4 - 5i$, find the value of $x^2 - xy + y^2$.

Solution

(a) Here, $x = 3 - 2i$, $y = 2 + 3i$

(i) $x^2 + y^2 = (3 - 2i)^2 + (2 + 3i)^2 = 9 - 12i + 4i^2 + 4 + 12i + 9i^2$

$= 9 - 12i + 4(-1)^2 + 4 + 12i + 9(-1)^2 = 9 - 12i - 4 + 4 + 12i - 9 = 0.$

(ii) $x^2 + xy + y^2 = (3 - 2i)^2 + (3 - 2i)(2 + 3i) + (2 + 3i)^2$

$= 9 - 12i + 4i^2 + 6 + 9i - 4i - 6i^2 + 4 + 12i + 9i^2$

$= 9 - 12i + 4(-1) + 6 + 9i - 4i - 6(-1) + 4 + 12i + 9(-1)$

$= 9 - 12i - 4 + 6 + 9i - 4i + 6 + 4 + 12i - 9 = 12 + 5i.$

(b) Here, $x = 4 + 5i$; $y = 4 - 5i$

$x^2 - xy + y^2$

$= (4 + 5i)^2 - (4 + 5i)(4 - 5i) + (4 - 5i)^2$

$= 16 + 40i + 25i^2 - 16 + 20i - 20i + 25i^2 + 16 - 40i + 25i^2$

$= 25(-1) + 25(-1) + 16 + 25(-1) = -25 - 25 + 16 - 25 = -59.$

7. Find the values of x & y if

(a) $6x + 14y\,i = 18 - 56i$

(b) $(10, 6x) = (2y, -12)$

(c) $10x + (6x - 2y)\,i = 20 + 4i$

(d) $5x + (3x - y)i = 10 + 2i$.

(e) If $(a + ib)(4 - 3i) = 3 + 4i$, find a & b.

Solution

(a) Given $6x + 14y\,i = 18 - 56i$

Comparing real and imaginary parts, we get

$6x = 18$ and $14\,y = -56$

or, $x = \dfrac{18}{6}$ and $y = \dfrac{-56}{14}$

$\therefore$ $x = 3$ and $y = -4$

(b) $(10, 6x) = (2y, -12)$

Comparing real and imaginary parts, we get

$10 = 2y$ and $6x = -12$

$\therefore$ $y = 5$ and $x = -2$

(c) $10x + (6x - 2y)\,i = 20 + 4\,i$

Comparing real and imaginary parts, we get

$10x = 20$

$\Rightarrow$ $x = 2$ and

and $6x - 2y = 4$

$\Rightarrow$ $6 \times 2 - 2y = 4$

$\Rightarrow$ $-2y = 4 - 12$

$\Rightarrow$ $y = \dfrac{-8}{-2} = 4$

$\therefore$ $x = 2$ and $y = 4$

(d) Equating real and imaginary parts, we get,

$5x = 10 \Rightarrow x = 2$

and $3x - y = 2$

$3 \times 2 - y = 2$

$y = 4$

$\therefore$ $x = 2,\ y = 4.$

(e) Solution

Given,

$(a + ib)(4 - 3i) = 3 + 4i$

or, $4a - 3ai + 4bi - 3bi^2 = 3 + 4i$

or, $4a + (4b - 3a)i + 3b = 3 + 4i$

or, $(4a + 3b) + (4b - 3a)i = 3 + 4i$

Comparing real and imaginary parts, we get

$4a + 3b = 3$

$4b - 3a = 4$

On solving, we get

$a = 0;\ b = 1$

8. If $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, find the values of

(a) $\dfrac{1}{z_1}$ (b) $(\overline{z_1})^2 + (\overline{z_2})^2$ (c) $\dfrac{\overline{z_1}}{\overline{z_2}}$

Solution

(a) $\dfrac{1}{z_1}$

$= \dfrac{1}{2+3i} = \dfrac{1}{2+3i} \times \dfrac{2-3i}{2-3i} = \dfrac{2-3i}{4+9} = \dfrac{2-3i}{13} = \dfrac{2}{13} - \dfrac{3}{13}\,i.$

(b) $(\overline{z_1})^2 + (\overline{z_2})^2$

$= (2 - 3i)^2 + (3 + 2i)^2 \qquad \left[\because \begin{array}{l} z_1 = a + ib \\ \overline{z_1} = a - ib \end{array}\right]$

$= 4 - 12i + 9i^2 + 9 + 12i + 4i^2 = 13 + 13i^2 = 13 + 13(-1) = 13 - 13 = 0.$

(c) $\dfrac{\overline{z_1}}{\overline{z_2}}$

$= \dfrac{2 - 3i}{3 + 2i} = \dfrac{2 - 3i}{3 + 2i} \times \dfrac{3 - 2i}{3 - 2i} = \dfrac{6 - 4i - 9i + 6i^2}{9 - 4i^2} = \dfrac{6 - 13i + 6(-1)}{9 - 4(-1)} = \dfrac{6 - 13i - 6}{13} = \dfrac{-13i}{13}$

$= -i.$

9. Find the modulus value of the complex numbers.

(a) $(2 + 3i)$ **(b)** $(3 - 2i)(2 - 3i)$ **(c)** $\dfrac{1 + 2i}{1 - 2i}$

(d) $\dfrac{3 + 4i}{12 - 5i}$ **(e)** $\dfrac{(2 + i)(1 + 2i)}{(1 - i)(2 + 3i)}$

Solution

(a) $2 + 3i$

Let, $z = 2 + 3i$

Here, $a = 2, b = 3$

$\therefore |z| = \sqrt{a^2 + b^2}$

$|z| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

(b) $(3 - 2i)(2 - 3i)$

$= 6 - 9i - 4i + 6i^2 = 6 - 13i + 6(-1) = -13i$

$= 0 + (-13)i = 0 + i(-13)$

Let $z = 0 + i(-13)$

Here, $a = 0, b = -13$

$\therefore |z| = \sqrt{a^2 + b^2} = \sqrt{0^2 + (-13)^2} = \sqrt{169} = 13$

(c) Let $z = \dfrac{1 + 2i}{1 - 2i}$

$|z| = \left|\dfrac{1 + 2i}{1 - 2i}\right| = \dfrac{|1 + 2i|}{|1 - 2i|} = \dfrac{\sqrt{1^2 + 2^2}}{\sqrt{1^2 + (-2)^2}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1.$

(d) Let, $|z| = \dfrac{3 + 4i}{12 - 5i}$

Then, $|z| = \left|\dfrac{3 + 4i}{12 - 5i}\right| = \dfrac{|3 + 4i|}{|12 - 5i|} = \dfrac{\sqrt{3^2 + 4^2}}{\sqrt{12^2 + (-5)^2}} = \dfrac{5}{13}$

(e) Let $z = \dfrac{(2 + i)(1 + 2i)}{(1 - i)(2 + 3i)}$

$|z| = \left|\dfrac{(2 + i)(1 + 2i)}{(1 - i)(2 + 3i)}\right| = \dfrac{|2 + i||1 + 2i|}{|1 - i||2 + 3i|} = \dfrac{\sqrt{2^2 + 1^2}\sqrt{1^2 + 2^2}}{\sqrt{1^2 + (-1)^2}\sqrt{2^2 + 3^2}} = \dfrac{\sqrt{5}\sqrt{5}}{\sqrt{2}\sqrt{13}} = \dfrac{5}{\sqrt{26}}.$

10. If $z_1 = 2 + 3i$ and $z_2 = 1 - i$, verify that:

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ **(b)** $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ **(c)** $|z_1 z_2| = |z_1||z_2|$

(d) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ **(e)** $|z_1 + z_2| \le |z_1 + z_2|$

Solution

(a) $\overline{z_1 + z_2} = \overline{(2 + 3i) + (1 - i)} = \overline{3 + 2i} = 3 - 2i$

And, $\bar{z}_1 + \bar{z}_2 = 2 - 3i + 1 + i = 3 - 2i$

$\therefore \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$

(b) $\overline{z_1 z_2} = \overline{(2 + 3i)(1 - i)} = \overline{2 + 3i - 2i - 3i^2} = \overline{5 + i} = 5 - i.$

$\bar{z}_1 \cdot \bar{z}_2 = (2 - 3i)(1 + i) = 2 - 3i + 2i - 3i^2 = 5 - i.$

$\therefore \quad \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2.$

(c) $|z_1 z_2| = |(2 + 3i)(1 - i)| = |5 - i| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$

$|z_1||z_2| = \sqrt{2^2 + 3^2} \cdot \sqrt{1^2 + (-1)^2} = \sqrt{13}\sqrt{2} = \sqrt{26}$

$\therefore \quad |z_1 z_2| = |z_1||z_2|.$

(d) $\left|\frac{z_1}{z_2}\right| = \left|\frac{2 + 3i}{1 - i} \times \frac{1 + i}{1 + i}\right| = \left|\frac{2 + 3i + 2i + 3i^2}{2}\right| = \left|\frac{-1 + 5i}{2}\right| = \frac{\sqrt{(-1)^2 + 5^2}}{2} = \frac{\sqrt{26}}{2}$

And, $\frac{|z_1|}{|z_2|} = \frac{|2 + 3i|}{|1 - i|} = \frac{\sqrt{2^2 + 3^2}}{\sqrt{1^2 + (-1)^2}} = \frac{\sqrt{13}}{\sqrt{2}} = \frac{\sqrt{13}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{26}}{2}$

$\therefore \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$

(e) $|z_1 + z_2| = |2 + 3i + 1 - i| = |3 + 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$

And, $|z_1| + |z_2| = |2 + 3i| + |1 - i| = \sqrt{2^2 + 3^2} + \sqrt{1^2 + (-1)^2} = \sqrt{13} + \sqrt{2}$

$\therefore |z_1 + z_2| < |z_1| + |z_2|.$

11. **(a)** **If $x - iy = \frac{2 - 3i}{2 + 3i}$, prove that $x^2 + y^2 = 1$.**

(b) **If $x + iy = \frac{a + ib}{a - ib}$, prove that $x^2 + y^2 = 1$.**

(c) **If $x + iy = \sqrt{\frac{1 + i}{1 - i}}$, show that $x^2 + y^2 = 1$.**

(d) **$\sqrt{x + iy} = a + ib$, prove that $\sqrt{x - iy} = a - ib$.**

Solution

(a) $x - iy = \frac{2 - 3i}{2 + 3i} = \frac{2 - 3i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i} = \frac{(2 - 3i)^2}{4 + 9} = \frac{4 - 12i - 9}{13} = \frac{-5 - 12i}{13} = \left(\frac{-5}{13}\right) + \left(\frac{-12}{13}\right) i$

Equating real and imaginary part.

$x = \frac{-5}{13}$ and $y = \frac{12}{13}$

Now, $x^2 + y^2 = \left(\frac{-5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = \frac{25}{169} + \frac{144}{169} = \frac{25 + 144}{169} = \frac{169}{169} = 1$ proved

Alternative method

We have,

$x - iy = \frac{2 - 3i}{2 + 3i}$

Taking modulus on both sides

$|x - iy| = \left|\frac{2 - 3i}{2 + 3i}\right|$

or, $|x - iy| = \frac{|2 - 3i|}{|2 + 3i|}$

or, $\sqrt{x^2+(-y)^2} = \frac{\sqrt{2^2+(-3)^2}}{\sqrt{2^2+3^2}}$

or, $\sqrt{x^2+y^2} = \frac{\sqrt{13}}{\sqrt{13}}$

Squaring on both sides,

$x^2+y^2 = 1$

(b) Here,

$$x+iy = \frac{a+ib}{a-ib}$$

$$= \frac{a+ib}{a-ib} \times \frac{a+ib}{a+ib} = \frac{a^2+a\,ib+a\,ib+(ib)^2}{a^2-(ib)^2} = \frac{a^2+2a\,ib+(-1)b^2}{a^2-(-1)\,b^2}$$

$$= \frac{a^2+2a\,ib-b^2}{a^2+b^2} = \frac{(a^2-b^2)+2abi}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i$$

Equating real and imaginary part,

We have

$$x = \frac{a^2-b^2}{a^2+b^2} \;\&\; y = \frac{2ab}{a^2+b^2}$$

$$\text{Now, } x^2+y^2 = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2 + \left(\frac{2ab}{a^2+b^2}\right)^2$$

$$= \frac{a^4-2a^2b^2+b^4+4a^2b^2}{(a^2+b^2)^2} = \frac{a^4+2a^2b^2+b^4}{(a^2+b^2)^2} = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$

Note: We may use modulus method.

(c) Here,

$$x+iy = \sqrt{\frac{1+i}{1-i}}$$

or, $x+iy = \sqrt{\frac{1+i}{1-i} \times \frac{1+i}{1-i}}$

or, $x+iy = \sqrt{\frac{(1+i)^2}{1-i^2}}$

or, $x+iy = \sqrt{\frac{(1+i)^2}{1-(-1)}}$

or, $x+iy = \sqrt{\frac{(1+i)^2}{2}}$

or, $x+iy = \frac{1+i}{\sqrt{2}}$

or, $x+iy = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

Equating real and imaging parts, we have

$$x = \frac{1}{\sqrt{2}};\ y\frac{1}{\sqrt{2}}$$

Now, $x^2+y^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}+\frac{1}{2} = 1$

$\therefore\quad x^2+y^2 = 1$ Proved

(d) $\sqrt{x+iy} = a+ib$

Squaring both sides,

$x+iy = (a+ib)^2$

or, $x+iy = a^2+2abi+i^2b^2$

or, $x+iy = a^2-b^2+2abi$

$\therefore\quad x = a^2-b^2,\ y = 2ab$

Now, $\sqrt{x-iy} = \sqrt{a^2-b^2-i\cdot 2ab} = \sqrt{a^2-2\cdot a\cdot ib+(ib)^2} = \sqrt{(a-ib)^2} = a-ib.$

12. Prove that $\frac{3+4i}{1-i}+\frac{3-4i}{1+i}$ is a real number.

Solution

$$\frac{3+4i}{1-i}+\frac{3-4i}{1+i}$$

$$= \frac{(3+4i)(1+i)+(3-4i)(1-i)}{(1-i)(1+i)} = \frac{3+4i+3i+4i^2+3-4i-3i+4i^2}{1^2-i^2}$$

$$= \frac{3+7i-4+3-7i-4}{2} = \frac{6-8}{2} = -1 \text{ which is a real number.}$$

Hence, $\frac{3+4i}{1-i}+\frac{3-4i}{1+i}$ is purely real.

13. If z_1 and z_2 are two complex number, prove that:

$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2Re\,(z_1\bar{z}_2)$

Solution

$|z_1+z_2|^2 = (z_1+z_2)\,(\overline{z_1+z_2})$ $\quad [\because |z_1|^2 = z_1\bar{z}_1]$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$ $\quad \left[\because \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2\right]$

$= z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+\bar{z}_1 z_2$

$= |z_1|^2+|\bar{z}_2|^2+z_1\bar{z}_2+\overline{z_1\bar{z}_2}$ $\quad \left[\because z_1\bar{z}_1 = |z_1|^2 \text{ and } \bar{\bar{z}}_2 = z_1\right]$

$= |z_1|^2+|z_2|^2+2\text{Re}\,(z_1\bar{z}_2)$ $\quad [\because z_1+\bar{z}_1 = 2\text{Re}(z_1)]$

$\therefore\ |z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\text{Re}\,(z_1\bar{z}_2)$.

Objective Questions

1. $\sqrt{-1}\sqrt{-4} =$

(a) 2 (b) −2 (c) 4 (d) −4

Ans: b

$\sqrt{-1}\sqrt{-4} = i\cdot 2i = 2i^2 = -2$

2. Real part of $(3\sqrt{2}+\sqrt{-4})$ is

(a) $3\sqrt{2}$ (b) 2 (c) $5\sqrt{2}$ (d) $\sqrt{2}$

Ans: a

$3\sqrt{2}+\sqrt{-4} = 3\sqrt{2}+2i$

Real part $= 3\sqrt{2}$

3. $(0, 1)^{99} =$

(a) (1, 0) (b) (−1, 0) (c) (0, 1) (d) (0, −1)

Ans: d

$(0,1)^{99} = i^{99} = i^{98}\cdot i = (i^2)^{49}\cdot i = (-1)^{49}\cdot i = -i = (0,-1)$

4. The number $\frac{2-\sqrt{-16}}{4-\sqrt{-9}}$ lies in

(a) 1st quadrant (b) 2nd quadrant (c) 3rd quadrant (d) 4th quadrant

Ans: d

$$\frac{2-\sqrt{-16}}{4-\sqrt{-9}} = \frac{2-4i}{4-3i}\times\frac{4+3i}{4+3i} = \frac{8+6i-16i-12i^2}{16-9i^2} = \frac{20-10i}{25} = \frac{4}{5}-\frac{2}{5}i \text{ (Fourth quadrant)}$$

5. The value of $3\sqrt{-4}+4\sqrt{-9}-5\sqrt{-16}$ is

(a) $-4i$ (b) $4i$ (c) $2i$ (d) $-2i$

Ans: d

$3\sqrt{-4}+4\sqrt{-9}-5\sqrt{-16}$

$= 3\cdot 2i+4\cdot 3i-5\cdot 4i = 6i+12i-20i = -2i$

6. The value of $\frac{1 + 2i + 3i^2}{1 - 2i + 3i^2} =$

(a) -1 (b) 1 (c) $-i$ (d) i

Ans: c

$$\frac{1 + 2i + 3i^2}{1 - 2i + 3i^2} = \frac{1 + 2i - 3}{1 - 2i - 3} = \frac{-2 + 2i}{-2 - 2i} = \frac{-1 + i}{-1 - i} = \frac{1 - i}{1 + i} \times \frac{1 - i}{1 - i} = \frac{1^2 - 2 \cdot 1 \cdot i + i^2}{1^2 - i^2}$$

$$= \frac{1 - 2i - 1}{2} = -i$$

7. The imaginary part of $\frac{i}{1 + i}$ is

(a) $-\frac{1}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{\sqrt{3}}{4}$

Ans: c

$$\frac{i}{1 + i} \times \frac{1 - i}{1 - i} = \frac{i - i^2}{1^2 - i^2} = \frac{i + 1}{2} = \frac{1}{2} + \frac{1}{2}i$$

Imaginary part of $\frac{i}{1 + i} = \frac{1}{2}$

8. The value of $i^{10} + i^{11} + i^{12} + i^{13} =$

(a) 0 (b) 1 (c) -1 (d) i

Ans: a

Sum of any four consecutive powers of i is zero.

9. If $(x + 2) + yi = (3 + i)(1 - 2i)$ then $x =$

(a) -3 (b) 3 (c) 5 (d) -5

Ans: b

$(x + 2) + yi = (3 + i)(1 - 2i)$

or, $(x + 2) + yi = 3 + i - 6i - 2i^2$

or, $(x + 2) + yi = 5 - 5i$

Equating real parts,

$x + 2 = 5$

$\therefore$ $x = 3$

10. The reciprocal of $(3, 1)$ is

(a) $\left(\frac{3}{10}, \frac{1}{10}\right)$ (b) $\left(-\frac{3}{10}, \frac{1}{10}\right)$ (c) $\left(\frac{3}{10}, -\frac{1}{10}\right)$ (d) $\left(-\frac{3}{10}, -\frac{1}{10}\right)$

Ans: c

Reciprocal of $(3, 1) = \frac{1}{3 + i} = \frac{1}{3 + i} \times \frac{3 - i}{3 - i} = \frac{3 - i}{3^2 - i^2} = \frac{3 - i}{10} = \frac{3}{10} - \frac{1}{10}i$

$$= \left(\frac{3}{10}, -\frac{1}{10}\right)$$

11. The conjugate of $\frac{1 + i}{1 - i}$ is

(a) 1 (b) -1 (c) i (d) $-i$

Ans: d

Let $z = \frac{1 + i}{1 - i} = \frac{1 + i}{1 - i} \times \frac{1 + i}{1 + i} = \frac{1 + 2i + i^2}{1^2 - i^2} = \frac{1 + 2i - 1}{2} = i$

$\bar{z} = -i.$

12. If $z = 2 + i$ and $w = 3$ then $|3z - 4w| =$

(a) $\sqrt{45}$ (b) $\sqrt{53}$ (c) $\sqrt{91}$ (d) $\sqrt{101}$

Ans: a

$|3z - 4w| = |3(2 + i) - 4 \cdot 3| = |6 + 3i - 12| = |3i - 6| = \sqrt{3^2 + (-6)^2} = \sqrt{45}$

$|x| < a \iff -a < x < a$

let $|x|<a$

but, $x \le |x|$

$\therefore x \le |x| < a$

$\Rightarrow x < a \ \cdots (i)$

similarly, $-x \le |x|$

So, $-x \le |x| < a$

$\Rightarrow -x < a$

$\Rightarrow x > -a \ \cdots (ii)$

$-a < x < a$

Conversely, let $-a < x < a$

First, $-a < x$

$\Rightarrow a > -x$

but $-x \le |x|$

$\Rightarrow |x| < a$

Second, $x < a$

but $x \le |x|$

so, $|x| < a$

13. The absolute value of $\dfrac{3-4i}{3+4i}$ is

(a) 5 (b) $\frac{1}{5}$ (c) 1 (d) 0

$2 < 3$

$-2 > -3$

Ans: c

$$\left|\frac{3-4i}{3+4i}\right| = \frac{|3-4i|}{|3+4i|} = \frac{\sqrt{3^2+(-4)^2}}{\sqrt{3^2+4^2}} = \frac{5}{5} = 1$$

14. If $x + iy = \dfrac{a+ib}{a-ib}$ then $x^2 + y^2 =$

(a) 0 (b) 1 (c) $a^2 + b^2$ (d) 5

Ans: b

$$|x + iy| = \frac{a+ib}{a-ib}$$

or, $\sqrt{x^2+y^2} = \dfrac{\sqrt{a^2+b^2}}{\sqrt{a^2+(-b)^2}}$

or, $\sqrt{x^2+y^2} = 1$

$\therefore$ $x^2 + y^2 = 1.$

prove that $|x+y| \le |x| + |y|$

Proof :- $|x+y|^2 = (x+y)^2$

$= x^2 + 2xy + y^2$

$= |x|^2 + 2 . x . y + |y|^2$

$\le |x|^2 + 2|x||y| + |y|^2$

$= (|x|+|y|)^2$

$x \le |x|$

$y \le |y|$

$|x+y| \le |x|+|y|$

Relation, Functions and Graphs

EXERCISE – 2 A

1. Find the values of x and y if

(a) $(x - y, x + y) = (2, 4)$ (b) $(2x - 1, -2) = (1, y + 1)$.

$|x-y| \ge |x|-|y|$

Solution

(a) By the definition of equal ordered pairs, we have

$x - y = 2$ ... (i)

$x + y = 4$ ... (ii)

Proof:-

$|x| = |x-y+y|$

$= |(x-y)+y|$

$\le |x-y| + |y|$

$|x|-|y| \le |x-y|$

Adding (i) & (ii), $2x = 6$

$\therefore$ $x = 3$

Substituting the value of x in eq^n. (i)

$3 - y = 2$

or, $3 - 2 = y$

or, $y = 1$

$|x+y| \le |x|+|y|$

$\therefore$ $x = 3$ and $y = 1$

(b) $2x - 1 = 1$ ... (i)

$y + 1 = -2$ ... (ii)

From (i),

$2x = 2$

$\therefore$ $x = 1$

From (ii)

$y = -2 - 1 = -3$

$\therefore$ $x = 1$ and $y = -3$

2. Let $A = \{a, b, c\}$ and $B = \{x, y, z\}$. Which of the following set of ordered pairs represent the relation from A to B?

(a) $\mathfrak{R}_1 = \{(a, x), (b, y), (c, x), (c, z)\}$ (b) $\mathfrak{R}_2 = \{(x, a), (y, b), (z, c)\}$

(c) $\mathfrak{R}_3 = \{(a, x), (b, x), (c, x)\}$.

Solution

$A = \{a, b, c\}$, $B = \{x, y, z\}$, $A \times B$

$= \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z), (c, x), (c, y), (c, z)\}$

(a) $R_1 = \{(a, x), (b, y), (c, x), (c, z)\}$,

$R_1 \subseteq A \times B$

$\therefore$ R_1 is the relation

(b) $R_2 = \{(x, a), (y, b), (z, c)\}$

$R_2 \nsubseteq A \times B$

$\therefore$ R_2 is not a relation

(c) $R_3 = \{(a, x), (b, x), (c, x)\}$

$R_3 \subseteq A \times B$

$\therefore$ R_3 is the relation from A to B.

3. Find the domain, range and inverse relation of the following:

(a) $\mathfrak{R}_1 = \{(a, a), (b, d), (c, f), (g, h)\}$ (b) $\mathfrak{R}_2 = \{(1, 2), (1, 3), (1, 4)\}$.

Solution

(a) Domain = $\{a, b, c, g\}$

Range = $\{a, d, f, h\}$

Inverse relation = $\{(a, a), (d, b), (f, c), (h, g)\}$

(b) Domain = $\{1\}$

Range = $\{2, 3, 4\}$

Inverse relation = $\{(2, 1), (3, 1), (4, 1)\}$

4. If $A = \{a, b, c\}$ and $B = \{1, 2\}$ find $A \times B$, $B \times A$, $A \times A$ and $B \times B$. Also show that $A \times B \neq B \times A$.

Solution

$A = \{a, b, c\}, B = \{1, 2\}$

Now,

$A \times B = \{a, b, c\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

$B \times A = \{1, 2\} \times \{a, b, c\} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$A \times A = \{a, b, c\} \times \{a, b, c\}$

$= \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

& $B \times B = \{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)$, we see that $A \times B \neq B \times A$

5. If $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$, find $A \times (B \cup C)$, $A \times (B \cap C)$ and $(A \times C) \cap (B \times B)$.

Solution

$B \cup C = \{2, 3\} \cup \{3, 4\} = \{2, 3, 4\}$, $B \cap C = \{3\}$

Now,

$A \times (B \cup C) = \{1, 2\} \times \{2, 3, 4\} = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$

$A \times (B \cap C) = \{1, 2\} \times \{3\} = \{(1, 3), (2, 3)\}$

$B \times B = \{2, 3\} \times \{2, 3\} = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$

& $A \times C = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

$\therefore$ $(B \times B) \cap (A \times C) = \{(2, 3)\}$

6. (a) Let $A = \{1, 2, 3\}$. Find the relation in $A \times A$ satisfying the condition $x + y < 4$, where $x, y \in A$.

(b) Let $A = \{2, 3, 4\}$. Find the relation in $A \times A$ satisfying the condition (i) $y = 2x$ (ii) $x > y$ (iii) $x + y \leq 4$ (iv) $x + y = 1$.

Solution

(a) $A \times A = \{1, 2, 3\} \times \{1, 2, 3\}$

$= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

The relation in $A \times A$ satisfying the condition $x + y < 4$ is

$\{(1, 1), (1, 2), (2, 1)\}$

(b) $A \times A = \{2, 3, 4\} \times \{2, 3, 4\}$

$= \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

(i) R_1 = relation from A to A satisfying $y = 2x$

$= \{(2, 4)\}$

(ii) R_2 = relation from A to A satisfying $x > y$

$= \{(3, 2), (4, 2), (4, 3)\}$

(iii) R_3 = relation from A to A satisfying $x + y \leq 4 = \{(2, 2)\}$.

(iv) R_4 = relation from A to A satisfying $x + y = 1 = \phi$

7. Determine whether the relation $\leq$ is or is not an equivalence relation for the set of real numbers.

Solution

Let $\Re$ be the relation denoted by $\leq$ on the real number. Let a and b be any two real numbers such that $a \leq b$.

That is $a \Re b \Rightarrow a \leq b$

$\Rightarrow \quad b - a \geq 0$

$\Rightarrow \quad -(a - b) \geq 0$

$\Rightarrow \quad a - b \leq 0$

$\Rightarrow \quad a \not> b$

$\not\Rightarrow \quad b \Re a$

Hence $\Re$ is not symmetric. Hence $\Re$ is not an equivalence relation on $\mathbb{R}$.

8. Let $\mathfrak{R}$ be the relation on the set of integers $\mathbb{Z}$ defined by $a \equiv b \pmod 2$. Prove that $\mathfrak{R}$ is an equivalence relation on $\mathbb{Z}$.

Solution

(i) **Reflexive:** Let $a \in \mathbb{Z}$. Then $a - a = 0$ which is divisible by 2 and hence $(a, a) \in \mathfrak{R}$. So, $\mathfrak{R}$ is reflexive.

(ii) **Symmetric:** Let $(a, b) \in \mathfrak{R}$. Then $a - b$ is divisible by 2.
Now, $b - a = -(a - b)$ which is divisible by 2.
Hence, $(b, a) \in \mathfrak{R}$.
So, $\mathfrak{R}$ is symmetric.

(iii) **Transitive:** Let $(a, b) \in \mathfrak{R}$ and $(b, c) \in \mathfrak{R}$. Then $(a - b)$ and $(b - c)$ are divisible by 2.
Now, $a - c = (a - b) + (b - c)$ is also divisible by 2.
Hence, $(a, c) \in \mathfrak{R}$.
So, $\mathfrak{R}$ is transitive.
Hence, $\mathfrak{R}$ is an equivalence relation on $\mathbb{Z}$.

9. Check whether the relation $\mathfrak{R}$ defined in the set $\{1, 2, 3, 4\}$ defined by $\mathfrak{R} = \{(x, y) : y = x + 1\}$ is reflexive, symmetric or transitive.

Solution

$\mathfrak{R} = (1, 2), (2, 3), (3, 4)\}$
Since 1 is not related to 1, so $\mathfrak{R}$ is not reflexive.
Since $1 \mathfrak{R} 2 \not\Rightarrow 2 \mathfrak{R} 1$, $\mathfrak{R}$ is not symmetric.
Since $1 \mathfrak{R} 2$, $2 \mathfrak{R} 3 \not\Rightarrow 1 \mathfrak{R} 3$, so $\mathfrak{R}$ is not transitive.
So, $\mathfrak{R}$ is not reflexive nor symmetric no transitive.

10. Show that the relation $\mathfrak{R}$ in the set $\{1, 2, 3, 4\}$ given by $\mathfrak{R} = \{(2, 3), (3, 2)\}$ is neither reflexive nor transitive but symmetric.

Solution

Let $A = \{1, 2, 3, 4\}$.
Here, $\mathfrak{R} = \{(2, 3), (3, 2)\}$.
Since 2 is not related to 2, $\mathfrak{R}$ is not reflexive.
Next $2 \mathfrak{R} 3 \Rightarrow 3 \mathfrak{R} 2$.
So, $\mathfrak{R}$ is reflexive.
And, $2 \mathfrak{R} 3$, $3 \mathfrak{R} 2 \not\Rightarrow 2 \mathfrak{R} 2$.
So, $\mathfrak{R}$ is not transitive.

Objective Questions

1. Which of the following ordered pairs are equal?
(a) (1, 1), (2, 2) (b) (3, 2), (2, 3) (c) (−2, −5), (2, 5) (d) (1, 5), (1, 5)
Ans: d

2. If $(2x + y, x - 3y) = (1, 11)$ then $y =$
(a) − 3 (b) − 2 (c) 2 (d) 3
Ans: a

$2x + y = 1$ ... (i)
$x - 3y = 11$ ... (ii)
Solving (i) and (ii), we get
$y = -3$

3. If $A = \{1, 2, 3\}$ and $B = \{a, b\}$ then $n(A \times B) = ?$
(a) 2 (b) 2 (c) 5 (d) 6
Ans: d

$n(A \times B) = n(A) \times n(B) = 3 \times 2 = 6$

4. If $A \times B = \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$ then $A =$
(a) {1, 3} (b) {1, 3, 5} (c) {2, 3} (d) {1, 2, 3}
Ans: b

$A \times B = \{(1, 2), (1, 3), (3, 2)\ (3, 3), (5.2)\ (5, 3)\}$
A = Set of all first elements of ordered pairs = {1, 3, 5}

5. Let A = {a, b, c} and B = {1, 2, 3}. Which of the following is a relation from A to B?
   (a) {(a, 1), (a, 2), (a, 3)} (b) {(a, 1), (2, b), (c, 3)}
   (c) {(a, a), (b, b), (c, c)} (d) {(1, a), (2, b), (3, c)}

Ans: a

6. The range of the relation {(a, d), (b, d), (c, d)} is
   (a) ϕ (b) {a} (c) {a, b, c} (d) {d}

Ans: d

Range = {d}

7. Let A = {1, 2, 3}. The total number of distinct relations on A is
   (a) 2^9 (b) 9^2 (c) 9 (d) 18

Ans: a

n(A) = 3

Total no. of district relations = $2^{n(A) \cdot n(A)} = 2 = 2^{3 \times 3} = 2^9$

8. If A and B are two sets containing n_1 and n_2 elements respectively, how many different relations can be defined from A to B?
   (a) $2^{n_1 n_2}$ (b) $n_1 n_2$ (c) $2^{n_1 + n_2}$ (d) 2

Ans: a

n(A) = n_1, n(B) = n_2

Total no. of relations = $2^{n(A) \cdot n(B)} = 2^{n_1 n_2}$

9. A relation on a set is said to be an equivalence relation if it is
   (a) reflexive (b) symmetric (c) transitive (d) all of above

Ans: d

Definition

10. The relation 'is less than' in the set of natural numbers is
    (a) symmetric (b) transitive
    (c) reflexive (d) equivalence relation

Ans: b

EXERCISE – 2 B

1. Let A = {a, b, c} and B = {d, e, f}. Determine which of the following relations from A to B are functions.
   (a) $\mathfrak{R}_1$ = {(a, d), (a, e), (b, e), (c, f)}. (b) $\mathfrak{R}_2$ = {(a, d), (b, e), (c, f)}.
   (c) $\mathfrak{R}_3$ = {(x, f) : x = a, b, c}.

Solution

A = {a, b, c}, B = {d, e, f}

Now,

A × B = {(a, d), (a, e), (a, f), (b, d), (b, e), (b, f), (c, d), (c, e), (c, f)}

(a) R_1 is not a function since 'a' has two images.

(b) R_2 is a function since each element of A has unique image in B.

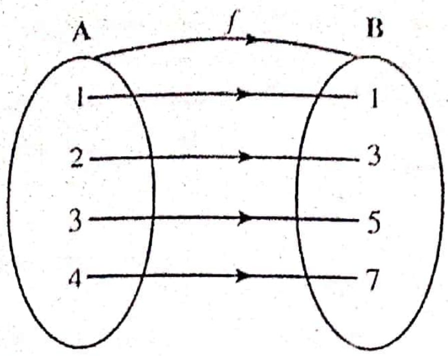
(c) R_3 is a function since each element of A has unique image in B.

2. What is the difference between a relation and a function? Is f = {(1, 1), (2, 3), (3, 5), (4, 7)} a function? If it is defined by $f(x) = ax + b$, what values should be assigned to a and b?

Solution

A relation from a set A to set B is any subset of A × B. A relation becomes a function when each element of A is uniquely associated with a element of B. Every function is a relation but a relation may not be a function.

Here, f = {(1, 1), (2, 3), (3, 5) (4, 7)}



Since each element of first set is associated with a unique element of second set, So f is a function.

By inspection $f(x) = 2x - 1$

$\therefore \quad a = 2, b = -1$

Alternatively,

Here, $f(1) = 1$

$a \cdot 1 + b = 1$

$a + b = 1 \quad \ldots (i)$

$f(2) = 3$

$2a + b = 3 \quad \ldots (ii)$

Solving (i) and (ii), $a = 2, b = -1$.

3. **Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2|x| + 1$. Find $f(0), f(1), \dfrac{f(1+h) - f(1)}{h}, h > 0$.**

Solution

$f(0) = 2|0| + 1 = 1$

$f(1) = 2|1| + 1 = 2 + 1 = 3$

$f(1 + h) = 2|1 + h| + 1 = 2(1 + h) + 1 \quad (\because h > 0)$

$= 2 + 2h + 1 = 3 + 2h$

Now, $\dfrac{f(1+h) - f(1)}{h} = \dfrac{3 + 2h - 3}{h} = 2.$

4. **Find the range of the following functions.**

(a) $f(x) = \dfrac{x}{x+2}$ **when domain = {–1, 0, 2, 4, 6}.**

(b) $g(x) = \dfrac{-x+1}{|x| - 2}$ **when domain = {– 3, 3, 4}.**

Solution

(a) $f(x) = \dfrac{x}{x+2}$

$f(-1) = \dfrac{-1}{-1+2} = -1$

$f(0) = \dfrac{0}{0+2} = 0$

$f(2) = \dfrac{2}{2+2} = \dfrac{1}{2}$

$f(4) = \dfrac{4}{4+2} = \dfrac{2}{3}$

$f(6) = \dfrac{6}{6+2} = \dfrac{3}{4}$

$\therefore$ Range of $f = \left\{-1, 0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}\right\}$.

(b) When $x = -3$, $g(-3) = \frac{-(-3)+1}{|-3|-2} = \frac{4}{1} = 4$

When $x = 3$, $g(3) = \frac{-3+1}{|3|-2} = \frac{-2}{1} = -2$

When $x = 4$, $g(4) = \frac{-4+1}{|4|-2} = \frac{-3}{2} = \frac{-3}{2}$

$\therefore$ Range of $g = \{4, -2, -3/2\}$.

5. **Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2-3x, & \text{for } -\frac{1}{2} \le x < 0 \\ 2+3x, & \text{for } 0 \le x < \frac{1}{2} \\ -2-3x, & \text{for } x \ge \frac{1}{2} \end{cases}$. Find $f\left(-\frac{1}{2}\right), f(0), f\left(\frac{1}{2}\right)$ and $\frac{f(h)-f(0)}{h}$ for $-\frac{1}{2} \le h < 0$.**

Solution

$f\left(-\frac{1}{2}\right) = 2 - 3\left(-\frac{1}{2}\right) = 2 + \frac{3}{2} = \frac{7}{2}$

$f(0) = 2 + 3 \times 0 = 2$

$f\left(\frac{1}{2}\right) = -2 - 3\left(\frac{1}{2}\right) = -2 - \frac{3}{2} = \frac{-4-3}{2} = \frac{-7}{2}$

$f(h) = 2 - 3h \qquad \left(\because -\frac{1}{2} \le h < 0\right)$

Now, $\frac{f(h)-f(0)}{h} = \frac{2-3h-2}{h} = \frac{-3h}{h} = -3$

6. **What do you mean by a function? Distinguish between the range and the co-domain of a function. Let A ={–2, –1, 0, 1, 2} and a function $f: A \to B$ be defined by $f(x) = \frac{x^2}{2}$. Find the range of f. Is the function one-one?**

Solution

Let A and B be two sets. A function f from a set A tc set B is a relation which associates each element of A with unique element of B and we write $f: A \to B$.

The element $f(x)$ of B is called the image of x under f while x is called the pre-image of $f(x)$ under f.

Distinguish between the range and the co-domain of a function

Let $f: A \to B$ be a function. The set B is known as the **co-domain** of f. The set of all images of elements of A is known as the **range** of f and is denoted by $f(A)$.

When $x = -2$, $f(-2) = \frac{(-2)^2}{2} = 2$

When $x = -1$, $f(-1) = \frac{(-1)^2}{2} = \frac{1}{2}$

When $x = 0$, $f(0) = \frac{0^2}{2} = 0$

When $x = 1$, $f(1) = \frac{1^2}{2} = \frac{1}{2}$

When $x = 2$, $f(2) = \frac{2^2}{2} = 2$

Range of $f = \left\{0, \frac{1}{2}, 2\right\}$

Here, $-1 \ne 1$ but $f(-1) = f(1)$ So, f is not one-one.

7. **Let a function f : A → B be defined by $f(x) = \frac{x-1}{x+2}$ with A = {–1, 0, 1, 2, 3, 4} and B = $\left\{-2, -\frac{1}{2}, 0, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, 2\right\}$. Find the range of f. Is the function f one-one and onto both? If not, how can you make it one-one and onto both?**

Solution

Here, the function f: A → B is defined by $f(x) = \frac{x-1}{x+2}$ and A = {–1, 0, 1, 2, 3, 4} and

$$B = \left\{-2, -\frac{1}{2}, 0, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, 2\right\}$$

when $x = -1$, $f(-1) = \frac{-1-1}{-1+2} = -2$

when $x = 0$, $f(0) = \frac{0-1}{0+2} = -\frac{1}{2}$

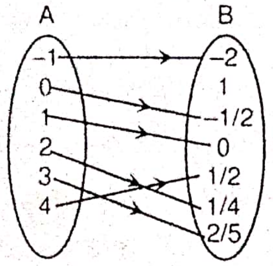
when $x = 1$, $f(1) = \frac{1-1}{1+2} = 0$

when $x = 2$, $f(2) = \frac{2-1}{2+2} = \frac{1}{4}$

when $x = 3$, $f(3) = \frac{3-1}{3+2} = \frac{2}{5}$

when $x = 4$, $f(4) = \frac{4-1}{4+2} = \frac{1}{2}$

Range of f = $\left\{-2, -\frac{1}{2}, 0, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}\right\}$



Since the different elements in A have different images in B, so f is one to one.
Again, the element 2 in B has no pre-image in A, i.e. f(A) ≠ B. So, f is not onto.
The function f can be made both one to one and onto when 2 is removed from B.

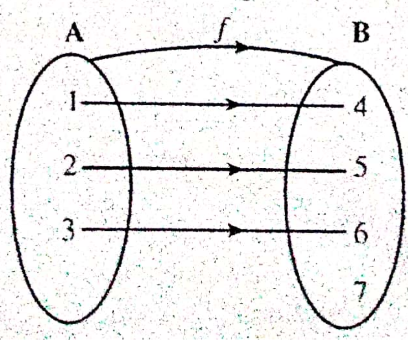
i.e. f : A → B – {2} defined by $f(x) = \frac{x-1}{x+2}$ is both one to one and onto.

8. **Let A = {1, 2, 3} and B = {4, 5, 6, 7}. Let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that (a) f is one to one (b) f is not onto.**

Solution

Given,
A = {1, 2, 3}
B = {4, 5, 6, 7}
f = {(1, 4), (2, 5), (3, 6)}

Arrow diagram

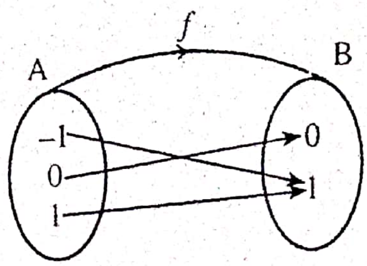


a. Since distinct elements in A have distinct images in B, so f is one to one.

b. Since $7 \in B$ has no pre-image in A, so f is not onto.

9. Examine whether the following functions are one-one, onto, both or neither.

(a)



(b) $f: \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 2x$.

(c) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 5$.

(d) $f: (-5, 5) \to \mathbb{R}$ defined by $f(x) = x^2$.

(e) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

Solution

a. Here,

$f(-1) = 1$ and $f(1) = 1$

$-1 \neq 1$ but $f(-1) = f(1)$

So, f is not one -one.

Again, $f(A) = B$, So f is onto.

b. Let, $x_1, x_2 \in \mathbb{N}$ (domain).

Then, $f(x_1) = 2x_1$ and $f(x_2) = 2x_2$

Now, $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Again, let $y \in \mathbb{N}$ (Co-domain). Then,

$f(x) = y = 2x$

or, $x = \frac{1}{2} y$

But for some $y \in \mathbb{N}$, $x \notin \mathbb{N}$.

For example, if $y = 9 \in \mathbb{N}$, then $x = \frac{9}{2} \notin \mathbb{N}$.

So, f is not onto.

c. Let, $x_1, x_2 \in \mathbb{R}$ (domain). Then,

$f(x_1) = 3x_1 + 5$ and $f(x_2) = 3x_2 + 5$

Now, $f(x_1) = f(x_2) \Rightarrow 3x_1 + 5 = 3x_2 + 5$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Again, let $y \in \mathbb{R}$ (Co-domain). Then, $y = f(x) = 3x + 5$

or, $3x = y - 5$

or, $x = \frac{y-5}{3} \in \mathbb{R}$ for all $y \in \mathbb{R}$.

and $f\left(\frac{y-5}{3}\right) = 3 \cdot \left(\frac{y-5}{3}\right) + 5 = y - 5 + 5 = y$

$\therefore$ f is onto.

Hence, f is both one-one and onto.

d. Given, $f: (-5, 5) \to \mathbb{R}$ be defined by $f(x) = x^2$.

Now, $f(-4) = (-4)^2 = 16$

$f(4) = 4^2 = 16$.

Here, $-4 \neq 4$ but $f(-4) = f(4)$

So, f is not one-one.

Again, let $y \in \mathbb{R}$. Then,

$y = f(x) = x^2$

or, $x = \sqrt{y}$

But for some $y \in \mathbb{R}$, $x \notin (-5, 5)$.

So, f is not onto.

c. Given,

$f: \mathbb{R} \to \mathbb{R}$ is defined by

$f(x) = x^2$

Consider $-1, 1 \in \mathbb{R}$ (domain). Then,

$f(-1) = (-1)^2 = 1$

$f(1) = 1^2 = 1$

$\therefore$ $f(-1) = f(1) \not\Rightarrow -1 = 1$

So, f is not one-one.

Again, consider $-1 \in \mathbb{R}$ (Co-domain).

Then, $f(x) = -1$

or, $x^2 = -1$

or, $x = \pm 1 \notin \mathbb{R}$

So, f is not onto.

10. **Prove that the function $f : \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = 2x + 3$ for all $x \in \mathbb{Q}$ is a bijective function. Find the inverse of f.**

Solution

One to one

Let $x_1, x_2 \in \mathbb{Q}$ (domain). Then,

$f(x_1) = 2x_1 + 3$ and $f(x_2) = 2x_2 + 3$

Now, $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3$

$\Rightarrow$ $2x_1 = 2x_2$

$\Rightarrow$ $x_1 = x_2$

$\therefore$ f is one to one.

Onto

Let $y \in \mathbb{Q}$ (Co-domain). Then,

$y = f(x) = 2x + 3$

or, $2x = y - 3$

or, $x = \dfrac{y-3}{2} \in \mathbb{Q}$ (domain) for all $y \in \mathbb{Q}$ (co-domain).

Also, $f\left(\dfrac{y-3}{2}\right) = 2\left(\dfrac{y-3}{2}\right) + 3 = y - 3 + 3 = y.$

$\therefore$ f is onto.

Hence f is bijective.

Inverse of f

Let, $y = f(x) = 2x + 3$

or, $y = f(x)$

$\therefore$ $x = f^{-1}(y)$ ... (i)

Also, $y = 2x + 3$

or, $y - 3 = 2x$

or, $x = \dfrac{y-3}{2}$ ... (ii)

From (i) and (ii), we get

$f^{-1}(y) = \dfrac{y-3}{2}$

Since y is dummy variable, replacing y by x, we get,

$f^{-1}(x) = \dfrac{x-3}{2}$

Alternative method for f^{-1}

Let $y = 2x + 3$

Interchanging the role of x and y, we get

$x = 2y + 3$

or, $x - 3 = 2y$

or, $y = \frac{x-3}{2}$

$\therefore$ $f^{-1}(x) = \frac{x-3}{2}$

11. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 7$ for all $x \in \mathbb{R}$ is bijective. Find the inverse of f.

Solution

Here,

f: $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 7$.

Let, $x_1, x_2 \in \mathbb{R}$ (domain). Then,

$f(x_1) = x_1^3 + 7$ and $f(x_2) = x_2^3 + 7$

Now, $f(x_1) = f(x_2) \Rightarrow x_1^3 + 7 = x_2^3 + 7$

$\Rightarrow$ $x_1^3 = x_2^3$

$\Rightarrow$ $x_1 = x_2$

$\therefore$ f is one to one.

Again, let $y \in \mathbb{R}$ (co-domain).

Then,

$y = f(x) = x^3 + 7$

or, $x^3 = y - 7$

or, $x = \sqrt[3]{y-7} \in \mathbb{R}$ (domain) for all $y \in \mathbb{R}$ (co-domain).

Also, $f\left(\sqrt[3]{y-7}\right) = \left(\sqrt[3]{y-7}\right)^3 + 7 = y - 7 + 7 = y$

$\therefore$ f is onto.

Hence f is bijective.

Again, let $y = f(x) = x^3 + 7$.

or, $y = f(x)$

$\therefore$ $x = f^{-1}(y)$ ... (i)

Also, $y = x^3 + 7$

or, $x^3 = y - 7$

or, $x = \sqrt[3]{y-7}$ ... (ii)

From (i) and (ii)

$f^{-1}(y) = \sqrt[3]{y-7}$

Since, y is dummy variable, so replacing y by x, we get

$f^{-1}(x) = \sqrt[3]{x-7}$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x^3 + 1$ and $g(x) = 5x$. Find

(a) $(2f + 3)(x)$ (b) $(f^2)(x)$ (c) $(f \cdot g)(x)$

Solution

a. $(2f + 3)(x) = 2(3x^3 + 1) + 3 = 6x^3 + 2 + 3 = 6x^3 + 5$

b. $(f^2)(x) = (3x^3 + 1)^2$

c. $(f \cdot g)(x) = f(x) \cdot g(x) = (3x^3 + 1)\, 5x = 5x\,(3x^3 + 1)$

13. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find:

(a) $(fog)(x)$ (b) $(gof)(x)$

(c) $(fof)(x)$ (d) $(gog)(x)$

Solution

a. $(fog)(x) = f(g(x)) = f(x^2 - 3) = x^2 - 3 + 5 = x^2 + 2$

b. $(gof)(x) = g(f(x)) = g(x + 5) = (x + 5)^2 - 3 = x^2 + 10x + 25 - 3 = x^2 + 10x + 22.$

c. $(fof)(x) = f(f(x)) = f(x + 5) = (x + 5) + 5 = x + 10$

d. $(gog)(x) = g(g(x)) = g(x^2 - 3) = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 9 - 3 = x^4 - 6x^2 + 6$

14. If $u(x) = 4x - 5$, $v(x) = x^2$ and $f(x) = \frac{1}{x}$, find (a) $u(v(f(x)))$, (b) $v(u(f(x)))$ and (c) $f(u(v(x)))$.

Solution

a. $u(v(f(x)) = u\left(v\left(\frac{1}{x}\right)\right) = u\left(\frac{1}{x^2}\right) = 4\left(\frac{1}{x^2}\right) - 5 = \frac{4}{x^2} - 5.$

b. $v(u(f(x)) = v\left(u\left(\frac{1}{x}\right)\right) = v\left(\frac{4}{x} - 5\right) = \left(\frac{4}{x} - 5\right)^2$

c. $f(u(v(x)) = f(u(x^2)) = f(4x^2 - 5) = \frac{1}{4x^2 - 5}$

15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 3$ and $g : \mathbb{R} \to \mathbb{R}$ be defined $g(x) = \frac{x+3}{2}$ show that $fog = gof$.

Solution

$(fog)(x) = f(g(x)) = f\left(\frac{x+3}{2}\right) = 2\left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$

$(gof)(x) = g(f(x)) = g(2x - 3) = \frac{2x - 3 + 3}{2} = x$

$\therefore$ $(fog)(x) = (gof)(x)$

16. Let $A = \{1, 2, 3\}$. Let $f = \{(1, 2), (2, 1), (3, 3)\}$ and $g = \{(1, 3), (2, 1), (3, 2)\}$. Find (a) gof (b) fog.

Solution

Given,

$A = \{1, 2, 3\}$

$f = \{(1, 2), (2, 1), (3, 3)\}$

$\therefore$ $f(1) = 2, f(2) = 1, f(3) = 3$

$g = \{(1, 3), (2, 1) (3, 2)\}$

$\therefore$ $g(1) = 3, g(2) = 1, g(3) = 2.$

a. $(gof)(1) = g(f(1)) = g(2) = 1$

$(gof)(2) = g(f(2)) = g(1) = 3$

$(gof)(3) = g(f(3)) = g(3) = 2$

$\therefore$ $gof = \{(1, 1), (2, 3) (3, 2)\}$

b. Now, $(fog)(1) = f(g(1)) = f(3) = 3$

$(fog)(2) = f(g(2)) = f(1) = 2$

$(fog)(3) = f(g(3)) = f(2) = 1$

$\therefore$ $fog = \{(1, 3), (2, 2), (3, 1)\}$

17. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = ax + b$, $a (\neq 0)$ and b are real numbers. Find f^{-1} and show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Solution

Let $y = f(x) = ax + b$

or, $y = f(x)$

$\therefore$ $x = f^{-1}(y)$ ... (i)

Also, $y = ax + b$

or, $ax = y - b$

$\therefore \quad x = \dfrac{y-b}{a}$ ... (ii)

From (i) and (ii), we get

$$f^{-1}(y) = \frac{y-b}{a}$$

Since y is dummy variable, so replacing y by x, we get

$$f^{-1}(x) = \frac{x-b}{a}$$

Now, $f(f^{-1}(x)) = f\left(\dfrac{x-b}{a}\right) = a\left(\dfrac{x-b}{a}\right) + b = x - b + b = x$

And, $f^{-1}(f(x)) = f^{-1}(ax+b) = \dfrac{(ax+b)-b}{a} = \dfrac{ax}{a} = x$

$\therefore \quad f(f^{-1}(x)) = f^{-1}(f(x)) = x.$

18. **Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x + 1$, fin $(fog)(x)$ and $(gof)(x)$. Are the functions $(fog)(x)$ and $(gof)(x)$ one-one?**

Solution

$f(x) = 2x$

$g(x) = x + 1$

Now, $(gof)(x) = g(f(x)) = g(2x) = 2x + 1$

$(fog)(x) = f(g(x) = f(x+1) = 2(x+1) = 2x + 2$

For (gof) (x)

Let x_1 and $x_2 \in \mathbb{R}$ (domain). Then $(gof)(x_1) = 2x_1 + 1$ and $(gof)(x_2) = 2x_2 + 1$

Now, $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow \quad 2x_1 + 1 = 2x_2 + 1$

$\Rightarrow \quad 2x_1 = 2x_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ gof is one to one.

Again, for (fog) (x)

Let $x_1, x_2 \in \mathbb{R}$ (domain).

$(fog)(x_1) = 2x_1 + 2$ and $(fog)(x_2) = 2x_2 + 2$

Now, $(fog)(x_1) = (fog)(x_2)$

$\Rightarrow \quad 2x_1 + 2 = 2x_2 + 2$

$\Rightarrow \quad 2x_1 = 2x_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ fog is one to one.

19. **Determine whether the function is even, odd or neither.**

(a) $f(x) = x^2 + x^4$ (b) $f(x) = x^3 + \dfrac{1}{x}$

(c) $h(t) = 2|t| + 1$ (d) $g(u) = u^2 + u$

Solution

a. $f(x) = x^2 + x^4$

Now, $f(-x) = (-x)^2 + (-x)^4 = x^2 + x^4 = f(x)$

So, f(x) is even function.

b. $f(x) = x^3 + \dfrac{1}{x}$

Now, $f(-x) = (-x)^3 + \dfrac{1}{(-x)} = -x^3 - \dfrac{1}{x} = -\left(x^3 + \dfrac{1}{x}\right) = -f(x)$

So, f(x) is odd function.

c. $h(t) = 2\,|t| + 1$

Now, $h(-t) = 2\,|-t| + 1 = 2|t| + 1 = h(t)$

So, $h(t)$ is even function.

d. $g(u) = u^2 + u$

Now, $g(-u) = (-u)^2 + (-u) = u^2 - u$ which is neither $g(u)$ nor $-g(u)$.

So, g is neither even nor odd function.

20. Find the domain and range of

(a) $f(x) = 2x + 3$ (b) $f(x) = 1 + x^2$ (c) $f(x) = \dfrac{1}{x+1}$

(d) $f(x) = \sqrt{4 - x^2}$ (e) $f(x) = \sqrt{6 - x - x^2}$

Solution

a. Let $y = f(x) = 2x + 3$

The given function is defined for all $x \in \mathbb{R}$. So, domain $= \mathbb{R} = (-\infty, \infty)$.

Also, $y = 2x + 3$

or, $x = \dfrac{y-3}{2}$

This shows that $x \in \mathbb{R}$ for all $y \in \mathbb{R}$. i.e. x is defined for all $y \in \mathbb{R}$.

Range $= \mathbb{R} = (-\infty, \infty)$

b. $y = f(x) = 1 + x^2$

For all real values of x, y is defined.

So, domain $= \mathbb{R} = (-\infty, \infty)$.

Again, $x^2 = y - 1$

i.e. $x = \pm\sqrt{y-1}$.

So, x is defined for $y - 1 \geq 0$

i.e. $y \geq 1$.

$\therefore$ Range $= [1, \infty)$

c. $y = f(x) = \dfrac{1}{x+1}$

Here, y is defined for all $x \in \mathbb{R}$ except $x = -1$.

So, domain $= \mathbb{R} - \{-1\}$

Again,

$$y = \frac{1}{x+1}$$

or, $x + 1 = \dfrac{1}{y}$

or, $x = \dfrac{1}{y} - 1 = \dfrac{1-y}{y}$ which is defined for all y except $y = 0$.

$\therefore$ Range $= \mathbb{R} - \{0\}$.

d. $y = f(x) = \sqrt{4 - x^2}$

Here, y is defined for $4 - x^2 \geq 0$. Otherwise, y will be imaginary.

or, $4 \geq x^2$

or, $x^2 \leq 4$

or, $|x|^2 \leq 2^2$

or, $|x| \leq 2$

or, $-2 \leq x \leq 2$

$\therefore$ Domain $= [-2, 2]$

Again, $y^2 = 4 - x^2$

or, $x^2 = 4 - y^2$

Since, $x^2 \geq 0$ for all $x \in \mathbb{R}$, so

$$4 - y^2 \geq 0$$

or, $4 \geq y^2$

or, $y^2 \leq 4$

or, $|y|^2 \leq 2^2$

or, $|y| \leq 2$

or, $-2 \leq y \leq 2$

But y is a positive square root, so $0 \leq y \leq 2$.

$\therefore$ Range = [0, 2].

e. Let $y = f(x) = \sqrt{6 - x - x^2}$

$$= \sqrt{6 - (x^2 + x)}$$

$$= \sqrt{6 - \left(x^2 + 2 \cdot x \cdot \frac{1}{2} + \frac{1}{4} - \frac{1}{4}\right)} = \sqrt{\frac{25}{4} - \left(x + \frac{1}{2}\right)^2} = \sqrt{\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}$$

Now, y will be defined if $\left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2 \geq 0$. Otherwise y will be imaginary.

or, $\left(\frac{5}{2}\right)^2 \geq \left(x + \frac{1}{2}\right)^2$

or, $\left(x + \frac{1}{2}\right)^2 \leq \left(\frac{5}{2}\right)^2$

or, $\left|x + \frac{1}{2}\right|^2 \leq \left(\frac{5}{2}\right)^2$

or, $\left|x + \frac{1}{2}\right| \leq \frac{5}{2}$

or, $\frac{-5}{2} \leq x + \frac{1}{2} \leq \frac{5}{2}$

or, $\frac{-5}{2} - \frac{1}{2} \leq x \leq \frac{5}{2} - \frac{1}{2}$

or, $-3 \leq x \leq 2$

$\therefore$ Domain = [-3, 2]

Again,

$$y^2 = \left(\frac{5}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2$$

or, $\left(x + \frac{1}{2}\right)^2 = \left(\frac{5}{2}\right)^2 - y^2$

Since $\left(x + \frac{1}{2}\right)^2 \geq 0$ for all $x \in \mathbb{R}$,

So, $\left(\frac{5}{2}\right)^2 - y^2 \geq 0$

or, $\left(\frac{5}{2}\right)^2 \geq y^2$

or, $y^2 \leq \left(\frac{5}{2}\right)^2$

or, $|y|^2 \leq \left(\frac{5}{2}\right)^2$

or, $-\frac{5}{2} \leq y \leq \frac{5}{2}$

Since y is a positive square root, so $0 \leq y \leq \frac{5}{2}$.

$\therefore$ Range = $\left[0, \frac{5}{2}\right]$

21. Find domain and range of $f, g, f+g$ and $f \cdot g$, when $f(x) = x, g(x) = \sqrt{x-1}$.

Solution

Here,

$f(x) = x,\ g(x) = \sqrt{x-1}$

Domain and Range of f

Let, $y = f(x) = x$

Here, y is defined for all x in $\mathbb{R}$.

So, domain $= (-\infty, \infty)$.

Again, $x = y$

So, x is defined for all y in $\mathbb{R}$.

Range $= (-\infty, \infty)$

Domain and range of g

Let, $y = g(x) = \sqrt{x-1}$

Here, y is defined for $x - 1 \geq 0$

i.e. $x \geq 1$

$\therefore$ Domain $= [1, \infty)$

Again, $y^2 = x - 1$

or, $x = y^2 + 1$ ... (i)

For all $x \geq 1$, from (i) we can write

$y^2 + 1 \geq 1$

or, $y^2 \geq 0$

or, $y \geq 0$.

$\therefore$ Range $= [0, \infty)$

Domain and Range of f + g and f ·g

$D_f : -\infty < x < \infty,\ D_g : x \geq 1$

$\Rightarrow D_{f+g} = D_{fg} : x \geq 1$

$\therefore$ Domain of $f + g = [1, \infty)$

Domain of $f \cdot g = [1, \infty)$

$R_f : -\infty < x < \infty,\ R_g : y \geq 0 \Rightarrow$

$R_{f+g} : y \geq 1$ and $R_{f \cdot g} : y \geq 0$.

Range of $f + g = [1, \infty)$

Range of $f \cdot g = [0, \infty)$

Objective Questions

1. If $f(x) = x + a$ maps 1 into 2, then the value of a is

(a) 1 (b) 2 (c) 3 (d) 4

Ans: a

$f(1) = 2$

or, $1 + a = 2$

$\therefore$ $a = 1$

2. If $f(x) = \dfrac{x - |x|}{|x|}$, then $f(-2) =$

(a) -2 (b) -1 (c) 0 (d) 2

Ans: a

$$f(x) = \frac{x - |x|}{|x|}$$

$$f(-2) = \frac{-2 - |-2|}{|-2|} = \frac{-2-2}{2} = -2$$

3. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) \begin{cases} 3 + 2x & \text{for } -\frac{3}{2} \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{for } x \ge \frac{3}{2} \end{cases} \text{, then } f\left(\frac{3}{2}\right) =$$

(a) 0 (b) 3 (c) −3 (d) −6

Ans: d

$$f\left(\frac{3}{2}\right) = -3 - 2 \times \frac{3}{2} = -6$$

4. If $f(x) = x^2$ and $g(x) = 2x + 1$ then $(f + g)(x) =$

(a) $(x + 1)^2$ (b) $(x - 1)^2$ (c) $x^2(2x + 1)$ (d) $2x^3 + x^2$

Ans: a

$(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1 = (x + 1)^2$

5. Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9, 16, 25\}$ and $f : A \to B$ be defined by $f(x) = x^2$. Range of f =

(a) {1, 4, 9, 16, 25} (b) {1, 4, 9, 16} (c) {1, 4, 9} (d) {1, 4}

Ans: c

Range of f = {f(−1), f(1), f(2), f(3)} = {1, 1, 4, 9} = {1, 4, 9}

6. The number of functions from $\{a, b, c\}$ into $\{1, 2\}$ is

(a) 2 (b) 4 (c) 6 (d) 8

Ans: d

The number of functions from a finite set A into a finite set B = $(n(B))^{n(A)}$

∴ Required no. of functions = $2^3 = 8$

7. If $f(x - 1) = x + 3$ then $f(x^2) =$

(a) x^2 (b) $x^2 + 3$ (c) $x^2 + 4$ (d) $x^2 - 1$

Ans: c

Here,

$f(x - 1) = x + 3 = (x - 1) + 4$

∴ $f(x^2) = x^2 + 4$

8. A function is defined as $f(1) = 3, f(2) = 4, f(3) = 5$, then f is

(a) one to one function (b) onto function

(c) both one to one and onto (d) neither onto nor one to one

Ans: c

Both one to one and onto.

9. Inverse function f^{-1} exists if f is

(a) injective (b) surjective (c) bijective (d) into

Ans: c

Bijective

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 5$. Then $f^{-1}(x) =$

(a) $3x - 5$ (b) $\frac{x-3}{5}$ (c) $\frac{x-5}{3}$ (d) $5 - 3x$

Ans: c

$y = f(x) = 3x + 5$

∴ $x = f^{-1}(y)$ ... (i)

Also, $y = 3x + 5$

or, $3x = y - 5$

∴ $x = \frac{y-5}{3}$ ... (ii)

From (i) and (ii),

$$f^{-1}(y) = \frac{y-5}{3}$$

$$\therefore \quad f^{-1}(x) = \frac{x-5}{3}$$

11. A function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 2$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$. Then $fog\,(x) =$

(a) $(x+2)^3$ (b) $x^3 + 8$ (c) $(x+2)^3$ (d) $x^3 + 2$

Ans: d

$fog\,(x) = f(g(x)) = f(x^3) = x^3 + 2$

12. If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ then $gof\,(x) =$

(a) $\sqrt{x+1}$ (b) $\sqrt{x} + 1$ (c) $x + \frac{1}{4}$ (d) $x + 2$

Ans: b

$gof\,(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$

13. The domain of the function $y = \frac{1}{x-1}$ is

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{1\}$ (c) $\mathbb{R} - \{-1\}$ (d) $[0, \infty)$

Ans: b

$y = \frac{1}{x-1}$ is defined for all $x \in \mathbb{R}$ except $x = 1$.

$\therefore$ Domain $= \mathbb{R} - \{1\}$.

14. The range of $y = \frac{1}{x}$ is

(a) $(-\infty, 0)$ (b) $(0, \infty)$ (c) $(-\infty, 0) \cup (0, \infty)$ (d) $(-\infty, \infty)$

Ans: c

$$y = \frac{1}{x}$$

or, $x = \frac{1}{y}$

So, x is defined for all $y \in \mathbb{R}$ except $y = 0$.

Range $= (-\infty, 0) \cup (0, \infty)$.

15. If $g(x) = \sqrt{1-x}$ then domain of $3g$ is

(a) $(-\infty, 1)$ (b) $(-\infty, 1]$ (c) $(-\infty, 3)$ (d) $(-\infty, 3]$

Ans: b

$g(x) = \sqrt{1-x}$

g is defined only for $1 - x \geq 0$

$1 \geq x$

$x \leq 1$

Domain of $g = (-\infty, 1]$

Domain of $3g = (-\infty, 1]$

EXERCISE – 2 C

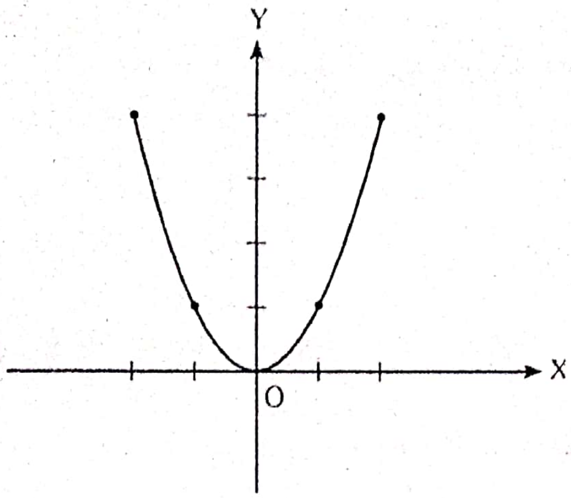
1. Graph the functions

(a) $y = x^2$ (b) $y = x^3$ (c) $y = \sin x \; (-\pi < x < \pi)$

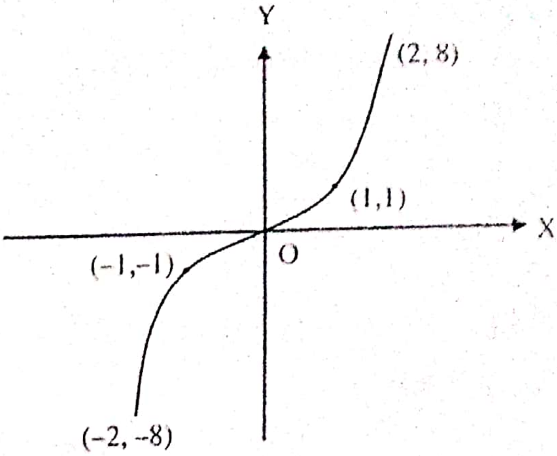
(d) $y = 3^x$ (e) $y = \log_e 2$.

Solution

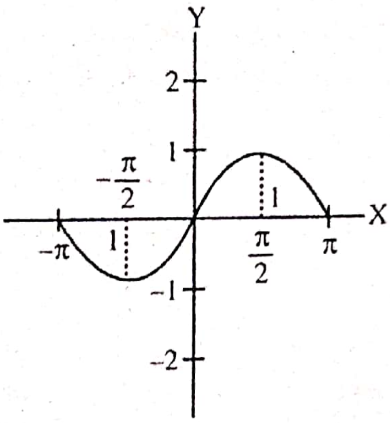
a.



b.

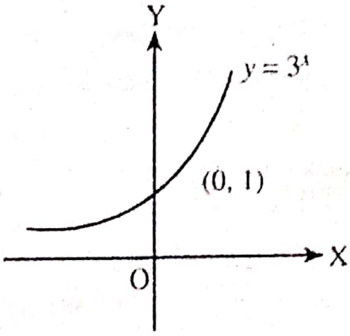


c.

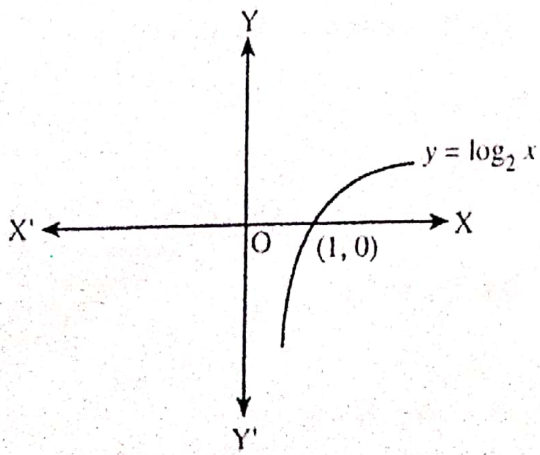


y = sin x

d.



e.



2. Prove that:

(a) $\log_a(x^2y^3z) = 2\log_a x + 3\log_a y + \log_a z$

(b) $\log_a\left(\frac{x^2}{y^3}\right) = 2\log_a x - 3\log_a y$

(c) $2\log_a\sqrt{x} = \log_a x$

(d) $a^{\log_a x} = x$

(e) $(\log x)^2 - (\log y)^2 = \log(xy) \cdot \log\left(\frac{x}{y}\right)$

(f) $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$

(g) $a^{\log b - \log c} \cdot b^{\log c - \log a} \cdot c^{\log a - \log b} = 1$

(h) $(yz)^{\log y - \log z} \cdot (zx)^{\log z - \log x} \cdot (xy)^{\log x - \log y} = 1.$

Solution

a. L.H.S. $= \log_a (x^2y^3z) = \log_a (x^2) + \log_a y^3 + \log_a z = 2 \log_a x + 3 \log_a y + \log_a z =$ R.H.S.

b. L.H.S. $= \log_a \left(\frac{x^2}{y^3}\right) = \log_a x^2 - \log_a y^3 = 2 \log_a x - 3 \log_a y$

c. L.H.S. $= 2 \log_a \sqrt{x} = 2 \log_a x^{1/2} = 2 \times \frac{1}{2} \log_a x = \log_a x =$ R.H.S.

d. Let, $\log_a x = y$

or, $x = a^y$

or, $x = a^{\log_a x}$

$\therefore$ $a^{\log_a x} = x$

e. L.H.S. $= (\log x)^2 - (\log y)^2 = (\log x + \log y)(\log x - \log y) = \log (xy) \log \left(\frac{x}{y}\right) =$ R.H.S.

f. L.H.S. $= \log(1 + 2 + 3) = \log (6) = \log (1 \cdot 2 \cdot 3) = \log 1 + \log 2 + \log 3$

g. Let $x = a^{\log b - \log c}\, b^{\log c - \log a}\, c^{\log a - \log b}$

Taking 'log' on both sides,

$\log x = (\log b - \log c) \log a + (\log c - \log a) \log b + (\log a - \log b) \log c$

$= \log a \log b - \log c \log a + \log b \log c - \log a \log b + \log c \log a - \log b \log c$

$= 0$

or, $\log x = \log 1$

$\therefore$ $x = 1$

$\therefore$ $a^{\log b - \log c} \cdot b^{\log c - \log a} \cdot c^{\log a - \log b} = 1$

h. Let $a = (yz)^{\log y - \log z} \cdot (zx)^{\log z - \log x} \cdot (xy)^{\log x - \log y}$

Taking log on both sides, we get

$\log a = \log \{(yz)^{\log y - \log z} \cdot (zx)^{\log z - \log x} \cdot (xy)^{\log x - \log y}\}$

$= (\log y - \log z)(\log y + \log z) + (\log z - \log z)(\log z + \log x) + (\log x - \log y)(\log x + \log y)$

or, $\log a = (\log y)^2 - (\log z)^2 + (\log z)^2 - (\log x)^2 + (\log x)^2 - (\log y)^2 = 0$

or, $\log a = \log 1$

or, $a = 1$

$\therefore$ $(yz)^{\log y - \log z} \cdot (zx)^{\log z - \log x} \cdot (xy)^{\log x - \log y} = 1$

3. Solve for x.

(a) $\left(\frac{1}{2}\right)^{-x} = 8^{3x-1}$ (b) $\log_3 (x - 1) = 2$

Solution

a. $\left(\frac{1}{2}\right)^{-x} = 8^{3x-1}$

or, $2^x = 2^{3(3x-1)}$

or, $x = 9x - 3$

or, $9x - x = 3$

or, $8x = 3$

$\therefore$ $x = \frac{3}{8}$

b. $\log_3 (x - 1) = 2$

or, $(x - 1) = 3^2$

or, $x = 9 + 1$

$\therefore$ $x = 10$

4. If $f(x) = \log\frac{1+x}{1-x}(-1 < x < 1)$, show that $f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right)$ where $|a| < 1, |b| < 1$.

Solution

Here,

$$f(x) = \log\frac{1+x}{1-x}$$

$$\text{So, } f(a) = \log\frac{1+a}{1-a}$$

$$\text{and } f(b) = \log\frac{1+b}{1-b}$$

$$\text{Now, } f(a) + f(b) = \log\left(\frac{1+a}{1-a}\right) + \log\left(\frac{1+b}{1-b}\right)$$

$$= \log\left(\frac{1+a}{1-a}\cdot\frac{1+b}{1-b}\right) = \log\left(\frac{1+a+b+ab}{1-a-b+ab}\right) = \log\left(\frac{(1+ab)+(a+b)}{(1+ab)-(a+b)}\right)$$

$$= \log\left\{\frac{\frac{(1+ab)+(a+b)}{1+ab}}{\frac{(1+ab)-(a+b)}{1+ab}}\right\} = \log\left(\frac{1+\frac{a+b}{1+ab}}{1-\frac{a+b}{1+ab}}\right) = f\left(\frac{a+b}{1+ab}\right)$$

5. If $f(x) = \log\frac{1-x}{1+x}(-1 < x < 1)$, show that: $f\left(\frac{2ab}{1+a^2b^2}\right) = 2f(ab)$ where $|ab| < 1$.

Solution

Here,

$$f(x) = \log\left(\frac{1-x}{1+x}\right)$$

$$\text{Now, } f\left(\frac{2ab}{1+a^2b^2}\right) = \log\left(\frac{1-\frac{2ab}{1+a^2b^2}}{1+\frac{2ab}{1+a^2b^2}}\right) = \log\left(\frac{\frac{1+a^2b^2-2ab}{1+a^2b^2}}{\frac{1+a^2b^2+2ab}{1+a^2b^2}}\right) = \log\left\{\frac{(1-ab)^2}{(1+ab)^2}\right\}$$

$$= \log\left(\frac{1-ab}{1+ab}\right)^2 = 2\log\left(\frac{1-ab}{1+ab}\right) = 2f(ab)$$

6. If $x^2 + y^2 = 11xy$, prove that: $\ln\left(\frac{x-y}{3}\right) = \frac{1}{2}(\ln x + \ln y)$.

Solution

Given,

$$x^2 + y^2 = 11xy$$

$$\text{or, } x^2 + y^2 - 2xy = 11xy - 2xy$$

$$\text{or, } (x-y)^2 = 9xy$$

$$\text{or, } \frac{(x-y)^2}{9} = xy$$

$$\text{or, } \left(\frac{x-y}{3}\right)^2 = xy$$

Taking '*ln*' on both sides, we get

$$2\ln\left(\frac{x-y}{3}\right) = \ln x + \ln y$$

$$\therefore \ln\left(\frac{x-y}{3}\right) = \frac{1}{2}(\ln x + \ln y)$$

7. If $x = \log_a bc, y = \log_b ca, z = \log_c ab$, prove that: $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$.

Solution

$x + 1 = \log_a bc + 1 = \log_a bc + \log_a a = \log_a (abc)$

or, $\frac{1}{x+1} = \frac{1}{\log_a abc}$

$= \log_{abc} (a)$ $\left[\because \log_a m = \frac{1}{\log_m a}\right]$

Similarly, $\frac{1}{y+1} = \log_{abc} b$

and $\frac{1}{z+1} = \log_{abc} c$

L.H.S. $= \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} abc = 1$

Objective Questions

1. The function $f(x) = ax^2 + bx + c, a \neq 0$ is
   (a) constant function (b) linear function
   (c) quadratic function (d) cubic function

Ans: c

2. The function $f(x) = 3^x, x \in \mathbb{R}$ is
   (a) linear function (b) trigonometric function
   (c) cubic function (d) exponential function

Ans: d

3. $\log_3 81 =$
   (a) 1 (b) 2 (c) 3 (d) 4

Ans: d

$\log_3 81 = \log_3 3^4 = 4 \log_3 3 = 4 \times 1 = 4$.

4. $\log_{\sqrt{2}} \frac{1}{16} =$
   (a) –8 (b) –4 (c) –2 (d) –1

Ans: a

$\log_{\sqrt{2}} \frac{1}{16} = \log_{\sqrt{2}} 16^{-1} = \log_{\sqrt{2}} ((\sqrt{2})^8)^{-1} = \log_{\sqrt{2}} (\sqrt{2})^{-8} = -8 \log_{\sqrt{2}} \sqrt{2} = -8$

5. $\log_x \sqrt{x\sqrt{x\sqrt{x^2}}} =$
   (a) x (b) x^2 (c) 1 (d) 0

Ans: c

$\log_x \sqrt{x\sqrt{x\sqrt{x^2}}} = \log_x \sqrt{x\sqrt{x \cdot x}} = \log_x \sqrt{x \cdot x} = \log_x x = 1$

6. $\log_a a^x =$
   (a) a (b) x (c) x^2 (d) a^x

Ans: b

$\log_a a^x = x \log_a a = x \times 1 = x$

7. $y = \log_a x$ is same as
   (a) $y = a^x$ (b) $x = a^y$ (c) $a = x^y$ (d) $a = y^x$

Ans: b

$y = \log_a x$ is same as $x = a^y$.

Unit – 3

Sequence and Series

EXERCISE – 3 A

1. (a) **Find the 10^{th} term of 2, 6, 10, 14...**

   (b) **Find the sum of the series: 2 + 4 + 6 + ... to 40 terms.**

   (c) **If $x + 2$, $3x$ and $4x + 1$ are in A.P., find x.**

Solution

(a) Given A.P. is 2, 6, 10, 14, ...

$\therefore$ $a = 2$, $d = t_2 - t_1 = 6 - 2 = 4$

$n = 10$, $t_{10} = ?$

We know that,

$t_n = a + (n - 1) d$

$t_{10} = 2 + (10 - 1) 4 = 2 + 9 \times 4 = 38$

(b) 2 + 4 + 6 + ... to 40 terms

Here, $a = 2$, $d = 2$, $n = 40$, $S_{40} = ?$

We have,

$$S_n = \frac{n}{2} \{2a + (n - 1) d\}$$

$$S_{40} = \frac{40}{2} \{2 \times 2 + (40 - 1) \times 2\} = 20 (4 + 78) = 20 \times 82 = 1640$$

(c) $a = t_1 = x + 2$

$t_2 = 3x$

$t_3 = 4x + 1$

By the definition of an A.P.

$t_2 - t_1 = t_3 - t_2$

or, $3x - (x + 2) = 4x + 1 - 3x$

or, $3x - x - 2 = x +$

or, $2x - x = 1 + 2$

$\therefore$ $x = 3$

2. (a) **If the 5^{th} and the 12^{th} terms of an A.P. are 14 and 35 respectively, find the first term and the common difference.**

   (b) **The 5^{th} and 11^{th} terms of an A.P. are 41 and 20 respectively, find the first term and the sum of the first 11 terms.**

Solution

(a) $t_5 = 14$, $t_{12} = 35$, $a = ?$, $d = ?$

Case I: $t_5 = 14$

$a + 4d = 14$ ...(i) $[\because t_n = a + (n - 1) d]$

Case II: $t_{12} = 35$

$a + 11d = 35$ ... (ii)

Now, from equation (i) and (ii)

$a + 11 d = 35$

$a + 4 d = 14$

$- \quad - \quad -$

$7 d = 21$

$d = 3$

Putting the value of d in (i), we get

$a + 4 \times 3 = 14$

$a + 12 = 14$

$a = 14 - 12$

$a = 2$

$\therefore$ $a = 2$ and $d = 3$

(b) $t_5 = 41$, $t_{11} = 20$ $a = ?$, $S_{11} = ?$

Case I: $t_5 = 41$

or, $a + 4d = 41$

or, $a = 41 - 4d$ ... (i)

Case II: $t_{11} = 20$

$a + 10d = 20$

or, $a = 20 - 10d$ ... (ii)

Now, from equation (i) and (ii)

$41 - 4d = 20 - 10d$

or, $10d - 4d = 20 - 41$

or, $6d = -21$

$\therefore$ $d = \frac{21}{6} = -\frac{7}{2}$

Putting the value of d in (i)

$$a + 4 \times \left(-\frac{7}{2}\right) = 41$$

or, $a - 14 = 41$

or, $a = 41 + 14 = 55$

Again,

$a = 55$, $d = -\frac{7}{2}$, $S_{11} = ?$, $n = 11$

We have,

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$S_{11} = \frac{11}{2}\{2a + (11-1)d\} = \frac{11}{2}\left\{2 \times 55 + 10 \times \left(-\frac{7}{2}\right)\right\} = \frac{11}{2}\{110 - 35\} = \frac{1}{2} \times 825$$

$$= 412\frac{1}{2}$$

3. Insert three arithmetic means between 5 and 405.

Solution

No. of A.M. (n) = 3

$a = 5$, $b = 405$

$\therefore$ $d = \frac{b-a}{n+1} = \frac{405-5}{3+1} = \frac{400}{4} = 100$

$\therefore$ Three A.M. are:

$m_1 = a + d = 5 + 100 = 105$

$m_2 = a + 2d = 5 + 2 \times 100 = 205$

$m_3 = a + 3d = 5 + 3 \times 100 = 305$

4. (a) The sum of three numbers in A.P. is 15 and the sum of their squares is 83, find them.

(b) How many terms of the series 24 + 20 + 16 + ... must be taken so that the sum may be 72? Explain the double answer.

Solution

(a) Let the three numbers of an A.P. be a – d, a, a + d

By first condition of question, we have,

$(a - d) + a + (a + d) = 15$

$3a = 15$

$a = 5$

Again, by second condition,

$(a - d)^2 + a^2 + (a + d)^2 = 83$

or, $a^2 - 2ad + d^2 + a^2 + a^2 + 2ad + d^2 = 83$

or, $3a^2 + 2d^2 = 83$

or, $3 \times 5^2 + 2d^2 = 83$

or, $2d^2 = 83 - 75$

or, $2d^2 = 8$

or, $d^2 = 4$

$\therefore$ $d = \pm 2$

The three numbers of an A.P. **when a = 5, d = 2** are

$a - d, a, a + d$

$5 - 2, 5, 5 + 2$

i.e. 3, 5, 7

The three numbers of an A.P. **when a = 5, d = – 2** are

$5 - (-2), 5, 5 + (-2)$

i.e. 7, 5, 3

(b) The given series is an Arithmetic series.

$a = 24; d = 20 - 24 = -4; S_n = 72; n = ?$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

or, $72 = \frac{n}{2}[2 \times 24 + (n - 1) \times (-4)]$

or, $72 = \frac{n}{2}(48 - 4n + 4)$

or, $72 = \frac{n}{2}(52 - 4n)$

or, $72 = 26n - 2n^2$

or, $n^2 - 13n - 36 = 0$

or, $n^2 - 9n - 4n - 36 = 0$

or, $n(n - 9) - 4(n - 9) = 0$

or, $(n - 9)(n - 4) = 0$

$\therefore$ $n = 9$ or 4

The first 9 terms of the given series are 24, 20, 16, 12,8, 4, 0, –4, –8 the sum of whose last 5 term is zero. Evidently, therefore the sum of the first 4 terms is equal to the sum of the first 9 terms. Hence, the double answer.

5. **(a) Find the 12th term of the sequence 1, 2, 4, 8, ...**

(b) Find the sum of the following series.

(i) 1 + 3 + 9 + 27 + ... to 10 terms. (ii) 6 + 12 + 24 + ... + 1536.

Solution

(a) Here,

First term (a) = 1

Common ratio $(r) = \frac{2}{1} = 2$

12^{th} term $(t_{12}) = ?$

Number of term (n) = 12

We have,

$t_n = a. r^{n-1}$

$t_{12} = 1 \times 2^{12-1} = 2^{11} = 2048$

(b) (i) First term (a) = 1

Common ratio (r) $= \frac{3}{1} = 3$

Number of terms (n) = 10

We have,

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad [\because r>1]$$

$$\therefore \quad S_{10} = \frac{1\,(3^{10} - 1)}{3 - 1} = \frac{59049 - 1}{2} = \frac{59048}{2} = 29524$$

(ii) First term (a) = 6

Last term (l) = 1536

Common ratio (r) $= \frac{12}{6} = 2$

We have,

$$S_n = \frac{lr - a}{r - 1} = \frac{1536 \times 2 - 6}{2 - 1} = 3066$$

6. (a) If the 5[th] and the 10[th] term of a G.P. are 32 and 1024 respectively, find the first term and the common ratio.

(b) The sum of a Geometric progression whose common ratio is 2 and the last term is 768 is 1533. Find the first term.

Solution

(a) Here, 5[th] term of G.P. = 32

i.e. $a\,r^4 = 32$ ... (i)

and 10[th] term of G.P. = 1024

i.e. $a\,r^9 = 1024$ ... (ii)

Dividing (ii) by (i), we get

or, $\frac{a\,r^9}{a\,r^4} = \frac{1024}{32}$

or, $r^5 = 32$

$\therefore$ $r = 2$

Putting the value of r in (i), we get,

$a \times 2^4 = 32$

$a = \frac{32}{2^4} = \frac{32}{16} = 2$

$\therefore$ First term = 2

(b) Last term (l) = 768

Sum of terms (S_n) = 1533

Common ratio = 2

or, $S_n = \frac{lr - a}{r - 1}$

or, $1533 = \frac{768 \times 2 - a}{2 - 1}$

or, $1533 - 1536 = -a$

$\therefore$ $a = 3$

7. **Insert 3 Geometric Means between 5 and 80.**

Solution

First term (a) = 5

Last term (l) = 80

No. of terms (n) = 3 + 2 = 5

Common ratio (r) = ?

$l = a\, r^{n-1}$

or, $80 = 5\, r^{5-1}$

or, $16 = r^4$

∴ $r = 2$

1st mean (m_1) = $ar = 5 \times 2 = 10$

2nd mean (m_2) = $m_1 \times r = 10 \times 2 = 20$

3rd mean (m_3) = $m_2 \times r = 20 \times 2 = 40$

Required three geometric means between 5 and 80 are 10, 20, 40.

8. **Find the three numbers in G.P. whose sum is 14, product is 64.**

Solution

Let the three numbers in G.P. be $\frac{a}{r}$, a, ar

We have,

Product of three number = 64

i.e. $\frac{a}{r} \times a \times ar = 64$

$a^3 = 46$

$a = 4$

Sum of numbers = 14

i.e. $\frac{a}{r} + a + ar = 14$

or, $\frac{4}{r} + 4 + 4r = 14$

or, $4r^2 - 10r + 4 = 0$

or, $2r^2 - 5r + 2 = 0$

or, $2r(r-2) - 11\,(r-2) = 0$

∴ $(r-2)(2r-1) = 0$

Either

$r - 2 = 0$ or, $2r - 1 = 0$

∴ $r = 2$ $\quad r = \frac{1}{2}$

When r = 2

$\frac{a}{r} = \frac{4}{2} = 2$

$a = 4$

$ar = 4 \times 2 = 8$

When r = 1/2

$\frac{a}{r} = \frac{4}{1/2} = \frac{4}{1} \times \frac{2}{1} = 8$

$a = 4$

$ar = 4 \times \frac{1}{2} = 2$

9. (a) **If G be the geometric mean between two distinct positive numbers a and b, show that:**

$$\frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2} = \frac{1}{G^2}.$$

(b) **If H be the harmonic mean between a and b, prove that:**

$$\frac{1}{H - a} + \frac{1}{H - b} = \frac{1}{a} + \frac{1}{b}.$$

Solution

a. If G be the G.M. between a and b, then $G^2 = ab$.

$$\text{L.H.S.} = \frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2}$$

$$= \frac{1}{ab - a^2} + \frac{1}{ab - b^2} = \frac{1}{a(b-a)} + \frac{1}{b(a-b)} = \frac{1}{a(b-a)} - \frac{1}{b(b-a)}$$

$$= \frac{b-a}{ab(b-a)} = \frac{1}{ab} = \frac{1}{G^2} = \text{R.H.S.}$$

b. Since H be harmonic mean between a & b, $H = \frac{2ab}{a+b}$

$$\text{L.H.S.} = \frac{1}{H-a} + \frac{1}{H-b}$$

$$= \frac{1}{\frac{2ab}{a+b} - a} + \frac{1}{\frac{2ab}{a+b} - b} = \frac{1}{\frac{2ab - a^2 - ab}{a+b}} + \frac{1}{\frac{2ab - ab - b^2}{a+b}}$$

$$= \frac{a+b}{ab - a^2} + \frac{a+b}{ab - b^2} = (a+b)\left\{\frac{1}{a(b-a)} + \frac{1}{b(a-b)}\right\}$$

$$= (a+b)\left\{\frac{1}{a(b-a)} + \frac{1}{a(b-a)}\right\} = (a+b)\left\{\frac{b-a}{ab(b-a)}\right\} = \frac{a+b}{ab}$$

$$= \frac{1}{a} + \frac{1}{b} = \text{R.H.S. proved.}$$

10. (a) **If $a^x = b^y = c^z$ and a, b, c are in G.P., then prove that: x, y, z are in H.P.**

(b) **If a, b, c are in AP; b, c, d are in G.P. and c, d, e are in H.P. then prove tha a, c, e are in G.P.**

Solution

(a) Here, $a^x = b^y = c^z$

So, $a^x = b^y \Rightarrow a = b^{y/x}$

and $c^z = b^y \Rightarrow c = b^{y/z}$

If a, b, c are in G.P., then

$b^2 = ac$

or, $b^2 = b^{y/x} . b^{y/z}$

or, $b^2 = b^{y/x + y/z}$

or, $2 = \frac{y}{x} + \frac{y}{z}$

or, $2 = \frac{yz + xy}{xz}$

or, $2xz = y(x + z)$

$\therefore$ $y = \frac{2xz}{x + z}$

Hence x, y, z are in H.P.

(b) a, b, c be in A.P.

$$b = \frac{a+c}{2} \quad \ldots \text{(i)}$$

b, c, d be in G.P

$$c^2 = bd \quad \ldots \text{(ii)}$$

c, d, e in H.P

$$d = \frac{2ce}{c+e} \quad \ldots \text{(iii)}$$

Now, from (ii)

$c^2 = bd$

or, $c^2 = \frac{a+c}{2} \cdot \frac{2ce}{c+e}$

$c = \frac{e(a+c)}{c+e}$

or, $c^2 + ce = ae + ce$

$c^2 = ae$

$\therefore$

i.e. a, c, e are in G.P.

11. (a) **The sum of three numbers in A.P. is 36. When the numbers are increased by 1, 4, 43 respectively, the resulting numbers are in G.P. Find the numbers.**

(b) **If the three consecutive term of a geometric series be increased by their middle term, then prove that the resulting terms will be in harmonic series.**

(c) **Prove that b^2 is greater than or equal to or less than ac according as three unequal positive numbers a, b, c are in A.P., G.P. or H.P.**

(d) **The A.M. between two numbers exceeds their G.M. by 1 and G.M. exceeds H.M. by 0.8. Find the numbers.**

Solution

a. Let three numbers in A.P. be a − d, a and a + d. By question,

$(a-d) + a + (a+d) = 36$

or, $3a = 36$

$\therefore$ $a = 12$

Again, by question,

$a - d + 1$, $a + 4$ and $a + d + 43$ are in G.P.

i.e. $12 - d + 1$, $12 + 4$ and $12 + d + 43$ are in G.P.

i.e. $13 - d$, 16 and $55 + d$ are in G.P.

$\Rightarrow$ $\frac{16}{13-d} = \frac{55+d}{16}$ (by definition of G.P.)

or, $256 = (55+d)(13-d)$

or, $256 = 715 - 55d + 13d - d^2$

or, $d^2 + 42d - 459 = 0$

or, $d^2 + 51d - 9d - 459 = 0$

or, $d(d+51) - 9(d+51) = 0$

or, $(d+51)(d-9) = 0$

$\therefore$ $d = -51, 9$

When d = −51, a = 12

$a - d = 12 + 51 = 63$, $a = 12$, $a + d = 12 - 51 = -39$

When d = 9, a = 12

$a - d = 12 - 9 = 3$, $a = 12$, $a + d = 9 + 12 = 21$

Required three numbers are 63, 12, −39 or 3, 12, 21

b. Let a, b and c be three term in G.S.

Then, $b^2 = ac$ ... (i)

We have to prove,

$a+b$, $b+b$, $b+c$ are in H.S.

i.e., $a+b$, $2b$, $b+c$ are in H.S.

These terms will be in H.S. if

$\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A.P.

i.e., if $\frac{1}{2b} - \frac{1}{a+b} = \frac{1}{b+c} - \frac{1}{2b}$

or, $\frac{1}{2b}+\frac{1}{2b}=\frac{1}{b+c}+\frac{1}{a+b}$

or, $\frac{1+1}{2b}=\frac{a+b+b+c}{(b+c)(a+b)}$

or, $\frac{1}{b}=\frac{a+2b+c}{ab+b^2+ac+bc}$

or, $ab+b^2+ac+bc=ab+2b^2+bc$

or, $ac=2b^2-b^2$

$\therefore$ $b^2=ac$. which is true by (i).

c. If a, b, c are in A.P then $b=\frac{a+c}{2}$

To prove $b^2>ac$.

Now, $b^2-ac=\left(\frac{a+c}{2}\right)^2-ac=\frac{(a+c)^2-4ac}{4}=\frac{(a-c)^2}{4}$ which is always positive.

$\therefore$ $b^2>ac$.

If a, b, c are in G.P. then

$\frac{t_2}{t_1}=\frac{t_3}{t_2}$

or, $\frac{b}{a}=\frac{c}{b}$

$\therefore$ $b^2=ac$.

If a, b, c are in H.P. then $b=\frac{2ac}{a+c}$

To prove, $b^2<ac$

i.e. $\frac{1}{b^2}>\frac{1}{ac}$

Now, $\frac{1}{b^2}-\frac{1}{ac}=\frac{1}{\left(\frac{2ac}{a+c}\right)^2}-\frac{1}{ac}=\frac{(a+c)^2}{4a^2c^2}-\frac{1}{ac}=\frac{(a+c)^2-4ac}{4a^2c^2}$

$=\frac{(a-c)^2}{4a^2c^2}=\left(\frac{a-c}{2ac}\right)^2$ which is always positive.

i.e. $\frac{1}{b^2}>\frac{1}{ac}$

Hence, $b^2<ac$.

d. Let two numbers be a and b.

Then, A.M. $=\frac{a+b}{2}$, G.M. $\sqrt{ab}$, H.M. $=\frac{2ab}{a+b}$.

By given,

A.M. – G.M. = 1

A.M. = G.M. + 1 ... (i)

And, G.M. – H.M. = 0.8

or, H.M. = G.M. – 0.8 ... (ii)

We have,

A.M. × H.M. = $(G.M.)^2$

or, (G.M. + 1) (G.M.– 0.8) = $(G.M.)^2$

or, $(G.M.)^2$ – 0.8 G.M. + G.M. – 0.8 = $(G.M.)^2$

or, 0.2 G.M. = 0.8

or, G.M. $=\frac{0.8}{0.2}=4$

or, $\sqrt{ab}=4$

$ab=16$ ... (iii)

Putting the value of G.M. in (i)

$$A.M. = 4 + 1$$

or, $\frac{a+b}{2} = 5$

or, $a + b = 10$

$a = 10 - b$ ... (iv)

Putting the value of a in (iii)

$(10 - b)\, b = 16$

or, $10b - b^2 = 16$

or, $0 = b^2 - 10b + 16$

or, $(b - 8)(b - 2) = 0$

$\therefore$ $b = 2, 8$

When $b = 2$, from (iv), $a = 10 - 2 = 8$

When $b = 8$, from (iv) $a = 10 - 8 = 2$

Required two numbers are 8, 2 or 2, 8.

12. If *a, b, c* are in A.P. & *x, y, z* in G.P. prove that $x^{b-c} \cdot y^{c-a} \cdot z^{a-b} = 1$

Solution

If a, b, c are in A.P, then $b - a = c - b = d$ (say)

Then, $a - b = -d$, $b - c = -d$ and $c - a = 2d$.

And, if x, y, z are in G.P, then $y^2 = xz$

L.H.S. $= x^{b-c} \cdot y^{c-a} \cdot z^{a-b}$

$= x^{-d} \cdot y^{2d} \cdot z^{-d} = x^{-d} \cdot (y^2)^d \cdot z^{-d} = x^{-d} \cdot (xz)^d \cdot z^{-d} = x^{-d} \cdot x^d \cdot z^d \cdot z^{-d}$

$= x^{-d+d} \cdot z^{d-d} = x^0 z^0 = 1 = $ R.H.S.

13. If *a, b, c* are three positive numbers in H.P., show that $a^2 + c^2 > 2b^2$.

Solution

A.M between a^2 and $c^2 = \frac{a^2 + c^2}{2}$

G.M between a^2 and $c^2 = \sqrt{a^2c^2} = ac$

Then, $\frac{a^2 + c^2}{2} > ac$ ... (i) [$\because$ A.M > G.M.]

G.M between a and c $= \sqrt{ac}$

H.M between a and c = b (given)

Then, $\sqrt{ac} > b$ [$\because$ G.M. > H.M]

$\therefore$ $ac > b^2$ ... (ii)

From (i) and (ii)

$$\frac{a^2 + c^2}{2} > b^2$$

$a^2 + c^2 > 2b^2$ proved.

14. (a) If *a, b, c* be in H. P. prove that: $a(b + c), b(c + a), c(a + b)$ are in A.P.

(b) If a^2, b^2, c^2 are in A.P., prove that $b + c, c + a, a + b$ are in H.P.

Solution

(a) If a, b, c are in H.P then

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow$ $\frac{1}{a} - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right), \frac{1}{b} - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right), \frac{1}{c} - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ are in A.P.

$\Rightarrow$ $-\left(\frac{1}{b} + \frac{1}{c}\right), -\left(\frac{1}{a} + \frac{1}{c}\right), -\left(\frac{1}{a} + \frac{1}{b}\right)$ are in A.P.

$\Rightarrow$ $\frac{1}{b} + \frac{1}{c}, \frac{1}{a} + \frac{1}{c}, \frac{1}{a} + \frac{1}{b}$ are in A.P.

$\Rightarrow$ $\frac{b+c}{bc}, \frac{c+a}{ca}, \frac{a+b}{ab}$ are in A.P

Multiplying each term by abc, we get

$$a(b + c), b(c + a), c(a + b) \text{ are in A.P.}$$

(b) Here, b + c, c + a, a + b will be in H.P if

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A.P.}$$

i.e. if $\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

or, $\frac{b+c-c-a}{(b+c)(c+a)} = \frac{c+a-a-b}{(a+b)(c+a)}$

or, $\frac{b-a}{c+b} = \frac{c-b}{b+a}$

or, $b^2 - a^2 = c^2 - b^2$

i.e. if a^2, b^2, c^2 are in A.P, which is given. Hence proved.

15. Sum to n terms the following series.

(a) $2 + 22 + 222 + \dots$ *n* **terms** **(b)** $5 + 55 + 555 + \dots$ *n* **terms**

(c) $0.6 + 0.66 + 0.666 + \dots$ *n* **terms** **(d)** $0.7 + 0.77 + 0.777 + \dots$ *n* **terms.**

Solution

(a) $S_n = 2 + 22 + 222 + \dots \text{ n terms}$

$= 2\,[1 + 11 + 111 + \dots \text{ n terms}]$

$= \frac{2}{9}\,[9 + 99 + 999 + \dots \text{ n terms}]$

$= \frac{2}{9}\,[(10-1) + (100-1) + (1000-1) + \dots \text{ n terms}]$

$= \frac{2}{9}\,[(10 + 100 + 1000 + \dots \text{ n terms}) - (1 + 1 + 1 \dots \text{ n terms}]$

$= \frac{2}{9}\left[\frac{10\,(10^n - 1)}{10-1} - n\right]$

$= \frac{2}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

(b) $S_n = 5 + 55 + 555 + \dots \text{ n terms}$

$= 5\,[1 + 11 + 111 + \dots \text{n terms}]$

$= \frac{5}{9}\,[9 + 99 + 999 + \dots \text{ n terms}]$

$= \frac{5}{9}\,[(10-1) + (100-1) + (1000-1) \dots \text{ n terms}]$

$= \frac{5}{9}\,[(10 + 100 + 1000 + \dots \text{ n terms}) - (1 + 1 + 1 \dots \text{ n terms})]$

$= \frac{5}{9}\left[\frac{10\,(10^n - 1)}{10 - 1} - n\right] = \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

(c) $S_n = 0.6 + 0.66 + 0.666 + \dots \text{ n terms}$

$= \frac{6}{10} + \frac{66}{100} + \frac{666}{1000} + \dots \text{ n terms}$

$= 6\left[\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \dots \text{ n terms}\right] = \frac{6}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \dots \text{ n terms}\right]$

$= \frac{6}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \dots \text{ n terms}\right]$

$= \frac{2}{3}\left[(1 + 1 + \dots \text{ n terms}) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \text{ n terms}\right)\right]$

$$= \frac{2}{3}\left[n - \frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^{n}\right\}}{1-\frac{1}{10}} \right] \qquad [\because r < 1]$$

$$= \frac{2}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$$

(d) $S_n = 0.7 + 0.77 + 0.777 \ldots$ n terms

$$= \frac{7}{10} + \frac{77}{100} + \frac{777}{1000} + \ldots \text{ n terms}$$

$$= 7\left(\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \ldots \text{ n terms}\right)$$

$$= \frac{7}{9}\left(\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ n terms}\right)$$

$$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \ldots \text{ n terms}\right]$$

$$= \frac{7}{9}\left[(1 + 1 + \ldots \text{ n terms})\left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots \text{ n terms}\right)\right]$$

$$= \frac{7}{9}\left[n - \frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^{n}\right\}}{1-\frac{1}{10}} \right] = \frac{4}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$$

Objective Questions

1. **If a, b, c are in A.P. then $\dfrac{(a-c)^2}{b^2 - ac} =$**

(a) 1 (b) 2 (c) 3 (d) 4

Ans: d

$$\frac{(a-c)^2}{b^2 - ac} = \frac{(a-c)^2}{\left(\frac{a+c}{2}\right)^2 - ac} = \frac{(a-c)^2}{\frac{(a+c)^2 - 4ac}{4}} = \frac{4(a-c)^2}{(a-c)^2} = 4$$

2. **If n^{th} term of A.P. is $4n + 1$, then common difference is**

(a) 2 (b) 3 (c) 4 (d) 5

Ans: c

$t_n = 4n + 1$

$t_1 = 4 \times 1 + 1 = 5,\ t_2 = 4 \times 2 + 1 = 9$

$d = t_2 - t_1 = 9 - 5 = 4$

3. **If three non-zero numbers a, b, c are in A.P. then $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in**

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

Ans: a

If a, b, c are in A.P. then

$\dfrac{a}{abc}, \dfrac{b}{abc}, \dfrac{c}{abc}$ in A.P.

i.e. $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ in A.P.

4. **If a, b, c are in G.P. then $\log a$, $\log b$, $\log c$ are in**

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

Ans: a

If a, b, c are in G.P. then $b^2 = ac$

Applying log on both sides,

$2 \log b = \log a + \log c$

$\log b = \dfrac{\log a + \log c}{2}$

i.e. log a, log b, log c are in A.P.

5. **If sum to n terms of an A.P. is n^2 then common difference is**

(a) 1 (b) 2 (c) 3 (d) 4

Ans: b

$S_n = n^2$

Then, $S_1 = 1^2 = 1$ $\therefore$ $t_1 = 1$

$S_2 = 2^2 = 4$

$\therefore$ $t_1 + t_2 = 4$

or, $t_2 = 4 - t_1 = 4 - 1 = 3$

$d = t_2 - t_1 = 3 - 1 = 2$

6. **If $S_n = n^3 - 100$ then $t_{10} =$**

(a) 900 (b) 1000 (c) 561 (d) 271

Ans: d

We have,

$t_n = S_n - S_{n-1}$

or, $t_{10} = S_{10} - S_9$

$= (10^3 - 100) - (9^3 - 100) = 271$

7. **If a, b, c are in G.P. then a^2, b^2, c^2 are in**

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

Ans: b

If a, b, c are in G.P. then a^2, b^2, c^2 are in G.P.

8. **If a, b, c are in A.P.; b, c, a are in H.P. then c, a, b are in**

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

Ans: b

If a, b, c are in A.P. then $b = \dfrac{a+c}{2}$ ... (i)

If b, c, a are in H.P then $c = \dfrac{2ab}{a+b}$ ... (ii)

Multiplying (i) and (ii),

$bc = \dfrac{a+c}{2} \times \dfrac{2ab}{a+b}$

or, $c = \dfrac{a(a+c)}{a+b}$

ro, $ac + bc = a^2 + ac$

or, $a^2 = bc$

i.e. c, a, b are in G.P.

9. **If a, b, c are in H.P. then $b =$**

(a) $\dfrac{a+c}{2}$ (b) $\sqrt{ac}$ (c) $\dfrac{2ac}{a+c}$ (d) $\dfrac{a+c}{2ac}$

Ans: c

If a, b, c are in H.P. then $b = \dfrac{2ac}{a+c}$.

10. If $\frac{x-y}{y-z}=\frac{x}{z}$ then x, y, z are in

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

Ans: c

If $\frac{x-y}{y-z}=\frac{x}{z}$ then

$$xz - yz = xy - xz$$

or, $xz + xz = yz + xy$

or, $y(x+z) = 2xz$

$\therefore$ $y = \frac{2xz}{x+z}$

i.e. x, y, z are in H.P.

11. **If 4^{th} term of H.P. is 5 and 5^{th} term is 4 then first term is**

(a) 1 (b) 4 (c) 5 (d) 20

Ans: d

$t_4 = 5$

or, $\frac{1}{a+3d} = 5$

$\therefore$ $a + 3d = \frac{1}{5}$ ... (i)

And, $t_5 = 4$

or, $\frac{1}{a+4d} = 4$

or, $a + 4d = \frac{1}{4}$ ... (ii)

Solving (i) and (ii), we get

$a = \frac{1}{20}$

$\therefore$ $t_1 = \frac{1}{a} = 20.$

12. **If x, y, z be in H.P., then $\frac{y+z}{y-z}+\frac{y+x}{y-x}=$**

(a) 1 (b) 2 (c) 3 (d) 4

Ans: b

If x, y, z are in H.P then $y = \frac{2xz}{x+z}$.

Now, $\frac{y+z}{y-z}+\frac{y+x}{y-x}$

$$= \frac{\frac{2xz}{x+z}+z}{\frac{2xz}{x+z}-z}+\frac{\frac{2xz}{x+z}+x}{\frac{2xz}{x+z}-x} = \frac{\frac{2xz+xz+z^2}{x+z}}{\frac{2xz-xz-z^2}{x+z}}+\frac{\frac{2xz+x^2+xz}{x+z}}{\frac{2xz-x^2-xz}{x+z}} = \frac{3xz+z^2}{xz-z^2}+\frac{3xz+x^2}{xz-x^2}$$

$$= \frac{3x+z}{x-z}+\frac{3z+x}{z-x} = \frac{3x+z}{x-z}-\frac{3z+x}{x-z} = \frac{3x+z-3z-x}{x-z} = \frac{2x-2z}{x-z} = \frac{2(x-z)}{x-z} = 2$$

13. **If three positive numbers a, b, c are in A.P., G.P. as well as in H.P. then**

(a) $a=b=c$ (b) $a \neq b = c$ (c) $a = b \neq c$ (d) $a \neq b \neq c$

Ans: a

If a, b, c are in A.P., G.P. as well as in H.P. then a = b = c.

EXERCISE – 3 B

1. **Decide which of the following infinite series have sums.**

(a) $1+\frac{1}{2}+\frac{1}{4}+\ \dots$ (b) $1+(-1)+1+(-1)+\dots$

(c) $1+\sqrt{3}+3+\dots$ (d) $1+\frac{1}{\sqrt{3}}+\frac{1}{3}+\dots$

Solution

a. $r=\frac{t_2}{t_1}=\frac{\frac{1}{2}}{1}=\frac{1}{2}$

$|r|=\left|\frac{1}{2}\right|=\frac{1}{2}<1$

So, the series has sum.

b. $r=\frac{t_2}{t_1}=\frac{-1}{1}=-1$

$|r|=|-1|=1$ which is not less than 1.

So, the series doesn't have sum.

c. $r=\frac{t_2}{t_1}=\frac{\sqrt{3}}{1}=\sqrt{3}$

$|r|=|\sqrt{3}|=\sqrt{3}>1$

So, the series doesn't have sum.

d. $r=\frac{t_2}{t_1}=\frac{\frac{1}{\sqrt{3}}}{1}=\frac{1}{\sqrt{3}}$

$|r|=\left|\frac{1}{\sqrt{3}}\right|=\frac{1}{\sqrt{3}}<1$

So, the series has sum.

2. **Find the sum of the following geometric series.**

(a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ \dots$ to ∞. (b) $8+4\sqrt{2}+4+\dots$ to ∞.

(c) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+\dots$ to ∞.

Solution

a. Here, $a=1$, $r=\frac{\frac{1}{2}}{1}=\frac{1}{2}$, $S_\infty=?$

We have,

$$S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2$$

b. Here, $a=8$, $r=\frac{t_2}{t_1}=\frac{4\sqrt{2}}{8}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$

$S_\infty=?$

We have,

$$S_\infty=\frac{a}{1-r}=\frac{8}{1-\frac{1}{\sqrt{2}}}=\frac{8\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}=\frac{8\sqrt{2}(\sqrt{2}+1)}{1}=8(2+\sqrt{2})$$

c. $\left(\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\dots\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\dots\right)$

$$=\frac{\frac{1}{2}}{1-\frac{1}{2^2}}+\frac{\frac{1}{3^2}}{1-\frac{1}{3^2}}=\frac{\frac{1}{2}}{\frac{3}{4}}+\frac{\frac{1}{9}}{\frac{9-1}{9}}=\frac{1}{2}\times\frac{4}{3}+\frac{1}{8}=\frac{16+3}{24}=\frac{19}{24}$$

3. (a) **Prove that:** $2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \ldots = 2$

(b) **If** $x = 1 + a + a^2 + \ldots$ **to** ∞ $y = 1 + b + b^2 + \ldots$ **to** ∞, **prove that** $1 + ab + a^2b^2 + \ldots$ **to** $\infty = \dfrac{xy}{x + y - 1}$ **where** $|a| < 1$ **and** $|b| < 1$.

Solution

a. L.H.S. $= 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \ldots$

$= 2^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots}$

$= 2^{\frac{\frac{1}{2}}{1 - \frac{1}{2}}} = 2^{\frac{\frac{1}{2}}{\frac{1}{2}}} = 2 =$ R.H.S. proved.

b. Here, $x = 1 + a + a^2 + \ldots$ to ∞

or, $x = \dfrac{1}{1 - a}$

or, $1 - a = \dfrac{1}{x}$

or, $1 - \dfrac{1}{x} = a$

$\therefore$ $a = \dfrac{x - 1}{x}$

And, $y = 1 + b + b^2 + \ldots +$ to ∞

or, $y = \dfrac{1}{1 - b}$

or, $1 - b = \dfrac{1}{y}$

or, $1 - \dfrac{1}{y} = b$

$\therefore$ $b = \dfrac{y - 1}{y}$

Now, $1 + ab + a^2b^2 + \ldots = \dfrac{1}{1 - ab} = \dfrac{1}{1 - \left(\dfrac{x - 1}{x}\right)\left(\dfrac{y - 1}{y}\right)} = \dfrac{xy}{xy - (x - 1)(y - 1)}$

$= \dfrac{xy}{xy - xy + x + y - 1} = \dfrac{xy}{x + y - 1}$

4. **The sum of first two terms of an infinite geometric series is 15 and each terms of the series is equal to the sum of all the terms following it. Find the series.**

Solution

Let the series be $a + ar + ar^2 + \ldots$

By given, $a + ar = 15$ ... (i)

And, $a = ar + ar^2 + ar^3 + \ldots$

or, $a = \dfrac{ar}{1 - r}$

or, $1 = \dfrac{r}{1 - r}$

or, $1 - r = r$

or, $1 = 2r$

$r = \dfrac{1}{2}$

Then from (i),

$$a\left(1+\frac{1}{2}\right)=15$$

$$a = 15\times\frac{2}{3} = 10$$

Required G.S. is

$$a + ar + ar^2 + \ldots$$

$$= \quad 10 + 10\times\frac{1}{2} + 10\times\left(\frac{1}{2}\right)^2 + \ldots = 10 + 5 + \frac{5}{2} + \ldots$$

5. The sum of an infinite geometric series is 8. If its second term is 2, find its common ratio.

Solution

$$S_\infty = 8$$

or, $\dfrac{a}{1-r} = 8$

or, $a = 8(1 - r)$ ...(i)

Also, $ar = 2$

$a = \dfrac{2}{r}$ ...(ii)

From (i) and (ii)

$$\frac{2}{r} = 8(1-r)$$

or, $\dfrac{1}{r} = 4(1 - r)$

or, $1 = 4r - 4r^2$

or, $4r^2 - 4r + 1 = 0$

or, $(2r - 1)^2 = 0$

or, $2r - 1 = 0$

$\therefore$ $r = \dfrac{1}{2}$.

6. The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the first term and common ratio.

Solution

Let the G.S. be $a + ar + ar^2 + \ldots$

By given, $a + ar + ar^2 + \ldots = 15$

or, $\dfrac{a}{1-r} = 15$

or, $a = 15(1 - r)$ ... (i)

Again, by question,

$$a^2 + a^2r^2 + a^2r^3 + \ldots = 45$$

or, $\dfrac{a^2}{1-r^2} = 45$

or, $a^2 = 45(1 - r^2)$ ... (ii)

From (i) and (ii)

$$45(1 - r^2) = 225(1 - r)^2$$

or, $45(1 - r)(1 + r) = 225(1 - r)^2$

or, $1 + r = 5(1 - r)$

or, $1 + r = 5 - 5r$

or, $6r = 4$

or, $r = \dfrac{2}{3}$

Putting the value of r in (i)

$$a = 15\left(1-\frac{2}{3}\right) = 15\times\frac{1}{3} = 5.$$

7. **The sides of a square are each 16 cm. A second square is drawn by joining the mid points of the sides, successively. In the second square the process is repeated to drawing the third square. If this process is continued indefinitely, find the sum of the perimeters of all the squares.**

Solution

Let ABCD be the square whose each side is 16 cm.

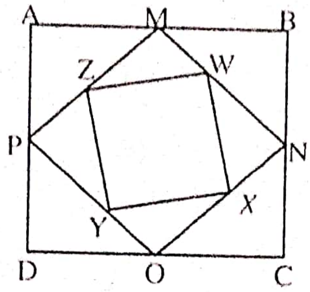
∴ Perimeter of ABCD = 4 × 16 = 64cm

Again, let MNOP be the square obtained by joining the mid points M, N, O, P of sides AB, BC, CD and DA of a square ABCD.

Then, $MN = \sqrt{(MB)^2 + (BN)^2} = \sqrt{8^2 + 8^2} = 8\sqrt{2}$ cm.

The perimeter of the square MNOP = $4(8\sqrt{2}) = 32\sqrt{2}$ cm

Also, WXYZ be the square obtained by joining the midpoints W, X, Y, Z of the sides MN, NO, OP and PQ of a square MNOP.



Then, $WX = \sqrt{(WN)^2 + (NX)^2} = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = 8$

Perimeter of square WXYZ = 4(8) = 32 cm and so on.

The series of the perimeter of all the squares is $64 + 32\sqrt{2} + 32 + \ldots$

$$a = 64,\ r = \frac{32\sqrt{2}}{64} = \frac{1}{\sqrt{2}}$$

$$S_\infty = ?$$

We have, $S_\infty = \dfrac{a}{1-r} = \dfrac{64}{1 - \frac{1}{\sqrt{2}}} = \dfrac{64\sqrt{2}}{\sqrt{2}-1} \times \dfrac{\sqrt{2}+1}{\sqrt{2}+1} = 64\sqrt{2}\,(\sqrt{2}+1) = 64\,(2+\sqrt{2})$ cm.

Objective Questions

1. **$0.\overline{45}$**

(a) $\frac{5}{9}$ (b) $\frac{5}{8}$ (c) $\frac{5}{7}$ (d) $\frac{5}{6}$

Ans: a

$0.\overline{45} = 0.454545\ldots$

$= 0.45 + 0.0045 + 0.000045 + \ldots$

$a = 0.45,\ r = 0.01$

$$S_\infty = \frac{a}{1-r} = \frac{0.45}{1-0.01} = \frac{0.45}{0.99} = \frac{45}{99} = \frac{5}{9}$$

2. **Sum to infinity of a G.S. with first term a and common ratio r, where $|r| < 1$, is**

(a) $\frac{ar}{1-r}$ (b) $\frac{ra}{r+1}$ (c) $\frac{a}{1-r}$ (d) $\frac{a}{r-1}$

Ans: c

Formula

3. $3^{\frac{1}{3}} \cdot 3^{\frac{1}{9}} \cdot 3^{\frac{1}{27}} \ldots$ to ∞ =

(a) $\sqrt{3}$ (b) 3

(c) $3^{[illegible]}$ (d) $3^{[illegible]}$

Ans: a

$$3^{\frac{1}{3}} \cdot 3^{\frac{1}{9}} \cdot 3^{\frac{1}{27}} \cdot$$

$$= 3^{\frac{1/3}{1-1/3}} = 3^{\frac{1/3}{2/3}} = \sqrt{3}$$

4. Sum to infinity $1 + (-1) + 1 + (-1) + \ldots$

(a) 1 (b) -1 (c) 0 (d) doesn't exist

Ans: d

$$r = \frac{-1}{1} = -1$$

$|r| = |-1| = 1 \not< 1$

So, sum of the series doesn't exist.

5. The sum to infinity of a G.S. is 15 and common ratio is $\frac{4}{5}$ then first term is

(a) 1 (b) 2 (c) 3 (d) 4

Ans: c

$$S_\infty = 15,\ r = \frac{4}{5}$$

We have,

$$S_\infty = \frac{a}{1-r}$$

or, $15 = \dfrac{a}{1-\frac{4}{5}}$

or, $15 = \dfrac{a}{\frac{5-4}{5}}$

or, $15 = \dfrac{a}{\frac{1}{5}}$ $\quad\therefore\quad a = 3$

EXERCISE – 3 C*

1. **Find the n^{th} term and sum to n terms of the series:**

(a) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots$ **(b)** $1^2 \cdot 1 + 2^2 \cdot 3 + 3^2 \cdot 5 + \ldots$

(c) $1 + (1 + 3) + (1 + 3 + 5) + \ldots$

Solution

(a) The n^{th} term of 1, 2, 3, . . . is n

The n^{th} term of 3, 4, 5, . . . is $(n + 2)$

Hence the n^{th} term of the given series is $n(n + 2)$

i.e. $t_n = n^2 + 2n$

Let S_n be the sum to n terms of the given series, then

$$S_n = \Sigma t_n = \Sigma(n^2 + 2n)$$

$$= \Sigma n^2 + 2\,\Sigma n = \frac{n(n+1)(2n+1)}{6} + \frac{2n\,(n+1)}{2}$$

$$= n(n+1)\left(\frac{2n+1}{6} + 1\right) = n\,(n+1)\left(\frac{2n+1+6}{6}\right) = \frac{n\,(n+1)\,(2n+7)}{6}$$

(b) The n^{th} term of $1^2, 2^2, 3^2, \ldots$ is n^2
and the n^{th} term of $1, 3, 5, \ldots$ is $2n - 1$
So, the n^{th} term of the given series $(t_n) = n^2 (2n - 1) = 2n^3 - n^2$
Let S_n be the sum to n terms of the given series then

$$S_n = \Sigma t_n$$
$$= 2\Sigma n^3 - \Sigma n^2$$
$$= \frac{2n^2 (n + 1)^2}{4} - \frac{n(n + 1)(2n + 1)}{6} = \frac{n^2 (n + 1)^2}{2} - \frac{n(n + 1)(2n + 1)}{6}$$
$$= \frac{n(n + 1)}{2}\left[n(n + 1) - \frac{2n + 1}{3}\right] = \frac{n(n + 1)}{2}\left(\frac{3n^2 + 3n - 2n - 1}{3}\right)$$
$$= \frac{n(n + 1)(3n^2 + n - 1)}{2}$$

(c) The n^{th} term $(t_n) = 1 + 3 + 5 + \ldots$ to n terms $= n^2$
And, $S_n = \Sigma t_n = \Sigma n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

2. Find the sum to n terms of the following series:

(a) $n \cdot 1 + (n - 1) \cdot 2 + (n - 2) \cdot 3 + \ldots$ **(b)** $3 + 7 + 13 + 21 + \ldots$

(c) $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots$

Solution

(a) Let t_r be the general term and S_n be the sum of first n terms.
Then, $t_r = (r^{th}$ term of $1,2,3,\ldots) \times (r^{th}$ term of $n, n - 1, n - 2, \ldots)$

$$= \{(1 + (r - 1) \cdot 1\} \times \{n + (r - 1) \cdot (-1)\} \quad [\because t_n = a + (n - 1) d] = r(n - r + 1)$$

We have,

$$S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} r(n - r + 1) = \sum_{r=1}^{n} \{r(n + 1) - r^2\} = \sum_{r=1}^{n} r(n + 1) - \sum_{r=1}^{n} r^2$$
$$= (n + 1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 = (n + 1) \cdot \frac{n(n + 1)}{2} - \frac{n(n + 1)(2n + 1)}{6}$$
$$= \frac{n(n + 1)}{2}\left\{(n + 1) - \frac{2n + 1}{3}\right\} = \frac{n(n + 1)}{2} \cdot \frac{(3n + 3 - 2n - 1)}{3} = \frac{n(n + 1)(n + 2)}{6}$$

(b) Given series is
$3 + 7 + 13 + 21 + 31 + \ldots$
$= (1^2 + 2) + (2^2 + 3) + (3^2 + 4) + (4^2 + 5) + \ldots$
The n^{th} term of $1^2, 2^2, 3^2, 4^2, 5^2 \ldots$ is n^2 & the n^{th} term of $2, 3, 4, 5, 6 \ldots$ is
$2 + (n - 1) . 1$ $[\because t_n = a + (n - 1) . 1] = 2 + n - 1 = n + 1$
The n^{th} term of given series $(t_n) = n^2 + n + 1$
We have,

$$S_n = \Sigma t_n$$
$$= \Sigma(n^2 + n + 1) = \Sigma n^2 + \Sigma n + \Sigma 1$$
$$= \frac{n(n + 1)(2n + 1)}{6} + \frac{n(n + 1)}{2} + n = n\left\{\frac{(n + 1)(2n + 1)}{6} + \frac{n + 1}{2} + 1\right\}$$
$$= n\left\{\frac{(n + 1)(2n + 1) + 3(n + 1) + 6}{6}\right\} = n\left(\frac{2n^2 + 2n + n + 1 + 3n + 3 + 6}{6}\right)$$
$$= \frac{n(2n^2 + 6n + 10)}{6} = \frac{n}{3}(n^2 + 3n + 5)$$

(c) Let S_n be the sum to n term of the given series.

Then,

$$S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots\ldots\ldots\ldots + \frac{3n-5}{5^{n-2}} + \frac{3n-2}{5^{n-1}}$$

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$$

Subtracting, we get

$$\frac{4}{5}S_n = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \ldots + \frac{3}{5^{n-1}} - \frac{3n-2}{5^n}$$

$$= 1 + 3\left\{\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots + \frac{1}{5^{n-1}}\right\} - \frac{3n-2}{5^n}$$

$$= 1 + 3\,\frac{\frac{1}{5}\left\{1 - \left(\frac{1}{5}\right)^{n-1}\right\}}{1 - \frac{1}{5}} - \frac{3n-2}{5^n} = 1 + \frac{3}{4} - \frac{3}{4}\cdot\frac{1}{5^{n-1}} - \frac{3n-2}{5^n}$$

$$= 1 + \frac{3}{4}\left\{1 - \frac{1}{5^{n-1}}\right\} - \frac{3n-2}{5^n} = \frac{7}{4} - \frac{3}{4}\cdot\frac{1}{5^{n-1}} - \frac{3n-2}{5^n} = \frac{7}{4} - \left\{\frac{15 + 12n - 8}{4.5^n}\right\}$$

$$\frac{4}{5}S_n = \frac{7}{4} - \frac{12n+7}{4.5^n}$$

$$S_n = \frac{35}{16} - \frac{5(12n+7)}{16.5^n}$$

$$\therefore \quad S_n = \frac{35}{15} - \frac{12n+7}{16.5^{n-1}}$$

3. Sum to infinity:

(a) $1 + 3x + 5x^2 + 7x^3 + \ldots\ (|x| < 1)$ **(b)** $1 + \frac{3}{5} + \frac{5}{5^2} + \frac{7}{5^3} + \ldots$

Solution

Let $S_\infty = 1 + 3x + 5x^2 + 7x^3 + \ldots$

Then $xS_\infty = x + 3x^2 + 5x^3 + \ldots$

By subtraction, we get

$(1-x)S_\infty = 1 + 2x + 2x^2 + 2x^3 + \ldots$

or, $(1-x)S_\infty = 1 + (2x + 2x^2 + 2x^3 + \ldots)$

or, $(1-x)S_\infty = 1 + \dfrac{2x}{1-x}$ $\left[\because S_\infty = \dfrac{a}{1-r},\ |r| < 1\right]$

or, $(1-x)S_\infty = \dfrac{1-x+2x}{1-x}$

or, $(1-x)S_\infty = \dfrac{1+x}{1-x}$

$\therefore$ $S_\infty = \dfrac{1+x}{(1-x)^2}$.

(b) Let $S_\infty = 1 + \frac{3}{5} + \frac{5}{5^2} + \frac{7}{5^3} + \ldots$

$$\frac{1}{5}S_\infty = \frac{1}{5} + \frac{3}{5^2} + \frac{5}{5^3} + \ldots$$

$$\frac{4}{5}S_\infty = 1 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \ldots$$

$$= 1 + \frac{2}{5}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \ldots\right) = 1 + \frac{2}{5}\cdot\frac{1}{1-\frac{1}{5}} = 1 + \frac{2}{5}\cdot\frac{1}{\frac{4}{5}} = 1 + \frac{2}{5}\times\frac{5}{4} = \frac{3}{2}$$

$$\therefore \quad S_\infty = \frac{15}{8}$$

Objective Questions

1. **The n^{th} term of the series $2 + 4 + 6 + \ldots$ is**

(a) n (b) $2n$ (c) $3n$ (d) $5n$

Ans: b

$a = 2, d = 4 - 2 = 2, t_n = ?$

We have,

$$t_n = a + (n - 1)d = 2 + (n - 1)\,2 = 2 + 2n - 2 = 2n$$

2. **If $\sum n = 55$ then $\sum n^3 =$**

(a) 55 (b) $(55)^2$ (c) $(55)^3$ (d) $(55)^4$

Ans: b

Here, $\Sigma n = 55$

We have,

$$\Sigma n^3 = (\Sigma n)^2 = (55)^2$$

3. **The n^{th} term of $1 + 3 + 6 + 10 + \ldots$ is**

(a) $\dfrac{n(n+1)}{2}$ (b) $\left[\dfrac{n(n+1)}{2}\right]^2$

(c) $\dfrac{n(n+1)(2n+1)}{6}$ (d) $\dfrac{n(n+1)(n+2)}{6}$

Ans: a

Given series is

$1 + 3 + 6 + 10 + \ldots$

$= 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \ldots$

n^{th} term (t_n) $= 1 + 2 + 3 + 4 + \ldots$ to n terms

$$= \frac{n(n+1)}{2}$$

4. **Sum to n terms: $1 + (3 + 5) + (7 + 9 + 11) + \ldots$**

(a) n^2 (b) $\dfrac{n(n+1)}{2}$

(c) $\left[\dfrac{n(n+1)}{2}\right]^2$ (d) $\dfrac{n(n+1)(n+2)}{6}$

Ans: c

Given, $1 + (3 + 5) + (7 + 9 + 11) + \ldots$ to n terms

$= 1 + 8 + 27 + \ldots$ to n terms

$= 1^3 + 2^3 + 3^3 + \ldots$ to n terms

= Sum of the cubes of first n natural numbers

$$= \left[\frac{n(n+1)}{2}\right]^2$$

5. **Sum to infinity $1 + \frac{3}{2} + \frac{5}{4} + \ldots$ is**

(a) 2 (b) 3 (c) 5 (d) 6

Ans: d

$$S_\infty = 1 + \frac{3}{2} + \frac{5}{4} + \ldots$$

$$\frac{1}{2}S_\infty = \quad \frac{1}{2} + \frac{3}{4} + \ldots$$

$$\frac{1}{2}S_\infty = 1 + \frac{2}{2} + \frac{2}{4} + \ldots$$

or, $\frac{1}{2}S_\infty = 1 + \left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right)$

or, $\frac{1}{2}S_\infty = 1 + \frac{1}{1 - \frac{1}{2}}$

or, $\frac{1}{2}\ S_\infty = 1 + 2$

$\therefore$ $S_\infty = 6$

EXERCISE – 3 D

1. (a) Find $\frac{1}{2}(e - e^{-1})$ (b) Prove that $\frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots = e.$

(c) Prove that $\dfrac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots}{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots} = \dfrac{e^2 + 1}{e^2 - 1}.$

Solution

(a) We have,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$$

Putting $x = 1$ and -1, we get,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} + \ldots$$

Now, $e - e^{-1} = \frac{2}{1!} + \frac{2}{3!} + \frac{2}{5!} + \ldots = 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\right)$

$\therefore$ $\dfrac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots$

(b) L.H.S. $= \frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots$

$$= \frac{1+1}{1!} + \frac{3+1}{3!} + \frac{5+1}{5!} + \ldots = \frac{1}{1!} + \frac{1}{1!} + \frac{3}{3!} + \frac{1}{3!} + \frac{5}{5!} + \frac{1}{5!} + \ldots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = e = \text{R.H.S.}$$

(c) We have,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$$

Putting $x = 1$ and -1, we get,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots$$

and $$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} + \ldots$$

Now,

$$e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \ldots = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\right)$$

$\therefore$ $\dfrac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots$

Also,

$$e - e^{-1} = \frac{2}{1!} + \frac{2}{3!} + \frac{2}{5!} + \ldots = 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\right)$$

$$\therefore \quad \frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots$$

Now,

$$\text{L.H.S.} = \frac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots}{1 + \frac{1}{3!} + \frac{1}{5!} + \ldots}$$

$$= \frac{\frac{e + e^{-1}}{2}}{\frac{e - e^{-1}}{2}} = \frac{e + \frac{1}{e}}{e - \frac{1}{e}} = \frac{e^2 + 1}{e^2 - 1} = \text{R.H.S.}$$

2. Find the Taylor series for the following function at $x = 0$. (Maclaurin series)

(a) e^{-x} **(b)** $\frac{1}{1+x}$

(c) $\sin x$ **(d)** $\cos x$.

Solution

(a) e^{-x}

Let $f(x) = e^{-x}$ $\qquad f(0) = e^{-0} = 1$

$f'(x) = -e^{-x}$ $\qquad f'(0) = -e^{-0} = -1$

$f''(x) = e^{-x}$ $\qquad f''(0) = e^{-0} = 1$

$f'''(x) = -e^{-x}$ $\qquad f'''(0) = -e^{-0} = -1$

$\vdots \qquad \vdots$

$f^n(x) = (-1)^n e^{-x}$ $\qquad f^n(0) = (-1)^n$

$\vdots \qquad \vdots$

The Taylor series generated by f at x = 0 is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots \frac{f^n(x)}{n!}x^n + \ldots$$

$$= 1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots + \frac{(-1)^n}{n!}x^n + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

(b) Let $f(x) = \frac{1}{1+x}$

$f(x) = (1+x)^{-1}$ $\qquad f(0) = 1$

$f'(x) = -(1+x)^{-2}$ $\qquad f'(0) = -1$

$f''(x) = 2!\,(1+x)^{-3}$ $\qquad f''(0) = 2!$

$f'''(x) = -3!\,(1+x)^{-4}$ $\qquad f'''(0) = -3!$

$\vdots \qquad \vdots$

$f^n(x) = (-1)^n\, n!\,(1+x)^{-(n+1)}$ $\qquad f^n(0) = (-1)^n \cdot n!$

$\vdots \qquad \vdots$

The Taylor series is

$$f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n.$$

(c) Let $f(x) = \sin x$

$f(x) = \sin x$ $\qquad$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\qquad$ $f'(0) = 1$

$f''(x) = -\sin x$ $\qquad$ $f''(0) = 0$

$f^{iv}(x) = -\cos x$ $\qquad$ $f^{iv}(0) = -1$

$\vdots$ $\qquad$ $\vdots$

$f^{2k}(x) = (-1)^k \sin x$ $\qquad$ $f^{2k}(0) = 0$

$f^{2k+1}(x) = (-1)^k \cos x$ $\qquad$ $f^{2k+1}(0) = (-1)^k$

The Taylor series at $x = 0$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$= \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

(d) Let $f(x) = \cos x$

$f(x) = \cos x$ $\qquad$ $f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ $\qquad$ $f'(0) = 0$

$f''(x) = -\cos x$ $\qquad$ $f''(0) = -1$

$f'''(x) = \sin x$ $\qquad$ $f'''(0) = 1$

$\vdots$ $\qquad$ $\vdots$

$f^{2k}(x) = (-1)^k \cos x$ $\qquad$ $f^{2k}(0) = (-1)^k$

$f^{2k+1}(0) = 0$

The Taylor series of $f(x) = \cos x$ at $x = 0$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

$$= \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$= \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

3. Find the Taylor series expansion for $f(x) = e^x$ at

(a) $x = 1$ $\qquad$ **(b) $x = 2$.**

Solution

(a) Here $f(x) = e^x$

$f(x) = e^x,$ $\qquad$ $f(1) = e$

$f'(x) = e^x,$ $\qquad$ $f'(1) = e$

$f''(x) = e^x$ $\qquad$ $f''(1) = e$

$\vdots$ $\qquad$ $\vdots$

$f^n(x) = e^x$ $\qquad$ $f^n(1) = e$

$\vdots$ $\qquad$ $\vdots$

The Taylor series generated by f at $x = 1$ is

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \dots$$

$$= e + e\cdot(x-1) + \frac{e\cdot(x-1)^2}{2!} + \frac{e\cdot(x-1)^3}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n.$$

(b) Here $f(x) = e^x$

$f(x) = e^x$, $\quad f(2) = e^2$

$f'(x) = e^x$, $\quad f'(2) = e^2$

$f''(x) = e^x$ $\quad f''(2) = e^2$

$\vdots \quad \vdots$

$f^n(x) = e^x$ $\quad f^n(2) = e^2$

$\vdots \quad \vdots$

The Taylor series generated by f at x = 2 is

$$f(x) = f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2!} + \ldots$$

$$= e^2 + e^2 \cdot (x-2) + \frac{e^2 \cdot (x-2)^2}{2!} + \frac{e^2 \cdot (x-2)^3}{3!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Objective Questions

1. **The Taylor series for $f(x) = e^x$ at $x = 0$ is**

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum_{n=0}^{\infty} n!\,x^n$ (c) $\sum_{n=0}^{\infty} \frac{n!}{x^n}$ (d) $\sum_{n=0}^{\infty} n \cdot x^n$

Ans: a

The Taylor series for $f(x) = e^x$ at $x = 0$ is

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. **The Taylor series for $f(x) = e^{-x}$ at $x = 0$ is**

(a) $1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$ (b) $-1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots$

(c) $1 - \frac{x}{1!} + \frac{x^2}{2!} - \ldots$ (d) $-1 + \frac{x}{1!} - \frac{x^2}{2!} + \ldots$

Ans: c

The Taylor series for $f(x) = e^{-x}$ at $x = 0$ is $1 - \frac{x}{1!} + \frac{x^2}{2!} - \ldots$

3. **The remainder of order n in Taylor's formula is**

(a) $\frac{f^{n+1}(x)}{n!}(x-a)^{n-1}$ (b) $\frac{f^{n+1}(x)}{(n+1)!}(x-a)^{n+1}$

(c) $\frac{f^{n}(x)}{(n+1)!}(x-a)^{n}$ (d) $\frac{f^{n-1}(x)}{(n+1)!}(x-a)^{n}$

Ans: b

Formula

4. **The value of $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$ is**

(a) e (b) $2e$ (c) $3e$ (d) $4e$

Ans: a

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e$$

5. **The geometric series with common ratio r converges for**

(a) $r > 1$ (b) $r < -1$ (c) $|r| = 1$ (d) $|r| < 1$

Ans: d

$|r| < 1$

♠

Unit – 4

Matrices and Determinants

EXERCISE – 4 A

1. What are the orders or sizes of the following matrices?

(a) $(1 \quad 2 \quad 3)$ (b) $\begin{pmatrix} a \\ b \end{pmatrix}$ (c) $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 3 & 1 & 5 \end{pmatrix}$.

Solution

(a) 1×3 (b) 2×1 (c) 2×3 (d) 3×3

2. Let $A = \begin{pmatrix} 10 & 2 & 13 \\ 4 & -2 & 1 \\ 0 & -2 & 11 \end{pmatrix}$ and $B = \begin{pmatrix} 11 & 3 & -2 \\ 1 & -4 & 2 \\ 10 & 1 & 4 \end{pmatrix}$. Write down the elements:

(a) a_{12}, a_{23}, a_{31} (b) b_{21}, b_{13}, b_{32}.

Solution

(a) $a_{12} = 2, a_{23} = 1, a_{31} = 0,$ (b) $b_{21} = 1, b_{13} = -2, b_{32} = 1$

3. Add the following matrices.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 1 \\ 4 & 1 & 2 \end{pmatrix}$ (b) $3\begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ -3 & 2 \\ 8 & 6 \end{pmatrix}$.

Solution

(a) $\begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 1 \\ 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1+6 & 2-2 & 3+1 \\ -4+4 & 3+1 & 2+2 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 4 \\ 0 & 4 & 4 \end{pmatrix}$

(b) $3\begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ -3 & 2 \\ 8 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 0 & 6 \\ 8 & 15 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ -3 & 2 \\ 8 & 6 \end{pmatrix} = \begin{pmatrix} 6+0 & 3+4 \\ 0-3 & 6+2 \\ 12+8 & 15+6 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ -3 & 8 \\ 20 & 21 \end{pmatrix}$

4. (a) Find a 2×2 matrix $A = [a_{ij}]$ where $a_{ij} = |2i + 3j - 6|$.

(b) Construct a 3×3 matrix $A = [a_{ij}]$ whose element a_{ij} are given by $a_{ij} = i + 2j$.

Solution

(a) $A = [a_{ij}]$, $a_{ij} = 2i + 3j - 6$; $i, j = 1, 2$

$a_{11} = 2 + 3 - 6 = -1$

$a_{12} = 2 + 6 - 6 = 2$

$a_{21} = 4 + 3 - 6 = 1$

$a_{22} = 4 + 6 - 6 = 4$

$\therefore \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 4 \end{pmatrix}$

(b) Let 3×3 matrix be $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Here, $a_{ij} = i + 2j$

$a_{11} = 1 + 2 \times 1 = 3$ $a_{12} = 1 + 2 \times 2 = 5$ $a_{13} = 1 + 2 \times 3 = 7$

$a_{21} = 2 + 2 \times 1 = 4$ $a_{22} = 2 + 2 \times 2 = 6$ $a_{23} = 2 + 2 \times 3 = 8$

$a_{31} = 3 + 2 \times 1 = 5$ $a_{32} = 3 + 2 \times 2 = 7$ $a_{33} = 3 + 2 \times 3 = 9$

Required matrix is $A = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{bmatrix}$

5. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$; find:

(a) $A + B$ (b) $A - B$ (c) $2A + B$ (d) $3A - 2B$.

Solution

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$

(a) $$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 5 & 5 & 9 \end{pmatrix}$$

(b) $$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

(c) $$2A + B = 2\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 8 \\ 7 & 8 & 13 \end{pmatrix}$$

(d) $$3A - 2B = 3\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} - 2\begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 6 & 9 \\ 6 & 9 & 12 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 4 \\ 6 & 4 & 10 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 5 & 2 \end{pmatrix}$$

6. (a) Find X, if $Y = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and $2X + Y = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$.

(b) If $2A + 3B = \begin{pmatrix} 2 & 7 & 12 \\ 13 & 12 & 23 \end{pmatrix}$ and $A - 2B = \begin{pmatrix} 1 & 0 & -1 \\ -4 & -1 & -6 \end{pmatrix}$, find the matrices A and B.

(c) Find the values of x, y and z if $\begin{pmatrix} x+y & z-x \\ y+2z & x \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 8 & 1 \end{pmatrix}$.

Solution

(a) $$2X + Y = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

or, $$2X + \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$$

or, $$2X = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$$

or, $$2X = \begin{pmatrix} -2 & -2 \\ -4 & -2 \end{pmatrix}$$

or, $$X = \frac{1}{2}\begin{pmatrix} -2 & -2 \\ -4 & -2 \end{pmatrix}$$

or, $$X = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$$

$$\therefore \quad X = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$$

(b) We have,

$$2A + 3B = \begin{pmatrix} 2 & 7 & 12 \\ 13 & 12 & 23 \end{pmatrix} \quad \text{...(i)}$$

$$A - 2B = \begin{pmatrix} 1 & 0 & -1 \\ -4 & -1 & -6 \end{pmatrix} \quad \text{...(ii)}$$

Multiplying eq^n. (ii) by 2, we get

$$2A - 4B = \begin{pmatrix} 2 & 0 & -2 \\ -8 & -2 & -12 \end{pmatrix} \qquad \text{... (iii)}$$

Subtracting eq^n. (iii) from (i)

$$\Rightarrow \quad 2A + 3B - (2A - 4B) = \begin{pmatrix} 2 & 7 & 12 \\ 13 & 12 & 23 \end{pmatrix} - \begin{pmatrix} 2 & 0 & -2 \\ -8 & -2 & -12 \end{pmatrix}$$

$$\Rightarrow \quad 2A + 3B - 2A + 4B) = \begin{pmatrix} 2-2 & 7-0 & 12+2 \\ -8-13 & 12+2 & 23+12 \end{pmatrix}$$

$$\Rightarrow \quad 7B = \begin{pmatrix} 0 & 7 & 14 \\ 21 & 14 & 35 \end{pmatrix}$$

$$\therefore \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$

From (ii)

$$A - 2B = \begin{pmatrix} 1 & 0 & -1 \\ -4 & -1 & -6 \end{pmatrix}$$

$$\text{or,} \quad A - \begin{pmatrix} 0 & 2 & 4 \\ 6 & 4 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -4 & -1 & -6 \end{pmatrix}$$

$$\therefore \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

(c) Given,

$$\begin{pmatrix} x+y & z-x \\ y+2z & x \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 8 & 1 \end{pmatrix}$$

By definition of equal matrices, we have,

$x + y = 3$ ... (i)

$y + 2z = 8$ ... (ii)

$z - x = 2$ ... (iii)

$x = 1$ ... (iv)

Substituting the value of x in (i), we get

$1 + y = 3$

$\therefore \quad y = 2$

Again, substituting the value of x in (iii), we get

$z - 1 = 2$

$\therefore \quad z = 3$

Therefore, $x = 1$, $y = 2$ and $z = 3$.

7. (a) Given $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}$. Calculate AB and BA.

(b) If $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$, show that $AB \neq BA$.

(c) If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$, show that AB is a null matrix.

Solution

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 4+10 & 5+12 \\ 8+15 & 10+18 \end{pmatrix} = \begin{pmatrix} 14 & 17 \\ 23 & 28 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4+10 & 8+15 \\ 5+12 & 10+18 \end{pmatrix} = \begin{pmatrix} 14 & 23 \\ 17 & 28 \end{pmatrix}$$

(b) $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2+0 & 1+0 \\ 4+12 & 2+6 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 16 & 8 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2+2 & 0+3 \\ 4+4 & 0+6 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix}$$

$\therefore$ $AB \neq BA$

(c) Given,

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\ B = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times(-1)+(-1)\times(-1) & 1\times 2+(-1)\times 2 \\ (-1)\times(-1)+1\times(-1) & (-1)\times 2+1\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1+1 & 2-2 \\ 1-1 & -2+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ AB is a null matrix.

8. (a) **If $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, show that $A^2 - 2A - 5I = O$ where I and O are 2 × 2 identity and null matrices respectively.**

(b) **If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, show that $A^2 - 4A + 3I = O$ where I and O are 2 × 2 identity and null matrix respectively.**

Solution

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

$$A^2 = A.A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1+6 & 2+2 \\ 3+3 & 6+1 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix}$$

$$2A = 2\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 2 \end{pmatrix}$$

$$5I = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\therefore\ A^2 - 2A - 5I = \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 6 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

= O where O is a null matrix of order 2×2.

(b) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$A^2 = A.A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

$$4A = 4\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix}$$

$$3I = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\therefore\ A^2 - 4A + 3I = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} - \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. (a) Given $A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}$, find AB and BA.

(b) Find 5 AB if $A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$.

Solution

(a) $A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 8-6+1 & 12+0-5 \\ 6+21-1 & 9-0+5 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 26 & 14 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix} = \begin{pmatrix} 8+9 & 4-21 & -2+3 \\ -12+0 & -6+0 & 3+0 \\ -4+15 & -2-35 & 1+5 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & -17 & 1 \\ -12 & -6 & 3 \\ 11 & -37 & 6 \end{pmatrix}$$

(b) $$AB = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0+1 & 4-1 & 6+0 \\ 0+0 & -6+0 & -9+0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ 0 & -6 & -9 \end{pmatrix}$$

$$5AB = 5\begin{pmatrix} 1 & 3 & 6 \\ 0 & -6 & -9 \end{pmatrix} = \begin{pmatrix} 5 & 15 & 30 \\ 0 & -30 & -45 \end{pmatrix}$$

10. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, verify that: A (BC) = (AB)C.

Solution

Here,

$$BC = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & -1+0 \\ 2+0 & -2-3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -5 \end{pmatrix}$$

$$\therefore \quad A(BC) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 1+4 & -1-10 \\ 3+8 & -3-20 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ 11 & -23 \end{pmatrix}$$

Again,

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 1+4 & 0-6 \\ 3+8 & 0-12 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 11 & -12 \end{pmatrix}$$

$$\therefore \quad (AB)C = \begin{pmatrix} 5 & -6 \\ 11 & -12 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5-0 & -5-6 \\ 11-0 & -11-12 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ 11 & -23 \end{pmatrix}$$

$\therefore$ A (BC) = (AB) C, verified.

11. If $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$,

verify that A (BC) = (AB)C.

Solution

$$BC = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-1+0 & 1+0+1 \\ 0-1+0 & 0+0+1 \\ 1+1+0 & 1+0+1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0+1+0 & 2-1+0 \\ 0-1+4 & 0+1+4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+0 & -1+1+0 & 1-1+0 \\ 0+0+2 & 0-1+2 & 0+1+2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+0 & 1+0+0 \\ 2+1+0 & 2+0+3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$$

$$\therefore \quad A(BC) = (AB)C = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$$

12. **If** $A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$**, find $A^2 - A - I$ where I be an identify matrix of order 3×3.**

Solution

$$A^2 = A.A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 4+8+6 & 0+4+0 & 0+6+3 \\ 2+0+2 & 0+0+0 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 18 & 4 & 9 \\ 4 & 0 & 1 \end{pmatrix}$$

$$A^2 - A - I = \begin{pmatrix} 1 & 0 & 0 \\ 18 & 4 & 9 \\ 4 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 14 & 1 & 6 \\ 2 & 0 & -1 \end{pmatrix}$$

13. **For the given matrices** $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ **and** $B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$**, show that $(A + B)^T = A^T + B^T$.**

Solution

$$A + B = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1+5 & 2-2 \\ 2-2 & 5+1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$(A + B)^T = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}^T = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \qquad \ldots \text{(i)}$$

$$A^T + B^T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^T + \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1+5 & 2-2 \\ 2-2 & 5+1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \qquad \ldots \text{(ii)}$$

From (i) and (ii)

$(A + B)^T = A^T + B^T$

14. **If** $A = \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$ **and $f(x) = x^2 - 5x + 3$, find $f(A)$.**

Solution

Here,

$$A = \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$f(x) = x^2 - 5x + 3$$

$$\therefore \quad f(A) = A^2 - 5A + 3I = A \cdot A - 5A + 3I$$

$$= \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix} - 5\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-12 & 3+6 \\ -4-8 & -12+4 \end{pmatrix} - \begin{pmatrix} 5 & 15 \\ -20 & 10 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -11-5+3 & 9-15+0 \\ -12+20+0 & -8-10+3 \end{pmatrix} = \begin{pmatrix} -13 & -6 \\ 8 & -15 \end{pmatrix}$$

15. **Express $A = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{pmatrix}$ as a sum of symmetric and skew symmetric matrix.**

Solution

Here,

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{pmatrix}$$

$$\therefore \quad A^T = \begin{pmatrix} 2 & 2 & 5 \\ 4 & 3 & 2 \\ 3 & 4 & 6 \end{pmatrix}$$

$$A + A^T = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 5 \\ 4 & 3 & 2 \\ 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 8 \\ 6 & 6 & 6 \\ 8 & 6 & 12 \end{pmatrix}$$

$$\text{Symmetric matrix} = \frac{1}{2}(A + A^T) = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{pmatrix}$$

$$A - A^T = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 2 & 5 \\ 4 & 3 & 2 \\ 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

$$\text{Skew-symmetric matrix} = \frac{1}{2}(A - A^T) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Objective Questions

1. **The size of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \end{bmatrix}$ is**

(a) 2×3 (b) 3×2 (c) 3×3 (d) 2×2

Ans: a

Since the matrix has 2 rows and 3 columns, the size is 2×3.

2. **If A is a square matrix then $A + A^T$ is a**

(a) diagonal matrix (b) scalar matrix

(c) symmetric matrix (d) skew-symmetric matrix

Ans: c

$A + A^T$ is a symmetric matrix.

3. **A square matrix A is symmetric if**

(a) $A^2 = A$ (b) $A^T = -A$ (c) $A^2 = -A$ (d) $A^T = A$

Ans: d

Definition

4. If $\begin{pmatrix} 0 & a+2 \\ 5 & 0 \end{pmatrix}$ is a skew-symmetric matrix then $a =$

(a) -7 (b) -5 (c) -3 (d) -2

Ans: a

If $\begin{pmatrix} 0 & a+2 \\ 5 & 0 \end{pmatrix}$ is a skew symmetric matrix then, $a + 2 = -5$

$\therefore \quad a = -7$

5. **Sum of diagonal elements of a skew-symmetric matrix is**

(a) 0 (b) 1 (c) 2 (d) 100

Ans: a

Sum of diagonal elements (trace) of a skew-symmetric matrix is 0.

6. **If A is a matrix of order $m \times n$, B is a matrix of order $n \times p$, then the order of AB is**

(a) $m \times n$ (b) $n \times p$ (c) $m \times p$ (d) $p \times m$

Ans: c

$$\underbrace{A_{m \times n} \cdot B_{n \times p}}_{\text{Equal}}$$

$\therefore \quad (AB)_{m \times p}$

7. **If A is a matrix of size $m \times n$ and B is a matrix of size $p \times q$ then BA is possible if**

(a) $m = n$ (b) $p = q$ (c) $m = q$ (d) $n = p$

Ans: c

$$\underbrace{B_{p \times q} \cdot A_{m \times n}}_{\text{Equal}}$$

$\therefore \quad m = q$

8. **If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $A^2 =$**

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Ans: c

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

9. **If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then AB =**

(a) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Ans: d

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

10. **If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $AA^T =$**

(a) 0 (b) I (c) A (d) A^T

Ans: b

$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

EXERCISE – 4 B

1. Evaluate the following determinants.

(a) $|-2|$ (b) $\begin{vmatrix} 5 & -3 \\ 2 & 5 \end{vmatrix}$ (c) $\begin{vmatrix} x+y & x \\ y & x-y \end{vmatrix}$.

Solution

(a) $|-2| = -2.$

(b) $\begin{vmatrix} 5 & -3 \\ 2 & 5 \end{vmatrix} = 5 \times 5 - 2 \times (-3) = 25 + 6 = 31.$

(c) $\begin{vmatrix} x+y & x \\ y & x-y \end{vmatrix} = (x + y)(x - y) - xy = x^2 - y^2 - xy.$

2. Solve for x.

(a) $\begin{vmatrix} 1 & 3 \\ 2 & x \end{vmatrix} = 4$ (b) $\begin{vmatrix} x-3 & 4 \\ x-3 & 7 \end{vmatrix} = 0.$

Solution

(a) $\begin{vmatrix} 1 & 3 \\ 2 & x \end{vmatrix} = 4$

or, $x - 6 = 4$

or, $x = 10$

$\therefore$ $x = 10$

(b) $\begin{vmatrix} x-3 & 4 \\ x-3 & 7 \end{vmatrix} = 0$

or, $7(x - 3) - 4(x - 3) = 0$

or, $7x - 21 - 4x + 12 = 0$

or, $3x - 9 = 0$

or, $3x = 9$

or, $x = \frac{9}{3}$

or, $x = 3$

$\therefore$ $x = 3$

3. Find the value of determinants by expanding along any row or column.

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$.

Solution

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$

Expanding along first row

$= 1\begin{vmatrix} 4 & 1 \\ 2 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 9 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

$= 1(36 - 2) - 2(18 - 3) + 3(4 - 12) = 34 - 30 - 24 = -20$

(b) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

Expanding along first row

$= 1\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}$

$= 1\,(4-6) - 2\,(3-4) + 3\,(9-8) = -2 + 2 + 3 = 3$

4. Find the value of determinants by using Sarrus rule.

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$.

Solution

(a)
$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 1 & 4 & 9 & 1 & 4 \end{matrix}$$

$= (1\times 2\times 9) + (1\times 3\times 1) + (1\times 1\times 4) - (1\times 2\times 1) - (4\times 3\times 1) - (9\times 1\times 1)$

$= 18 + 3 + 4 - 2 - 12 - 9 = 25 - 23 = 2$

(b)
$$\begin{matrix} 1 & 4 & 7 & 1 & 4 \\ 2 & 5 & 8 & 2 & 5 \\ 3 & 6 & 9 & 3 & 6 \end{matrix}$$

$= (1\times5\times9) + (4\times8\times3) + (7\times2\times6) - (3\times5\times7) - (6\times8\times1) - (9\times2\times4)$

$= 45 + 96 + 84 - 105 - 48 - 72 = 0$

5. Prove that the value of following determinants is zero by using their properties

(a) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 4 & 8 & 12 \end{vmatrix}$ (b) $\begin{vmatrix} 55 & 1 & 7 \\ 30 & 2 & 4 \\ 10 & 6 & 2 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

(d) $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$ (e) $\begin{vmatrix} b-c & b+c & b \\ c-a & c+a & c \\ a-b & a+b & a \end{vmatrix}$ (f) $\begin{vmatrix} a & a^2 & ab+ac \\ b & b^2 & bc+ab \\ c & c^2 & ca+cb \end{vmatrix}$.

Solution

(a) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 4 & 8 & 12 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 4 & 8 & 3 \end{vmatrix} = 4\times 0\ (\because R_1 = R_3)$

$= 0.$

(b) $\begin{vmatrix} 55 & 1 & 7 \\ 30 & 2 & 4 \\ 10 & 6 & 2 \end{vmatrix}$

$C_3 \to 8C_3 - C_1$

$= \begin{vmatrix} 55 & 1 & 1 \\ 30 & 2 & 2 \\ 10 & 6 & 6 \end{vmatrix} = 0\ (\because C_2 = C_3)$

(c) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$C_2 \to C_2 + C_3$

$= \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix}$

Taking (a + b + c) common from C_2

$= (a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix} = (a+b+c) \times 0 \ (\because C_1 = C_2) = 0$

(d) $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} \frac{abc}{a} & a^2 & bc \\ \frac{abc}{b} & b^2 & ca \\ \frac{abc}{c} & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} bc & a^2 & bc \\ ac & b^2 & ca \\ ab & c^2 & ab \end{vmatrix}$

$= \frac{1}{abc} \times 0 \ (\because C_1 = C_3) = 0$

(e) $\begin{vmatrix} b-c & b+c & b \\ c-a & c+a & c \\ a-b & a+b & a \end{vmatrix}$

$C_1 \to C_1 + C_2$

$= \begin{vmatrix} 2b & b+c & b \\ 2c & c+a & c \\ 2a & a+b & a \end{vmatrix}$

Taking 2 common from C_1

$= 2\begin{vmatrix} b & b+c & b \\ c & c+a & c \\ a & a+b & a \end{vmatrix} = 2 \times 0 \ (\because C_1 = C_2)$

$= 0$

(f) $\begin{vmatrix} a & a^2 & ab+ac \\ b & b^2 & bc+ab \\ c & c^2 & ca+cb \end{vmatrix}$

Taking a, b and c common from R_1, R_2 and R_3 respectively

$= abc\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$C_2 \to C_2 + C_3$

$= abc\begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix}$

Taking (a + b + c) common from C_2

$= abc\,(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix} = abc\,(a+b+c) \times 0 \ (\because C_1 = C_2)$

$= 0$

6. **Without expanding, prove that:**

(a) $\begin{vmatrix} -2 & -7 & 4 \\ 3 & 13 & 2 \\ 4 & 6 & 11 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 11 \\ -2 & 3 & 4 \\ -7 & 13 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

(c) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Solution

a. LHS. $= \begin{vmatrix} -2 & -7 & 4 \\ 3 & 13 & 2 \\ 4 & 6 & 11 \end{vmatrix} = \begin{vmatrix} -2 & 3 & 4 \\ -7 & 13 & 6 \\ 4 & 2 & 11 \end{vmatrix}$ $[\because |A| = |A^T|]$

$= -\begin{vmatrix} -2 & 3 & 4 \\ 4 & 2 & 11 \\ -7 & 13 & 6 \end{vmatrix}$ $[R_2 \leftrightarrow R_3]$

$= \begin{vmatrix} 4 & 2 & 11 \\ -2 & 3 & 4 \\ -7 & 13 & 6 \end{vmatrix}$ $[R_1 \leftrightarrow R_2]$

b. RHS $= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \dfrac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$

Taking abc common from c_3

$= \dfrac{1}{abc} \cdot abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} (C_2 \leftrightarrow C_3) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (C_1 \leftrightarrow C_2)$

= L.H.S. proved.

c. LHS $= \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \dfrac{1}{abc}\begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$

Taking common abc from C_1

$= \dfrac{1}{abc} \cdot abc \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ = RHS proved

d. LHS $= \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & bc & a+b+c-a \\ 1 & ca & a+b+c-b \\ 1 & ab & a+b+c-c \end{vmatrix}$

$= \begin{vmatrix} 1 & bc & a+b+c \\ 1 & ca & a+b+c \\ 1 & ab & a+b+c \end{vmatrix} - \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix}$

$= (a+b+c)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} - \dfrac{1}{abc}\begin{vmatrix} 1 & abc & a^2 \\ b & abc & b^2 \\ c & abc & c^2 \end{vmatrix}$

$$= (a+b+c)\times 0 - \frac{1}{abc}\times abc\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad (C_1 \leftrightarrow C_2)$$

= RHS proved

7. Prove that

(a) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(b) $$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)$$

(c) $$\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

(d) $$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

(e) $$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$$

(f) $$\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$$

(g) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

(h) $$\begin{vmatrix} a+b+2c & a & b \\ c & c+b+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

(i) $$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

(j) $$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2.$$

Solution

(a) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

Taking (a − b) and (b − c) common from C_1 and C_2 respectively

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix} = (a-b)(b-c).1\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$$

$$= (a-b)(b-c)(b+c-a-b) = (a-b)(b-c)(c-a) \text{ Proved}$$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ -z(x-y) & -x(y-z) & xy \end{vmatrix}$$

Taking (x − y) and (y − z) common from C_1 and C_2 respectively

$$= (x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ -z & -x & xy \end{vmatrix} = (x-y)(y-z).1\begin{vmatrix} 1 & 1 \\ -z & -x \end{vmatrix}$$

$$= (x-y)(y-z)(-x+z) = (x-y)(y-z)(z-x)$$

(c) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= \begin{vmatrix} 0 & x-y & x^3-y^3 \\ 0 & y-z & y^3-z^3 \\ 1 & z & z^3 \end{vmatrix}$$

Taking (x − y) and (y − z) common from R_1 and R_2 respectively

$$= (x-y)(y-z)\begin{vmatrix} 0 & 1 & x^2+yz+y^2 \\ 0 & 1 & y^2+yz+z^2 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)\,1\begin{vmatrix} 1 & x^2+xy+y^2 \\ 1 & y^2+yz+z^2 \end{vmatrix}$$

$$= (x-y)(y-z)(y^2+yz+z^2-x^2-xy-y^2) = (x-y)(y-z)(yz+z^2-x^2-xy)$$

$$= (x-y)(y-z)[(z-x)(z+x)+y(z-x)] = (x-y)(y-z)(z-x)(x+y+z)$$

(d) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ b-a & c-b & a+b \end{vmatrix}$$

Taking (a − b) and (b − c) common from C_1 and C_2 respectively

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -1 & -1 & a+b \end{vmatrix}$$

$R_1 \to R_1 + R_3$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & a+b+c \\ a+b & b+c & c^2 \\ -1 & -1 & a+b \end{vmatrix} = (a-b)(b-c)(a+b+c)\begin{vmatrix} a+b & b+c \\ -1 & -1 \end{vmatrix}$$

$$= (a-b)(b-c)(a+b+c)(-a-b+b+c) = (a-b)(b-c)(c-a)(a+b+c)$$

(e) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= \begin{vmatrix} x & -x & 0 \\ 0 & x & -x \\ a & b & x+c \end{vmatrix}$$

$C_2 \to C_2 + C_1$

$$= \begin{vmatrix} x & 0 & 0 \\ 0 & x & -x \\ a & a+b & x+c \end{vmatrix} = x \begin{vmatrix} x & -x \\ a+b & x+c \end{vmatrix}$$

$$= x\,[x(x+c) + x\,(a+b)] = x^2\,[x + c + a + b] = x^2(x + a + b + c)$$

(f) $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+a+b+c & b & c \\ 1+a+b+c & 1+b & c \\ 1+a+b+c & b & 1+c \end{vmatrix}$$

Taking $(1 + a + b + c)$ common from C_1

$$= (1+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$$= (1+a+b+c) \begin{vmatrix} 0 & -1 & 0 \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix} = (1+a+b+c)\,1 \begin{vmatrix} 1 & c \\ 1 & 1+c \end{vmatrix}$$

$$= (1+a+b+c)(1+c-c) = (1+a+b+c)$$

(g) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a + b + c)$ common from R_1

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

Taking $(a + b + c)$ common from C_1 and C_2 respectively

$$= (a+b+c)^3 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix} = (a+b+c)^3 .\, 1 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= (a+b+c)^3$$

(h) $\begin{vmatrix} a+b+2c & a & b \\ c & c+b+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & c+b+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

Taking $(2a + 2b + 2c)$ common from C_1

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & c+b+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$

$$= 2(a+b+c)\begin{vmatrix} 0 & -c-b-a & 0 \\ 0 & c+b+a & -c-a-b \\ 1 & a & c+a+2b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1 and R_2 respectively

$$= 2(a+b+c)^3\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3 \cdot 1\begin{vmatrix} 0 & -1 \\ 1 & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)^3 \cdot 1 = 2(a+b+c)^3$$

(i) $$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking a, b and c common from C_1, C_2 and C_3 respectively

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & a \end{vmatrix}$$

$R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$

$$= abc\begin{vmatrix} -b & c-b & c \\ a & -c & a-c \\ b & b+c & c \end{vmatrix}$$

$R_1 \rightarrow R_1 + R_3$

$$= abc\begin{vmatrix} 0 & 2c & 2c \\ a & -c & a-c \\ b & b+c & c \end{vmatrix}$$

Taking 2c common from R_1

$$= 2abc^2\begin{vmatrix} 0 & 1 & 1 \\ a & -c & a-c \\ b & b+c & c \end{vmatrix}$$

$C_2 \rightarrow C_2 - C_3$

$$= 2abc^2\begin{vmatrix} 0 & 0 & 1 \\ a & -a & a-c \\ b & b & c \end{vmatrix} = 2abc^2 \cdot 1\begin{vmatrix} a & -a \\ b & b \end{vmatrix}$$

$$= 2abc^2(ab+ab) = 2abc^2(2ab) = 4a^2b^2c^2$$

(j) $$\text{LHS} = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ ac^2 & bc^2 & c(c^2+1) \end{vmatrix}$$

Taking common a, b, c from C_1, C_2, C_3 respectively,

$$= \frac{1}{abc} \cdot abc\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} \quad R_1 \rightarrow R_1 + R_2 + R_3$$

Taking common $(1+a^2+b^2+c^2)$ from R_1

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 0 & 1 & 1 \\ 0 & b^2+1 & b^2 \\ -1 & c^2 & c^2+1 \end{vmatrix} \quad C_1 \rightarrow C_1 - C_3$$

Expanding along C_1

$$= (1 + a^2 + b^2 + c^2)(-1)\begin{vmatrix} 1 & 1 \\ b^2+1 & b^2 \end{vmatrix}$$

$$= (1 + a^2 + b^2 + c^2)(-1)(b^2 - b^2 - 1) = 1 + a^2 + b^2 + c^2 = \text{R.H.S}$$

8. **Solve the following equations**

(a) $\begin{vmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 2x & x^2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0.$

Solution

(a) $\begin{vmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 2x & x^2 \end{vmatrix} = 0$

Taking 2 common from C_2

or, $2\begin{vmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & x & x^2 \end{vmatrix} = 0$

$R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$

or, $2\begin{vmatrix} 0 & 3 & 3 \\ 0 & -1-x & 1-x^2 \\ 1 & x & x^2 \end{vmatrix} = 0$

Taking 3 and $(1 + x)$ common from R_1 and R_2 respectively

or, $6(1+x)\begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1-x \\ 1 & x & x^2 \end{vmatrix} = 0$

or, $6(1+x) \cdot 1\begin{vmatrix} 1 & 1 \\ -1 & 1-x \end{vmatrix} = 0$

or, $6(1 + x)(1 - x + 1) = 0$

or, $6(1 + x)(2 - x) = 0$

or, $(1 + x)(2 - x) = 0$

$\therefore$ Either $1 + x = 0 \Rightarrow x = -1$

or, $2 - x = 0 \Rightarrow x = 2$

$\therefore$ $x = -1, 2$

b) $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$

$R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$

or, $\begin{vmatrix} x-1 & -x+1 & 0 \\ 0 & x-1 & -x+1 \\ 2 & 3 & x+4 \end{vmatrix} = 0$

$C_2 \rightarrow C_2 + C_1$

or, $\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & -x+1 \\ 2 & 5 & x+4 \end{vmatrix} = 0$

or, $(x-1)\begin{vmatrix} x-1 & -x+1 \\ 5 & x+4 \end{vmatrix} = 0$

or, $(x - 1)[(x - 1)(x + 4) - 5(1 - x)] = 0$

or, $(x - 1)(x^2 + 4x - x - 4 - 5 + 5x) = 0$

or, $(x - 1)(x^2 + 8x - 9) = 0$

Either, $(x - 1) = 0 \quad \Rightarrow x = 1$

or, $x^2 + 8x - 9 = 0$

or, $x^2 + 9x - x - 9 = 0$

or, $x(x + 9) - (x + 9) = 0$

or, $(x + 9)(x - 1) = 0$

$\therefore \quad x + 9 = 0 \quad \Rightarrow x = -9$

$(x - 1) = 0 \quad \Rightarrow x = 1$

$\therefore \quad x = 1, -9.$

Objective Questions

1. **If $A = [-2]$ then $|A| =$**

(a) 2 (b) -2 (c) 0 (d) 1

Ans: b

$A = [-2]$ then $|A| = |-2| = -2$ [$\because$ Here $|\;|$ is a determinant]

2. **The cofactor of 2 in the matrix $\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}$ is**

(a) -5 (b) 5 (c) 4 (d) 1

Ans: b

Cofactor of $2 = (-1)^{1+2}(-5) = 5$

3. $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} =$

(a) 8 (b) 7 (c) 6 (d) 5

Ans: d

$$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 8 - 3 = 5$$

4. $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} =$

(a) 0 (b) 5 (c) 41 (d) 66

Ans: a

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0$$

5. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} =$

(a) $a + b + c$ (b) $3abc$ (c) $1 + a + b + c$ (d) 0

Ans: d

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 + C_2$

$$\begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= (a + b + c) \times 0$ [$\because C_1 = C_3$]

$= 0$

6. $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} =$

(a) $a(a+3)$ (b) $a^2(a+3)$ (c) $3a(a+3)$ (d) $2a(a+3)$

Ans: b

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix} = (a+3) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$= (a+3) \begin{vmatrix} 0 & -a & 0 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = (a+3) \cdot a \begin{vmatrix} 1 & 1 \\ 1 & 1+a \end{vmatrix} = a^2 (a+3)$$

7. If $\begin{vmatrix} -3-x & 4 \\ 2 & -1-x \end{vmatrix} = 0$ then $x =$

(a) 1, 5 (b) $-1, 5$ (c) $1, -5$ (d) $-1, -5$

Ans: c

$$\begin{vmatrix} -3-x & 4 \\ 2 & -1-x \end{vmatrix} = 0$$

$(x+3)(x+1) - 8 = 0$

or, $x^2 + 4x + 3 - 8 = 0$

or, $x^2 + 4x - 5 = 0$

$\therefore$ $x = -5, 1$

8. If $\begin{vmatrix} -4 & x-4 \\ -2 & x+1 \end{vmatrix} = 0$ then $x =$

(a) -6 (b) 6 (c) 3 (d) -3

Ans: a

$$\begin{vmatrix} -4 & x-4 \\ -2 & x+1 \end{vmatrix} = 0$$

$-4x - 4 + 2x - 8 = 0$

or, $-2x = 12$

$\therefore$ $x = -6$

9. If A is a square matrix then which of the following is always true?

(a) $|A| = 0$ (b) $|A| = |A^T|$ (c) $|A| = 1$ (d) $|A| = -|A^T|$

Ans: b

$|A| = |A^T|$

10. If A is a square matrix of order n and $A = kB$ where k is a scalar then $|A| =$

(a) $|B|$ (b) $k|B|$ (c) $n|B|$ (d) $k^n|B|$

Ans: d

$\therefore$ $|A| = |kB| = k^n|B|$

EXERCISE – 4 C

1. **Find the adjoint of the following matrices.**

(a) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & -7 \\ -3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -4 & 3 \\ 5 & 3 & 6 \\ 1 & 2 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

Solution

a. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

The cofactor are

$A_{11} = (-1)^{1+1}(3) = 3$ $A_{12} = (-1)^{1+2}(2) = -2$

$A_{21} = (-1)^{2+1}(-1) = 1$ $A_{22} = (-1)^{2+2}(1) = 1$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$$

b. Let $A = \begin{pmatrix} 5 & -7 \\ -3 & 2 \end{pmatrix}$

The cofactor are

$A_{11} = (-1)^{1+1}(2) = 2$ $A_{12} = (-1)^{1+2}(-3) = 3$

$A_{21} = (-1)^{2+1}(-7) = 7$ $A_{22} = (-1)^{2+2}(5) = 5$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} 2 & 3 \\ 7 & 5 \end{pmatrix}^T = \begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}$$

c. Let $A = \begin{pmatrix} 2 & -4 & 3 \\ 5 & 3 & 6 \\ 1 & 2 & -1 \end{pmatrix}$

The cofactor are

$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 6 \\ 2 & -1 \end{vmatrix} = -3 - 12 = -15$ $A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 6 \\ 1 & -1 \end{vmatrix} = -(-5 - 6) = 11$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$ $A_{21} = (-1)^{2+1}\begin{vmatrix} -4 & 3 \\ 2 & -1 \end{vmatrix} = -(4 - 6) = 2$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -2 - 3 = -5$ $A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -4 \\ 1 & 2 \end{vmatrix} = -(4 + 4) = -8$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -4 & 3 \\ 3 & 6 \end{vmatrix} = -24 - 9 = -33$ $A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -(12 - 15) = 3$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -4 \\ 5 & 3 \end{vmatrix} = 6 + 20 = 26$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \begin{pmatrix} -15 & 11 & 7 \\ 2 & -5 & -8 \\ -33 & 3 & 26 \end{pmatrix}^T = \begin{pmatrix} -15 & 2 & -33 \\ 11 & -5 & 3 \\ 7 & -8 & 26 \end{pmatrix}$$

d. Let $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

The cofactor are

$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3 + 0 = 3$

$A_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$

$A_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2 - 0 = 2$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = -(2-4) = 2$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 + 0 = 1$

$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -(-2-0) = 2$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 0 + 6 = 6$

$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = -(0-2) = 2$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 + 2 = 5$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{pmatrix}^T = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

2. **Find the inverse of the following matrices if possible.**

(a) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 4 \\ -3 & 0 & 1 \end{pmatrix}$

Solution

a. Let $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$

$|A| = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 6 - 5 = 1 \neq 0$

So, A^{-1} exists.

The cofactors are

$A_{11} = (-1)^{1+1}(2) = 2$ $A_{12} = (-1)^{1+2}(1) = -1$

$A_{21} = (-1)^{2+1}(5) = -5$ $A_{22} = (-1)^{2+2}(3) = 3$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj. } A = \frac{1}{1} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

b. Let $A = \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$

$|A| = \begin{vmatrix} 5 & 4 \\ -3 & -2 \end{vmatrix} = -10 + 12 = -2 \neq 0$

So, A^{-1} exists.

The cofactors are

$A_{11} = (-1)^{1+1}(-2) = -2$ $A_{12} = (-1)^{1+2}(-3) = 3$

$A_{21} = (-1)^{2+1}(4) = -4$ $A_{22} = (-1)^{2+2}(5) = 5$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix}^T = \begin{pmatrix} -2 & -4 \\ 3 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj. } A = \frac{1}{2} \begin{pmatrix} -2 & -4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ \frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

c. $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$

$= 1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4 \neq 0$

$\therefore$ A^{-1} exists

The cofactors are:

$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1(1-1) = 0$ $\quad A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1(-1-1) = 2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1(1+1) = 2$ $\quad A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1(-1-1) = 2$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1(-1-1) = -2$ $\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -1(0) = 0$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1(1+1) = 2$ $\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -1(0) = 0$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1(-1-1) = -2$

$$\text{Adj. } A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

Now,

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj. } A = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

d. $|A| = \begin{vmatrix} 1 & -1 & 0 \\ 2 & -1 & 4 \\ -3 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix}$

$= 1(-1) + 1(2+12) = -1 + 14 = 13 \neq 0$

$\therefore$ A^{-1} exists.

The cofactors are:

$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} = 1(-1) = -1$ $\quad A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = -1(2+12) = -14$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} = 1(-3) = -3$ $\quad A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1(-1) = 1$

$A_{22} = (-1)^{2+2}\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = 1(1) = 1$ $\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} = -1(-3) = 3$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix} = 1(-4) = -4$ $\quad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = -1(4) = -4$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1(-1+2) = 1(1) = 1$

$$\text{Matrix of cofactors} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & -14 & -3 \\ 1 & 1 & 3 \\ -4 & -4 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & -4 \\ -14 & 1 & -4 \\ -3 & 3 & 1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|}\cdot \text{Adj. } A = \frac{1}{13}\begin{bmatrix} -1 & 1 & -4 \\ -14 & 1 & -4 \\ -3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} & \frac{1}{13} & -\frac{4}{13} \\ -\frac{14}{13} & \frac{1}{13} & -\frac{4}{13} \\ -\frac{3}{13} & \frac{3}{13} & \frac{1}{13} \end{bmatrix}$$

3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ be a matrix. Show that $AA^{-1} = A^{-1}A = I$.

Solution

Given,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 8 - 6 = 2 \neq 0$$

So, A^{-1} exists.

The cofactors are

$A_{11} = (-1)^{1+1}\,(8) = 8$

$A_{12} = (-1)^{1+2}\,(3) = -3$

$A_{21} = (-1)^{2+1}\,(2) = -2$

$A_{22} = (-1)^{2+2}\,(1) = 1$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} 8 & -3 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{ Adj. } A = \frac{1}{2}\begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}$$

$$\text{Now, } A\cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 8-6 & -2+2 \\ 24-24 & -6+8 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Similarly, } A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore \quad AA^{-1} = A^{-1}A = I.$

4. Prove that the matrices $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ are inverses of each other.

Solution

Let $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

Two matrices A and B are inverses of each other if $AB = BA = I$.

$$\text{Now, } AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\cdot\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 15-14 & -6+6 \\ 35-35 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\text{And } BA = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 15-14 & 10-10 \\ -21+21 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Here, $AB = BA = I$

Hence A and B are inverses of each other.

5. **Given a matrix $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$, find a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that they are inverse to each other.**

Solution

Let $A = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

$|A| = \begin{vmatrix} 3 & -1 \\ 5 & -2 \end{vmatrix} = -6 + 5 = -1 \neq 0$

So, A^{-1} exists.

$\text{Adj. } A = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj. } A) = \frac{1}{-1}\begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$

$\therefore \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$

6. **If $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.**

Solution

Here, $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$

$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1 \neq 0$

$\text{Adj. } A = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}\text{ Adj. } A = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

$\text{Adj. } B = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$B^{-1} = \frac{1}{|B|}\text{ Adj. } B = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$

$B^{-1}A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -20 - 14 & 8 + 6 \\ 15 + 7 & -6 - 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -34 & 14 \\ 22 & -9 \end{pmatrix} = \begin{pmatrix} -17 & 7 \\ 11 & -\frac{9}{2} \end{pmatrix} \quad \ldots \text{(i)}$

$AB = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 + 6 & 6 + 8 \\ 7 + 15 & 14 + 20 \end{pmatrix} = \begin{pmatrix} 9 & 14 \\ 22 & 34 \end{pmatrix}$

$|AB| = \begin{vmatrix} 9 & 14 \\ 22 & 34 \end{vmatrix} = 306 - 308 = -2 \neq 0$

$\text{Adj. } (AB) = \begin{pmatrix} 34 & -14 \\ -22 & 9 \end{pmatrix}$

$(AB)^{-1} = \frac{1}{|AB|}(\text{Adj. } AB)$

$= \frac{1}{-2}\begin{pmatrix} 34 & -14 \\ -22 & 9 \end{pmatrix} = \begin{pmatrix} -17 & 7 \\ 11 & \frac{-9}{2} \end{pmatrix} \quad \ldots \text{(ii)}$

From (i) and (ii) $(AB)^{-1} = B^{-1}A^{-1}$, Hence verified.

7. If $A = \begin{pmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$, verify that: $A \cdot (\text{adj. } A) = (\text{adj. } A) \cdot A = |A|\, I$.

Solution

$$|A| = \begin{vmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -5\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 9\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -5(-2) - 9(1) = 10 - 9 = 1$$

The cofactors are:

$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -2$ $\qquad$ $A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1(1) = -1$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1(2) = 2$ $\qquad$ $A_{21} = (-1)^{2+1}\begin{vmatrix} 9 & 0 \\ 2 & 1 \end{vmatrix} = -1(9) = -9$

$A_{22} = (-1)^{2+2}\begin{vmatrix} -5 & 0 \\ 0 & 1 \end{vmatrix} = 1(-5) = -5$ $\qquad$ $A_{23} = (-1)^{2+3}\begin{vmatrix} -5 & 9 \\ 0 & 2 \end{vmatrix} = -1(-10) = 10$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 9 & 0 \\ 0 & 1 \end{vmatrix} = 1(9) = 9$ $\qquad$ $A_{32} = (-1)^{3+2}\begin{vmatrix} -5 & 0 \\ 1 & 1 \end{vmatrix} = -1(-5) = 5$

$A_{33} = (-1)^{3+3}\begin{vmatrix} -5 & 9 \\ 1 & 0 \end{vmatrix} = 1(-9) = -9$

$$\text{Adj. } A = \begin{bmatrix} -2 & -1 & 2 \\ -9 & -5 & 10 \\ 9 & 5 & -9 \end{bmatrix}^T = \begin{bmatrix} -2 & -9 & 9 \\ -1 & -5 & 5 \\ 2 & 10 & -9 \end{bmatrix}$$

$$A.\,(\text{Adj. } A) = \begin{pmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} -2 & -9 & 9 \\ -1 & -5 & 5 \\ 2 & 10 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} 10-9+0 & 45-45+0 & -45+45-0 \\ -2+0+2 & -9+0+10 & 9+0-9 \\ 0-2+2 & 0-10+10 & 0+10-9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots \text{(i)}$$

$$\text{Similarly } (\text{Adj. } A) \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{And } |A|\, I = 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore \quad A \cdot (\text{Adj. } A) = (\text{Adj. } A) \cdot A = |A|\, I$

Objective Questions

1. **Two matrices A and B are inverses of each other if and only if**

(a) $AB = BA = 0$ (b) $AB = BA$
(c) $AB = I,\ BA = 0$ (d) $AB = BA = I$

Ans: d

$AB = BA = I$

2. **A square matrix A is said to be singular if**

(a) $|A| = 0$ (b) $|A| = 1$ (c) $|A| \neq 0$ (d) $|A| = 100$

Ans: a

Definition

3. **If $A = \begin{pmatrix} 3 & -2 \\ 5 & 5 \end{pmatrix}$ then $A^{-1} =$**

(a) $\frac{1}{5}\begin{pmatrix} 5 & 2 \\ -5 & 3 \end{pmatrix}$ (b) $\frac{1}{10}\begin{pmatrix} 5 & -2 \\ -5 & 3 \end{pmatrix}$

(c) $\frac{1}{25}\begin{pmatrix} 5 & 2 \\ -5 & 3 \end{pmatrix}$ (d) $\frac{1}{15}\begin{pmatrix} 3 & -2 \\ 5 & 5 \end{pmatrix}$

Ans: c

$$A = \begin{pmatrix} 3 & -2 \\ 5 & 5 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & -2 \\ 5 & 5 \end{vmatrix} = 15 + 10 = 25$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj. } A = \frac{1}{25}\begin{pmatrix} 5 & 2 \\ -5 & 3 \end{pmatrix}$$

4. **The adjoint of $\begin{pmatrix} 5 & -7 \\ -3 & 2 \end{pmatrix}$ is**

(a) $\begin{pmatrix} 2 & 7 \\ -3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -7 \\ -3 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & 7 \\ -3 & -5 \end{pmatrix}$

Ans: c

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then Adj. $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

So, adjoint of $\begin{pmatrix} 5 & -7 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}$

5. **If A is a square matrix then $A \cdot (\text{adj. } A) = (\text{adj. } A) \cdot A =$**

(a) $|A|$ (b) $|A|I$ (c) $2|A|$ (d) $4A$

Ans: b

6. **If inverse of $\begin{bmatrix} 1 & 1 & 2 \\ x & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ doesn't exist then $x =$**

(a) 0 (b) 2 (c) 3 (d) 1

Ans: d

If inverse of $\begin{bmatrix} 1 & 1 & 2 \\ x & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ doesn't exist

then $\begin{vmatrix} 1 & 1 & 2 \\ x & 2 & 5 \\ 2 & 1 & 1 \end{vmatrix} = 0$

or, $1\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} x & 5 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} x & 2 \\ 2 & 1 \end{vmatrix} = 0$

or, $1(2 - 5) - 1(x - 10) + 2(x - 4) = 0$

or, $-3 - x + 10 + 2x - 8 = 0$

or, $x - 1 = 0$

$\therefore$ $x = 1$.

EXERCISE – 4 D

1. **Find the rank of the following matrices.**

(a) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & 0 \\ -5 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 4 & 8 & 12 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & -5 \\ 7 & 0 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & -1 \\ 2 & 1 & 0 & 5 \end{bmatrix}$

Solution

a. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

So, rank of A is less than 2.

Consider a submatrix of order 1; say [2]

$|2| = 2 \neq 0$

So, rank of A = 1

b. Let $A = \begin{bmatrix} 2 & 3 & 0 \\ -5 & 2 & 1 \end{bmatrix}$

The size of A is 2 × 3. So, consider a submatrix of order 2 namely $\begin{bmatrix} 2 & 3 \\ -5 & 2 \end{bmatrix}$

$\begin{vmatrix} 2 & 3 \\ -5 & 2 \end{vmatrix} = 4 + 15 = 19 \neq 0$

So, rank of A = 2.

c. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 4 & 8 & 12 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 4 & 8 & 12 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 4 \times 0 \ (\because R_1 = R_3) = 0$

So, rank of A is not 3

Consider a submatrix of order 2 namely $\begin{bmatrix} 1 & 2 \\ -4 & 7 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ -4 & 7 \end{vmatrix} = 7 + 8 = 15 \neq 0$

Rank A = 2.

d. Let $A = \begin{bmatrix} 1 & 2 & -5 \\ 7 & 0 & 2 \\ 1 & 3 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & -5 \\ 7 & 0 & 2 \\ 1 & 3 & 2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 2 \\ 3 & 2 \end{vmatrix} - 2 \begin{vmatrix} 7 & 2 \\ 1 & 2 \end{vmatrix} - 5 \begin{vmatrix} 7 & 0 \\ 1 & 3 \end{vmatrix}$

$= 1\,(0 - 6)\,(-2\,(14 - 2) - 5\,(21 - 0) = -6 - 24 - 105 = -135 \neq 0$

So, rank of A = 3

e. Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & -1 \\ 2 & 1 & 0 & 5 \end{bmatrix}$

The matrix A is of size 3 × 4. Consider a submatrix of order 3 namely $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 1 & 0 \end{bmatrix}$

Now, $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 2\,(14 - 15) - 1\,(7 - 6) = -2 - 1 = -3 \neq 0$

$\therefore$ Rank of A = 3.

2. **Given $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Transform u, v, u + v and u − v by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.**

Solution

Let, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Here, $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$Au = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 - 1 \\ 4 + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$$Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-3 \\ 2+0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$A(u+v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4+2 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0-4 \\ 6+0 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

$$A(u-v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4-2 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0+2 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

3. **Transform $u = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.**

Solution

$$Au = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+0+0 \\ 0+4+0 \\ 0+0+6 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$$

$$Av = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0+0 \\ 0+0+0 \\ 0+0+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

4. (a) **Transform $u = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and check whether th transformation is linear or not?**

(b) **Transform $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and show tha this transformation is linear.**

Solution

a. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$Au = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Let, $T(x) = Ax$

Now, $T(u+v) = A(u+v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2+2 \\ -2+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 4-0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{... (i)}$$

$$T(u) + T(v) = Au + Av = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{... (ii)}$$

From (i) and (ii),

$T(u+v) = T(u) + T(v)$

Let α be a scalar,

$T(\alpha u) = A(\alpha u)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2\alpha \\ -2\alpha \end{bmatrix} = \begin{bmatrix} 2\alpha+0 \\ 0+2\alpha \end{bmatrix} = \begin{bmatrix} 2\alpha \\ 2\alpha \end{bmatrix} = \alpha\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \alpha \cdot Au = \alpha T(u)$$

And, $T(\alpha v) = A(\alpha v)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 2\alpha \\ -\alpha \end{bmatrix} = \alpha\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \alpha\, Av = \alpha\, T(v)$$

$\therefore$ T is linear.

Here,

b. $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Let, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$Au = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-3 \\ 2+0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 4+0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Let us define $T(x) = Ax$

Now, $T(u+v) = A(u+v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2+4 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 0-4 \\ 6+0 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$ ... (i)

$T(u) + T(v) = Au + Av = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$ ... (ii)

From (i) and (ii), $T(u+v) = T(u) + T(v)$

Let α be scalar,

$T(\alpha u) = A(\alpha u)$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2\alpha \\ 3\alpha \end{bmatrix} = \begin{bmatrix} 0-3\alpha \\ 2\alpha+0 \end{bmatrix} = \begin{bmatrix} -3\alpha \\ 2\alpha \end{bmatrix} = \alpha\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \alpha \cdot Au = \alpha T(u).$

$T(\alpha v) = A(\alpha v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 0-\alpha \\ 4\alpha+0 \end{bmatrix} = \alpha\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \alpha Av = \alpha T(v).$

$\therefore$ T is linear.

5. **Show that $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by**

(a) $T(x, y, z) = (x, y, 0)$ is linear. **(b) $T(x, y, z) = (0, y, z)$ is linear.**

Solution

a. Let $u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$ in $\mathbb{R}^3$

Let α and β be the scalars.

$T(\alpha u + \beta v) = T(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2))$

$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, 0)$

$= (\alpha x_1, \alpha y_1, 0) + (\beta x_2, \beta y_2, 0) = \alpha(x_1, y_1, 0) + \beta(x_2, y_2, 0) = \alpha T(u) + \beta T(v)$

$\therefore$ T is linear.

b. Let $u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$ in $\mathbb{R}^3$

Let α and β be the scalars.

$T(\alpha u + \beta v) = T(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2))$

$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$= (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) = (0, \alpha y_1, \alpha z_1) + (0, \beta y_2, \beta z_2) = \alpha(0, y_1, z_1) + \beta(0, y_2, z_2)$

$= \alpha T(u) + \beta T(v)$

$\therefore$ T is linear.

6. **Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.**

Solution

Consider $(1, -1), (0, 1) \in \mathbb{R}^2$.

$T(1, -1) = (4 \cdot 1 - 2 \cdot (-1), 3|-1|) = (4+2, 3) = (6, 3)$

$T(0, 1) = (4 \cdot 0 - 2 \cdot 1, 3|1|) = (-2, 3)$

$T(1, -1) + T(0, 1) = (6, 3) + (-2, 3) = (4, 6)$ ... (1)

Again,

$T((1, -1) + (0, 1)) = T(1, 0) = (4 \cdot 1 - 2 \cdot 0, 3|0|) = (2, 0)$ ... (2)

From (1) and (2)

$T(1, -1) + T(0, 1) \neq T((1, -1) + (0, 1))$

i.e. $T(u) + T(v) \neq T(u, v)$

So, T is not linear.

7. Find the standard matrix A for the transformation $T(x) = 3x$ for x in $\mathbb{R}^2$.

Solution

Here,

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(e_1) = 3e_1 = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$T(e_2) = 3e_2 = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

The standard matrix is $[T(e_1)\ T(e_2)] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

8. Prove that the matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal.

Solution

$$AA^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly, $A^TA = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ $AA^T = A^TA = I$

Thus, A is orthogonal matrix.

9. If a transformation T is defined by $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that T is orthogonal transformation.

Solution

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ $AA^T = A^TA = I.$

So, A is orthogonal.

Hence, T is orthogonal transformation.

Objective Questions

1. The image of $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ by the matrix $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Ans: d

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+6 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$. Then the image under T of $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is

(a) $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ans: a

$$T(u) = Au = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 0+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

3. A transformation T is linear if

(a) $T(u + v) = T(u) + T(v)$, where u and v are in domain of T.

(b) $T(ku) = k\,T(u)$, where u is in domain of T and k is a scalar.

(c) both a and b.

(d) neither a nor b.

Ans: c

Definition

4. A matrix transformation $T(x) = Ax$ is orthogonal transformation if

(a) $A^1 = I$ (b) $A^T = A$ (c) $A^T = -A$ (d) $A = I$

Ans: b

Definition

5. The rank of matrix $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Ans: b

Let $|A| = \begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix} = 8 - 8 = 0$

rank $(A) \neq 2$

$|2| = 2 \neq 0$

So, rank $(A) = 1$

6. The rank of matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Ans: c

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$|A| = 0$

$\therefore$ Rank $(A) \neq 3$

Consider a 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$

Now, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

$\therefore$ rank (A) = 2.

7. The rank of matrix $\begin{bmatrix} 10 & 2 & 5 \\ -1 & 3 & 1 \\ 2 & 4 & 5 \end{bmatrix}$ **is**

(a) 0 (b) 1 (c) 2 (d) 3

Ans: d

$$\begin{vmatrix} 10 & 2 & 5 \\ -1 & 3 & 1 \\ 2 & 4 & 5 \end{vmatrix} = 10\begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 2 & 5 \end{vmatrix} + 5\begin{vmatrix} -1 & 3 \\ 2 & 4 \end{vmatrix}$$

$= 10(15 - 4) - 2(-5 - 2) + 5(-4 - 6) = 110 + 14 - 50 = 74 \neq 0$

So, rank of matrix = 3.

♠

Unit – 5

Analytical Geometry

EXERCISE – 5 A

1. **Find the equation of the circle with**

(a) **centre (0, 0) and radius 5.** **(b)** **centre (4, 5) and radius 7.**

(c) **centre (2, 3) and touches x-axis.**

Solution

(a) Centre $(h, k) = (0, 0)$

Radius $(r) = 5$

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

or, $(x - 0)^2 + (y - 0)^2 = 5^2$

$\therefore$ $x^2 + y^2 = 25.$

(b) Centre $(h, k) = (4, 5)$

Radius $(r) = 7$

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

or, $(x - 4)^2 + (y - 5)^2 = 7^2$

or, $x^2 - 8x + 16 + y^2 - 10y + 25 = 49$

$\therefore$ $x^2 + y^2 - 8x - 10y - 8 = 0.$

(c) Since the circle touches the x-axis, so $r = k = 3$

Centre $(h, k) = (2, 3)$

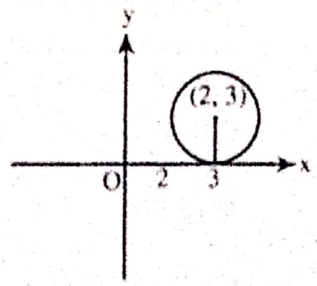
$\therefore k = 3$

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

or, $(x - 2)^2 + (y - 3)^2 = 3^2$

or, $x^2 - 4x + 4 + y^2 - 6y + 9 = 9$

$\therefore$ $x^2 + y^2 - 4x - 6y + 4 = 0.$



2. **Find the equation of the circle.**

(a) **whose centre is (a, b) and which passes through the origin.**

(b) **whose centre is the point (2, 3) and which passes through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.**

Solution

(a) Centre $(h, k) = (a, b)$

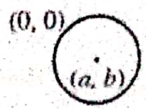
Radius (r) = distance between (a, b) and $(0, 0) = \sqrt{a^2 + b^2}$

Equation of circle is $(x - h)^2 + (y - k)^2 = r^2$

or, $(x - a)^2 + (y - b)^2 = (\sqrt{a^2 + b^2})^2$

or, $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = a^2 + b^2$

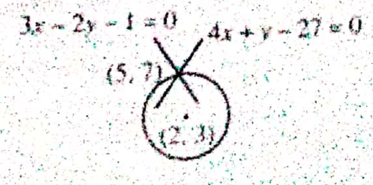
$\therefore$ $x^2 + y^2 - 2ax - 2by = 0$



(b) Given equations are

$3x - 2y - 1 = 0$ ... (i)

$4x + y - 27 = 0$ ... (ii)



Solving (i) and (ii), we get,

$x = 5, y = 7$

Centre $(h, k) = (2, 3)$

Radius (r) = Distance between (2, 3) and (5, 7) $= \sqrt{(5-2)^2+(7-3)^2} = \sqrt{9+16} = 5$

Equation of circle is $(x-h)^2+(y-k)^2 = r^2$

or, $(x-2)^2+(y-3)^2 = 5^2$

or, $x^2-4x+4+y^2-6y+9 = 25$

$\therefore$ $x^2+y^2-4x-6y-12 = 0.$

3. Find the equation of the circle which has (1, 3) and (4, 5) as ends of a diameter.

Solution

Given, $(x_1, y_1) = (1, 3)$

$(x_2, y_2) = (4, 5)$

Equation of circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$

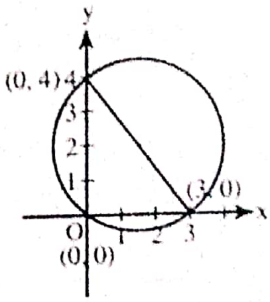
or, $(x-1)(x-4)+(y-3)(y-5) = 0$

or, $x^2-4x-x+4+y^2-3y-5y+15 = 0$

$\therefore$ $x^2+y^2-5x-8y+19 = 0.$

4. Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the x-axis respectively.

Solution



Here, (3, 0) and (0, 4) are end points of diameter of circle since $\angle XOY = 90°$.

$\therefore$ $(x_1, y_1) = (3, 0)$ and $(x_2, y_2) = (0, 4)$

Equation of circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$

or, $(x-3)(x-0)+(y-0)(y-4) = 0$

or, $x^2-3x+y^2-4y = 0$

$\therefore$ $x^2+y^2-3x-4y = 0.$

5. One end of the diameter of the circle $x^2+y^2-6x+5y-7 = 0$ is (7, –8). Find the coordinates of other end.

Solution

Given equation of circle is

$x^2+y^2-6x+5y-7 = 0$ ... (i)

Comparing (i) with

$x^2+y^2+2gx+2fy+c = 0$

We get, $2g = -6 \Rightarrow g = -3$

$2f = 5 \Rightarrow f = \frac{5}{2}$

$c = -7$

Centre $= (-g, -f) = \left(3, -\frac{5}{2}\right)$

One end of diameter $(x_1, y_1) = (7, -8)$

Other end $(x_2, y_2) = ?$

Centre is the mid-point of the diameter.

so, $\left(3, -\frac{5}{2}\right) = \left(\frac{7+x_2}{2}, \frac{-8+y_2}{2}\right)$

or, $3 = \frac{7+x_2}{2} \Rightarrow x_2 = 6-7 = -1$

and $\frac{-8+y_2}{2} = -\frac{5}{2} \Rightarrow y_2 = 3$

$\therefore$ Required coordinates of other end = (–1, 3).

6. **Find the equation of circle concentric with $x^2 + y^2 + x + 2y + 3 = 0$ and through the point (1, 1).**

Solution

Equation of any circle concentric with

$x^2 + y^2 + x + 2y + 3 = 0$ is

$x^2 + y^2 + x + 2y + k = 0$ ... (i)

If equation (i) passes through the point (1, 1)

Then, $1^2 + 1^2 + 1 + 2 \times 1 + k = 0$

or, $k = -5$

Putting the value of k in (i), we get,

$x^2 + y^2 + x + 2y - 5 = 0.$

7. **Find the equation of circle passing through the following points.**
   (a) (0, 0), (3, 0), (0, 4) **(b) (5, 5), (6, 4), (–2, 4)**

Solution

(a) Let the equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

If equation (i) passes through the points

(0, 0), (4, 0) and (0, 3), so

$0^2 + 0^2 + 2g \cdot 0 + 2f \cdot 0 + c = 0 \Rightarrow c = 0.$

Also, $4^2 + 0^2 + 2g \cdot 4 + 2f \cdot 0 + c = 0$

$\Rightarrow$ $16 + 8g + 0 = 0$ $[\because c = 0]$

$\Rightarrow$ $g = -2.$

And, $0^2 + 3^2 + 2g \cdot 0 + 2f \cdot 3 + c = 0$

$\Rightarrow$ $9 + 6f + 0 = 0$ $[\because c = 0]$

$\Rightarrow$ $f = -\frac{3}{2}.$

Putting the value of g, f and c in (i)

$$x^2 + y^2 + 2 \cdot (-2) \cdot x + 2 \cdot \left(-\frac{3}{2}\right) \cdot y + 0 = 0$$

$\therefore$ $x^2 + y^2 - 4x - 3y = 0.$

(b) Let the equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

If equation (i) passes through the points

(5, 5), (6, 4) and (–2, 4), then

$5^2 + 5^2 + 2g \cdot 5 + 2f \cdot 5 + c = 0 \Rightarrow 10g + 10f + c = -50$ ...(ii)

Also, $6^2 + 4^2 + 2g \cdot 6 + 2f \cdot 4 + c = 0 \Rightarrow 12g + 8f + c = -52$ ...(iii)

And, $(-2)^2 + 4^2 + 2g \cdot (-2) + 2f \cdot 4 + c = 0 \Rightarrow -4g + 8f + c = -20$ ...(iv)

Solving (ii), (iii) and (iv), we get,

$g = -2, f = -1, c = -20$

Putting the value of g, f and c in (i), we get,

$x^2 + y^2 - 4x - 2y - 20 = 0.$

Objective Questions

1. **Which of the following can not be the equation of circle?**
   (a) $x^2 + y^2 - 7 = 0$ (b) $x^2 + xy + y^2 = 5$
   (c) $x^2 + y^2 - 7x + 5y = 2$ (d) $x^2 + y^2 = 3x - 2y + 4$

Ans: b

The equation of circle doesnot contains any term involving xy.

2. **The equation of circle in diameter form is**
   (a) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = r$
   (b) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
   (c) $(x - x_1)(y - y_1) + (x - x_2)(y - y_2) = r$
   (d) $(x - x_1)(y - y_1) + (x - x_2)(y - y_2) = 0$

Ans: b

Formula

3. **The equation of circle having centre (0, 0) and diameter 4 is**
   (a) $x^2 + y^2 = 0$ (b) $x^2 + y^2 = 2$ (c) $x^2 + y^2 = 1$ (d) $x^2 + y^2 = 4$

Ans: d

Radius $= \dfrac{\text{diameter}}{2} = \dfrac{4}{2} = 2$

Equation of circle is $x^2 + y^2 = a^2$

or, $x^2 + y^2 = 2^2$

$\therefore$ $x^2 + y^2 = 4$

4. **The centre of the circle $x^2 + y^2 + 4x - 6y + 4 = 0$ is**
   (a) (2, 3) (b) (– 2, 3) (c) (– 2, – 3) (d) (– 2, – 3)

Ans: b

Given equation is $x^2 + y^2 + 4x - 6y + 4 = 0$ ... (i)

Comparing (i) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$g = 2, f = -3$

Centre $= (-g, -f) = (-2, 3)$

5. **The radius of the circle $x^2 + y^2 + 4x + 6y + 4 = 0$ is**
   (a) 1 (b) 2 (c) 3 (d) 4

Ans: c

Given $x^2 + y^2 + 4x + 6y + 4 = 0$ ... (i)

Comparing (i) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$g = 2, f = 3, c = 4$

Radius (r) $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 - 4} = 3$

6. **The equation of circle concentric with $x^2 + y^2 + 2x + 6y - 39 = 0$ and passing throug the point (2, 0) is**
   (a) $x^2 + y^2 + 2x + 6y - 8 = 0$ (b) $x^2 + y^2 + 2x + 6y - 12 = 0$
   (c) $x^2 + y^2 + 2x + 6y - 6 = 0$ (d) $x^2 + y^2 + 2x + 6y - 39 = 0$

Ans: a

The equation of circle concentric with

$x^2 + y^2 + 2x + 6y - 39 = 0$ is

$x^2 + y^2 + 2x + 6y + k = 0$ ... (i)

If equation (i) passes through the point (2, 0) then

$4 + 0 + 4 + 0 + k = 0$

$\therefore$ $k = -8$

The equation of circle is

$x^2 + y^2 + 2x + 6y - 8 = 0$

EXERCISE – 5 B

1. **Find focus and directrix of following parabolas. Sketch the parabola including focus and directrix.**
   (a) $y^2 = 12x$ (b) $x = -3y^2$
   (c) $x^2 = 6y$ (d) $y = -8x^2$.

Solution

(a) $y^2 = 12x$ ... (i)

Comparing (i) with $y^2 = 4ax$, we get,

$4a = 12$

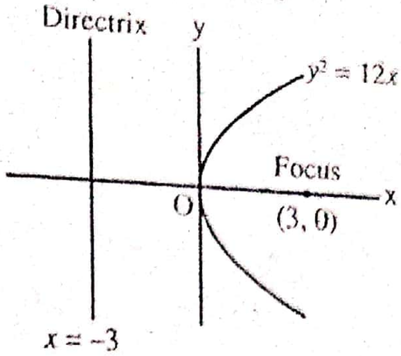
$\Rightarrow a = 3.$

Focus $= (a, 0) = (3, 0)$

Directrix : $x = -a$

$x = -3.$

Sketch



(b) $x = -3y^2$

or, $y^2 = -\frac{1}{3}x$ ... (i)

Comparing (i) with $y^2 = -4ax$, we get,

$4a = \frac{1}{3}$

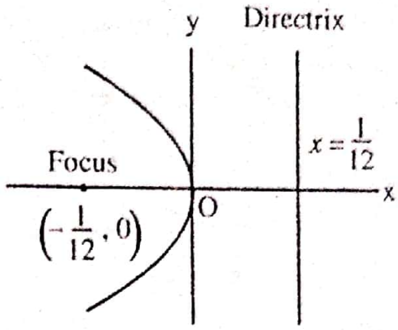
or, $a = \frac{1}{12}$

Focus $= (-a, 0) = \left(-\frac{1}{12}, 0\right)$

Directrix : $x = a$

$x = \frac{1}{12}$

Sketch



(c) $x^2 = 6y$ ...(i)

Comparing (i) with $x^2 = 4ay$, we get,

$4a = 6$

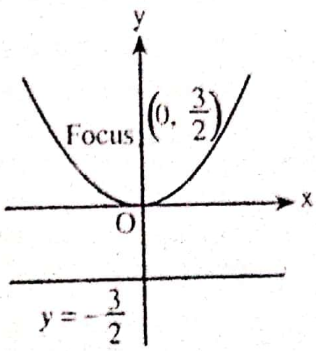
or, $a = \frac{3}{2}$

Focus $= (0, a) = \left(0, \frac{3}{2}\right)$

Directrix : $y = -a$

$y = -\frac{3}{2}$

Sketch



(d) $y = -8x^2$

or, $x^2 = -\frac{1}{8}y$ ... (i)

Comparing (i) with $x^2 = -4ay$, we get,

$4a = \frac{1}{8}$

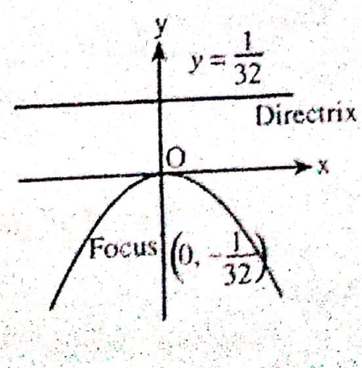
or, $a = \frac{1}{32}$

Focus $= (0, -a) = \left(0, -\frac{1}{32}\right)$

Directrix : $y = a$

$y = \frac{1}{32}$

Sketch



2. **Find the vertices, centre-to-focus distance, foci of the ellipse and sketch the graph.**

(a) $16x^2 + 25y^2 = 400$ **(b)** $2x^2 + 3y^2 = 12$

(c) $3x^2 + 2y^2 = 6.$

Solution

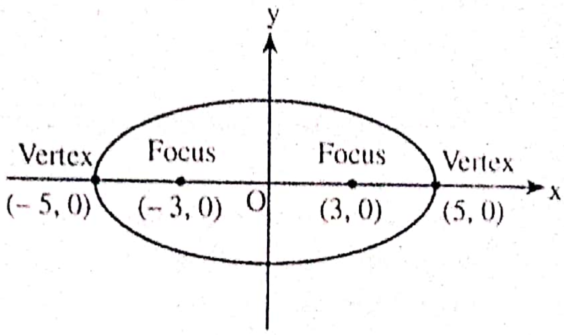
(a) $16x^2 + 25y^2 = 400$

Dividing both sides by 400

$$\frac{16x^2}{400} + \frac{25y^2}{400} = \frac{400}{400}$$

or, $$\frac{x^2}{25} + \frac{y^2}{16} = 1 \qquad \ldots(\text{i})$$

Sketch



Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

$a^2 = 25 \Rightarrow a = 5$

$b^2 = 16 \Rightarrow b = 4.$

Since $a > b$, so major axis is along x-axis.

Vertices = $(\pm a, 0) = (\pm 5, 0)$

Centre-to-focus distance $(c) = \sqrt{a^2 - b^2}$

$= \sqrt{5^2 - 4^2}$

$= 3$

Foci = $(\pm c, 0) = (\pm 3, 0)$

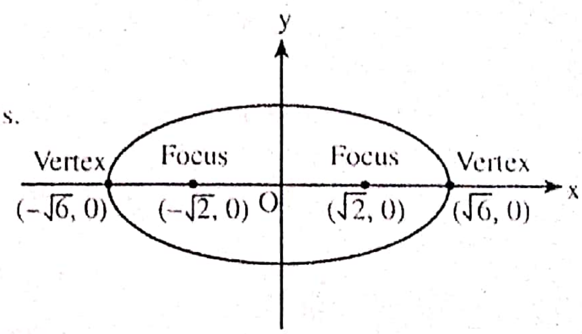
(b) $2x^2 + 3y^2 = 12$

Dividing both sides by 12,

$$\frac{x^2}{6} + \frac{y^2}{4} = 1 \qquad \ldots(\text{i})$$

Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

Sketch



$a^2 = 6 \Rightarrow a = \sqrt{6}$

$b^2 = 4 \Rightarrow b = 2$

Since $a > b$, the major axis is along x-axis.

Vertices = $(\pm a, 0) = (\pm\sqrt{6}, 0)$

Centre-to-focus distance $(c) = \sqrt{a^2 - b^2}$

$= \sqrt{6 - 4}$

$= \sqrt{2}$

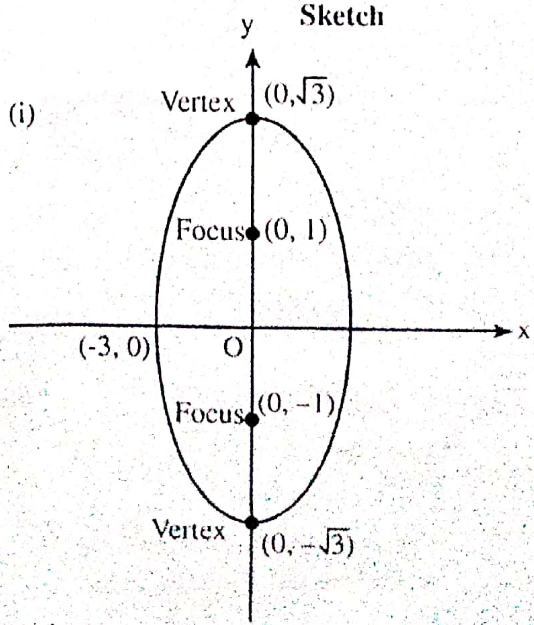
Foci = $(\pm c, 0) = (\pm\sqrt{2}, 0)$

(c) $3x^2 + 2y^2 = 6$

Dividing both sides by 6, we get

$$\frac{x^2}{2} + \frac{y^2}{3} = 1 \qquad \ldots(\text{i})$$

Sketch



Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

$a^2 = 2 \Rightarrow a = \sqrt{2}$

$b^2 = 3 \Rightarrow b = \sqrt{3}$

Since $b > a$, the major axis is along y-axis.

Vertices = $(0, \pm b) = (0, \pm\sqrt{3})$

Centre-to-focus distance $(c) = \sqrt{b^2 - a^2}$

$= \sqrt{3 - 2}$

$= 1$

Foci = $(0, \pm c) = (0, \pm 1)$

3. Find the centre-to-focus distance, vertices, foci and asymptotes of the following hyperbola. Sketch the graph.

(a) $x^2 - y^2 = 1$ (b) $9x^2 - 16y^2 = 144$

(c) $y^2 - x^2 = 8$ (d) $8y^2 - 2x^2 = 16$.

Solution

(a) $x^2 - y^2 = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get,

$a^2 = 1, b^2 = 1$

$\therefore$ $a = 1, b = 1$

Centre-to-focus distance $(c) = \sqrt{a^2 + b^2}$

$= \sqrt{1^2 + 1^2} = \sqrt{2}$

Vertices $= (\pm a, 0) = (\pm 1, 0)$

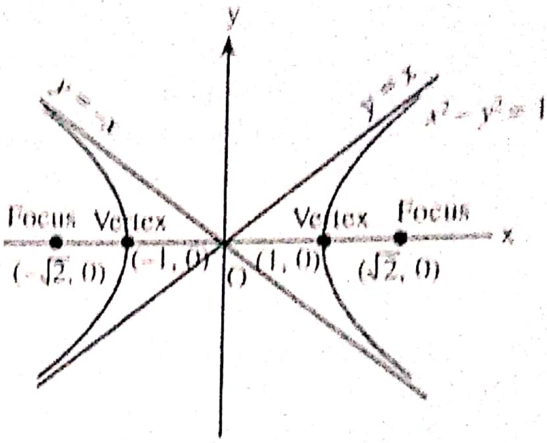
Foci $= (\pm c, 0) = (\pm\sqrt{2}, 0)$

Asymptotes : $x^2 - y^2 = 0$

or, $y^2 = x^2$

$\therefore$ $y = \pm x$.

Sketch



(b) $9x^2 - 16y^2 = 144$

Dividing both sides by 144, we get,

$\frac{x^2}{16} - \frac{y^2}{9} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get,

$a^2 = 16, b^2 = 9$

$\therefore$ $a = 4, b = 3$

Centre-to-focus distance $(c) = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$

Vertices $= (\pm a, 0) = (\pm 4, 0)$

Centre-to-focus distance $(c) = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$

Foci $= (\pm c, 0) = (\pm 5, 0)$

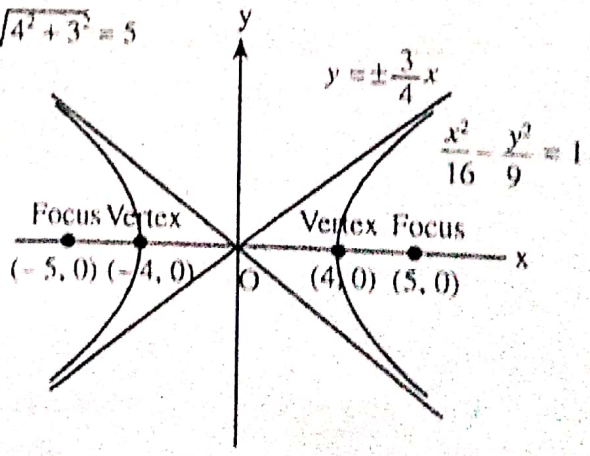
Asymptotes : $\frac{x^2}{16} - \frac{y^2}{9} = 0$

or, $\frac{y^2}{9} = \frac{x^2}{16}$

or, $y^2 = \frac{9}{16}x^2$

$\therefore$ $y = \pm\frac{3}{4}x$.

Sketch



(c) $y^2 - x^2 = 8$

Dividing both sides by 4

$\frac{y^2}{4} - \frac{x^2}{4} = 1$ ... (i)

Comparing (i) with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get,

$b^2 = 4, b = 2$

$a^2 = 4, a = 2$

Centre-to-focus distance $(c) = \sqrt{a^2 + b^2} = \sqrt{4 + 4} = 2\sqrt{2}$

Vertices $= (0, \pm b) = (0, \pm 2)$

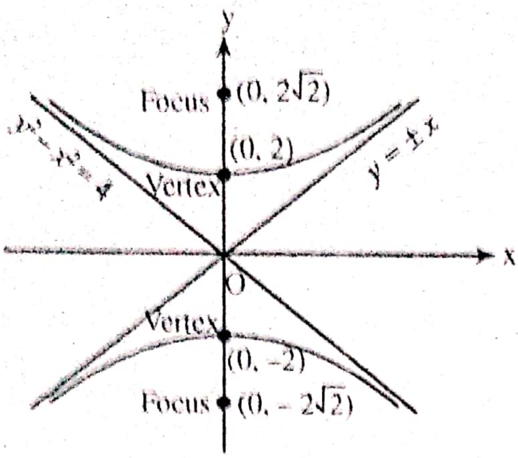
Foci = $(0, \pm c) = (0 \pm 5, 2\sqrt{2})$

Asymptotes : $\frac{y^2}{4} - \frac{x^2}{4} = 0$

or, $\quad y^2 = x^2$

$\therefore \quad y = \pm x.$

Sketch



(d) $\quad 8y^2 - 2x^2 = 16$

or, $\quad \frac{y^2}{2} - \frac{x^2}{8} = 1 \qquad \ldots$ (i)

Comparing (i) with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get,

$b^2 = 2 \Rightarrow b = \sqrt{2}$

$a^2 = 8 \Rightarrow a = 2\sqrt{2}$

Centre-to-focus distance $(c) = \sqrt{a^2 + b^2} = \sqrt{10}$

Vertices = $(0, \pm b) = (0, \pm\sqrt{2})$

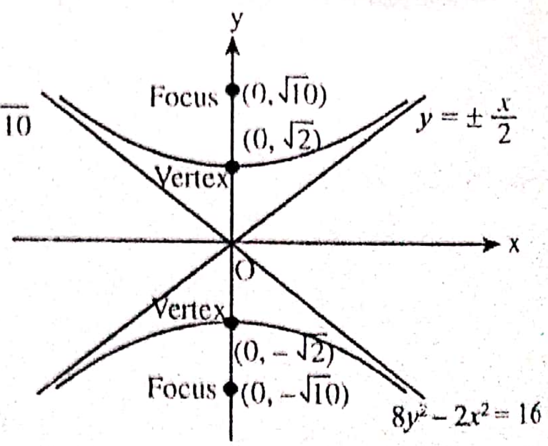
Foci = $(0, \pm c) = (0, \pm\sqrt{10})$

Asymptotes : $\frac{y^2}{2} - \frac{x^2}{8} = 0$

or, $\quad y^2 = \frac{x^2}{4}$

$\therefore \quad y = \pm\frac{x}{2}.$

Sketch



Objective Questions

1. **The parabola $y^2 = -4ax$ with positive 'a' opens**

(a) right (b) left (c) up (d) down

Ans: b

Left

2. **The directrix of the parabola $x^2 = -4ay$ is**

(a) $x = -a$ (b) $x = a$ (c) $y = -a$ (d) $y = a$

Ans: d

Equation of directrix of the parabola $x^2 = -4ay$ is $y = a$.

3. **The focus of the parabola $y^2 = 4ax$ is**

(a) $(a, 0)$ (b) $(-a, 0)$ (c) $(0, a)$ (d) $(0, -a)$

Ans: a

Formula

4. **Centre to focus distance of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a>b$ is**

(a) $\sqrt{a^2+b^2}$ (b) $\sqrt{a^2-b^2}$ (c) $\sqrt{b^2-a^2}$ (d) a

Ans: b

Formula

5. **The equations of asymptotes of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ are**

(a) $\frac{x^2}{4}-\frac{y^2}{9}=-1$ (b) $\frac{x^2}{4}-\frac{y^2}{9}=0$ (c) $\frac{x^2}{4}-\frac{y^2}{9}=1$ (d) $\frac{x^2}{4}+\frac{y^2}{9}=0$

Ans: b

Equation of asymptotes of the hyperbola, $\frac{x^2}{4}-\frac{y^2}{9}=1$ are $\frac{x^2}{4}-\frac{y^2}{9}=0$.

6. **The foci of the hyperbola $\frac{y^2}{4}-\frac{x^2}{5}=1$ are**

(a) $(0,\pm 1)$ (b) $(0,\pm 2)$ (c) $(0,\pm 3)$ (d) $(0,\pm 4)$

Ans: c

Given, $\frac{y^2}{4}-\frac{x^2}{5}=1$ ... (i)

Comparing (i) with $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, we get

$b^2=4 \Rightarrow b=2$

$a^2=5 \Rightarrow a=\sqrt{5}$

$c=\sqrt{a^2+b^2}=3$

Foci $=(0,\pm c)=(0,\pm 3)$

EXERCISE – 5 C

1. **Find eccentricity, foci and directrices of the following ellipse.**

(a) $6x^2+9y^2=54$ (b) $2x^2+y^2=2$ (c) $3x^2+2y^2=6$.

Solution

(a) $6x^2+9y^2=54$

or, $\frac{x^2}{9}+\frac{y^2}{6}=1$ (i)

Comparing (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get,

$a^2=9 \Rightarrow a=3$

$b^2=6 \Rightarrow b=\sqrt{3}$

$$e=\frac{\sqrt{a^2-b^2}}{a}=\frac{\sqrt{9-6}}{3}=\frac{1}{\sqrt{3}}$$

$$\text{Foci}=(\pm ae,0)=\left(\pm 3\cdot\frac{1}{\sqrt{3}},0\right)=(\pm\sqrt{3},0)$$

$$\text{Directrix}: x=\pm\frac{a}{e}=\pm\frac{3}{\frac{1}{\sqrt{3}}}$$

$\therefore x=\pm 3\sqrt{3}.$

(b) $2x^2 + y^2 = 2$

or, $\frac{x^2}{1} + \frac{y^2}{2} = 1$ (i)

Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

$$a^2 = 1 \Rightarrow a = 1$$
$$b^2 = 2 \Rightarrow b = \sqrt{2}$$
$$c = \sqrt{b^2 - a^2} = \sqrt{2-1} = 1$$
$$e = \frac{c}{b} = \frac{1}{\sqrt{2}}$$

Foci = $(0, \pm be) = \left(0, \pm\sqrt{2}\cdot\frac{1}{\sqrt{2}}\right) = (0, \pm 1)$

Directrix : $y = \pm\frac{b}{e} = \pm\frac{\sqrt{2}}{\frac{1}{\sqrt{2}}}$

$\therefore \quad y = \pm 2.$

(c) $3x^2 + 2y^2 = 6$

or, $\frac{x^2}{2} + \frac{y^2}{3} = 1$ (i)

Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

$$a^2 = 2 \Rightarrow a = \sqrt{2}$$
$$b^2 = 3 \Rightarrow b = \sqrt{3}$$
$$e = \frac{\sqrt{b^2 - a^2}}{b} = \frac{\sqrt{3-2}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Foci = $(0, \pm be) = \left(0, \pm\sqrt{3}\cdot\frac{1}{\sqrt{3}}\right)$

$= (0, \pm 1)$

Directrix : $y = \pm\frac{b}{e} = \pm\frac{\sqrt{3}}{\frac{1}{\sqrt{3}}}$

$\therefore \quad y = \pm 3.$

2. Find the equation of ellipse in standard form:

(a) Foci: $(\pm 8, 0)$

Eccentricity: 0.2

(b) Focus: $(\sqrt{5}, 0)$

Directrix: $x = \frac{9}{\sqrt{5}}$

(c) Foci : $(0, \pm 3)$

Eccentricity: 0.5.

Solution

(a) Foci: $(\pm 8, 0)$

or, $(\pm ae, 0) = (\pm 8, 0)$

$ae = 8$ ...(i)

Given, $e = 0.2$ ...(ii)

Putting $e = 0.2$ in (i), we get,

$$a \times 0.2 = 8$$
$$a = \frac{8}{0.2} = 40$$

Also, $e^2 = 1 - \frac{b^2}{a^2}$

or, $(0.2)^2 = 1 - \frac{b^2}{1600}$

or, $\frac{1}{25} = 1 - \frac{b^2}{1600}$

or, $\frac{b^2}{1600} = 1 - \frac{1}{25} = \frac{24}{25}$

$$b^2 = \frac{24}{25} \times 1600 = 1536$$

Required equation of ellipse is

$$\frac{x^2}{1600} + \frac{y^2}{1536} = 1$$

(b) Focus (ac, 0) = $(\sqrt{5}, 0)$

or, $ae = \sqrt{5}$ ...(i)

Directrix : $x = \frac{9}{\sqrt{5}}$

or, $\frac{a}{e} = \frac{9}{\sqrt{5}}$

or, $a = \frac{9e}{\sqrt{5}}$

Putting the value of a in (i)

$$\frac{9e^2}{\sqrt{5}} = \sqrt{5}$$

$$e^2 = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$$

Putting the value of e in (i)

$$a \cdot \frac{\sqrt{5}}{3} = \sqrt{5}$$

$$a = 3$$

We have,

$$a^2e^2 = a^2 - b^2$$
$$5 = 9 - b^2$$
$$b^2 = 9 - 5$$
$$= 4$$

Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(c) Foci : $(0, \pm be) = (0, \pm 3)$

$be = 3$ ... (i)

$e = 0.5$ ... (ii)

From (i) and (ii),

$$b \times 0.5 = 3$$

$$b = \frac{3}{0.5} = 6$$

We have,

$$e^2 = 1 - \frac{a^2}{b^2}$$

or, $b^2e^2 = b^2 - a^2$

or, $36 \times (0.5)^2 = 36 - a^2$

or, $a^2 = 36 - 9 = 27$

Equation of ellipse is

$$\frac{x^2}{27} + \frac{y^2}{36} = 1$$

3. **Find the eccentricity of the hyperbola, foci and directrices of the hyperbola.**
(a) $x^2 - y^2 = 1$ **(b)** $8x^2 - 2y^2 = 16$ **(c)** $y^2 - x^2 = 8$.

Solution

(a) $x^2 - y^2 = 1$

or, $\frac{x^2}{1} - \frac{y^2}{1} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get,

$a^2 = 1 \Rightarrow a = 1$

$b^2 = 1 \Rightarrow b = 1$

We have,

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{1}} = \sqrt{2}$$

Foci $= (\pm\, ae, 0)$

$= (\pm 1 \cdot \sqrt{2}, 0) = (\pm\sqrt{2}, 0)$

Directrix: $x = \pm \frac{a}{e} = \pm \frac{1}{\sqrt{2}}$.

(b) $8x^2 - 2y^2 = 16$.

or, $\frac{x^2}{2} - \frac{y^2}{8} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get,

$a^2 = 2 \Rightarrow a = \sqrt{2}$

$b^2 = 8 \Rightarrow b = 2\sqrt{2}$

We have,

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{8}{2}} = \sqrt{5}$$

Foci $= (\pm\, ae, 0)$

$= (\pm \sqrt{2} \cdot \sqrt{5}, 0)$

$= (\pm \sqrt{10}, 0)$

Directrix: $x = \pm \frac{a}{e} = \pm \frac{\sqrt{2}}{\sqrt{5}}$

$\therefore \quad x = \pm \sqrt{\frac{2}{5}}$.

(c) $y^2 - x^2 = 8$

or, $\frac{y^2}{8} - \frac{x^2}{8} = 1$ ... (i)

Comparing (i) with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get,

$$a^2 = 8 \Rightarrow a = 2\sqrt{2}$$
$$b^2 = 8 \Rightarrow b = 2\sqrt{2}$$

We have,

$$e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{8}{8}} = \sqrt{2}$$

Foci $= (0, \pm be) = (0, \pm 2\sqrt{2} \cdot \sqrt{2}) = (0, \pm 4)$

Directrix: $y = \pm \frac{b}{e} = \pm \frac{2\sqrt{2}}{\sqrt{2}} = \pm 2.$

4. **Find the equation of hyperbola in standard form.**

(a) **Eccentricity: $\frac{5}{4}$**

Vertices: $(\pm 4, 0)$

(b) **Eccentricity: 3**

Vertices: $(0, \pm 1)$

(c) **Focus: $(-2, 0)$**

Directrix: $x = -\frac{1}{2}$.

Solution

(a) Vertices $= (\pm 4, 0)$

or, $(\pm a, 0) = (\pm 4, 0)$

$\therefore$ $a = 4$

and $e = \frac{5}{4}$

We have,

$$e^2 = 1 + \frac{b^2}{a^2}$$

or, $\frac{25}{16} = 1 + \frac{b^2}{16}$

or, $\frac{25}{16} = \frac{16 + b^2}{16}$

or, $b^2 = 9$

Equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\therefore$ $9x^2 - 16y^2 = 144.$

(b) Vertices $= (0, \pm 1)$

or, $(0, \pm b) = (0, \pm 1)$

$\therefore$ $b = 1$

Eccentricity $(e) = 3$

We have,

$$e^2 = 1 + \frac{a^2}{b^2}$$

or, $9 = 1 + \frac{a^2}{1}$

or, $a^2 = 8$

Equation of hyperbola is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1$$

or, $\frac{y^2}{1} - \frac{x^2}{8} = 1$

$\therefore$ $8y^2 - x^2 = 1.$

(c) Focus $= (-2, 0)$

or, $(-ae, 0) = (-2, 0)$

$ae = 2$ ... (i)

Directrix $= -\frac{1}{2}$

or, $-\frac{a}{e} = -\frac{1}{2}$ $\left[\because x = \pm\frac{a}{e}\right]$

or, $2a = e$

$a = \frac{e}{2}$ ... (ii)

From (i) and (ii), we get,

$$\frac{e}{2}\cdot e = 2$$

or, $e^2 = 4$

$\therefore$ $e = 2$

Putting the value of e in (ii)

$$a = \frac{2}{2} = 1$$

We have,

$$e^2 = 1 + \frac{b^2}{1^2}$$

or, $4 = 1 + \dfrac{b^2}{1}$

or, $b^2 = 3$

Equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or, $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$

$\therefore$ $3x^2 - y^2 = 3.$

5. **Find the vertex, focus and directrix of the parabolas.**

(a) $(y + 2)^2 = 8(x - 1)$ **(b)** $(x - 1)^2 = 4(y + 2)$

Solution

(a) $(y + 2)^2 = 8(x - 1)$ ...(i)

Comparing (i) with

$(y - k)^2 = 4a(x - h)$, we get,

$h = 1, k = -2, 4a = 8$

$\Rightarrow a = 2$

Vertex $= (h, k) = (1, -2)$

Focus $= (h + a, k) = (1 + 2, -2)$

$= (3, -2)$

Directrix : $x = h - a$

or, $x = 1 - 2$

$\therefore$ $x = -1$

(b) $(x - 1)^2 = 4(y + 2)$ ... (i)

Comparing (i) with

$(x - h)^2 = 4a(y - k)$, we get,

$h = 1, k = -2, 4a = 4$

$\Rightarrow a = 1$

Vertex $= (h, k) = (1, -2)$

Focus $= (h, k + a) = (1, -2 + 1)$

$= (1, -1)$

Directrix : $y = k - a$

or, $y = -2 - 1$

$\therefore$ $y = -3$

6. **Find the equation of parabola with focus at (2, 1) and directrix $x = -1$.**

Solution

Let P(x, y) be any point on the parabola. Then,

distance of P from focus = Length of perpendicular from P on the directrix

or, $(x - 2)^2 + (y - 1)^2 = (x + 1)^2$

or, $x^2 - 4x + 4 + y^2 - 2y + 1 = x^2 + 2x + 1$

or, $y^2 - 6x - 2y + 4 = 0.$

7. **Find the equation of the parabola in which the ends of the latus rectum have the coordinates (– 1, 5) and (– 1, –11) and the vertex is (– 5, – 3).**

Solution

Since the end points of latus rectum are (–1, 5) and (–1, –11), so the equation of latus rectum is x = –1. So it is parallel to x-axis.

Here, vertex (h, k) = (–5, –3)

We know that the equation of parabola is

$(y - k)^2 = 4a(x - h)$

or, $(y + 3)^2 = 4a(x + 5)$ ....(i)

If (i) passes through the point (– 1, 5), then

$(5 + 3)^2 = 4a(-1 + 5)$

$64 = 16a$

$\therefore \quad a = 4$

Putting the value of a in (i), we get

$(y + 3)^2 = 4 \times 4(x + 5)$

or, $\quad y^2 + 6y + 9 = 16x + 80$

$\therefore \quad y^2 + 6y - 16x - 71 = 0$

8. **Find centre, vertices, eccentricity and foci of the ellipses.**

(a) $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$ (b) $9x^2 + 6y^2 + 36y = 0$

Solution

(a) Given, $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$ ... (i)

Comparing (i) with $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, we get,

$h = -2, k = 1$

$a^2 = 9 \Rightarrow a = 3$

$b^2 = 4 \Rightarrow b = 2$

$\therefore$ a > b. So major axis is along x-axis.

Center = (h, k) = (– 2, 1)

Vertices = $(h \pm a, k) = (-2 \pm 3, 1) = (-2 + 3, 1)$ and $(-2 - 3, 1)$

$= (1, 1)$ and $(-5, 1)$

Eccentricity (e) $= \sqrt{1 - \dfrac{b^2}{a^2}} = \sqrt{1 - \dfrac{4}{9}} = \sqrt{\dfrac{5}{9}} = \dfrac{\sqrt{5}}{3}$

Foci $= (h \pm ae, k) = \left(-2 \pm 3 \cdot \dfrac{\sqrt{5}}{3}, 1\right) = (-2 \pm \sqrt{5}, 1)$.

(b) Given, $9x^2 + 6y^2 + 36y = 0$

or, $\quad 9x^2 + 6(y^2 + 6y) = 0$

or, $\quad 9x^2 + 6(y^2 + 6y + 9 - 9) = 0$

or, $\quad 9x^2 + 6(y + 3)^2 - 54 = 0$

or, $\quad 9x^2 + 6(y + 3)^2 = 54$

or, $\quad \dfrac{x^2}{6} + \dfrac{(y+3)^2}{9} = 1$ ... (i)

Comparing (i) with $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, we get,

$h = 0, k = -3$

$a^2 = 6 \Rightarrow a = \sqrt{6}$

$b^2 = 9 \Rightarrow b = 3$

Since b > a, so major axis is along y-axis.

Center = (h, k) = (0, – 3)

Vertices = $(h, k \pm b) = (0, -3 \pm 3) = (0, -3 + 3)$ and $(0, -3 - 3)$

$= (0, 0)$ and $(0, -6)$

Eccentricity (e) $= \sqrt{1 - \dfrac{a^2}{b^2}} = \sqrt{1 - \dfrac{6}{9}} = \sqrt{\dfrac{3}{9}} = \dfrac{1}{\sqrt{3}}$

Foci $= (h, k \pm ae) = \left(0, -3 \pm \sqrt{6} \cdot \dfrac{1}{\sqrt{3}}\right) = (0, -3 \pm \sqrt{2})$.

9. Find centre, vertices, eccentricity and foci of the hyperbolas.

(a) $\frac{(x-2)^2}{16}-\frac{y^2}{9}=1$ (b) $9(x-1)^2-16(y+2)^2=144$

Solution

(a) $\frac{(x-2)^2}{16}-\frac{y^2}{9}=1$ ... (i)

Comparing (i) with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get,

$h=2,\ k=0$

$a^2=16 \Rightarrow a=4$

$b^2=9 \Rightarrow b=3$

Center = (h, k) = (2, 0)

Vertices = $(h \pm a, k) = (2 \pm 4, 0) = (2+4, 0)$ and $(2-4, 0)$

$= (6, 0)$ and $(-2, 0)$

Eccentricity (e) $=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$

Foci $= (h \pm ae, k) = \left(2 \pm 4\cdot\frac{5}{4}, 0\right) = (2+5, 0)$ and $(2-5, 0)$

$= (7, 0)$ and $(-3, 0)$.

(b) $9(x-1)^2-16(y+2)^2=144$

Given hyperbola is

$$9(x-1)^2-16(y+2)^2=144$$

Dividing both sides by 144 we get,

$$\frac{9(x-1)^2}{144}-\frac{16(y+2)^2}{144}=\frac{144}{144}$$

or, $\frac{(x-1)^2}{16}-\frac{(y+2)^2}{9}=1$ ... (1)

Comparing (1) with

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get,

$h=1,\ k=-2$

$a^2=16,\ b^2=9$

$\therefore$ $a=4,\ b=3$

Centre = $(h, k) = (1, -2)$

Vertices = $(h \pm a, k) = (1 \pm 4, -2) = (5, -2)$ and $(-3, -2)$

Eccentricity (e) $=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{16+9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$

Foci $= (h \pm ae, k) = (1 \pm 4.\frac{5}{4}, -2) = (1 \pm 5, -2) = (1+5, -2)$ and $(1-5, -2)$

$= (6, -2)$ and $(-4, -2)$

Objective Questions

1. The conic section becomes a parabola if

(a) $e=0$ (b) $e=1$ (c) $e>1$ (d) $e<1$

Ans: b

$e=1$

2. The eccentricity of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

(a) $\frac{3}{5}$ (b) $\frac{3}{7}$ (c) $\frac{4}{5}$ (d) $\frac{5}{7}$

Ans: a

$a^2 = 25, b^2 = 16$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

3. The eccentricity of hyperbola $21x^2 - 4y^2 = 84$ is

(a) $\frac{2}{5}$ (b) $\frac{5}{2}$ (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

Ans: b

$21x^2 - 4y^2 = 8y$

or, $\frac{x^2}{4} - \frac{y^2}{21} = 1$

$a^2 = 4, b = 21$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{21}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

4. The equation of directrix of $\frac{x^2}{3} - \frac{y^2}{6} = 1$ is

(a) $x = 4$ (b) $x = 3$ (c) $x = 2$ (d) $x = 1$

Ans: d

$a^2 = 3, b^2 = 6$

$\therefore$ $a = \sqrt{3}, b = \sqrt{6}$

We have,

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{6}{3}} = \sqrt{\frac{9}{3}} = \sqrt{3}$$

Equation of directrix: $x = \pm\frac{a}{e}$

or, $x = \pm\frac{\sqrt{3}}{\sqrt{3}}$

$\therefore$ $x = \pm 1$

5. The vertices of an ellipse whose foci are $(0, \pm 7)$ and eccentricity $\frac{4}{5}$ is

(a) $(0, \pm 4.75)$ (b) $(0, \pm 5.75)$ (c) $(0, \pm 7.75)$ (d) $(0, \pm 8.75)$

Ans: d

Foci = $(0, \pm 7)$

$be = 7$ ... (i)

and $e = \frac{4}{5}$ ... (ii)

From (i) and (ii)

$b \cdot \frac{4}{5} = 7$

$\therefore$ $b = \frac{35}{4}$

Vertices = $(0, \pm b) = (0, \pm 8.75)$

EXERCISE – 5 D

1. Find the cartesian equations for

(a) $r\cos\left(\theta-\frac{\pi}{4}\right)=\sqrt{2}$ **(b)** $r\cos\left(\theta+\frac{2\pi}{3}\right)=3.$

Solution

(a) $r\cos\left(\theta-\frac{\pi}{4}\right)=\sqrt{2}$

or, $r\left(\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}\right)$

or, $r\cos\theta\cdot\frac{1}{\sqrt{2}}+r\sin\theta\cdot\frac{1}{\sqrt{2}}=\sqrt{2}$

or, $r\cos\theta+r\sin\theta=2$

$\therefore$ $x+y=2.$

(b) $r\cos\left(\theta+\frac{2\pi}{3}\right)=3$

or, $r\left(\cos\theta\cos\frac{2\pi}{3}-\sin\theta\sin\frac{2\pi}{3}\right)=3$

or, $r\cos\theta\left(-\frac{1}{2}\right)-r\sin\theta\frac{\sqrt{3}}{2}=3$

or, $-\frac{x}{2}-y\frac{\sqrt{3}}{2}=3$

or, $0=\frac{x}{2}+\frac{\sqrt{3}y}{2}+3$

or, $x+\sqrt{3}\,y+6=0.$

2. Find the polar equations in the form $r\cos(\theta-\theta_0)=r_0$ for

(a) $\sqrt{2}\,x+\sqrt{2}\,y=6$ **(b)** $y=-5.$

Solution

(a) $\sqrt{2}\,x+\sqrt{2}\,y=6$

or, $\sqrt{2}\cdot r\cos\theta+\sqrt{2}\cdot r\sin\theta=6$

Dividing both sides by 2

$r\frac{1}{\sqrt{2}}\cos\theta+r\frac{1}{\sqrt{2}}\sin\theta=3$

or, $r\left(\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}\right)=3$

$\therefore$ $r\cos\left(\theta-\frac{\pi}{4}\right)=3.$

(b) $y=-5$

or, $r\sin\theta=-5$

or, $-r\sin\theta=5$

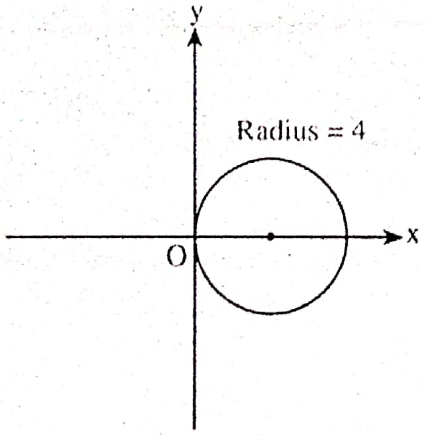
or, $0\cdot r\cos\theta-r\cdot 1\cdot\sin\theta=5$

or, $r\cos\theta\cos\frac{\pi}{2}-r\cdot\sin\theta\cdot\sin\frac{\pi}{2}=5$

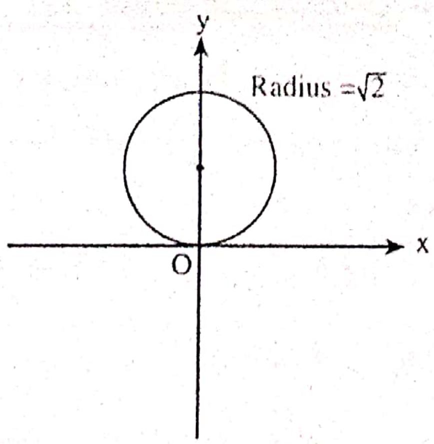
or, $r\cos\left(\theta+\frac{\pi}{2}\right)=5.$

3. Find polar equations for the following circles.

(a)



(b)



Solution

(a) We know that the equation of circle centre at positive x-axis is

$r = 2a\cos\theta$ ... (i)

Here, a = 4 so, equation (i) becomes

$r = 2 \times 4\cos\theta$

$r = 8\cos\theta.$

(b) The equation of circle having centre at positive y-axis is

$r = 2a\sin\theta$ ... (i)

Here, $a = \sqrt{2}$

The equation (i) becomes,

$r = 2 \times \sqrt{2}\sin\theta$

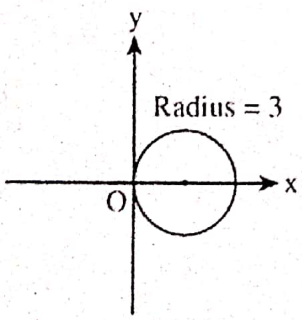
$\therefore$ $r = 2\sqrt{2}\sin\theta.$

4. Sketch the circle

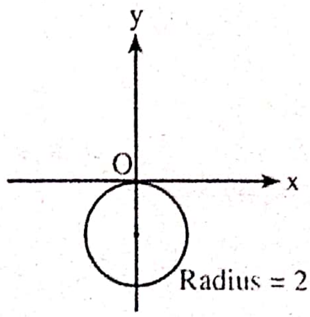
(a) $r = 3\cos\theta$ **(b)** $r = -2\sin\theta.$

Solution

(a)



(b)



5. Find the polar equation for

(a) $(x-6)^2 + y^2 = 36$ **(b)** $x^2 + (y-5)^2 = 25.$

Solution

(a) $(x-6)^2 + y^2 = 36$

or, $x^2 - 12x + 36 + y^2 = 36$

or, $x^2 + y^2 = 12x$

or, $r^2 = 12 \cdot r\cos\theta$

$\therefore$ $r = 12\cos\theta.$

(b) $x^2 + (y-5)^2 = 25$

or, $x^2 + y^2 - 10y = 0$

or, $r^2 = 10 \cdot r\sin\theta$

$\therefore$ $r = 10\sin\theta.$

6. **Find polar equation for each conic section, where the eccentricities of conic section with one focus at origin along with the directrix corresponding to that focus are given.**

(a) $e = 1, x = 2$ (b) $e = \frac{1}{2}, x = 1$ (c) $e = 5, y = -6$.

Solution

(a) $e = 1, x = 2$

$\therefore$ $k = 2$ and $e = 1$

The eq^n of conic section is

$$r = \frac{ke}{1 + e\cos\theta}$$

or, $$r = \frac{2 \cdot 1}{1 + 1 \cdot \cos\theta}$$

$\therefore$ $$r = \frac{2}{1 + \cos\theta}$$

(b) $e = \frac{1}{2}, x = 1$

$\therefore$ $k = 1$ and $e = \frac{1}{2}$

The eq^n of conic section is

$$r = \frac{ke}{1 + e\cos\theta}$$

or, $$r = \frac{1 \cdot \frac{1}{2}}{1 + \frac{1}{2} \cdot \cos\theta}$$

$\therefore$ $$r = \frac{1}{2 + \cos\theta}$$

(c) $e = 5, y = -6$

$\therefore$ $k = 6$ and $e = 5$

The eq^n of conic section is

$$r = \frac{ke}{1 - e\sin\theta}$$

or, $$r = \frac{6 \cdot 5}{1 - \sin\theta}$$

$\therefore$ $$r = \frac{30}{1 - 5\sin\theta}.$$

7. **Find the eccentricity and directrix of the following conic.**

(a) $r = \frac{1}{1 + \cos\theta}$ (b) $r = \frac{25}{10 - 5\cos\theta}$ (c) $r = \frac{400}{16 + 8\sin\theta}$.

Solution

(a) $$r = \frac{1}{1 + \cos\theta} \quad \ldots (i)$$

Comparing (i) with $\frac{ke}{1 + e\cos\theta}$, we get,

$e = 1, k = 1$

$\therefore$ The eq^n of directrix is $x = 1$ and $e = 1$.

(b) $$r = \frac{25}{10 - 5\cos\theta} \quad \ldots (i)$$

Dividing both numerator and denominator by 10,

$$r = \frac{\frac{25}{10}}{1 - \frac{5}{10}\cos\theta} = \frac{\frac{5}{2}}{1 - \frac{1}{2}\cos\theta} = \frac{5 \cdot \frac{1}{2}}{1 - \frac{1}{2}\cos\theta} \quad \ldots (ii)$$

Comparing (ii) with $\frac{ke}{1 - e\cos\theta}$, we get,

$e = \frac{1}{2}, x = -5$

$\therefore$ The eq^n of directrix is $x = -5$ and eccentricity is $e = \frac{1}{2}$.

(c) $r = \dfrac{400}{16 + 8 \sin \theta}$.

Dividing both numerator and denominator by 16,

$$r = \frac{\frac{400}{16}}{1 + \frac{8}{16} \sin \theta} = \frac{25}{1 + \frac{1}{2} \sin \theta} = \frac{50 \cdot \frac{1}{2}}{1 + \frac{1}{2} \sin \theta} \quad \ldots \text{(ii)}$$

Comparing (ii) with $r = \dfrac{ke}{1 + e \sin \theta}$, we get,

$$k = 50,\ e = \frac{1}{2}$$

∴ The eq^n of directrix is $y = 50$ and $e = \frac{1}{2}$.

Objective Questions

1. **The cartesian equation of $r \sin \theta = 2$ is**

(a) $x = 2$ (b) $y = 2$ (c) $xy = 2$ (d) $y = x$

Ans: b

$y = 2$ ($\because r \sin \theta = y$)

2. **The polar equation of $x = -4$ is**

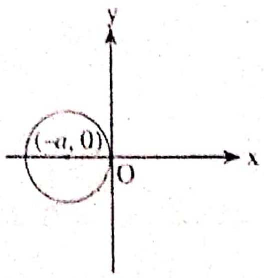
(a) $r \cos \theta = -4$ (b) $r \sin \theta = -4$ (c) $r \cos \theta = 4$ (d) $r \sin \theta = 4$

Ans: a

$x = -4$

$r \cos \theta = -4$

3. **The polar equation of the circle**



is

(a) $r = 2a \cos \theta$ (b) $r = 2a \sin \theta$ (c) $r = -2a \cos \theta$ (d) $r = -2a \sin \theta$

Ans: c

4. **The radius of the circle $r = 6 \cos \theta$ is**

(a) 1 (b) 2 (c) 3 (d) 4

Ans: c

$r = 3$

5. **The equation $r = \dfrac{5}{1 + \cos \theta}$ represents**

(a) circle (b) ellipse (c) hyperbola (d) parabola

Ans: d

$e = 1$ (parabola)

6. **The equation $r = \dfrac{400}{16 + 8 \cos \theta}$ represents**

(a) circle (b) ellipse (c) hyperbola (d) parabola

Ans: b

$$r = \frac{400}{16 + 8 \cos \theta} = \frac{400}{16\left(1 + \frac{1}{2} \cos \theta\right)} = \frac{25}{1 + \frac{1}{2} \cos \theta}$$

$e = \frac{1}{2} < 1$ (ellipse)

EXERCISE – 5 E

1. **Find a vector with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$.**

(a) **A(–1, 1), B(3, 2)** **(b)** **A(–1, 3, 4), B(2, 2, 2).**

Solution

(a) Here, $(x_1, y_1) = (-1, 1)$

$(x_2, y_2) = (3, 2)$

$\overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} = (3 + 1)\vec{i} + (2 - 1)\vec{j} = 4\vec{i} + \vec{j}.$

(b) Here, $(x_1, y_1, z_1) = (-1, 3, 4)$

$(x_2, y_2, z_2) = (2, 2, 2)$

$\overrightarrow{AB} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$

$= (2 + 1)\vec{i} + (2 - 3)\vec{j} + (2 - 4)\vec{k} = 3\vec{i} - \vec{j} - 2\vec{k}.$

2. **Find $\vec{a} + \vec{b}$, $2\vec{a} + 3\vec{b}$, $|\vec{a}|$ and $|\vec{a} - \vec{b}|$**

(a) $\vec{a} = (5, -12), \vec{b} = (-3, -6)$ **(b)** $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}, \vec{b} = -2\vec{i} - \vec{j} + 5\vec{k}.$

Solution

(a) Given, $\vec{a} = (5, -12), \vec{b} = (-3, -6)$

$\vec{a} + \vec{b} = (5, -12) + (-3, -6) = (2, -18)$

$2\vec{a} + 3\vec{b} = 2(5, -12) + 3(-3, -6) = (10, -24) + (-9, -18)$

$= (10 - 9, -24 - 18) = (1, -42)$

$|\vec{a}| = \sqrt{5^2 + (-12)^2} = 13$

$|\vec{a} - \vec{b}| = |(5, -12) - (-3, -6)| = |(8, -6)|$

$= \sqrt{8^2 + (-6)^2} = \sqrt{100} = 10.$

(b) $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}, \vec{b} = -2\vec{i} - \vec{j} + 5\vec{k}$

$\vec{a} + \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k} + (-2\vec{i} - \vec{j} + 5\vec{k}) = -\vec{i} + \vec{j} + 2\vec{k}.$

$2\vec{a} + 3\vec{b} = 2(\vec{i} + 2\vec{j} - 3\vec{k}) + 3(-2\vec{i} - \vec{j} + 5\vec{k})$

$= 2\vec{i} + 4\vec{j} - 6\vec{k} - 6\vec{i} - 3\vec{j} + 15\vec{k} = -4\vec{i} + \vec{j} + 9\vec{k}.$

$|\vec{a}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

$|\vec{a} - \vec{b}| = |\vec{i} + 2\vec{j} - 3\vec{k} - (-2\vec{i} - \vec{j} + 5\vec{k})| = |3\vec{i} + 3\vec{j} - 8\vec{k}|$

$= \sqrt{3^2 + 3^2 + (-8)^2} = \sqrt{82}.$

3. **Find a unit vector that has the same direction as the given vector.**

(a) $-3\vec{i} + 7\vec{j}$ **(b)** $8\vec{i} - \vec{j} + 4\vec{k}.$

Solution

(a) Let $\vec{a} = -3\vec{i} + 7\vec{j}$

$|\vec{a}| = \sqrt{(-3)^2 + 7^2} = \sqrt{58}$

Unit vector $(\hat{a}) = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{-3}{\sqrt{58}}\vec{i} + \dfrac{7}{\sqrt{58}}\vec{j}$

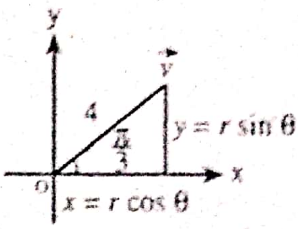
(b) Let $\vec{a} = 8\vec{i} - \vec{j} + 4\vec{k}$

$|\vec{a}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{64 + 1 + 16} = 9$

Unit vector $(\hat{a}) = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{8\vec{i} - \vec{j} + 4\vec{k}}{9} = \dfrac{8}{9}\vec{i} - \dfrac{1}{9}\vec{j} + \dfrac{4}{9}\vec{k}$

4. If $\vec{v}$ lies in the first quadrant and makes an angle $\frac{\pi}{3}$ with the positive x-axis and $|\vec{v}| = 4$, find v in component form.

Solution



x-component $= 4\cos\dfrac{\pi}{3} = 4 \cdot \dfrac{1}{2} = 2$

y-component $= 4\sin\dfrac{\pi}{3} = 4 \cdot \dfrac{\sqrt{3}}{2} = 2\sqrt{3}$

$\therefore \quad \vec{v} = 2\vec{i} + 2\sqrt{3}\,\vec{j}$.

5. Find scalar product of the following pair of vectors.

(a) (1, 2) and (3, 4) (b) (1, 2, 0) and (3, 2, 1)

(c) $3\vec{i} - 7\vec{j} + \vec{k}$ and $-\vec{i} + \vec{j} - 2\vec{k}$.

Solution

(a) Let $\vec{a} = (1, 2)$ and $\vec{b} = (3, 4)$

$\vec{a} \cdot \vec{b} = (1, 2) \cdot (3, 4) = 3 + 8 = 11$.

(b) Let $\vec{a} = (1, 2, 0)$ and $\vec{b} = (3, 2, 1)$

$\vec{a} \cdot \vec{b} = (1, 2, 0) \cdot (3, 2, 1) = 3 + 4 + 0 = 7$.

(c) Let $\vec{a} = 3\vec{i} - 7\vec{j} + \vec{k}$ and $\vec{b} = -\vec{i} + \vec{j} - 2\vec{k}$

$\vec{a} \cdot \vec{b} = (3\vec{i} - 7\vec{j} + \vec{k}) \cdot (-\vec{i} + \vec{j} - 2\vec{k}) = -3 - 7 - 2 = -12$.

6. Find the angle between the following vectors.

(a) $\vec{a} = \vec{i} - 2\vec{j} - 2\vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

(b) $\vec{a} = 2\vec{j} + 4\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$.

Solution

(a) $\vec{a} \cdot \vec{b} = (\vec{i} - 2\vec{j} - 2\vec{k}) \cdot (6\vec{i} + 3\vec{j} + 2\vec{k}) = 6 - 6 - 4 = -4$

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$

$|\vec{b}| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$

If θ be the angle between $\vec{a}$ and $\vec{b}$ then $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{-4}{(3)(7)} = \dfrac{-4}{21}$

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{-4}{21}\right)$

(b) $\vec{a}\cdot\vec{b} = (2\vec{j}+4\vec{k})\cdot(3\vec{i}-2\vec{j}+\vec{k}) = 0-4+4=0$

$|\vec{a}| = \sqrt{0^2+2^2+4^2} = \sqrt{20}$

$|\vec{b}| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$

If θ be the angle between $\vec{a}$ and $\vec{b}$ then $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{0}{\sqrt{20}\sqrt{14}} = 0 = \cos\dfrac{\pi}{2}$

$\therefore \quad \theta = \dfrac{\pi}{2}.$

7. **Find the direction cosines of the following vectors.**

(a) (2, 1, 2) **(b) (c, c, c), where $c > 0$.**

Solution

(a) (2, 1, 2)

Let $\vec{a} = (a_1, a_2, a_3) = (2, 1, 2)$

$|\vec{a}| = \sqrt{2^2+1^2+2^2} = 3.$

If α, β, γ be the direction angles then

$\cos\alpha = \dfrac{a_1}{|\vec{a}|} = \dfrac{2}{3}$

$\cos\beta = \dfrac{a_2}{|\vec{a}|} = \dfrac{1}{3}$

$\cos\gamma = \dfrac{a_3}{|\vec{a}|} = \dfrac{2}{3}.$

(b) (c, c, c)

Let $\vec{a} = (a_1, a_2, a_3) = (c, c, c)$

$|\vec{a}| = \sqrt{c^2+c^2+c^2} = \sqrt{3c^2} = c\sqrt{3}$

If α, β, γ be the direction angles then

$\cos\alpha = \dfrac{a_1}{|\vec{a}|} = \dfrac{c}{c\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\cos\beta = \dfrac{a_2}{|\vec{a}|} = \dfrac{c}{c\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\cos\gamma = \dfrac{a_3}{|\vec{a}|} = \dfrac{c}{c\sqrt{3}} = \dfrac{1}{\sqrt{3}}.$

8. **Find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$.**

(a) $\vec{a} = (-5, 12)$, $\vec{b} = (4, 6)$ **(b) $\vec{a} = (3, 6, -2)$, $\vec{b} = (1, 2, 3)$**

Solution

(a) Here, $\vec{a} = (-5, 12)$

$\vec{b} = (4, 6)$

$|\vec{a}| = \sqrt{(-5)^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13.$

$\vec{a}\cdot\vec{b} = (-5, 12)\cdot(4, 6) = -20+72 = 52.$

Scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{52}{13} = 4$

Vector projection of $\vec{b}$ on $\vec{a} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \left(\dfrac{52}{13^2}\right)(-5, 12) = \dfrac{4}{13}(-5, 12) = \left(-\dfrac{20}{13}, \dfrac{48}{13}\right).$

(b) Here, $\vec{a} = (3, 6, -2)$

$\vec{b} = (1, 2, 3)$

$|\vec{a}| = \sqrt{3^2+6^2+(-2)^2} = \sqrt{9+36+4} = 7.$

$\vec{a}\cdot\vec{b} = (3, 6, -2)\cdot(1, 2, 3) = 3+12-6 = 9.$

Scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{9}{7}$

Vector projection of $\vec{b}$ on $\vec{a} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

$$= \left(\frac{9}{7^2}\right)(3, 6, -2) = \frac{9}{49}(3, 6, -2) = \left(\frac{27}{49}, \frac{54}{49}, -\frac{18}{49}\right).$$

9. **Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.**

(a) $\vec{a} = (6, 0, -2), \vec{b} = (0, 8, 0)$ (b) $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k},\ \vec{b} = -\vec{i} + 5\vec{k}$

(c) $\vec{a} = \vec{i} - \vec{j} - \vec{k},\ \vec{b} = \frac{1}{2}\vec{i} + \vec{j} + \frac{1}{2}\vec{k}$.

Solution

(a) $\vec{a} = (6, 0 - 2)$ $\vec{b} = (0, 8, 0)$

For $\vec{a} \times \vec{b}$,

$$\begin{array}{cccccc} 6 & 0 & -2 & 6 & 0 \\ 0 & 8 & 0 & 0 & 8 \end{array}$$

For $\vec{a} \times \vec{b} = (0 + 16, 0 - 0, 48 - 0) = (16, 0, 48)$

Now, $(\vec{a} \times \vec{b}) \cdot \vec{a} = (16, 0, 48) \cdot (6, 0, -2) = 96 + 0 - 96 = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = (16, 0, 48) \cdot (0, 8, 0) = 0 + 0 + 0 = 0$

$\therefore$ $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

(b) $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}\vec{k}$

$= 15\vec{i} - 3\vec{j} + 3\vec{k}$

Now, $(\vec{a} \times \vec{b}) \cdot \vec{a} = (15\vec{i} - 3\vec{j} + 3\vec{k}) \cdot (\vec{i} + 3\vec{j} - 2\vec{k}) = 15 - 9 - 6 = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = (15\vec{i} - 3\vec{j} + 3\vec{k}) \cdot (-\vec{i} + 5\vec{k}) = -15 - 0 + 15 = 0$

$\therefore$ $\vec{a} \times \vec{b}$ is perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$.

(c) $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & -1 \\ 1 & \frac{1}{2} \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}$

$= \left(-\frac{1}{2} + 1\right)\vec{i} - \left(\frac{1}{2} + \frac{1}{2}\right)\vec{j} + \left(1 + \frac{1}{2}\right)\vec{k} = \frac{1}{2}\vec{i} - \vec{j} + \frac{3}{2}\vec{k}$

Now, $(\vec{a} \times \vec{b}) \cdot \vec{a} = \left(\frac{1}{2}\vec{i} - \vec{j} + \frac{3}{2}\vec{k}\right) \cdot (\vec{i} - \vec{j} - \vec{k}) = \frac{1}{2} + 1 - \frac{3}{2} = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = \left(\frac{1}{2}\vec{i} - \vec{j} + \frac{3}{2}\vec{k}\right) \cdot \left(\frac{1}{2}\vec{i} + \vec{j} + \frac{1}{2}\vec{k}\right) = \frac{1}{4} - 1 + \frac{3}{4} = 0$

This shows that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

10. Find a unit vector orthogonal to both (3, 2, 1) and (–1, 1, 0).

Solution

Let $\vec{a} = (3, 2, 1) = 3\vec{i} + 2\vec{j} + \vec{k}$

$\vec{b} = (-1, 1, 0) = -\vec{i} + \vec{j}$

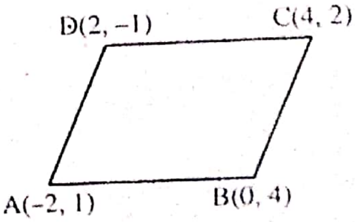
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= (0-1)\vec{i} - (0+1)\vec{j} + (3+2)\vec{k} = -\vec{i} - \vec{j} + 5\vec{k} = (-1, -1, 5)$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (-1)^2 + 5^2} = \sqrt{27} = 3\sqrt{3}$$

Unit vector orthogonal to both $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \dfrac{1}{3\sqrt{3}}(-1, -1, 5) = -\dfrac{1}{3\sqrt{3}}, -\dfrac{1}{3\sqrt{3}}, \dfrac{5}{3\sqrt{3}}$

11. Find the area of the parallelogram with vertices A(–2, 1), B(0, 4), C(4, 2) and D(2, –1).



Solution

$\vec{AB} = (0+2)\vec{i} + (4-1)\vec{j} = 2\vec{i} + 3\vec{k}$

$\vec{AD} = (2+2)\vec{i} + (-1-1)\vec{j} = 4\vec{i} - 2\vec{j}$

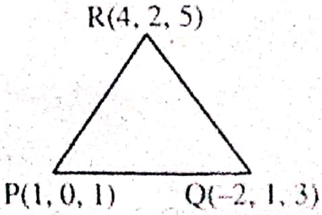
$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -2 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix}\vec{k}$$

$$= (0-0)\vec{i} - (0-0)\vec{j} + (-4-12)\vec{k} = 0 \cdot \vec{i} - 0 \cdot \vec{j} - 16\vec{k}$$

$$\vec{AB} \times \vec{AD} = \sqrt{0^2 + 0^2 + (-16)^2} = \sqrt{256} = 16$$

Area of ABCD = 16 sq. units

12. Find a nonzero vector orthogonal to the plane through the points P(1, 0, 1), Q(–2, 1, 3), and R(4, 2, 5), and also find the area of triangle PQR.



Solution

$\vec{PQ} = (-2-1)\vec{i} + (1-0)\vec{j} + (3-1)\vec{k} = -3\vec{i} + \vec{j} + 2\vec{k}$

$\vec{PR} = (4-1)\vec{i} + (2-0)\vec{j} + (5-1)\vec{k} = 3\vec{i} + 2\vec{j} + 4\vec{k}$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (4-4)\vec{i} - (-12-6)\vec{j} + (-6-3)\vec{k} = 18\vec{j} - 9\vec{k}$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{18^2 + (-9)^2} = \sqrt{324 + 81} = \sqrt{405}$$

Area of triangle PQR $= \dfrac{1}{2}|\vec{PQ} \times \vec{PR}| = \dfrac{1}{2}\sqrt{405}$ sq. units

13. Determine whether the given vectors are orthogonal, parallel or neither.

(a) $\vec{a} = (4, 6), \vec{b} = (-3, 2)$ (b) $\vec{a} = (-5, 3, 7), \vec{b} = (6, -8, 2)$

(c) $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}, \vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$.

Solution

(a) $\vec{a} = (4, 6)$

$\vec{b} = (-3, 2)$

$\vec{a} \cdot \vec{b} = (4, 6) \cdot (-3, 2) = -12 + 12 = 0$

$\therefore$ $\vec{a}$ is perpendicular to $\vec{b}$.

(b) $\vec{a} = (-5, 3, 7)$

$\vec{b} = (6, -8, 2)$

$\vec{a} \cdot \vec{b} = (-5, 3, 7) \cdot (6, -8, 2) \quad = -30 - 24 + 14 = -40 \neq 0$.

So, $\vec{a}$ is not perpendicular to $\vec{b}$.

For $\vec{a} \cdot \vec{b}$

$$\begin{array}{cccccc} -5 & 3 & 7 & -5 & 3 \\ 6 & -8 & 2 & 6 & -8 \end{array}$$

For $\vec{a} \times \vec{b} = (6 + 56, 42 + 10, 40 - 18) = (62, 52, 22) \neq 0$

So, $\vec{a}$ is not parallel to $\vec{b}$.

Hence, $\vec{a}$ and $\vec{b}$ are neither perpendicular nor parallel.

(c) $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}$

$\vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$

$\vec{a} \cdot \vec{b} = -6 - 54 - 24 = -84 \neq 0$

So, $\vec{a}$ is not perpendicular to $\vec{b}$.

Again,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 6 & -4 \\ -3 & -9 & 6 \end{vmatrix} = \vec{i}\begin{vmatrix} 6 & -4 \\ -9 & 6 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -4 \\ -3 & 6 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 6 \\ -3 & -9 \end{vmatrix}$$

$$= (36 - 36)\vec{i} - (12 - 12)\vec{j} + (-18 + 18)\vec{k} = 0\vec{i} - 0\vec{j} + 0\vec{k} = \text{zero vector}$$

i.e. $\vec{a} \times \vec{b} = \vec{0}$.

Hence $\vec{a}$ and $\vec{b}$ are parallel.

Alternatively

$$\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k} = -2(-\vec{i} - 3\vec{j} + 2\vec{k}) = -2 \times \frac{1}{3}(-3\vec{i} - 9\vec{j} + 6\vec{k}) = -\frac{2}{3}\vec{b}.$$

$\therefore$ $\vec{a} \,/\!/\, \vec{b}$ since $\vec{a} = k\vec{b}$ where k is scalar.

14. (a) If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, prove that $\vec{a}$ is perpendicular to $\vec{b}$.

Solution

Here,

$$|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$$

or, $|\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$

or, $(\vec{a} + \vec{b})^2 = (\vec{a} - \vec{b})^2$

or, $a^2 + 2\vec{a} \cdot \vec{b} + b^2 = a^2 - 2\vec{a} \cdot \vec{b} + b^2$

or, $4\vec{a} \cdot \vec{b} = 0$

or, $\vec{a} \cdot \vec{b} = 0$

$\therefore$ $\vec{a}$ is perpendicular to $\vec{b}$.

(b) Show that: $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$.

Solution

L.H.S. $= |\vec{a} \times \vec{b}|^2 = (ab \sin\theta)^2 = a^2b^2 \sin^2\theta = a^2b^2(1 - \cos^2\theta) = a^2b^2 - a^2b^2\cos^2\theta$

$= |\vec{a}|^2 |\vec{b}|^2 - (ab\cos\theta)^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 =$ R.H.S.

15. If $\vec{a} \cdot \vec{b} = \sqrt{3}$ and $\vec{a} \times \vec{b} = (1, 2, 2)$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution

$$\vec{a} \cdot \vec{b} = \sqrt{3}$$

or, $ab\cos\theta = \sqrt{3}$ ... (i)

Also, $\vec{a} \times \vec{b} = (1, 2, 2)$

$|\vec{a} \times \vec{b}| = |(1, 2, 2,)|$

$ab\sin\theta = 3$ ... (ii)

Dividing equation (ii) by equation (i)

$$\frac{ab\sin\theta}{ab\cos\theta} = \frac{3}{\sqrt{3}}$$

or, $\tan\theta = \sqrt{3} = \tan 60°$

$\therefore$ $\theta = 60°$.

16. $\vec{a} + \vec{b} + \vec{c} = 0$, show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

Solution

Given, $\vec{a} + \vec{b} + \vec{c} = 0$ ... (i)

From (i)

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

or, $0 + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$ $\quad (\because \vec{a} \times \vec{a} = 0)$

or, $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(ii)

Again, from (i)

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{b} + \vec{c} \times \vec{b} = 0$$

or, $\vec{a} \times \vec{b} + 0 = -\vec{c} \times \vec{b}$

or, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ...(iii)

From (ii) and (iii),

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

17. **By vector method, prove in any ΔABC that**

(a) $c = a \cos B + b \cos A$ (b) $c^2 = a^2 + b^2 - 2ab \cos C$

Solution

(a) In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{b}$
By the definition of vector addition,

$$\vec{AB} = \vec{AC} + \vec{BC}$$

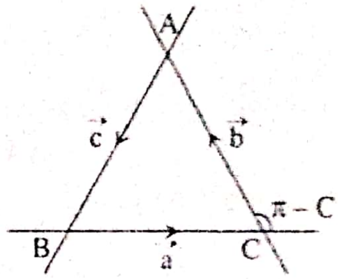
or, $\vec{AB} = -\vec{CA} - \vec{BC}$

or, $\vec{c} = -\vec{a}$

Multiplying each term scalarly by $\vec{c}$, we have

$$\vec{c}\,.\,\vec{c} = -\vec{b}\,.\,\vec{c} - \vec{a}\,.\,\vec{c}$$

or, $c^2 = -bc \cos(\pi - A) - ac \cos(\pi - B)$
or, $c^2 = -bc(-\cos A) - ac(-\cos B)$
or, $c^2 = bc \cos A + ac \cos B$
$\therefore$ $c = b \cos A + a \cos B.$



(b) In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$.
By definition of vector addition,

$$AB = \vec{AC} + \vec{CB}$$

or, $\vec{c} = -\vec{b} - \vec{a}$

or, $\vec{c}^2 = (-\vec{b} - \vec{a})^2$

$= (\vec{a} + \vec{b})^2 = a^2 + 2\vec{a} \cdot \vec{b} + b^2$

$= a^2 + b^2 + 2ab \cos(\pi - C)$

$\therefore$ $c^2 = a^2 + b^2 - 2ab \cos C.$

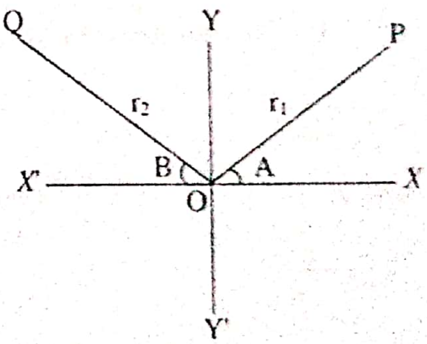
18. **Using vector method, prove that**

(a) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(b) $\sin(A - B) = \sin A \cos B - \cos A \sin B.$

Solution

(a) Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let $\angle XOP = A$ and $\angle QOX' = B$ so that $\angle POQ = \pi - (A + B)$. Also, let $OP = r_1$ and $OQ = r_2$. Then the coordinates of P and Q are $(r_1 \cos A, r_1 \sin A)$ and $(r_2 \cos(\pi - B), r_2 \sin(\pi - B)) = (-r_2 \cos B, r_2 \sin B)$.



So $\vec{OP} = (r_1 \cos A, r_1 \sin A)$

& $\vec{OQ} = (-r_2 \cos B, r_2 \sin B)$

Now, $\vec{OP}\,.\,\vec{OQ} = (r_1 \cos A, r_1 \sin A).(-r_2 \cos B, r_2 \sin B)$

$= -r_1 r_2 \cos A \cos B + r_1 r_2 \sin A \sin B$

$= -r_1 r_2 (\cos A \cos B - \sin A \sin B)$

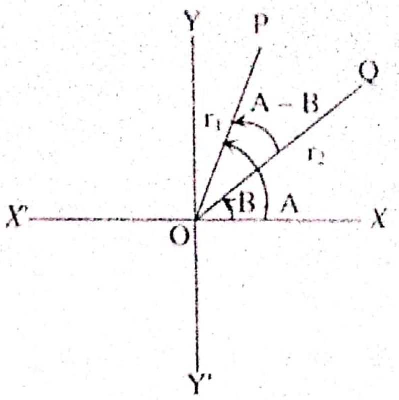
Since $\pi - (A + B)$ is the angle between $\vec{OP}$ and $\vec{OQ}$, so

$$\cos[\pi - (A + B)] = \frac{\vec{OP}.\vec{OQ}}{|\vec{OP}||\vec{OQ}|}$$

or, $-\cos(A + B) = \dfrac{-r_1 r_2 (\cos A \cos B - \sin A \sin B)}{r_1 r_2}$

$\therefore$ $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(b) Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let $\angle XOQ = B$ and $\angle XOP = A$ so that $\angle QOP = A - B$. Again, let $OP = r_1$ and $OQ = r_2$. Then the coordinates of P and Q are $(r_1 \cos A, r_1 \sin A)$ and $(r_2 \cos B, r_2 \sin B)$ respectively. So,



$$\vec{OP} = (r_1 \cos A, r_1 \sin A) = (r_1 \cos A, r_1 \sin A, 0)$$

$$\vec{OQ} = (r_2 \cos B, r_2 \sin B) = (r_2 \cos B, r_2 \sin B, 0)$$

Now,

$r_2 \cos B$	$r_2 \sin B$	0	$r_2 \cos B$	$r_2 \sin B$
$r_1 \cos A$	$r_1 \sin A$	0	$r_1 \cos A$	$r_1 \sin A$

$$\vec{OQ} \times \vec{OP} = (0, 0, r_1 r_2 \sin A \cos B - r_1 r_2 \cos A \sin B)$$

$$|\vec{OQ} \times \vec{OP}| = r_1 r_2 (\sin A \cos B - \cos A \sin B)$$

Since $(A - B)$ is the angle between OQ and OP, so

$$\sin (A - B) = \frac{|\vec{OQ} \times \vec{OP}|}{|\vec{OQ}||\vec{OP}|}$$

$$= \frac{r_1 r_2 (\sin A \cos B - \cos A \sin B)}{r_1 r_2} = \sin A \cos B - \cos A \sin B$$

19. Find $[\vec{a}\ \vec{b}\ \vec{c}]$.

(a) $\vec{a} = 3\vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = 2\vec{j} - 3\vec{k}$.

(b) $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{b} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + \vec{j} + \vec{k}$.

Solution

(a) $$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & -3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 3(-3-2) - 1(-3-2) = -15 + 5 = -10$$

(b) $$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(-1-1) - 2(1-1) - 1(1+1) = -2 - 0 - 2 = -4$$

20. $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$, find $\vec{a} \times (\vec{b} \times \vec{c})$ and verify that find $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

Solution

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (-1-2)\vec{i} - (-2-1)\vec{j} + (4-1)\vec{k} = -3\vec{i} + 3\vec{j} + 3\vec{k}$$

$$\therefore \quad \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 1 \\ 3 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix}$$

$$= (-6-3)\vec{i} - (3+3)\vec{j} + (3-6)\vec{k}$$

$$= -9\vec{i} - 6\vec{j} - 3\vec{k} \qquad \ldots \text{(i)}$$

Again, $\vec{a}\cdot\vec{c} = (\vec{i} - 2\vec{j} + \vec{k})\cdot(\vec{i} + 2\vec{j} - \vec{k}) = 1 - 4 - 1 = -4$

$\vec{a}\cdot\vec{b} = (\vec{i} - 2\vec{j} + \vec{k})\cdot(2\vec{i} + \vec{j} + \vec{k}) = 2 - 2 + 1 = 1$

$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -4(2\vec{i} + \vec{j} + \vec{k}) - 1(\vec{i} + 2\vec{j} - \vec{k})$

$= -8\vec{i} - 4\vec{j} - 4\vec{k} - \vec{i} - 2\vec{j} + \vec{k} = -9\vec{i} - 6\vec{j} - 3\vec{k}$

Hence, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ verified

21. If $\vec{a} = (1, 0, 1)$, $\vec{b} = (2, 1, -1)$, and $\vec{c} = (0, 1, 3)$, show that $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$.

Solution

$\vec{a} = (1, 0, 1) = \vec{i} + \vec{k}$

$\vec{b} = (2, 1, -1) = 2\vec{i} + \vec{j} - \vec{k}$

$\vec{c} = (0, 1, 3) = \vec{j} + 3\vec{k}$

Now, $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= (3 + 1)\vec{i} - (6 - 0)\vec{j} + (2 - 0)\vec{k} = 4\vec{i} - 6\vec{j} + 2\vec{k}.$

Again, $\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 4 & -6 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ -6 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 4 & -6 \end{vmatrix}$

$= (0 + 6)\vec{i} - (2 - 4)\vec{j} + (-6 - 0)\vec{k}$

$= 6\vec{i} + 2\vec{j} - 6\vec{k}$ ... (i)

Next, $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$

$= (0 - 1)\vec{i} - (-1 - 2)\vec{j} + (1 - 0)\vec{k} = -\vec{i} + 3\vec{j} + \vec{k}$

$\therefore$ $\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}$

$= (9 - 1)\vec{i} - (-3 - 0)\vec{j} + (-1 - 0)\vec{k}$

$= 8\vec{i} + 3\vec{j} - \vec{k}$ ... (ii)

From (i) and (ii), we get that

$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

22. **Find the volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ where $\vec{a} = (6, 3, -1)$, $\vec{b} = (0, 1, 2)$, $\vec{c} = (4, -2, 5)$.**

Solution

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix}$$

$$= 6(5 + 4) - 3(0 - 8) - 1(0 - 4) = 54 + 24 + 4 = 82$$

Volume of parallelopiped = 82 cubic units.

23. **Find the volume of the parallelepiped with adjacent edges PQ, PR and PS, where P(–2, 1, 0), Q(2, 3, 4), R(1, 4, –1), S(3, 6, 1).**

Solution

$$\vec{PQ} = (2 + 2)\vec{i} + (3 - 1)\vec{j} + (4 - 0)\vec{k} = 4\vec{i} + 2\vec{j} + 4\vec{k}$$

$$\vec{PR} = (1 + 2)\vec{i} + (4 - 1)\vec{j} + (-1 - 0)\vec{k} = 3\vec{i} + 3\vec{j} - \vec{k}$$

$$\vec{PS} = (3 + 2)\vec{i} + (6 - 1)\vec{j} + (1 - 0)\vec{k} = 5\vec{i} + 5\vec{j} + \vec{k}$$

$$[\vec{PQ}\ \vec{PR}\ \vec{PS}] = \begin{vmatrix} 4 & 2 & 4 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix}$$

$$= 4(3 + 5) - 2(3 + 5) + 4(15 - 15) = 32 - 16 + 0 = 16$$

Volume of parallelopiped = 16 cubic units.

Objective Questions

1. **If $\vec{a} = 3\vec{i} + 2\vec{j} + 2\vec{k}$ then $|\vec{a}| =$**

(a) 1 (b) $2\sqrt{2}$ (c) 3 (d) $\sqrt{17}$

Ans: d

$$|\vec{a}| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$$

2. **If $\vec{a}$ is a non-zero vector then unit vector in the direction of $\vec{a}$ is**

(a) $\vec{a}$ (b) $\dfrac{\vec{a}}{|\vec{a}|}$ (c) $\vec{a}\,|\vec{a}|$ (d) $|\vec{a}|$

Ans: b

$$\frac{\vec{a}}{|\vec{a}|}$$

3. **Scalar projection of $\vec{b}$ on $\vec{a}$ =**

(a) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ (b) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ (c) $\dfrac{|\vec{a}|}{\vec{a}\cdot\vec{b}}$ (d) $\dfrac{|\vec{b}|}{\vec{a}\cdot\vec{b}}$

Ans: a

Formula

4. **Vector projection of (1, 1, 2) onto (–2, 3, 1) is**

(a) $\left(\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right)$ (b) $\left(-\frac{3}{7}, -\frac{9}{14}, \frac{3}{14}\right)$

(c) $\left(-\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right)$ (d) $\left(-\frac{3}{7}, -\frac{9}{14}, -\frac{3}{14}\right)$

Ans: c

Let, $\vec{a} = (1, 1, 2)$

$\vec{b} = (-2, 3, 1)$

Vector projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$

$$= \frac{-2+3+2}{(\sqrt{4+9+1})^2}(-2, 3, 1) = \frac{3}{14}(-2, 3, 1) = \left(-\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right)$$

5. If $\vec{a} = 3\vec{i} + \vec{k}$ and $\vec{b} = \lambda\vec{i} + \vec{j} + 3\vec{k}$ are orthogonal then $\lambda =$

(a) 0 (b) 1 (c) -1 (d) 2

Ans: c

$\vec{a} \cdot \vec{b} = 0$

$3\lambda + 0 + 3 = 0$

$\therefore \quad \lambda = -1$

6. If $\vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 2\vec{i} + 4\vec{j} - 4\vec{k}$ then $\vec{a} \cdot \vec{b} =$

(a) 14 (b) 18 (c) 10 (d) 6

Ans: b

$\vec{a} \cdot \vec{b} = (3)(2) + (2)(4) + (-1)(-4) = 6 + 8 + 4 = 18$

7. If $\vec{a} = \vec{i} + \vec{j} - 3\vec{k}$ and $\vec{b} = -\vec{i} + 3\vec{j} - \vec{k}$ then $\vec{a} \times \vec{b} =$

(a) $\vec{i} - 4\vec{j} - 4\vec{k}$ (b) $-8\vec{i} - 4\vec{j} - 4\vec{k}$

(c) $8\vec{i} + 4\vec{j} + 4\vec{k}$ (d) $8\vec{i} - 4\vec{j} + 4\vec{k}$

Ans: c

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 3 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}\vec{k}$$

$$= (-1+9)\vec{i} - (-1-3)\vec{j} + (3+1)\vec{k} = 8\vec{i} + 4\vec{j} + 4\vec{k}$$

8. If $\vec{a} = (1, 2, 3)$, $\vec{b} = (5, -2, 3)$ and $\vec{c} = (2, 4, 6)$ then $[\vec{a}\ \vec{b}\ \vec{c}] =$

(a) -4 (b) -2 (c) 3 (d) 0

Ans: d

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -2 & 3 \\ 2 & 4 & 6 \end{vmatrix} = 1\begin{vmatrix} -2 & 3 \\ 4 & 6 \end{vmatrix} - 2\begin{vmatrix} 5 & 3 \\ 2 & 6 \end{vmatrix} + 3\begin{vmatrix} 5 & -2 \\ 2 & 4 \end{vmatrix}$$

$= -24 - 48 + 72 = 0$

9. The vector triple product $\vec{a} \times (\vec{b} \times \vec{c})$ is given by

(a) $(\vec{a} \cdot \vec{c})\vec{b} + (\vec{a} \cdot \vec{b})\vec{c}$ (b) $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

(c) $(\vec{a} \times \vec{c}) \cdot \vec{b} + (\vec{a} \times \vec{b}) \cdot \vec{c}$ (d) $(\vec{a} \times \vec{c}) \cdot \vec{b} - (\vec{a} \times \vec{b}) \cdot \vec{c}$

Ans: b

Formula

Permutation and Combination

EXERCISE– 6 A

1. Evaluate the following:

(a) $\frac{10! - 9!}{9!}$ (b) ${}^{8}P_{8}$ (c) ${}^{14}P_{4} \div {}^{12}P_{2}$

Solution

a. $\frac{10! - 9!}{9!} = \frac{10 \times 9! - 9!}{9!} = \frac{9!\,(10-1)}{9!} = 9$

b. ${}^{8}P_{8} = \frac{8!}{(8-8)!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{0!} = 40320$

c. ${}^{14}P_4 \div {}^{12}P_2 = \frac{14!}{(14-4)!} \div \frac{12!}{(12-2)!} = \frac{14!}{10!} \times \frac{10!}{12!} = \frac{14!}{12!} = \frac{14 \times 13 \times 12!}{12!} = 14 \times 13 = 182$

2. (a) If ${}^{n}P_{2} = 20$, find the value of n.

(b) If ${}^{n}P_{5} : {}^{n}P_{3} = 2 : 1$, find the value of n.

Solution

(a) Given,

${}^{n}P_2 = 20$

or, $\frac{n!}{(n-2)!} = 20$

or, $\frac{n(n-1)(n-2)!}{(n-2)!} = 20$ $[\because n! = n(n-1)(n-2)!]$

or, $n(n-1) = 20$

or, $n^2 - n - 20 = 0$

or, $n^2 - 5n + 4n - 20 = 0$

or, $n(n-5) + 4(n-5) = 0$

or, $(n-5)(n+4) = 0$

Either $n - 5 = 0 \Rightarrow n = 5$

or, $n + 4 \Rightarrow n = -4$

Since n cannot be negative so, we reject $n = -4$

Hence $n = 5$

(b) Given,

${}^{n}P_5 : {}^{n}P_3 = 2:1$

or, $\frac{n!}{(n-5)!} : \frac{n!}{(n-3)!} = 2:1$

or, $\frac{n!}{(n-5)!} \times \frac{(n-3)!}{n!} = \frac{2}{1}$

or, $\frac{(n-3)!}{(n-5)!} = 2$

or, $\frac{(n-3)(n-4)(n-5)!}{(n-5)!} = 2$

or, $(n-3)(n-4) = 2$

or, $n^2 - 7n + 12 = 2$

or, $n^2 - 7n + 10 = 0$

or, $n^2 - 5n - 2n + 10 = 0$

or, $n(n-5) - 2(n-5) = 0$

or, $(n-5)(n-2) = 0$

$\therefore$ $n = 5$ or 2

Since n cannot be equal to 0, 1, 2, 3, 4 i.e. $n \nless 5$ because $n \geq r$ [i.e. $n \geq 5$]

$\therefore$ $n = 5$

3. (a) There are 7 entrance doors and 4 exit doors in a stadium. In how many ways can a person enter the stadium and exit?

(b) There are 8 local taxies plying between Dhulikhel and Banepa. In how many ways can Mr. Dinesh go from Dhulikhel to Banepa and return by a different taxies?

Solution

a. A person can enter the stadium in 7 different ways and can exit in 4 different ways. By basic principle of counting, total no. of ways = $7 \times 4 = 28$.

b. Mr. Dinesh can go from Dhulikhel to Banepa in 8 different ways and can return from Banepa to Dhulikhel in 7 different ways. By basic principle of counting, total no. of ways $= 8 \times 7 = 56$.

4. (a) How many ways can 6 passengers sit in a compartment having 10 vacant seats?

(b) If 4 people enter a bus in which there are 7 vacant seats, in how many ways can they take their seats?

Solution

a. Here, $n = 10$, $r = 6$

Total no. of arrangements $= {}^nP_r = {}^{10}P_6 = \frac{10!}{(10-6)!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!} = 151200$

b. Here, $n = 7$, $r = 4$

Total no. of arrangements $= {}^nP_r = {}^7P_4 = \frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 840$

5. How many numbers of plates of vehicles consisting of 4 different digits be made out of integers 4, 5, 6, 7, 8, 9?

Solution

Here, $n = 6$, $r = 4$

$\therefore$ No. of plates of vehicles with different 4 digits out of 6 integers

$\therefore$ $= {}^nP_r = {}^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$

6. How many numbers between 4000 and 5000 can be formed with the digits 2, 3, 4, 5, 6, 7; if no digits being repeated? How many of these numbers are divisible by 5?

Solution

The numbers formed must lie between 4000 and 5000 and so each numbers must contain 4 digits and each number must start from 4. So we fix the digit 4, in thousand's place. Then, $n = 6 - 1 = 5$ and $r = 4 - 1 = 3$. The number of 4 digits numbers between 4000 and 5000 $= {}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$

If these numbers must be divisible by 5, the digit in the unit's place must be 5. So we fix the digit 5 again.

Now, $n = 6 - 2 = 4$, $r = 4 - 2 = 2$

The required number of 4 digits $= {}^4P_2 = \frac{4!}{(4-2)!} = \frac{4 \times 3 \times 2!}{2!} = 12$

7. **In how many ways can 6 people be seated in a row of 6 seats so that two particular persons always come together?**

Solution

Since two people come together, so we consider two people as one. Then,

$n = 5, r = 5$

5 people in 5 seats can be arranged in ${}^5P_5$ ways.

i.e. ${}^5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 1 \times 2 \times 3 \times 4 \times 5 = 120$ ways

The two people can interchange their positions in 2 ways.

$\therefore$ Total no. of arrangements $= 120 \times 2 = 240$

8. **In how many ways can 5 girls and 3 boys stand in a row so that all the boys stand together?**

Solution

No. of girls = 5

No. of boys = 3

If all the three boys stand together we consider three boys as one. Then,

$n = 5 + 1 = 6$

$r = 6$

6 people can be arranged in a row in ${}^6P_6$ ways

$= \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1} = 720$

Again, the three boys can be arranged in ${}^3P_3$ ways

$= \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3 \times 2 \times 1}{1} = 6$ ways

$\therefore$ Total no. of arrangements $= 720 \times 6 = 4320$

9. **Find the number of arrangements that can be made out of the letters of the following words.**

(a) **EXAMINATION** (b) **KATHMANDU** (c) **NEPAL**

(d) **HUMANITIES** (e) **MATHEMATICS**

Solution

(a) EXAMINATION

n = no. of letters = 11

p = no. of A = 2

q = no. of I = 2

r = no. of N = 2

$\therefore$ Total no. of arrangement $= \frac{n!}{p!\,q!\,r!} = \frac{11!}{2!\,2!\,2!} = 4989600$

(b) KATHMANDU

n = no. of letters = 9

p = no. of A = 2

$\therefore$ Total no. of arrangements $= \frac{n!}{p!} = \frac{9!}{2!} = 181440$

(c) NEPAL

There are 5 letters in the word NEPAL. All the letters are different. So. $n = 5$

$\therefore$ Total no. of arrangement $= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

(d) HUMANITIES

There are 10 letters in the word HUMANITIES. Also there are 2 I's and rest are single.

$\therefore$ $n = 10, p = 2$

Total no. of arrangements $= \frac{n!}{p!} = \frac{10!}{2!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2!} = 1814400$

(e) MATHEMATICS

n = no. of letters = 11

p = no. of M = 2

q = no. of A = 2

r = no. of T = 2

$\therefore$ Total no. of arrangement $= \frac{n!}{p!\,q!\,r!} = \frac{11!}{2!\,2!\,2!} = 4989600$

10. There are 5 red balls, 4 white balls and 3 yellow balls. In how many ways can they be arranged in a row?

Solution

Total no. of balls, $n = 5 + 4 + 3 = 12$

No. of red balls, $p = 5$

No. of white balls, $q = 4$

No. of yellow balls, $r = 3$

$\therefore$ Total no. of arrangements $= \frac{n!}{p!\,q!\,r!} = \frac{12!}{5!\,4!\,3!} = 27720$

11. (a) In how many ways can 10 girls be arranged at a round table?

(b) In how many ways can 7 boys be arranged at a round table so that two particular boys can be together?

(c) In how many ways 4 BCA students and 4 BBA students be arranged alternatively at a round table?

Solution

a. The arrangement of 10 girls at a round table $= (10 - 1)! = 9! = 362880$ ways

b. Here, n = no. of boys = 7

If two particular boys be together then $n = 7 - 1 = 6$

The arrangements of 6 boys at a round table $= (n - 1)! = 5! = 120$ ways

c. 4 BCA students at a round table can be arranged in $(4 - 1)! = 3!$ ways $= 3 \times 2 \times 1 = 6$ ways.

Since BCA and BBA students must be arranged alternatively, so 4 BBA students in 4 seats can be arranged in $P(4, 4)$ ways $= 4!$ ways $= 4 \times 3 \times 2 \times 1 = 24$ ways

Total number of arrangements $= 6 \times 24 = 144$.

12. In how many ways can 10 different beads form a bracelet?

Solution

Here, $n = 10$

Total no. of arrangements $= \frac{(n-1)!}{2} = \frac{(10-1)!}{2} = \frac{9!}{2!}$

$= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1!}{2!} = 181440$

13. (a) In how many ways can 3 letters be posted in 5 letter boxes?

(b) How many 3 digits numbers can be formed by using the digits 3, 4, 5, 6 when each digits may be repeated any number of times?

(c) There are 3 prizes to be distributed among 5 BCA students. In how many ways can it be done if:

(i) no students gets more than one prize?

(ii) a student may get any number of prizes?

(iii) no students get all the prizes?

Solution

a. The first letter can be posted in 5 ways.

Similarly, second letter also can be posted in 5 ways and the third letter can be posted in 5 ways.

Total no. of ways = $5 \times 5 \times 5 = 125$

b. Given digits are 3, 4, 5, 6.

$\therefore \quad n = 4$

For 3-digit numbers, we have $r = 3$.

Since the repetition of digits are allowed, so total no. of permutations = $n^r = 4^3 = 64$.

c. i. The first prize can be given in 5 different ways.

Since, no student gets more than one prize, so the second prize can be distributed in 4 different ways.

Similarly, the third prize can be distributed in 3 different ways.

Total no. of ways = $5 \times 4 \times 3 = 60$

ii. The first prize can be distributed in 5 ways because it may be given to any one of 5 students.

Since, a student may get more than one prize, so second prize also can be distributed in 5 ways and third prize also can be distributed in 5 ways.

Total no. of ways = $5 \times 5 \times 5 = 125$

iii. The no. of ways in which a student gets all the 3 prizes is 5 since any one of 5 students may get all prizes.

$\therefore$ Total no. of ways in which a student does not get all prizes = $125 - 5 = 120$.

14. In how many ways can the letters of the word ARRANGE be arranged so that no two R's come together?

Solution

Given word is 'ARRANGE'

Total number of letters $(n) = 7$

Number of A's $(p) = 2$

Number of R's $(q) = 2$

And the rest are single.

$$\text{Total number of arrangements} = \frac{n!}{p!\,q!} = \frac{7!}{2!\,2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2! \times 2 \times 1} = 1260$$

To find the number of arrangements in which no two 'R' come together, we first find the number of arrangements in which two R's come together. For this, consider two R's as a single letter, then the number of letters will be 6.

$$\text{Total number of arrangements in which two R's come together} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!}$$

$$= 360$$

$\therefore$ Required number of arrangements in which no two 'R' come together = $1260 - 360$
$= 900$.

15. In how many ways can the letter of the word 'SUNDAY' be arranged? How many of these arrangements do not begin with S? How many begin with S and do not end with Y?

Solution

There are 6 letters in the word 'SUNDAY'. Total number of arrangements

$= P(6, 6)$ ways $= 6!$ ways $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways

To find the number of arrangements which do not begin with 'S', we first find the number of arrangements that begin with 'S'. For, fix 'S' at first place, then remaining 5 letters can be arranged in $= P(5, 5)$ ways $= 5!$ ways $= 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

The number of arrangements that do not begin with 'S' $= 720 - 120 = 600$

Again, fix 'S' at first place and 'Y' at last, then remaining 4 letters can be arranged in $P(4,4)$ ways $= 4!$ ways $= 4 \times 3 \times 2 \times 1 = 24$ ways

$\therefore$ Total number of arrangements that begin with S and do not end with Y $= 120 - 24$

$= 96$

16. In how many ways can the letters of the word "TUESDAY" be arranged? How many of these arrangements do not begin with T? How many begin with T and do not end with Y?

Solution

There are 7 letters in the word 'TUESDAY'.

So total no. of arrangements $= P(7,7) = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

To find the arrangements that do not begin with 'T', we find the arrangements that begin with 'T'. For, fix 'T' at first place.

Then, remaining 6 letters can be arranged in $P(6, 6)$ ways $= 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

$\therefore$ Number of arrangements that do not begin with 'T' $= 5040 - 720 = 4320$

Again, fix 'T' at first place and 'Y' at last place. Then remaining 5 letters can be arranged in $P(5, 5)$ ways $= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

$\therefore$ The number of arrangements begin with 'T' and do not end with 'Y' $= 720 - 120$

$= 600$

17. In how many ways can the letters of the word "COMPUTER" be arranged so that:

(a) all vowels are always together?

(b) the relative positions of the vowels and consonants are not changed?

(c) the vowels may occupy only the odd positions?

Solution

There are 8 letters in the word 'COMPUTER'

a. There are 3 vowels in the word 'COMPUTER'. Consider 3 vowels as one letter. Then 6 letters namely C, M, P, T, R, (O, U, E) can be arranged in $P(6, 6)$ ways $= 6!$ ways

Also, 3 vowels among themselves can be arranged in $P(3, 3)$ ways $= 3!$ ways

$\therefore$ Total number of arrangements $= 6! \times 3!$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 = 4320$ ways

b. There are 3 vowels and 5 consonants in the word 'COMPUTER'.

The 3 vowels in 3 places can be arranged in $P(3, 3)$ ways $= 3!$

And 5 consonants in 5 places can be arranged in $P(5, 5)$ ways $= 5!$

$\therefore$ Total number of arrangements $= 3! \times 5!$

$= 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ words

c. If the three vowels O, U, E occupy only in odd positions namely 1^{st}, 3^{rd}, 5^{th} and 7^{th} i.e. 4 positions, then the total number of arrangements of 3 vowels in 4 places $= P(4, 3)$

Again, the remaining 5 letters in remaining 5 places can be arranged in $P(5,5)$ ways $= 5!$ ways

$\therefore$ Total number of arrangements $= P(4, 3) \times 5! = \frac{4!}{1!} \times 5!$

$= 4 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 = 2880$ ways

Objective Questions

1. $\dfrac{100!}{98!\ 2!} =$

(a) 9900 (b) 4950 (c) 3300 (d) 2475

Ans: b

$$\frac{100!}{98!\ 2!} = \frac{100 \times 99 \times 98!}{98! \times 2 \times 1} = 4950$$

2. $\dfrac{11! - 10!}{10} =$

(a) 10 (b) 10! (c) 11 (d) 11!

Ans: b

$$\frac{11! - 10!}{10} = \frac{11 \times 10! - 10!}{10} = \frac{10!\,(11-1)}{10} = 10!$$

3. **Which of the following is true?**

(a) $(2 + 5)! = 2! + 5!$ (b) $(5 - 3)! = 5! - 3!$

(c) $(5 \times 3)! = 5! \times 3!$ (d) none

Ans: d

None

4. **How many different numbers of five digits can be formed with the digits 0, 1, 2, 3, 4?**

(a) 120 (b) 114 (c) 96 (d) 60

Ans: c

$5! - 4! = 96$

5. **The no. of ways in which 4 boys and 3 girls can be seated in a row if they sit alternatively?**

(a) 12! (b) 7! (c) $4! \times 3!$ (d) $4! + 3!$

Ans: c

B G B G B G B

4 boys in 4 seats can be arranged in 4! ways

3 girls in 3 seats can be arranged in 3! ways.

Total no. of arrangement $= 4! \times 3!$

6. **The number of ways can the letters of the word ELEMENT be arranged?**

(a) 420 (b) 840 (c) 210 (d) 1680

Ans: b

There are 7 letters in the word ELEMENT. Also there are 3 E's and rest are single.

$\therefore \quad n = 7, p = 3$

Total no. of arrangements $= \dfrac{n!}{p!} = \dfrac{7!}{3!} = \dfrac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 840$

7. **The number of 6 digit numbers that can be formed form 2, 2, 2, 5, 5, 0?**

(a) 30 (b) 40 (c) 60 (d) 50

Ans: d

$$\frac{6!}{3!\,2!} - \frac{5!}{3!\,2!} = 50$$

8. **In how many ways can 6 BCA students be seated in a round table?**

(a) 720 (b) 360 (c) 120 (d) 60

Ans: c

Total no. of arrangement $= (6-1)! = 5! = 120$

9. The number of ways in which 6 different beads be strung on a necklace?

(a) 120 (b) 60 (c) 30 (d) 15

Ans: b

Total no. of arrangements $= \frac{(6-1)!}{2} = \frac{5!}{2} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2} = 60$

10. In how many ways can 7 letters be posted in 3 letter boxes?

(a) 21 (b) 3^7 (c) 7^3 (d) 42

Ans: b

Total no. of arrangements $= 3^7$

EXERCISE– 6 B

1. **Find the value of:**

(a) $^7C_5$ (b) $^{100}C_2$ (c) $^{10}C_4 + {^{10}C_3}$

Solution

a. $^7C_5 = \frac{7!}{(7-5)!\,5!} = \frac{7!}{2!\,5!} = \frac{7 \times 6 \times 5!}{2 \times 1 \times 5!} = 21.$

b. $^{100}C_2 = \frac{100!}{2!\,(100-2)!} = \frac{100!}{2!\,98!} = \frac{100 \times 99 \times 98!}{98! \times 2!} = \frac{100 \times 99}{2 \times 1} = 4950$

c. $^{10}C_4 + {^{10}C_3} = \frac{10!}{4!\,(10-4)!} + \frac{10!}{3!\,(10-3)!} = \frac{10!}{4!\,6!} + \frac{10!}{3!\,7!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4! \times 6!} + \frac{10 \times 9 \times 8 \times 7!}{3!\,7!}$

$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} + \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 210 + 120 = 330$

Alternative,

$^{10}C_4 + {^{10}C_3} = {^{10+1}C_4}$ $[\because {^nC_r} + {^nC_{r-1}} = {^{n+1}C_r}]$

$= {^{11}C_4} = \frac{11!}{4!\,7!} = \frac{11 \times 10 \times 9 \times 8 \times 7!}{7! \times 4 \times 3 \times 2 \times 1} = 330$

2. **State the meaning of ^nP_r and ^nC_r. Give the relation between them. Find the value of $^3P_2 + {^7C_2}$.**

Solution

^nP_r means permutations (arrangements) of n objects taken r at a time which is given by the formula: $^nP_r = \frac{n!}{(n-r)!}$

^nC_r means combinations (selections) of n objects taken r at a time which is given by the formula: $^nC_r = \frac{n!}{(n-r)!\,r!}$

$^3P_2 + {^7C_2} = \frac{3!}{(3-2)!} + \frac{7!}{(7-2)!\,2!} = \frac{3!}{1!} + \frac{7!}{5!\,2!} = 3 \times 2 \times 1 + \frac{7 \times 6 \times 5!}{5! \times 2 \times 1} = 6 + \frac{42}{2} = 27$

3. **(a) If $^{18}C_r = {^{18}C_{r+2}}$ find the value of r.**

(b) If $^nP_r = 336$ and $^nC_r = 56$, find n and r.

Solution

(a) Given,

$^{18}C_r = {^{18}C_{r+2}}$

or, $\frac{18!}{r!\,(18-r)!} = \frac{18!}{(r+2)!\,(18-r-2)!}$

or, $\frac{1}{r!(18-r)!} = \frac{1}{(r+2)!(16-r)!}$

or, $(r+2)!(16-r)! = r!(18-r)!$

or, $(r+2)(r+1)r! \times (16-r)! = r!(18-r)(18-r-1)(18-r-2)!$

or, $(r+2)(r+1)(16-r)! = (18-r)(17-r)(16-r)!$

or, $r^2 + 3r + 2 = 306 - 35r + r^2$

or, $3r + 2 = 306 - 35r$

or, $3r + 35r = 306 - 2$

or, $38r = 304$

$\therefore$ $r = 8$

Alternatively,

Given,

$^{18}C_r = {}^{18}C_{r+2}$

$\therefore$ $r + r + 2 = 18$ [$\because$ If $^nC_x = {}^nC_y$ then either $x = y$ or $x + y = n$]

or, $2r = 16$

$\therefore$ $r = 8$

(b) Given,

$^nP_r = 336$

or, $\frac{n!}{(n-r)!} = 336$ ... (i)

and $^nC_r = 56$

or, $\frac{n!}{r!(n-r)!} = 56$

or, $\frac{336}{r!} = 56$ (Using (i))

or, $r! = \frac{336}{56}$

or, $r! = 6$

or, $r! = 1 \times 2 \times 3$

$\therefore$ $r! = 3!$

$\therefore$ $r = 3$

Again, from eq^n. (i)

$\frac{n!}{(n-r)!} = 336$

or, $\frac{n!}{(n-3)!} = 336$

or, $\frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 336$

or, $n(n-1)(n-2) = 6 \times 7 \times 8$

or, $n(n-1)(n-2) = 8(8-1)(8-2)$

$\Rightarrow$ $n = 8$

Hence, $n = 8$ and $r = 3$

4. (a) **In the question paper of an examination 12 questions were asked and the students had to give the answer of only 10 questions. In how many ways can the students select the questions?**

(b) **In an entrance test, 22 questions are set. In how many ways can you choose 18 questions to answer?**

(c) **How many different committees of 7 members may be formed from 8 Nepalese and 4 Chinese?**

Solution

(a) Here, n = 12, r = 10

$\therefore$ Required no. of ways $= {}^nC_r = {}^{12}C_{10} = \dfrac{12!}{2!\ 10!} = \dfrac{12 \times 11 \times 10!}{2 \times 1 \times 10!} = 66$

(b) Here, n = 22, r = 18

No. of selection of 18 questions out of 22 $= C(22, 18) = \dfrac{22!}{(22-18)!\ 18!}$

$= \dfrac{22 \times 21 \times 20 \times 19 \times 18!}{4 \times 3 \times 2 \times 1 \times 18!} = 7315$

(c) Total no. of members = 8 Nepalese + 4 Chinese = 12

No. of members in committees = 7

$\therefore$ Required no. of ways $= {}^nC_r = {}^{12}C_7 = \dfrac{12!}{7!\ 5!} = \dfrac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{7! \times 4 \times 3 \times 2 \times 1} = 792$

5. (a) A committee is to be chosen from 12 men and 8 women and is to consist of 3 men and 2 women. How many committees can be formed?

(b) In how many ways can a committee of 5 members be selected from 6 men and 5 women consisting of 3 men and 2 women?

(c) A bag contains 8 red balls and 5 blue balls. In how many ways can 3 red balls and 4 blue balls be drawn?

Solution

a. **Selection of men**

No. of selection of 3 men out of 12 $= {}^{12}C_3 = \dfrac{12!}{(12-3)!\ 3!} = \dfrac{12 \times 11 \times 10 \times 9!}{9! \times 3 \times 2 \times 1} = 220$

Selection of women

No. of selection of 2 women out of 8 $= {}^8C_2 = \dfrac{8!}{(8-2)! \times 2!} = \dfrac{8 \times 7 \times 6!}{6! \times 2!} = \dfrac{8 \times 7}{2 \times 1} = 28$

$\therefore$ Total no. of committees that can be formed = 220 × 28 = 6160

b. **Selection of men**

No. of selection of 3 men out of 6 $= {}^6C_3 = \dfrac{6!}{(6-3)!\ 3!} = \dfrac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \dfrac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$

Selection of women

No. of selection of 2 women out of 5 $= {}^5C_2 = \dfrac{5!}{(5-2)!\ 2!} = \dfrac{5 \times 4 \times 3!}{3! \times 2 \times 1} = 10$

$\therefore$ Total no. of committees that can be formed = 20 × 10 = 200

c. No. of red balls = 8

No. of blue balls = 5

Selection of 3 red balls and 4 blue balls $= {}^8C_3 \times {}^5C_4 = \dfrac{8!}{3!\ 5!} \times \dfrac{5!}{4!\ 1!} = \dfrac{6 \times 7 \times 8}{1 \times 2 \times 3} \times 5 = 280$

6. Ram has got 15 friends of whom 10 are relatives. In how many ways can he invite 9 guests so that 7 of them be his relatives?

Solution

Total no. of friends = 15

No. of relatives = 10

$\therefore$ No. of non–relatives = 15 − 10 = 5

Now,

7 relatives can be selected out of 10 relatives in ${}^{10}C_7$

i.e. ${}^{10}C_7 = \dfrac{10!}{7!\ 3!} = \dfrac{8 \times 9 \times 10}{1 \times 2 \times 3} = 120$ ways and

2 non–relatives can be selected out of 5 non–relatives in ${}^5C_2$

i.e. $\quad {}^5C_2 = \dfrac{5!}{2!\,3!} = \dfrac{4 \times 5}{1 \times 2} = 10$ ways

Hence, the no. of ways in which 9 guests can be invited including 7 relatives = 120 × 10 = 1200 ways

7. **From 6 men and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include 2 ladies?**

Solution

No. of men = 6

No. of ladies = 4

A committee of 5 members is to be formed

When two ladies are included, no. of men = 5 – 2 = 3.

2 ladies out of 4 can be selected in ${}^4C_2$ ways

i.e. $\quad {}^4C_2 = \dfrac{4!}{2!\,2!} = \dfrac{3 \times 4}{2} = 6$ ways

3 men out of 6 can be selected in ${}^6C_3$ ways

i.e. $\quad {}^6C_3 = \dfrac{6!}{3!\,3!} = \dfrac{4 \times 5 \times 6}{1 \times 2 \times 3} = 20$ ways

∴ $\quad$ Total no. of committees = 6 × 20 = 120

Alternatively,

Ladies (4)	Men (6)	Selection
2	3	${}^4C_2 \times {}^6C_3$

∴ $\quad$ Total no. of committees = ${}^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

8. **Mr. A has five friends. In how many ways can he invite one or more of them to a dinner?**

Solution

Mr. A may invite one or more friends by selecting either 1 friend or 2 friends or 3 friends or 4 friends or 5 friends out of 5.

∴ $\quad$ The required no. of ways = ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$$= \frac{5!}{1!\,4!} + \frac{5!}{2!\,3!} + \frac{5!}{3!\,2!} + \frac{5!}{4!\,1!} + \frac{5!}{5!\,0!}$$

$$= 5 + \frac{4 \times 5}{2} + \frac{4 \times 5}{2} + 5 + 1 = 5 + 10 + 10 + 6 = 31$$

9. **In an examination, a candidate has to pass in each of the five subjects. In how many ways can the candidate fail?**

Solution

A candidate fails in an examination if he fails either in 1 or 2 or 3 or 4 or 5 subjects.

Total no. of ways by which he fails = ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$$= \frac{5!}{4!\,1!} + \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} + \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!}$$

$$= 5 + 10 + 10 + 5 + 1 = 31$$

10. **In an examination question paper 9 questions were asked. In how many different ways can a student choose 5 questions to answer? If question number 1 is made compulsory, in how many ways can the student select to answer 5 questions in all?**

Solution

n = 9, r = 5

5 questions can be selected in ${}^9C_5$ ways

i.e. $\quad {}^9C_5 = \dfrac{9!}{5!\,4!} = \dfrac{6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4} = 126$ ways

If question no. 1 (one) is made compulsory, then

$n = 9 - 1 = 8, r = 5 - 1 = 4$

The remaining questions can be selected in $^8C_4$ ways

i.e. $^8C_4 = \frac{8!}{4!\,4!} = \frac{5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4} = 70$ ways

11. In how many ways can 4 students be selected out of 12 students if (a) 2 particular students are excluded? (b) 2 particular students are included?

Solution

(a) If 2 particular students are not included at all. Then,

$n = 12 - 2 = 10$

$r = 4$

$\therefore$ No. of selection $= {}^{10}C_4 = \frac{10!}{4!\,6!} = \frac{7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4} = 210$ ways

(b) If 2 particular students are always included. Then,

$n = 12 - 2 = 10$

$r = 4 - 2 = 2$

$\therefore$ No. of selection $= {}^{10}C_2 = \frac{10!}{2!\,8!} = \frac{9 \times 10}{2} = 45$

12. In how many ways a committee of 8 members be selected from 8 men and 6 ladies, if the committee is to include not more than three ladies?

Solution

The selection of the members in the committee can be made as follows

Men (8)	Ladies (6)	Selection
5	3	C(8, 5) × C(6, 3)
6	2	C(8, 6) × C(6, 2)
7	1	C(8, 7) × C(6, 1)
8	0	C(8, 8) × C(6, 0)

The required number of committees

$= C(8, 5) \times C(6, 3) + C(8, 6) \times C(6, 2) + C(8, 7) \times C(6, 1) + C(8, 8) \times C(6, 0)$

$= \frac{8!}{3!5!} \times \frac{6!}{3!3!} + \frac{8!}{2!6!} \times \frac{6!}{4!2!} + \frac{8!}{1!7!} \times \frac{6!}{5!1!} + \frac{8!}{0!8!} \times \frac{6!}{6!0!} = 1120 + 420 + 48 + 1 = 1589$

13. From 3 men and 7 women a committee of 5 is to be formed. In how many ways can this be done so as to include at least one man?

Solution

The selection of the members in the committee can be made as

Men (3)	Women (7)	Selection
1	4	C(3,1) × C(7,4)
2	3	C(3,2) × C(7,3)
3	2	C(3,3) × C(7,2)

$\therefore$ The required no. of committees

$= C(3, 1) \times C(7, 4) + C(3, 2) \times C(7, 3) + C(3, 3) \times C(7, 2)$

$= \frac{3!}{2!1!} \times \frac{7!}{3!4!} + \frac{3!}{1!2!} \times \frac{7!}{4!3!} + \frac{3!}{0!3!} \times \frac{7!}{5!2!} = 105 + 105 + 21 = 231$

14. An examination paper consisting of 10 questions is divided into two groups A and B. Group A contains 6 questions. In how many ways can an examinee attempt 7 questions selecting at least two questions from each group?

Solution

Total no. of questions = 10

No. of questions in group A = 6

No. of questions in group B = 10 – 6 = 4

The selection of questions from two groups can be made as follows:

Group A (6)	Group B (4)	Selection
5	2	${}^6C_5 \times {}^4C_2$
4	3	${}^6C_4 \times {}^4C_3$
3	4	${}^6C_3 \times {}^4C_4$

Total no. of selections

$$= {}^6C_5 \times {}^4C_2 + {}^6C_4 \times {}^4C_3 + {}^6C_3 \times {}^4C_4$$

$$= \frac{6!}{1!\,5!} \times \frac{4!}{2!\,2!} + \frac{6!}{2!\,4!} \times \frac{4!}{1!\,3!} + \frac{6!}{3!\,3!} \times \frac{4!}{0!\,4!} = 36 + 60 + 20 = 116.$$

Objective Questions

1. Which of the following is not true?

(a) ${}^nC_r = r! \cdot {}^nP_r$ (b) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(c) ${}^nC_0 = 1$ (d) ${}^nC_n = n$

Ans: d

Option d is wrong because ${}^nC_n = 1$

2. If ${}^nP_r = 336$ and ${}^nC_r = 56$ then $r =$

(a) 2 (b) 3 (c) 4 (d) 5

Ans: b

We know that,

$${}^nP_r = r! \cdot {}^nC_r$$

or, $336 = r! \cdot 56$

or, $r! = \frac{336}{56} = 6$

or, $r! = 3!$

$\therefore$ $r = 3$

3. If there are 10 persons in a party and each two of them shakes hands with each other, how many hand shakes happen in the party?

(a) 90 (b) 45 (c) 30 (d) 15

Ans: b

$n = 10$, $r = 2$

Total no. of hand shakes = ${}^{10}C_2 = 45$

4. A man has 4 friends. In how many ways can he invite one or more of them to a dinner?

(a) 16 (b) 15 (c) 12 (d) 4

Ans: b

Total no. of ways = ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$

5. **If ${}^9C_{2r} = {}^9C_{3r-1}$ then $r =$**

(a) 2, 4 (b) 1, 3 (c) 2, 3 (d) 1, 2

Ans: d

$${}^9C_{2r} = {}^9C_{3r-1}$$

Either $2r = 3r - 1$

or, $1 = r$

or, $2r + (3r - 1) = 9$

or, $5r = 10$

or, $r = 2$

$\therefore$ $r = 1, 2$

6. **From 10 persons, in how many ways can a committee of 4 be made whe particular person is always included?**

(a) 72 (b) 84 (c) 92 (d) 100

Ans: b

$n = 10 - 1 = 9$

$r = 4 - 1 = 3$

Total no. of committees $= {}^9C_3 = 84$

7. **How many different sums of money can be made from 4 coins of diffe denominations?**

(a) 4 (b) 8 (c) 15 (d) 16

Ans: c

Total no. of ways $= {}^4C_1 + {}^4C_2 + {}^4C^3 + {}^4C^4 = 15$

8. **In an examination, a candidate has to pass in each of the 5 subjects. In how m ways can the candidate fail?**

(a) 31 (b) 30 (c) 32 (d) 1

Ans: a

Total no. of ways $= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 31$