BCA
Fifth Semester
“ Computer Graphics and Animation“
Chapter: 2

Transformation

Changing Position, shape, size, or orientation of an object on display is known as transformation

Basic Transformation

- Basic transformation includes three transformations **Translation**, **Rotation**, and **Scaling**.
- These three transformations are known as basic transformation because with combination of these three transformations we can obtain any transformation.

Translation

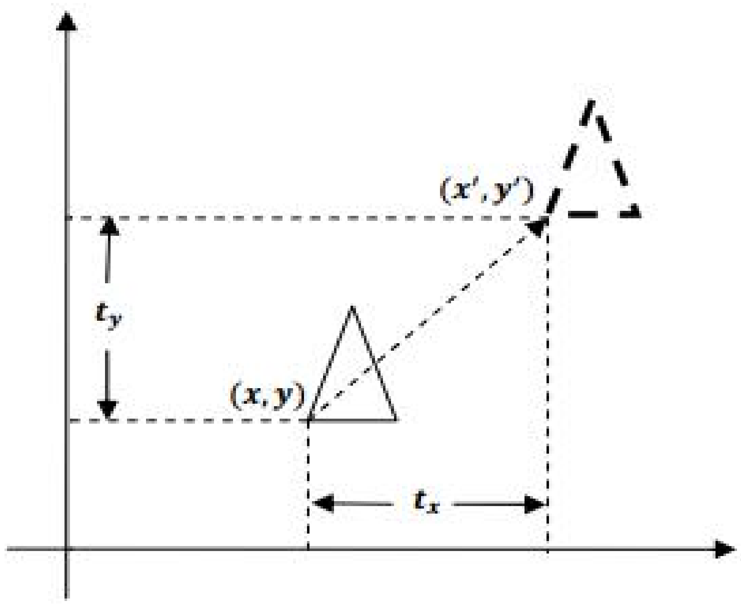


Fig. 3.1: - Translation.

- It is a transformation that used to reposition the object along the straight line path from one coordinate location to another.

- It is rigid body transformation so we need to translate whole object.
- We translate two dimensional point by adding translation distance t_x and t_y to the original coordinate position (x, y) to move at new position (x', y') as:

  $x' = x + t_x$ & $y' = y + t_y$
- Translation distance pair (t_x, t_y) is called a **Translation Vector** or **Shift Vector**.
- We can represent it into single matrix equation in column vector as;

  $P' = P + T$

  $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$
- We can also represent it in row vector form as:

  $P' = P + T$

  $$[x' \quad y'] = [x \quad y] + [t_x \quad t_y]$$
- Since column vector representation is standard mathematical notation and since many graphics package like **GKS** and **PHIGS** uses column vector we will also follow column vector representation.
- **Example**: - Translate the triangle [A (10, 10), B (15, 15), C(20, 10)] 2 unit in x direction and one unit in y direction.

  We know that

  $P' = P + T$

  $$P' = [P] + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

For point (10, 10)

$$A' = \begin{bmatrix} 10 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$$

For point (15, 15)

$$B' = \begin{bmatrix} 15 \\ 15 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} 17 \\ 16 \end{bmatrix}$$

For point (10, 10)

$$C' = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$C' = \begin{bmatrix} 22 \\ 11 \end{bmatrix}$$

- Final coordinates after translation are [A' (12, 11), B' (17, 16), C' (22, 11)].

Rotation

- It is a transformation that used to reposition the object along the circular path in the XY - plane.
- To generate a rotation we specify a rotation angle θ and the position of the **Rotation Point (Pivot Point)** (x_r, y_r) about which the object is to be rotated.
- Positive value of rotation angle defines counter clockwise rotation and negative value of rotation angle defines clockwise rotation.

- We first find the equation of rotation when pivot point is at coordinate origin$(\mathbf{0}, \mathbf{0})$.

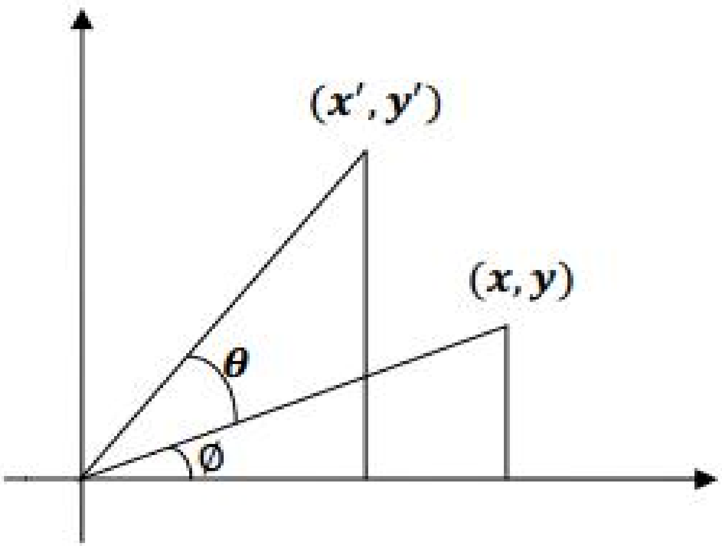


Fig. 3.2: - Rotation.

- From figure we can write.

  $x = r\cos\emptyset$

  $y = r\sin\emptyset$

  and

  $x' = r\cos(\theta + \emptyset) = r\cos\emptyset\cos\theta - r\sin\emptyset\sin\theta$

  $y' = r\sin(\emptyset + \theta) = r\cos\emptyset\sin\theta + r\sin\emptyset\cos\theta$

- Now replace $r\cos\emptyset$ with x and $r\sin\emptyset$ with y in above equation.

  $x' = x\cos\theta - y\sin\theta$

  $y' = x\sin\theta + y\cos\theta$

- We can write it in the form of column vector matrix equation as;

  $P' = R \cdot P$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- Rotation about arbitrary point is illustrated in below figure.

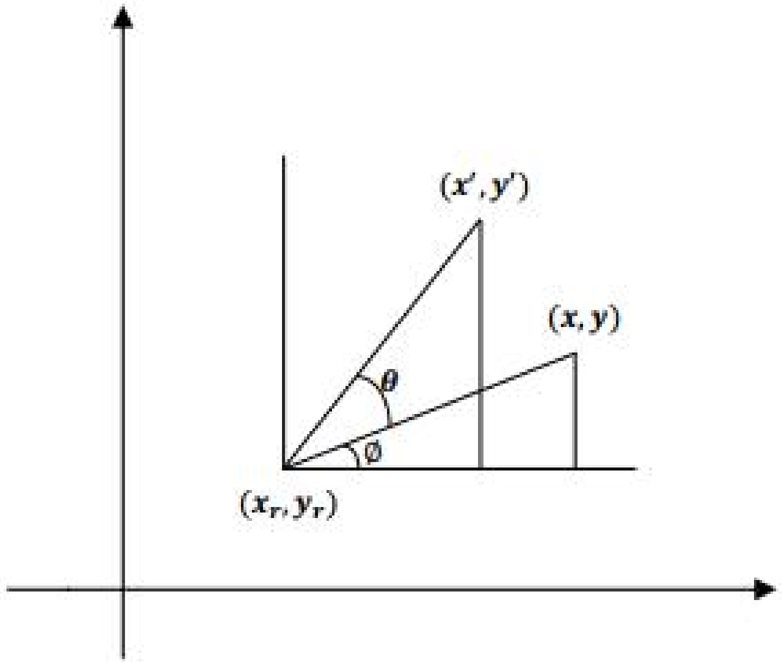


Fig. 3.3: - Rotation about pivot point.

- Transformation equation for rotation of a point about pivot point (x_r, y_r) is:

  $x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$

  $y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$

- These equations are differing from rotation about origin and its matrix representation is also different.
- Its matrix equation can be obtained by simple method that we will discuss later in this chapter.
- Rotation is also rigid body transformation so we need to rotate each point of object.

- **Example**: - Locate the new position of the triangle [A (5, 4), B (8, 3), C (8, 8)] after its rotation by 90° clockwise about the origin.

  As rotation is clockwise we will take $\theta = -90^{\circ}$.

  $P' = R \cdot P$

  $$P' = \begin{bmatrix} \cos(-90) & -\sin(-90) \\ \sin(-90) & \cos(-90) \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 4 & 3 & 8 \\ -5 & -8 & -8 \end{bmatrix}$$

- Final coordinates after rotation are [A′ (4, -5), B′ (3, -8), C′ (8, -8)].

Scaling

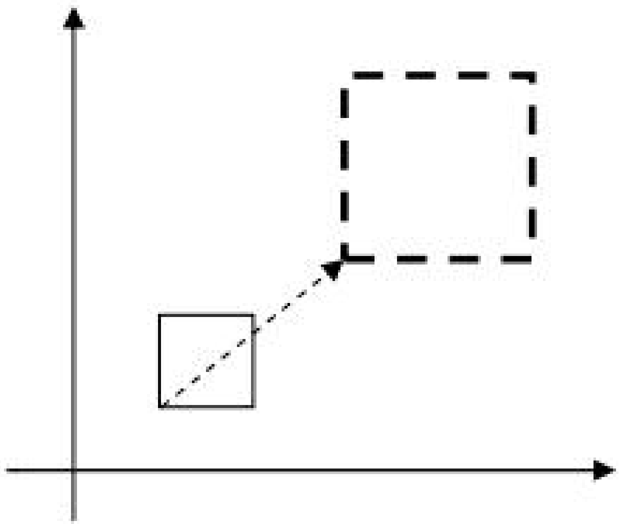


Fig. 3.4: - Scaling.

- It is a transformation that used to alter the size of an object.
- This operation is carried out by multiplying coordinate value (x, y) with scaling factor (s_x, s_y) respectively.
- So equation for scaling is given by:

  $x' = x \cdot s_x$

  $y' = y \cdot s_y$
- These equation can be represented in column vector matrix equation as:

  $P' = S \cdot P$

  $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$
- Any positive value can be assigned to (s_x, s_y).
- Values less than 1 reduce the size while values greater than 1 enlarge the size of object, and object remains unchanged when values of both factor is 1.
- Same values of s_x and s_y will produce **Uiform Scaling**. And different values of s_x and s_y will produce **Differential Scaling**.
- Objects transformed with above equation are both scale and repositioned.
- Scaling factor with value less than 1 will move object closer to origin, while scaling factor with value greater than 1 will move object away from origin.

- We can control the position of object after scaling by keeping one position fixed called **Fix point** (x_f, y_f) that point will remain unchanged after the scaling transformation.

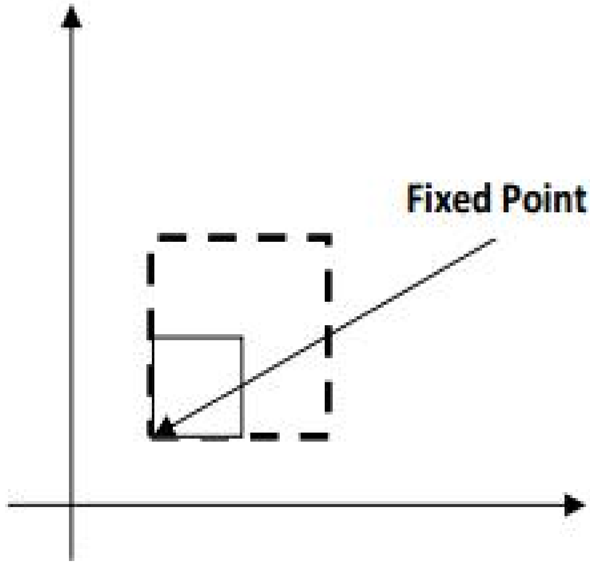


Fig. 3.5: - Fixed point scaling.

- Equation for scaling with fixed point position as (x_f, y_f) is:

$$x' = x_f + (x - x_f)s_x \qquad y' = y_f + (y - y_f)s_y$$
$$x' = x_f + xs_x - x_f s_x \qquad y' = y_f + ys_y - y_f s_y$$
$$x' = xs_x + x_f(1 - s_x) \qquad y' = ys_y + y_f(1 - s_y)$$

- Matrix equation for the same will discuss in later section.
- Polygons are scaled by applying scaling at coordinates and redrawing while other body like circle and ellipse will scaling using its defining parameters. For example ellipse will scale using its semi major axis, semi minor axis and center point scaling and redrawing at that position.

- **Example**: - Consider square with left-bottom corner at (2, 2) and right-top corner at (6, 6) apply the transformation which makes its size half.
  As we want size half so value of scale factor are $s_x = 0.5, s_y = 0.5$ and Coordinates of square are [A (2, 2), B (6, 2), C (6, 6), D (2, 6)].
  $P' = S \cdot P$

$$P' = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \end{bmatrix}$$
$$P' = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \end{bmatrix}$$
$$P' = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

- Final coordinate after scaling are [A' (1, 1), B' (3, 1), C' (3, 3), D' (1, 3)].

Matrix Representation and homogeneous coordinates

- Many graphics application involves sequence of geometric transformations.
- For example in design and picture construction application we perform Translation, Rotation, and scaling to fit the picture components into their proper positions.
- For efficient processing we will reformulate transformation sequences.
- We have matrix representation of basic transformation and we can express it in the general matrix form as:
  $\boldsymbol{P'} = \boldsymbol{M_1} \cdot \boldsymbol{P} + \boldsymbol{M_2}$
  Where $\boldsymbol{P}$ and $\boldsymbol{P'}$ are initial and final point position, $\boldsymbol{M_1}$ contains rotation and scaling terms and $\boldsymbol{M_2}$ contains translation al terms associated with pivot point, fixed point and reposition.
- For efficient utilization we must calculate all sequence of transformation in one step and for that reason we reformulate above equation to eliminate the matrix addition associated with translation terms in matrix $\boldsymbol{M_2}$.

- We can combine that thing by expanding 2X2 matrix representation into 3X3 matrices.
- It will allows us to convert all transformation into matrix multiplication but we need to represent vertex position (x, y) with homogeneous coordinate triple (x_h, y_h, h) Where $x = \frac{x_h}{h}$, $y = \frac{y_h}{h}$ thus we can also write triple as $(h \cdot x, h \cdot y, h)$.
- For two dimensional geometric transformation we can take value of h is any positive number so we can get infinite homogeneous representation for coordinate value (x, y).
- But convenient choice is set $h = 1$ as it is multiplicative identity, than (x, y) is represented as $(x, y, 1)$.
- Expressing coordinates in homogeneous coordinates form allows us to represent all geometric transformation equations as matrix multiplication.
- Let's see each representation with $h = 1$

Translation

$$P' = T_{(t_x, t_y)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

NOTE: - Inverse of translation matrix is obtain by putting $-t_x$ & $-t_y$ instead of t_x & t_y.

Rotation

$$P' = R_{(\theta)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

NOTE: - Inverse of rotation matrix is obtained by replacing θ by $-\theta$.

Scaling

$$P' = S_{(s_x, s_y)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

NOTE: - Inverse of scaling matrix is obtained by replacing $\boldsymbol{s_x}$ & $\boldsymbol{s_y}$ by $\frac{1}{s_x}$ & $\frac{1}{s_y}$ respectively.

Composite Transformation

- We can set up a matrix for any sequence of transformations as a **composite transformation matrix** by calculating the matrix product of individual transformation.
- For column matrix representation of coordinate positions, we form composite transformations by multiplying matrices in order from right to left.

Translations

- Two successive translations are performed as:

$$P' = T(t_{x2}, t_{y2}) \cdot \{T(t_{x1}, t_{y1}) \cdot P\}$$

$$P' = \{T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1})\} \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = T(t_{x1} + t_{x2}, t_{y1} + t_{y2}) \cdot P\}$$

Here $\boldsymbol{P'}$ and $\boldsymbol{P}$ are column vector of final and initial point coordinate respectively.

- This concept can be extended for any number of successive translations.

Rotations

- Two successive Rotations are performed as:

$$P' = R(\theta_2) \cdot \{R(\theta_1) \cdot P\}$$

$$P' = \{R(\theta_2) \cdot R(\theta_1)\} \cdot P$$

$$P' = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} \cos\theta_2 \cos\theta_1 - \sin\theta_2 \sin\theta_1 & -\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1 & 0 \\ \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1 & \cos\theta_2 \cos\theta_1 - \sin\theta_2 \sin\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = R(\theta_1 + \theta_2) \cdot P$$

Here P' and P are column vector of final and initial point coordinate respectively.

- This concept can be extended for any number of successive rotations.

Scaling

- Two successive scaling are performed as:

$$P' = S(s_{x2}, s_{y2}) \cdot \{S(s_{x1}, s_{y1}) \cdot P\}$$

$$P' = \{S(s_{x2}, s_{y2}) \cdot S(s_{x1}, s_{y1})\} \cdot P$$

$$P' = \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

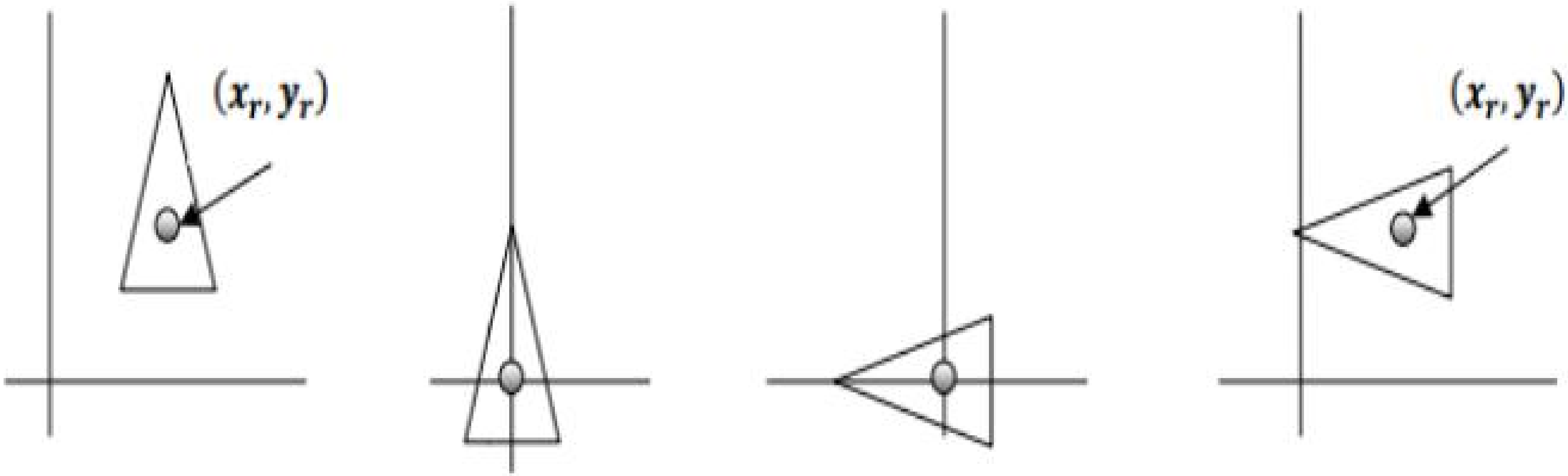
$$P' = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = S(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2}) \cdot P$$

Here P' and P are column vector of final and initial point coordinate respectively.

- This concept can be extended for any number of successive scaling.

General Pivot-Point Rotation



(a) Original Position of Object and Pivot Point.

(b) Translation of Object so that Pivot Point (x_r, y_r) is at Origin.

(c) Rotation about Origin.

(d) Translation of Object so that Pivot Point is Return to Position (x_r, y_r) .

Fig. 3.6: - General pivot point rotation.

- For rotating object about arbitrary point called pivot point we need to apply following sequence of transformation.
  1. Translate the object so that the pivot-point coincides with the coordinate origin.
  2. Rotate the object about the coordinate origin with specified angle.
  3. Translate the object so that the pivot-point is returned to its original position (i.e. Inverse of step-1).
- Let's find matrix equation for this

$$P' = T(x_r, y_r) \cdot [R(\theta) \cdot \{T(-x_r, -y_r) \cdot P\}]$$

$$P' = \{T(x_r, y_r) \cdot R(\theta) \cdot T(-x_r, -y_r)\} \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r \sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = R(x_r, y_r \theta) \cdot P$$

Here P' and P are column vector of final and initial point coordinate respectively and (x_r, y_r) are the coordinates of pivot-point.

- **Example**: - Locate the new position of the triangle [A (5, 4), B (8, 3), C (8, 8)] after its rotation by 90° clockwise about the centroid.

  Pivot point is centroid of the triangle so:

$$x_r = \frac{5+8+8}{3} = 7, \qquad y_r = \frac{4+3+8}{3} = 5$$

As rotation is clockwise we will take $\theta = -90^\circ$.

$$P' = R_{(x_r, y_r, \theta)} \cdot P$$

$$P' = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r \sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} \cos(-90) & -\sin(-90) & 7(1-\cos(-90)) + 5\sin(-90) \\ \sin(-90) & \cos(-90) & 5(1-\cos(-90)) - 7\sin(-90) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

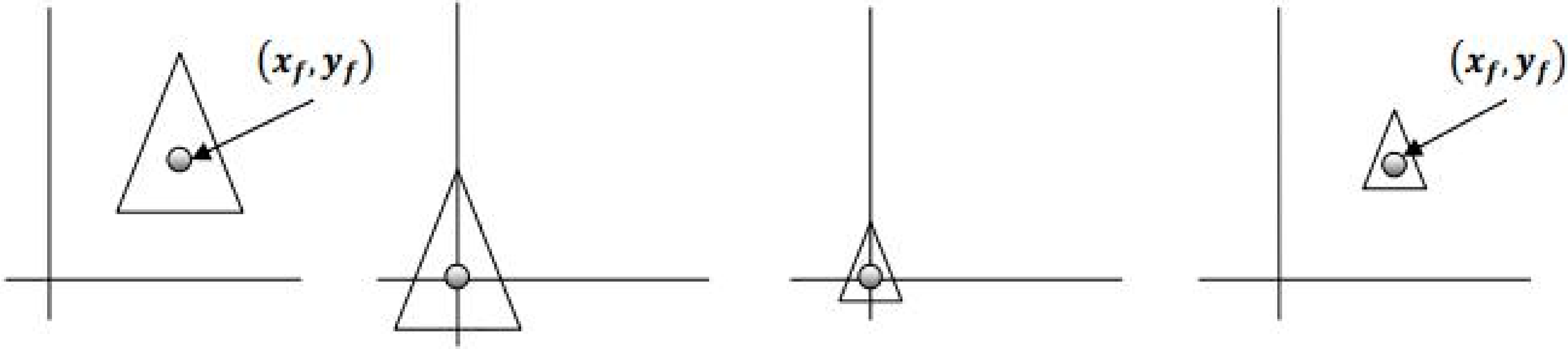
$$P' = \begin{bmatrix} 0 & 1 & 7(1-0)-5(1) \\ -1 & 0 & 5(1-0)+7(1) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 12 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 11 & 13 & 18 \\ 7 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

- Final coordinates after rotation are [A' (11, 7), B' (13, 4), C' (18, 4)].

General Fixed-Point Scaling



(a)
Original Position of Object and Fixed Point

(b)
Translate Object so that Fixed Point (x_f, y_f) is at Origin

(c)
Scale Object with Respect to Origin

(d)
Translate Object so that Fixed Point is Return to Position (x_f, y_f) .

Fig. 3.7: - General fixed point scaling.

- For scaling object with position of one point called fixed point will remains same, we need to apply following sequence of transformation.
  1. Translate the object so that the fixed-point coincides with the coordinate origin.
  2. Scale the object with respect to the coordinate origin with specified scale factors.
  3. Translate the object so that the fixed-point is returned to its original position (i.e. Inverse of step-1).
- Let's find matrix equation for this

$$P' = T(x_f, y_f) \cdot [S(s_x, s_y) \cdot \{T(-x_f, -y_f) \cdot P\}]$$

$$P' = \{T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f)\} \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} s_x & 0 & x_f(1 - s_x) \\ 0 & s_y & y_f(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = S(x_f, y_f, s_x, s_y) \cdot P$$

Here P' and P are column vector of final and initial point coordinate respectively and (x_f, y_f) are the coordinates of fixed-point.

- **Example**: - Consider square with left-bottom corner at (2, 2) and right-top corner at (6, 6) apply the transformation which makes its size half such that its center remains same.

Fixed point is center of square so:

$$x_f = 2 + \frac{6-2}{2}, \quad y_f = 2 + \frac{6-2}{2}$$

As we want size half so value of scale factor are $s_x = 0.5, s_y = 0.5$ and Coordinates of square are [A (2, 2), B (6, 2), C (6, 6), D (2, 6)].

$$P' = S(x_f, y_f, s_x, s_y) \cdot P$$

$$P' = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.5 & 0 & 4(1-0.5) \\ 0 & 0.5 & 4(1-0.5) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.5 & 0 & 2 \\ 0 & 0.5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 3 & 5 & 5 & 3 \\ 3 & 3 & 5 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- Final coordinate after scaling are [A' (3, 3), B' (5, 3), C' (5, 5), D' (3, 5)]

General Scaling Directions

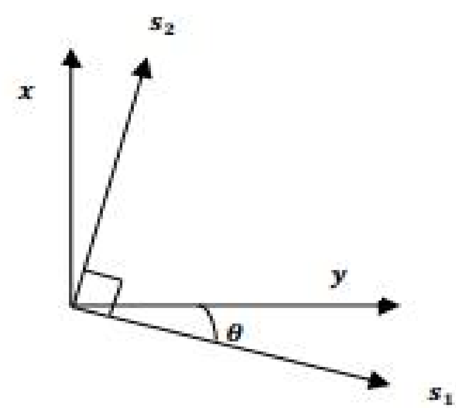


Fig. 3.8: - General scaling direction.

- Parameter s_x and s_y scale the object along x and y directions. We can scale an object in other directions by rotating the object to align the desired scaling directions with the coordinate axes before applying the scaling transformation.
- Suppose we apply scaling factor s_1 and s_2 in direction shown in figure than we will apply following transformations.
  1. Perform a rotation so that the direction for s_1 and s_2 coincide with x and y axes.
  2. Scale the object with specified scale factors.
  3. Perform opposite rotation to return points to their original orientations. (i.e. Inverse of step-1).
- Let's find matrix equation for this

$$P' = R^{-1}(\theta) \cdot [S(s_1, s_2) \cdot \{R(\theta) \cdot P\}]$$

$$P' = \{R^{-1}(\theta) \cdot S(s_1, s_2) \cdot R(\theta)\} \cdot P$$

$$P' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} s_1 \cos^2\theta + s_2 \sin^2\theta & (s_2 - s_1)\cos\theta\sin\theta & 0 \\ (s_2 - s_1)\cos\theta\sin\theta & s_1 \sin^2\theta + s_2 \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

Here P' and P are column vector of final and initial point coordinate respectively and θ is the angle between actual scaling direction and our standard coordinate axes.

Other Transformation

- Some package provides few additional transformations which are useful in certain applications. Two such transformations are reflection and shear.

Reflection

- A reflection is a transformation that produces a mirror image of an object.
- The mirror image for a two –dimensional reflection is generated relative to an **axis of reflection** by rotating the object 180° about the reflection axis.
- Reflection gives image based on position of axis of reflection. Transformation matrix for few positions are discussed here.

Transformation matrix for reflection about the line $y = 0$ *, the x axis.*

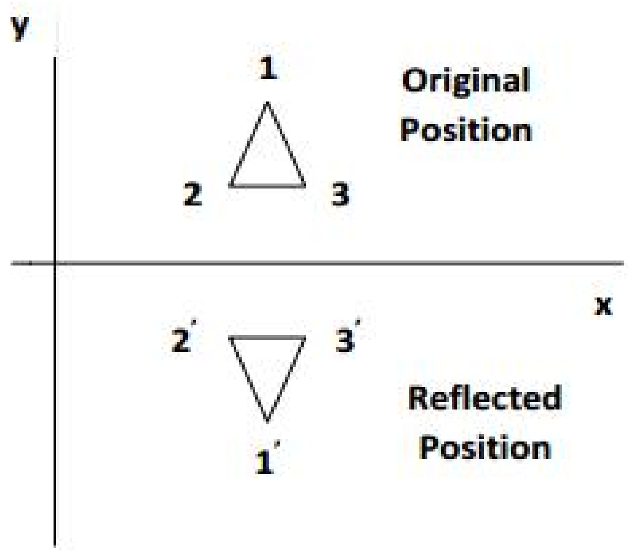


Fig. 3.9: - Reflection about x - axis.

- This transformation keeps x values are same, but flips (Change the sign) y values of coordinate positions.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Transformation matrix for reflection about the line $x = 0$, *the y axis*.</u>

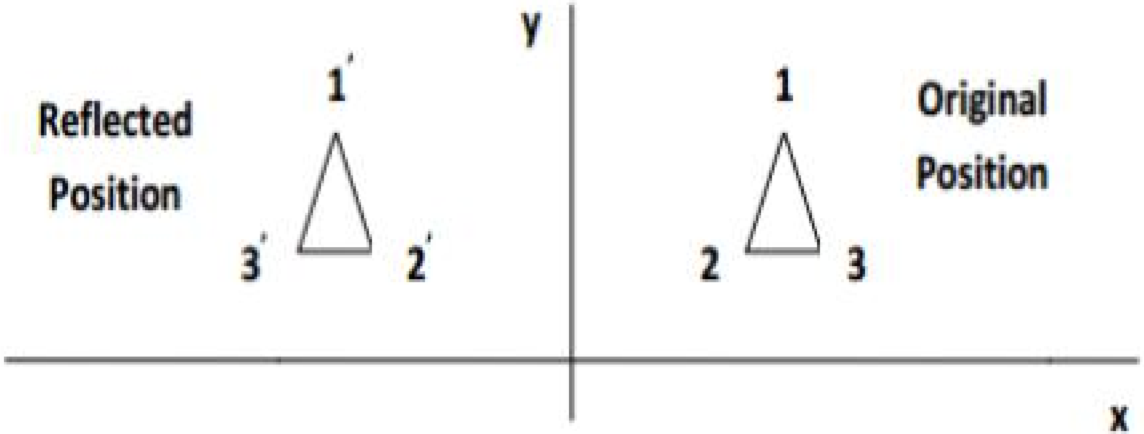


Fig. 3.10: - Reflection about y - axis.

- This transformation keeps y values are same, but flips (Change the sign) x values of coordinate positions.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Transformation matrix for reflection about the ***Origin***.</u>

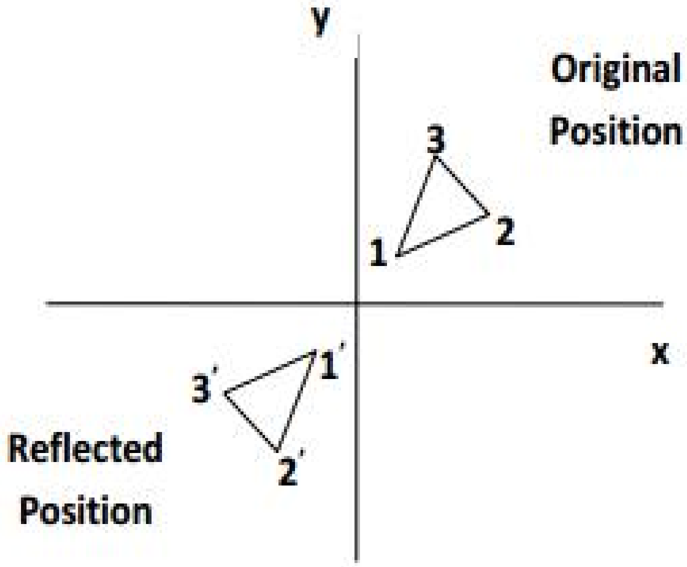


Fig. 3.11: - Reflection about origin.

- This transformation flips (Change the sign) x and y both values of coordinate positions.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation matrix for reflection about the line $x = y$.

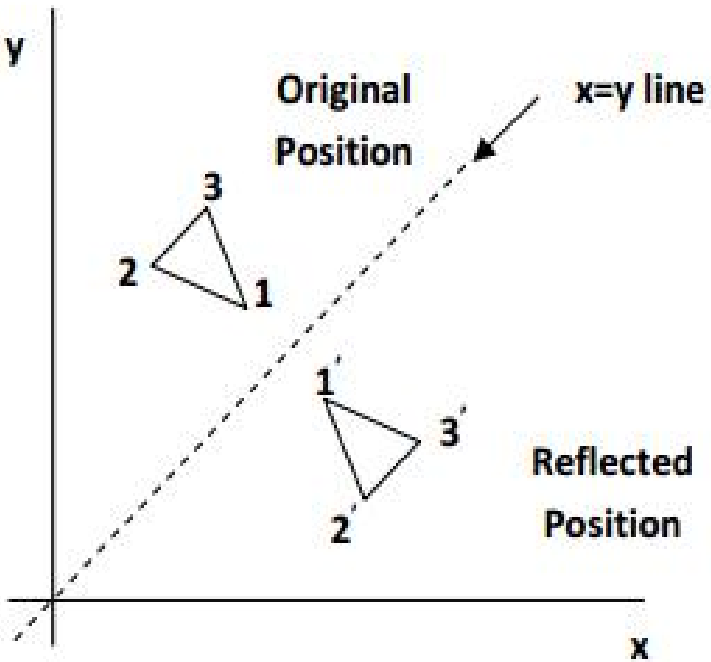


Fig. 3.12: - Reflection about x=y line.

- This transformation interchange x and y values of coordinate positions.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation matrix for reflection about the line $x = -y$.

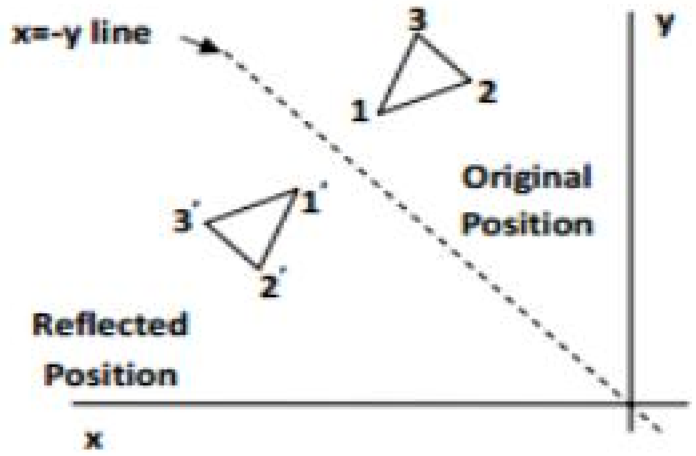


Fig. 3.12: - Reflection about x=-y line.

- This transformation interchange x and y values of coordinate positions.

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Example**: - Find the coordinates after reflection of the triangle [A (10, 10), B (15, 15), C (20, 10)] about x axis.

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 15 & 20 \\ 10 & 15 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 10 & 15 & 20 \\ -10 & -15 & -10 \\ 1 & 1 & 1 \end{bmatrix}$$

- Final coordinate after reflection are [A' (10, -10), B' (15, -15), C' (20, -10)]

Shear

- A transformation that distorts the shape of an object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other is called **shear**.
- Two common shearing transformations are those that shift coordinate x values and those that shift y values.

<u>Shear in $x - direction$.</u>

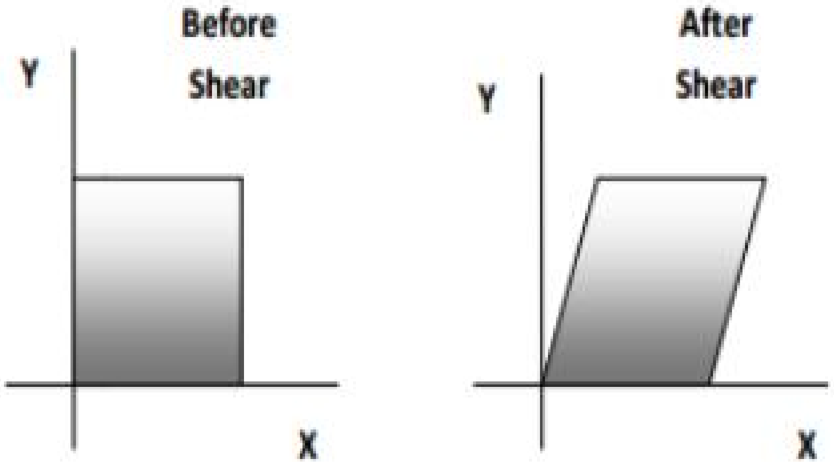


Fig. 3.13: - Shear in x-direction.

- Shear relative to $x - axis$ that is $y = 0$ line can be produced by following equation:

  $$x' = x + sh_x \cdot y, \qquad y' = y$$

- Transformation matrix for that is:

  $$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  Here sh_x is shear parameter. We can assign any real value to sh_x.

- We can generate $x - direction$ shear relative to other reference line $y = y_{ref}$ with following equation:

  $$x' = x + sh_x \cdot (y - y_{ref}), \qquad y' = y$$

- Transformation matrix for that is:

  $$\begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Example**: - Shear the unit square in x direction with shear parameter ½ relative to line $y = -1$.

  Here $y_{ref} = -1$ and $sh_x = 0.5$

  Coordinates of unit square are [A (0, 0), B (1, 0), C (1, 1), D (0, 1)].

  $$P' = \begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 1 & 0.5 & -0.5 \cdot (-1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 0.5 & 1.5 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- Final coordinate after shear are [A' (0.5, 0), B' (1.5, 0), C' (2, 1), D' (1, 1)]

Shear in $y - direction$.

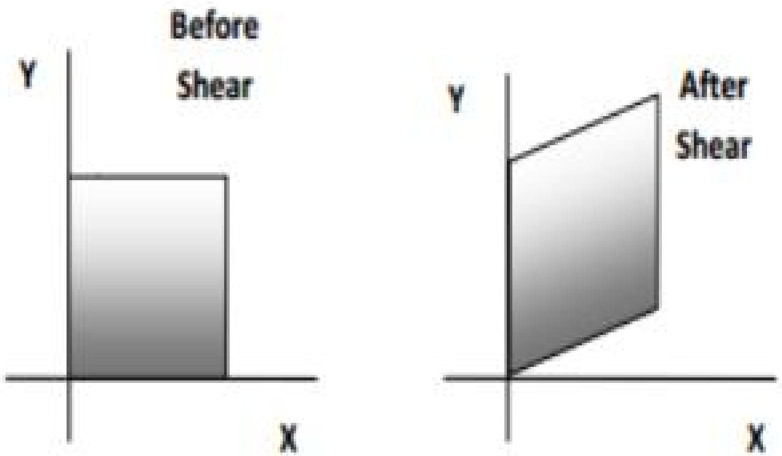


Fig. 3.14: - Shear in y-direction.

- Shear relative to $y - axis$ that is $x = 0$ line can be produced by following equation:

  $x' = x$, $\quad y' = y + sh_y \cdot x$

- Transformation matrix for that is:

  $$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  Here sh_y is shear parameter. We can assign any real value to sh_y.

- We can generate $y - direction$ shear relative to other reference line $x = x_{ref}$ with following equation:

  $$x' = x, \qquad y' = y + sh_y \cdot (x - x_{ref})$$

- Transformation matrix for that is:

  $$\begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

- **Example**: - Shear the unit square in y direction with shear parameter ½ relative to line $x = -1$.

  Here $x_{ref} = -1$ and $sh_y = 0.5$

  Coordinates of unit square are [A (0, 0), B (1, 0), C (1, 1), D (0, 1)].

  $$P' = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & -0.5 \cdot (-1) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

  $$P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0.5 & 1 & 2 & 1.5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- Final coordinate after shear are [A' (0, 0.5), B' (1, 1), C' (1, 2), D' (0, 1.5)]

3D Transformation

3D Translation

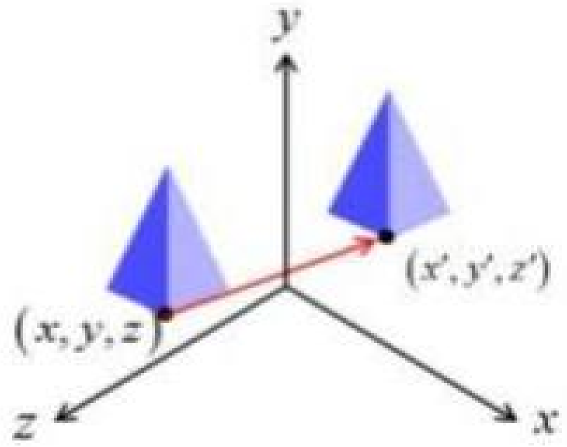


Fig. 5.1: - 3D Translation.

- Similar to 2D translation, which used 3x3 matrices, 3D translation use 4X4 matrices (X, Y, Z, h).
- In 3D translation point (X, Y, Z) is to be translated by amount tx, ty and tz to location (X', Y', Z').

  $x' = x + tx$

  $y' = y + ty$

  $z' = z + tz$

- Let's see matrix equation

  $P' = T \cdot P$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Example : - Translate the given point P (10,10,10) into 3D space with translation factor T (10,20,5).

$$P' = T \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 10 \\ 10 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 15 \\ 1 \end{bmatrix}$$

Final coordinate after translation is P′ (20, 30, 15).

Rotation

- For 3D rotation we need to pick an axis to rotate about.
- The most common choices are the X-axis, the Y-axis, and the Z-axis

Coordinate-Axes Rotations

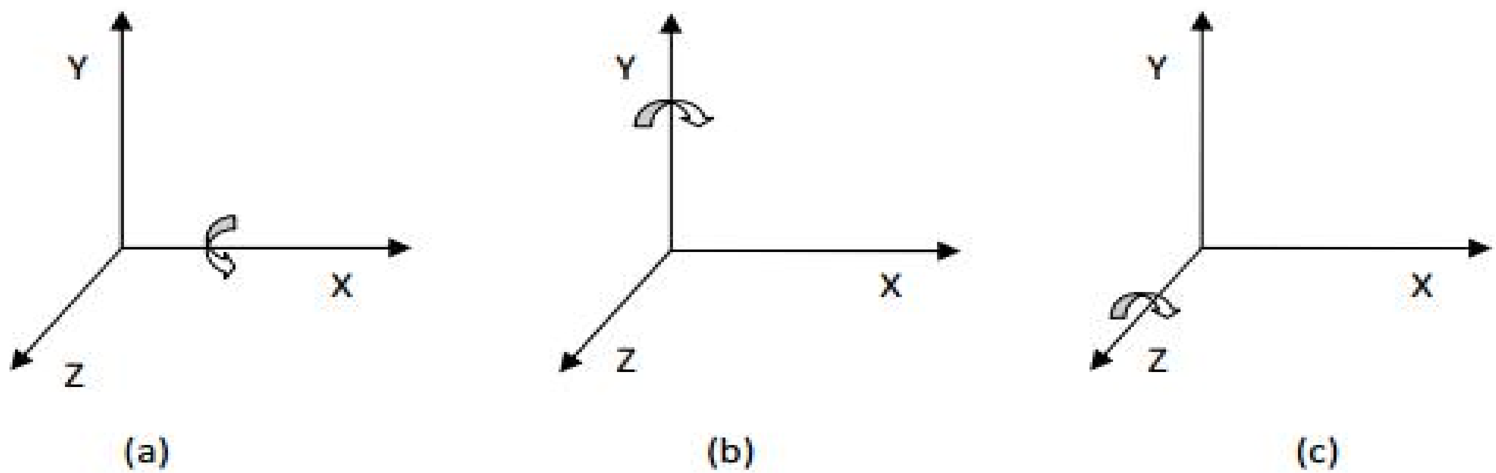


Fig. 5.2: - 3D Rotations.

Z-Axis Rotation

- Two dimension rotation equations can be easily convert into 3D Z-axis rotation equations.
- Rotation about z axis we leave z coordinate unchanged.

  $x' = x\cos\theta - y\sin\theta$

  $y' = x\sin\theta + y\cos\theta$

  $z' = z$

  Where Parameter θ specify rotation angle.
- Matrix equation is written as:

  $P' = R_z(\theta) \cdot P$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

X-Axis Rotation

- Transformation equation for x-axis is obtain from equation of z-axis rotation by replacing cyclically as shown here

  $x \rightarrow y \rightarrow z \rightarrow x$

- Rotation about x axis we leave x coordinate unchanged.

  $y' = y\cos\theta - z\sin\theta$

  $z' = y\sin\theta + z\cos\theta$

  $x' = x$

  Where Parameter θ specify rotation angle.

- Matrix equation is written as:

  $P' = R_x(\theta) \cdot P$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Y-Axis Rotation

- Transformation equation for y-axis is obtain from equation of x-axis rotation by replacing cyclically as shown here

  $x \rightarrow y \rightarrow z \rightarrow x$

- Rotation about y axis we leave y coordinate unchanged.

$$z' = z\cos\theta - x\sin\theta$$
$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$

Where Parameter θ specify rotation angle.

- Matrix equation is written as:

$$P' = R_y(\theta) \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Example: - Rotate the point P(5,5,5) 90° about Z axis.

$$P' = R_z(\theta) \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 90 & -\sin 90 & 0 & 0 \\ \sin 90 & \cos 90 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 5 \\ 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 5 \\ 1 \end{bmatrix}$$

Final coordinate after rotation is P' (-5, 5, 5).

General 3D Rotations when rotation axis is parallel to one of the standard axis

- Three steps require to complete such rotation
  1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.
  2. Perform the specified rotation about that axis.
  3. Translate the object so that the rotation axis is moved back to its original position.
- This can be represented in equation form as:

  $P' = T^{-1} \cdot R(\theta) \cdot T \cdot P$

General 3D Rotations when rotation axis is inclined in arbitrary direction

- When object is to be rotated about an axis that is not parallel to one of the coordinate axes, we need rotations to align the axis with a selected coordinate axis and to bring the axis back to its original orientation.
- Five steps require to complete such rotation.
  1. Translate the object so that the rotation axis passes through the coordinate origin.
  2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes.
  3. Perform the specified rotation about that coordinate axis.
  4. Apply inverse rotations to bring the rotation axis back to its original orientation.
  5. Apply the inverse translation to bring the rotation axis back to its original position.
- We can transform rotation axis onto any of the three coordinate axes. The Z-axis is a reasonable choice.

- We are given line in the form of two end points P1 (x1,y1,z1), and P2 (x2,y2,z2).
- We will see procedure step by step.

1) **Translate the object so that the rotation axis passes through the coordinate origin.**

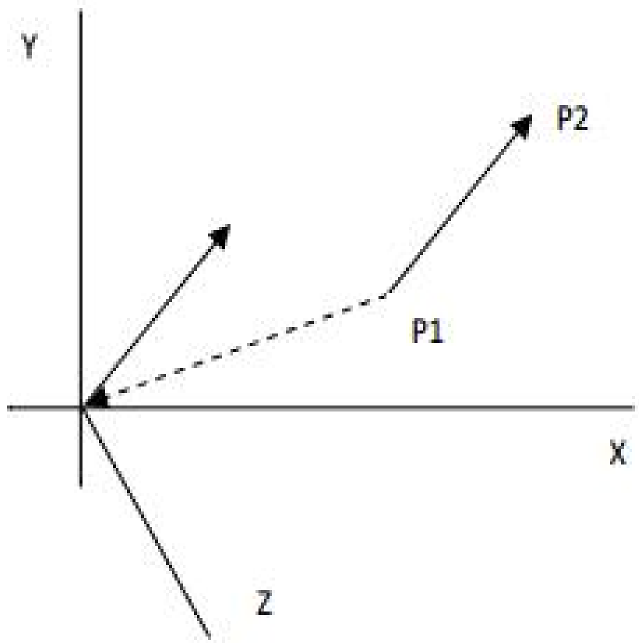


Fig. 5.3: - Translation of vector V.

- For translation of step one we will bring first end point at origin and transformation matrix for the same is as below

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) **Rotate the object so that the axis of rotation coincides with one of the coordinate axes.**

- This task can be completed by two rotations first rotation about x-axis and second rotation about y-axis.
- But here we do not know rotation angle so we will use dot product and vector product.
- Lets write rotation axis in vector form.

$$V = P_2 - P_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

- Unit vector along rotation axis is obtained by dividing vector by its magnitude.

$$u = \frac{V}{|V|} = \left(\frac{x_2 - x_1}{|V|}, \frac{y_2 - y_1}{|V|}, \frac{z_2 - z_1}{|V|}\right) = (a, b, c)$$

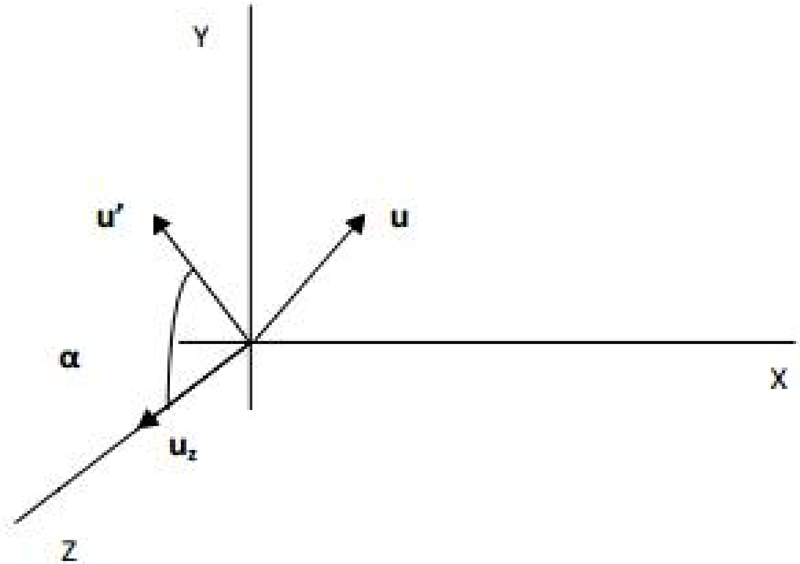


Fig. 5.4: - Projection of u on YZ-Plane.

- Now we need cosine and sin value of angle between unit vector '**u**' and XZ plane and for that we will take projection of u on YZ-plane say '**u'**' and then find dot product and cross product of '**u'**' and '**u_z**'.
- Coordinate of '**u'**' is (0,b,c) as we will take projection on YZ-plane x value is zero.

$$u' \cdot u_z = |u'||u_z| \cos\alpha$$

$$\cos\alpha = \frac{u' \cdot u_z}{|u'||u_z|} = \frac{(0,b,c)(0,0,1)}{\sqrt{b^2+c^2}} = \frac{c}{d} \quad where\ d = \sqrt{b^2+c^2}$$

And

$$u' \times u_z = u_x|u'||u_z|\sin\alpha = u_x \cdot b$$

$$u_x|u'||u_z|\sin\alpha = u_x \cdot b$$

Comparing magnitude

$$|u'||u_z|\sin\alpha = b$$

$$\sqrt{b^2+c^2} \cdot (1)\sin\alpha = b$$

$$d\sin\alpha = b$$

$$\sin\alpha = \frac{b}{d}$$

- Now we have $\sin\alpha$ and $\cos\alpha$ so we will write matrix for rotation about X-axis.

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{d} & -\frac{b}{d} & 0 \\ 0 & \frac{b}{d} & \frac{c}{d} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- After performing above rotation '**u**' will rotated into '**u'''** in XZ-plane with coordinates (a, 0, $\sqrt{(b^2+c^2)}$). As we know rotation about x axis will leave x coordinate unchanged, '**u'''** is in XZ=plane so y coordinate is zero, and z component is same as magnitude of '**u'**'.

- Now rotate '**u''**' about Y-axis so that it coincides with Z-axis.

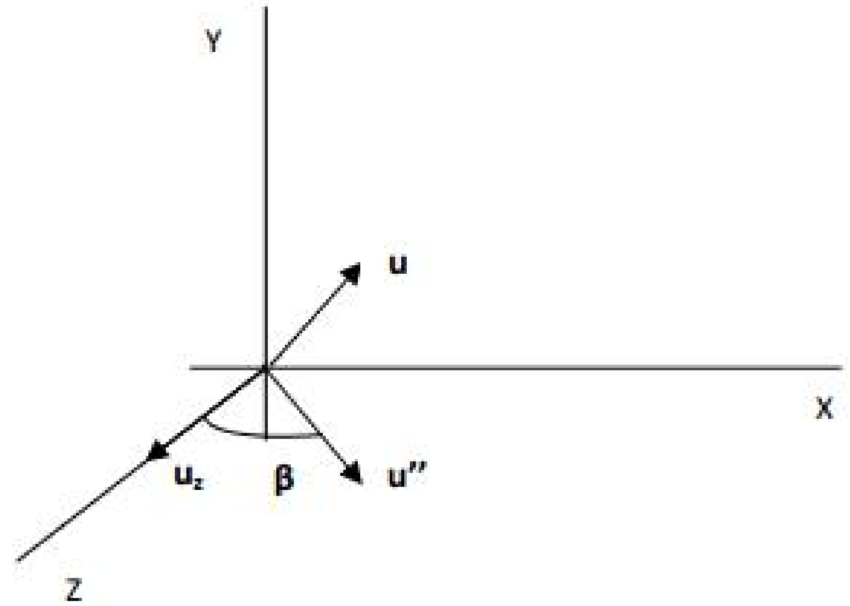


Fig. 5.5: - Rotation of u about X-axis.

- For that we repeat above procedure between '**u''**' and '**u_z**' to find matrix for rotation about Y-axis.

$$\boldsymbol{u'' \cdot u_z = |u''||u_z| \cos\beta}$$

$$\boldsymbol{\cos\beta = \frac{u' \cdot u_z}{|u'||u_z|} = \frac{\left(a, 0, \sqrt{b^2+c^2}\right)(0,0,1)}{1} = \sqrt{b^2+c^2} = d \quad where\ \ d = \sqrt{b^2+c^2}}$$

And

$$\boldsymbol{u'' \times u_z = u_y |u''||u_z| \sin\beta = u_y \cdot (-a)}$$

$$\boldsymbol{u_y |u''||u_z| \sin\beta = u_y \cdot (-a)}$$

Comparing magnitude

$$\boldsymbol{|u''||u_z| \sin\beta = (-a)}$$

$(1)\ \sin\beta = -a$

$\sin\beta = -a$

- Now we have $\sin\beta$ and $\cos\beta$ so we will write matrix for rotation about Y-axis.

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Now by combining both rotation we can coincides rotation axis with Z-axis

3) **Perform the specified rotation about that coordinate axis.**

- As we know we align rotation axis with Z axis so now matrix for rotation about z axis

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) **Apply inverse rotations to bring the rotation axis back to its original orientation.**

- This step is inverse of step number 2.

5) **Apply the inverse translation to bring the rotation axis back to its original position.**

6) This step is inverse of step number 1.

So finally sequence of transformation for general 3D rotation is

$$P' = T^{-1} \cdot R_x{}^{-1}(\alpha) \cdot R_y{}^{-1}(\beta) \cdot R_z(\theta) \cdot R_y(\beta) \cdot R_x(\alpha) \cdot T \cdot P$$

Scaling

- It is used to resize the object in 3D space.
- We can apply uniform as well as non uniform scaling by selecting proper scaling factor.
- Scaling in 3D is similar to scaling in 2D. Only one extra coordinate need to consider into it.

Coordinate Axes Scaling

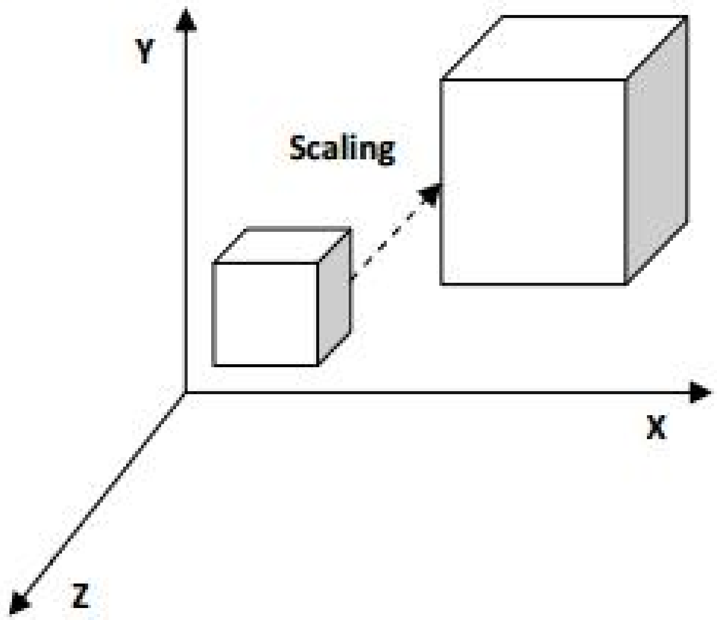


Fig. 5.6: - 3D Scaling.

- Simple coordinate axis scaling can be performed as below.

  $P' = S \cdot P$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Example: - Scale the line AB with coordinates (10,20,10) and (20,30,30) respectively with scale factor S(3,2,4).

$P' = S \cdot P$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A_x' & B_x' \\ A_y' & B_y' \\ A_z' & B_z' \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 10 & 20 \\ 20 & 30 \\ 10 & 30 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_x' & B_x' \\ A_y' & B_y' \\ A_z' & B_z' \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 60 \\ 40 & 60 \\ 40 & 120 \\ 1 & 1 \end{bmatrix}$$

Final coordinates after scaling are A' (30, 40, 40) and B' (60, 60, 120).

Fixed Point Scaling

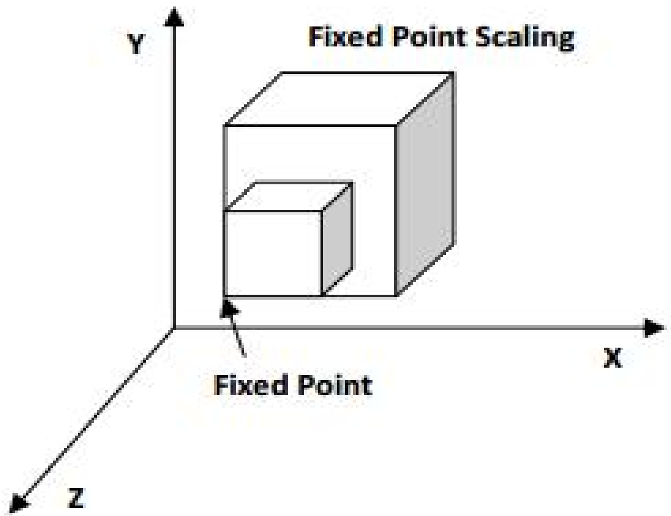


Fig. 5.7: - 3D Fixed point scaling.

- Fixed point scaling is used when we require scaling of object but particular point must be at its original position.
- Fixed point scaling matrix can be obtained in three step procedure.
  1. Translate the fixed point to the origin.
  2. Scale the object relative to the coordinate origin using coordinate axes scaling.
  3. Translate the fixed point back to its original position.
- Let's see its equation.

$$P' = T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f) \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot P$$

Other Transformations

Reflections

- Reflection means mirror image produced when mirror is placed at require position.
- When mirror is placed in XY-plane we obtain coordinates of image by just changing the sign of z coordinate.
- Transformation matrix for reflection about XY-plane is given below.

$$RF_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Similarly Transformation matrix for reflection about YZ-plane is.

$$RF_x = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Similarly Transformation matrix for reflection about XZ-plane is.

$$RF_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shears

- Shearing transformation can be used to modify object shapes.
- They are also useful in 3D viewing for obtaining general projection transformations.
- Here we use shear parameter **'a'** and **'b'**
- Shear matrix for Z-axis is given below

$$SH_z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Similarly Shear matrix for X-axis is.

$$SH_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Similarly Shear matrix for X-axis is.

$$SH_y = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

END