Similarly, putting $x = x_1$, $y = f_1$ in eq^n ②,
we get,

$$a_1 = \frac{f_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

processing in the same way, we find $a_1, a_2 \cdots a_n$.
Substituting the value in ②, we get eq^n ①.

Note: With $(n+1)$ sample points, we can generate n^{th} degree polynomial.

the equation ① may be represented in general form as

$$f(x) = \sum_{i=0}^{n} f_i l_i(x) \quad \text{-------- (3)}$$

$$\text{where, } l_i(x) = \prod_{j=0, j\neq i}^{n-1} \frac{(x - x_j)}{(x_i - x_j)}$$

Equation ③ is called Lagrange interpolation polynomials.

Q. The table below gives the square root of integers.

x	1	2	3	4	5
$f(x)$	1	1.4142	1.7321	2	2.2361

find the square root of 2.5 using second order Lagrange interpolation polynomial.

Sol^n

Let us consider the following three points
$x_0 = 2$, $x_1 = 3$ and $x_2 = 4$
So, $f_0 = 1.4142$ $f_1 = 1.7321$ and $f_2 = 2$
and $x = 2.5$.

Then,
for $x = 2.5$, we have

$$l_0(2.5) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$
$$= \frac{(2.5-3)(2.5-4)}{(2-3)(2-4)}$$
$$= \frac{(-0.5)(-1.5)}{(-1)(-2)}$$
$$= 0.375$$

Again, $$l_1(2.5) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$
$$= \frac{(2.5-2)(2.5-4)}{(3-2)(3-4)}$$
$$= \frac{(0.5)(-1.5)}{1(-1)}$$
$$= 0.75$$

Again $$l_2(2.5) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$
$$= \frac{(2.5-2)(2.5-3)}{(4-2)(4-3)}$$
$$= \frac{(0.5)(-0.5)}{2\times 1}$$
$$= -0.125$$

Then, the second order lagrange interpolation is given by we have,

$$P_2(2.5) = f_0 l_0 + f_1 l_1 + f_2 l_2$$
$$= 1.4142 \times 0.375 + 1.7321 \times 0.75 + (-0.125) \times 2$$
$$= 0.530325 + 1.299075 - 0.25$$
$$= 1.579425$$
$$\approx 1.5794.$$

Q. Find the lagrange interpolation polynomial to fit the following data :

i	0	1	2	3
x_i	0	1	2	3
$e^{x_i}-1$	0	1.7183	6.3891	19.0855

Use the polynomial to estimate the value of $x.e^{1.5}$

Soln

Here, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

Now,

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}$$

$$= \frac{(x-1)(x-2)(x-3)}{-6} \qquad = (x^2-2x-x+2)(x-3)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$= \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}$$

$$= \frac{(x-0)(x-2)(x-3)}{2} \qquad (x^2-2x)(x-3)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$= \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}$$

$$= \frac{x(x-1)(x-3)}{-2} \qquad (x^2-x)(x-3)$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$$

$$= \frac{x(x-1)(x-2)}{6}$$

the, interpolating polynomial is,

$$p(x) = f_0L_0(x) + f_1L_1(x) + f_2L_2(x) + f_3L_3(x)$$

$$= 1.7183 \times \frac{x(x-2)(x-3)}{2} + 6.3891 \times \frac{x(x-1)(x-3)}{-2} +$$

$$19.0855 \times \frac{x(x-1)(x-2)}{6}$$

$$= 1.7183 \times \frac{(x^2-2x)(x-3)}{2} + \frac{6.3891}{-2} \times (x^2-1)(x-3) +$$

$$\frac{19.0855}{6} \times (x^2-1)(x-2)$$

$$= \frac{1.7183}{2} \times (x^3-2x^2-3x^2+6x) + \frac{6.3891}{-2} \times (x^3-x-3x+3) +$$

$$\frac{19.0855}{6} \times (x^3-x-2x^2+2)$$

$=$

Netwon's Interpolation Polynomial:
Divide Difference:

let $(x_0, f_0)\ (x_1, f_1)\ (x_2, f_2) \ldots\ldots (x_n, f_n)$ be the given points then divided difference can be defined as follows.

⇒ First divided difference for x_0, x_1 is given by

$$F[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

⇒ Second divide difference is given by.

$$F[x_0, x_1, x_2] = \frac{F[x_1, x_2] - F[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0}$$

⇒ Third divide difference is given by

$$f(x_0, x_1, x_2, x_3] = \frac{F(x_1 x_2 x_3) - F(x_0, x_1, x_2] ~~F[x_0 x_1]~~}{x_3 - x_0}$$

$$= \frac{F(x_2, x_3) - F(x_1, x_2) - F(x_1 x_2) - F(x_0 x_1)}{x_3 - x_0}$$

The netwon's polynomial is,

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots\ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \ldots \quad (1)$$

where the interpolation points $x_0, x_1 \ldots x_{n-1}$ acts as centres.

Proof :-

To construct the interpolation polynomial, we need to determine the coefficient $a_0, a_1 \ldots\ldots a_n$.
let us assume that, $(x_0, f_0)\ (x_1, f_1) \ldots\ldots (x_{n-1}, f_{n-1})$ are the interpolating points.

That is,

$$P_n(x_k) = f_k \quad , k = 0, 1, \ldots n-1$$

Now,

at $x = x_0$ we have from eq^n 1

$$P_n(x_0) = f_0 = a_0 \quad \text{----} \quad (2)$$

~~$P_n(x_1) = f_1$~~,

Similarly at $x = x_1$ we have,

$$P_n(x_1) = a_0 + a_1(x_1 - x_0) = f_1$$

substating for a_0 from eq^n 2 we get,

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} \quad \text{----} \quad (3)$$

at $x = x_2$,

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$$

substuting for a_0, a_1 and soloing we get

$$a_2 = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} \quad \text{----} \quad (4)$$

let us define a notation,

$$f[x_k] = f_k$$

$$f[x_k, x_{k+1}] = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

Similarly,

$$f(x_k, x_{k+1}, x_{k+2}] = \frac{f(x_{k+1}, x_{k+2}) - f(x_k, x_{k+1}]}{x_{k+2} - x_k}$$

$$f(x_k, x_{k+1} \ldots x_i, x_{i+1}] = \frac{f(x_{k+1} \ldots x_{i+1}] - f(x_k \ldots x_i)}{x_{i+1} - x_k} \quad \text{----} \quad (5)$$

These quantities are called divided differences. Now we can express the coefficients a_i in terms of divided differences,

$$a_0 = f_0 = f[x_0]$$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$$

$$a_2 = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} = f(x_0, x_1, x_2)$$

Thus,

$$a_n = f[x_0, x_1, \text{------} x_n]$$

Here a_1 represent the first divided difference, a_2 represent the second divided difference and so on, substituting for a_i coefficient in eq^n ① we get,

$$P_n(x) = f[x_0] + f[x_0, x_1][x - x_0] + f(x_0, x_1, x_2)(x - x_0)(x - x_1) + \text{------------} + f[x_0, x_1 \text{---} x_n][x - x_0]\ldots[x - x_{n-1}]$$

which can be written as,

$$P_n(x) = \sum_{i=1}^{n} f[x_0, \ldots x_i] \prod_{j=0}^{i-1} (x - x_j] \text{ ---} ⑥$$

which is the required netwon's interpolation polynomial.

Q) Given below table for $\log x$. Estimate 2.5 using second order Netwons interpolation polynomial.

i	0	1	2	3
x_i	1	2 x_0	3 x_1	4 x_2
$\log x_i$	0	0.3010	0.4771	0.6021

f_0 $\quad$ f_1 $\quad$ f_2

Construction of divide difference table:

As we know that Netwon's interpolation polynomial are evaluated using the divide difference at the interpolating points. we also have used the lower order divided difference to construct the higher order divide difference.

For example, consider the second order divided difference,

$$a_2 = f(x_0, x_1, x_2)$$

$$= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

where $f[x_1, x_2]$ and $f[x_0, x_1]$ are first order divided difference, and are given by,

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

and, $$f(x_2, x_1] = \frac{f_2 - f_1}{x_2 - x_1}$$

This shows that, given the interpolating points, we can obtain recursively a higher order divided difference starting from the first order difference.

For example, a divided difference table for 5 points is given below.

i	x_i	$f[x_i]$	first difference	Second difference	Third difference	Fourth difference	~~fifth difference~~
0	$f[x_0]$	$f[x_0]$					
			$f[x_0, x_1]$				
1	$f(x_1)$	$f[x_1]$		$f[x_0, x_1, x_2]$			
			$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$		
2	$f(x_2)$	$f[x_2]$		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$	
			$f[x_2, x_3]$				
3	$f(x_3)$	$f[x_3]$		$f[x_2, x_3, x_4]$			
			$f(x_3, x_4)$		$f(x_1, x_2, x_3, x_4)$		
4	$f(x_4)$	$f(x_4)$					

Q1 Given the following set of data points, obtain the table of divided difference. Use the table to estimate the value of $f(1.5)$

i	0	1	2	3	4
x_i	1	2	3	4	5
$f(x_i)$	0	7	26	63	124

Soln,

The required table of divided difference is given below.

i	x_i	$f(x_i)$	first difference	Second difference	Third difference
0	1	0			
			7		
1	2	7		6	
			19		1
2	3	26		9	
			37		1
3	4	63		12	
			61		
4	5	124			

Now, we know,

$$P_n(x) = a_0 + a_1[x-x_0] + a_2[x-x_0](x-x_1] + \dots$$

from the table, we have,

$a_0 = 0$

$a_1 = 7$

$a_2 = 6$

$a_3 = 1$

Now, from the formula

$$P_3[1.5] = a_0 + a_1[x-x_0] + a_2(x-x_0)(x-x_1] + a_3[x-x_0](x-x_1][x-x_2]$$

$$= 0 + 7[1.5-1] + 6[1.5-1][1.5-2] + 1[1.5-1][1.5-2][1.5-3]$$

$$= 2.375$$

Q. Using the following table find $f(x)$ as a polynomial in x.

x	$f(x)$
-1	3
0	-6
3	39
6	822
7	1611

Soln,

constructing the divided difference table, we have,

i	x_i	$f(x_i)$	first difference	Second difference	Third difference	fourth difference
0	-1	3				
			-9			
1	0	-6		6		
			15		5	
2	3	39		41		1
			261		13	
3	6	822		132		
			789			
4	7	1611				

Now, we know,

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots$$

and from the table, we know,

$a_0 = 3$

$a_1 = -9$

$a_2 = 6$

$a_3 = 5$

$a_4 = 1$

Then, we have,

$$P_4(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$= 3 + -9(x+1) + a_2(x+1)(x-0) + a_3(x+1)(x-0)(x-3) + a_4(x+1)(x-0)(x-3)(x-6)$$

$= 3 + -9(x+1) + a_2 x(x+1) + a_3 x(x+1)(x-3) + a_4 x$
$(x+1)(x-3)(x-6)$

$= 3 + 9x + 9 + 6x(x+1) + 5x(x^2-3x+x-3) + x[x^2-3x+x-3]$
$[x-6]$

$= -6 - 9x + 6x^2 + 6x + 5x^3 - 15x^2 + 5x^2 - 15x + x[x^3 - 3x^2$
$+ x^2 - 3x - 6x^2 + 18x + 6x + 18]$

$= -6 - 9x + 6x^2 + 6x + 5x^3 - 15x^2 + 5x^2 - 15x + x^4 - 3x^3$
$+ x^3 - 3x^2 - 6x^3 + 18x^2 - 6x^2 + 18x$

$= x^4 - 3x^3 + 5x^2 + 6$

Interpolation with equidistant points:

⇒ Equally spaced data can be represented by,

$x_{i+1} = x_i + h$ or $x_k = x_0 + kh$

where h is the step size and k is any integer (-ve or +ve)

⇒ Also forward difference can be defined as,

$\Delta^j f_i = \Delta^{j-1} f_{i+1} - \Delta^{j-1} f_i$

first forward difference

$\Delta^1 f_i = \Delta f_{i+1} - \Delta f_i$

second forward difference

$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$

In tabular form, we have

x	f	1st diff (Δf_i)	Second diff ($\Delta^2 f_i$)	3rd diff ($\Delta^3 f_i$)	4th diff ($\Delta^4 f_i$)
x_0	f_0				
		$\Delta f_0 = f_1 - f_0$			
x_1	f_1		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$		
		$\Delta f_1 = f_2 - f_1$		$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$	
x_2	f_2		$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$		$\Delta^4 f_0 = \Delta^3 f_1 - \Delta^3 f_0$
		$\Delta f_2 = f_3 - f_2$		$\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1$	
x_3	f_3		$\Delta^2 f_2 = \Delta f_3 - \Delta f_2$		
		$\Delta f_3 = f_4 - f_3$			
x_4	f_4				

Here,

$x_1 = x_0 + h$

$x_2 = x_1 + h$ and so on..

We have,

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots\dots\dots$$

Where,

$$a_0 = f_0$$

$$a_1 = f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{\Delta f_0}{h}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} =$$

$$= \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{2h}$$

$$= \frac{\frac{\Delta f_1 - \Delta f_0}{h}}{2h} \quad [\because \Delta f_1 - \Delta f_0 = \Delta^2 f$$

$$= \frac{\Delta^2 f_0}{2h^2}$$

Simillarly,

$$a_3 = \frac{\Delta^3 f_0}{3!\, h^3}$$

So,

The general term is,

$$a_n = \frac{\Delta^n f_0}{n!\, h^n}$$

Substituting these value in Netwon's divided difference interpolation polynomial, we get.

$$P_n(x) = f_0 + \frac{\Delta f_0}{h}(x - x_0) + \frac{\Delta^2 f_0}{2h^2}(x - x_0)(x - x_1] + \dots\dots$$

This equation is known as Gregory-Newtons forward difference formula (Evenly space data)

Q) Estimate the value of $\sin\theta$ at $\theta = 25°$ using Newton-Gregory forward difference formula with the help of following table.

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5000	0.6428	0.7660.

Soln;

In order to use Newton-Gregory forward difference formula, we need the value of, $\Delta^i f_0$. These coefficients can be obtained from the difference table as follows.

$x(\theta)$	$f(x)\sin\theta$	1st diff Δf_i	2nd diff $\Delta^2 f_i$	3rd diff $\Delta^3 f_i$	4th diff $\Delta^4 f_i$
10	0.1736				
		0.1684			
20	0.3420		-0.0104		
		0.1580		-0.0048	
30	0.5000		-0.0152		0.0004
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

Then we have;

$$P_n(x) = f_0 + \frac{\Delta f_0}{h}(x-x_0) + \frac{\Delta^2 f_0}{2h^2}(x-x_0)(x-x_1)$$

$$+ \frac{\Delta^3 f_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) + \frac{\Delta^4 f_0}{4!h^4}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$= 0.1736 + \frac{0.1684}{10}(25-10) + \frac{(-0.0104)}{2\times 10^2}(25-10)(25-20)$$

$$+ \frac{(-0.0048)}{3!\,10^3}(25-10)(25-20)(25-30) + \frac{0.0004}{4!\times 10^4}(25-10)(25-20)(25-30)(25-40)$$

$= 0.1736 + 0.2526 - 0.0039 + 0.003 + 0.0009375$

$= 0.4256376.$

Backward difference.

If the table is too long and if the required point is closed to the end of the table, we can use another formula known as Newton Gregory backward difference formula. The table for Newton Gregory backward difference is as follows.

x	f	∇f	$\nabla^2 f_i$	$\nabla^3 f_i$	$\nabla^4 f_i$
x_0	f_0	∇f_1			
x_1	f_1	∇f_2	$\nabla^2 f_2$		
				$\nabla^3 f_3$	
x_2	f_2	∇f_3	$\nabla^2 f_3$		$\nabla^4 f_4$
				$\nabla^3 f_4$	
x_3	f_3	∇f_4	$\nabla^2 f_4$		
x_4	f_4	∇f_5			

cubic spline Interpolation.

Till now we have known how an interpolation polyno[mial] of degreen can be constructed from the given set of values of function. It has been proved that when n is large, th[e] interpolation polynomial doesnot provide accurate result. This illustrate in figure below.

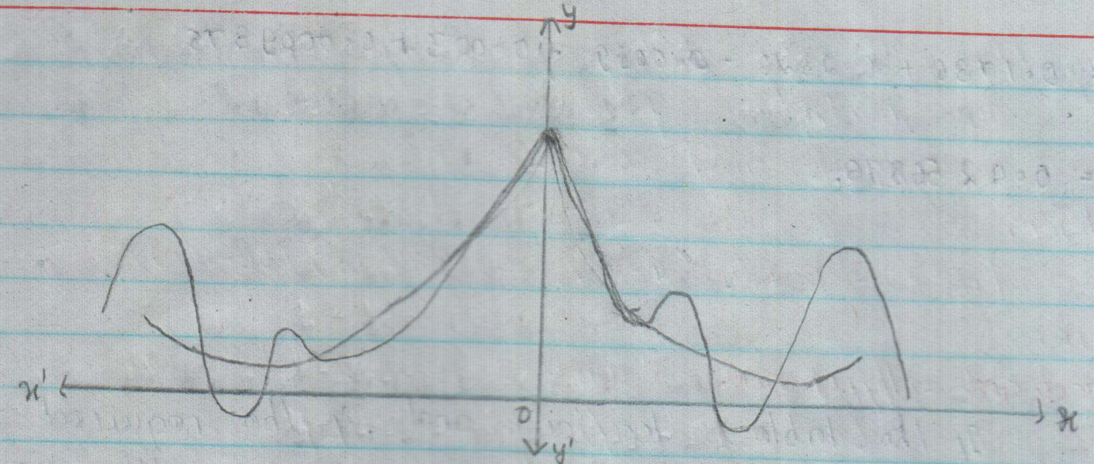


fig: Interpolation polynomial of degree 11 of the function $\frac{1}{1+x^2}$

To overcome this problem, we divide the entire range of points into sub intervals, and use local low order polynomial to interpolate each sub interval. Such polynomial are called piecewise polynomial.

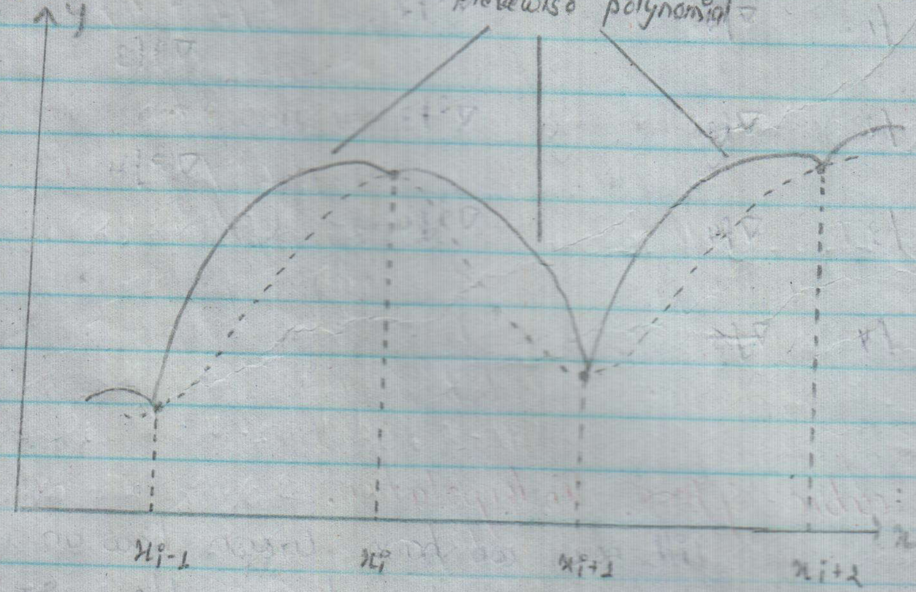


fig:- piecewise polynomial interpolation.

The piecewise polynomial that prevent the discontinuities at the connecting points are called. spline function (or simply splines) The connecting points are called knots or nodes.

Consider the construction of cubic spline function which would interpolate the points $(x_0, f_0), (x_1, f_1), \ldots \ldots (x_n, f_n)$. The cubic spline $S(x)$ consist of $(n-1)$ cubes corresponds to $(n-1)$ sub intervals. If we denote such cubes by $S_i(x)$ then,

$$S(x) = S_i(x)$$

where $i = 1, 2 \ldots \ldots n$.

These cubics must satisfy the following condition.

① $S(x)$ must interpolate f at all the points $x_0, x_1 \ldots x_n$ i.e for each i,

$$S(x_i) = f_i \quad \text{-----} \; ①$$

② The function values must be equal at all the interior knots.

$$S_i(x_i) = S_{i+1}(x_i) \quad \text{---} \; ②$$

③ The first derivatives at the interior knots must be equal.

$$S_i'(x_i) = S'_{i+1}(x_i) \quad \text{-----} \; ③$$

④ The second derivative at the interior knots must be equal.

$$S_i''(x_i) = S''_{i+1}(x_i) \quad \text{---} \; ④$$

⑤ The second derivative at the end points are zero.

$$S''(x_0) = S''(x_n) = 0 \quad \text{---} \; ⑤$$

Step 1:

Let us consider the second derivative. Since $S_i(x)$ is a cubic function, its second derivative $S_i''(x)$ is a striaght line. This striaght line can be represented by a first order lagrange interpolation polynomial. Since the line passes through the points $(x_i, S_i''(x))$ and $(x_{i-1}, S_i''(x_{i-1}))$. We have

$$S_i''(x) = S_i''(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + S_i''(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}} \quad \text{---} \; ⑥$$

The unknown $S_i''(x_{i-1})$ and $S_i''(x_i)$ are to be determined.

for the sake of simplicity, let us denote,

$$s_i''(x_{i-1}) = a_{i-1} \quad \text{and,}$$

$$s_i''(x_i) = a_i$$

$$x - x_i = u_i$$

$$x - x_{i-1} = u_{i-1}$$

$$x_i - x_{i-1} = h_i = u_{i-1} - u_i$$

Then equation ⑥ becomes,

$$s_i''(x) = a_{i-1}\frac{u_i}{-h_i} + a_i\frac{u_{i-1}}{h_i}$$

$$= \frac{a_i u_{i-1} - a_{i-1} u_i}{h_i} \quad \text{----} ⑦$$

Step 2:

Now we can obtain $s_i(x)$ by integrating eqⁿ ⑦ twice, thus

$$s_i(x) = \frac{a_i u_{i-1}^3 - a_{i-1} u_i^3}{6h_i} + c_1 x + c_2 \quad \text{----} ⑧$$

where c_1 and c_2 are constants of integration. The linear part $c_1 x + c_2$ can be expressed as,

$$b_1(x - x_{i-1}) + b_2(x - x_i)$$

with the suitable choice of b_1 and b_2. Therefore,

$$c_1 x + c_2 = b_1(x - x_{i-1}) + b_2(x - x_i)$$

Then equation ⑧ becomes,

$$s_i(x) = \frac{a_i u_{i-1}^3 - a_{i-1} u_i^3}{6h_i} + b_1(u_{i-1}) + b_2(u_i) \quad \text{----} ⑨$$

Now, we must determine the coefficient b_1 and b_2,
we know, by condition i,

$$s(x_i) = f_i \text{ and}$$
$$s(x_{i-1}) = f_{i-1}$$

At $x = x_i$

$$u_i = 0$$
$$u_{i-1} = h_i$$

and,

$$f_i = \frac{a_i h_i^2}{6} + b_1 h_i$$

$$\therefore b_1 = \frac{f_i}{h_i} - \frac{a_i h_i}{6} \quad \text{----} \ (10)$$

similarly,
at $x = x_{i-1}$

$$v_i = -h_i$$
$$u_{i-1} = 0$$

and,

$$f_{i-1} = \frac{a_{i-1} h_i^2}{6} - b_2 h_i \text{ ----}$$

$$b_2 = \frac{-f_{i-1}}{h_i} + \frac{a_{i-1} h_i}{6} \quad \text{----} \ (11)$$

substituting for b_1 and b_2 in eq^n (9) & solving we get.

$$S_i(x) = \frac{a_{i-1}}{6h_i} \cdot (h_i^2 v_i - v_i^3) + \frac{a_i}{6h_i} (u_{i-1}^3 - h_i^2 v_{i-1})$$
$$+ \frac{1}{h_i} (f_i v_{i-1} - f_{i-1} v_i) \text{ ------} \ (12)$$

Equation 12 has only two unknowns a_{i-1} and a_i

Step-4

The final step is to evaluate this constants. This can be done by invoking the conditions
$$s_i'(x_i) = s_{i+1}'(x_i)$$
Differentiating eq^n (12) we get

$$s_i'(x) = \frac{a_{i-1}}{6h_i}(h_i^2 - 3u_i^2) + \frac{a_i}{6h_i}(3u_{i-1}^2 - h_i^2) + \frac{1}{h_i}(f_i - f_{i-1})$$

setting $x = x_i$
$$s_i'(x_i) = \frac{a_{i-1}h_i}{6} + \frac{a_ih_i}{3} - \frac{f_i - f_{i-1}}{h_i}$$
similarly,
$$s_{i+1}(x_i) = -\frac{a_ih_{i+1}}{3} - \frac{a_{i+1}h_{i+1}}{6} + \frac{f_{i+1} - f_i}{h_{i+1}}$$

since $s_i'(x_i) = s_{i+1}'(x_i)$
we have,
$$h_i a_{i-1} + 2(h_i + h_{i+1})a_i + h_{i+1}a_{i+1}$$
$$= 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right] \text{---------} (13)$$

Here, since the 2^nd derivatives at the end points are zero, we get,
$$a_0 = a_n = 0$$
Also the system of $n-1$ equations combined in eq^n (13) can be expressed as,

$$\begin{bmatrix} 2(h_1+h_2) & h_2 & 0 \cdots 0 & 0 & 0 \\ h_2 & 2(h_2+h_3) & h_3 \cdots 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots 0 & h_{n-1} & 2[h_{n-1}+h_n] \end{bmatrix} \cdots$$

$$\times \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ \vdots \\ D_{n-1} \end{bmatrix} \quad \text{------} \quad (14)$$

where,

$$D_i = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$$

$$h_i = x_i - x_{i-1}$$

$$i = 1, 2, 3 \; --- \; n-1$$

Q11 Given the data points, estimate the function of value f at $x = 7$ using cubic spline

i	0	1	2
x_i	4	9	16
f_i	2	3	4

soln,

Here $h_1 = x_1 - x_0 = 9 - 4 = 5$ $\qquad$ $h_2 = x_2 - x_1 = 16 - 9 = 7$

$f_0 = 2, \quad f_1 = 3 \quad f_2 = 4$

Therefore,

$$h_1 a_0 + 2(h_1 + h_2)a_1 + h_2 a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

we know that.

$a_0 = a_2 = 0$ Then

$$2(h_1 + h_2)a_1 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

or $2[5+7]\,q_1 = 6\left[\frac{4-3}{7} - \frac{3-2}{5}\right]$

or, $2 \times 12\, q_1 = 6\left[\frac{1}{7} - \frac{1}{5}\right]$

or, $\frac{24}{6}\, q_1 = \left[\frac{5-7}{35}\right]$

or, $4 q_1 = \frac{-2}{35}$

$\therefore q_1 = \frac{-2}{24 \times 35}$

$= \frac{-1}{70}$

$= -0.0143$

Since $n = 3$, there are two cubic splines namely,

$s_1(x) : x_0 \le x \le x_1$

$s_2(x) : x_1 \le x \le x_2$

the target point $x = 7$ is in the domain of $s_1(x)$ and therefore we need to use only $s_1(x)$ for estimate

$$\therefore s_1(x) = \frac{q_1\,[u_0^3 - h_1^2 u_0]}{6h_1} + \frac{1}{h_1}\,[f_1 u_0 - f_0 u_1]$$

and,

$u_0 = x - x_0 \qquad u_1 = x - x_1$

$= 7 - 4 \qquad\quad = 7 - 9$

$= 3 \qquad\qquad = -2$

So,

$$s_1(7) = \frac{-0.0143\,(3^3 - 5^2 \cdot 3)}{6 \times 5} + \frac{1}{5}\left[3 \times 3 - 2 \times -2\right]$$

$$= -0.0143\,\frac{(27 - 75)}{30} + \frac{1}{5}\,[9 + 4]$$

$$= 0.02288 + 2.6$$

$$= 2.62288$$

Hence $y = 2.62288$.

Q, Find the cubic splines for the following table of values.

$x =$	1	2	3
$y =$	-6	-1	16

Hence evaluate $y(1.5)$ and $y'(2)$.

Sol^n, Here we have,

$x_0 = 1$, $x_1 = 2$, $x_2 = 3$

$h_1 = x_1 - x_0 = 2 - 1 = 1$

$h_2 = x_2 - x_1 = 3 - 2 = 1$.

$f_0 = -6$, $f_1 = -1$, $f_2 = 16$

Therefore,

$$h_1 q_0 + 2(h_1 + h_2) q_1 + h_2 q_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

or, $$1 \times 0 + 2(1 + 1) q_1 + 1 \times 0 = 6\left[\frac{16 - -1}{1} - \frac{(-1) - (-6)}{1}\right]$$

or, $$4q_1 = 6\,[17 - 5]$$

or $$4q_1 = 6\,[22]$$

or $$q_1 = 18$$

Since n=3, there are two cubic splines only,

$$S_1(x) = x_0 \le x \le x_1$$
$$S_2(x) = x_1 \le x_0 \le x_2$$

Here the given point $x = 1.5$ in in the domain of $S_1(x)$ & therefore we we only $S_1(x)$ for estimate,

$$u_0 = x - x_0 \qquad v_1 = x - x_1$$
$$= 1.5 - 1 \qquad = 1.5 - 2$$
$$= 0.5 \qquad = -0.5$$

then,

$$y(1.5) = \frac{18[0.5^3 - 1^2 \cdot 0.5]}{6} + \frac{1}{1}[-1 \times 0.5 - (-6) \times (-0.5)]$$

$$= -1.125 + (-3.5)$$
$$= -4.625$$

$$S_1(x) = \frac{a_1(u_0^3 - h_1^2 u_0)}{6h_1} + \frac{1}{h_1}[f_1 \times u_0 - f_0 \times u_1)$$

$f_1 \times v_1 - f_0 \times v_1$

$$S_1'(2) = 11$$

least Square Approximation.

The process of establishigg the relationship between dependent variable & independent variable in the form of mathematical equation is known as curve fitting (regression)

Suppose the value of y for the different values of x are given. If we want to know effect of x on y then we may write a functional relationship

$$y = f(x)$$

Here y is called dependent variable.
x is independent variable.

The relationship may be linear or non linear as shown in the figure below.

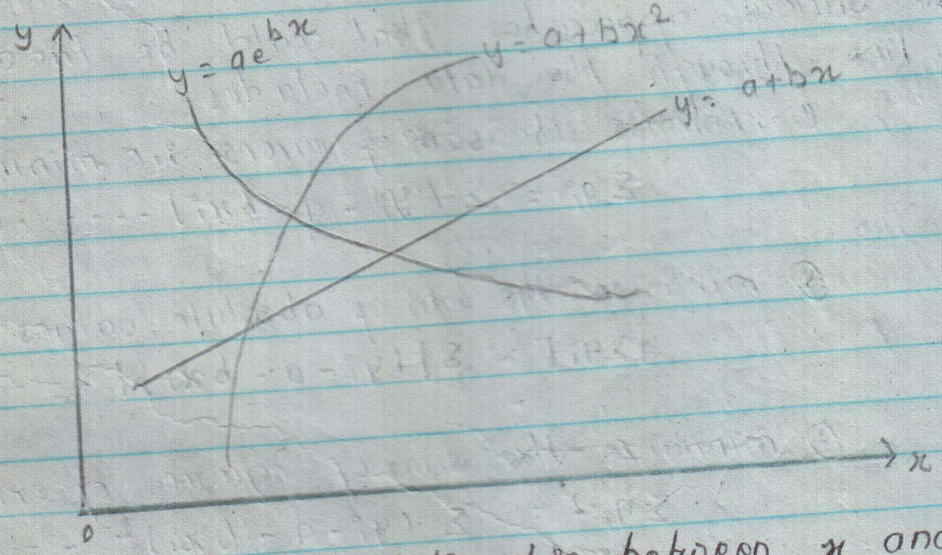


fig:- various relationship between x and y.

fitting linear equation:

let $y = a + bx = f(x)$ be the mathematical equation for a straight line.

from this equation we know that a is the intercept of the line and b its slope.

Consider a point (x_i, y_i) as shown in the figure below.

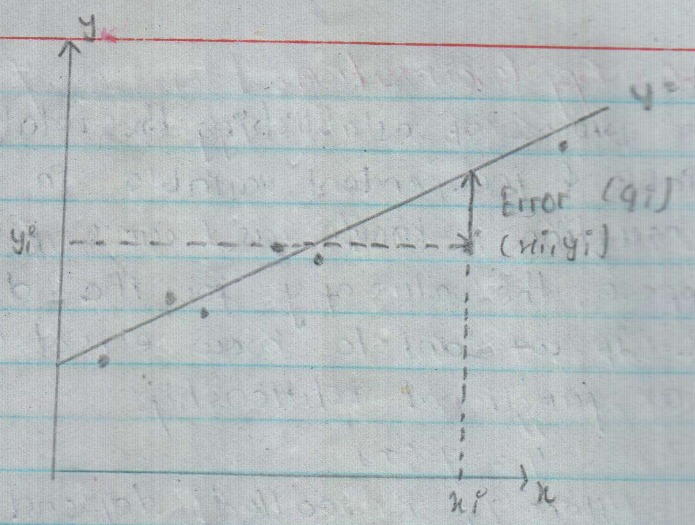


fig: scatter diagram.

Here if q_i is the error, then

$$q_i = y_i - f(x_i)$$

The various approaches that could be tried for fitting a best line through the data includes

① minimize the sum of errors. i.e minimize Σq_i

$$\Sigma q_i = \Sigma (y_i - a - bx_i) \text{ ----- } ①$$

② minimize the sum of absolute values of errors.

$$|\Sigma q_i| = \Sigma |(y_i - a - bx_i)| \text{ ----- } ②$$

③ minimize the sum of squares of errors.

$$\Sigma q_i^2 = \Sigma (y_i - a - bx_i)^2 \text{ --- } ③$$

From above equations, only eqn ③ gurantees a unique line. Hence the technique of minimizing the sum of squares of errors is known as least square regression.

least square regression.

let the sum of squares of individual errors be expressed as

$$Q = \sum_{i=1}^{n} q_i^2$$

$$= \sum_{i=1}^{n} [y_i - f(x_i)]^2$$

$$= \sum_{i=1}^{n} [y_i - a - bx_i]^2 \quad \text{------} \; ①$$

In the method of least squares we choose a and b such that Q is minimum. since Q depends on a and b, a necessary condition for Q to be minimum is,

$$\frac{\partial Q}{\partial a} = 0 \quad \text{and} \quad \frac{\partial Q}{\partial b} = 0$$

Then,

$$\frac{\partial Q}{\partial a} = -2\sum_{i=1}^{n} [y_i - a - bx_i] = 0$$

$$\frac{\partial Q}{\partial b} = -2\sum_{i=1}^{n} x_i [y_i - a - bx_i] = 0$$

Thus,

$$\Sigma y_i = na + b\,\Sigma x_i$$

and

$$\Sigma x_i y_i = a\,\Sigma x_i + b\,\Sigma x_i^2$$

These are called normal equations. solving for a and b we get

$$b = \frac{n\,\Sigma x_i y_i - \Sigma x_i\,\Sigma y_i}{n\,\Sigma x_i^2 - (\Sigma x_i)^2}$$

$$a = \frac{\Sigma y_i}{n} - b\frac{\Sigma x_i}{n} = \bar{y} - b\bar{x}$$

where $\bar{x}$ and $\bar{y}$ are the average of x values and y values respectively.

Q/1 Fit a straight line to the following set of data.

x	1	2	3	4	5
y	3	4	5	6	8

Soln

	x_i	y_i	x_i^2	$x_i y_i$
	1	3	1	3
	2	4	4	8
	3	5	9	15
	4	6	16	24
	5	8	25	40
sum =	15	26	55	90

Then, $b = \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}$

$= \dfrac{5 \times 90 - 15 \times 26}{5 \times 55 - (15)^2}$

$= \dfrac{450 - 390}{275 - 225}$

$= \dfrac{60}{50}$

$= 1.2.$

$a = \dfrac{\sum y_i}{n} - b\dfrac{\sum x_i}{n}.$

$= \dfrac{26}{5} - 1.2 \times \dfrac{15}{5}$

$= 5.2 - 3.6$

$= 1.6$

Thus the linear equation is,

$$y = 1.6 + 1.2x$$

The regression line along with the above data is shown in the figure below.

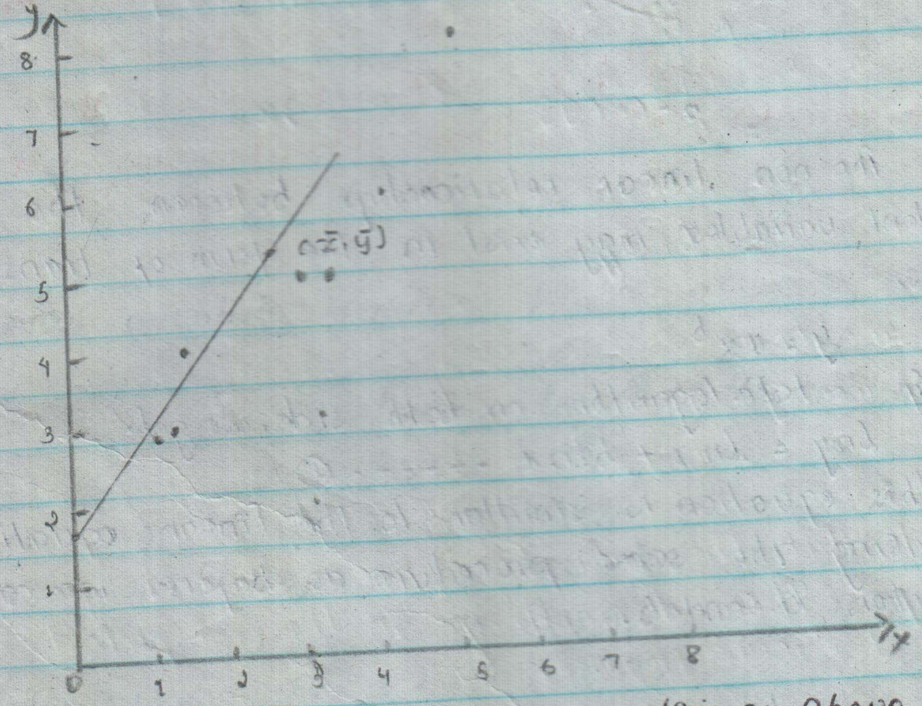


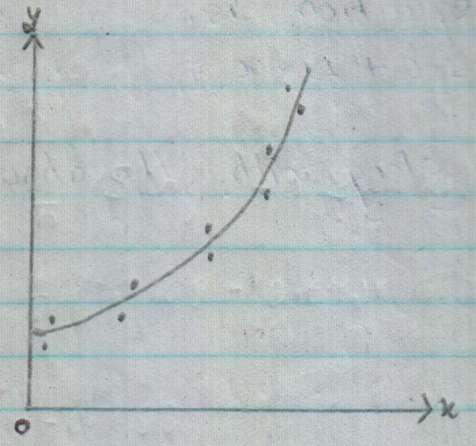
fig:- plot of data and regression line of above example.

Algorithm for linear regression

1. Read data values.
2. Compute the sum of powers & products i.e Σx_i, Σy_i, Σx_i^2, $\Sigma x_i y_i$
3. check weather the denomentator for the equation of b is 0.
4. Compute b and a.
5. printout the equation.
6. Interpolate data if required.

Fitting Transcendental Equation (exponential) :-

The relationship between dependent & independent variables is not always linear, as shown in the figure below



The non linear relationship between the dependent & independent variables may exist in the form of transcendental equation as,

$$y = ax^b$$

If we take logarithm on both side, we get.

$$\ln y = \ln a + b \ln x \quad \text{------} \quad (1)$$

This equation is simillar to the linear equation and therefore using the same procedure as before, we can evaluate the parameters a and b.

$$b = \frac{n \sum \ln x_i \ln y_i - \sum \ln x_i \sum \ln y_i}{n \sum (\ln x_i)^2 - (\sum \ln x_i)^2}$$

$$\ln a = R = \frac{1}{n} [\sum \ln y_i - b \sum \ln x_i]$$

$$a = e^R.$$

There is another form of non linear model known as saturation growth rate, equation as shown below,

$$p = \frac{k_1 t}{k_2 + t}$$

This can be linearized by taking inversion of the terms,

i.e. $\frac{1}{P} = \frac{K_2}{K_1}\frac{1}{t} + \frac{1}{K_1}$

This is again simillar to linear equation

$y = a + bx$ where,

$y = \frac{1}{P}$, $x = \frac{1}{t}$, $a = \frac{1}{K_1}$, $b = \frac{K_2}{K_1}$

Q/ Given the data table,

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

fit a power function model of the form $y = ax^b$

Soln,

various quantities required for the above equations are tabula below,

x_i	y_i	$\ln x_i$	$\ln y_i$	$(\ln x_i)^2$	$(\ln x_i)(\ln y_i)$
1	0.5	0	-0.693	0	0
2	2	0.693	0.693	0.48	0.48
3	4.5	1.0986	1.504	1.207	1.652
4	8	1.386	2.079	1.921	2.881
5	12.5	1.609	2.526	2.589	4.064
Sum = 15	27.5	4.787	6.109	6.197	9.077

Now,

$$b = \frac{n\sum \ln x_i \ln y_i - \sum \ln x_i \sum \ln y_i}{n\sum(\ln x_i)^2 - (\sum \ln x_i)^2}$$

$$= \frac{5 \times 9.077 - 4.787 \times 6.109}{5 \times (6.197)^2 - (4.787)^2}$$

$$= \frac{45.385 - 29.243}{\cancel{114.5}\ 26.198}$$

$$= \frac{16.141}{\cancel{28.198}} \quad \frac{16.141}{8.069}$$

$$= \cancel{0.616} = 2$$

Then,

$$a = e^{l}$$

$$= e^{\frac{1}{n}(\Sigma \ln y_i - b \Sigma \ln x_i)}$$

$$= e^{\frac{1}{5}(6.109 - \cancel{0.616}2 \times 4.787)}$$

$$= e^{-0.693}$$

$$= 0.5$$

Now, we know

$$y = ax^{b}$$

$$= 0.5x^{2}$$

Note that the data have been derived from the eqn $y = x^2/2$. The discrepancy in the computed coefficients is due to roundoff error.

Q) The temperature of a metal strip was measured at various ~~interval~~ during heating and the value are given in a table below

time t (min)	1	2	3	4
temp T (°c)	70	83	100	124

If the relationship between the temperature T and time t is of the form

$T = be^{t/4} + a$ estimate the temperature at $t = 6$ min.

Solⁿ.

The relationship between temperature & time can be written as,

$$y = b f(x) + a$$

This is simillar to the linear eqⁿ except that the variable x is replaced by function $f(x)$ therefore we can solve for the parameter a & b by replacing x_i by $f x_i$, Σx_i by $\Sigma f x_i$, Σx_i^2 by $\Sigma (f x_i)^2$.

fitting a polynomial function.

consider a polynomial of degree m-1, y = a

$$y = a_1 + a_2 x + a_3 x^2 + \text{----------} \; a_m x^{m-1} = f(x)$$

if there is n set of data for x and y values, then the sum of squares of errors is given by,

$$Q = \sum_{i=1}^{n} [y_i - f(x_i)]^2$$

since $f(x)$ is polynomial and contains coefficients a_1, a_2, a_3 ------- etc, we have to estimate all the m coefficients.

following m equations solve these coefficients.

$$\frac{\partial Q}{\partial a_1} = 0$$

$$\frac{\partial Q}{\partial a_2} = 0$$

$$\frac{\partial Q}{\partial a_m} = 0$$

Consider a general term,

$$\frac{\partial Q}{\partial a_j} = -2 \sum_{i=1}^{n} [y_i - f(x_i)] \frac{\partial f(x_i)}{\partial a_j} = 0$$

$$\frac{\partial f(x_i)}{\partial a_j} = x_i^{j-1}$$

Thus we have,

$$\sum_{i=1}^{n} [y_i - f(x_i)] x_i^{j-1} = 0 \quad , \; j = 1, 2 \text{ ---- } m.$$

$$\sum_{i=1}^{n} [y_i x_i^{j-1} - x_i^{j-1} f(x_i)] = 0$$

Substating for $f(x_i)$

$$\sum_{i=1}^{n} x_i^{j-1} \left[a_1 + a_2 x_i + a_3 x_i^2 + \text{--------} a_m x_i^{m-1}\right] = \sum_{i=1}^{n} y_i x_i^{j-1}$$

These are m equations ($j = 1, 2, 3 \text{------} m$) and each summation is for $i = 1$ to n

$$a_1 n + a_2 \sum x_i + a_3 \sum x_i^2 + \text{--------} + a_m \sum x_i^{m-1} = \sum y_i$$
$$a_1 \sum x_i + a_2 \sum x_i^2 + a_3 \sum x_i^3 + \text{--------} + a_m \sum x_i^m = \sum y_i x_i$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_1 \sum x_i^{m-1} + a_2 \sum x_i^m + a_3 \sum x_i^{m+1} + \text{------} + a_m \sum x_i^{2m-2} = \sum y_i x_i^{m-1}$$

The set of m equations can be represented in matrix notation as follows,

$$CA = B$$

where,

$$C = \begin{bmatrix} n & \sum x_i & \sum x_i^2 & \cdots & \sum x_i^{m-1} \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \cdots & \sum x_i^m \\ \vdots & \vdots & \vdots & & \vdots \\ \sum x_i^{m-1} & \sum x_i^m & \text{---} & \text{---} & \sum x_i^{2m-2} \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \\ \vdots \\ \sum y_i x_i^{m-1} \end{bmatrix}$$

Imp

Q1 Fit a second order polynomial to the data in the table below,

x	1.0	2.0	3.0	4.0
y	6.0	11.0	18.0	27.0

Ans

Soln,

The order of polynomial is two and therefore we will have three simultaneous eqn as shown below,

$$a_1 n + a_2 \Sigma x_i + a_3 \Sigma x_i^2 = \Sigma y_i$$
$$a_1 \Sigma x_i + a_2 \Sigma x_i^2 + a_3 \Sigma x_i^3 = \Sigma x_i y_i$$
$$a_1 \Sigma x_i^2 + a_2 \Sigma x_i^3 + a_3 \Sigma x_i^4 = \Sigma x_i^2 y_i$$

The sum of powers & product can be evaluated in tabular form as shown below.

	x	y	x^2	x^3	x^4	xy	x^2y
	1	6	1	1	1	6	6
	2	11	4	8	16	22	44
	3	18	9	27	81	54	162
	4	27	16	64	256	108	432
sum =	10	62	30	100	354	190	644

substituting these value we get,

$$4a_1 + 10a_2 + 30a_3 = 62$$
$$10a_1 + 30a_2 + 100a_3 = 190$$
$$30a_1 + 100a_2 + 354a_3 = 644$$

Solving we get
$a_1 = 3 \quad a_2 = 2, \quad a_3 = 1$

Therefore the least square quadratic polynomial is,

$$y = 3 + 2x + x^2$$

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Solution of linear Algebric Equation.

A linear equation involving two variables x and y has the standard form,

$$ax + by = c$$

where a, b and c are real numbers and a & b cannot be both equal to zero. $(a, b \neq 0)$

Analysis of linear equation is important because mathematical model of many of the real world problems are linear.

In practise linear equations occurs in more than two variables. A linear eqn with n variables has the form,

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \text{-------} a_n x_n = b.$$

where a_i $(i = 1, 2, 3 \text{----} n)$ are real numbers and atleast one of them is not zero. The main concern here is to solve for x_i $(i = 1, 2, \text{-----} n)$, given the values of a_i and b.

If we need a unique soln of an eqn with n variables, then we need a set of n such independent equations which is known as system of simultaneous equations [or simply system of equations]. A system of m linear equation is represented as

$$a_{11} x_1 + a_{12} x_2 + \text{--------} a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \text{--------} a_{2n} x_n = b_2$$
$$\text{--------------------}$$
$$\text{--------------------}$$
$$a_{n1} x_1 + a_{n2} x_2 + \text{--------} a_{nn} x_n = b_n$$

In matrix form the above equation can be expressed as

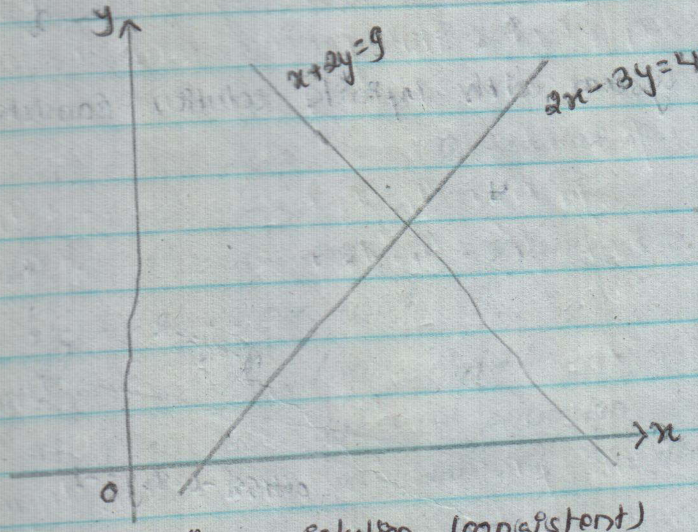
$$Ax = b$$

where A is an $n \times n$ matrix, b is an n vector & x is a vector of n unknowns.

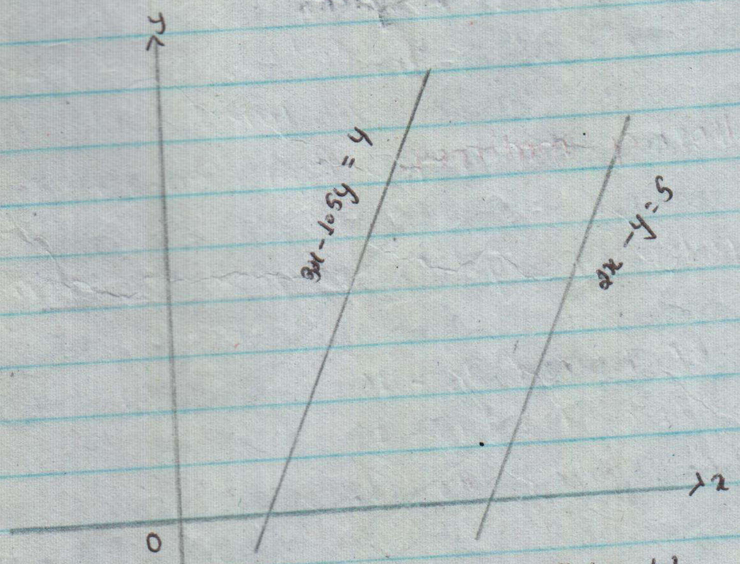
Existance of solution.

Given an arbiratory system of equation. It is difficult to say weather the system has a soln or not. There are 4 possibilities.

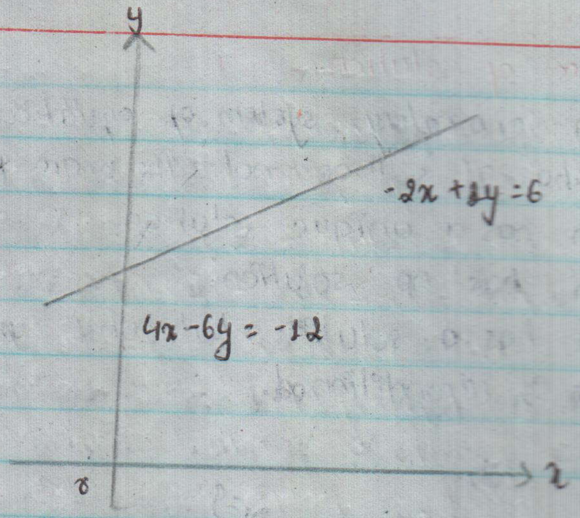
1. System has a unique solution.
2. System has no solution.
3. System has a solution but not a unique one. (infinite soln)
4. System is ill conditioned.



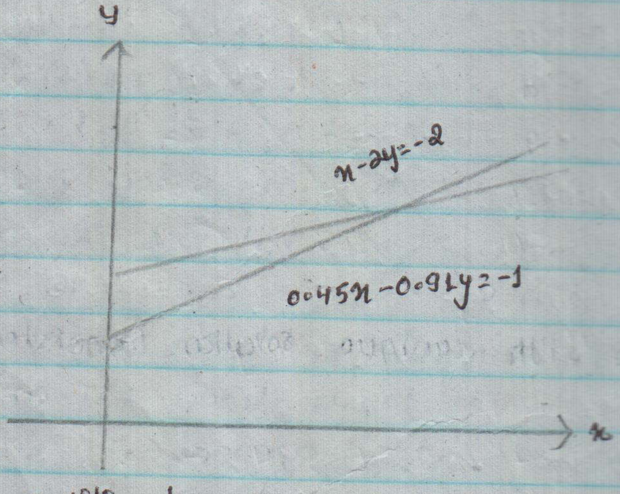
(a) System with unique solution (consistent)



(b) System with no solution (inconsistent)



(c) System with infinite solution (consistent)



(d) ill-conditioned system.

Properties of matrices.

Gaussian Elimination method:

Gauss elimination method reduces the system of equations to an equivalent upper triangular system which can be solved by backward substitution.

Let us consider a general set of n equation and n unknown variable.

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots a_{nn}x_n = b_n$$

Let us assume that a solution exist and that is unique. Following algorithms are used.

① Arrange equations such that $a_{11} \neq 0$.

② Eliminate x_1 from all but except 1st equation as follows.

ⓐ Normalize the first equation by dividing it by a_{11}

ⓑ Substract from the second equation a_{21} times the normalized 1st equation. The result is

$$\left[a_{21} - a_{21}\frac{a_{11}}{a_{11}}\right]x_1 + \left[a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right]x_2 + \cdots\cdots$$
$$\cdots\cdots\cdots = b_2 - a_{21}\frac{b_1}{a_{11}}$$

We can see that $a_{21} - a_{21}\frac{a_{11}}{a_{11}} = 0$

Thus the resultant equation doesnot contain x_1. The new second equation is

$$0 + a_{22}'x_2 + a_{23}'x_3 + \cdots\cdots a_{2n}'x_n = b_2'$$

③ Similarly substract from the third equation a_{31} times the normalized 1st equation. The result will be

$$0 + a_{32}'x_2 + \cdots\cdots a_{3n}x_n = b_3'$$

Continuing this process we get,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n = b_1'$$
$$a_{22}'x_2 + \cdots\cdots a_{2n}'x_n = b_2'$$

$a_{n2}x_2 + \cdots\cdots a_{nn}x_n = b_n'$

③ eliminate x_2 from the third to the last equation in the new set, again we assume that $a_{22}' \neq 0$

④ substract from the third equation a_{32}' times the normalized second equation.

⑤ substract from the fourth equation a_{42}' times the normalized second equation & so on.

This process will continue till the last equation contains only one unknown, namely x_n. The final form is

$$a_{11}x_n + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{22}'x_2 + \cdots\cdots + a_{2n}'x_n = b_2'$$
$$\cdots\cdots\cdots\cdots$$
$$a_{nn}^{(n-1)}x_n = b_n^{n-1}$$

This process is called ~~tran~~ triangularisation. The number of primes indicate the number of times the coefficient have been modified.

④ Obtain the solution by back substution. The solution is as follows,

$$x_n = \frac{b_n^{n-1}}{a_{nn}^{n-1}}$$

This can be substuted back in the $n-1^{th}$ eq^n to obtained the solution for $x_{n-1}, x_{n-2}, \cdots\cdots x_1$.

Q/1 Solve the following system of equations using Gauss elimination method

$$3x_1 + 6x_2 + x_3 = 16$$
$$2x_1 + 4x_2 + 3x_3 = 13$$
$$x_1 + 3x_2 + 2x_3 = 9.$$

Solution,

The given equations are,

$$3x_1 + 6x_2 + x_3 = 16$$

$$2x_1 + 4x_2 + 3x_3 = 13$$
$$x_1 + 3x_2 + 2x_3 = 9$$

Now:
the first normalized equation is
$$3/3\, x_1 + 6/3\, x_2 + 1/3\, x_3 = 16/3$$

$$x_1 + 2x_2 + 1/3\, x_3 = 16/3$$

multiplying this equation by a_{21} i.e 2 we get,
$$2x_1 + 4x_2 + 2/3\, x_3 = 32/3$$

substracting this equation from equation 2 we get

$$(2x_1 + 4x_2 + 3x_3) - (2x_1 + 4x_2 + 2/3\, x_3) = 13 - \frac{32}{3}$$
$$0 + 0 + 3x_3 - \frac{2}{3} x_3 = \frac{39-32}{3}$$

$$\text{or, } \frac{7x_3}{3} = \frac{7}{3}$$

$$\therefore x_3 = 1,$$

for 3rd eq^n, multiplying 1st normalized equation by a_{31} i.e 1, we g
$$x_1 + 2x_2 + \frac{1}{3} x_3 = 16/3$$

substracting this equation from eq^n 3 we get
$$(x_1 + 3x_2 + 2x_3) - (x_1 + 2x_2 + \frac{1}{3} x_3) = 9 - \frac{16}{3}$$

$$\text{or, } x_2 + \frac{5x_3}{3} = \frac{11}{3}$$

Now we know the value of x_3, so by back substitution, we get.
$$x_2 + \frac{5 \cdot 1}{3} = \frac{11}{3}$$

$$x_2 = \frac{11}{3} - 5/3$$

$$x_2 = \frac{11 - 5}{3}$$

$$= 6/3$$

$$= 2$$

and,

$$3x_1 + 6x_2 + x_3 = 16$$

or $3x_1 + 6 \times 2 + 1 = 16$

or $3x_1 + 13 = 16$

or $3x_1 = 16 - 13$

or, $3x_1 = 3$

$\therefore \; x_1 = 1$

Therefore

$x_1 = 1$

$x_2 = 2$

$x_3 = 1.$

Gauss Elimination with pivoting.

In basic gaussian elimination method, the elimination a_{ij} where $i=j$ is known as pivot element. Each row is normalized by dividing the coefficient of that row by its pivot element. If a_{kk} is zero, k^{th} row cannot be normalized and the procedure fails. One way to overcome this problem is to interchange this row with another row below it which doesnot have a zero element and have largest (absolute value) coefficient in that position.

The procedure of re-ordering involves the following steps.

(i) search & locate the largest absolute value among the coefficients in the first column.

(ii) Exchange the first row with the row containing that element.

(iii) Then eliminate the 1st variable in the second equation as explained earlier.

(iv) When the second row becomes pivot row, search for the coefficients in the 2nd column from the 2nd row to the n^{th} row and locate the largest coefficient. Exchange the 2nd row with the row containing largest coefficient

(v) Continue this process untill $n-1$ unknown are eliminated.

Q/1 Solve the following system of equations using partial pivoting technique,

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 - x_2 + x_3 = 0$$

Soln,

The given equations are,

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$

$$x_1 - x_2 + x_3 = 0$$

Here in these three equations, the highest coefficient of x_1 is in second equation, so interchanging second equation with first equation we get,

$$4x_1 + 2x_2 + 3x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = 6$$
$$x_1 - x_2 + x_3 = 0$$

Now Normalized 1st equation we get,

$$\frac{4}{4}x_1 + \frac{2}{4}x_2 + \frac{3}{4}x_3 = \frac{4}{4}$$

or $$x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3 = 1,$$

multiplying this equation by 2 we get,

$$2x_1 + x_2 + \frac{3}{2}x_3 = 2.$$

Substracting this equation from second equation we get,

$$(2x_1 + 2x_2 + x_3) - (2x_1 + x_2 + \frac{3}{2}x_3) = 6 - 2$$

or, $$0 + x_2 + x_3 - \frac{3}{2}x_3 = 4$$

or $$x_2 - \frac{x_3}{2} = 4$$

for third equation multiplying the normalized 1st eq^n by 1 we get,

$$x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3 = 1,$$

Substracting this eq^n from third equation we get,

$$(x_1 - x_2 + x_3) - (x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3) = 0 - 1$$

or, $$-x_2 - \frac{1}{2}x_2 + x_3 - \frac{3}{4}x_3 = -1.$$

or, $\frac{-3x_2}{2} + \frac{x_3}{4} = -1$

or, $-6x_2 + x_3 = -4$

or $6x_2 - x_3 = 4.$

The system of linear equations becomes,

$$4x_1 + 2x_2 + 3x_3 = 4$$
$$2x_2 - x_3 = 8$$
$$6x_2 - x_3 = 4.$$

Now comparing the ~~eq~~ coefficients of x_2 in second & 3rd row we get the greater coefficient in 3rd row, so we interchange 2nd & 3rd row,

$$4x_1 + 2x_2 + 3x_3 = 4$$
$$6x_2 - x_3 = 4$$
$$2x_2 - x_3 = 8$$

Now Normalized second eq^n is

$$\frac{6}{6}x_2 - \frac{1}{6}x_3 = \frac{4}{6}$$

or $x_2 - \frac{1}{6}x_3 = \frac{2}{3}$

multiplying this equation by 2 we get

$$2x_2 - \frac{1}{3}x_3 = \frac{4}{3}$$

Substracting this eq^n from row 3 we get

$$(2x_2 - x_3) - (2x_2 - \frac{1}{3}x_3) = 8 - 4/3$$

$$-x_3 + \frac{1}{3}x_3 = \frac{24-4}{3}$$

$$-\frac{2x_3}{3} = \frac{20}{3}$$

$$2x_3 = -20$$

$$x_3 = -10$$

by back substitution, of row 2 we got

$$6x_2 - x_3 = 4$$

$$6x_2 - (-10) = 4$$

$$6x_2 + 10 = 4$$

$$6x_2 = -6$$

$$x_2 = -1 \,//$$

and,

$4x_1 + 2x_2 + 3x_3 = 4$

or $4x_1 - 2 - 3\times10 = 4$

or $4x_1 - 2 - 30 = 4$

or $4x_1 - 32 = 4$

or $4x_1 = 36$

or $x_1 = \frac{36}{4}$

$= 9 \,//$

$\therefore$ $x_1 = 9$

$x_2 = -1$

$x_3 = -10 \,//$

Gauss-Jordan method:

It is another method used for solving a system of linear equations. In this method a variable is eliminated from the rows below & above the pivot equation. This process thus eliminates all the off diagonal terms producing a diagonal matrix rather than a triangular matrix. Further all rows are normalized by dividing them by their pivot elements. This is illustrated in figure as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3'' \end{bmatrix}$$

Result of Gauss elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix}$$

Result of Gauss-Jordan elimi[nation]

The algorithm for Gauss-Jordan method is as follows.

1. Normalized first eqn by dividing it by its pivot element.
2. Eliminate x_1 term from all other equations.
3. Now, normalized the 2nd equation by dividing it by [its] pivot element.
4. Eliminate x_2 from all the equations above & below the nor[ma]-lized pivotal equation.
5. Repeate this process until x_n is eliminated from all but from the last equation.

⑥ The resultant b vector is the solution vector

The gauss - Jordan method requires approximately 50 % more arithmatic operation compare to gauss method therefore this method is rarely used.

Q. Solve the system of linear equations by Gauss - Jordan method.

$$2x_1 + 4x_2 - 6x_3 = -8$$
$$x_1 + 2x_2 + x_3 = 10$$
$$2x_1 - 4x_2 - 2x_3 = -12.$$

Solution,

The Given equations are,

$$2x_1 + 4x_2 - 6x_3 = -8 \quad \text{---} \quad (i)$$
$$x_1 + 2x_2 + x_3 = 10 \quad \text{---} \quad (ii)$$
$$2x_1 - 4x_2 - 2x_3 = -12 \quad \text{---} \quad (iii)$$

Normalized 1st equation is,

$$x_1 + 2x_2 - 3x_3 = -4$$

multiplying this eqn by 1 we get

$$x_1 + 2x_2 - 3x_3 = -4$$

Substracting this eqn from eqn (ii) we get.

$$(x_1 + 2x_2 + x_3) - (x_1 + 2x_2 - 3x_3) = 10 - (-4)$$

or $\quad 0 + 0 + x_3 + 3x_3 = 14$

or $\quad 4x_3 = 14$

$$x_3 = \frac{14}{4}$$
$$= 3.5$$

Again multiplying normalized 1st eqn by 2 we get,

$$2x_1 + 4x_2 - 6x_3 = -8$$

substracting from 3rd we get

$$-4x_2 - 4x_2 - 2x_3 + 6x_3 = -12 - (-8)$$

or, $-8x_2 + 4x_3 = -4$

Now, the system of linear eq^n becomes,

$$x_1 + 2x_2 - 3x_3 = -4 \quad \text{--- (i)}$$
$$x_3 = 3.05 \quad \text{--- (ii)}$$
$$-8x_2 + 4x_3 = 4 \quad \text{--- (iii)}$$

Interchanging eq^n (ii) & (3)

$$x_1 + 2x_2 - 3x_3 = -4 \quad \text{--- (i)}$$
$$-8x_2 + 4x_3 = \cancel{3.05}\ -4 \quad \text{--- (ii)}$$
$$x_3 = 3.05 \quad \text{--- (iii)}$$

Now, dividing eq^n (ii) by -8 to get normaliced form

$$x_2 - 0.5x_3 = -0.4375$$

multiplying this eq^n by 2 and substracting from eq^n (1)

$$(x_1 + 2x_2 - 3x_3) - (2x_2 - x_3) = -4 - (-0.875)$$

or $x_1 - 0 - 3x_3 + x_3 = -4 + 0.875$

or $x_1 - 2x_3 = -3.0125$,,

Hence the system of equations becomes,

$$x_1 - 2x_3 = -3.0125 \quad \text{--- (i)}$$
$$x_2 - 0.5x_3 = -0.4375 \quad \text{--- (ii)}$$
$$x_3 = -4 \quad \text{--- (iii)}$$

Now multiplying 3rd eq^n by -0.5 and ~~adding~~ substracting with eq^n (2) we get.

$$x_2 - 0.5x_3 - (-0.5x_3) = -0.4375 - (2)$$

or $x_2 = 1.5625$

Now multiplying 3rd eqn by -2 & substracting from eqn (1)

$$(x_1 - 2x_3) - (-2x_3) = -3.125 - (-8)$$

or, $x_1 = -11.128.$

Hence the system of linear eqn becomes

$$x_1 + 0x_2 + 0x_3 = -11.128$$
$$0x_1 + x_2 + 0x_3 = 1.5625$$
$$0x_1 + 0x_2 + \cancel{0}x_3 = -4.5$$

which can be represented as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11.128 \\ 1.5625 \\ -4 \end{bmatrix}$$

Hence the solution vector is

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} -11.128 \\ 1.5625 \\ -4 \end{bmatrix} //$$