1

UNIT

Set Theory and Real & Complex Number

Set Theory

1.1 Introduction

The idea of a set theory is fundamental in mathematics. All mathematical objects and constructions go back to set theory. It was developed by German mathematician George Cantor (1845–1915).

A **set** is defined as a well defined collection of objects. By well defined, we mean clearly defined or clearly described. The objects are called the elements of the set. Words like collection, group, family, aggregate are often used to convey the idea of set in everyday life.

The examples of sets are given below:

(a) The set of BCA students of Tribhuvan University.

(b) The set of letters of the word "COMPUTER".

(c) The set of prime numbers = {2, 3, 5, 7, 11 ...}.

(d) The set of square numbers = {1, 4, 9, 16, 25 ...}.

(e) The set of vowels in English alphabet = $\{a, e, i, o, u\}$.

The collections given below are not sets as they are not well defined.

(a) Collection of smart students of a college.

(b) Collection of rich people of the world.

(c) Collection of good computer programmings.

1.2 Notation

The objects of a set are called elements or members of the set. Sets are usually denoted by capital letters A, B, C ... X, Y, Z but the elements of sets are usually denoted by small letters *a, b, c, ... x, y, z*.

1.3 Meaning of Symbol

$\in$ an element of or belongs to or is a member of

$\notin$ not an element of or does not belong to or is not a member of

$\Rightarrow$ implies that (conditional)

$\Leftrightarrow$ if and only if (Biconditional)

/or : such that

ϕ null or empty or void set

$\subseteq$ subset

$\subset$ proper subset

1.4 Specification of Sets

The following are the method of representing a set.

(a) Listing method or Roster method or Tabulation method.

(b) Descriptive phrase method.

(c) Rule method or set builder method.

1. **Listing Method or Roster Method or Tabulation Method**

This method is used in listing each element of the set within the brackets.

For example

A = {1, 2, 3, 4, 5}.

The order in which the elements are listed is not important in sets. Therefore, B = {*a, b, c, d*} and C = {*a, d, b, c*} represent the same set. Note that an element of a set is not written more than once.

2. **Descriptive Phrase Method**

This method consists in placing a phrase describing the elements of the set.

For examples

A = The set of odd numbers between 1 and 20.

B = The set of stars in the sky.

This method may be used when there is a large number of elements or when all the elements cannot be named.

3. **Rule Method or Set Builder Method**

In this method, set is described by defining the property that characterizes or specifies all elements of the set.

For example

V = $\{a, e, i, o, u\}$ can be written as V = $\{x : x$ is a vowel in English alphabet$\}$.

1.5 Types of Sets

1. **Finite and Infinite Sets**

A set having finite number of elements is called finite set.

For examples

A = $\{1, 2, 3\}$.

B = $\{x : x$ is a month of the year$\}$.

A set having infinite number of elements is called infinite set.

For examples

A = $\{x : x$ is a star in the sky$\}$.

B = $\{x : x$ is a prime number$\} = \{2, 3, 5, 7, \ldots\}$.

2. **Null Set or Empty Set or Void Set**

A set having no element is called a null or empty or void set. It is denoted by ϕ (phi) or $\{\ \}$.

For examples

A = $\{x : x$ is a prime number between 25 and 27$\} = \phi$.

B = $\{x : x^2 + 1 = 0, x \in \mathbb{R}\}$.

3. **Singleton Set or Unit Set**

A set having only one element is called singleton set.

For examples

A = $\{ 0 \}$.

B = The set of even prime numbers = $\{2\}$.

4. **Universal Set**

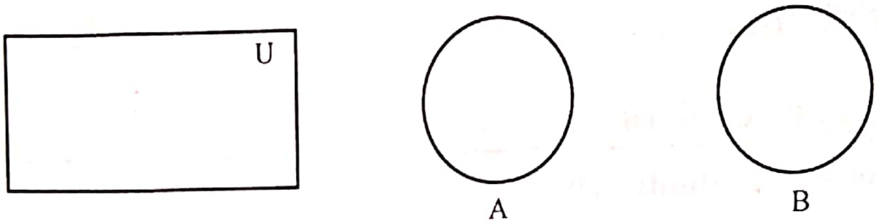
Universal set is the totality of elements under the consideration as elements of any set. It is denoted by the letter U.

For example

The set of all real numbers is the universal set in the real number system.

1.6 Venn Diagram

The diagramatic representation of sets is called Venn diagram. It was developed by British mathematician John Venn (1834 – 1923). The universal set U is usually represented by rectangle and any other given set is represented by a circle.



1.7 Relations between Sets

1. Subsets

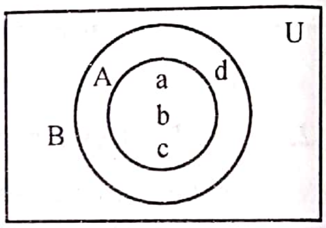
If A and B are two sets such that each element of A is also an element of B then A is called a subset of B. It is denoted by $A \subseteq B$.

Here, B is called **superset** of A.

For example

Let $A = \{a, b, c\}$

$B = \{a, b, c, d\}$. Here, all the elements of A are in B. So, $A \subseteq B$.



$A \subseteq B$

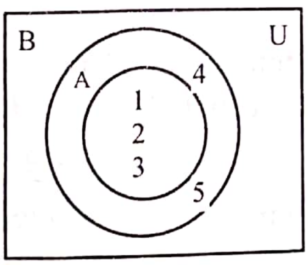
Proper Subset

If each element of set A is an element of B but at least one element of B does not belong to A (i.e. $A \neq B$), then A is called a **proper subset of** B. It is denoted by $A \subset B$.

For example

Let $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5\}$. Here, each elements of A are in B but $4, 5 \in B$ are not in A. So, $A \subset B$.



$A \subset B$

Note:

(i) *Every set is a subset of itself.*

(ii) *The null set ϕ is the subset of every set.*

(iii) *Every set is a subset of universal set U.*

(iv) *The number of subsets of a set having n distinct elements is 2^n, number of proper subsets $= 2^n - 1$ and number of proper non–empty subsets $= 2^n - 2$.*

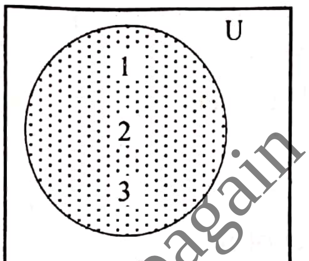
2. Equal and Equivalent Sets

Two given sets A and B are called equal sets if they have the same elements. It is denoted by A = B.

For example

Let, A = {1, 2, 3}, B = {3, 2, 1}

Then, A = B



A = B

Two sets A and B are said to be equivalent if they have equal number of elements. It is denoted by A ~ B.

For example

A = {*a*, *b*, *c*} and B = {1, 2, 3} are equivalent sets.

3. Joint and Disjoint Sets

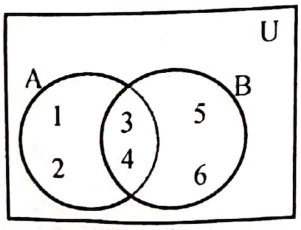
Two given sets are said to be joint (overlapping) sets if they have at least one element in common. Otherwise the sets are disjoint.

For example

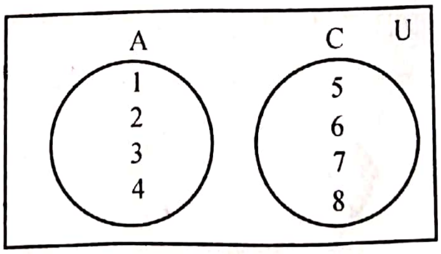
Let, A = {1, 2, 3, 4}

B = {3, 4, 5, 6}

C = {5, 6, 7, 8}



Joint sets



Disjoint sets

Here, A and B are joint sets and A and C are disjoint sets.

1.8 Power Set

The set of all the subsets of the given set S is called power set of S. It is denoted by 2^S or P(S).

For example

Set	No. of elements	No. of subsets
ϕ	0	$2^0 = 1$
$\{a\}$	1	$2^1 = 2$
$\{a, b\}$	2	$2^2 = 4$
$\{a, b, c\}$	3	$2^3 = 8$
$\vdots$	$\vdots$	$\vdots$
$\{a, b, c, \ldots$, to n elements$\}$	n	2^n

Example: Write the power set of the set S = $\{a, b, c\}$.

Solution

No. of elements in set S, $n = 3$

Total no. of possible subsets $= 2^n = 2^3 = 8$

Then, the possible subsets of S are

$\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}$.

Hence, the power set of S,

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$.

1.9 Operations on Sets

We can combine given sets to produce new sets by using operations on sets defined as follows:

1. Union

The union of two sets A and B is the set of all elements belonging either to A or to B or to both. It is denoted by $A \cup B$.

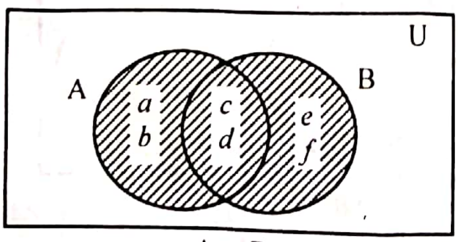
In set builder form, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

For examples

(i) If $A = \{a, b, c, d\}$ and

$B = \{c, d, e, f\}$ then

$A \cup B = \{a, b, c, d, e, f\}$.

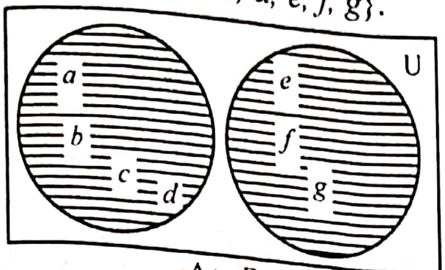


$A \cup B$

(ii) If A $= \{a, b, c, d\}$ and

B $= \{e, f, g\}$ then

$A \cup B = \{a, b, c, d, e, f, g\}$.



A ∪ B

Note: The words like either–or, at least one refer to union.

2. Intersection

The intersection of two sets A and B is the set of all elements belonging to both A and B. It is denoted by $A \cap B$.

In set builder form, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

For examples

(i) If A $= \{a, b, c, d\}$

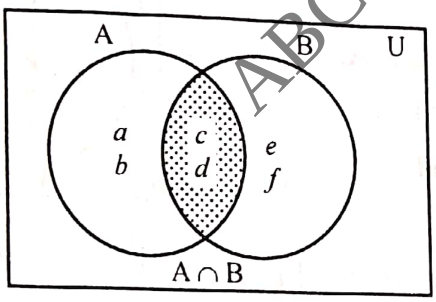
B $= \{c, d, e, f\}$ then

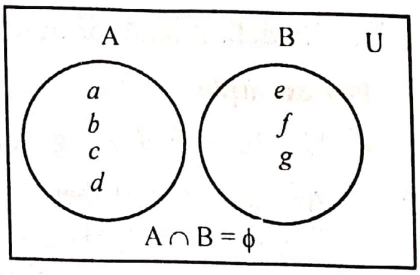
$A \cap B = \{c, d\}$.

(ii) If A $= \{a, b, c, d\}$

B $= \{e, f, g\}$ then

$A \cap B = \phi$.





Note: The words like– and both, common refer to intersection.

3. Difference

The difference of sets A and B denoted by $A - B$ is the set of all the elements that belongs to A but does not belong to B.

In set builder form, $A - B = \{x : x \in A, x \notin B\}$.

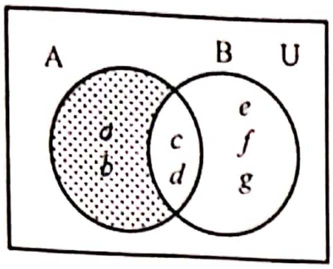
Similarly, $B - A = \{x : x \in B, x \notin A\}$.

For examples

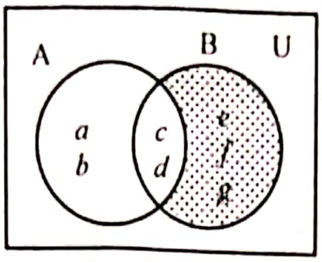
(i) If $A = \{a, b, c, d\}$

$B = \{c, d, e, f, g\}$ then

$A - B = \{a, b\}$ and $B - A = \{e, f, g\}$.



A − B



B − A

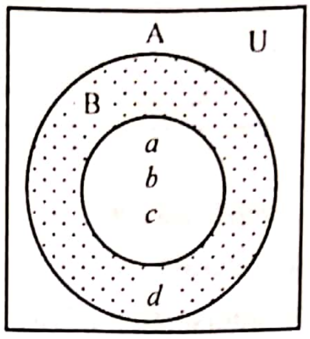
(ii) If A = $\{a, b, c, d\}$

B = $\{a, b, c\}$ then

A − B = $\{d\}$

Similarly, B − A = ϕ.

Note: *The word only refers to difference.*



A − B

4. Complement

If U be a universal set and $A \subseteq U$, then the complement of A is the set of all elements that belong to U but not to A. It can be written as A′ or $\bar{A}$ or A^c.

In set builder form,

$\bar{A} = U - A = \{x : x \in U, x \notin A\}$

$\therefore$ By definition A′ or $\bar{A}$ or $A^c = U - A$.

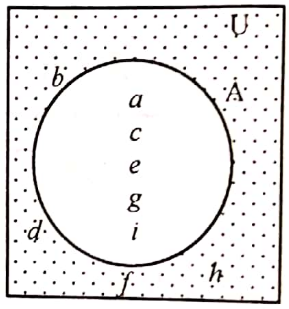
For example

If U = $\{a, b, c, d, e, f, g, h, i\}$ and

A= $\{a, c, e, g, i\}$ then

$A^c = U - A = \{a, b, c, d, e, f, g, h, i\} - \{a, c, e, g, i\}$

$= \{b, d, f, h\}$.



$A^c = U - A$

Note: *The words like–none of them, neither, nor refer to complement.*

5. Symmetric Difference

The symmetric difference of two sets A and B, denoted by AΔB (read as A delta B) is defined by A Δ B = (A − B) ∪ (B − A).

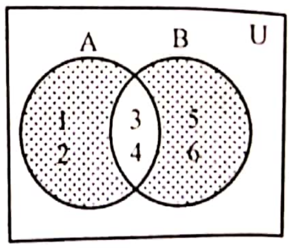
For example

If A = $\{1, 2, 3, 4\}$ and B = $\{3, 4, 5, 6\}$, then

AΔB = (A − B) ∪ (B − A)

= $\{1, 2\} \cup \{5, 6\}$

= $\{1, 2, 5, 6\}$.



A Δ B

1.10 Laws of Algebra of Sets

There are various laws (properties) of the operations on sets which are given below.

1. **Idempotent laws**

   (a) $A \cup A = A$ (b) $A \cap A = A$.

2. **Identity laws**

   (a) $A \cup U = U$ (b) $A \cap U = A$

   (c) $A \cup \phi = A$ (d) $A \cap \phi = \phi$.

3. **Commutative laws**

   (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$.

4. **Associative laws**

   (a) $A \cup (B \cup C) = (A \cup B) \cup C$ (b) $A \cap (B \cap C) = (A \cap B) \cap C$.

5. **Distributive laws**

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

6. **Complementation laws**

   (a) $(A^c)^c = A$ (b) $\phi^c = U$

   (c) $U^c = \phi$ (d) $A \cup A^c = U$

   (e) $A \cap A^c = \phi$.

7. **De–Morgan's laws**

   (a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$.

1.11 Cardinality of a Set

The number of elements in a finite set A is called the cardinality of the set. It is denoted by $n(A)$ or $|A|$.

Thus, if $A=\{x_1, x_2, x_3 \ldots x_n\}$ then $|A| = n$.

For example

If $A = \{1, 2, 3\}$ then $n(A) = 3$.

The cardinality of the null set ϕ is zero. The cardinality of singleton set is one.

Note: *We can say that the cardinality of an infinite set is undefined.*

List of Important Formulae

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ if A and B are joint sets.

(ii) $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint sets.

(iii) $n(A \cup B) = n_o(A) + n_o(B) + n(A \cap B)$, where $n_o(A)$ and $n_o(B)$ denote the number of elements in A only and B only respectively.

(iv) $n(A^c) = n(U) - n(A)$

(v) $n_o(A) = n(A) - n(A \cap B)$

(vi) $n(A - B) = n(A) - n(A \cap B)$

(vii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - (C \cap A) + n(A \cap B \cap C)$

* *Besides these formulae, we can find the number of elements* in *sets using Venn diagrams.*

WORKED OUT EXAMPLES

Example 1. Rewrite the following in set notation form.

(a) A is a subset of B.

(b) x belongs to set A.

(c) a is not an element of set A.

(d) x belongs to G implies that x belongs to H.

Solution

(a) $A \subseteq B$

(b) $x \in A$

(c) $a \notin A$

(d) $x \in G \Rightarrow x \in H$.

Example 2. Interpret the following.

(a) $a \notin A$

(b) $A \subseteq B$

(c) $M \not\subset N$

(d) $x \in R \Rightarrow x \in H$.

Solution

(a) a does not belong to A.

(b) A is a subset of B.

(c) M is not a proper subset of N.

(d) x belongs to R implies that x belongs to H.

Example 3. Write the following sets in tabular form

(a) Set of vowels in English alphabets.

(b) Set of single digit even positive numbers.

Solution

(a) $\{a, e, i, o, u\}$

(b) $\{2, 4, 6, 8\}$.

Example 4. Write the following sets in set builder form

(a) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(b) $B = \{3, 6, 9, 12, 15, 18\}$

(c) $C = \{-9, -8, -7, -6, -5\}$.

Solution

(a) $A = \{x : x$ is a set of natural numbers less than $10\}$.

(b) $B = \{x : x$ is a set of natural numbers less than 19 and divisible by $3\}$.

(c) $C = \{x : -9 \le x \le -5, x \in \mathbb{Z}\}$.

Example 5. If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{7, 8, 9\}$, find

(a) $A \cup B$ (b) $A \cap B$

(c) $A \cup C$ (d) $A \cap C$

(e) $A - B$ (f) $A \Delta B$.

Solution

(a) $A \cup B = \{1, 2, 3, 4\} \cup \{3, 4, 5, 6\}$
$= \{1, 2, 3, 4, 5, 6\}$.

(b) $A \cap B = \{1, 2, 3, 4\} \cap \{3, 4, 5, 6\}$
$= \{3, 4\}$.

(c) $A \cup C = \{1, 2, 3, 4\} \cup \{7, 8, 9\}$
$= \{1, 2, 3, 4, 7, 8, 9\}$.

(d) $A \cap C = \{1, 2, 3, 4\} \cap \{7, 8, 9\}$
$= \phi$.

(e) $A - B = \{1, 2, 3, 4\} - \{3, 4, 5, 6\}$
$= \{1, 2\}$.

(f) $A \Delta B = (A - B) \cup (B - A)$
$= \{1, 2\} \cup \{5, 6\}$
$= \{1, 2, 5, 6\}$.

Example 6. If $A = \{a, b, c, d, e\}$, $B = \{c, d, e, f, g\}$ and $C = \{a, c, e, g\}$, show that

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) $A - (B \cup C) = (A - B) \cap (A - C)$.

Solution

(a) $A \cup B = \{a, b, c, d, e\} \cup \{c, d, e, f, g\}$
$= \{a, b, c, d, e, f, g\}$

$A \cup C = \{a, b, c, d, e\} \cup \{a, c, e, g\}$
$= \{a, b, c, d, e, g\}$

$B \cap C = \{c, d, e, f, g\} \cap \{a, c, e, g\}$
$= \{c, e, g\}$

Now, $A \cup (B \cap C) = \{a, b, c, d, e\} \cup \{c, e, g\}$
$= \{a, b, c, d, e, g\}$

And,

$(A \cup B) \cap (A \cup C) = \{a, b, c, d, e, f, g\} \cap \{a, b, c, d, e, g\}$

$= \{a, b, c, d, e, g\}$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$

(b) $B \cup C = \{c, d, e, f, g\} \cup \{a, c, e, g\}$

$= \{a, c, d, e, f, g\}$

$A - B = \{a, b, c, d, e\} - \{c, d, e, f, g\}$

$= \{a, b\}$

$A - C = \{a, b, c, d, e\} - \{a, c, e, g\}$

$= \{b, d\}$

Now, $A - (B \cup C) = \{a, b, c, d, e\} - (a, c, d, e, f, g\}$

$= \{b\}$

And,

$(A - B) \cap (A - C) = \{a, b\} \cap \{b, d\}$

$= \{b\}$

$\therefore A - (B \cup C) = (A - B) \cap (A - C).$

Example 7. If $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 3, 4, 6\}$ and $B = \{2, 3, 5, 7\}$, verify that $(A \cap B)^c = A^c \cup B^c$.

Solution

$(A \cap B) = \{1, 3, 4, 6\} \cap \{2, 3, 5, 7\}$

$= \{3\}$

$A^c = U - A = \{1, 2, 3, 4, 5, 6, 7\} - \{1, 3, 4, 6\}$

$= \{2, 5, 7\}$

$B^c = U - B$

$= \{1, 2, 3, 4, 5, 6, 7\} - \{2, 3, 5, 7\}$

$= \{1, 4, 6\}$

And,

$(A \cap B)^c = U - (A \cap B)$

$= \{1, 2, 3, 4, 5, 6, 7\} - \{3\}$

$= \{1, 2, 4, 5, 6, 7\}$

Then,

$A^c \cup B^c = \{2, 5, 7\} \cup \{1, 4, 6\}$

$= \{1, 2, 4, 5, 6, 7\}$

$(A \cap B)^c = A^c \cup B^c.$

Example 8. Given $U = \{1, 2, 3 \ldots, 15\}$; $A = \{x : x \geq 8\}$; $B = \{x : x \leq 4\}$; $C = \{x : 4 < x < 12\}$. Find $(A \cap B)$, $(A \cup B)$, $(\overline{A \cup C})$, $(A - C)$.

Solution

$U = \{1, 2, 3, \ldots, 15\}$

$A = \{x : x \geq 8\} = \{8, 9, 10, \ldots, 15\}$

$B = \{x : x \leq 4\} = \{1, 2, 3, 4\}$

$C = \{x : 4 < x < 12\} = \{5, 6, 7, \ldots, 11\}$

Now,

$A \cap B \quad = \{8, 9, 10, 11, 12, 13, 14, 15\} \cap \{1, 2, 3, 4\}$

$\quad = \phi$

$A \cup B \quad = \{8, 9, 10, 11, 12, 13, 14, 15\} \cup \{1, 2, 3, 4\}$

$\quad = \{1, 2, 3, 4, 8, 9, 10, \ldots, 15\}$

$A \cup C \quad = \{8, 9, 10, 11, 12, 13, 14, 15\} \cup \{5, 6, 7, 8, 9, 10, 11\}$

$\quad = \{5, 6, 7, 8, 9, \ldots, 15\}$

$(A \cup C)^c \quad = U - (A \cup C)$

$\quad = \{1, 2, 3, \ldots, 15\} - \{5, 6, 7, \ldots, 15\}$

$\quad = \{1, 2, 3, 4\}$

$A - C \quad = \{8, 9, 10, \ldots 15\} - \{5, 6, 7, \ldots 11\}$

$\quad = \{12, 13, 14, 15\}$.

Example 9. In a statistical investigation of 500 families in certain town, it was found that 40 families had neither a radio nor a TV, and 320 families had a radio and 190 a TV. How many families in that group had both radio and TV?

Solution

Total number of families, $n(U) = 500$

Number of families having radio, $n(R) = 320$

Number of families having TV, $n(T) = 190$

Number of families having neither radio nor TV, $n(\overline{R \cup T}) = 40$

Number of families having both radio and TV, $n(R \cap T) = ?$

We have,

$n(R \cup T) \quad = n(U) - n(\overline{R \cup T})$

$\quad = 500 - 40$

$\quad = 460$

Also,

$n(R \cap T) \quad = n(R) + n(T) - n(R \cup T)$

$\quad = 320 + 190 - 460$

$\quad = 50$.

Example 10. In a class containing 100 students, 50 study mathematics, 40 study computer and 25 study both. Find out (i) how many students study at least one subject (ii) how many students study mathematics only (iii) how many students study computer only (iv) how many students study neither mathematics nor computer.

Solution

Total no. of students, $n(U) = 100$

No. of mathematics students, $n(M) = 50$

No. of computer students, $n(C) = 40$

No. of students who read both the subjects, $n(M \cap C) = 25$

(i) No. of students who study at least one subject.

$n(M \cup C) = n(M) + n(C) - n(M \cap C)$

$= 50 + 40 - 25$

$= 65.$

(ii) No. of students who study mathematics only.

$n_0(M) = n(M) - n(M \cap C)$

$= 50 - 25$

$= 25.$

(iii) No. of students who study computer only.

$n_0(C) = n(C) - n(M \cap C)$

$= 40 - 25$

$= 15.$

(iv) No. of students who study neither mathematics nor computer.

$n(M \cup C)^c = n(U) - n(M \cup C)$

$= 100 - 65$

$= 35.$

Example 11. In a survey, it was found that 400 people were randomly selected out of which 182 used Nike shoes, 169 used Shikhar shoes and 90 persons used both Nike and Shikhar. Find the number of persons who used exactly one of these brands.

Solution

Let N and S represent the set of people using Nike and Shikhar shoes respectively. Then,

$n(N) = 182$

$n(S) = 169$

$n(N \cap S) = 90$

$n_o(N)$ = No. of persons who used Nike only.

$= n(N) - n(N \cap S)$

$= 182 - 90$

$= 92$

$n_o(S)$ = No. of persons who used Shikhar only.

$= n(S) - n(N \cap S)$

$= 169 - 90$

$= 79$

$\therefore$ Total no. of persons using exactly one brand of shoes

$= n_0(N) + n_0(S)$

$= 92 + 79$

$= 171.$

Example 12. In a town of 25,000 population in Kathmandu, 14000 use motorbike, 2500 use bicycle and 500 use both. What percentage use neither motorbike nor bicycle?

Solution

Total population, $n(U) = 25{,}000$

No. of people who use motorbike, $n(M) = 14000$

No. of people who use bicycle, $n(B) = 2500$

No. of people who use both motorbike and bicycle,

$n(M \cap B) = 500$

No. of people who use either motorbike or bicycle,

$$\begin{aligned} n(M \cup B) &= n(M) + n(B) - n(M \cap B) \\ &= 14000 + 2500 - 500 \\ &= 16000 \end{aligned}$$

No. of people who use neither motorbike nor bicycle,

$$\begin{aligned} n(M \cup B)^c &= n(U) - n(M \cup B) \\ &= 25000 - 16000 \\ &= 9000 \end{aligned}$$

$\therefore$ Percentage of people who use neither motorbike nor bicycle

$$\begin{aligned} &= \frac{9000}{25000} \times 100 \\ &= 36\%. \end{aligned}$$

Example 13. If A and B be two subsets of universal set U such that $n(U) = 350$, $n(A) = 100$, $n(B) = 150$ and $n(A \cap B) = 50$, then find $n(\overline{A} \cap \overline{B})$.

Solution

From De-Morgan's law of set theory, we have,

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

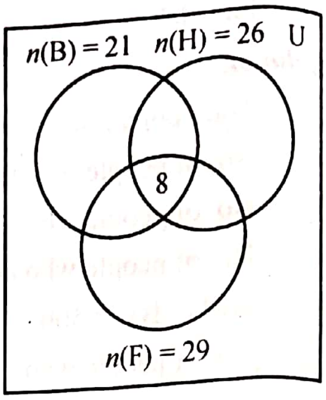
$$\therefore\ n(\overline{A} \cap \overline{B}) = n(\overline{A \cup B})$$

$$\begin{aligned} \text{Now, } n(\overline{A} \cap \overline{B}) &= n(\overline{A \cup B}) \\ &= n(U) - n(A \cup B) \\ &= n(U) - [n(A) + n(B) - n(A \cap B)] \\ &= 350 - (100 + 150 - 50) \\ &= 350 - 200 = 150. \end{aligned}$$

Example 14. Of the number of three athletic teams, 21 are in the basketball team, 26 in hockey team and 29 in football team, 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. How many members are there in all?

Solution

Let B, H, F be the sets of members playing basketball, hockey and football respectively.



Then,

$n(B)$ = No. of members playing basketball = 21

$n(H)$ = No. of members playing hockey = 26

$n(F)$ = No. of members playing football = 29

$n(H \cap B)$ = No. of members playing hockey and basketball = 14

$n(H \cap F)$ = No. of members playing hockey and football = 15

$n(F \cap B)$ = No. of members playing football and basketball = 12

$n(B \cap H \cap F)$ = No. of members playing all three games = 8

$n(U) = n(B \cup H \cup F)$ = Total no. of members = ?

We know,

$$n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(B \cap H) - n(H \cap F) - n(F \cap B) + n(B \cap H \cap F)$$
$$= 21 + 26 + 29 - 14 - 15 - 12 + 8 = 84 - 41$$
$$= 43.$$

$\therefore$ Total number of members = 43.

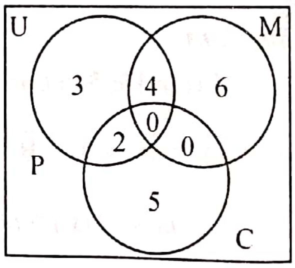
Example 15. Out of group of 20 teachers in a school, 10 teach Maths, 9 teach Physics, 7 teach Chemistry, 4 teach Maths and Physics, but none teach both Maths and Chemistry:

(i) How many teach Physics and Chemistry?

(ii) How many teach only Physics?

(iii) How many teach only Chemistry?

Solution



Let U be the set of all teachers. Also, let M, P and C denote the set of teachers who teaches Maths, Physics and Chemistry respectively. Then,

$n(U) = 20$ $\quad$ $n(M) = 10$

$n(P) = 9$ $\quad$ $n(C) = 7$

$n(M \cap P) = 4$ $\quad$ $n(M \cap C) = 0$

Then, $n(M \cap C \cap P) = 0$ $\quad$ $[\because (M \cap C) = 0]$

(i) $n(U) = n(P) + n(M) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(P \cap M \cap C)$

or, $20 = 9 + 10 + 7 - 4 - 0 - n(P \cap C) + 0$

or, $20 = 22 - n(P \cap C)$

$\therefore$ $n(P \cap C) = 22 - 20 = 2$

(ii) $n(P) = n_0(P) + n(M \cap P) + n(P \cap C) - n(P \cap M \cap C)$

or, $9 = n_0(P) + 4 + 2 - 0$

or, $n_0(P) = 9 - 6$

or, $n_0(P) = 3$

$\therefore$ 3 teachers teach only Physics.

(iii) $n(C) = n_0(C) + n(P \cap C) + n(M \cap C) - n(P \cap M \cap C)$

or, $7 = n_0(C) + 2 + 0 - 0$

or, $n_0(C) = 7 - 2 = 5$

$\therefore$ 5 teachers teach only Chemistry.

Example 16. In an examination conducted by T.U., 55% failed in English, 35% failed in Account and 30% failed in Economics, 16% failed in English and Economics, 10% failed in Economics and Account, 15% failed in English and Account and 7% failed in all three subjects. Find

(i) The pass percentage in all subjects

(ii) The fail percentage in one subject

(iii) The fail percentage in exactly two subjects.

Solution

Let the total number of students be 100.

If E, A, E_c represent the sets of students failed in English, Account and Economics respectively. Then

$n(E) = 55$, $n(A) = 35$,

$n(E_c) = 30$, $n(E \cap E_c) = 16$

$n(E_c \cap A) = 10$, $n(E \cap A) = 15$,

$n(E \cap A \cap E_c) = 7$

(i) We have

$$n(E \cup A \cup E_c) = n(E) + n(A) + n(E_c) - n(E \cap A) - n(A \cap E_c) - n(E_c \cap E) + n(E \cap A \cap E_c)$$

$$= 55 + 35 + 30 - 15 - 10 - 16 + 7$$

$$= 86$$

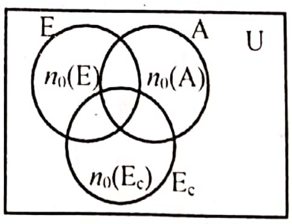
$\therefore$ 86% failed in at least one subject. Thus the percentage of students pass in all subjects $= n(U) - n(E \cup A \cup E_c)$

$= 100 - 86 = 14.$

(ii) No. of students who failed in economics only,

$n_0(E_c) = n(E_c) - n(E \cap E_c) - n(E_c \cap A) + n(E \cap A \cap E_c)$

$= 30 - 16 - 10 + 7 = 11$



No. of students who failed in Account only,

$n_0(A) = n(A) - n(E_C \cap A) - n(E \cap A) + n(E \cap A \cap E_c)$

$= 35 - 10 - 15 + 7 = 17$

No. of students who failed in English only,

$n_0(E) = n(E) - n(E \cap A) - n(E \cap E_c) + n(E \cap A \cap E_c)$

$= 55 - 15 - 16 + 17 = 31$

Thus, total number of students who failed in exactly one subject

$= n_0(E) + n_0(A) + n_0(E)$

$= 11 + 17 + 31 = 59$

$\therefore$ % failed in exactly one subject is 59%.

(iii) No. of students who failed in English and Economics only

$n_0(A \cap E_c) = n(E \cap E_c) - n(E \cap A \cap E_c)$

$= 16 - 7 = 9$

No. of students who failed in English and Account only

$n_0(E \cap A) = n(E \cap A) - n(E \cap A \cap E_c)$

$= 15 - 7 = 8$

No. of students who failed in Account and Economics only

$n_0(A \cap E_c) = n(A \cap E_c) - n(E \cap A \cap E_c)$

$= 10 - 7 = 3$

$\therefore$ The total no. of students who failed in exactly two subjects

$n_0(E \cap E_c) + n_0(E \cap A) + n_0(A \cap E_c) = 9 + 8 + 3 = 20$

$\therefore$ 20% students failed in exactly two subjects.

EXERCISE – 1 A

1. (a) Given U = {1, 2, 3, 4, …, 12}, A = {2, 3, 5, 6, 8, 10}, B = {1, 4, 6, 9, 10} and C = {2, 3, 6, 8, 12}. Find:

   (i) $A \cup B$ (ii) $A - B$

   (iii) $(A \cup C)^c$ (iv) $(B \cap C)^c$ (v) $A \Delta B$

   (b) Given:

   U = {1, 2, 3, 4, …, 10}, A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6}. Find:

   (i) $(A \cup B) \cup C$ (ii) $(A \cup B) - C$ (iii) $(A - B) \cap C$

   (c) Given U = {1, 2, 3, …, 20}, A = $\{x : x \geq 10\}$, B = $\{x : x \leq 14\}$ where, A and B are subsets of the universal set U. Find:

   (i) $A - B$ (ii) $A^c \cup B^c$ (iii) $(B - A)^c$

   (d) If U = $\{x : 2 \leq x + 1 \leq 11$, x is an integer$\}$ A = $\{x : x$ is an even number$\}$ and B = $\{x : x$ is a prime number$\}$; find the followings.

   (i) $A \cap B$ (ii) $(A \cap B)^c$ (iii) $(A \cup B)^c$

2. (a) If A = $\{a, b, x, y\}$ and B = $\{c, d, x, y\}$ then find the following by Venn diagram (i) $B - A$ (ii) $A \cup B$.

   (b) Given set U = $\{x : x$ is a positive integer less than 11$\}$, A = {2, 3, 6, 7, 9}, B = {2, 4, 6, 8} and C = {2, 3, 4, 5, 9}. Find by Venn-diagram.

   (i) $B \cup C$ (ii) $A - B$

   (iii) $A \cup (B \cup C)$ (iv) $\overline{A \cap (B \cap C)}$.

3. Let $U = \{a, b, c, d, e\}$; $A = \{a, b, c\}$; $B = \{c, d, e\}$; $C = \{b, c\}$.
Verify that
(a) $(A \cap B)^c = A^c \cup B^c$
(b) $(A \cup B)^c = A^c \cap B^c$
(c) $A - (B \cap C) = (A - B) \cup (A - C)$.

4. If $n(U) = 200$, $n(A) = 150$, $n(B) = 80$, $n(A \cup B) = 160$ find $n(A \cap B)$, $n(A - B)$, $n(\overline{A \cup B})$. Also if $B \subset A$, find $n(A \cup B)$ and $n(A \cap B)$.

5. If $n(U) = 100$, $n(A) = 70$ and $n(B) = 40$, find:
(a) maximum value of $n(A \cup B)$ (b) maximum value of $n(A \cap B)$
(c) minimum value of $n(A \cup B)$ (d) minimum value of $n(A \cap B)$.

6. 20 students play football and 15 students play hockey. It is found that 5 students play both games. Find the number of students playing at least one game.

7. In a survey of a city market, it was found that 143 families used Colgate toothpaste, 135 used Everest toothpaste and 70 families used both. Find the number of families using at least one type of toothpaste.

8. In a college of 500 students, 400 use Facebook, 300 use Twitter and 50 use neither of them. Find the number of students who use both Facebook and Twitter.

9. 32 students play basketball and 25 students play volleyball. It is found that 20 students play both the games. Find the number of students playing at least one game. Also, find total number of students if 13 students play none of these games.

10. In a city of 26000 populations, 5000 read English newspaper, 12000 read Nepali newspaper and 1000 read both. What percentage read neither English nor Nepali newspaper?

11. In a survey of a city market, it was noted that 300 families were randomly selected, out of which 142 used Laptop and 139 used Desktop computers and 70 families used both. Find the number of families who used exactly one of these types of computers.

12. In a market survey of 1000 consumers of tea, it was found that 500 purchased Soktim Tea, 400 purchased Tokla Tea and 150 purchased both brands. How many purchased (a) Soktim only (b) Tokla only (c) exactly one of these brands and (d) neither of them.

13. Of the number of three athletic teams, 25 are in the basketball team, 30 in hockey team and 28 in football team, 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. How many members are there in all?

14. In a group of twenty eight teachers of a school, 15 teach English, 15 teach Maths, 14 teach Nepali, 7 teach English and Maths, 6 teach English and Nepali, 5 teach Maths and Nepali. Find how many teach (a) all three subjects, (b) Maths only (c) Nepali only.

15. In a group of students, 24 study Maths, 30 study Biology, 22 study Physics, 8 study Math only, 14 study Biology only, 6 study Biology and physics only and 2 study Maths and Biology only. Find:
   (a) How many study all three subjects?
   (b) How many students were in the group?

16. In a city of 50,000 population, 20,000 read The Rising Nepal, 25,000 read The Kathmandu Post, 30,000 read The Annapurna, 10,000 read none of these newspapers, 5,000 read The Rising Nepal and The Kathmandu Post 15,000 read The Rising Nepal and The Annapurna and 20,000 read The Kathmandu Post and The Annapurna. Find
   (a) The number of readers reading all newspapers.
   (b) The number of readers reading The Rising Nepal only.
   (c) The number of readers reading The Kathmandu Post only.
   (d) The number of readers reading The Kathmandu Post and The Annapurna only.

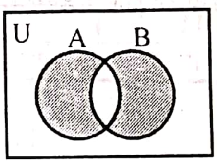
Answers

1. (a) (i) {1, 2, 3, 4, 5, 6, 8, 9, 10} (ii) {2, 3, 5, 8}
   (iii) {1, 4, 7, 9, 11} (iv) {1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12}
   (v) {1, 2, 3, 4, 5, 8, 9}.
   (b) (i) {1, 2, 3, 4, 5, 6, 8} (ii) {1, 2, 8} (iii) {3}.
   (c) (i) {15, 16, 17, 18, 19, 20}
   (ii) {1, 2, 3, 4, 5, 6, 7, 8, 9, 15, 16, 17, 18, 19, 20}
   (iii) {10, 11, 12, 13, ..., 20}.
   (d) (i) {2} (ii) {1, 3, 4, 5, 6, 7, 8, 9, 10} (iii) {1, 9}.
2. (a) (i) $\{c, d\}$ (ii) $\{a, b, c, d, x, y\}$.
   (b) (i) {2, 3, 4, 5, 6, 8, 9} (ii) {3, 7, 9}
   (iii) {2, 3, 4, 5, 6, 7, 8, 9} (iv) {1, 3, 4, 5, 6, 7, 8, 9, 10}.
4. 70, 80, 40, 150, 80
5. (a) 100 (b) 40 (c) 70 (d) 10
6. 30 7. 208 8. 250
9. 37, 50 10. 38.46% 11. 141
12. (a) 350 (b) 250 (c) 600 (d) 250
13. 50 14. (a) 2 (b) 5 (c) 5
15. (a) 8 (b) 46
16. (a) 5,000 (b) 5,000 (c) 5,000 (d) 15,000

24

Objective Questions

1. Which of the following collection is not a set?
   (a) $\{x : x^2 + 1 = 0, x \in \mathbb{R}\}$
   (b) collection of smart BCA students of Tribhuvan University.
   (c) collection of rivers of Nepal.
   (d) collection of prime numbers.
2. The set $\{x : x^2 + 4 = 0, x \in \mathbb{R}\}$ is a / an
   (a) null set (b) singleton set
   (c) infinite set (d) set having two students.
3. The number of subsets of set $\{a, b, c\}$ is
   (a) 3 (b) 4
   (c) 6 (d) 8
4. The number of proper subsets of A = $\{a, b, c, d\}$ is
   (a) 15 (b) 16
   (c) 31 (d) 32
5. If A = {1, 2, 3} then $n(P(A))$ =
   (a) 3 (b) 6
   (c) 8 (d) 12
6. The shaded region in the Venn-diagram is



   (a) $A \cup B$ (b) $A \cap B$
   (c) $A - B$ (d) $A \Delta B$
7. Which of the following is true?
   (a) $A - B = B - A$ (b) $A - B \subseteq A \cup B$
   (c) $A - B \subseteq A \cap B$ (d) $A \cup B \subseteq A \cap B$
8. If A and B are two subsets of universal set U, then $\overline{A \cup B} = \bar{A} \cap \bar{B}$ is called
   (a) Identity laws (b) Associate laws
   (c) Distributive laws (d) De-Morgans' law
9. Let A = $\{x : x^2 = 16, x \in \mathbb{R}\}$ and B = $\{x : 2x = 8\}$ then $A \cap B$ =
   (a) ϕ (b) {4}
   (c) {–4, 4} (d) {8, 16}
10. If A = $\{x : x = 2n + 1, x \leq 4, n \in \mathbb{N}\}$ and B = {7, 9, 11} then A – B =
    (a) {3, 5} (b) {3, 5, 7}
    (c) {7} (d) {9, 11}

11. If $n(A) = 30$, $n(B) = 32$ and $n(A \cap B) = 12$ then $n(A \cup B) =$

(a) 30 (b) 32

(c) 2 (d) 50

12. If $n(U) = 150$, $n(A) = 90$, $n(B) = 70$, what is the maximum value of $n(A \cup B)$?

(a) 70 (b) 90

(c) 150 (d) 160

13. In a group of 130 people, 80 people like football, 20 like both football and cricket. How many like cricket and not football?

(a) 50 (b) 40

(c) 30 (d) 15

14. If A and B are two subsets of universal set U such that $n(U) = 350$, $n(A) = 100$, $n(B) = 150$ and $n(A \cap B) = 50$, then $n(\bar{A} \cap \bar{B}) =$

(a) 200 (b) 250

(c) 300 (d) 150

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	a	d	a	c	d	b	d	b	a
11	12	13	14						
d	c	a	d						

Real Number

1.12 Introduction

A study of the order and regularity among numbers is an important feature of Mathematics. Discovery of pattern in numbers can help to predict future behavior through chains of reasoning.

1.13 Natural Numbers

The most familiar numbers are the counting numbers or the natural numbers. They are 1, 2, 3, 4, ...

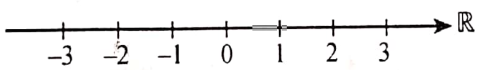
They are also called cardinal numbers or positive integers. Natural numbers are closed under the operation of addition and multiplication. The set of natural numbers is denoted by $\mathbb{N}$. So, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Note that 0 is not a natural number.

1.14 Integers

The set of integers contains all natural numbers, negatives of natural numbers and zero defined as,

$$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

An integer is a whole number and does not have a fractional part.



We can associate the concepts of distance and measurement with numbers.

Take a point 'O' on the real line, on the right of O mark all positive integers with a certain unit of measurement. With the same unit, mark all negative integers on the left of O. Manipulation of integers by adding, subtracting, multiplying, dividing etc. gives rise to integers as well as fractions, decimals numbers.

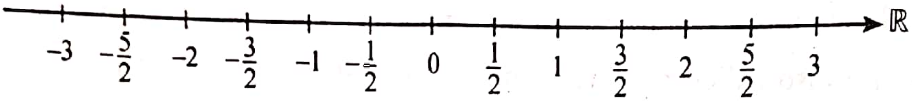
Integers are closed under the operation of addition, subtraction and multiplication. The set of integers is denoted by $\mathbb{I}$ or $\mathbb{Z}$.

1.15 Rational Numbers

A rational number is a ratio of two integers, positive, or negative, the denominator in the quotient being non–zero.

If x and y are integers and $y \neq 0$ then $\frac{x}{y}$ is a rational number.

All integers are rational numbers but all rational numbers are not integers. The fraction $\frac{8}{4}$ is a rational number. Being equal to 2, it is an integer but the fraction $\frac{1}{4}$ is not an integer.



A rational number $\frac{x}{y}$ can be represented by a point on the number line shown as a distance from the origin O. There is an infinite number of rational number on any line segment.

$\frac{1}{4} = 0.25, \frac{1}{3} = 0.333..., \frac{1}{6} = 0.1666$ ... are examples of rational numbers. A rational number can thus be expressed as a terminating decimal or as a repeating decimal. Note that $\frac{1}{3} = 0.333$ ... can be written as $0.\bar{3}$ or $0.\dot{3}$.

Likewise 0.324324324 ... can be written as $0.\overline{324}$ etc.

Rational numbers are closed under the operation of addition, subtraction, multiplication, and division (excluding division by zero). The set of rational numbers is denoted by $\mathbb{Q}$.

1.16 Irrational Numbers

Rational numbers do not constitute the set of all real numbers. A number, which cannot be expressed as the ratio of two integers, is called an irrational number. For example, $\sqrt{2} = 1.4142$ ... is an irrational number as it cannot be expressed as the ratio of two integers and the process of finding the square root in decimal never terminates.

Other examples are $\sqrt[3]{7}$, $\sqrt[5]{2}$, π etc.

An irrational number does not have a rational co–ordinate on the number line. The co–ordinate for an irrational number exists but its distance from origin cannot be taken as the ratio of two integers. Numbers represented by non–terminating and non–repeating decimals are known as irrational numbers.

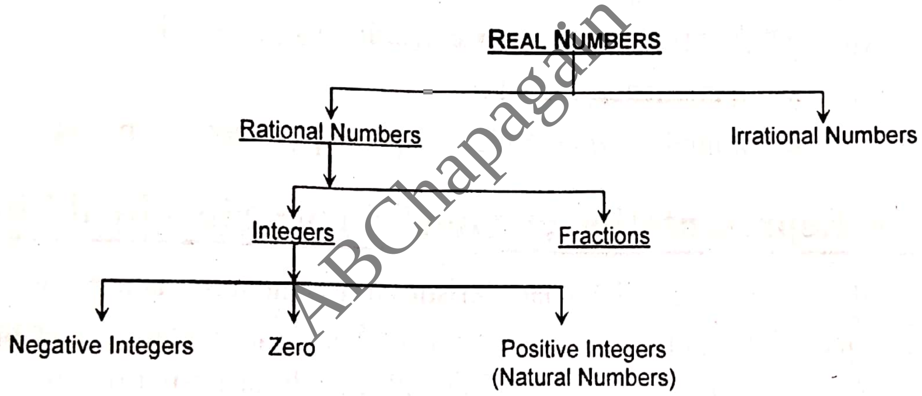
The set of irrational numbers is denoted by $\bar{\mathbb{Q}}$.

1.17 Real Numbers

Rational and irrational numbers taken together constitute the real number system. The set of real numbers is denoted by $\mathbb{R}$.

Any real number is either rational or irrational. Any point on the number line is related to some corresponding real point on the number line uniquely and vice versa.

The classification of the set of all real numbers is as follows:



1.18 Properties of Real Numbers

The real numbers satisfy the following properties. Let a, b and c be the three real numbers.

1. **Additional Property**

   (i) $a + b$ is a real number. [Closure property]

   (ii) $a + b = b + a$. [Commutative property]

   (iii) $a + (b + c) = (a + b) + c$. [Associative property]

   (iv) $a + 0 = 0 + a = a$ for all a. [Existence of identity element]

   (v) $a + (-a) = (-a) + a = 0$ for all a. [Existence of inverse element]

2. **Multiplication Property**

(i) ab is a real number. [Closure property]

(ii) $ab = ba$. [Commutative property]

(iii) $a\,(bc) = (ab)\,c$. [Associative property]

(iv) $a \cdot 1 = 1 \cdot a = a$ for all a. [Existence of identity element]

(v) $a \cdot a^{-1} = a^{-1} \cdot a = 1, a \neq 0$. [Existence of inverse element]

3. **Multiplication Over Addition Property (Distributive Property)**

(i) $a\,(b + c) = ab + ac$. [Left distributive property]

(ii) $(b + c)\,a = ba + ca$. [Right distributive property]

4. **Order Property**

(i) Only one of the relations,

$a = b$, $a > b$, $a < b$ is true. [Trichotomy property]

(ii) $a > b$ and $b > c$ implies $a > c$. [Transitivity property]

(iii) $a > b$ implies $a + c > b + c$. [Additivity property]

(iv) $a > b$ implies $ca > cb$ if $c > 0$.

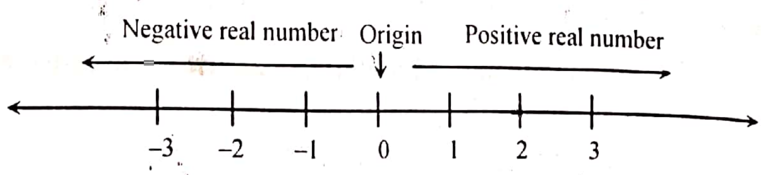
$a < b$ implies $ca > cb$ if $c < 0$. [Multiplicativity property]

1.19 Representation of a Real Number in a Real Line

One of the most interesting characteristics of real numbers is that they can be identified with a point on a real line, called the real number line or the real axis. A real number can be identified with exactly one point on the real line and conversely, a point on the line can be identified with exactly one real number.

For this reason we often use the words "point" and "number" interchangeable. Thus the real numbers leave no gap on the real number line. The point representing the number 0 is called origin which is shown in the figure given below.

The numbers to the right of zero are positive and the numbers to the left of zero are negative. The real numbers are arranged on the number line in increasing order of magnitude of the numbers from left to right.



Example: Prove that $\sqrt{5}$ is an irrational number.

Solution

If possible, suppose $\sqrt{5}$ is a rational number.
Then, by definition, we can write

$$\sqrt{5} = \frac{p}{q}, \quad q \neq 0 \qquad \dots \text{(i)}$$

where p and q are integers and have no common factors.
Squaring both sides of equation (i), we get

$$5 = \frac{p^2}{q^2} \Rightarrow p^2 = 5q^2 \qquad \dots \text{(ii)}$$

$\therefore$ p^2 is multiple of 5 and hence p is also multiple of 5
Let $p = 5k$ where k is an integer.
Then equation (ii) can be written as $(5k)^2 = 5q^2$
or, $25k^2 = 5q^2$
or, $q^2 = 5k^2 \qquad \dots$ (iii)

$\therefore$ q^2 is multiple of 5 and hence q is also multiple of 5. Thus, p and q both are multiple of 5 and hence have a common factor which contradicts our assumption.

Hence, $\sqrt{5}$ is an irrational number.

1.20 Inequality

Let a and b be any two real numbers. The possible relation between a and b are as follows: (i) $a > b$ (ii) $a < b$ (iii) $a = b$. The first two relations (i.e. $a > b$ and $a < b$) are known as inequalities and the last one (i.e. $a = b$) is known as equality or equation. For example $3 > 2$ since $3 - 2 = 1$ is positive, $2 < 4$ since $2 - 4 = -2$ is negative, $8 = 8$ since $8 - 8 = 0$.

The inequality $a > b$ means the difference $a - b$ is positive. Similarly, the inequality $a < b$ implies that the difference $a - b$ is negative.

1.21 Properties of Inequalities

The following are the properties of inequalities

Let a, b and c be three real numbers

(i) If $a > b$ then $a + c > b + c$

(ii) If $a > b$, then $a - c > b - c$

(iii) If $a > b$, then $ca > cb$ if $c > 0$

(iv) If $a > b$, then $ca < cb$ if $c < 0$

(v) If $a < b$, then $a + c < b + c$

(vi) If $a < b$, then $a - c < b - c$

(vii) If $a < b$, then $ca < cb$ if $c > 0$

(viii) If $a < b$, then $ca > cb$ if $c < 0$

(ix) If $a > b$, then $-a < -b$

(x) If $a > 0$, then $-a < 0$.

Example: Solve: $6 + 4x \leq 12$.

Solution

$$6 + 4x \leq 12$$

Adding -6 on both sides

$$-6 + 6 + 4x \leq 12 - 6$$

or, $4x \leq 6$

Dividing both sides by 4

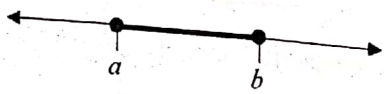
or, $\frac{4x}{4} \leq \frac{6}{4}$

$\therefore$ $x \leq \frac{3}{2}$.

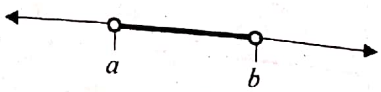
1.22 Intervals

Let 'a' and 'b' be any two points on the real line with $a < b$. The set of all points between a and b is called an interval. The points 'a' and 'b' are called the end points of the interval. An interval may or may not contain end points. There are four types of intervals. They are:

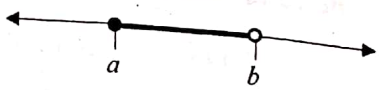
Closed interval : $[a, b] = \{x : a \leq x \leq b\}$



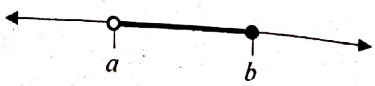
Open interval : $(a, b) = \{x : a < x < b\}$



Right open interval : $[a, b) = \{x : a \leq x < b\}$



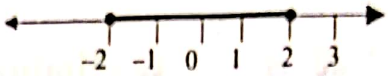
Left open interval : $(a, b] = \{x : a < x \leq b\}$



Some examples of intervals are as follows:

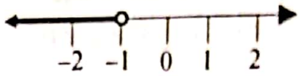
The figure alongside represents the half closed interval $[0, \infty)$. This is written as $[0, \infty) = \{x: 0 \le x < \infty\}$.

The figure alongside represents the closed interval $-2 \le x \le 2$,



i.e., $[-2, 2] = \{x : -2 \le x \le 2\}$.

The figure alongside represents the half open interval $x < -1$,



i.e., $(-\infty, -1) = \{x : -\infty < x < -1\}$.

The real number line is represented by

$\mathbb{R} = (-\infty, \infty) = \{x : -\infty < x < \infty\}$.

Note: '( )' is used for the open interval, where end points are excluded and '[ ]' is used for the closed interval where the end points are included.

Example : Let A = [–3, 1] and B = [–2, 4]. Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$.

Solution

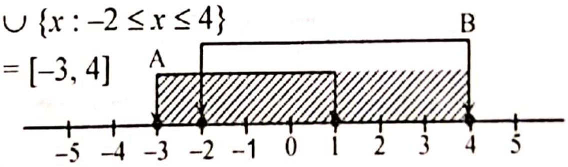
Here, $A = [-3, 1]$

$B = [-2, 4]$

$A \cup B = [-3, 1] \cup [-2, 4]$

$= \{x : -3 \le x \le 1\} \cup \{x : -2 \le x \le 4\}$

$= \{x : -3 \le x \le 4\} = [-3, 4]$

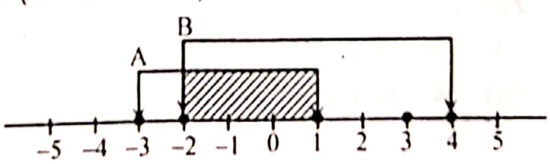


$A \cap B = [-3, 1] \cap [-2, 4]$

$= \{x : -3 \le x \le 1\} \cap \{x : -2 \le x \le 4\}$

$= \{x : -2 \le x \le 1\}$

$= [-2, 1]$

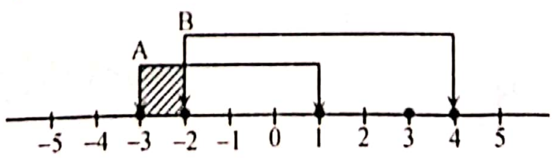


$A - B = [-3, 1] - [-2, 4]$

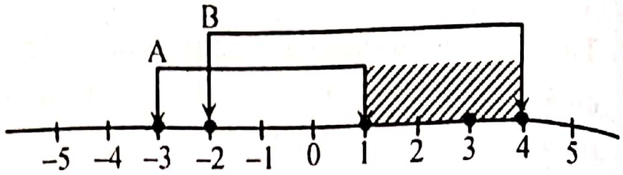
$= \{x : -3 \le x \le 1\} - \{x : -2 \le x \le 4\}$

$= \{x : -3 \le x < -2\}$

$= [-3, -2)$



$$
\begin{aligned}
B - A &= [-2, 4] - \{-3, 1] \\
&= \{x : -2 \le x \le 4\} - \{x : -3 \le x \le 1\} \\
&= \{x : 1 < x \le 4\} \\
&= (1, 4].
\end{aligned}
$$

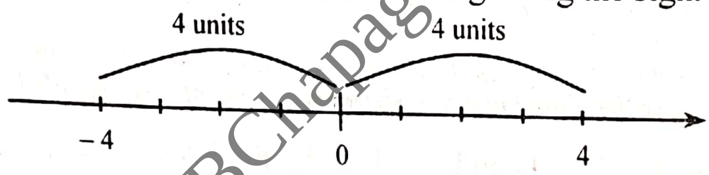


1.23 Absolute (Modulus) Value

The absolute value or modulus of a real number x, denoted by $|x|$, is defined as $|x| = \begin{cases} x \text{ for } \times \ge 0 \\ -x \text{ for } x < 0 \end{cases}$

Thus, it is clear that the absolute value of a real number is always non–negative i.e. $|x| \ge 0$. So, $|4| = 4$, $|-4| = -(-4) = 4$, $|0| = 0$.

Geometrically, the absolute value of a real number is its distance from the origin. It is the magnitude of the number disregarding the sign.



1.24 Some Properties of Absolute Values

1. For any two real numbers x and y,

   (i) $x \le |x|$ and $-x \le |x|$

   (ii) $|x\,y| = |x|\;|y|$

   (iii) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ where $y \ne 0$

   (iv) $|x + y| \le |x| + |y|$ [Triangle inequality]

   (v) $|x - y| \ge |x| - |y|$

   (vi) $|x - y| \le |x| + |y|$.

Proof

(i) Obvious

(ii) We have $(x\,y)^2 = x^2\,y^2$

But $x^2 = |x|^2$, $y^2 = |y|^2$

and $(xy)^2 = |xy|^2$

or, $|xy|^2 = |x|^2\,|y|^2$

$\therefore$ $|xy| = |x|\,|y|$ [Since both $|xy|$ and $|x|$, $|y|$ are non negative.].

(iii) $\left|\frac{x}{y}\right| = \left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2} = \frac{|x|^2}{|y|^2} = \left(\frac{|x|}{|y|}\right)^2 = \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

(iv) $|x+y|^2 = (x+y)^2$

$|x+y|^2 = x^2 + y^2 + 2xy$

$|x+y|^2 \le x^2 + y^2 + 2\,|x|\;|y|$ $\quad [\because\ |x| \ge x \text{ and } |y| \ge y]$

$= |x|^2 + |y|^2 + 2|x|\,|y|$

$= [\,|x| + |y|\,]^2$

$\therefore\ |x+y| \le |x| + |y|$.

(v) $|x| = |x - y + y|$

or, $|x| \le |x-y| + |y|$ [Using Triangle inequality]

or, $|x| - |y| \le |x-y|$

$\therefore\ |x-y| \ge |x| - |y|$.

(vi) $|x-y| = |x + (-y)| \le |x| + |-y|$ [Using Triangle inequality]

$= |x| + |y|$

$\therefore\ |x-y| \le |x| + |y|$.

2. If a be any positive real number and $x \in \mathbb{R}$, prove that $|x| < a \Leftrightarrow -a < x < a$.

Proof

For all $x \in \mathbb{R}$, $x \le |x|$

Given $|x| < a$

$\therefore\ x \le |x| < a$

$\Rightarrow\ x < a$ ... (i)

Again, for all $x \in \mathbb{R}$, $-x \le |x|$

Given $|x| < a$

$\therefore\ -x \le |x| < a$

$\Rightarrow\ -x < a$

$\Rightarrow\ x > -a$

$\Rightarrow\ -a < x$ ... (ii)

Combining (i) and (ii) $-a < x < a$.

Conversely, let $-a < x < a$

At first, $x < a$

If $x \geq 0$, then $|x| = x$

$\therefore \quad |x| = x < a$ ...(iii)

Again, $-a < x$

$\Rightarrow \quad a > -x$

$\Rightarrow \quad -x < a$

But for $x < 0$, $|x| = -x$

$\therefore \quad |x| = -x < a$ ...(iv)

Hence, from (iii) and (iv), for all $x \in R$, $|x| < a$.

WORKED OUT EXAMPLES

Example 1. Find the value of $|-16| + |3| + |-5|$.

Solution

$$\begin{aligned} & |-16| + |3| + |-5| \\ = \ & -(-16) + 3 + \{-(-5)\} \\ = \ & 16 + 3 + 5 = 24. \end{aligned}$$

Example 2. If $|x| < a$, then prove that $-a < x < a$.

Solution

Given,

$$|x| < a$$

$$\Rightarrow \begin{cases} x < a & \text{if } x \geq 0 \\ -x < a & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < a \\ x > -a \end{cases}$$

$$\Rightarrow \quad x > -a, x < a$$

$$\Rightarrow \quad -a < x, x < a$$

$$\Rightarrow \quad -a < x < a.$$

Example 3. If $x = -4$ and $y = 3$ then verify that

(i) $|x - y| \geq |x| - |y|$ (ii) $|xy| = |x| \cdot |y|$

(iii) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ (iv) $|x + y| \leq |x| + |y|$.

Solution

(i) Now,

$$\begin{aligned} |x| - |y| &= |-4| - |3| \\ &= -(-4) - 3 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

and $|x - y| = |-4 - 3|$
$= |-7|$
$= -(-7)$
$= 7$

$\therefore \quad |x - y| > |x| - |y|.$

(ii) $|xy| = |(-4) \times 3|$
$= |-12|$
$= 12$

$|x| \cdot |y| = |-4|\ |3|$
$= 4 \times 3$
$= 12$

$\therefore |xy| = |x| \cdot |y|.$

(iii) $\dfrac{x}{y} = \dfrac{-4}{3}$

$\therefore \quad \left|\dfrac{x}{y}\right| = \left|\dfrac{-4}{3}\right| = \dfrac{4}{3}$

$\dfrac{|x|}{|y|} = \dfrac{|-4|}{|3|} = \dfrac{4}{3}$

$\therefore \quad \left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}.$

(iv) $|x + y| = |-4 + 3|$
$= |-1|$
$= 1$

$|x| + |y| = |-4| + |3|$
$= 4 + 3$
$= 7$

$\therefore \quad |x + y| < |x| + |y|.$

Example 4. Solve the equation $|3x + 5| = 7$.

Solution

Let, $3x + 5$ be positive.
Then $3x + 5 = 7$
or, $3x = 7 - 5$
or, $3x = 2$
or, $x = \dfrac{2}{3}$

Again, let, $3x + 5$ be negative.
Then $3x + 5 = -7$
or, $3x = -7 - 5$
or, $3x = -12$
or, $x = -\dfrac{12}{3}$
$= -4$

$\therefore$ The solution set $= \dfrac{2}{3}, -4.$

Example 5. Rewrite the following without absolute sign

(i) $|x-1|<6$ (ii) $|3x-2|\le 7$.

Solution

(i) $|x-1|<6$

$\Rightarrow \quad -6<x-1<6$ [By definition]

Adding 1 to each side.

$\Rightarrow \quad -6+1<x-1+1<6+1$

$\Rightarrow \quad -5<x<7.$

(ii) $|3x-2|\le 7$

$\Rightarrow \quad -7\le 3x-2\le 7$

Adding 2 to each side.

$\Rightarrow \quad -7+2\le 3x+2-2\le 7+2$

$\Rightarrow \quad -5\le 3x\le 9$

Dividing each side by 3.

$\Rightarrow \quad -\frac{5}{3}\le x\le 3.$

Example 6. Solve the following inequalities

(i) $|4x-3|<6$ (ii) $|3x-15|\le\frac{3}{2}$.

Solution

(i) Here, $|4x-3|<6$

$\Rightarrow \quad -6<4x-3<6$

$\Rightarrow \quad -6+3<4x-3+3<6+3$

$\Rightarrow \quad -3<4x<9$

$\Rightarrow \quad -\frac{3}{4}<\frac{4x}{4}<\frac{9}{4}$

$\Rightarrow \quad \frac{-3}{4}<x<\frac{9}{4}.$

(ii) Here

$|3x-15|\le\frac{3}{2}$

$\Rightarrow \quad \frac{-3}{2}\le 3x-15\le\frac{3}{2}$

$\Rightarrow \quad \frac{-3}{2}+15\le 3x-15+15\le\frac{3}{2}+15$

$\Rightarrow \quad \frac{27}{2}\le 3x\le\frac{33}{2}$

$\Rightarrow \quad \frac{27}{2\times 3}\le\frac{3x}{3}\le\frac{33}{2\times 3}$

$\Rightarrow \quad \frac{9}{2}\le x\le\frac{11}{2}.$

Example 7. Rewrite so that x is alone between the inequality sign $-7 < -2x + 3 < 5$.

Solution

$$-7 < -2x + 3 < 5$$

Adding -3 to each side of the given inequality, we get

$$-10 < -2x < 2$$

Dividing each side by -2, we have

$$5 > x > -1$$

$\therefore \quad -1 < x < 5.$

Example 8. Rewrite the following using absolute value sign

(i) $-3 < x < 5$ (ii) $-1 < x < 7$ (iii) $-3 \le x \le 8$.

Solution

(i) $-3 < x < 5$

Adding -1 to each side

$\Rightarrow \quad -3 - 1 < x - 1 < 5 - 1$

$\Rightarrow \quad -4 < x - 1 < 4$

$\therefore \quad |x - 1| < 4.$ $\quad (\because -a < x < a \Rightarrow |x| < a)$

(ii) $-1 < x < 7$

Adding -3 to each side, we get

$\Rightarrow \quad -1 - 3 < x - 3 < 7 - 3$

$\Rightarrow \quad -4 < x - 3 < 4$

$\therefore \quad |x - 3| < 4.$ $\quad (\because -a < x < a \Rightarrow |x| < a)$

(iii) $-3 \le x \le 8$

Adding $-\frac{5}{2}$ to each side, we get

$\Rightarrow \quad -3 - \frac{5}{2} \le x - \frac{5}{2} \le 8 - \frac{5}{2}$

$\Rightarrow \quad \frac{-11}{2} \le x - \frac{5}{2} \le \frac{11}{2}$

$\Rightarrow \quad \frac{-11}{2} \le \frac{2x-5}{2} \le \frac{11}{2}$

$\therefore \quad |2x - 5| \le 11.$ $\quad (\because -a \le x \le a \Rightarrow |x| \le a)$

Alternative Method:

Multiplying each side by 2

$$-6 \le 2x \le 16$$

Again, adding -5 to each side

$$-11 \le 2x - 5 \le 11$$

$\therefore \quad |2x - 5| \le 11.$

Note: To write $a \le x \le b$ using absolute value sign we add $-\left(\frac{a+b}{2}\right)$ to each side.

Example 9. Rewrite $|2x - 1| \leq 5$ without using absolute value sign and draw the graph.

Solution

$$|2x - 1| \leq 5$$

or, $-5 \leq 2x - 1 \leq 5$ $\quad [\because |x| \leq a \Leftrightarrow -a \leq x \leq a]$

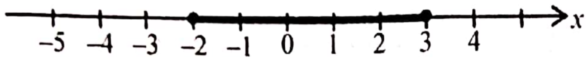
or, $-5 + 1 \leq 2x - 1 + 1 \leq 5 + 1$

or, $-4 \leq 2x \leq 6$

or, $\frac{-4}{2} \leq \frac{2x}{2} \leq \frac{6}{2}$

$\therefore$ $-2 \leq x \leq 3$.

The graph is as follows.



Example 10. Represent the solution set of x of $-2 \leq 3x + 4 \leq 10$ in interval form.

Solution

Given, $-2 \leq 3x + 4 \leq 10$

Subtracting 4 from each side, we get

$$-2 - 4 \leq 3x + 4 - 4 \leq 10 - 4$$

or, $-6 \leq 3x \leq 6$

Dividing each side by 3

$$-2 \leq x \leq 2$$

In interval form, this can be written as $[-2, 2]$.

Example 11. Rewrite the following relation without using absolute value sign $|2x - 1| \leq 5$. Also, draw the graph of the inequality.

Solution

$$|2x - 1| \leq 5$$

or, $-5 \leq 2x - 1 \leq 5$ $\quad [\because |x| \leq a \Leftrightarrow -a \leq x \leq a]$

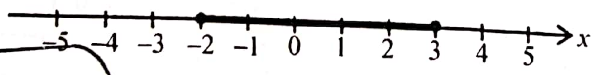
or, $-5 + 1 \leq 2x - 1 + 1 \leq 5 + 1$

or, $-4 \leq 2x \leq 6$

or, $\frac{-4}{2} \leq \frac{2x}{2} \leq \frac{6}{2}$

$\therefore$ $-2 \leq x \leq 3$.

The graph is as follows.



Example 12. Solve the inequality: $6 + 5x - x^2 \geq 0$.

Solution

The corresponding equation of given inequality is

$$6 + 5x - x^2 = 0$$

or, $(6 - x)(x + 1) = 0$

$\therefore$ $x = -1, x = 6$

These two points divide the whole real line into 3 sub-intervals $(-\infty, -1)$, $(-1, 6)$ and $(6, \infty)$

Intervals	Sign of		
	$(6 - x)$	$(x + 1)$	$(6 - x)(x + 1)$
$(-\infty, -1)$	+ ve	– ve	– ve
$(-1, 6)$	+ ve	+ ve	+ ve
$(6, \infty)$	– ve	+ ve	– ve

Also, at $x = -1$, and $x = 6$, $6 + 5x - x^2 = 0$

The possible interval is $(-1, 6) \cup \{-1, 6\} = [-1, 6]$.

Example 13. Solve: $x^2 - 2x - 3 > 0$.

Solution

The corresponding equation of given inequality is

$x^2 - 2x - 3 = 0$

or, $x^2 - 3x + x - 3 = 0$

or, $x(x - 3) + 1(x - 3) = 0$

or, $(x + 1)(x - 3) = 0$

$\therefore$ $x = -1, 3$

Now, these two points divide the real line into 3 sub-intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$

Intervals	Sign of		
	$(x + 1)$	$(x - 3)$	$(x + 1)(x - 3)$
$(-\infty, -1)$	–ve	– ve	+ ve
$(-1, 3)$	+ ve	– ve	– ve
$(3, \infty)$	+ ve	+ ve	+ ve

$\therefore$ The possible intervals is $(-\infty, -1) \cup (3, \infty)$.

Example 14. Solve the following inequalities and graph their solution sets on the real line

(i) $2x - 1 < x + 3$ (ii) $\frac{6}{x - 1} \geq 5$.

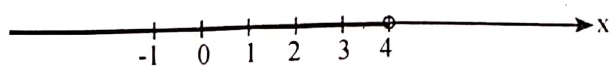
Solution

(i) $2x - 1 < x + 3$

or, $2x - x < 3 + 1$

or, $x < 4$

The solution set is the interval $(-\infty, 4)$. The graph is,



(ii) $\frac{6}{x-1} \geq 5$

The inequality $\frac{6}{x-1} \geq 5$ will be hold if $x > 1$, because if it not so $\frac{6}{x-1}$ will be undefined or negative.

Therefore the inequality will be preserved if we multiply both sides by $(x - 1)$.

Multiply both sides by $x - 1$.

$$6 \geq 5(x-1)$$

or, $6 \geq 5x - 5$

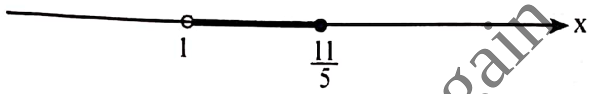
or, $6 + 5 \geq 5x$

or, $\frac{11}{5} \geq x$

or, $x \leq \frac{11}{5}$

Hence, combining $x > 1$ and $x \leq \frac{11}{5}$, we get $1 < x \leq \frac{11}{5}$.

Thus the solution set is the half open interval $\left(1, \frac{11}{5}\right]$.



Example 15. Solve $|x - 5| < 9$ and graph the solution set on the real line.

Solution

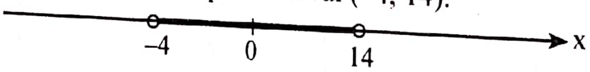
$$|x - 5| < 9$$

or, $-9 < x - 5 < 9$ $[\because |x| < a \Leftrightarrow -a < x < a]$

or, $-9 + 5 < x < 9 + 5$ [Adding 5 to each side]

or, $-4 < x < 14$

The solution set is the open interval $(-4, 14)$.



Example 16. Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$ and graph the solution set on the real line.

Solution

$$\left|5 - \frac{2}{x}\right| < 1$$

or, $-1 < 5 - \frac{2}{x} < 1$

or, $-1 - 5 < 5 - \frac{2}{x} - 5 < 1 - 5$

or, $-6 < -\frac{2}{x} < -4$

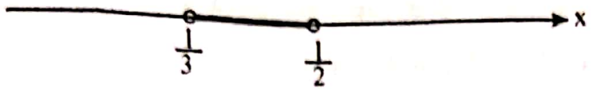
Multiplying each side by $-\frac{1}{2}$

$$3 > \frac{1}{x} > 2.$$

Note: *Multiplying by negative number reverses the inequality.*

Taking reciprocals, we get, $\frac{1}{3} < x < \frac{1}{2}$ [Taking reciprocals in an inequality in which both sides are positive also reverses the inequality sign]

Thus the solution set is $\left(\frac{1}{3}, \frac{1}{2}\right)$.



Example 17. Solve the inequality and graph the solution set

(i) $|2x - 3| \le 1$ (ii) $|2x - 3| \ge 1$.

Solution:

(i) $|2x - 3| \le 1$

or, $-1 \le 2x - 3 \le 1$

or, $-1 + 3 \le 2x - 3 + 3 \le 1 + 3$

or, $2 \le 2x \le 4$

or, $\frac{2}{2} \le \frac{2x}{2} \le \frac{4}{2}$

$\therefore$ $1 \le x \le 2$

The solution set is the closed set [1, 2].

(ii) If $2x - 3$ is positive, then

$2x - 3 \ge 1$

or, $2x \ge 1 + 3$

or, $2x \ge 4$

or, $x \ge 2$

i.e. $x \in [2, \infty)$

Again, if $2x - 3$ is negative, then

$-(2x - 3) \ge 1$

or, $2x - 3 \le -1$

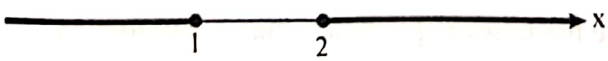
or, $2x \le 3 - 1$

or, $2x \le 2$

or, $x \le 1$

i.e. $x \in (-\infty, 1]$

Thus the solution set is $(-\infty, 1] \cup [2, \infty)$.



EXERCISE – 1 B

1. (a) If $-10 < 5x + 10 < 5$, prove that $-4 < x < -1$.
   (b) If $-5 < 7x + 9 < 30$, prove that $-2 < x < 3$.
   (c) If $-8 \le 2x + 2 < -2$, prove that $-5 \le x < -2$.
   (d) If $0 \le 3x + 9 \le 27$, prove that $-3 \le x \le 6$.
2. If $x = -3, y = 5$, verify that:
   (a) $|x + y| \le |x| + |y|$ (b) $|x| - |y| \le |x - y|$
   (c) $|xy| = |x|\ |y|$ (d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
3. Express $\frac{1}{9}$ as a repeating decimal, using a bar to indicate the repeating digits. What are the representations of $\frac{2}{9}, \frac{7}{9}$ and $\frac{8}{9}$?
4. If $2 < x < 6$, which of the following statements about x are necessarily true and which are not necessarily true?
   (a) $0 < x < 4$ (b) $1 < \frac{x}{2} < 3$
   (c) $1 < \frac{6}{x} < 3$ (d) $|x - 4| < 2$.
5. Solve the inequalities.
   (a) $-2x > 4$ (b) $5x - 3 \le 7 - 3x$
   (c) $2x - \frac{1}{2} \ge 7x + \frac{7}{6}$ (d) $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$
   (e) $-\frac{x + 5}{2} \le \frac{12 + 3x}{4}$.
6. Solve:
   (a) $|x| = 3$ (b) $|2x + 5| = 4$
   (c) $|8 - 3x| = \frac{9}{2}$.
7. (i) If $A = [-2, 1)$ and $B = (-1, 3]$ then find
   (a) $A \cup B$ (b) $A \cap B$
   (c) $A - B$ (d) $B - A$.
   (ii) Rewrite the following without using modulus sign.
   (a) $|x + 2| \le 1$ (b) $|2x - 3| \le 1$.

(iii) Solve the inequalities expressing the solution sets as intervals or unions of intervals. Also, graph each solution set on the real line.

(a) $|x| < 2$ (b) $|x - 2| < 6$

(c) $|3x - 7| < 4$ (d) $\left|3 - \frac{1}{y}\right| < \frac{1}{2}$

(e) $|3 - 5x| \le 2$ (f) $|1 - x| > 1$

(g) $\left|\frac{x+1}{2}\right| \ge 1.$

8. Rewrite the following by using the modulus sign.

(a) $-3 < x < 3$ (b) $-3 < x < 9$

(c) $-4 \le x \le 1$ (d) $-3 \le x \le -2$

(e) $-5 \le x \le -2.$

9. Solve:

(a) $x^2 < 2$ (b) $(x - 1)^2 < 4$

(c) $x^2 - x < 0$ (d) $x^2 - x - 2 \ge 0.$

Answers

3. $0.\bar{1}, 0.\bar{2}, 0.\bar{7}, 0.\bar{8}$

4. (a) not necessarily true (b) true

(c) true (d) true

5. (a) $x < -2$ (b) $x \le \frac{5}{4}$ (c) $x \le -\frac{1}{3}$

(d) $x < -\frac{6}{7}$ (e) $x \ge -\frac{2}{5}$

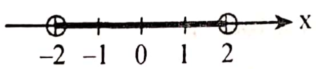
6. (a) ± 3 (b) $-\frac{1}{2}, -\frac{9}{2}$ (c) $\frac{7}{6}, \frac{25}{6}$

7. (i) (a) $[-2, 3]$ (b) $(-1, 1)$

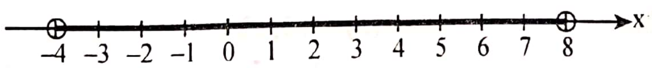
(c) $[-2, -1]$ (d) $[1, 3]$

(ii) (a) $-3 \le x \le -1$ (b) $1 \le x \le 2$

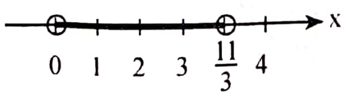
(iii) (a) $(-2, 2)$;



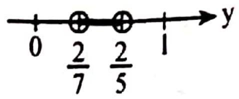
(b) $(-4, 8)$;



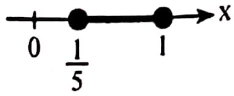
(c) $\left(1, \frac{11}{3}\right)$;



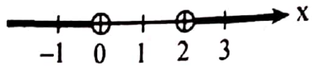
(d) $\left(\frac{2}{7}, \frac{2}{5}\right)$;



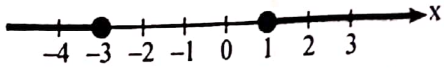
(e) $\left[\frac{1}{5}, 1\right]$;



(f) $(-\infty, 0) \cup (2, \infty)$;



(g) $(-\infty, -3] \cup [1, \infty)$;



8. (a) $|x| < 3$ (b) $|x - 3| < 6$ (c) $|2x + 3| \leq 5$
   (d) $|2x + 5| \leq 1$ (e) $|2x + 7| \leq 3$.
9. (a) $x \in (-2, 2)$ (b) $x \in (-1, 3)$
   (c) $x \in [0, 1]$ (d) $x \in (-\infty, -1] \cup [2, \infty)$.

Objective Questions

1. π is
   (a) a natural number. (b) an integer.
   (c) a rational number. (d) an irrational number.

2. $0.\overline{45}$ is
   (a) a natural number (b) an integer
   (c) a rational number (d) an irrational number

3. $[2, 3] =$
   (a) $\{x : 2 < x < 3\}$ (b) $\{x : 2 \leq x \leq 3\}$
   (c) $\{x : 2 < x \leq 3\}$ (d) $\{x : 2 \leq x < 3\}$

4. The shaded region in the figure in interval form is

   (a) $(-1, 2)$ (b) $(-1, 2]$
   (c) $[-1, 2)$ (d) $[-1, 2]$

5. If $A = [-3, 1)$ and $B = (0, 3]$ then $A \cup B =$
   (a) $\{-3, -2, -1, 0, 1, 2, 3\}$ (b) $[-3, 3]$
   (c) $(0, 1)$ (d) $[0, 1]$

6. Which of the following is not true?

(a) $|-a| = |a|$
(b) $|ab| = |a|\,|b|$
(c) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
(d) $|a+b| = |a| + |b|$

7. If $2 < x < 4$ and $3 < y < 7$, the largest integer value of $x + y$ is

(a) 9
(b) 10
(c) 11
(d) 12

8. The number of integers that satisfy the inequality $|x| < \pi$ is

(a) 2
(b) 5
(c) 7
(d) 11

9. The solution set of $|x| = 2$ is

(a) $\{2\}$
(b) $\{-2\}$
(c) $\{-2, 2\}$
(d) ϕ

10. $-7 < -2x + 3 < 5$ implies

(a) $-1 < x < 5$
(b) $1 < x < 5$
(c) $-5 < x < 1$
(d) $-5 < x < -1$

11. $-1 \le x \le 4$ implies

(a) $|x| \le 4$
(b) $|x| \le 4$
(c) $|2x - 3| \le 5$
(d) $|2x - 3| \le 2$

12. $1 \le x \le 2$ implies

(a) $|2x - 3| \le 1$
(b) $|2x + 3| \le 4$
(c) $|2x - 3| \le 5$
(d) $|2x - 3| \le 2$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
d	c	b	b	b	d	b	c	c	a
11	12								
c	a								

Complex Numbers

1.25 Introduction

Let us consider an equation $x^2 + 1 = 0$ or, $x^2 = -1$. This equation has no real solution because **square of any real number is not negative**. In order to provide the solution for such equation, the number system is extended. The extended number system is the complex number system. In the complex number $\sqrt{-1}$ exists which does not exist in the real number system. We used to represent $\sqrt{-1}$ by i (iota), i.e.

$$x^2 = -1$$

or, $x^2 = i^2$

or, $x = \pm i$

So, the solutions of $x^2 + 1 = 0$ are i and $-i$ (known as imaginary numbers).

Thus, $i^2 = -1$. Then,

$$\begin{aligned} i^3 &= i^2 i \\ &= (-1)\, i \\ &= -i \end{aligned} \qquad\qquad \begin{aligned} i^4 &= i^2 i^2 \\ &= (-1)(-1) \\ &= 1 \text{ etc.} \end{aligned}$$

Definition

An ordered pair of real number in the form of $a + ib$ is called a complex number where a is called the real part and b is called the imaginary part of the complex number and $i = \sqrt{-1}$. A complex number is generally denoted by z or w.

For example

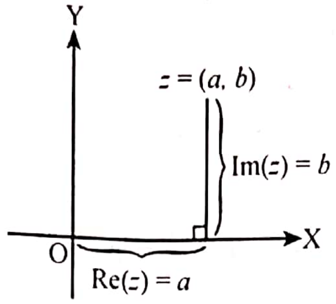
$5 + 6i$, $4 - 3i$, $5i$. (i.e. $0 + 5i$) are complex numbers.

Note that $a + ib$ can be written as (a, b).

1.26 Geometrical Representation of Complex Number

If $z = (a, b)$ be a complex number then 'a' is called the real part of z and 'b' is called the imaginary part of z. The real and imaginary parts of z are denoted by Re(z) and Im(z) respectively. Thus, Re(z) = a and Im(z) = b.

Complex numbers are to be interpreted as points in the complex plane with rectangular coordinates. The x-axis is called real axis and y-axis is called imaginary axis. The complex number $z = (a, b)$ is displayed in the complex plane as shown in the figure.



Integral Powers of *i*

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = i^2.i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$ and so on.

In general: $i^{4n} = 1,\ i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i$, where $n \in \mathbb{Z}$.

Thus, any integral power of i is one of the four numbers, $1, -1, i$ or $-i$.

Note: *1. Sum of any four consecutive integral powers of i is zero.*

2. For any two real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one a or b is non-negative.

Thus, $\sqrt{-1}\sqrt{-16} = \sqrt{(-1)(-16)} = \sqrt{16} = 4$ is wrong.

In fact, $\sqrt{-1}\sqrt{-16} = i \cdot 4i = 4i^2 = -4$.

Example: Explain, why

$$-1 = i \times i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1 \text{ is wrong?}$$

Solution

We know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one a or b is non-negative.

$\therefore \quad \sqrt{-1}\sqrt{-1} \neq \sqrt{(-1)(-1)}$.

Example: Evaluate:

(a) $(1, 0)^{10}$ (b) $(0, 1)^{101}$.

Solution

(a) $(1, 0)^{10} = 1^{10} = 1 = (1, 0)$.

(b) $(0, 1)^{101} = i^{101}$

$$= i^{100} \cdot i$$
$$= (i^4)^{25} \cdot i$$
$$= 1 \cdot i = i = (0, 1).$$

Example: Show that: $1 + i^2 + i^4 + i^6 + i^8 = 1$

Solution

$$i^2 = -1$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$
$$i^6 = i^4\, i^2 = 1\,(-1) = -1$$
$$i^8 = i^4\, i^4 = 1 \times 1 = 1$$

Now,

$$1 + i^2 + i^4 + i^6 + i^8$$
$$= 1 - 1 + 1 - 1 + 1 = 1.$$

1.27 Algebra of Complex Numbers

Let $z_1 = a + ib$, $z_2 = c + id$. We define the following algebra of complex numbers.

(a) Addition

$$z_1 + z_2 = a + ib + c + id$$
$$= (a + c) + i\,(b + d).$$

(b) Subtraction

$$z_1 - z_2 = a + ib - (c + id)$$
$$= (a - c) + i\,(b - d).$$

(c) Multiplication

$$z_1\, z_2 = (a + ib)\,(c + id)$$
$$= ac + i^2\, bd + iad + ibc$$
$$= (ac - bd) + i\,(ad + bc).$$

(d) Division

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

$$= \frac{a+ib}{c+id} \times \frac{c-id}{c-id} \qquad = \frac{ac - i^2 bd + ibc - iad}{c^2 - i^2 d^2}$$

$$= \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \qquad = \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right).$$

1.28 Properties of Complex Numbers

1. If $a + ib = 0$, then $a = 0, b = 0$.
2. If $a + ib = c + id$, then $a = c, b = d$.
3. Let z_1, z_2, z_3 be any three complex numbers, then
   (a) $z_1 + z_2 = z_2 + z_1$ (Commutative of addition).
   (b) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Associative of addition).
   (c) $z_1 z_2 = z_2 z_1$ (Commutative of multiplication).
   (d) $z_1 (z_2 z_3) = (z_1 z_2) z_3$ (Associative of multiplication).
   (e) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (Distributive).

1.29 Inverse

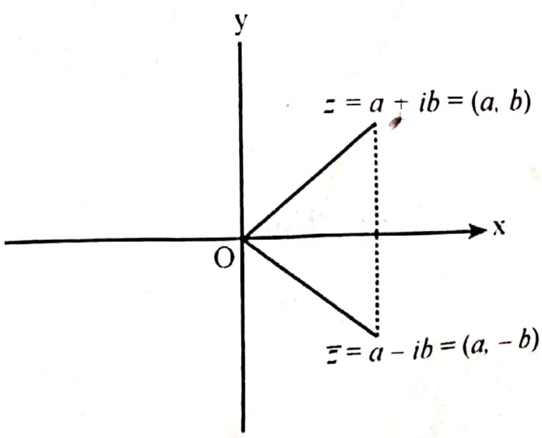
(a) The additive inverse of $z = a + ib$ is $-z = -a - ib$.

(b) The multiplicative inverse (reciprocal) of non-zero complex number $z = a + ib$ is $\frac{1}{z} = \frac{a}{a^2+b^2} + \frac{(-b)}{a^2+b^2} i$.

1.30 Conjugate of a Complex Number

If $z = a + ib$ be any complex number then the conjugate of z, denoted by $\bar{z}$ is defined as $\bar{z} = a - ib$.

Geometrically, the conjugate of a complex number is the reflection of that number about x–axis (real axis).



For example

If $z = 3 + 2i$, then $\bar{z} = 3 - 2i$.

Properties of Conjugate of Complex Numbers

If z, z_1 and z_2 be the Complex numbers then

(a) $\dfrac{z+\bar{z}}{2}$ = Real part of z = Re (z)

(b) $\dfrac{z-\bar{z}}{2i}$ = Imaginary part of z = Im (z)

(c) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(d) $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$

(e) $\bar{\bar{z}} = z.$

Proof

(a) Let $z = a + ib$. Then $\bar{z} = a - ib$.

Now, $\dfrac{z+\bar{z}}{2} = \dfrac{a+ib+a-ib}{2}$

$= \dfrac{2a}{2} = a = \text{Re}(z).$

(b) Let $z = a + ib$. Then $\bar{z} = a - ib$.

Now, $\dfrac{z-\bar{z}}{2i} = \dfrac{a+ib-(a-ib)}{2i}$

$= \dfrac{a+ib-a+ib}{2i} = \dfrac{2ib}{2i} = b = \text{Im}(z).$

(c) Let $z_1 = a + ib$ and $z_2 = c + id$.

Now, $\overline{z_1+z_2} = \overline{(a+ib)+(c+id)}$

$= \overline{a+c+i(b+d)}$

$= a+c-i(b+d)$

$= a-ib+c-id$

$= \bar{z}_1 + \bar{z}_2\,.$

(d) Let $z_1 = a + ib$ and $z_2 = c + id$.

Now, $\overline{z_1\, z_2} = \overline{(a+ib)\cdot(c+id)}$

$= \overline{ac+iad+ibc+i^2bd}$

$= \overline{(ac-bd)+i(ad+bc)}$

$= (ac - bd) - i(ad + bc)$

$= ac - bd - iad - ibc$

$= ac - ibc - iad - bd$

$= ac - ibc - iad + i^2bd$

$= c(a - ib) - id(a - ib)$

$= (a - ib)(c - id)$

$= \overline{z_1}\ \overline{z_2}$.

(e) Let $z = a + ib$.

Then, $\overline{z} = a - ib$.

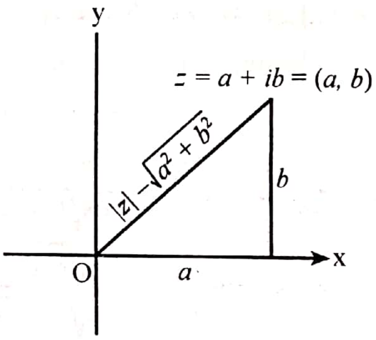
$\therefore \quad \overline{\overline{z}} = \overline{(\overline{z})} = a + ib = z$.

1.31 Modulus of a Complex Number

If $z = a + ib$ be a complex number, then the modulus (absolute value) of z denoted by $|z|$ is defined by

$$|z| = \sqrt{a^2 + b^2}$$

$$= \sqrt{(\text{real part})^2 + (\text{imaginary part})^2}$$



Geometrically, $|z|$ gives the distance of Complex number from origin.

For example

If $z = 3 + 4i$ then

$$|z| = \sqrt{(3)^2 + (4)^2}$$

$$= \sqrt{9 + 16} = \sqrt{25} = 5.$$

1.32 Properties of Modulus of a Complex Number

If z and w be two complex numbers, then

(i) $|z| = |\overline{z}| = |-z|$

(ii) $z\overline{z} = |z|^2$

(iii) $|z\,w| = |z| \cdot |w|$

(iv) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$, $w \neq 0$.

(v) $|z + w| \leq |z| + |w|$ [Triangle inequality]

Proof

(i) Let $z = a + ib$. Then $\overline{z} = a - ib$ and $-z = -a - ib$.

Now, $|z| = |a + ib| = \sqrt{a^2 + b^2}$

$|\overline{z}| = |a - ib| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$

$|-z| = |-a - ib| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

$\therefore \quad |z| = |\overline{z}| = |-z|$.

(ii) Let $z = a + ib$. Then $\overline{z} = a - ib$ and $|z| = \sqrt{a^2 + b^2}$.

$$
\begin{aligned}
\text{Now, } z\overline{z} &= (a + ib)(a - ib) \\
&= a^2 - i^2b^2 \\
&= a^2 + b^2 \\
&= \left(\sqrt{a^2 + b^2}\right)^2 \\
&= |z|^2.
\end{aligned}
$$

(iii) Let $z = a + ib$ and $w = c + id$.

$$
\begin{aligned}
\text{Now, } |zw| &= |(a + ib)(c + id)| \\
&= |ac + iad + ibc + i^2bd| \\
&= |ac - bd + i(ad + bc)| \\
&= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\
&= \sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} \\
&= \sqrt{a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2} \\
&= \sqrt{c^2(a^2 + b^2) + d^2(a^2 + b^2)} \\
&= \sqrt{(a^2 + b^2)\cdot(c^2 + d^2)} \\
&= \sqrt{(a^2 + b^2)}\ \sqrt{(a^2 + b^2)} \\
&= |z||w|.
\end{aligned}
$$

(iv) Let $z = a + ib$ and $w = c + id$.

$$
\begin{aligned}
\text{Now, } \left|\frac{z}{w}\right| &= \left|\frac{(a + ib)}{(c + id)}\right| \\
&= \left|\frac{a + ib}{c + id} \times \frac{c - id}{c - id}\right| \\
&= \left|\frac{ac + ibc - iad - i^2bd}{c^2 + d^2}\right| \\
&= \left|\frac{ac + bd + i(bc - ad)}{c^2 + d^2}\right| \\
&= \left|\frac{ac + bd}{c^2 + d^2} \times i\,\frac{bc - ad}{c^2 + d^2}\right|
\end{aligned}
$$

$$= \sqrt{\left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{bc-ad}{c^2+d^2}\right)^2}$$

$$= \sqrt{\frac{a^2c^2+2abcd+b^2d^2+b^2c^2-2abcd+a^2d^2}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{a^2c^2+b^2c^2+a^2d^2+b^2d^2}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{c^2(a^2+b^2)+d^2(a^2+b^2)}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}}$$

$$= \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$$

$$= \frac{|z|}{|w|}.$$

(v) We have,

$$|z+w|^2 = (z+w)\overline{(z+w)} \qquad [\because |z|^2 = z\bar{z}]$$

$$= (z+w)(\bar{z}+\bar{w}) \qquad \left[\because \overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2\right]$$

$$= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w$$

$$= |z|^2 + |w|^2 + z\bar{w} + \overline{z\bar{w}} \qquad \left[\because z\bar{z} = |z|^2 \text{ and } \bar{\bar{w}} = w\right]$$

$$= |z|^2 + |w|^2 + 2\text{Re}(z\bar{w}) \qquad [\because z+\bar{z} = 2\text{Re}(z)]$$

$$\leq |z|^2 + |w|^2 + 2|z\bar{w}|$$

$$= |z|^2 + |w|^2 + 2|z||w| \qquad [\because |w| = |\bar{w}|]$$

$$= (|z| + |w|)^2$$

$\therefore \quad |z+w| \leq |z| + |w|.$

Alternatively,

Let $z = a + ib$ and $w = c + id$. Then,

$$z + w = (a+ib) + (c+id) = (a+c) + i(b+d)$$

Also, $|z| = \sqrt{a^2+b^2}$, $|w| = \sqrt{c^2+d^2}$

$$|z+w| = \sqrt{(a+c)^2+(b+d)^2}.$$

Now, $|z| + |w| \geq |z + w|$ will be true if

$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \geq \sqrt{(a + c)^2 + (b + d)^2}$$

i.e., $a^2 + b^2 + 2\sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2} + c^2 + d^2 \geq a^2 + 2ac + c^2 + b^2 + 2bd + d^2$

i.e., $2\sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2} \geq 2(ac + bd)$

i.e., $\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \geq ac + bd$

i.e., $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$

i.e., $a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \geq a^2c^2 + 2abcd + b^2d^2$

i.e., $a^2d^2 - 2abcd + b^2c^2 \geq 0$

i.e., $(ad - bc)^2 \geq 0$ which is true for all real numbers a, b, c and d.

Hence, $|z| + |w| \geq |z + w|$.

WORKED OUT EXAMPLES

Example 1. Express the following complex numbers in the form of A + *i*B.

(i) $(2+ 3i)(1 - i)$ (ii) $\dfrac{1}{1 + i}$

(iii) $\dfrac{5 + 4i}{4 + 5i}$ (iv) $\dfrac{2 - \sqrt{-16}}{1 - \sqrt{-25}}$

Solution

(i) $(2 + 3i)(1 - i) = 2 - 2i + 3i - 3i^2$

$= 2 - 2i + 3i + 3$

$= 5 + i$ which is in form of A + *i*B. where, A = 5, B = 1.

(ii) $\dfrac{1}{1 + i} = \dfrac{1}{1 + i} \times \dfrac{1 - i}{1 - i}$

$= \dfrac{1 - i}{1^2 - i^2} = \dfrac{1 - i}{1 + 1}$

$= \dfrac{1}{2} - \dfrac{i}{2}$

$= \dfrac{1}{2} + \left(\dfrac{-1}{2}\right) i$ which is of the form A + *i*B. where, $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$.

(iii) $\dfrac{5 + 4i}{4 + 5i} = \dfrac{5 + 4i}{4 + 5i} \times \dfrac{4 - 5i}{4 - 5i}$

$= \dfrac{20 - 20i^2 - 25i + 16i}{4^2 + 5^2}$

$= \dfrac{40 - 9i}{41}$

$= \dfrac{40}{41} - \dfrac{9i}{41}$ which is of the form A + *i*B. where, $A = \dfrac{40}{41}$ and $B = \dfrac{-9}{41}$.

(iv) $\dfrac{2-\sqrt{-16}}{1-\sqrt{-25}} = \dfrac{2-\sqrt{16i^2}}{1-\sqrt{25i^2}}$

$= \dfrac{2-4i}{1-5i}$

$= \dfrac{2-4i}{1-5i} \times \dfrac{1+5i}{1+5i}$

$= \dfrac{2-4i+10i-20i^2}{(1)^2-(5i)^2}$

$= \dfrac{22+6i}{26}$

$= \dfrac{22}{26} + \dfrac{6}{26}\, i$

$= \dfrac{11}{13} + \dfrac{3}{13}\, i$ which is of the form A + iB. where, $A = \dfrac{11}{13}$, $B = \dfrac{3}{13}$.

Example 2. Reduce $\left(\dfrac{1-i}{1+i}\right)^3$ in the form of A + iB.

Solution

$\left(\dfrac{1-i}{1+i}\right)^3 = \left(\dfrac{1-i}{1+i} \times \dfrac{1-i}{1-i}\right)^3$

$= \left(\dfrac{1-2i-1}{1-i^2}\right)^3$

$= \left(\dfrac{-2i}{1-(-1)}\right)^3$

$= \left(\dfrac{-2i}{2}\right)^3$

$= -i^3$

$= -(i^2 . i)$

$= -(-1. i) = i$

$= 0 + 1.i$, which is of the form A + iB where A = 0 & B = 1.

Example 3. Simplify: $\dfrac{3-i}{2+i} + \dfrac{3+i}{2-i}$.

Solution

$\dfrac{3-i}{2+i} + \dfrac{3+i}{2-i} = \dfrac{(3-i)(2-i)+(2+i)(3+i)}{(2+i)(2-i)}$

$= \dfrac{6-3i-2i+i^2+6+2i+3i+i^2}{2^2-i^2}$

$= \dfrac{12+2i^2}{4-(-1)}$

$= \dfrac{12+2\times(-1)}{5}$

$= \dfrac{10}{5}$

$= 2.$

Example 4. Find the conjugates of the complex numbers

(i) $\dfrac{4+5i}{3-4i}$ (ii) $\dfrac{1}{(2-3i)^2}$.

Solution

(i) Let $z = \dfrac{4+5i}{3-4i}$

$$= \frac{4+5i}{3-4i} \times \frac{3+4i}{3+4i}$$

$$= \frac{12+16i+15i+20i^2}{9+16}$$

$$= \frac{12+16i+15i-20}{25}$$

$$= \frac{-8+31i}{25}$$

$$= \frac{-8}{25} + \frac{31}{25}i$$

where, $A = \dfrac{-8}{25}$ and $B = \dfrac{31}{25}$

$\therefore \bar{z} = A - iB$

$$= \frac{-8}{25} - \frac{31}{25}i.$$

(ii) Let $z = \dfrac{1}{(2-3i)^2}$

$$= \frac{1}{4-12i+9i^2}$$

$$= \frac{1}{-5-12i} \times \frac{-5+12i}{-5+12i}$$

$$= \frac{-5+12i}{25+144}$$

$$= \frac{-5+12i}{169}$$

$$= \frac{-5}{169} + \frac{12}{169}i$$

where, $A = \dfrac{-5}{169}$ and $B = \dfrac{12}{169}$

$\therefore \bar{z} = A - iB$

$$= \frac{-5}{169} - \frac{12}{169}i.$$

Example 6. Find the modulus of

(i) $4 + 3i$ (ii) $(3 - 4i)(2 - i)$ (iii) $\frac{8 + 6i}{5 - 12i}$.

Solution

(i) Let $z = 4 + 3i$

Here, $a = 4$ and $b = 3$

$$\therefore\ |z| = \sqrt{a^2 + b^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{25} = 5.$$

(ii) Let $z = (3 - 4i)(2 - i)$

$$= 6 - 3i - 8i + 4i^2$$
$$= 6 - 3i - 8i - 4$$
$$= 2 - 11i$$

Here, $a = 2$ and $b = -11$

$$\therefore\ |z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{(2)^2 + (-11)^2}$$
$$= \sqrt{4 + 121}$$
$$= \sqrt{125}$$
$$= 5\sqrt{5}.$$

(iii) Let $z = \frac{8 + 6i}{5 - 12i}$

$$= \frac{(8 + 6i)}{(5 - 12i)} \times \frac{5 + 12i}{5 + 12i}$$
$$= \frac{40 + 96i + 30i + 72i^2}{(5)^2 - (12i)^2}$$
$$= \frac{40 + 126i - 72}{169}$$
$$= \frac{-32 + 126i}{169}$$
$$= \frac{-32}{169} + \frac{126}{169}i$$

Here, $a = \frac{-32}{169}$, $b = \frac{126}{169}$

$$\therefore\ |z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{\left(\frac{-32}{169}\right)^2 + \left(\frac{126}{169}\right)^2}$$
$$= \sqrt{\frac{16900}{(169)^2}}$$
$$= \sqrt{\frac{100}{169}} = \frac{10}{13}.$$

Alternatively,

$$z = a + ib$$

$$|z| = \sqrt{a^2 + b^2}$$

$$\therefore \quad |z| = \left|\frac{8 + 6i}{5 - 12i}\right|$$

$$= \frac{|8 + 6i|}{|5 - 12i|}$$

$$= \frac{\sqrt{8^2 + 6^2}}{\sqrt{5^2 + (-12)^2}}$$

$$= \frac{\sqrt{100}}{\sqrt{169}}$$

$$= \frac{10}{13}.$$

Example 6. If $x = 2 - 3i$, prove that $x^2 - 4x + 13 = 0$.

Solution

$$\text{L.H.S.} = x^2 - 4x + 13$$

$$= (2 - 3i)^2 - 4(2 - 3i) + 13$$

$$= 4 - 12i + 9i^2 - 8 + 12i + 13$$

$$= 4 - 9 - 8 + 13$$

$$= 17 - 17$$

$$= 0 = \text{R.H.S.}$$

Example 7. Find the values of x and y if $5x + (3x - y)\, i = 10 + 2i$.

Solution

Equating real and imaginery parts

$$5x = 10$$

$$\therefore \quad x = 2$$

$$\& \quad 3x - y = 2$$

$$\text{or,} \quad 3 \times 2 - y = 2$$

$$\therefore \quad y = 4.$$

Example 8. If $z_1 = 3 - 2i$, $z_2 = 2 + 3i$, find the values of

(i) $\frac{1}{z_1}$ (ii) $\frac{1}{z_2}$ (iii) $\frac{z_1}{z_2}$ (iv) $\frac{\overline{z_1}}{\overline{z_2}}$.

Solution

(i) $$\frac{1}{z_1} = \frac{1}{3 - 2i} \times \frac{3 + 2i}{3 + 2i}$$

$$= \frac{3 + 2i}{9 + 4}$$

$$= \frac{3 + 2i}{13}$$

$$= \frac{3}{13} + \frac{2}{13}i.$$

(ii) $\dfrac{1}{z_2} = \dfrac{1}{2+3i} \times \dfrac{2-3i}{2-3i}$

$= \dfrac{2-3i}{4+9}$

$= \dfrac{2-3i}{13}$

$= \dfrac{2}{13} - \dfrac{3}{13}i.$

(iii) $\dfrac{z_1}{z_2} = \dfrac{3-2i}{2+3i}$

$= \dfrac{3-2i}{2+3i} \times \dfrac{2-3i}{2-3i}$

$= \dfrac{6-9i-4i+6i^2}{4+9}$

$= \dfrac{6-13i-6}{13}$

$= \dfrac{-13i}{13} = -i$

$= 0 + (-1)\, i.$

(iv) $\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{\overline{3-2i}}{\overline{2+3i}} = \dfrac{3+2i}{2-3i}$

$= \dfrac{3+2i}{2-3i} \times \dfrac{2+3i}{2+3i}$

$= \dfrac{6+9i+4i+6i^2}{4+9}$

$= \dfrac{13i}{13} = i.$

Example 9. If $x - iy = \dfrac{5-6i}{5+6i}$, prove that $x^2 + y^2 = 1$.

Solution

Here,

$x - iy = \dfrac{5-6i}{5+6i} \times \dfrac{5-6i}{5-6i}$

$= \dfrac{25-30i-30i+36i^2}{25+36}$

$= \dfrac{25-60i-36}{25+36}$

$= \dfrac{-11-60i}{61}$

$= \dfrac{-11}{61} - \dfrac{60}{61}i$

Equating real and imaginary parts, we have,

$$x = \frac{-11}{61} \text{ and } y = \frac{-60}{61}$$

Now,

$$x^2 + y^2 = \left(\frac{-11}{61}\right)^2 + \left(\frac{-60}{61}\right)^2$$

$$= \left(\frac{121 + 3600}{(61)^2}\right)$$

$$= \frac{3721}{3721} = 1$$

Hence, $x^2 + y^2 = 1$.

Alternatively

We have,

$$x - iy = \frac{5-6i}{5+6i}$$

Taking modulus on both sides

$$|x - iy| = \left|\frac{5-6i}{5+6i}\right|$$

or, $|x - iy| = \frac{|5-6i|}{|5+6i|}$

or, $\sqrt{x^2+y^2} = \frac{\sqrt{5^2 + (-6)^2}}{\sqrt{5^2 + (6)^2}}$

or, $\sqrt{x^2+y^2} = \frac{\sqrt{25+36}}{\sqrt{25+36}}$

or, $\sqrt{x^2+y^2} = \frac{\sqrt{61}}{\sqrt{61}}$

or, $\sqrt{x^2+y^2} = 1$

Squaring both sides, we have

$$x^2 + y^2 = 1.$$

Example 10. If $a + ib = \frac{x+iy}{x-iy}$, prove that: $a^2 + b^2 = 1$.

Solution

$$a + ib = \frac{x+iy}{x-iy}$$

$$= \frac{x+iy}{x-iy} \times \frac{x+iy}{x+iy}$$

$$= \frac{x^2 + i\,xy + i\,xy + i^2y^2}{x^2+y^2}$$

$$= \frac{x^2 - y^2 + 2xyi}{x^2+y^2}$$

$$= \frac{x^2 - y^2}{x^2+y^2} + \frac{2xy}{x^2+y^2}\, i$$

Equating real and imaginary parts, we have

$$a = \frac{x^2 - y^2}{x^2 + y^2},\ b = \frac{2xy}{x^2 + y^2}$$

Now,

$$a^2 + b^2 = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2 + \left(\frac{2xy}{x^2 + y^2}\right)^2$$

$$= \frac{(x^2 - y^2)^2 + 4x^2y^2}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2}$$

$$\therefore\ a^2 + b^2 = 1$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i.$$

Note: We may also use modulus method.

Example 11. If $a + ib = \sqrt{\frac{1 + i}{1 - i}}$, show that $a^2 + b^2 = 1$.

Solution

Here, $$a + ib = \sqrt{\frac{1 + i}{1 - i} \times \frac{1 + i}{1 + i}}$$

$$= \sqrt{\frac{(1 + i)^2}{1 - i^2}}$$

$$= \frac{1 + i}{\sqrt{1 + 1}}$$

$$\therefore\ a + ib = \frac{1 + i}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

Equating real and imaginary parts, we have,

$$a = \frac{1}{\sqrt{2}};\ b = \frac{1}{\sqrt{2}}$$

Now,

$$a^2 + b^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1.$$

Note: We may also use modulus method.

Example 12. If $\sqrt{x - iy} = a - ib$, prove that $\sqrt{x + iy} = a + ib$.

Solution

We have,

$$\sqrt{x - iy} = a - ib$$

Squaring both sides

$x - iy = (a - ib)^2$

$= (a^2 - b^2) - 2abi$

Equating real and imaginary parts.

$x = a^2 - b^2$ and $y = 2ab$

$\therefore \quad x + iy = (a^2 - b^2) + 2abi$

$= a^2 + 2abi + i^2b^2$

$= (a + ib)^2$

$\therefore \quad \sqrt{x + iy} = a + ib.$

Example 13. Find the additive inverse of the complex number $z = 3 + 5i$.

Solution

Let $w = x + iy$ be the additive inverse of z. Then,

$z + w = 0$

or, $3 + 5i + x + iy = 0$

or, $x + 3 + iy + 5i = 0$

or $(x + 3) + i(y + 5) = 0 + 0.i$

Equating real and imaginary parts, we get

$x + 3 = 0$

$\therefore \quad x = -3$

and $y + 5 = 0$

$\therefore \quad y = -5$

$\therefore \quad w = x + iy$

$= -3 - 5i.$

Example 14. Find the multiplicative inverse of $z = \dfrac{i}{1+i}$.

Solution

Let w be the multiplicative inverse of z. Then,

$z \cdot w = 1$

or, $\dfrac{i}{1+i} \cdot w = 1$

or, $w = \dfrac{1+i}{i} \cdot \dfrac{i}{i}$

or, $w = \dfrac{i + i^2}{i^2}$

or, $w = \dfrac{i - 1}{-1}$

or, $-w = i - 1$

$\therefore \quad w = 1 - i.$

EXERCISE – 1 C

1. Find the values of

(a) i^8 (b) i^{15}

(c) $(1, 0)^{100}$ (d) $(0, 1)^{405}$

(e) $i^2 + i^4 + i^6 + i^8$ (f) $i^{10} + i^{11} + i^{12} + i^{13}$.

2. Find the values of

(a) i^{19} (b) $(3 + 4i)(7 + i)$

(c) $(3 - 2i)(5 - 4i)$ (d) $\dfrac{(1 + i)^2}{(1 - i)^2}$

(e) $\dfrac{(1 + i)^2 + (1 - i)^2}{(1 + i)^2 - (1 - i)^2}$. (f) $(1 - i)^3 + (1 + i)^3$

(g) $(1 + i)^4 \left(1 + \dfrac{1}{i}\right)^4$.

3. Express each of the following complex number in the form of A + iB.

(a) $(3 + 4i) + (4 + 3i)$ (b) $(3 - 2i)(6 - 8i)$

(c) $\dfrac{1 + 2i}{1 - i} + \dfrac{1 - 2i}{1 + i}$ (d) $\dfrac{1 - i}{1 + i}$

(e) $\left(\dfrac{1 - i}{1 + i}\right)^3$ (f) $\sqrt{-1}\sqrt{-4}$ (g) $\dfrac{3 - \sqrt{-25}}{2 - \sqrt{-16}}$.

4. Find the additive inverse of

(a) $3 + 2i$ (b) $-1 + 5i$.

5. Find the multiplicative inverse of

(a) $2 + i$ (b) $\dfrac{3 + 4i}{12 - 5i}$.

6. (a) If $x = 3 - 2i$, $y = 2 + 3i$, find the values of

(i) $x^2 + y^2$ (ii) $x^2 + xy + y^2$.

(b) If $x = 4 + 5i$ and $y = 4 - 5i$, find the value of $x^2 - xy + y^2$.

7. Find the values of x and y if

(a) $6x + 14y\,i = 18 - 56i$ (b) $(10, 6x) = (2y, -12)$

(c) $10x + (6x - 2y)\,i = 20 + 4i$ (d) $5x + (3x - y)i = 10 + 2i$.

(e) If $(a + ib)(4 - 3i) = 3 + 4i$, find a and b.

8. If $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, find the values of

(a) $\dfrac{1}{z_1}$ (b) $(\overline{z_1})^2 + (\overline{z_2})^2$ (c) $\dfrac{\overline{z_1}}{\overline{z_2}}$.

9. Find the modulus of the complex numbers.

(a) $(2+3i)$ (b) $(3-2i)(2-3i)$ (c) $\dfrac{1+2i}{1-2i}$

(d) $\dfrac{3+4i}{12-5i}$ (e) $\dfrac{(2+i)(1+2i)}{(1-i)(2+3i)}$.

10. If $z_1 = 2+3i$ and $z_2 = 1-i$, verify that:

(a) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ (b) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

(c) $|z_1 z_2| = |z_1|\,|z_2|$ (d) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(e) $|z_1+z_2| \le |z_1+z_2|$.

11. (a) If $x - iy = \dfrac{2-3i}{2+3i}$, prove that $x^2+y^2 = 1$.

(b) If $x+iy = \dfrac{a+ib}{a-ib}$, prove that $x^2+y^2=1$.

(c) If $x+iy = \sqrt{\dfrac{1+i}{1-i}}$, show that $x^2+y^2 = 1$.

(d) $\sqrt{x+iy} = a+ib$, prove that $\sqrt{x-iy} = a-ib$.

12. Prove that $\dfrac{3+4i}{1-i} + \dfrac{3-4i}{1+i}$ is a real number.

13. If z_1 and z_2 are two complex number, prove that:

$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re\,(z_1\bar{z}_2)$.

Answers

1. (a) 1 (b) $-i$ (c) 1
   (d) 1 (e) 0 (f) 0
2. (a) $-i$ (b) $17+31i$ (c) $7-22i$
   (d) -1 (e) 0 (f) -4
   (g) 16
3. (a) $A=7,\ B=7$ (b) $A=2, B=-36$ (c) $A=-1, B=0$
   (d) $A=0, B=-1$ (e) $A=0, B=1$ (f) $A=-2, B=0$
   (g) $A=\dfrac{13}{10}, B=\dfrac{1}{10}$
4. (a) $-3-2i$ (b) $1-5i$
5. (a) $\dfrac{2}{5}-\dfrac{1}{5}i$ (b) $\dfrac{16}{25}-\dfrac{63}{25}i$
6. (a) (i) 0 (ii) $12+5i$ (b) -59

7. (a) $x = 3, y = -4$ (b) $x = -2, y = 5$
(c) $x = 2, y = 4$ (d) $x = 2, y = 4$
(e) $a = 0, b = 1$

8. (a) $\frac{2}{13} - \frac{3}{13}i$ (b) 0 (c) $-i$

9. (a) $\sqrt{13}$ (b) 13 (c) 1
(d) $\frac{5}{13}$ (e) $\frac{5}{\sqrt{26}}$

Objective Questions

1. $\sqrt{-1}\sqrt{-4} =$
(a) 2 (b) -2
(c) 4 (d) -4

2. Real part of $(3\sqrt{2} + \sqrt{-4})$ is
(a) $3\sqrt{2}$ (b) 2
(c) $5\sqrt{2}$ (d) $\sqrt{2}$

3. $(0, 1)^{99} =$
(a) $(1, 0)$ (b) $(-1, 0)$
(c) $(0, 1)$ (d) $(0, -1)$

4. The number $\frac{2-\sqrt{-16}}{4-\sqrt{-9}}$ lies in
(a) 1st quadrant (b) 2nd quadrant
(c) 3rd quadrant (d) 4th quadrant

5. The value of $3\sqrt{-4} + 4\sqrt{-9} - 5\sqrt{-16}$ is
(a) $-4i$ (b) $4i$
(c) $2i$ (d) $-2i$

6. The value of $\frac{1+2i+3i^2}{1-2i+3i^2} =$
(a) -1 (b) 1
(c) $-i$ (d) i

7. The imaginary part of $\frac{i}{1+i}$ is
(a) $-\frac{1}{4}$ (b) $\frac{1}{4}$
(c) $-\frac{\sqrt{3}}{4}$ (d) $\frac{\sqrt{3}}{4}$

1/2

8. The value of $i^{10} + i^{11} + i^{12} + i^{13} =$
(a) 0 (b) 1
(c) -1 (d) i

9. If $(x+2)+yi=(3+i)(1-2i)$ then $x=$

(a) -3
(b) 3
(c) 5
(d) -5

10. The reciprocal of $(3, 1)$ is

(a) $\left(\frac{3}{10}, \frac{1}{10}\right)$
(b) $\left(-\frac{3}{10}, \frac{1}{10}\right)$
(c) $\left(\frac{3}{10}, -\frac{1}{10}\right)$
(d) $\left(-\frac{3}{10}, -\frac{1}{10}\right)$

11. The conjugate of $\frac{1+i}{1-i}$ is

(a) 1
(b) -1
(c) i
(d) $-i$

12. If $z=2+i$ and $w=3$ then $|3z-4w|=$

(a) $\sqrt{45}$
(b) $\sqrt{53}$
(c) $\sqrt{91}$
(d) $\sqrt{101}$

13. The absolute value of $\frac{3-4i}{3+4i}$ is

(a) 5
(b) $\frac{1}{5}$
(c) 1
(d) 0

14. If $x+iy=\frac{a+ib}{a-ib}$ then $x^2+y^2=$

(a) 0
(b) 1
(c) a^2+b^2
(d) 5

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	a	d	d	d	c	c	a	b	c
11	12	13	14						
d	a	c	b						

2

UNIT

Relation, Functions and Graphs

2.1 Ordered Pairs

A pair of elements a and b is said to be an ordered pair if 'a' is fixed as the first element and 'b' as the second element. It is denoted by (a, b). We note that $(a, b) \neq (b, a)$.

Two ordered pairs (a, b) and (c, d) are said to be equal if and only if $a = c$ and $b = d$.

2.2 Cartesian Product

Let A and B be two sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B. It is denoted by $A \times B$ and read as A cross B.

Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Similarly, $B \times A = \{(b, a): b \in B \text{ and } a \in A\}$.

Example: If $A = \{a, b, c\}$ and $B = \{2, 3\}$, then

$$A \times B = \{(a, 2), (a, 3), (b, 2), (b, 3), (c, 2), (c, 3)\}.$$

$$B \times A = \{(2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}.$$

Note: (i) $A \times B \neq B \times A$

(ii) $A \times B = B \times A$ *if and only if* $A = B$.

If the number of elements in A is m and number of elements in B is n, then the number of elements in $A \times B$ is mn. In the above example, A contains 3 elements and B contains 2 elements, the total number of elements in $A \times B$ is $2 \times 3 = 6$.

2.3 Relation

Let A and B be two sets. Any subset of A × B is called a relation from A to B, denoted by $a \,\mathfrak{R}\, b$, $a \in A$, $b \in B$ or simply by $\mathfrak{R}$. The relation from A to A is called a relation on A.

For example

Let A = $\{a, b\}$ and B = $\{1, 2\}$.

Then A × B = $\{(a, 1), (a, 2), (b, 1), (b, 2)\}$

Let $\mathfrak{R} = \{(a, 1), (b, 2)\}$. Then $\mathfrak{R} \subseteq A \times B$.

So, $\mathfrak{R}$ is a relation from A to B.

Again, let $\mathfrak{R}_1 = \{(1, a), (b, 2)\}$. Then $\mathfrak{R}_1 \nsubseteq A \times B$.

So, $\mathfrak{R}_1$ is not a relation from A to B.

Note: *Total no. of relations from a set A to B with n(A) = m and n(B) = n is* $2^{m \cdot n}$.

2.4 Domain and Range of a Relation

If $\mathfrak{R}$ is a relation from A to B then the set of all first element of ordered pairs in $\mathfrak{R}$ is called the domain of $\mathfrak{R}$ and the set of all second elements is called the range of the relation $\mathfrak{R}$.

Thus, Dom $(\mathfrak{R}) = \{a : (a, b) \in \mathfrak{R}\}$ and Range $(\mathfrak{R}) = \{b : (a, b) \in \mathfrak{R}\}$.

2.5 Inverse of a Relation

Let $\mathfrak{R}$ be a relation from a set A to set B. The inverse of the relation $\mathfrak{R}$ from the set B to A, denoted by $\mathfrak{R}^{-1}$, is the set of all ordered pairs obtained by interchanging the first and second element in each ordered pairs of the given relation $\mathfrak{R}$. Thus, if

$$\mathfrak{R} = \{(a, b) : a \in A, b \in B\} \text{ then } \mathfrak{R}^{-1} = \{(b, a) : (a, b) \in \mathfrak{R}\}.$$

Example: Find the domain, range and inverse of $\mathfrak{R} = \{(a, b), (b, c), (c, d)\}$.

Solution

Domain = $\{a, b, c\}$;

Range = $\{b, c, d\}$ and $\mathfrak{R}^{-1} = \{(b, a), (c, b), (d, c)\}$.

2.6 Types of Relation

1. **Reflexive**

A relation $\mathfrak{R}$ on a set A is called reflexive if every element x of A is related to itself, i.e. $x \,\mathfrak{R}\, x$ or $(x, x) \in \mathfrak{R}$ for every $x \in A$.

2. Symmetric

A relation $\Re$ on a set A is called symmetric if

$(x, y) \in \Re \Rightarrow (y, x) \in \Re$ for every $x, y \in A$.

or, $x \Re y \Rightarrow y \Re x$ for every $x, y \in A$.

3. Transitive

A relation $\Re$ on a set A is called transitive if

$(x, y) \in \Re, (y, z) \in \Re \Rightarrow (x, z) \in \Re$ for every $x, y, z \in A$.

or, $x \Re y, y \Re z \Rightarrow x \Re z$ for every $x, y, z \in A$.

4. Antisymmetric

A relation $\Re$ on A is called antisymmetric if

$x, y \in A, x \Re y, y \Re x \Rightarrow x = y$

or, $x, y \in A, (x, y) \in \Re, (y, x) \in \Re \Rightarrow x = y$.

2.7 Equivalence Relation

A relation on a set A is called an equivalence relation on A if it is reflexive, symmetric and transitive. It is generally denoted by ~ symbol.

Example: If A be the set of all parallel lines in a plane and $\Re = \{(x, y): x, y \in A \text{ and } x \parallel y\}$ and $\Re \subseteq A \times A$ then show that $\Re$ is an equivalence relation on A.

Solution

(i) R is reflexive since $x \parallel x$ for all $x \in A$.

(ii) R is symmetric since $x \parallel y \Rightarrow y \parallel x$ for all $x, y \in A$.

(iii) R is transitive since $x \parallel y, y \parallel z \Rightarrow x \parallel z$ for all $x, y, z \in A$.

Thus, '$\parallel$' is an equivalence relation of A.

Example: Show that the relation "is perpendicular to" over the set of all straight lines in the plane is symmetric but neither reflexive nor transitive.

Solution

Let A be the set of all straight lines in a plane and

$\Re = \{(x, y) : x \perp y; x, y \in A\}$ and $\Re \subseteq A \times A$

(i) For every $x \in A$, x is not perpendicular to itself. So, $(x, x) \notin \Re$. Hence, $\Re$ is not reflexive on A.

(ii) If $x \perp y$ then $y \perp x$ for all $x, y \in A$.

i.e. $(x, y) \in \Re \Rightarrow (y, x) \in \Re$.

Thus $\Re$ is symmetric.

(iii) If $x \perp y, y \perp z$ then x is not $\perp z$ for all $x, y, z \in A$.

i.e. $(x, y) \in \Re, (y, z) \in \Re \Rightarrow (x, z) \notin \Re$.

Thus $\Re$ is not transitive.

In this case $\Re$ is not an equivalence relation on A.

WORKED OUT EXAMPLES

Example 1. Find x and y if $(x - y, 2x + y) = (5, 3)$.

Solution

$$(x - y, 2x + y) = (5, 3)$$

By the equality of two ordered pairs, we have,

$$x - y = 5 \quad \text{... (i)}$$

$$2x + y = 3 \quad \text{... (ii)}$$

Adding (i) and (ii)

$$\begin{array}{l} x - y = 5 \\ \underline{2x + y = 3} \\ 3x \quad = 8 \end{array}$$

$$\therefore \quad x = \frac{8}{3}$$

Substituting the value of x in (i), we have,

$$x - y = 5$$

or, $\frac{8}{3} - y = 5$

or, $\frac{8}{3} - 5 = y$

or, $y = -\frac{7}{3}$

$$\therefore \quad x = \frac{8}{3}, y = \frac{-7}{3}.$$

Example 2. If A = {1, 3} and B = {–2, 1}, find A × B and B × A.

Solution

A = {1, 3} and B = {– 2, 1}

Now, A × B = {1, 3} × {– 2, 1}

= {(1, – 2), (1, 1), (3, – 2), (3, 1)}

and B × A = {– 2, 1} × {1, 3}

= {(– 2, 1), (– 2, 3), (1, 1), (1, 3)}.

Example 3. If $A = \{3, 5\}$ and $B = \{y : y = x + 1, x \in A\}$, find $A \times B$.

Solution

When $x = 3, y = x + 1 = 3 + 1 = 4$

When $x = 5, y = x + 1 = 5 + 1 = 6$

$\therefore$ $A = \{3, 5\}$ and $B = \{4, 6\}$.

Now, $A \times B = \{3, 5\} \times \{4, 6\}$
$= \{(3, 4), (3, 6), (5, 4), (5, 6)\}$.

Example 4. If $A = \{a, b, c\}$, $B = \{a, c\}$ and $C = \{c, d\}$, find $A \times (B \cup C)$ and $A \times (B \cap C)$.

Solution

$B \cup C = \{a, c\} \cup \{c, d\} = \{a, c, d\}$

$B \cap C = \{a, c\} \cap \{c, d\} = \{c\}$

Now, $A \times (B \cup C) = \{a, b, c\} \times \{a, c, d\}$
$= \{(a, a), (a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, c), (c, d)\}$

and $A \times (B \cap C) = \{a, b, c\} \times \{c\}$
$= \{(a, c), (b, c), (c, c)\}$.

Example 5. Let $A = \{b, d\}$ and $B = \{a, c, e\}$. Which of the following set of ordered pairs represent the relation from A to B?

(i) $\Re_1 = \{(b, a), (b, e), (d, c)\}$ (ii) $\Re_2 = \{(d, c), (a, d)\}$

Solution

$A = \{b, d\}$ and $B = \{a, c, e\}$,

$A \times B = \{(b, a), (b, c), (b, e), (d, a), (d, c), (d, e)\}$

(i) $\Re_1 = \{(b, a), (b, e), (d, c)\}$

Here, $\Re_1 \subseteq A \times B$

$\therefore$ $\Re_1$ is the relation from A to B.

(ii) $\Re_2 = \{(d, c), (a, d)\}$

Here, $\Re_2 \not\subseteq A \times B$

$\therefore$ $\Re_2$ is not the relation from A to B.

Example 6. Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$. Find the relation from A to B determined by the relation.

(i) $y = x$ (ii) $x < y$ (iii) $x + y \geq 5$.

Solution

$A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ then

$A \times B = \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$

(i) $\Re_1$ = relation from A to B satisfying $y = x$ is $\{(1, 1), (3, 3)\}$.

(ii) $\Re_2$ = relation from A to B satisfying $x < y$
$= \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$.

(iii) $\Re_3$ = relation from A to B satisfying $x + y \geq 5$
$= \{(1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$.

Example 7. Let A = {1, 2, 3}. Find the relation in A × A satisfying the condition $x > y$ for all $(x, y) \in$ A × A. Find the domain and range of the relation.

Solution

$$A \times A = \{1, 2, 3\} \times \{1, 2, 3\}$$
$$= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$
$$\mathfrak{R} = \text{Relation in } A \times A \text{ satisfying } x > y$$
$$= \{(2, 1), (3, 1), (3, 2)\}$$

Domain of the relation = {2, 3}

Range of the relation = {1, 2}.

Example 8. Find the domain, range and inverse relation of:

$\mathfrak{R} = \{(a, b), (c, e), (d, f), (g, h)\}$.

Solution

$\mathfrak{R} = \{(a, b), (c, e), (d, f), (g, h)\}$

Domain of the relation = $\{a, c, d, g\}$

Range of the relation = $\{b, e, f, h\}$

Inverse relation $(\mathfrak{R}^{-1}) = \{(b, a), (e, c), (f, d), (h, g)\}$.

Example 9. Let $\mathfrak{R}$ be the relation in the set of integers defined by $a \equiv b \pmod 5$. Show that $\mathfrak{R}$ is an equivalence relation on $\mathbb{Z}$.

***Note:** $a \equiv b \pmod 5$ is read as 'a is congruent to b modulo 5'. It means $a - b$ is divisible by 5.*

Solution

(i) **Reflexive:** Let $a \in \mathbb{Z}$. Then $a - a = 0$ which is divisible by 5 and hence $(a, a) \in \mathfrak{R}$. So, $\mathfrak{R}$ is reflexive.

(ii) **Symmetric:** Let $(a, b) \in \mathfrak{R}$. Then $a - b$ is divisible by 5.
Now, $b - a = -(a - b)$ which is divisible by 5.
Hence, $(b, a) \in \mathfrak{R}$.
So, $\mathfrak{R}$ is symmetric.

(iii) **Transitive:** Let $(a, b) \in \mathfrak{R}$ and $(b, c) \in \mathfrak{R}$. Then $(a - b)$ and $(b - c)$ are divisible by 5.
Now, $a - c = (a - b) + (b - c)$ is also divisible by 5.
Hence, $(a, c) \in \mathfrak{R}$.
So, $\mathfrak{R}$ is transitive.
Hence, $\mathfrak{R}$ is an equivalence relation on $\mathbb{Z}$.

EXERCISE – 2 A

1. Find the values of x and y if

   (a) $(x - y, x + y) = (2, 4)$ (b) $(2x - 1, -2) = (1, y + 1)$.

2. Let A = $\{a, b, c\}$ and B = $\{x, y, z\}$. Which of the following set of ordered pairs represent the relation from A to B?

   (a) $\mathfrak{R}_1 = \{(a, x), (b, y), (c, x), (c, z)\}$

(b) $\Re_2 = \{(x, a), (y, b), (z, c)\}$

(c) $\Re_3 = \{(a, x), (b, x), (c, x)\}$.

3. Find the domain, range and inverse relation of the following:

(a) $\Re_1 = \{(a, a), (b, d), (c, f), (g, h)\}$

(b) $\Re_2 = \{(1, 2), (1, 3), (1, 4)\}$.

4. If $A = \{a, b, c\}$ and $B = \{1, 2\}$ find $A \times B$, $B \times A$, $A \times A$ and $B \times B$. Also show that $A \times B \neq B \times A$.

5. If $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$, find $A \times (B \cup C)$, $A \times (B \cap C)$ and $(A \times C) \cap (B \times B)$.

6. (a) Let $A = \{1, 2, 3\}$. Find the relation in $A \times A$ satisfying the condition $x + y < 4$, where $x, y \in A$.

(b) Let $A = \{2, 3, 4\}$. Find the relation in $A \times A$ satisfying the condition (i) $y = 2x$ (ii) $x > y$ (iii) $x + y \leq 4$ (iv) $x + y = 1$.

7. Determine whether the relation $\leq$ is or is not an equivalence relation for the set of real numbers.

8. Let $\Re$ be the relation on the set of integers $\mathbb{Z}$ defined by $a \equiv b \pmod 2$. Prove that $\Re$ is an equivalence relation on $\mathbb{Z}$.

9. Check whether the relation $\Re$ defined in the set $\{1, 2, 3, 4\}$ defined by $\Re = \{(x, y) : y = x + 1\}$ is reflexive, symmetric or transitive.

10. Show that the relation $\Re$ in the set $\{1, 2, 3, 4\}$ given by $\Re = \{(2, 3), (3, 2)\}$ is neither reflexive nor transitive but symmetric.

Answers

1. (a) $x = 3, y = 1$ (b) $x = 1, y = -3$.

2. (a) $\Re_1$ is a relation (b) $\Re_2$ is not a relation (c) $\Re_3$ is a relation.

3. (a) Domain = $\{a, b, c, g\}$, Range = $\{a, d, f, h\}$,

Inverse relation = $\{(a, a), (d, b), (f, c), (h, g)\}$.

(b) Domain = $\{1\}$, Range = $\{2, 3, 4\}$, Inverse relation = $\{(2, 1), (3, 1), (4, 1)\}$.

4. $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$,

$B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$,

$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

$B \times B = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

5. $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$; $\{(1, 3) (2, 3)\}$; $\{(2, 3)\}$.

6. (a) $\{(1, 1), (1, 2), (2, 1)\}$

(b) (i) $\{(2, 4)\}$, (ii) $\{(3, 2), (4, 2), (4, 3)\}$ (iii) $\{(2, 2)\}$ (iv) ϕ

7. Not an equivalence relation

9. Neither.

Objective Questions

1. Which of the following ordered pairs are equal?

(a) (1, 1), (2, 2) (b) (3, 2), (2, 3)

(c) (–2, –5), (2, 5) (d) (1, 5), (1, 5)

2. If $(2x + y, x - 3y) = (1, 11)$ then $y =$

(a) – 3 (b) – 2

(c) 2 (d) 3

3. If A = {1, 2, 3} and B = {a, b} then $n(A \times B) = ?$

(a) 2 (b) 2

(c) 5 (d) 6

4. If A × B = {(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)} then A =

(a) {1, 3} (b) {1, 3, 5}

(c) {2, 3} (d) {1, 2, 3}

5. Let A = {a, b, c} and B = {1, 2, 3}. Which of the following is a relation from A to B?

(a) {(a, 1), (a, 2), (a, 3)} (b) {(a, 1), (2, b), (c, 3)}

(c) {(a, a), (b, b), (c, c)} (d) {(1, a), (2, b), (3, c)}

6. The range of the relation {(a, d), (b, d), (c, d)} is

(a) ϕ (b) {a}

(c) {a, b, c} (d) {d}

7. Let A = {1, 2, 3}. The total number of distinct relations on A is

(a) 2^9 (b) 9^2

(c) 9 (d) 18

8. If A and B are two sets containing n_1 and n_2 elements respectively, how many different relations can be defined from A to B?

(a) $2^{n_1 n_2}$ (b) $n_1 n_2$

(c) $2^{n_1 + n_2}$ (d) 2

9. A relation on a set is said to be an equivalence relation if it is

(a) reflexive (b) symmetric

(c) transitive (d) all of above

10. The relation 'is less than' in the set of natural numbers is

(a) symmetric (b) transitive

(c) reflexive (d) equivalence relation

Answer Sheet

1	2	3	4	5	6	7	8	9	10
d	a	d	b	a	d	a	c	d	b

2.8 Function

Definition

Let A and B be two sets. A function f from a set A to set B is a relation which associates each element of A with unique element of B and we write $f: A \rightarrow B$.

The element $f(x)$ of B is called the image of x under f while x is called the pre-image of $f(x)$ under f.

2.9 Domain, Co-domain and Range of a Function

Let $f: A \rightarrow B$ be a function. The set A is known as the **domain** of f and the set B is known as the **co-domain** of f. The set of all images of elements of A is known as the **range** of f and is denoted by $f(A)$.

Thus $f(A) = \{f(x) : x \in A\}$ = Range of f.

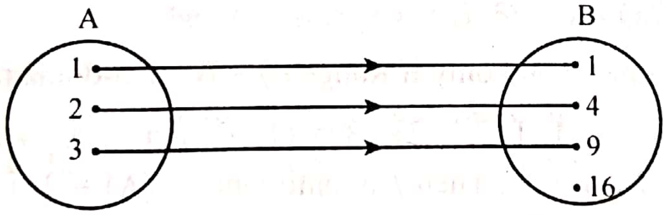
Clearly, $f(A) \subseteq B$ = Co-domain of f.

Example : Let A = {1, 2, 3} and B = {1, 4, 9, 16}. Define $f(x) = x^2$. Is f a function? Also, find domain, co-domain and range.

Solution

Here, $f(1) = 1, f(2) = 4, f(3) = 9$.

Clearly, each element in A has a unique image in B, so f is a function.



Domain (f) = {1, 2, 3}, Range (f) = {1, 4, 9} and co-domain = {1, 4, 9, 16}

Example: Which of the following relations are functions? Give reasons.

If it is a function, find its domain and range.

(i) f = {(1, 3), (1, 5), (2, 3)}

(ii) g = {(2, 1), (5, 1), (8, 1)}.

Solution

(i) f = {1, 3), (1, 5), (2, 3)}

Here, one element namely 1 has two images 3 and 5 under f.

$\therefore$ f is not a function.

(ii) $g = \{(2, 1), (5, 1), (8, 1)\}$

Since each element in the domain have unique image, g is a function.

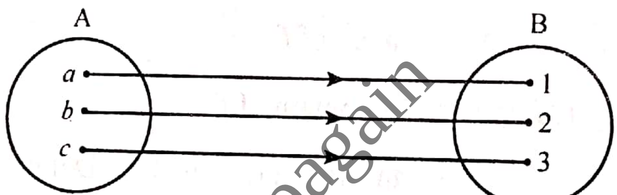
$\therefore$ dom $(g) = \{2, 5, 8\}$ and range $(g) = \{1\}$.

2.10 Various Types of Functions

1. One-One or One to One Function (Injective)

Let $f : A \to B$ be a function. The function f is said to be one-one function or an injective if different elements of A have different images in B. Thus $f: A \to B$ is one-one if and only if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in A$. Equivalently $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in A$.

Example: Let A = $\{a, b, c\}$ and B = $\{1, 2, 3\}$. Let us define $f: A \to B$ by $f(a) = 1, f(b) = 2$ and $f(c) = 3$. Clearly, f is a one to one function from A to B, since different elements in A have different images in B.

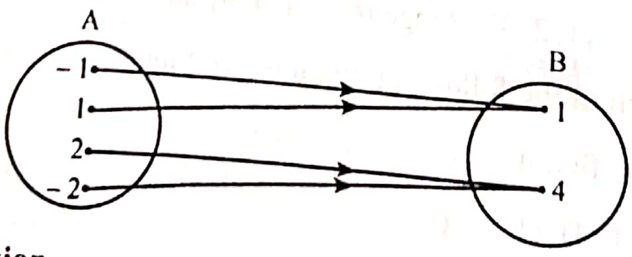


2. Onto Function (Surjective)

Let $f : A \to B$ be a function. If every element in B has at least one pre-image in A then f is said to be an onto function (surjective). Thus $f: A \to B$ is a surjective if and only if for each y $\in$ B, there exists $x \in A$ such that $f(x) = y$.

Thus, f is onto if and only if Range (f) = B = Co-domain f.

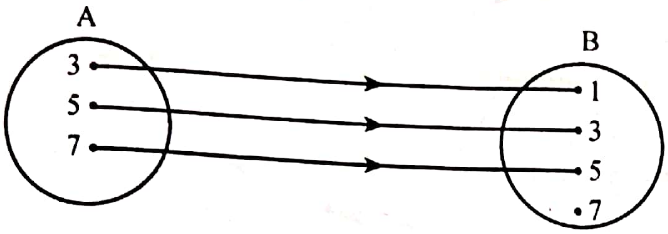
Example: Let A = $\{-1, 1, 2, -2\}$, B = $\{1, 4\}$ and $f: A \to B$ be a function defined by $f(x) = x^2$. Then f is onto since $f(A) = \{f(-1), f(1), f(2), f(-2)\} = \{1, 4\}$ = B.



3. Into Function

Let $f : A \to B$ be a function. If there exists at least one element in B having no pre-image in A then f is said to be an into function.

Example : Let A = {3, 5, 7} and B = {1, 3, 5, 7}. Let f : A → B be defined by $f(x) = x - 2$. Then $f(3) = 1$, $f(5) = 3$ and $f(7) = 5$. Clearly, f is a function from A to B. Here, 7 ∈ B, has no pre-image in A. So f is an into function.

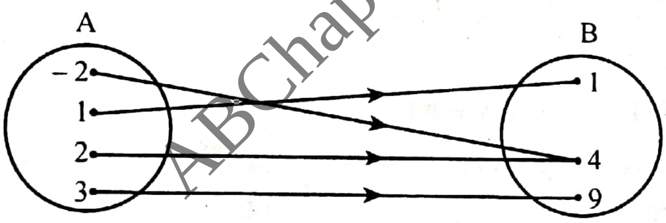


4. **Many to One Function**

Let f: A → B be a function. If two or more than two elements of A have the same image in B, then f is said to be many to one function.

Thus f: A → B is a many-one function if there exist $x_1, x_2 \in A$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

Example : Let A = {–2, 1, 2, 3} and B = {1, 4, 9}. Define f : A → B by $f(x) = x^2$. Then each element in A has a unique image in B. So f is a function from A to B.



Here, $f(-2) = 4$, $f(1) = 1$, $f(2) = 4$ and $f(3) = 9$. Clearly, two elements namely 2 and –2 have the same image 4 ∈ B. So f is many-one function.

5. **Bijective Function**

A function which is both one-one and onto function is said to be bijective. A bijective function is also known as a one-to-one correspondence.

Note: Let $f: A \to B$

(i) f is one-one $\Rightarrow n(A) \leq n(B)$.

(ii) f is onto $\Rightarrow n(B) \leq n(A)$.

(iii) f is both one-one and onto $\Rightarrow n(A) = n(B)$.

2.11 Inverse Function

Let f : A → B be both one-one and onto function. Then f^{-1}: B → A is a function which associates to each y of B, a unique x of A such that $f(x) = y$ is called the inverse function of f.

$\therefore\ f^{-1}(y) = x$ if and only if $f(x) = y$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 5$. Show that f is bijective. Find f^{-1}.

Solution

f is one-one:

Let, $x_1, x_2 \in \mathbb{R}$ (domain).

Now, $f(x_1) = f(x_2)$

$\Rightarrow \quad 3x_1 + 5 = 3x_2 + 5$

$\Rightarrow \quad x_1 = x_2$

$\therefore \quad f$ is one-one.

f is onto:

Let y be an element of $\mathbb{R}$ (co-domain). Then $y = 3x + 5$

or, $\quad x = \dfrac{y-5}{3}$

Clearly, $x = \dfrac{y-5}{3} \in \mathbb{R}$ for all $y \in \mathbb{R}$ and

$$f(x) = f\left(\frac{y-5}{3}\right) = 3\left(\frac{y-5}{3}\right) + 5 = y$$

$\therefore \quad f$ is onto.

Here, $y = f(x) = 3x + 5$, we have

$$y = f(x) \Rightarrow x = f^{-1}(y) \qquad \dots \text{(i)}$$

Again, $y = 3x + 5$

$$y - 5 = 3x$$

$$x = \frac{y-5}{3} \qquad \dots \text{(ii)}$$

From (i) and (ii), we get,

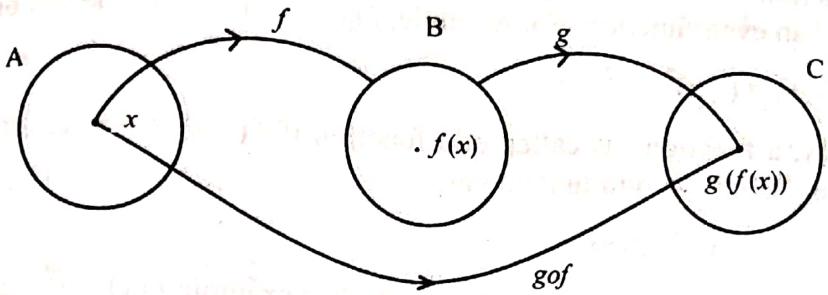
$$f^{-1}(y) = \frac{y-5}{3}$$

Since y is dummy, so replacing y by x.

$$\therefore \quad f^{-1}(x) = \frac{x-5}{3}.$$

2.12 Composite Function

Let, A, B and C be three sets. Let $f: A \to B$ and $g: B \to C$ be two functions. Since $f: A \to B$, for each $x \in A$ there exists a unique element $f(x) \in B$. Again since $g: B \to C$, corresponding to $f(x) \in B$, there exists a unique element $g(f(x))$ of C. Thus for each $x \in A$, there is a unique element $g(f(x))$ of C. Thus from f and g, we can define a new function from A to C. This function is called the composite of f and g denoted by gof and defined by (gof): A → C: $(gof)(x) = g(f(x))$, for all $x \in$ A.



Note: 1. If fog exists then gof may or may not exist and vice-versa.

2. In general fog ≠ gof.

3. If gof is one-one then f is one-one.

4. If gof is onto then g is onto.

5. If f and g are one-one and onto then gof and fog are one-one and onto.

Example: Find gof and fog, if $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are given by $f(x) = 2x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution

We have, $gof(x) = g(f(x))$

$= g(2x)$

$= 3(2x)^2$

$= 3 \times 4x^2$

$= 12x^2$.

Again, $fog(x) = f(g(x))$

$= f(3x^2)$

$= 2(3x^2)$

$= 6x^2$

Hence, $gof \neq fog$.

Example: If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 2$, $g(x) = 2 - x$, find $fog(5)$.

Solution

$fog(x) = f(g(x))$

$= f(2 - x)$

$= 2 - x + 2 = 4 - x$

$\therefore \quad fog(5) = 4 - 5 = -1$.

Even and Odd Functions

If a function f satisfies $f(-x) = f(x)$ for all x in its domain, then the function f is called an even function. For example, $f(x) = x^2$ is even function because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Similarly, a function f is called odd function if $f(-x) = -f(x)$ for all x. For example, $f(x) = x$ is odd function as,

$$f(-x) = -x = -f(x).$$

A function could be neither odd nor even. For example $f(x) = x^2 + x$ is neither odd nor even.

WORKED OUT EXAMPLES

Example 1. Let $f: A \to R$ be given by $f(x) = 2|x| + 3$ where $A = \{-2, 0, 1, 2\}$. Find the range of f.

Solution

When $x = -2$, $f(-2) = 2|-2| + 3 = 2 \times 2 + 3 = 7$

When $x = 0$, $f(0) = 2|0| + 3 = 3$

When $x = 1$, $f(1) = 2|1| + 3 = 2 \times 1 + 3 = 5$

When $x = 2$, $f(2) = 2|2| + 3 = 2 \times 2 + 3 = 7$

$\therefore$ Range of $f = \{f(-2), f(0), f(1), f(2)\}$

$= \{7, 3, 5, 7\} = \{3, 5, 7\}$.

Example 2. Let $A = \{-1, 0, 2, 4, 6\}$ and a function $f: A \to R$ be defined by $y = f(x) = \frac{x}{x+2}$. Find the range of f.

Solution

Here, $f(x) = \frac{x}{x+2}$ and $A = \{-1, 0, 2, 4, 6\}$

When $x = -1$, $f(-1) = \frac{-1}{-1+2} = -1$

When $x = 0$, $f(0) = \frac{0}{0+2} = 0$

When $x = 2$, $f(2) = \frac{2}{2+2} = \frac{1}{2}$

When $x = 4$, $f(4) = \frac{4}{4+2} = \frac{2}{3}$

When $x = 6$, $f(6) = \frac{6}{6+2} = \frac{3}{4}$

$\therefore$ Range of $f = \left\{-1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}$.

Example 3. Let a function $f: A \rightarrow B$ be defined by $f(x) = \frac{x^2}{6}$ with $A = \{-2, -1, 0, 1, 2\}$ and $B = \left\{0, \frac{1}{6}, \frac{2}{3}\right\}$. Find the range of f. Is the function f one to one and onto both?

Solution

Here, $f(x) = \frac{x^2}{6}$, $A = \{-2, -1, 0, 1, 2\}$ and $B = \left\{0, \frac{1}{6}, \frac{2}{3}\right\}$

when $x = -2$, $\quad f(-2) = \frac{(-2)^2}{6} = \frac{4}{6} = \frac{2}{3}$

when $x = -1$, $\quad f(-1) = \frac{(-1)^2}{6} = \frac{1}{6}$

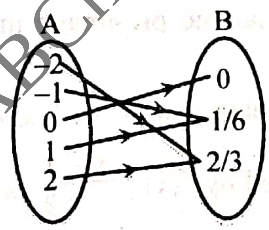
when $x = 0$, $\quad f(0) = \frac{0^2}{6} = 0$

when $x = 1$, $\quad f(1) = \frac{1^2}{6} = \frac{1}{6}$

when $x = 2$, $\quad f(2) = \frac{2^2}{6} = \frac{2}{3}$

Range of $f = \left\{0, \frac{1}{6}, \frac{2}{3}\right\}$.

Arrow diagram



Since range of f = B i.e. every elements in B has at least one pre–image in A, so f is onto.

Here, $-2 \neq 2$ but $f(-2) = \frac{2}{3} = f(2)$

So, f is not one to one.

Example 4. Let a function $f: A \rightarrow B$ be defined by $f(x) = \frac{x-1}{x+2}$ with $A = \{-1, 0, 1, 2, 3, 4\}$ and $B = \left\{-2, 1, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, \frac{2}{5}\right\}$. Find the range of f. Is the function f one to one and onto both? If not, how can you make it one to one and onto both?

Solution

Here, the function $f: A \rightarrow B$ is defined by $f(x) = \frac{x-1}{x+2}$ and $A = \{-1, 0, 1, 2, 3, 4\}$ and $B = \left\{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{4}, \frac{2}{5}\right\}$.

when $x = -1$, $\quad f(-1) = \dfrac{-1-1}{-1+2} = -2$

when $x = 0$, $\quad f(0) = \dfrac{0-1}{0+2} = -\dfrac{1}{2}$

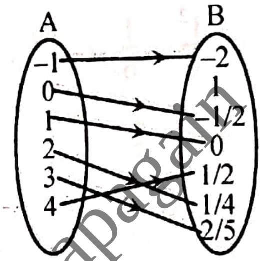
when $x = 1$, $\quad f(1) = \dfrac{1-1}{1+2} = 0$

when $x = 2$, $\quad f(2) = \dfrac{2-1}{2+2} = \dfrac{1}{4}$

when $x = 3$, $\quad f(3) = \dfrac{3-1}{3+2} = \dfrac{2}{5}$

when $x = 4$, $\quad f(4) = \dfrac{4-1}{4+2} = \dfrac{1}{2}$

Range of $f = \left\{-2, -\dfrac{1}{2}, 0, \dfrac{1}{4}, \dfrac{2}{5}, \dfrac{1}{2}\right\}$.



Since the different elements in A have different images in B, so f is one to one.

Again, the element 1 in B has no pre-image in A, i.e. $f(A) \neq B$. So, f is not onto.

The function f can be made both one to one and onto when 1 is removed from B.

i.e. $f: A \to B - \{1\}$ defined by $f(x) = \dfrac{x-1}{x+2}$ is both one to one and onto.

Example 5. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is both one-one and onto i.e. bijective.

Solution

To prove f is one-one (Injective)

Let, $x_1, x_2 \in \mathbb{R}$ (domain) and let $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad x_1^3 - x_2^3 = 0$

$\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$\Rightarrow \quad (x_1 - x_2)\left[\left(x_1 + \dfrac{x_2}{2}\right)^2 + \dfrac{3x_2^2}{4}\right] = 0$

$\Rightarrow \quad x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad f$ is one-one.

To prove f is onto (Surjective)

Let $y \in \mathbb{R}$ (Co-domain).

Now, $y = f(x) = x^3 \Rightarrow x = y^{\frac{1}{3}}$

$\therefore$ For $y \in \mathbb{R}$, there exists $x = y^{\frac{1}{3}} \in \mathbb{R}$ (domain)

Also, $f(x) = f\left(y^{\frac{1}{3}}\right)$

$= \left(y^{\frac{1}{3}}\right)^3 = y.$

$\therefore$ f is onto.

Since f is both one–one and onto, so f is bijective.

Example 6. Determine which of the following $f : \mathbb{R} \to \mathbb{R}$ are (i) one to one (ii) onto

(a) $f(x) = x + 5$ (b) $f(x) = x + |x|$.

Solution

(a) $f(x) = x + 5$

Let x_1 and x_2 be elements in domain $(f) = \mathbb{R}$ such that

$f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 + 5 = x_2 + 5$

$\Rightarrow$ $x_1 = x_2$

$\therefore$ f is one-one function.

Let, $y \in \mathbb{R}$ (Co-domain), $y = f(x) = x + 5$

$x = y - 5 \in \mathbb{R}$ (domain) for all $y \in \mathbb{R}$ (co-domain)

Also, $f(x) = f(y - 5) = y - 5 + 5$

$= y$

Hence f is onto function.

(b) $f(x) = x + |x| = \begin{cases} x + x = 2x \text{ if } x \geq 0 \\ -x + x = 0 \text{ if } x < 0 \end{cases}$

Let, x_1 and $x_2 \in \mathbb{R}^-$. Then, $f(x_1) = 0$ and $f(x_2) = 0$

$\therefore$ $f(x_1) = f(x_2)$ but x_1 may not equal to x_2. Hence f is not one–one function.

Let, $y = f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$\therefore$ Range $f \subseteq \mathbb{R}$

$\therefore$ Range $f \neq$ Co-domain f. Hence, f is not onto.

Example 7. Let $A = \{1, 2, 3\}$. Determine whether the function $f : A \to A$ defined as below have inverse. Find f^{-1}, if it exists.

(a) $f = \{(1, 1), (2, 2), (3, 3)\}$ (b) $f = \{(1, 2), (2, 1), (3, 1)\}$

Solution:

(a) $f = \{(1, 1), (2, 2), (3, 3)\}$

Here, Range of $f = \{1, 2, 3\} = A =$ Co-domain of f.

$\therefore$ f is onto. Also, no two elements of A have same image under f. Thus, f is one–one.

$\therefore$ f is one-one and onto, so f^{-1} exists.

$f^{-1} = \{(1, 1), (2, 2), (3, 3)\} = f$.

(b) $f = \{(1, 2), (2, 1), (3, 1)\}$

Here, $f(2) = 1 = f(3)$

$\Rightarrow$ f is not one-one.

$\Rightarrow$ f is not invertible.

Example 8. Let $\mathbb{Q}$ be the set of all rational numbers. Show that the function $f: \mathbb{Q} \rightarrow \mathbb{Q}$ such that $f(x) = 3x + 5$ for all $x \in \mathbb{Q}$ is one to one and onto. Find f^{-1}.

Solution

One to one:

Let $a, b \in \mathbb{Q}$ (domain).

Then $f(a) = 3a + 5$

$f(b) = 3b + 5$

Now, $f(a) = f(b)$

$\Rightarrow$ $3a + 5 = 3b + 5$

$\Rightarrow$ $3a = 3b$

$\Rightarrow$ $a = b$

Since for $f(a) = f(b) \Rightarrow a = b$, f is one to one function.

Onto function:

Let $y \in \mathbb{Q}$ (Co-domain) be arbitrarily chosen.

Then, $y = 3x + 5$

or, $y - 5 = 3x$

$\therefore$ $x = \dfrac{y-5}{3} \in \mathbb{Q}$ for all $y \in \mathbb{Q}$.

Thus, for every $y \in \mathbb{Q}$, there exists $\dfrac{y-5}{3} \in \mathbb{Q}$ such that

$$f\left(\frac{y-5}{3}\right) = 3\left(\frac{y-5}{3}\right) + 5 = y$$

So, f is onto function.

To find f^{-1}:

Since f is one to one and onto, f^{-1} exists.

Let, $y = f(x) = 3x + 5$

Here, $y = f(x)$

$\Rightarrow$ $x = f^{-1}(y)$ ... (i)

Again, solving for x in terms of y, we have

$$y = 3x + 5$$

or, $y - 5 = 3x$

or, $x = \frac{y-5}{3}$ ... (ii)

From (i) and (ii), we get,

$$f^{-1}(y) = \frac{y-5}{3}$$

Here y is a dummy variable and can be replaced by x.

Thus, $f^{-1}(x) = \frac{x-5}{3}$.

Example 9. Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = cx + d$ where $c\ (\neq 0)$ and d are real numbers, is one to one and onto. Find $f^{-1}(x)$.

Solution

Here,

$f(x) = cx + d\ (c \neq 0)$

One to one

Let $x_1, x_2 \in \mathbb{R}$ (domain).

Then $f(x_1) = cx_1 + d$ and $f(x_2) = cx_2 + d$.

Now, $f(x_1) = f(x_2) \Rightarrow cx_1 + d = cx_2 + d$

$\Rightarrow \quad cx_1 = cx_2$

$\Rightarrow \quad x_1 = x_2$

$\therefore \quad f$ is one to one.

Onto

Let $y \in \mathbb{R}$ (Co-domain). Then,

$$y = f(x) = cx + d$$

or, $y - d = cx$

or, $x = \frac{y-d}{c} \in \mathbb{R}$ (domain) for all $y \in \mathbb{R}$ (Co-domain).

Also, $f(x) = f\left(\frac{y-d}{c}\right) = c\left(\frac{y-d}{c}\right) + d = y.$

$\therefore \quad f$ is onto.

To find f^{-1}

Let $y = f(x) = cx + d$.

or, $x = f^{-1}(y)$ ... (i)

Again, $y = cx + d$

or, $y - d = cx$

or, $x = \frac{y-d}{c}$ ... (ii)

From (i) and (ii), $f^{-1}(y) = \frac{y-d}{c}$

Since y is dummy variable, replacing y by x, we get,

$f^{-1}(x) = \frac{x-d}{c}$.

Example 10. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 3$ and $g: R \to R$ be defined by $g(x) = x^2$. find $(gof)(x)$ and $(fog)(x)$.

Solution

The given functions are

$$f(x) = 2x + 3 \text{ and } g(x) = x^2$$

Now,

(i) $(gof)(x) = g(f(x))$
$= g(2x + 3)$
$= (2x + 3)^2$.

(ii) $(fog)(x) = f(g(x))$
$= f(x^2)$
$= 2(x^2) + 3$
$= 2x^2 + 3$.

Example 11. Let, $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 1$ and $g(x) = x^5$, find f^{-1}, $(gof)(x)$ and $(fog)(x)$.

Solution

We have,

$f(x) = x^3 + 1$ ... (i)

$g(x) = x^5$ ... (ii)

To find f^{-1}; let $y = f(x) = x^3 + 1$

Now, we interchange the role of x and y,

Then the value of y thus obtained is $f^{-1}(x)$.

$\Rightarrow \quad x = y^3 + 1$

or, $\quad y^3 = x - 1$

or, $\quad y = \sqrt[3]{x - 1}$

$\therefore \quad f^{-1}(x) = \sqrt[3]{x - 1}$

Again, $(gof)(x) = g(f(x))$
$= g(x^3 + 1)$
$= (x^3 + 1)^5$

and, $(fog)(x) = f(g(x))$
$= f(x^5)$
$= (x^5)^3 + 1 = x^{15} + 1$.

Example 12. Let $f: \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g: \{3, 4, 5, 9\} \to \{7, 11, 13\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Solution:

Here, $\quad R_f = \{3, 4, 5\} \subset D_g \Rightarrow gof$ is defined.

We have, $gof(2) = g(f(2)) = g(3) = 7$

$gof(3) = g(f(3)) = g(4) = 7$

$gof(4) = g(f(4)) = g(5) = 11$

and $\quad gof(5) = g(f(5)) = g(5) = 11$

$\therefore \quad gof = \{(2, 7), (3, 7), (4, 11), (5, 11)\}$.

Note: *When $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ then fog, gof, fof, gog are all defined from $\mathbb{R}$ to $\mathbb{R}$.*

Example 13. Find the domain of the real function $f(x) = \dfrac{2x}{x^2 - 7x + 12}$.

Solution

Clearly, $f(x)$ is not defined when $x^2 - 7x + 12 = 0$

i.e. $\quad$ when $(x - 3)(x - 4) = 0$

i.e. $\quad$ when $x = 3$ or $x = 4$

$\therefore \quad$ Domain $(f) = \mathbb{R} - \{3, 4\}$.

Example 14. Find the domain and range of the real function $f(x) = x^2$.

Solution

Clearly, $f(x)$ is defined for all $x \in \mathbb{R}$.

$\therefore \quad$ dom $(f) = \mathbb{R}$.

Let, $y = f(x) = x^2$.

$\Rightarrow \quad x = \pm\sqrt{y}$.

Clearly, x is not a real number for negative real values of y.

$\therefore$ Range $(f) = [0, \infty)$.

Example 15. Find the domain and range of the real function $f(x) = \dfrac{1}{x - 3}$.

Solution:

We have, $f(x) = \dfrac{1}{x - 3}$

Clearly, $f(x)$ is not defined when $x - 3 = 0$ i.e. $x = 3$

$\therefore \quad$ Domain $(f) = \mathbb{R} - \{3\}$.

Now, $y = \dfrac{1}{x - 3}$

or, $\quad x - 3 = \dfrac{1}{y}$

or, $\quad x = \dfrac{1}{y} + 3$

$\therefore \quad x = \dfrac{1 + 3y}{y}$

Clearly, x is not defined when $y = 0$.

$\therefore \quad$ Range $(f) = \mathbb{R} - \{0\}$.

Example 16. Find the domain and range of the real function $f(x) = \sqrt{9 - x^2}$.

Solution

Clearly, $f(x)$ is not defined if $9 - x^2 < 0$ i.e. $f(x)$ is defined only if $9 - x^2 \geq 0$.

or, $x^2 - 9 \leq 0$

or, $x^2 \leq 9$

or, $|x|^2 \leq 3^2$

or, $|x| \leq 3$

or, $-3 \leq x \leq 3$

or, $x \in [-3, 3]$

$\therefore$ Domain (f) = $[-3, 3]$.

For each $x \in [-3, 3]$, $f(x) = \sqrt{9 - x^2}$ will have a value in the interval $[0, 3]$

$\therefore$ Range $(f) = [0, 3]$.

Example 17. Find the domain and range of the function $f(x) = \sqrt{2 - x - x^2}$.

Solution

We have,

$$y = \sqrt{2 - x - x^2}$$

$$= \sqrt{2 - (x + x^2)}$$

$$= \sqrt{2 - \left\{x^2 + 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}$$

$$= \sqrt{2 - \left(x + \frac{1}{2}\right)^2 + \frac{1}{4}}$$

$$= \sqrt{\frac{9}{4} - \left(x + \frac{1}{2}\right)^2}$$

Now, $y \geq 0$ for $\frac{9}{4} - \left(x + \frac{1}{2}\right)^2 \geq 0$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 \leq \frac{9}{4}$$

$$\Rightarrow \left|x + \frac{1}{2}\right|^2 \leq \left(\frac{3}{2}\right)^2$$

$$\Rightarrow \left|x + \frac{1}{2}\right| \leq \frac{3}{2}$$

$$\Rightarrow -\frac{3}{2} \leq x + \frac{1}{2} \leq \frac{3}{2}$$

$$\Rightarrow -2 \leq x \leq 1$$

Domain of the function is [– 2 , 1].

Again, $\frac{9}{4}-\left(x+\frac{1}{2}\right)^2=y^2$

$\Rightarrow \quad \left(x+\frac{1}{2}\right)^2 \leq \frac{9}{4}-y^2$

As $\left(x+\frac{1}{2}\right)^2 \geq 0$, for all $x \in \mathbb{R}$,

$$\frac{9}{4}-y^2 \geq 0 \Rightarrow y^2 \leq \frac{9}{4}$$

Since y cannot be negative, $0 \leq y \leq \frac{3}{2}$. Hence range of $f = \left[0, \frac{3}{2}\right]$.

EXERCISE – 2 B

1. Let A = $\{a, b, c\}$ and B = $\{d, e, f\}$. Determine which of the following relations from A to B are functions.
   (a) $\mathfrak{R}_1 = \{(a, d), (a, e), (b, e), (c, f)\}$.
   (b) $\mathfrak{R}_2 = \{(a, d), (b, e), (c, f)\}$.
   (c) $\mathfrak{R}_3 = \{(x, f) : x = a, b, c\}$.
2. What is the difference between a relation and a function? Is $f = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If it is defined by $f(x) = ax + b$, what values should be assigned to a and b?
3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2|x| + 1$. Find $f(0), f(1), \frac{f(1+h)-f(1)}{h}$, $h > 0$.
4. Find the range of the following functions.
   (a) $f(x) = \frac{x}{x+2}$ when domain = $\{-1, 0, 2, 4, 6\}$.
   (b) $g(x) = \frac{-x+1}{|x|-2}$ when domain = $\{-3, 3, 4\}$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2-3x, & \text{for } -\frac{1}{2} \leq x < 0 \\ 2+3x, & \text{for } 0 \leq x < \frac{1}{2} \\ -2-3x, & \text{for } x \geq \frac{1}{2} \end{cases}$. Find $f\left(-\frac{1}{2}\right), f(0), f\left(\frac{1}{2}\right)$ and $\frac{f(h)-f(0)}{h}$ for $-\frac{1}{2} \leq h < 0$.

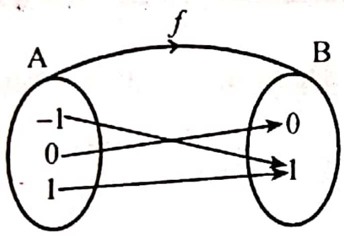
6. What do you mean by a function? Distinguish between the range and the co-domain of a function. Let A ={–2, –1, 0, 1, 2} and a function $f: A \to B$ be defined by $f(x) = \frac{x^2}{2}$. Find the range of f. Is the function one-one?

7. Let a function $f: A \to B$ be defined by $f(x) = \frac{x-1}{x+2}$ with A = {–1, 0, 1, 2, 3, 4} and B = $\left\{-2, -\frac{1}{2}, 0, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, 2\right\}$. Find the range of f. Is the function f one-one and onto both? If not, how can you make it one-one and onto both?

8. Let A = {1, 2, 3} and B = {4, 5, 6, 7}. Let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that (a) f is one to one (b) f is not onto.

9. Examine whether the following functions are one-one, onto, both or neither.

(a)



(b) $f: \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 2x$.

(c) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 5$.

(d) $f: (-5, 5) \to \mathbb{R}$ defined by $f(x) = x^2$.

(e) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

10. Prove that the function $f: \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = 2x + 3$ for all $x \in \mathbb{Q}$ is a bijective function. Find the inverse of f.

11. Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 7$ for all $x \in \mathbb{R}$ is bijective. Find the inverse of f.

12. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x^3 + 1$ and $g(x) = 5x$. Find

(a) $(2f + 3)(x)$ (b) $(f^2)(x)$

(c) $(f \cdot g)(x)$

13. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find:

(a) $(fog)(x)$ (b) $(gof)(x)$

(c) $(fof)(x)$ (d) $(gog)(x)$

14. If $u(x) = 4x - 5$, $v(x) = x^2$ and $f(x) = \frac{1}{x}$, find (a) $u(v(f(x)))$, (b) $v(u(f(x)))$ and (c) $f(u(v(x)))$.

15. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 3$ and $g: \mathbb{R} \to \mathbb{R}$ be defined $g(x) = \frac{x+3}{2}$ show that $fog = gof$.

16. Let A = {1, 2, 3}, Let $f = \{(1, 2), (2, 1), (3, 3)\}$ and $g = \{(1, 3), (2, 1), (3, 2)\}$. Find (a) gof (b) fog.

17. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = ax + b$, $a\ (\neq 0)$ and b are real numbers. Find f^{-1} and show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

18. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x + 1$, find $(fog)(x)$ and $(gof)(x)$. Are the functions $(fog)(x)$ and $(gof)(x)$ one-one?

19. Determine whether the function is even, odd or neither.

(a) $f(x) = x^2 + x^4$ (b) $f(x) = x^3 + \frac{1}{x}$

(c) $h(t) = 2|t| + 1$ (d) $g(u) = u^2 + u$

20. Find the domain and range of

(a) $f(x) = 2x + 3$ (b) $f(x) = 1 + x^2$

(c) $f(x) = \frac{1}{x+1}$ (d) $f(x) = \sqrt{4 - x^2}$

(e) $f(x) = \sqrt{6 - x - x^2}$

21. Find domain and range of f, g, $f + g$ and $f \cdot g$, when $f(x) = x$, $g(x) = \sqrt{x - 1}$.

Answers

1. (a) $\mathfrak{R}_1$ is not a function, (b) $\mathfrak{R}_2$ is a function,
(c) $\mathfrak{R}_3$ is a function.

2. Yes, $a = 2$, $b = -1$

3. 1, 3, 2

4. (a) $\left\{-1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right\}$ (b) $\{4, -2, -\frac{3}{2}\}$.

5. $\frac{7}{2}, 2, -\frac{7}{2}, -3$

6. Range $= \left\{0, \frac{1}{2}, 2\right\}$, No.

7. Range $= \left\{-2, -\frac{1}{2}, 0, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}\right\}$, one-one but not onto. It can be made onto by redefining $f: A \to B - \{2\}$.

9. (a) Onto but not one-one (b) One-one but not onto
(c) Both (d) Neither
(e) Neither.

10. $f^{-1}(x) = \frac{x-3}{2}$

11. $f^{-1}(x) = \sqrt[3]{x - 7}$

12. (a) $6x^3 + 5$ (b) $(3x^3 + 1)^2$
(c) $5x(3x^3 + 1)$

13. (a) $x^2 + 2$ (b) $x^2 + 10x + 22$
(c) $x + 10$ (d) $x^4 - 6x^2 + 6$

14. (a) $\frac{4}{x^2} - 5$ (b) $\left(\frac{4}{x} - 5\right)^2$
(c) $\frac{1}{4x^2 - 5}$.

16. (a) $\{(1, 1), (2, 3), (3, 2)\}$ (b) $\{(1, 3), (2, 2), (3, 1)\}$.

17. $f^{-1}(x) = \frac{x - b}{a}$ 18. $2(x + 1)$, $2x + 1$. Yes

19. (a) Even (b) Odd
(c) Even (d) Neither.

20. (a) Domain = $(-\infty, \infty)$, Range = $(-\infty, \infty)$
(b) Domain = $(-\infty, \infty)$, Range = $[1, \infty)$
(c) Domain = $\mathbb{R} - \{-1\}$, Range = $\mathbb{R} - \{0\}$
(d) Domain = $[-2, 2]$, Range = $[0, 2]$
(e) Domain = $[-3, 2]$, Range = $\left[0, \frac{5}{2}\right]$.

21. $D_f = (-\infty, \infty)$, $R_f = (-\infty, \infty)$; $D_g = [1, \infty)$, $R_g = [0, \infty)$
$D_{f \cdot g} = [1, \infty)$, $R_{f \cdot g} = [1, \infty)$ $D_{f \cdot g} = [1, \infty)$, $R_{f \cdot g} = [0, \infty)$.

Objective Questions

1. If $f(x) = x + a$ maps 1 into 2, then the value of a is
(a) 1 (b) 2
(c) 3 (d) 4

2. If $f(x) = \frac{x - |x|}{|x|}$, then $f(-2) =$
(a) -2 (b) -1
(c) 0 (d) 2

3. A function $f: \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x)\begin{cases} 3 + 2x & \text{for } -\frac{3}{2} \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{for } x \ge \frac{3}{2} \end{cases}, \text{ then } f\left(\frac{3}{2}\right) =$$

(a) 0 (b) 3
(c) -3 (d) -6

4. If $f(x) = x^2$ and $g(x) = 2x + 1$ then $(f + g)(x) =$

(a) $(x + 1)^2$ (b) $(x - 1)^2$
(c) $x^2(2x + 1)$ (d) $2x^3 + x^2$

5. Let A = $\{-1, 1, 2, 3\}$ and B = $\{1, 4, 9, 16, 25\}$ and $f: A \to B$ be defined by $f(x) = x^2$. Range of f =

(a) $\{1, 4, 9, 16, 25\}$ (b) $\{1, 4, 9, 16\}$
(c) $\{1, 4, 9\}$ (d) $\{1, 4\}$

6. The number of functions from $\{a, b, c\}$ into $\{1, 2\}$ is

(a) 2 (b) 4
(c) 6 (d) 8

7. If $f(x - 1) = x + 3$ then $f(x^2) =$

(a) x^2 (b) $x^2 + 3$
(c) $x^2 + 4$ (d) $x^2 - 1$

8. A function is defined as $f(1) = 3, f(2) = 4, f(3) = 5$, then f is

(a) one to one function (b) onto function
(c) both one to one and onto (d) neither onto nor one to one

9. Inverse function f^{-1} exists if f is

(a) injective (b) surjective
(c) bijective (d) into

10. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 5$. Then $f^{-1}(x) =$

(a) $3x - 5$ (b) $\frac{x-3}{5}$
(c) $\frac{x-5}{3}$ (d) $5 - 3x$

11. A function $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 2$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$. Then $fog(x) =$

(a) $(x + 2)^3$ (b) $x^3 + 8$
(c) $(x + 2)^3$ (d) $x^3 + 2$

12. If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ then $gof(x) =$

(a) $\sqrt{x + 1}$ (b) $\sqrt{x} + 1$
(c) $x + \frac{1}{4}$ (d) $x + 2$

13. The domain of the function $y = \frac{1}{x - 1}$ is

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{1\}$
(c) $\mathbb{R} - \{-1\}$ (d) $[0, \infty)$

14. The range of $y = \frac{1}{x}$ is

(a) $(-\infty, 0)$ (b) $(0, \infty)$

(c) $(-\infty, 0) \cup (0, \infty)$ (d) $(-\infty, \infty)$

15. If $g(x) = \sqrt{1 - x}$ then domain of $3g$ is

(a) $(-\infty, 1)$ (b) $(-\infty, 1]$

(c) $(-\infty, 3)$ (d) $(-\infty, 3]$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
a	c	d	a	c	d	c	c	c	c
11	12	13	14	15					
a	b	b	c	b					

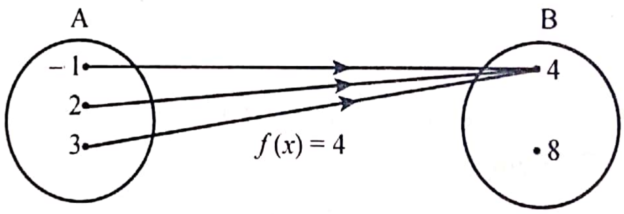
2.13 Algebraic Functions

1. Constant Function

Let, $f: A \to B$ be defined in such a way that all the elements in A have the same image in B then f is said to be a constant function. i.e. $f(x) = c$ for all $x \in A$ and c is a constant.

Example: Let A = {1, 2, 3} and B = {4, 8}.

Let, $f: A \to B$ be defined by $f(x) = 4$ for all $x \in A$. Then all the elements in A have the same image namely 4 in B. Hence, f is a constant function.



Note: The range set of a constant function is a singleton set.

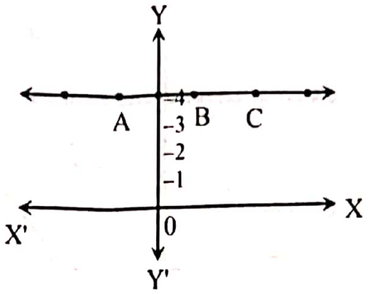
Example: Draw the graph of the following constant function: $f(x) = 4$, for all $x \in \mathbb{R}$.

Solution

Let $f(x) = 4$ then dom $(f) = \mathbb{R}$ and range $(f) = \{4\}$.

x	-1	0	1
$f(x) = 3$	4	4	4

Plot the points A (–1, 3), B (0, 3), C (1, 4) on a graph paper. Join these points to obtain the graph line as shown in the given figure.



2. Identity Function

Let A be a set. Then the function I_A, defined by $I_A: A \to A$ given by $I_A(x) = x$ for all $x \in A$ is called an Identity function on A.

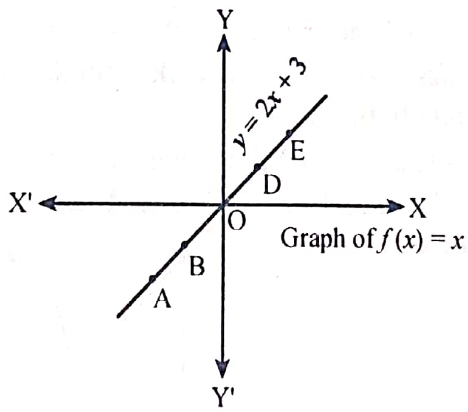
Example: Draw the graph of the identity function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ for all $x \in \mathbb{R}$.

Solution

Here, $f(x) = x$

x	–2	–1	0	1	2
$f(x) = x$	–2	–1	0	1	2

On a graph paper plot the points A (–2, –2), B (–1, –1), O (0, 0), C (1, 1) and D (2, 2). Join these points to obtain the graph line as shown in the given figure.

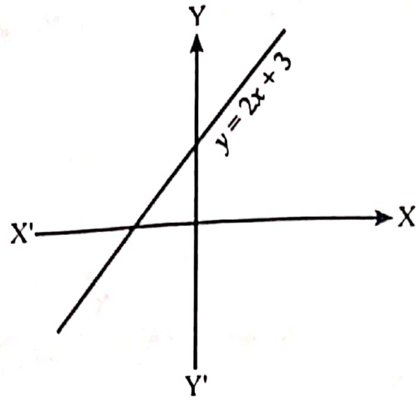


*Note: The constant and identity functions are sometimes called **special functions.***

3. Linear Function

A function $f: A \to B$ defined by $y = f(x) = ax + b$, $a \neq 0$ is said to be a linear function.

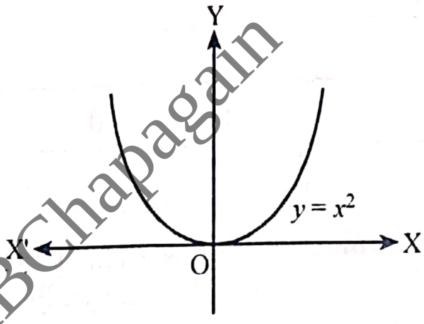
A function $f: \mathbb{R} \to \mathbb{R}$ defined by $y = f(x) = 2x + 3$ is a linear function.



4. **Quadratic Function**

A function $f: A \to B$ defined by $f(x) = y = ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$ is called quadratic function. For example, the function $y = x^2$ be a quadratic function.

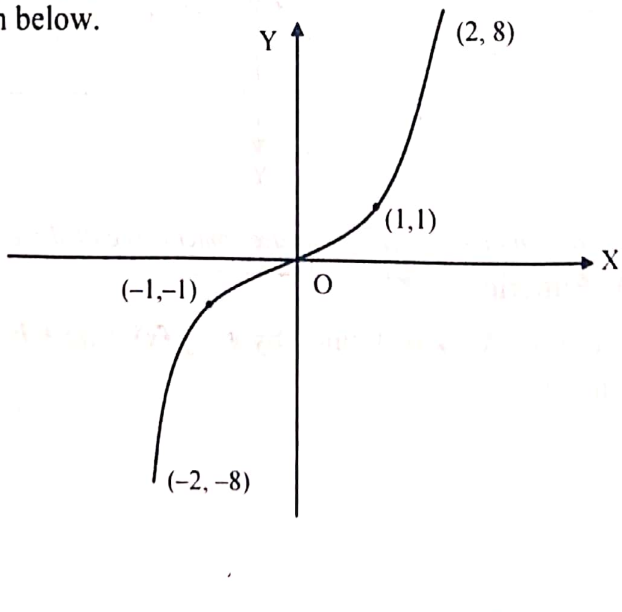
Now, we plot the point (0, 0), ($\pm$ 1, 1) ($\pm$ 2, 4), ($\pm$ 3, 9). The parabola whose vertex is at the origin which is shown in the figure below.



5. **Cubic Function**

The function $f: A \to B$ defined by $f(x) = y = ax^3 + bx^2 + cx + d$, where a, b, c, d are constants and $a \neq 0$ is called cubic function. For example, $y = x^3$ is a cubic function.

Now, if $x = 0, 1, -1, 2, -2$ then the values of $y = 0, 1, -1, 8, -8$. Plot the points (0, 0), (1, 1), (2, 8) and (–2, –8). Draw a curve through these points as shown below.



6. The Polynomial Function

A function f : A $\rightarrow$ B defined by $y = f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ for $x \in \mathbb{R}$, where $a_n, a_{n-1}, \ldots, a_2, a_1$ and a_0 are constants, is called a polynomial function.

The constant function, the linear function, the quadratic function and the cubic function described above are the special cases of the polynomial function.

Rational Function

A function f defined by $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are the polynomials in x and $q(x) \neq 0$ is known as a rational function.

Example : $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \dfrac{3x}{x+1}$, $x \neq -1$ is a rational function.

Absolute Value Function

A function $f(x)$ defined by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is known as the absolute value function. The domain of the function is the set of real numbers $\mathbb{R}$ and the range is the set of non negative real numbers. That is, domain of f = D $(f) = \mathbb{R}$ and the range of $f = \mathbb{R}(f) = [0, \infty)$.

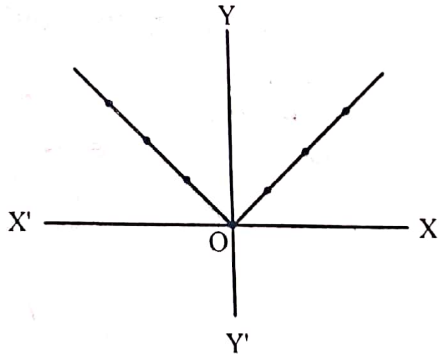
The graph of the absolute value function $f(x) = |x|$ is given below:

When $x \geq 0$, $y = f(x) = x$

x	0	1	2	3
y	0	1	2	3

Again, when $x < 0$, $y = f(x) = -x$

x	0	1	2	3
y	0	–1	–2	–3

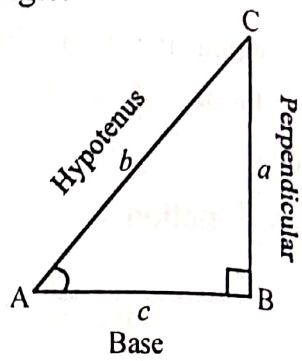


Elementary Transcendental Functions

Here we will discuss about trigonometric, exponential and logarithmic functions.

2.14 Trigonometric Functions and Graphs

There are the functions related to the angles and their measurements. Trigonometric function of the angles of a right angled triangle are defined in terms of the ratios of the sides of a right angled triangle.



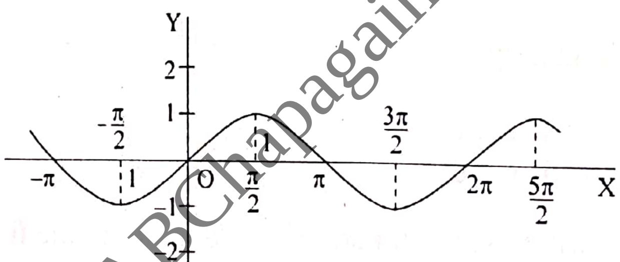
Now,

$$\sin\theta = \frac{a}{b}, \quad \cos\theta = \frac{c}{b}$$

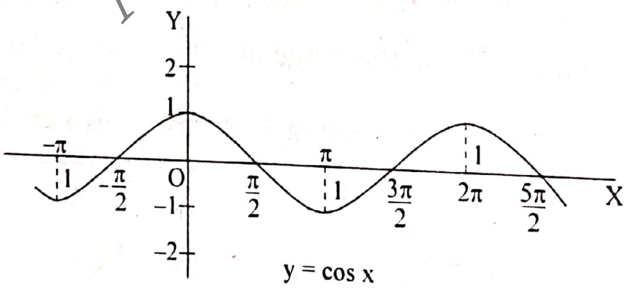
$$\tan\theta = \frac{a}{c}, \quad \cot\theta = \frac{c}{a}$$

$$\operatorname{cosec}\theta = \frac{b}{a} \text{ and } \sec\theta = \frac{b}{c}$$

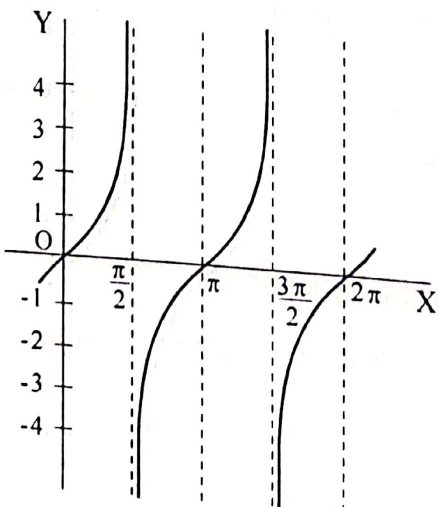
The six-trigonometric ratios are known as the trigonometric functions. The graph of $y = \sin x$, $y = \cos x$ and $y = \tan x$ are shown below.



y = sin x



y = cos x



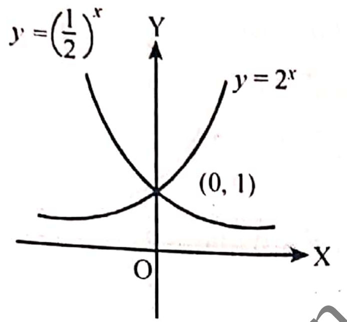
y = tan x

2.15 Exponential Function

A function of the form $y = a^x$ is said to be an exponential function where $a > 0$, $a \neq 1$ and $x \in \mathbb{R}$.

For example

$y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ are exponential functions. Their graphs are as follows:



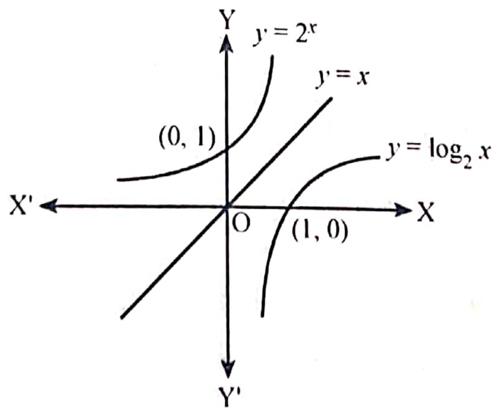
The function e^x is the special type of exponential function. The number e is an irrational number where $e = 2.718281$ ..... The number e is defined as the value of $\left(1 + \frac{1}{n}\right)^n$ as n increase indefinitely. (i.e. $n \to \infty$).

2.16 Logarithmic Function

It is the inverse of an exponential function. If $y = a^x$ then $x = \log_a y$ where $a \neq 0$ and 1. Thus, if y is an exponential function of x then x is a logarithmic function of y.

Let $y = \log_2 x$. Then $x = 2^y$.

The graph of exponential and logarithmic functions are given below:



Logarithmic to the base 10 is known as **common logarithm**, and that to the base e is known as **natural logarithm**. If no confusion arise, we write $\log_e x = \log x = \ell nx$.

Some Results of Logarithmic Functions

Let $a > 0$, m, n be any positive integers.

Theorem 1: $\log_a mn = \log_a m + \log_a n$.

Proof

Let $\log_a m = x$ so that $a^x = m$ . . . . (i)

and $\log_a n = y$ so that $a^y = n$ . . . . (ii)

Multiplying (i) and (ii), we get,

$$m \times n = a^x \times a^y = a^{x+y}$$

or, $\log_a mn = x + y$ (by definition)

$\therefore$ $\log_a mn = \log_a m + \log_a n$.

Theorem 2: $\log_a \frac{m}{n} = \log_a m - \log_a n$.

Proof

Let $\log_a m = x$ so that $a^x = m$ . . . . (i)

and $\log_a n = y$ so that $a^y = n$ . . . . (ii)

Dividing (i) by (ii), we get,

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}.$$

Then, by definition, we get

$$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

Theorem 3: $\log_a m^n = n \log_a m$.

Proof

Let $\log_a m = x$ so that $a^x = m$ . . . (i)

or, $(a^x)^n = (m)^n$

or, $a^{nx} = m^n$

or, $\log_a m^n = nx$

$\therefore$ $\log_a m^n = n \log_a m$.

Theorem 4: $\log_a m = \log_a b \cdot \log_b m.$

Proof

Let, $\log_a m = x$, $\log_b m = y$ & $\log_a b = z$. Then by definition,

$$a^x = m,\ b^y = m\ \&\ a^z = b$$

Also, $a^x = b^y = (a^z)^y = a^{yz}$

or, $x = yz$

$\therefore$ $\log_a m = \log_a b \cdot \log_b m.$

WORKED OUT EXAMPLES

Example 1. Prove that $\log_a x^2 - 2\log_a \sqrt{x} = \log_a x.$

Solution

$$\begin{aligned} \text{L.H.S.} &= \log_a x^2 - 2\log_a \sqrt{x} \\ &= \log_a x^2 - \log_a (\sqrt{x})^2 \\ &= \log_a x^2 - \log_a x \\ &= \log_a \left(\frac{x^2}{x}\right) \\ &= \log_a x = \text{R.H.S.} \end{aligned}$$

Example 2. If $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$, prove that $xyz = 1$.

Solution

Given,

$$\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k \text{ (say)}$$

Then, $\log x = k(y-z)$

$\log y = k(z-x)$

$\log z = k(x-y)$

On adding, we get,

$$\log x + \log y + \log z = k(y-z) + k(z-x) + k(x-y)$$

or, $\log(x\,y\,z) = ky - kz + kz - kx + kx - ky$

or, $\log(xyz) = 0$

or, $\log(xyz) = \log 1$ ($\because \log 1 = 0$)

$\therefore$ $xyz = 1.$

Example 3. If $a^2 + b^2 = 7ab$, prove that $\ln\left(\frac{a+b}{3}\right) = \frac{1}{2}(\ln a + \ln b)$.

Solution

We have, $a^2 + b^2 = 7ab$

or, $(a+b)^2 = 9ab$

or, $\left(\frac{a+b}{3}\right)^2 = ab$

or, $\frac{a+b}{3} = (ab)^{\frac{1}{2}}$

Taking ln on both sides, we get,

$$\ln\left(\frac{a+b}{3}\right) = \ln(ab)^{\frac{1}{2}}$$

or, $\ln\left(\frac{a+b}{3}\right) = \frac{1}{2}\ln(ab)$

$\therefore$ $\ln\left(\frac{a+b}{3}\right) = \frac{1}{2}(\ln a + \ln b)$.

Example 4. If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $xyz + 1 = 2yz$.

Solution

$$\begin{aligned}
xyz + 1 &= \log_{2a} a \log_{3a} 2a \log_{4a} 3a + 1 \\
&= \log_{2a} a (\log_{4a} 3a \cdot \log_{3a} 2a) + 1 \\
&= \log_{2a} a \log_{4a} 2a + 1 && [\because \log_a b \log_b x = \log_a x] \\
&= \log_{4a} 2a \log_{2a} a + 1 \\
&= \log_{4a} a + 1 && [\because \log_a b \log_b x = \log_a x] \\
&= \log_{4a} a + \log_{4a}(4a) && [\because \log_x x = 1] \\
&= \log_{4a}(a \cdot 4a) \\
&= \log_{4a}(2a)^2 \\
&= 2\log_{4a}(2a) \\
&= 2\log_{4a} 3a \cdot \log_{3a} 2a && [\because \log_a b \log_b x = \log_a x] \\
&= 2yz
\end{aligned}$$

$\therefore$ $xyz + 1 = 2yz$.

EXERCISE – 2 C

1. Graph the functions

(a) $y = x^2$

(b) $y = x^3$

(c) $y = \sin x \ (-\pi < x < \pi)$

(d) $y = 3^x$

(e) $y = \log_e 2$.

2. Prove that:

(a) $\log_a(x^2y^3z) = 2\log_a x + 3\log_a y + \log_a z$

(b) $\log_a\left(\frac{x^2}{y^3}\right) = 2\log_a x - 3\log_a y$

(c) $2\log_a\sqrt{x} = \log_a x$

(d) $a^{\log_a x} = x$

(e) $(\log x)^2 - (\log y)^2 = \log(xy)\cdot\log\left(\frac{x}{y}\right)$

(f) $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$

(g) $a^{\log b - \log c}\cdot b^{\log c - \log a}\cdot c^{\log a - \log b} = 1$

(h) $(yz)^{\log y - \log z}\cdot(zx)^{\log z - \log x}\cdot(xy)^{\log x - \log y} = 1.$

3. Solve for x.

(a) $\left(\frac{1}{2}\right)^{-x} = 8^{3x-1}$

(b) $\log_3(x-1) = 2$

4. If $f(x) = \log\frac{1+x}{1-x}\ (-1 < x < 1)$, show that $f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right)$ where $|a| < 1, |b| < 1.$

5. If $f(x) = \log\frac{1-x}{1+x}\ (-1 < x < 1)$ show that: $f\left(\frac{2ab}{1+a^2b^2}\right) = 2f(ab)$ where $|ab| < 1.$

6. If $x^2 + y^2 = 11xy$, prove that: $\ln\left(\frac{x-y}{3}\right) = \frac{1}{2}(\ln x + \ln y).$

7. If $x = \log_a bc,\ y = \log_b ca,\ z = \log_c ab$, prove that:

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1.$$

Answers

3. (a) $\frac{3}{8}$ (b) 10

Objective Questions

1. The function $f(x) = ax^2 + bx + c,\ a \neq 0$ is

(a) constant function (b) linear function
(c) quadratic function (d) cubic function

2. The function $f(x) = 3^x,\ x \in \mathbb{R}$ is

(a) linear function (b) trigonometric function
(c) cubic function (d) exponential function

3. $\log_3 81 =$

(a) 1 (b) 2
(c) 3 (d) 4

4. $\log_{\sqrt{2}} \frac{1}{16} =$

(a) −8 (b) −4
(c) −2 (d) −1

5. $\log_x \sqrt{x\sqrt{x\sqrt{x^2}}} =$

(a) x (b) x^2
(c) 1 (d) 0

6. $\log_a a^x =$

(a) a (b) x
(c) x^2 (d) a^x

7. $\log_a x$ is same as

(a) $y = a^x$ (b) $x = a^y$
(c) $a = x^y$ (d) $a = y^x$

Answer Sheet

1	2	3	4	5	6	7		
c	d	d	a	c	b	b		

○○○

3

UNIT

Sequence and Series

3.1 Introduction

In this chapter, we shall discuss about the concept of sequence and series. In particular, we discuss the arithmetic, geometric and harmonic sequences and also their corresponding series. There are wide range of applications of the sequence and series. Installments of loan payment of finance company, premiums of life insurance, fixed deposits in a bank, delay of radioactive materials, etc. are some of the examples where the concept of sequence and series are applied. We will also discuss the concepts of convergence and divergence.

3.2 Sequence

A set of numbers occurring in a definite order or by a rule is called a sequence. Thus any set of numbers a_1, a_2, a_3, ... a_n ... such that to each positive integer n, there corresponds a number a_n is a sequence. A sequence is a function whose domain is the set of natural numbers and range is a subset of real or complex numbers. The numbers a_1, a_2, ... a_n, ... are called elements (or terms) of the sequence.

Finite Sequence

A sequence is said to be finite if the number of elements in that sequence is finite. In this case, the last term is specified.

For example

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Infinite Sequence

A sequence is said to be infinite, if the number of elements is infinite.

For example

1, 2, 3, 4, 5 ...

3.3 Series

When the terms of a sequence are connected with plus or minus signs, a series is formed. In other word, a series is an expression consisting of the sum of the terms in a sequence. Thus if a_n is the n^{th} term of a sequence then

$a_1 + a_2 + a_3 + \dots + a_n$ is a series of n terms.

3.4 Progression

A sequence which follows a certain rule (pattern) is called a progression.

(i) The sequence 2, 4, 6, 8, ... is a progression in which each term differs from its preceding term by 2.

(ii) The sequence, 3, 9, 27, ... is a progression as the ratio of any term to its preceeding is 3.

(iii) The sequence $1, \frac{1}{2}, \frac{1}{3} \dots$ of numbers are in progression as the reciprocals of the terms of the sequence differ by a constant number.

(iv) The sequence 1, 1, 2, 3, 5 ... is not a progression because it follows definite order but not definite rule.

3.5 Arithmetic Progression

The sequence whose terms increase or decrease by a constant number is called a arithmetic progression. The constant difference is called common difference.

For example: 1, 3, 5, 7 ... is an infinite arithmetic progression whose first term is 1 and the common difference is 2. Thus, if the first term and common difference are known, the A.P. is completely known.

Thus, if for a sequence $\{a_n\}$, $a_{n+1} - a_n$ remains constant for all positive integers n, then the sequences is called the A.P. and the difference between two consecutive terms is called the common difference of the A.P.

The n^{th} term of an A.P.

Let a be the first term and d be the common difference then $t_n = a + (n - 1)\, d$ which is also the term indicated by the last term l.

This n^{th} term is called the general term of an A.P. The A.P. is in the form: $a, a + d, a + 2d, \dots$

Sum of a Series in A.P.

The formula used for finding out the sum of a series in A.P. is

$$S_n = \frac{n}{2}(a + l)$$

or, $S_n = \frac{n}{2}[2a + (n - 1)d]$, putting $l = t_n = a + (n - 1)d$.

3.6 Properties of Arithmetic Progression

The properties of arithmetic progression are as follows:

(i) If each term of an A.P. be increased or decreased by a constant number, the resulting numbers are in A.P.

(ii) If each term of an A.P. be multiplied or divided by a constant number (denominator not being 0), the resulting numbers are in A.P.

Let, $a, b, c, d, \ldots$ be an A.P. and k be any constant number not equal to zero then

(a) $a + k, b + k, c + k, d + k, \ldots$

(b) $a - k, b - k, c - k, d - k, \ldots$

(c) $ak, bk, ck, dk, \ldots$

(d) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}, \ldots$ are in A.P.

Example: What term of the A.P. 2, 5, 8 … is 56 ?

Solution

$a = 2, d = 3, t_n = 56$ and $n = ?$

We have,

$t_n = a + (n - 1)d$

or, $56 = 2 + (n - 1)3$

or, $56 = 3n - 1$

or, $3n = 56 + 1$

$\therefore$ $n = 19.$

Example: Find the sum of the following series

(i) $1 + 4 + 7 + 10 + \ldots$ to 40 terms

(ii) $2 + 7 + 12 + 17 + \ldots + 102.$

Solution

(i) Here $a = 1, d = 3, n = 40$

We have,

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{40} = \frac{40}{2}[2 \times 1 + (40-1)3]$$

$$= 20(2 + 117)$$

$$= 20 \times 119 = 2380.$$

(ii) Here $t_n = 102, a = 2, d = 7 - 2 = 5$

We have,

$$t_n = a + (n-1)d$$

or, $102 = 2 + (n-1)5$

or, $102 - 2 = 5n - 5$

or, $5n = 105$

$\therefore$ $n = 21$

Then

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{11} = \frac{21}{2}[2 \times 2 + (21-1)5]$$

$$= \frac{21}{2}[4 + 100]$$

$$= \frac{21}{2} \times 104$$

$$= 21 \times 52 = 1092.$$

3.7 Geometric Progression (G.P.)

A sequence whose terms increase or decrease by a constant ratio is called geometric progression. The constant ratio is called common ratio.

For example: 1, 2, 4, 8, 16, 32, ... is an infinite G.P., whose first term is 2 and the common ratio is 2. The G.P. is in the form: $a, ar, ar^2, ar^3, \ldots$

Thus, if for a sequence $\frac{a_{n+1}}{a_n}$ remains constant for all natural numbers n, then the sequence is called G.P. The n^{th} term of G.P is $t_n = ar^{n-1}$ and its sum of the first n terms is given by

$$S_n = \begin{cases} na, \text{when } r = 1; \\ \frac{a(1-r^n)}{(1-r)}, \text{ when } r < 1; \\ \frac{a(r^n - 1)}{(r-1)}, \text{ when } r > 1 \end{cases} \qquad S_n = \frac{a(\ell r - a)}{r-1}, r \neq 1.$$

3.8 Properties of Geometric Progression

The important properties of geometric progression are as follows:

(i) If each term of a G.P. is multiplied or divided by a constant number, the resulting sequence is also a G.P.

i.e. $ka, kb, kc, \ldots$ is a G.P. and

$\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \ldots$ is a G.P., if $a, b, c \ldots$ is a G.P.

(ii) If each term of a G.P. is raised to a constant power the resulting sequence is again a G.P. i.e.

$a^k, b^k, c^k \ldots$ is a G.P., if $a, b, c \ldots$ is a G.P.

Example : If the 4^{th} and 9^{th} terms of a G.P. are 54 and 13122 respectively, find the G.P.

Solution

Let a be the first term and r the common ratio of the given G.P. Then,

$t_4 = 54$

or, $ar^3 = 54$ ... (i)

$t_9 = 13122$

or, $ar^8 = 13122$ ... (ii)

On dividing (ii) by (i), we get,

$$\frac{ar^8}{ar^3} = \frac{13122}{54}$$

or, $r^5 = 243 = 3^5$

or, $r = 3$

Putting $r = 3$ in (i), we get,

$a \times 3^3 = 54$

or, $27a = 54$

or, $a = 2$

Thus, $a = 2$ and $r = 3$

Hence, the required G.P. is 2, 6, 18, 54, ...

Example: The first term of a G.P. is 1. The sum of its third and fifth term is 90. Find the common ratio of the G.P.

Solution

The first term of the given G.P. is 1.

Let r be the common ratio of this G.P. Then,

$t_3 = 1 \cdot r^{(3-1)} = r^2$ and $t_5 = 1 \cdot r^{(5-1)} = r^4$

$\therefore \quad t_3 + t_5 = 90$

or, $r^2 + r^4 = 90$

or, $r^4 + r^2 - 90 = 0$

or, $(r^2 + 10)(r^2 - 9) = 0$

or, $r^2 = 9$ $\qquad [\because r^2 \neq -10]$

or, $r = \pm 3$

Hence, the common ratio of the given G.P. is 3 or – 3.

Example: Find three numbers in G.P. whose sum is 52 and the sum of whose products in pairs is 624.

Solution

Let the required numbers be a, ar, ar^2. Then,

$$a + ar + ar^2 = 52$$

$$\Rightarrow \quad a(1 + r + r^2) = 52 \qquad \text{... (i)}$$

And, $a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624$

$$\Rightarrow \quad a^2 r(1 + r + r^2) = 624 \qquad \text{... (ii)}$$

On dividing (ii) by (i), we get,

$$ar = 12$$

or, $a = \dfrac{12}{r}$ $\qquad$ ...(iii)

Putting $a = \dfrac{12}{r}$ in (i), we get,

$$\frac{12}{r} \cdot (1 + r + r^2) = 52$$

or, $3(1 + r + r^2) = 13r$

or, $3r^2 - 10r + 3 = 0$

or, $(3r - 1)(r - 3) = 0$

or, $r = \dfrac{1}{3}$ or $r = 3$

Putting the value of r in (iii), we get,

$a = 36$ or $a = 4$.

3.9 Harmonic Progression (H.P.)

The sequence of number formed by taking the reciprocals of the terms forms an A.P. is called Harmonic Progression.

For example

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$ is a H.P. because the reciprocals of the terms of this progression 1, 2, 3, 4, … is a A.P.

3.10 Properties of Harmonic Progression

If each term of a H.P. be multiplied or divided by a constant number, the sequence of the resulting numbers is also in H.P. Let, $a, b, c, d \ldots$ be in H.P. and k be any non zero constant then

(i) $ak, bk, ck, dk \ldots$ are also in H.P.

(ii) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}, \ldots$ are also in H.P.

3.11 Mean

Any term between the first term and last term of an arithmetic sequence is called an Arithmetic Mean (A.M.).

Any term between the first and last term of a geometric sequence is called a Geometric Mean (G.M.).

Any term in between the first and last term of a harmonic sequence is called a Harmonic Mean (H.M.).

Theorem 1: Let a and b be any two positive numbers. Then A.M., G.M. and H.M. between them are given by

(i) $\text{A.M.} = \text{A} = \frac{a+b}{2}$

(ii) $\text{G.M.} = \text{G} = \sqrt{ab}$

(iii) $\text{H.M.} = \text{H} = \frac{2ab}{a+b}$.

Proof

(i) Let A be the single A.M. between a and b. Then a, A, b be are in A.P. By definition of A.P.

$$\text{A} - a = b - \text{A}$$

or, $2\text{A} = a + b$

$\therefore \quad \text{A} = \frac{a+b}{2}$.

(ii) Let G be the single G.M. between a and b. Then a, G, b be are in G.P. By definition of G.P.

$$\frac{\text{G}}{a} = \frac{b}{\text{G}}$$

or, $\text{G}^2 = ab$

$\therefore \quad \text{G} = \sqrt{ab}$.

(iii) Let H be the single H.M. between a and b. Then a, H, b be are in H.P.
By definition of H.P.

$$\frac{1}{a}, \frac{1}{H}, \frac{1}{b} \text{ are in A.P.}$$

$$\therefore \quad \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\text{or,} \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

$$\therefore \quad H = \frac{2ab}{a+b}.$$

Theorem 2: Given any two numbers a and b, the n A. M.'s between them are given by $a + b$, $a + 2d$, $a + 3d$, …, $a + nd$, where, $d = \dfrac{b-a}{n+1}$.

Proof

Let, $m_1, m_2, m_3 \ldots, m_n$ be n A. M's to be inserted between a and b. then, $a, m_1, m_2, m_3 \ldots, m_n, b$ are in A. P.

The number of terms in the above A.P. is $n + 2$ of which the first term is a and the last term is b.

If d is the common difference, then

$$b = a + (n + 2 \ldots 1)\, d$$

$$\text{or,} \quad b - a = (n + 1)\, d$$

$$\therefore \quad d = \frac{b-a}{(n+1)}$$

Now, $m_1 = t_2 = a + d$

$m_2 = t_3 = a + 2d$

$m_3 = t_4 = a + 3d$

. . . .

. . . .

$m_n = t_{n+1} = a + nd.$

Theorem 3: Given any two numbers a and b, the n G. M.'s between them are given by $ar, ar^2, ar^3, \ldots, ar^n$, where $r = \left(\dfrac{b}{a}\right)^{\frac{1}{(n+1)}}$.

Proof

Let $G_1, G_2, G_3, \ldots, G_n$ be n G. M.'s between a and b then,

$a, G_1, G_2, G_3, \ldots, G_n, b$ form a G.P.

The number of terms in the above G.P. is $(n + 2)$ of which the first term is a and the last term is b.

If r be the common ratio, then

$$b = ar^{n+2-1}$$

or, $b = ar^{n+1}$

or, $r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

Now,

$G_1 = t_2 = ar$

$G_2 = t_3 = ar^2$

$G_3 = t_4 = ar^3$

. . . .

. . . .

$G_n = t_{n+1} = ar^n$.

Theorme 4: The A.M. G.M. and H.M. between any two unequal positive numbers satisfy the relations:

(i) A.M. × H.M. = (G.M.)^2 (ii) A.M. > G.M. > H.M.

Proof

Let a and b be two unequal positive numbers. Then,

$$\text{A.M.} = \frac{a+b}{2}$$

$$\text{G.M.} = \sqrt{ab}$$

and $\text{H.M.} = \frac{2ab}{a+b}$

(i) $\text{A.M.} \times \text{H.M.} = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab$

$$= \left(\sqrt{ab}\right)^2$$

$$= (\text{G.M.})^2.$$

i.e. A.M., G.M., H.M. are in G.P.

(ii) $\text{A.M.} - \text{G.M.} = \frac{a+b}{2} - \sqrt{ab}$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$= \frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2$ which is always positive.

∴ A.M. > G.M. ... (i)

Again from (a), A.M. × H.M. = G.M. × G.M.

or, $\frac{\text{A.M.}}{\text{G.M.}} = \frac{\text{G.M.}}{\text{H.M.}}$

Since, A.M. > G.M., so we have, G.M. > H.M. ... (ii)

Combining (i) and (ii), we get, A.M. > G.M. > H.M.

i.e. A.M., G.M. and H.M. are in descending order of magnitudes.

WORKED OUT EXAMPLES

Example 1. The third term of an A.P. is $\frac{1}{5}$ and the 5^{th} term is $\frac{1}{3}$. Show that the sum to 15 terms of the A.P. is 8.

Solution

Let a be the first term and d be the common difference of the A.P. Then,

$$t_n = a + (n-1)d$$

or, $t_3 = a + (3-1)d$

or, $t_3 = a + 2d$

$$\frac{1}{5} = a + 2d \qquad \ldots(i)$$

Similarly, $t_5 = a + (5-1)d$

or, $\frac{1}{3} = a + 4d \qquad \ldots(ii)$

Subtracting (i) from (ii)

$$\frac{1}{3} = a + 4d$$

$$\frac{1}{5} = a + 2d$$

$$- \quad - \quad -$$

$$\frac{1}{3} - \frac{1}{5} = 2d$$

or, $\frac{2}{15} = 2d$

$$d = \frac{1}{15}$$

Putting $d = \frac{1}{15}$ in equation (i), we get,

$$a + 2\left(\frac{1}{15}\right) = \frac{1}{5}$$

or, $a = \frac{1}{15}$

Now, $S_n = \frac{n}{2}\,[2a + (n-1)d]$

$$S_{15} = \frac{15}{2}\left[2\times\frac{1}{15} + (15-1)\frac{1}{15}\right]$$

$$= \frac{15}{2}\left(\frac{2}{15} + \frac{14}{15}\right) = \frac{15}{2} \times \frac{16}{15}$$

$$= 8.$$

Example 2. Insert 4 arithmetic means between 5 and 20.

Solution

Let m_1, m_2, m_3, m_4 be the 4 arithmetic means between 5 and 20.

Then $5, m_1, m_2, m_3, m_4, 20$ are in A.P. Here $a = 5$, $t_6 = 20$, $n = 6$

Let 'd' be the common difference. Then

$t_n = a + (n - 1)d$

or, $t_6 = 5 + (6 - 1)d$

or, $20 = 5 + 5d$

or, $15 = 5d$

$\therefore$ $d = 3$

Now.

$m_1 = a + d = 5 + 3 = 8$

$m_2 = a + 2d = 5 + 2 \times 3 = 5 + 6 = 11$

$m_3 = a + 3d = 5 + 3 \times 3 = 5 + 9 = 14$

$m_4 = a + 4d = 5 + 4 \times 3 = 5 + 12 = 17$

$\therefore$ The required arithmetic means are 8, 11, 14 and 17.

Example 3. How many terms of the series 20 + 16 + 12 + ... amounts to 48? Explain the double answers.

Solution

$a = 20$, $d = -4$; $S_n = 48$, $n = ?$

We have,

$$S_n = \frac{n}{2}\{2a + (n - 1)d\}$$

or, $48 = \frac{n}{2}\{2 \times 20 + (n - 1)(-4)\}$

or, $48 = \frac{n}{2}(40 - 4n + 4)$

or, $48 = \frac{n}{2}(44 - 4n)$

or, $48 = \frac{n}{2} \times 4(11 - n)$

or, $48 = 2n(11 - n)$

or, $48 = 22n - 2n^2$

or, $2n^2 - 22n + 48 = 0$

or, $2(n^2 - 11n + 24) = 0$

or, $n^2 - 11n + 24 = 0$

or, $n^2 - 8n - 3n + 24 = 0$

or, $n(n - 8) - 3(n - 8) = 0$

or, $(n - 8)(n - 3) = 0$

$\therefore$ $n = 8$ or 3

The first 8 terms of the given series are 20, 16, 12, 8, 4, 0, −4, −8 the sum of whose last 5 term is zero. Therefore the sum of the first 3 terms is equal to the sum of the first 8 terms. Hence, the double answer.

Example 4. If a^2, b^2, c^2 are in A.P. prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A.P.

Solution

Here, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ will be in A.P.

$$\text{if } \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

i.e. $\text{if } \frac{b+c-c-a}{(c+a)(b+c)} = \frac{c+a-a-b}{(a+b)(c+a)}$

i.e. if $(b-a)(a+b) = (c-b)(b+c)$

i.e. if $b^2 - a^2 = c^2 - b^2$

i.e. if a^2, b^2, c^2 are in A.P. which is given. Proved

Example 5. Find the 4^{th} term of the G.P. whose 5^{th} term is 32 and 8^{th} term is 256.

Solution

We have,

$t_n = ar^{n-1}$

or, $t_5 = ar^{5-1}$

$\therefore$ $32 = ar^4$ ...(i)

Similarly,

$t_8 = ar^{8-1}$

$\therefore$ $256 = ar^7$ ... (ii)

Dividing (ii) by (i)

$$\frac{256}{32} = \frac{ar^7}{ar^4}$$

or, $8 = r^3$

or, $r^3 = (2)^3$

$r = 2$

Putting $r = 2$ in equation (i)

$32 = a(2)^4$

or, $a = \frac{32}{16}$

$\therefore$ $a = 2$

Now,

$t_4 = ar^{4-1}$

$= 2(2)^{4-1} = 16.$

Example 6. Find the sum of the geometric series: $3 + 6 + 12 + 24 + \ldots + 384$.

Solution

Here, $a = 3$, $r = \frac{t_2}{t_1} = \frac{6}{3} = 2$, $l = 384$, $S_n = ?$

We know,

$$S_n = \frac{lr - a}{r - 1} = \frac{384 \times 2 - 3}{2 - 1} = 768 - 3 = 765$$

Hence, sum of the given series = 765.

Example 7. Insert 5 geometric mean between $3\frac{5}{9}$ and $40\frac{1}{2}$.

Solution

Let 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$ be G_1, G_2, G_3, G_4 and G_5

Then $\frac{32}{9}$, G_1, G_2, G_3, G_4, G_5, $\frac{81}{2}$ be in G.P.

So, $a = \frac{32}{9}$, $l = \frac{81}{2}$, $n = 5 + 2 = 7$, $r = ?$

We have

$$l = ar^{n-1}$$

or, $$\frac{81}{2} = \frac{32}{9} r^{7-1}$$

or, $$\frac{729}{64} = r^6$$

or, $$\left(\frac{3}{2}\right)^6 = r^6$$

$$\therefore \quad r = \frac{3}{2}$$

So, $$G_1 = ar = \frac{32}{9} \times \frac{3}{2} = \frac{16}{3} = 5\frac{1}{3}$$

$$G_2 = ar^2 = \frac{32}{9} \times \left(\frac{3}{2}\right)^2 = \frac{32}{9} \times \frac{9}{4} = 8$$

$$G_3 = ar^3 = \frac{32}{9} \times \left(\frac{3}{2}\right)^3 = \frac{32}{9} \times \frac{27}{8} = 12$$

$$G_4 = ar^4 = \frac{32}{9} \times \left(\frac{3}{2}\right)^4 = \frac{32}{9} \times \frac{81}{16} = 18$$

$$G_5 = ar^5 = \frac{32}{9} \times \left(\frac{3}{2}\right)^5 = \frac{32}{9} \times \frac{243}{32} = 27$$

So, the required 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$ are $5\frac{1}{3}$, 8, 12, 18 and 27.

Example 8. The common ratio, the last term and the sum of G.P. are 3,486 and 728 respectively. Find the first term and the number of terms.

Solution

Here, $r = 3$, $t_n = 486$, $S_n = 728$, $a = ?$ and $n = ?$

We have, $t_n = ar^{n-1}$

$$486 = a\,(3)^{n-1} \qquad \ldots\text{(i)}$$

Similarly,

$$S_n = \frac{a(r^n - 1)}{r-1}$$

or, $728 = \dfrac{a(3^n - 1)}{3-1}$

or, $728 = \dfrac{a(3^n - 1)}{2}$

or, $728 \times 2 = a3^n - a$

or, $1456 = a3^n - a$

or, $1456 = a\,3^{n-1}(3) - a$

or, $1456 = 486(3) - a$ [$\because$ From (i), $a(3)^{n-1} = 486$]

$\therefore$ $a = 1458 - 1456$

$= 2$

Putting $a = 2$ in equation (i), we get

$486 = 2(3)^{n-1}$

or, $\dfrac{486}{2} = (3)^{n-1}$

or, $243 = (3)^{n-1}$

or, $(3)^5 = (3)^{n-1}$

or, $5 = n - 1$

$\therefore$ $n = 6$

Hence, the first term = 2 and the numbers of terms = 6.

Example 9. The sum of 3 numbers in G.P. is 35 and their product is 1000. Find the numbers.

Solution

Let, the three numbers in G.P. be $\frac{a}{r}$, a, ar. Then by given condition,

$\frac{a}{r} + a + ar = 35$ ... (i)

$\frac{a}{r} \times a \times ar = 1000$ ... (ii)

From (ii) $a^3 = 1000 = (10)^3$

$\therefore$ $a = 10$

Putting, $a = 10$ in equation (i)

or, $\dfrac{10}{r} + 10 + 10r = 35$

or, $10 + 10r + 10r^2 = 35r$

or, $5(2 + 2r + 2r^2) = 5(7r)$

or, $2 + 2r + 2r^2 - 7r = 0$

or, $2r^2 - 4r - r + 2 = 0$

or, $2r(r-2) - 1(r-2) = 0$

or, $(r-2)(2r-1) = 0$

$\therefore$ $r = 2$ or $\frac{1}{2}$

If $r = 2$ then three numbers in G.P. are $\frac{a}{r}$, a, ar, ie. $\frac{10}{2}$, 10, 10×2

i.e. 5, 10, 20

If $r = \frac{1}{2}$ then three numbers in G.P. are $\frac{a}{r}$ a, ar, ie. $\frac{10}{\frac{1}{2}}$, 10, $10 \times \frac{1}{2}$

i.e. 20, 10, 5.

Example 10. Find the sum to n terms of $4 + 44 + 444 + \ldots$

Solution

Let. $S_n = 4 + 44 + 444 + \ldots$ to n terms

$= 4(1 + 11 + 111 + \ldots$ to terms)

$= \frac{4}{9}(9 + 99 + 999 + \ldots$ to n terms)

$= \frac{4}{9}[(10 - 1) + (100 - 1) + (1000 - 1) + \ldots$ to n terms]

$= \frac{4}{9}[(10 + 100 + 1000 + \ldots$ to n terms) $-$ ($1 + 1 + 1 \ldots$ to n terms)]

$$= \frac{4}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$$

$$= \frac{4}{9}\left[\frac{10}{9}(10^n - 1) - n\right].$$

Example 11. Find the sum to n terms of the series, $0{\cdot}3 + 0{\cdot}33 + 0{\cdot}333 + \ldots$

Solution

Let, $S_n = 0{\cdot}3 + 0{\cdot}33 + 0{\cdot}333 + \ldots$ to n terms

$= 3(0{\cdot}1 + 0{\cdot}11 + 0{\cdot}111 + \ldots$ to n terms)

$= \frac{3}{9}(0{\cdot}9 + 0{\cdot}99 + 0{\cdot}999 + \ldots$ to n terms)

$= \frac{3}{9}[1 - 0{\cdot}1) + (1 - 0{\cdot}01) + (1 - 0{\cdot}001) \ldots$ to n terms]

$= \frac{1}{3}[(1 + 1 + 1 + \ldots$ to n terms) $-$ ($0{\cdot}1 + 0{\cdot}01 + 0{\cdot}001 + \ldots$ to n terms)]

$$= \frac{1}{3}\left[n - 0{\cdot}1\left\{\frac{1 - (0{\cdot}1)^n}{1 - 0{\cdot}1}\right\}\right] = \frac{1}{3}\left[n - \frac{1}{9}\{1 - (0{\cdot}1)^n\}\right].$$

Example 12. Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, then the resulting numbers are in a G.P. Find the numbers.

Solution

Let the three numbers in A.P. be $a - d$, a, $a + d$.

According to the question

$(a - d) + a + (a + d) = 15$

or $3a = 15$

$\therefore$ $a = 5$

The three numbers become $5 - d, 5, 5 + d$.

If 1, 3, 9 be added to these numbers, we get $6 - d, 8, 14 + d$ which are in G.P.

Then,

$$\frac{8}{6-d} = \frac{14+d}{8}$$

or, $64 = 84 - 8d - d^2$

or, $d^2 + 8d - 20 = 0$

or, $(d + 10)(d - 2) = 0$

$\therefore$ $d = -10, d = 2$

If $d = -10$, then the three numbers are $5 - (-10), 5, 5 + (-10)$

i.e. 15, 5, – 5

If $d = 2$, then the three numbers are $5 - 2, 5, 5 + 2$

i.e. 3, 5, 7.

Example 13. If b^2, a^2, c^2 are in A.P. prove that $a + b$, $b + c$ and $c + a$ are in H.P.

Solution

Here, $a + b, b + c, c + a$ will be in H.P.,

if $\frac{1}{a+b}, \frac{1}{b+c}, \frac{1}{c+a}$ are in A.P.

i.e. if $\frac{1}{b+c} - \frac{1}{a+b} = \frac{1}{c+a} - \frac{1}{b+c}$

i.e. if $\frac{a+b-b-c}{(b+c)(a+b)} = \frac{b+c-c-a}{(c+a)(b+c)}$

i.e. if $(a - c)(a + c) = (b - a)(b - a)$

i.e. if $a^2 - c^2 = b^2 - a^2$

i.e. b^2, a^2, c^2 are in A.P.

Example 14. If p, q, r be in H.P. prove that $\frac{q+p}{q-p} + \frac{q+r}{q-r} = 2$.

Solution

If, p, q, r be in H.P., then

$$q = \frac{2pr}{p+r}$$

$$\text{Now, } \frac{q+p}{q-p} + \frac{q-r}{q-r} = \frac{\frac{2pr}{p+r}+p}{\frac{2pr}{p+r}-p} + \frac{\frac{2pr}{p+r}+r}{\frac{2pr}{p+r}-r}$$

$$= \frac{2pr + p^2 + pr}{2pr - p^2 - pr} + \frac{2pr + pr + r^2}{2pr - pr - r^2}$$

$$= \frac{3r + p - 3p - r}{r - p}$$

$$= \frac{2(r-p)}{(r-p)}$$

$$= 2.$$

EXERCISE – 3 A

1. (a) Find the 10^{th} term of 2, 6, 10, 14...
   (b) Find the sum of the series: $2 + 4 + 6 + \ldots$ to 40 terms.
   (c) If $x + 2$, $3x$ and $4x + 1$ are in A.P., find x.
2. (a) If the 5^{th} and the 12^{th} terms of an A.P. are 14 and 35 respectively, find the first term and the common difference.
   (b) The 5^{th} and 11^{th} terms of an A.P. are 41 and 20 respectively, find the first term and the sum of the first 11 terms.
3. Insert three arithmetic means between 5 and 405.
4. (a) The sum of three numbers in A.P. is 15 and the sum of their squares is 83, find them.
   (b) How many terms of the series $24 + 20 + 16 + \ldots$ must be taken so that the sum may be 72? Explain the double answer.
5. (a) Find the 12^{th} term of the sequence 1, 2, 4, 8, ...
   (b) Find the sum of the following series.
      (i) $1 + 3 + 9 + 27 + \ldots$ to 10 terms.
      (ii) $6 + 12 + 24 + \ldots + 1536$.
6. (a) If the 5^{th} and the 10^{th} term of a G.P. are 32 and 1024 respectively, find the first term and the common ratio.
   (b) The sum of a Geometric progression whose common ratio is 2 and the last term is 768 is 1533. Find the first term.
7. Insert 3 Geometric Means between 5 and 80.
8. Find the three numbers in G.P. whose sum is14, product is 64.
9. (a) If G be the geometric mean between two distinct positive numbers a and b, show that:
   $$\frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2} = \frac{1}{G^2}.$$
   (b) If H be the harmonic mean between a and b, prove that:
   $$\frac{1}{H - a} + \frac{1}{H - b} = \frac{1}{a} + \frac{1}{b}.$$
10. (a) If $a^x = b^y = c^z$ and a, b, c are in G.P., then prove that: x, y, z are in H.P.
    (b) If a, b, c are in AP; b, c, d are in G.P. and c, d, e are in H.P. then prove that a, c, e are in G.P.
11. (a) The sum of three numbers in A.P. is 36. When the numbers are increased by 1, 4, 43 respectively, the resulting numbers are in G.P. Find the numbers.

(b) If the three consecutive term of a geometric series be increased by their middle term, then prove that the resulting terms will be in harmonic series.

(c) Prove that b^2 is greater than or equal to or less than ac according as three unequal positive numbers a, b, c are in A.P., G.P. or H.P.

(d) The A.M. between two numbers exceeds their G.M. by 1 and G.M. exceeds H.M. by 0.8. Find the numbers.

12. If a, b, c are in A.P. & x, y, z in G.P. prove that $x^{b-c}\cdot y^{c-a}\cdot z^{a-b} = 1$

13. If a, b, c are three positive numbers in H.P., show that $a^2 + c^2 > 2b^2$.

14. (a) If a, b, c be in H. P. prove that: $a(b + c)$, $b(c + a)$, $c(a + b)$ are in A.P.

 (b) If a^2, b^2, c^2 are in A.P., prove that $b + c$, $c + a$, $a + b$ are in H.P.

15. Sum to n terms the following series.

 (a) $2 + 22 + 222 + \ldots$ n terms

 (b) $5 + 55 + 555 + \ldots$ n terms

 (c) $0.6 + 0.66 + 0.666 + \ldots$ n terms

 (d) $0.7 + 0.77 + 0.777 + \ldots$ n terms.

Answers

1. (a) 38 (b) 1640 (c) 3
2. (a) 2, 3 (b) 55 and $412\frac{1}{2}$
3. 105, 205, 305
4. (a) 3, 5, 7 or 7, 5, 3 (b) 4 or 9
5. (a) 2048 (b) (i) 29524 (ii) 3066
6. (a) 2, 2 (b) 3
7. 10, 20, 40
8. 2, 4, 8 or 8, 4, 2
11. (a) 3, 12, 21 or 63, 12, – 39 (d) 2, 8 or 8, 2
15. (a) $\frac{2}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$ (b) $\frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

 (c) $\frac{2}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$ (d) $\frac{7}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$

Objective Questions

1. If a, b, c are in A.P. then $\frac{(a-c)^2}{b^2 - ac} =$

 (a) 1 (b) 2

 (c) 3 (d) 4

2. If n^{th} term of A.P. is $4n + 1$, then common difference is

 (a) 2 (b) 3

 (c) 4 (d) 5

3. If three non-zero numbers a, b, c are in A.P. then $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) A.G.P.
4. If a, b, c are in G.P. then $\log a$, $\log b$, $\log c$ are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) A.G.P.
5. If sum to n terms of an A.P. is n^2 then common difference is
   (a) 1 (b) 2
   (c) 3 (d) 4
6. If $S_n = n^3 - 100$ then $t_{10} =$
   (a) 900 (b) 1000
   (c) 561 (d) 271
7. If a, b, c are in G.P. then a^2, b^2, c^2 are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) A.G.P.
8. If a, b, c are in A.P.; b, c, a are in H.P. then c, a, b are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) A.G.P.
9. If a, b, c are in H.P. then $b =$
   (a) $\frac{a+c}{2}$ (b) $\sqrt{ac}$
   (c) $\frac{2ac}{a+c}$ (d) $\frac{a+c}{2ac}$
10. If $\frac{x-y}{y-z} = \frac{x}{z}$ then x, y, z are in
    (a) A.P. (b) G.P.
    (c) H.P. (d) A.G.P.
11. If 4th term of H.P. is 5 and 5th term is 4 then first term is
    (a) 1 (b) 4
    (c) 5 (d) 20
12. If x, y, z be in H.P., then $\frac{y+z}{y-z} + \frac{y+x}{y-x} =$
    (a) 1 (b) 2
    (c) 3 (d) 4
13. If three positive numbers a, b, c are in A.P., G.P. as well as in H.P. then
    (a) $a = b = c$ (b) $a \neq b = c$
    (c) $a = b \neq c$ (d) $a \neq b \neq c$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
d	c	a	a	b	d	b	b	c	c
11	12	13							
d	b	a							

3.12 Infinite Geometric Series

In general, an infinite series is said to have a sum, if the sum of the first n terms s_n gets closer and closer to some real number as the number of terms gets larger and larger. The arithmetic series has no such sum. The geometric series may have a sum.

A geometric series will have a sum and only if the numerical value of the common ratio r is less than unity.

i.e. $|r| < 1$ thus

Let $a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = S_n$.

We have,

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

or, $$s_n = \frac{a - ar^n}{1 - r}$$

$\because$ $|r| < 1, r^n \to 0$ as $n \to \infty$

$$\therefore \quad s_\infty = \frac{a}{1 - r}.$$

WORKED OUT EXAMPLES

Example 1. Find the sum of the infinite series.

(a) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \ldots \infty$ (b) $\frac{-5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots \infty.$

Solution

(a) The given series is an infinite geometric series in which $a = 1$, $r = \frac{-1}{3}$ and $|r| = \frac{1}{3} < 1$

Hence, the sum of the given infinites geometric series is

$$s_\infty = \frac{a}{1 - r} = \frac{1}{1 + \frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$$

(b) The given series is an infinite geometric series in which $a = -\frac{5}{4}$,

$$r = \frac{5}{16} \times \frac{4}{-5} = -\frac{1}{4} \text{ and } |r| = \frac{1}{4} < 1.$$

Hence, the sum of the given infinites geometric series is

$$s_\infty = \frac{a}{1 - r} = \frac{-\frac{5}{4}}{1 + \frac{1}{4}} = -\frac{5}{4} \times \frac{4}{5} = -1.$$

Example 2. Prove that: $6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \ldots \infty$.

Solution

$$6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \ldots \infty$$

$$= 6^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty\right)}$$

$$= 6^1 = 6 \left[\because \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty \text{ is an infinite G.S., so, } s_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1\right].$$

Example 3. If $y = x + x^2 + x^3 + \ldots \infty$ where $|x| < 1$, prove that, $x = \frac{y}{1+y}$.

Solution

By summing the given infinite G.S., we get,

$$y = \frac{x}{1-x}$$

or, $x = (1-x)y$

or, $x = y - xy$

or, $x + xy = y$

or, $x(1+y) = y$

$\therefore \quad x = \frac{y}{1+y}$.

Example 4. The sum of an infinite G.P. is $\frac{80}{9}$ and its common ratio is $-\frac{4}{5}$. Find its first term.

Solution

Let a be the first term of the given infinite G.P.

Here, $r = -\frac{4}{5}$ and $|r| = \left|-\frac{4}{5}\right| = \frac{4}{5} < 1$

and $s_\infty = \frac{80}{9}$

Now, $s_\infty = \frac{a}{1-r}$

or, $\frac{a}{1+\frac{4}{5}} = \frac{80}{9}$

$\therefore \quad a = 16.$

Example 5. The sides of a square are each 16 cm. A second square is drawn by joining the mid points of the sides, successively. In the second square the process is repeated to drawing the third square. If this process is continued indefinitely, find the sum of the areas of all the squares.

Solution

Let ABCD be the square whose each side is 16 cm.

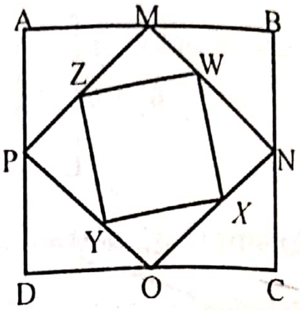
$\therefore$ Area of ABCD $= (16)^2 = 256 \text{ cm}^2$

Again, let MNOP be the square obtained by joining the mid points M, N, O, P of sides AB, BC, CD and DA of a square ABCD, respectively.

Then, $MN = \sqrt{(MB)^2 + (BN)^2} = \sqrt{8^2 + 8^2} = 8\sqrt{2}$ cm.

The area of the square $MNOP = (8\sqrt{2})^2 = 128 \text{ cm}^2$

Also, WXYZ be the square obtained by joining the midpoints W, X, Y, Z of the sides MN, NO, OP and PQ of a square MNOP.



Then, $WX = \sqrt{(WN)^2 + (NX)^2}$

$= \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = 8$

Area of square $WXYZ = 8^2 = 64 \text{ cm}^2$ and so on.

The series of the areas of all the squares is $256 + 128 + 64 + \ldots$

Here, $a = 256$, $r = \frac{128}{256} = \frac{1}{2}$, $S_\infty = ?$

We have, $S_\infty = \frac{a}{1-r} = \frac{256}{1-\frac{1}{2}} = \frac{256}{\frac{1}{2}}$

$= 256 \times 2 = 512 \text{ cm}^2$.

EXERCISE – 3 B

1. Decide which of the following infinite series have sums.

(a) $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ (b) $1 + (-1) + 1 + (-1) + \ldots$

(c) $1 + \sqrt{3} + 3 + \ldots$ (d) $1 + \frac{1}{\sqrt{3}} + \frac{1}{3} + \ldots$

2. Find the sum of the following geometric series.

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ to ∞.

(b) $8 + 4\sqrt{2} + 4 + \ldots$ to ∞.

(c) $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots$ to ∞.

3. (a) Prove that: $2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \ldots = 2$

(b) If $x = 1 + a + a^2 + \ldots$ to ∞

$y = 1 + b + b^2 + \ldots$ to ∞, prove that

$1 + ab + a^2b^2 + \ldots$ to $\infty = \frac{xy}{x + y - 1}$

where $|a| < 1$ and $|b| < 1$.

4. The sum of first two terms of an infinite geometric series is 15 and each terms of the series is equal to the sum of all the terms following it. Find the series.
5. The sum of an infinite geometric series is 8. If its second term is 2, find its common ratio.
6. The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the first term and common ratio.
7. The sides of a square are each 16 cm. A second square is drawn by joining the mid points of the sides, successively. In the second square the process is repeated to drawing the third square. If this process is continued indefinitely, find the sum of the perimeters of all the squares.

Answers

1. (a) sum exists (b) sum doesn't exist
   (c) sum doesn't exist (d) sum exists
2. (a) 2 (b) $8\left(2+\sqrt{2}\right)$ (c) $\frac{19}{24}$
4. $10+5+\frac{5}{2}+\frac{5}{4}+\ldots\infty$ 5. $\frac{1}{2}$
6. $a=5, r=\frac{2}{3}$ 7. $64(2+\sqrt{2})$ cm

Objective Questions

1. $0.\overline{45}$
   (a) $\frac{5}{9}$ (b) $\frac{5}{8}$
   (c) $\frac{5}{7}$ (d) $\frac{5}{6}$
2. Sum to infinity of a G.S. with first term a and common ratio r, where $|r|<1$, is
   (a) $\frac{ar}{1-r}$ (b) $\frac{ra}{r+1}$
   (c) $\frac{a}{1-r}$ (d) $\frac{a}{r-1}$
3. $3^{\frac{1}{3}} \cdot 3^{\frac{1}{9}} \cdot 3^{\frac{1}{27}} \ldots$ to $\infty =$
   (a) $\sqrt{3}$ (b) 3
   (c) 3^2 (d) 3^3
4. Sum to infinity $1+(-1)+1+(-1)+\ldots$
   (a) 1 (b) -1
   (c) 0 (d) doesn't exist

5. The sum to infinity of a G.S. is 15 and common ratio is $\frac{4}{5}$ then first term is

(a) 1 (b) 2

(c) 3 . (d) 4

Answer Sheet

1	2	3	4	5	6	7	8	9	10
a	c	a	d	c					

3.13 Sum of the Natural Numbers

1. Sum of the first n natural numbers

The sum of the first n natural numbers is

$$S_1 = 1 + 2 + 3 + \dots r + \dots + n = \sum_{1}^{n} r = \frac{n(n+1)}{2}.$$

Proof

We have the identity

$$(a+1)^2 - a^2 \equiv 2a + 1 \qquad \dots \text{(i)}$$

By putting $a = 1, 2, 3, \dots, n$ in (i), we get

$$2^2 - 1^2 = 2\cdot 1 + 1$$
$$3^2 - 2^2 = 2\cdot 2 + 1$$
$$4^2 - 3^2 = 2\cdot 3 + 1$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$n^2 - (n-1)^2 = 2(n-1) + 1$$
$$(n+1)^2 - n^2 = 2\cdot n + 1$$

Adding column-wise, we get,

$$(n+1)^2 - 1^2 = 2(1 + 2 + 3 + \dots + n) + (1 + 1 + 1 + \dots \text{to } n \text{ terms})$$

$\therefore \quad n^2 + 2n = 2S_1 + n$

$\Rightarrow \quad 2S_1 = n^2 + n = n(n+1)$

$\therefore \quad S_1 = \frac{n(n+1)}{2}.$

Alternative Method

Consider the series: $1 + 2 + 3 + \dots + n$

This is an A.S. in which $a = 1$, $d = 1$ and $\ell = n$

$\therefore \quad S_n = \frac{n}{2}(\ell + n)$

or, $\quad S_n = \frac{n}{2}(n+1)$

Hence, $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1).$

2. Sum of the squares of the first n natural numbers

The sum of the squares of the first n natural numbers is

$$S_2 = 1^2 + 2^2 + 3^2 + \ldots + r^2 + \ldots + n^2 = \sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof

We know the identity

$$(a+1)^3 - a^3 \equiv 3a^2 + 3a + 1 \quad \ldots \text{(i)}$$

By putting $a = 1, 2, 3, \ldots n$ in (i), we get,

$$\begin{aligned}
2^3 - 1^3 &= 3\cdot1^2 + 3\cdot1 + 1\\
3^3 - 2^3 &= 3\cdot2^2 + 3\cdot2 + 1\\
4^3 - 3^3 &= 3\cdot3^2 + 3\cdot3 + 1\\
&\vdots\\
n^3 - (n-1)^3 &= 3(n-1)^2 + 3(n-1) + 1\\
(n+1)^3 - n^3 &= 3n^2 + 3n + 1
\end{aligned}$$

Adding these n equalities column-wise, we get,

$$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + (1 + 1 + 1 + \ldots \text{ to } n \text{ terms})$$

or, $(n+1)^3 - 1 = 3S_2 + 3S_1 + n$

or, $n^3 + 3n^2 + 3n = 3S_2 + 3\cdot\dfrac{n(n+1)}{2} + n$

or, $3S_2 = n^3 + 3n^2 + 3n - \dfrac{3}{2}n(n+1) - n$

or, $3S_2 = \dfrac{n}{2}[2n^2 + 6n + 6 - 3(n+1) - 2]$

or, $3S_2 = \dfrac{n}{2}[2n^2 + 3n + 1] = \dfrac{n}{2}(n+1)(2n+1)$

$\therefore \quad S_2 = \dfrac{n(n+1)(2n+1)}{6}.$

3. Sum of the cubes of the first n natural number

The sum of the cubes of the first n natural numbers is

$$S_3 = 1^3 + 2^3 + 3^3 + \ldots + r^3 + \ldots + n^3 = \sum_{1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Proof

We know the identity

$$(a+1)^4 - a^4 = 4a^3 + 6a^2 + 4a + 1$$

Putting $a = 1, 2, 3, \ldots, (n-1), n$ successively in the identity, we get,

$$2^4 - 1^4 = 4\cdot 1^3 + 6\cdot 1^2 + 4\cdot 1 + 1$$
$$3^4 - 2^4 = 4\cdot 2^3 + 6\cdot 2^2 + 4\cdot 2 + 1$$
$$4^4 - 3^4 = 4\cdot 3^3 + 6\cdot 3^2 + 4\cdot 3 + 1$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$
$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$
$$(n+1)^4 - n^4 = 4n^2 + 6n^2 + 4n + 1$$

Adding these n equalities column-wise, we get

$$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + 3^3 + \ldots + n^3) + 6(1^2 + 2^2 + 3^2 + \ldots + n^2) + 4(1 + 2 + 3 + \ldots + n) + (1 + 1 + 1 + \ldots \text{ to } n \text{ terms})$$

or, $(n+1)^4 - 1 = 4S_3 + 6S_2 + 4S_1 + n$

or, $4S_3 = n^4 + 4n^3 + 6n^2 + 4n - 6S_2 - 4S_1 - n$

or, $4S_3 = n^4 + 4n^3 + 6n^2 + 4n - 6\dfrac{n(n+1)(2n+1)}{6} - 4\dfrac{n(n+1)}{2} - n$

or, $4S_3 = n^4 + 4n^3 + 6n^2 + 4n - 2n^3 - 3n^2 - n - 2n^2 - 2n - n$

or, $4S_3 = n^4 + 2n^3 + n^2$

or, $4S_3 = n^2(n^2 + 2n + 1) = n^2(n+1)^2$

$$\therefore \quad S_3 = \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Arithmetico-Geometric Series

The term of A. S. are

$$a, a + d, a + 2d, \ldots$$

The terms of a G.S. are

$$1, r, r^2, \ldots$$

A series of the type

$$a\cdot 1 + (a+b)r + (a+2d)r^2 + \ldots$$

whose each term is the product of the corresponding terms of an A.S. and a G.S. is known as the arithmetic-geometric series.

The following example will illustrate the method of finding the sum of the above type of series.

Example: Sum to n terms of the following series:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots$$

Solution

The given series may be written as follows:

$$1\cdot\frac{1}{2}+2\cdot\frac{2}{4}+3\cdot\frac{3}{8}+4\cdot\frac{4}{16}+\ldots$$

Clearly, the above series is the arithmetico-geometric series. The common ratio of the G.S. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16},\ \ldots$ is $\frac{1}{2}$

Let $\quad S = \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots+\frac{n}{2^n}$

Then, $\quad \frac{1}{2}S = \qquad \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\ldots+\frac{n-1}{2^n}+\frac{n}{2^{n+1}}$

$$\frac{1}{2}S = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{2^n}-\frac{n}{2^{n+1}}$$

$$\frac{1}{2}S = \frac{\frac{1}{2}\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}}-\frac{n}{2^{n+1}} \qquad \left(\because S_n = \frac{a(1-r^n)}{1-r}\right)$$

$$\frac{1}{2}S = 1-\frac{1}{2^n}-\frac{n}{2^{n+1}}$$

$$\therefore \quad S = 2-\frac{1}{2^{n-1}}-\frac{n}{2^n}.$$

WORKED OUT EXAMPLES

Example 1. Find the sum of the series: $1\cdot4+3\cdot7+5\cdot10+\ldots$ to n terms.

Solution

The r^{th} terms of the series is equal to the product of the r^{th} terms of the two series.

$$\begin{aligned}\therefore \quad t_r &= [1+(r-1)\,2]\cdot[4+(r-1)\,3]\\ &= (2r-1)(3r+1) = 6r^2-r-1\end{aligned}$$

Hence,

$$S_n = \sum_{r=1}^{n}(6r^2-r-1) = 6\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

or, $\quad S_n = \frac{6n\,(n+1)(2n+1)}{6}-\frac{n\,(n+1)}{2}-n$

$$= \frac{1}{2}\,n\,(4n^2+5n-1).$$

Example 2. Find the sum of the series: $1^2 + 3^2 + 5^2 + 7^2 + \ldots$ to n terms.

Solution

$$t_r = [1 + (r-1)2]^2 = (2r-1)^2 = 4r^2 - 4r + 1$$

$$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} (4r^2 - 4r + 1)$$

$$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{n(4n^2-1)}{3}.$$

Example 3. Find the sum of the series: $1 + (1+2) + (1+2+3) \ldots$ to n terms.

Solution

$$t_r = 1 + 2 + 3 + \ldots + r = \frac{r(r+1)}{2} = \frac{r^2 + r}{2}$$

$$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \frac{1}{2}\left[\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right]$$

$$= \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(n+2)}{6}.$$

Example 4. Find the n^{th} term and sum to n terms of the series: $4 + 6 + 9 + 13 + 18 + \ldots$

Solution

Let S_n denote the sum of the first n terms and t_n the n^{th} term of a given series then

$$S_n = 4 + 6 + 9 + 13 + 18 + \ldots + t_n \quad \ldots(i)$$

Also $S_n = 4 + 6 + 9 + 13 + \ldots + t_{n-1} + t_n \quad \ldots(ii)$

The same series is written again with each term shifted by one place to the right.

By subtraction, we have,

$$0 = 4 + 2 + 3 + 4 + 5 + \ldots + (t_n - t_{n-1}) - t_n$$

$$\Rightarrow \quad t_n = 4 + (2 + 3 + 4 + 5 + \ldots + \text{to } n-1 \text{ terms})$$

$$= 4 + \frac{n-1}{2}[4 + (n-2)1]$$

$$= 4 + \frac{n-1}{2}(n+2)$$

$$= \frac{1}{2}(n^2 + n + 6)$$

$$\therefore \quad S_n = \Sigma t_n = \frac{1}{2}[\Sigma n^2 + \Sigma n + 6n]$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + 6n\right]$$

$$= \frac{n}{6}(n^2 + 3n + 20).$$

Example 5. Sum to n-terms of the series: $\frac{2}{5}+\frac{6}{5^2}+\frac{10}{5^3}+\frac{14}{5^4}+\ldots, \ldots, \ldots$

Solution

Let. $S_n=\frac{2}{5}+\frac{6}{5^2}+\frac{10}{5^3}+\frac{14}{5^4}+\ldots+\frac{4n-6}{5^{n-1}}+\frac{4n-2}{5^n}$

$$\frac{1}{5}S_n=\frac{2}{5^2}+\frac{6}{5^3}+\frac{10}{5^4}+\ldots+\frac{4n-6}{5^n}+\frac{4n-2}{5^{n+1}}$$

Subtracting, we get,

$$\left(1-\frac{1}{5}\right)S_n=\frac{2}{5}+\underbrace{\frac{4}{5^2}+\frac{4}{5^3}+\frac{4}{5^4}+\ldots+\frac{4}{5^n}}_{(n-1)\text{ terms}}-\frac{4n-2}{5^{n+1}}$$

or, $$\frac{4}{5}S_n=\frac{2}{5}+\frac{\frac{4}{5^2}\left\{1-\left(\frac{1}{5}\right)^{n-1}\right\}}{1-\frac{1}{5}}-\frac{4n-2}{5^{n+1}}$$

or, $$\frac{4}{5}S_n=\frac{2}{5}+\frac{\frac{4}{5^2}\left(1-\frac{1}{5^{n-1}}\right)}{\frac{4}{5}}-\frac{4n-2}{5^{n+1}}$$

or, $$\frac{4}{5}S_n=\frac{2}{5}+\frac{1}{5}\left(1-\frac{1}{5^{n-1}}\right)-\frac{4n-2}{5^{n+1}}$$

or, $$\frac{4}{5}S_n=\frac{2}{5}+\frac{1}{5}-\frac{1}{5^n}-\frac{4n-2}{5^{n+1}}$$

or, $$\frac{4}{5}S_n=\frac{3}{5}-\left(\frac{1}{5^n}+\frac{4n-2}{5^{n+1}}\right)$$

or, $$\frac{4}{5}S_n=\frac{3}{5}-\frac{5+4n-2}{5^{n+1}}$$

or, $$\frac{4}{5}S_n=\frac{3}{5}-\frac{4n+3}{5^{n+1}}$$

or, $$S_n=\frac{5}{4}\left(\frac{3}{5}-\frac{4n+3}{5^{n+1}}\right)$$

$$\therefore\quad S_n=\frac{3}{4}-\frac{5(4n+3)}{4.5^{n+1}}$$

$$=\frac{3}{4}-\frac{4n+3}{4.5^n}.$$

Example 6. Sum to infinity the following series.

$1-5a+9a^2-13a^3+\ldots$ to ∞ $(-1<a<1)$.

Solution

Let $S_\infty=1-5a+9a^2-13a^3+\ldots$

$\underline{aS_\infty= \quad a-5a^2+9a^3-\ldots}$

$(1+a)S_\infty=1-4a+4a^2-4a^3+\ldots$

or, $(1+a)\,S_x = 1 - 4a\,(1 - a + a^2 \ldots)$

or, $(1+a)\,S_x = 1 - 4a \cdot \dfrac{1}{1-(-a)}$ $\qquad \left[\because S_x = \dfrac{a}{1-r}\right]$

or, $(1+a)\,S_x = 1 - \dfrac{4a}{1+a}$

or, $(1+a)S_x = \dfrac{1+a-4a}{1+a}$

or, $(1+a)S_x = \dfrac{1-3a}{1+a}$

$\therefore\ S_x = \dfrac{1-3a}{(1+a)^2}$

EXERCISE – 3 C*

1. Find the n^{th} term and sum to n terms of the series:
   (a) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots$
   (b) $1^2 \cdot 1 + 2^2 \cdot 3 + 3^2 \cdot 5 + \ldots$
   (c) $1 + (1+3) + (1+3+5) + \ldots$
2. Find the sum to n terms of the following series:
   (a) $n \cdot 1 + (n-1) \cdot 2 + (n-2) \cdot 3 + \ldots$
   (b) $3 + 7 + 13 + 21 + \ldots$
   (c) $1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + \ldots$
3. Sum to infinity:
   (a) $1 + 3x + 5x^2 + 7x^3 + \ldots\ (|x| < 1)$.
   (b) $1 + \dfrac{3}{5} + \dfrac{5}{5^2} + \dfrac{7}{5^3} + \ldots$

Answers

1. (a) $n^2 + 2n, \dfrac{n(n+1)(2n+7)}{6}$ (b) $n^2(2n-1), \dfrac{1}{6}n(n+1)(3n^2+n-1)$
   (c) $n^2, \dfrac{n(n+1)(2n+1)}{6}$
2. (a) $\dfrac{n(n+1)(n+2)}{6}$ (b) $\dfrac{n}{3}(n^2+3n+5)$
   (c) $\dfrac{35}{16} - \dfrac{12n+7}{16 \cdot 5^{n-1}}$
3. (a) $\dfrac{1+x}{(1-x)^2}$ (b) $\dfrac{15}{8}$

Objective Questions

1. The n^{th} term of the series $2 + 4 + 6 + \ldots$ is
   (a) n (b) $2n$
   (c) $3n$ (d) $5n$
2. If $\sum n = 55$ then $\sum n^3 =$
   (a) 55 (b) $(55)^2$
   (c) $(55)^3$ (d) $(55)^4$
3. The n^{th} term of $1 + 3 + 6 + 10 + \ldots$ is
   (a) $\frac{n(n+1)}{2}$ (b) $\left[\frac{n(n+1)}{2}\right]^2$
   (c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{n(n+1)(n+2)}{6}$
4. Sum to n terms: $1 + (3 + 5) + (7 + 9 + 11) + \ldots$
   (a) n^2 (b) $\frac{n(n+1)}{2}$
   (c) $\left[\frac{n(n+1)}{2}\right]^2$ (d) $\frac{n(n+1)(n+2)}{6}$
5. Sum to infinity $1 + \frac{3}{2} + \frac{5}{4} + \ldots$ is
   (a) 2 (b) 3
   (c) 5 (d) 6

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	b	a	c	d					

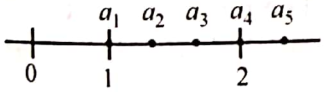
3.14 Limits of Sequence of Numbers

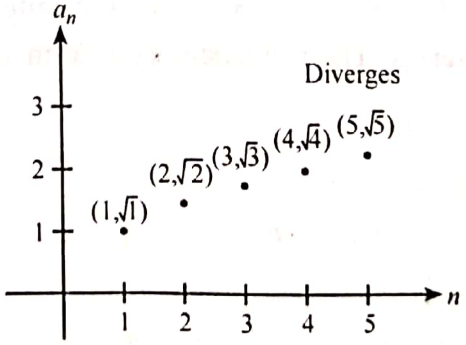
Consider the following sequences

1. $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots \sqrt{n}, \ldots$

   Its n^{th} term $(t_n) = \sqrt{n}$

   Let us graph the above sequence in two different ways. One by plotting the number a_n on a horizontal axis and next by plotting (n, a_n) in the coordinate plane.

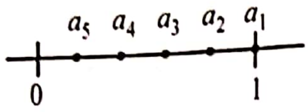


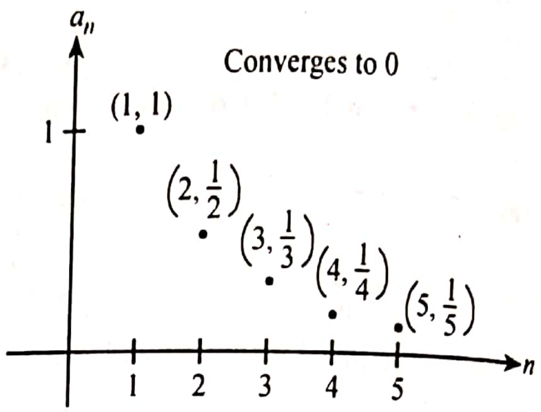


The terms $a_n = \sqrt{n}$ surpass every integer. So, the sequence $\{a_n\}$ diverges.

2. $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

Its n^{th} term $(a_n) = \frac{1}{n}$





Here the terms $a_n = \frac{1}{n}$ decrease steadily and get arbitrarily close to 0 as n increases. So, the sequence $\{a_n\}$ converges to 0.

Convergence and Divergence

The sequence $\{a_n\}$ converges to the number L if to every positive number $\in$ (Epsilon), there corresponds an integer N, such that for all n,

$$n > N \Rightarrow |a_n - L| < \in$$

If no such number L exists, we say $\{a_n\}$ diverges.

If $\{a_n\}$ converges to L, we write

$\lim\limits_{n \to \infty} a_n = L$ or simply $a_n \to L$ and say L the limit of the sequence.

Infinite Series

Given a sequence $\{a_n\}$ of numbers, an expression of the form $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is an infinite series. The number a_n is the n^{th} term of the series. The sequences $\{s_n\}$ defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots \quad \vdots$$

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k.$$

is the sequence of partial sums of the series and s_n is the n^{th} partial sum. If the sequence of partial sums converges to a limit L, we say that the series converges and that its sum is L. In this case, we write

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

Convergence of Geometric Series

If $|r| < 1$, the geometric series $a + ar + ar^2 + \ldots$ converges to $\frac{a}{1-r}$.

That is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $|r| < 1$

If $|r| = 1$ or $|r| > 1$, the series diverges.

For example

The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is convergent as $|r| = \frac{1}{2} < 1$.

It converges to $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$.

Absolute Convergence

A series Σa_n converges absolutely if the corresponding series of absolute values $\Sigma|a_n|$ converges

Note: If $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Power Series

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots$$

in which the centre a and coefficients $c_0, c_1, \ldots c_n, \ldots$ are constants.

Factorial Notation

The notation $n!$ ('n factorial') means the product $1 \cdot 2 \cdot 3 \ldots n$ of the integers from 1 to n.

For example

$$3! = 1 \cdot 2 \cdot 3 = 6$$

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

We note that $0! = 1$ but $0!$ has no meaning by definition.

Review of Derivatives

Let $y = f(x)$ be a given continuous function. If Δy be an increment in y corresponding to an increment Δx in x, then $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$, if it exists, is called derivative of y with respect to x and is denoted by $\dfrac{dy}{dx}$. It may be denoted by $\dfrac{df(x)}{dx}$ or $f'(x)$. The second, third order derivatives are denoted by f'', f''' respectively. The n^{th} order derivative of f is denoted by f^n.

List of Important Formulae

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$
2. $\dfrac{d}{dx}$ (constant) $= 0$
3. $\dfrac{d}{dx}(u \pm v) = \dfrac{d}{dx}(u) \pm \dfrac{d}{dx}(v)$
4. $\dfrac{d}{dx}(u \cdot v) = u\dfrac{d}{dx}(v) + v\dfrac{d}{dx}(u)$
5. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{d}{dx}(u) - u\dfrac{d}{dx}(v)}{v^2}$
6. $\dfrac{d}{dx}(e^x) = e^x$
7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax}$
8. $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$
9. $\dfrac{d(\sin x)}{dx} = \cos x$
10. $\dfrac{d(\cos x)}{dx} = -\sin x$
11. $\dfrac{d(\tan x)}{dx} = \sec^2 x$
12. $\dfrac{d(\sec x)}{dx} = \sec x \tan x$
13. $\dfrac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$
14. $\dfrac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

Example: If $y = x^3 - 3x^2 + 7$ then find $\frac{dy}{dx}$.

Solution

Here, $y = x^3 - 3x^2 + 7$

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 3x^2 + 7)$$

$$= \frac{dx^3}{dx} - 3\frac{dx^2}{dx} + \frac{d(7)}{dx}$$

$$= 3x^2 - 6x + 0$$

$$= 3x^2 - 6x.$$

Example: If $y = \sin x + e^x + 7$ then find $\frac{dy}{dx}$.

Solution

Here, $y = \sin x + e^x + 7$

Differentiating both sides with respect to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + e^x + 7)$$

$$= \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) + \frac{d}{dx}(7)$$

$$= \cos x + e^x.$$

Example: If $f(x) = e^x$ then find $f'(x), f''(x), f'''(x)$.

Solution

Here, $f(x) = e^x$.

Then, $f'(x) = e^x$.

$f''(x) = e^x$.

$f'''(x) = e^x$.

3.15 Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. The Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n + \dots$$

which is the Taylor series generated by f at $x = 0$.

Example: Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Find, where, if any, does the series converge to $\frac{1}{x}$?

Solution

Here, $f(x) = \frac{1}{x} = x^{-1}$, $\qquad f(2) = \frac{1}{2}$

$f'(x) = -x^{-2}$ $\qquad f'(2) = -2^{-2} = -\frac{1}{2^2}$

$f''(x) = 2x^{-3} = 2!\,x^{-3}$, $\qquad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}$

$f'''(x) = -3!\,x^{-4}$, $\qquad \frac{f'''(2)}{3!} = -2^{-4} = -\frac{1}{2^4}$

$\vdots \qquad \vdots \qquad \vdots$

$f^n(x) = (-1)^n n!\,x^{-(n+1)}$, $\qquad \frac{f^n(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$.

$\vdots \qquad \vdots \qquad \vdots$

The Taylor series of $f(x) = \frac{1}{x}$ at $x = 2$

is $f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \dots + \frac{f^n(x-2)^n}{n!} + \dots$

$$= \frac{1}{2} - \frac{x-2}{2^2} + \frac{(x-2)^2}{2^3} - \dots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \dots$$

This is a geometric series with first term $\frac{1}{2}$ and ratio $(r) = \frac{t_2}{t_1} = \frac{-\frac{(x-2)}{2^2}}{\frac{1}{2}}$

$$= \frac{-(x-2)}{2}$$

It converges absolutely for $|r| < 1$ i.e. $|x-2| < 2$

i.e. $\quad -2 < x - 2 < 2$

i.e. $\quad 0 < x < 4$

And its sum is $\frac{a}{1-r} = \dfrac{\frac{1}{2}}{1 + \frac{(x-2)}{2}}$

$$= \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

Hence the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$ converges to $\frac{1}{x}$ for $0 < x < 4$.

Taylor Polynomials

Let f be a function with derivatives of order k for $k = 1, 2, \ldots$ N is some interval containing a as an interior point. Then for any integer n from 0 through N, the Taylor Polynomial of order n generated by f at $x = a$ is the polynomial.

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^k(a)}{k!}(x-a)^k + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

Example: Find the Taylor series and Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Solution

$f(x) = e^x$, $\qquad f(0) = e^0 = 1$

$f'(x) = e^x$, $\qquad f'(0) = e^0 = 1$

$f''(x) = e^x$ $\qquad f''(0) = e^0 = 1$

$\vdots \qquad \vdots$

$f^n(x) = e^x$ $\qquad f^n(0) = 1$

$\vdots \qquad \vdots$

The Taylor series generated by f at $x = 0$ is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^n(0)}{n!} + \ldots$$

$$= \quad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^2}{n!} + \ldots$$

$$= \quad \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This is the exponential series. By definition it is also the Maclaurin series for e^x. The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}.$$

3.16 Exponential Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}. \qquad \dots \text{(i)}$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Replacing x by $-x$, we get,

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots \qquad \dots \text{(ii)}$$

Adding (i) and (ii), we get,

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

$$\therefore \quad \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Again, subtracting (ii) from (i)

$$e^x - e^{-x} = 2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

$$\therefore \quad \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Example: Find the value of $\frac{1}{2}(e + e^{-1})$.

Solution

We have $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Putting $x = 1$ and -1, we have,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

and $$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$$

Adding, we get,

$$e + \frac{1}{e} = 2 + \frac{2}{2!} + \frac{2}{4!} + \dots$$

$$= 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \dots\right)$$

$$\therefore \quad \frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots$$

3.17 Taylor's Theorem

If f and its first n derivatives $f', f'', \ldots f^n$ are continuous on $[a, b]$ or $[b, a]$ and f^n is differentiable on (a, b) or (b, a), then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^n(a)}{n!}(b-a)^n + \frac{f^{n+1}(c)}{(n+1)}(b-a)^{n+1}$$

Corollary to Taylor's Theorem

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x) \qquad \ldots \text{(i)}$$

where, $R_n(x) = \dfrac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x. ... (ii)

Equation (i) is called Taylor's formula. The function $R_n(x)$ is called the remainder of order n. If $R_n(x) \to 0$ as $n \to \infty$ for all x in I, then we say that the Taylor series generated by f at $x = a$ converges to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k.$$

WORKED OUT EXAMPLES

Example 1. Prove that $\dfrac{2}{1!} + \dfrac{4}{3!} + \dfrac{6}{5!} + \ldots = e$.

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots \\
&= \frac{1+1}{1!} + \frac{3+1}{3!} + \frac{5+1}{5!} + \ldots \\
&= \frac{1}{1!} + \frac{1}{1!} + \frac{3}{3!} + \frac{1}{3!} + \frac{5}{5!} + \frac{1}{5!} + \ldots \\
&= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = e = \text{R.H.S.}
\end{aligned}$$

Example 2. Show that: $\dfrac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots} = \dfrac{e-1}{e+1}$.

Solution

We have,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$$

Putting $x = 1$ and -1, we get,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots$$

and $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} + \ldots$

Now,

$$e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \ldots$$

$$= 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\right)$$

$$\therefore \quad \frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots$$

Also,

$$e - e^{-1} = \frac{2}{1!} + \frac{2}{3!} + \frac{2}{5!} + \ldots = 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\right)$$

$$\therefore \quad \frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots$$

Now,

$$\text{L.H.S.} = \frac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots}$$

$$= \frac{\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\right) - 1}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots}$$

$$= \frac{\frac{e + e^{-1}}{2} - 1}{\frac{e - e^{-1}}{2}} = \frac{e + e^{-1} - 2}{e - e^{-1}} = \frac{e + \frac{1}{e} - 2}{e - \frac{1}{e}}$$

$$= \frac{e^2 - 2e + 1}{e^2 - 1} = \frac{(e - 1)^2}{(e + 1)(e - 1)}$$

$$= \frac{e - 1}{e + 1} = \text{R.H.S.}$$

Example 3. Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution

The function $f(x) = e^x$ has derivatives of all orders throughout the interval $I = (-\infty, \infty)$.

Then $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n(x) \qquad \text{[From previous example of } e^x\text{]}$$

and $R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}$ for some c between 0 and x.

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c and $e^c < 1$.

When $x = 0$, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c, and $e^c < e^x$. Thus,

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \qquad \text{when } x \le 0.$$

$$\text{and } |R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \qquad \text{when } x > 0.$$

Since, $\lim\limits_{n \to \infty} \dfrac{x^{n+1}}{(n+1)!} = 0$ for every x,

So, $\lim\limits_{n \to \infty} R_n(x) = 0$ and the series converges to e^x for every x.

Thus, $e^x = \sum\limits_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots + \dfrac{x^k}{k!} + \ldots$

EXERCISE – 3 D

1. (a) Find $\frac{1}{2}(e - e^{-1})$

   (b) Prove that $\dfrac{2}{1!} + \dfrac{4}{3!} + \dfrac{6}{5!} + \ldots = e.$

   (c) Prove that $\dfrac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots}{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \ldots} = \dfrac{e^2 + 1}{e^2 - 1}.$

2. Find the Taylor series for the following function at $x = 0$. (Maclaurin series)

   (a) e^{-x} (b) $\dfrac{1}{1+x}$

   (c) $\sin x$ (d) $\cos x$.

3. Find the Taylor series expansion for $f(x) = e^x$ at

   (a) $x = 1$ (b) $x = 2$.

Answers

1. (a) $\dfrac{1}{1!} + \dfrac{1}{3!} + \dfrac{1}{5!} + \ldots$

2. (a) $\sum\limits_{n=0}^{\infty} \dfrac{(-x)^n}{n!} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \ldots$

   (b) $\sum\limits_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \ldots$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

3. (a) $\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$ (b) $\sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

Objective Questions

1. The Taylor series for $f(x) = e^x$ at $x = 0$ is

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum_{n=0}^{\infty} n!\, x^n$

(c) $\sum_{n=0}^{\infty} \frac{n!}{x^n}$ (d) $\sum_{n=0}^{\infty} n \cdot x^n$

2. The Taylor series for $f(x) = e^{-x}$ at $x = 0$ is

(a) $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ (b) $-1 - \frac{x}{1!} - \frac{x^2}{2!} - \dots$

(c) $1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots$ (d) $-1 + \frac{x}{1!} - \frac{x^2}{2!} + \dots$

3. The remainder of order n in Taylor's formula is

(a) $\frac{f^{n+1}(x)}{n!}(x-a)^{n-1}$ (b) $\frac{f^{n+1}(x)}{(n+1)!}(x-a)^{n+1}$

(c) $\frac{f^n(x)}{(n+1)!}(x-a)^n$ (d) $\frac{f^{n-1}(x)}{(n+1)!}(x-a)^n$

4. The value of $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$ is

(a) e (b) $2e$

(c) $3e$ (d) $4e$

5. The geometric series with common ratio r converges for

(a) $r > 1$ (b) $r < -1$

(c) $|r| = 1$ (d) $|r| < 1$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
a	c	b	a	d					

○○○

4

UNIT

Matrices and Determinants

Matrices

4.1 Introduction

The matrices are the most important tools in modern mathematics developed by Sylvester (1814 – 1897) and Hamilton. Later it was propounded by A. Caley in 1958. Today, it has wide applications in many fields such as Engineering, Computers, Statistics, Economics, etc. It is also used to solve the system of linear equations.

4.2 Definition of a Matrix

A matrix is a rectangular arrangement of numbers in rows and columns and enclosed by a pair of brackets [ ] or ( ).

Each number of a matrix is known as the element or entry of the matrix. The number of rows followed by the number of columns gives the **order** or **size** or **dimension** of a matrix. For example, if a matrix consists of 3 rows and 4 columns, then its order is 3×4 which is read as '3 by 4'.

For example

Marks obtained by two students John and Alan in English, Mathematics and Computer are as follows.

	English	Mathematics	Computer
John	65	76	80
Alan	71	78	78

These marks may be represented by the following rectangular arrangement enclosed by a pair of brackets [ ] or ( ).

$$\begin{bmatrix} 65 & 76 & 80 \\ 71 & 78 & 78 \end{bmatrix}_{2\times 3} \quad \text{or} \quad \begin{pmatrix} 65 & 76 & 80 \\ 71 & 78 & 78 \end{pmatrix}_{2\times 3}$$

Here 2×3 indicates the two rows and three columns. The first row indicates the marks obtained by John in English, Mathematics and Computer respectively. Similarly, second row by Alan in the three subjects. The first column refers to the marks obtained by John and Alan in English, the second column in Mathematics and the third in Computer.

4.3 Notation

A matrix is denoted by a capital letters A, B, C, ... X, Y, Z and its respective elements are denoted by small letters a, b, c, ... by two suffixes, i.e. a_{ij}, b_{ij}, The first suffix indicates the number of rows and second indicates the number of columns of any element.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

Here $m \times n$ indicates the m rows and n columns. Similarly, a_{13} indicates the element in first row and third column.

For example

$$A = \begin{array}{c} \begin{pmatrix} 3 & 5 & 8 \\ 2 & -1 & 3 \end{pmatrix}_{2\times 3} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{1st col} & \text{2nd col} & \text{3rd col} \end{array} \end{array} \begin{array}{l} \longrightarrow \text{First row} \\ \longrightarrow \text{Second row} \\ \\ \\ \end{array}$$

has two rows and three columns and hence A is a 2 × 3 matrix.

Similarly, $B = \begin{bmatrix} 5 & 9 \\ 7 & 0 \\ 3 & 5 \end{bmatrix}_{3\times 2}$ has three rows and two columns and hence B is a 3 × 2 matrix.

4.4 Types of Matrices

1. **Row Matrix**

A matrix having only one row is called a row matrix (or a row vector).

For example

$[11 \quad 12 \quad 13]_{1\times 3}$, $[3 \quad 7 \quad 8 \quad 9]_{1\times 4}$ are row matrices.

2. **Column Matrix**

A matrix having only one column is called a column matrix (or a column vector).

For example

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times 1}$, $\begin{pmatrix} 3 \\ 7 \\ 8 \\ 9 \end{pmatrix}_{4\times 1}$ are column matrices.

3. **Null (or Zero) Matrix**

A matrix of any order (rectangular or square) whose elements are zero is called null (or Zero) matrix and is denoted by O.

For example

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times 2}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times 3}$ are null matrices.

4. **Square matrix**

A matrix having equal number of rows and columns is called square matrix.

For example

$\begin{pmatrix} 10 & 5 \\ 2 & 2 \end{pmatrix}_{2\times 2}$, $\begin{pmatrix} 3 & 7 & 3 \\ 4 & 8 & 2 \\ 9 & 6 & 8 \end{pmatrix}_{3\times 3}$ are square matrices.

5. **Rectangular Matrix**

A matrix in which the number of rows and columns are not equal, is called a rectangular matrix.

For example

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 1 & 2 \\ -1 & 2 \end{bmatrix}$ are rectangular matrices.

6. Diagonal Matrix

If a square matrix in which the main (principal) diagonal elements are not zero and the rest of all the elements are zero, then the matrix is called diagonal matrix.

For example

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}_{2\times 2}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}_{3\times 3}$$ are diagonal matrices.

7. Scalar Matrix

A diagonal matrix whose all the main diagonal elements are equal is called scalar matrix.

For example

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}_{2\times 2}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}_{3\times 3}$$ are scalar matrices.

8. Identity (or unit) Matrix

If a square matrix whose main diagonal elements are all identity and the other elements are all zero, then it is called identity (or unit matrix). It is denoted by I.

For example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$$ are identity matrices.

9. Triangular Matrix

If in a square matrix all the elements above the leading (main) diagonal are zero, then it is called a **lower triangular matrix**. A square matrix all of whose elements below the leading (main) diagonal are zero is called an **upper triangular matrix.** That is, a square matrix $A = [a_{ij}]$ is said to be upper triangular matrix if $a_{ij} = 0$ for $i > j$ and lower triangular matrix if $a_{ij} = 0$ for $i < j$.

For example

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 6 & 1 \end{bmatrix}$$ is a lower triangular matrix

and $$\begin{bmatrix} 5 & 7 & 6 \\ 0 & 9 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$ is an upper triangular matrix.

10. Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a symmetric matrix, if $a_{ij} = a_{ji}$ for all i and j.

For example

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 3 & 4 & 2 \\ -5 & 2 & 1 \end{bmatrix} \text{ is a symmetric matrix.}$$

Here, $a_{11} = 2, a_{22} = 4, a_{33} = 1, a_{12} = a_{21} = 3, \ a_{13} = a_{31} = -5, \ a_{23} = a_{32} = 2.$

11. Skew Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a skew symmetric matrix if $a_{ij} = -a_{ji}$ for all i and j.

For example

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \text{ is a skew symmetric matrix.}$$

Here, $a_{11} = a_{22} = a_{33} = 0, a_{12} = 1, a_{21} = -1, a_{13} = 2, a_{31} = -2, a_{23} = -3, a_{32} = 3.$

Note: In a skew symmetric matrix, all the diagonal elements are zero.

4.5 Equality of Two Matrices

Two matrices A and B are said to be equal if and only if A and B have the same order and each element of A is equal to the corresponding elements of B.

For example

The matrices $A = \begin{pmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \end{pmatrix}_{2 \times 3}$ and $B = \begin{pmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \end{pmatrix}$ are equal matrices.

Sub matrix

A matrix obtained by omitting at least one row or column or both from a given matrix is called a sub–matrix.

For example

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 2 \end{pmatrix}$ then $B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ are sub matrices of matrix A.

4.6 Operations on Matrices (Algebra of Matrices)

1. Addition and Subtraction of Two Matrices

If two matrices A and B are of the same order i.e., they have the same number of rows and columns, then A and B can be added or subtracted. It can be written as A + B and A − B.

Let $A = \begin{pmatrix} 2 & 5 & -1 \\ 4 & 1 & 5 \end{pmatrix}_{2\times 3}$ and $B = \begin{pmatrix} 3 & 2 & 4 \\ -1 & 5 & 2 \end{pmatrix}_{2\times 3}$.

Then
$$A + B = \begin{pmatrix} 2 & 5 & -1 \\ 4 & 1 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 4 \\ -1 & 5 & 2 \end{pmatrix}_{2\times 3}$$
$$= \begin{pmatrix} 2+3 & 5+2 & -1+4 \\ 4-1 & 1+5 & 5+2 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 7 & 3 \\ 3 & 6 & 7 \end{pmatrix}.$$

Similarly,
$$A - B = \begin{pmatrix} 2 & 5 & -1 \\ 4 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 4 \\ -1 & 5 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2-3 & 5-2 & -1-4 \\ 4-(-1) & 1-5 & 3-2 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 3 & -5 \\ 5 & -4 & 1 \end{pmatrix}.$$

Properties of Matrix Addition

(i) Matrix addition is commutative : A + B = B + A.

(ii) Matrix addition is associative : (A+ B) + C = A + (B + C).

(iii) Matrix addition is distributive : $k(A + B) = kA + kB$, where k is a scalar.

(iv) Existence of additive identity : A + 0 = 0 + A = A.

(v) Existence of additive inverse : A + (−A) = (−A) + A = 0.

2. Scalar Multiplication of Matrix

Let k be a scalar and A be a matrix. The product of the scalar k and the matrix A, written as kA, is the matrix obtained by multiplying each elements of A by k.

If $k = 3$ and $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix}_{2\times 3}$

then $kA = 3\begin{pmatrix} 1 & 2 & 5 \\ 2 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 15 \\ 6 & -9 & 12 \end{pmatrix}_{2\times 3}$.

3. Matrix Multiplication

Two matrices can be multiplied if and only if the number of columns of the first matrix is equal to the number rows of the second matrix. The resultant matrix will have order equal to the number of rows of the first matrix by the number of columns of the second matrix.

Let A = (a_{ij}) be the matrix having i rows and j columns and B = (b_{jk}) be the matrix having j rows and k columns then the product AB is possible and have i rows and k columns.

In order words $(i, k)^{th}$ element of the product AB is obtained by multiplying the elements in the i^{th} row of A with the corresponding elements in the j^{th} column of B and adding the resulting product.

For example

$$\text{Consider } A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 1 & 5 \end{bmatrix}_{2\times 3} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}.$$

$$\text{Then } AB = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 1 & 5 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}_{3\times 2}.$$

Since the no. of columns of matrix A is equal to the no. of rows of matrix B, so the product AB is defined. Also, the order of matrix AB will be 2 × 2.

We can get the elements of AB as follows:

First step: $$\begin{bmatrix} \boxed{2 \;\; 3 \;\; 4} \\ 4 \;\; 1 \;\; 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2\times 2 + 3\times 1 + 4\times 4 & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Second step: $$\begin{bmatrix} \boxed{2 \;\; 3 \;\; 4} \\ 4 \;\; 1 \;\; 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} \cdots & 2\times 3 + 3\times 2 + 4\times 1 \\ \cdots & \cdots \end{bmatrix}$$

Third step: $$\begin{bmatrix} 2 \;\; 3 \;\; 4 \\ \boxed{4 \;\; 1 \;\; 5} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ 4\times 2 + 1\times 1 + 5\times 4 & \cdots \end{bmatrix}$$

Fourth step: $$\begin{bmatrix} 2 \;\; 3 \;\; 4 \\ \boxed{4 \;\; 1 \;\; 5} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} \cdots & \cdots \\ \cdots & 4\times 3 + 1\times 2 + 5\times 1 \end{bmatrix}$$

Thus,

$$AB=\begin{bmatrix} 2\times2+3\times1+4\times4 & 2\times3+3\times2+4\times1 \\ 4\times2+1\times1+5\times4 & 4\times3+1\times2+5\times1 \end{bmatrix}$$

$$=\begin{bmatrix} 23 & 16 \\ 29 & 19 \end{bmatrix}_{2\times2}.$$

Properties of Matrix Multiplication

(i) In general, the product of matrices is not commutative i.e. if two matrices A, B are conformable for the products AB and BA, then it is not necessary that AB ≠ BA.

(ii) If A, B, C are three matrices such that AB = AC, then in general B ≠ C.

(iii) If A, B, C be three matrices of order $m \times n$, $n \times p$, $n \times p$ respectively, then A(B + C) = AB + AC. *[Distributive Law]*

(iv) If A, B, C be three matrices of order $m \times n$, $n \times p$, $p \times q$ respectively then A(BC) = (AB)C. *[Associative Law]*

(v) If AB = O where A, B are two matrices, then, in general A ≠ O, B ≠ O.

(vi) AI = IA = A.

4.7 Transpose of a Matrix

A matrix obtained by interchanging rows and columns of given matrix is called transpose of a matrix. It is denoted by A^T or A'.

$$\text{If } A=\begin{pmatrix} 5 & 3 & -2 \\ 2 & 0 & -5 \end{pmatrix}_{2\times3} \text{ then } A^T=\begin{pmatrix} 5 & 2 \\ 3 & 0 \\ -2 & -5 \end{pmatrix}_{3\times2}.$$

Properties of Transpose

Let A and B be two matrices which have sizes for which indicated sums and products are defined.

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(kA)^T = kA^T$ where k is scalar.

(iv) $(AB)^T = B^TA^T$.

Note: *(i) For any square matrix A, $A + A^T$ is symmetric matrix and $A - A^T$ is skew-symmetric matrix.*

(ii) Every square matrix A can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

WORKED OUT EXAMPLES

Example 1. If $A = \begin{pmatrix} 8 & 3 & 2 \\ 11 & -5 & 31 \end{pmatrix}$, find $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$.

Solution

We have, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$\therefore \quad a_{11} = 8, a_{12} = 3, a_{13} = 2$

$a_{21} = 11, a_{22} = -5, a_{23} = 31.$

Example 2. Find a 2×3 matrix $A = [a_{ij}]$, where $a_{ij} = \frac{1}{2}(i-j)^2$.

Solution

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ be a 2×3 matrix.

Here, $a_{ij} = \frac{1}{2}(i-j)^2$

$\therefore \quad a_{11} = \frac{1}{2}(1-1)^2 = 0 \qquad a_{12} = \frac{1}{2}(1-2)^2 = ½$

$a_{13} = \frac{1}{2}(1-3)^2 = 2 \qquad a_{21} = \frac{1}{2}(2-1)^2 = ½$

$a_{22} = \frac{1}{2}(2-2)^2 = 0 \qquad a_{23} = \frac{1}{2}(2-3)^2 = ½$

$\therefore \quad A = \begin{pmatrix} 0 & \frac{1}{2} & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$

Example 3. If $X = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$ and $Y = \begin{pmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{pmatrix}$, find the value of $2X + 3Y$.

Solution

$$2X = 2\begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{pmatrix}$$

and $$3Y = 3\begin{pmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{pmatrix}$$

Then, $$2X + 3Y = \begin{pmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{pmatrix} + \begin{pmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 0+21 & 4+18 & 6+9 \\ 4+3 & 2+12 & 8+15 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{pmatrix}.$$

Example 4. Find X and Y if $X + Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$ and $X - Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Solution

$$X + Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix} \quad \ldots \text{(i)}$$

$$X - Y = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad \ldots \text{(ii)}$$

Adding (i) and (ii), we get,

$$2X = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

or, $$2X = \begin{pmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{pmatrix}$$

or, $$X = \frac{1}{2}\begin{pmatrix} 10 & 0 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix}$$

Putting $X = \begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix}$ in equation (i), we get,

$$\begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix} + Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix}$$

or, $$Y = \begin{pmatrix} 7 & 0 \\ 2 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix}$$

or, $$Y = \begin{pmatrix} 7-5 & 0-0 \\ 2-1 & 5-4 \end{pmatrix}$$

$$\therefore \quad Y = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Example 5. If $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}$, find AB and BA. Is AB = BA?

Solution

$$AB = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+5(-3) & 2(-1)+5\times2 \\ 1\times1+3(-3) & 1(-1)+3\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 2-15 & -2+10 \\ 1-9 & -1+6 \end{pmatrix} = \begin{pmatrix} -13 & 8 \\ -8 & 5 \end{pmatrix}$$

and $$BA = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times2+(-1)\times1 & 1\times5+(-1)\times3 \\ (-3)\times2+2\times1 & (-3)\times5+2\times3 \end{pmatrix}$$

$$= \begin{pmatrix} 2-1 & 5-3 \\ -6+2 & -15+6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -4 & -9 \end{pmatrix}$$

Hence $AB \neq BA$.

Example 6. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, find A^2.

Solution

$$A^2 = A.A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & 2+2 \\ 3+3 & 6+1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix}.$$

Example 7. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, show that $A^2 - 2A - 3I = 0$.

Solution

$$A^2 = A \times A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 1 + 2 \times 2 & 1 \times 2 + 2 \times 1 \\ 2 \times 1 + 1 \times 2 & 2 \times 2 + 1 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$2A = 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

$$\text{and } 3I = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\text{Now, } A^2 - 2A - 3I = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= 0.$$

Example 8. Using matrices A, B and C where $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$,

$B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, verify the rule $(AB)C = A(BC)$.

Solution

$$AB = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+1 & -1-1+1 & 0+1+1 \\ 0+0+1 & 0+2+1 & 0-2+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix}$$

$$\therefore \quad (AB)C = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0+2 & 0-1+2 \\ 1+0-1 & 0+3-1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}.$$

$$\text{Again, } BC = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 0-1+0 \\ 0+0-1 & 0+1-1 \\ 1+0+1 & 0+1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 2 & 2 \end{pmatrix}$$

$$\therefore \quad A(BC) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1+2 & -1+0+2 \\ 0-2+2 & 0+0+2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}.$$

Hence, (AB) C = A (BC).

Example 9. Three shops A, B and C sell three items X, Y, Z. The following matrix P shows the stock of X, Y and Z in three shops. Matrix Q shows the number of each item received at the beginning of a week. Matrix R shows the sales during the week.

$$P = \begin{array}{c} \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix} \end{array}, \quad Q = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} \end{array}, \quad R = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 3 & 3 & 4 \\ 3 & 4 & 4 \\ 4 & 4 & 5 \end{pmatrix} \end{array}$$

Find: (i) the number of items after receiving the goods.

(ii) the number of items at the end of the week.

(iii) the number of items to be ordered so that the stock of all items in all shops will be 6.

Solution

(i) The number of items after receiving the goods will be given by

$$P + Q = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2+3 & 3+4 & 4+5 \\ 3+2 & 4+3 & 5+4 \\ 4+1 & 5+2 & 6+3 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \\ 5 & 7 & 9 \\ 5 & 7 & 9 \end{pmatrix}.$$

(ii) The number of items at the end of the week is given by

$$(P + Q) - R = \begin{pmatrix} 5 & 7 & 9 \\ 5 & 7 & 9 \\ 5 & 7 & 9 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 4 \\ 3 & 4 & 4 \\ 4 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5-3 & 7-3 & 9-4 \\ 5-3 & 7-4 & 9-4 \\ 5-4 & 7-4 & 9-5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 5 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix}.$$

(iii) The required order matrix

$$= \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 5 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 6-2 & 6-4 & 6-5 \\ 6-2 & 6-3 & 6-5 \\ 6-1 & 6-3 & 6-4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ 4 & 3 & 1 \\ 5 & 3 & 2 \end{pmatrix}.$$

Example 10. If $A = \begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}$, find $(A^T)^T$.

Solution

Here, $A = \begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 6 & 1 \end{pmatrix}$$

$$(A^T)^T = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 6 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ -3 & 6 \\ 0 & 1 \end{pmatrix}.$$

Example 11. If $A = \begin{pmatrix} 4 & x+3 \\ 2x-1 & -1 \end{pmatrix}$ be a symmetric matrix, find the value of x.

Solution

If $A = \begin{pmatrix} 4 & x+3 \\ 2x-1 & -1 \end{pmatrix}$ be a symmetric matrix then $A^T = A$.

i.e. $\begin{pmatrix} 4 & x+3 \\ 2x-1 & -1 \end{pmatrix}^T = \begin{pmatrix} 4 & x+3 \\ 2x-1 & -1 \end{pmatrix}$

or, $\begin{pmatrix} 4 & 2x-1 \\ x+3 & -1 \end{pmatrix} = \begin{pmatrix} 4 & x+3 \\ 2x-1 & -1 \end{pmatrix}$

Comparing corresponding elements, we get,

$2x - 1 = x + 3$

or, $2x - x = 3 + 1$

$\therefore$ $x = 4$.

Example 12. Express the following matrix as a sum of a symmetric and skew symmetric matrix: $\begin{bmatrix} -2 & 6 & 1 \\ 2 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} -2 & 6 & 1 \\ 2 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix}$.

Then $A^T = \begin{bmatrix} -2 & 2 & 3 \\ 6 & 2 & 4 \\ 1 & 4 & 0 \end{bmatrix}$.

We know that if A is a square matrix, then $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, where first is a symmetric matrix and second is a skew-symmetric matrix.

Now, $A + A^T = \begin{bmatrix} -2 & 6 & 1 \\ 2 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 2 & 3 \\ 6 & 2 & 4 \\ 1 & 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} -4 & 8 & 4 \\ 8 & 4 & 8 \\ 4 & 8 & 0 \end{bmatrix}$

$\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} -4 & 8 & 4 \\ 8 & 4 & 8 \\ 4 & 8 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ 4 & 2 & 4 \\ 2 & 4 & 0 \end{bmatrix}$,

which is a symmetric matrix.

And, $A - A^T = \begin{bmatrix} -2 & 6 & 1 \\ 2 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 2 & 3 \\ 6 & 2 & 4 \\ 1 & 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

$\frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

which is a skew-symmetric matrix.

Hence, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$\therefore \begin{bmatrix} -2 & 6 & 1 \\ 2 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 \\ 4 & 2 & 4 \\ 2 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

EXERCISE – 4 A

1. What are the orders or sizes of the following matrices?

   (a) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} a \\ b \end{pmatrix}$

   (c) $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 3 & 1 & 5 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 10 & 2 & 13 \\ 4 & -2 & 1 \\ 0 & -2 & 11 \end{pmatrix}$ and $B = \begin{pmatrix} 11 & 3 & -2 \\ 1 & -4 & 2 \\ 10 & 1 & 4 \end{pmatrix}$.

   Write down the elements: (a) a_{12}, a_{23}, a_{31} (b) b_{21}, b_{13}, b_{32}.

3. Add the following matrices.

   (a) $\begin{pmatrix} 1 & 2 & 3 \\ -4 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 6 & -2 & 1 \\ 4 & 1 & 2 \end{pmatrix}$

   (b) $3\begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ -3 & 2 \\ 8 & 6 \end{pmatrix}$.

4. (a) Find a 2×2 matrix $A = [a_{ij}]$ where $a_{ij} = |2i + 3j - 6|$.

   (b) Construct a 3×3 matrix $A = [a_{ij}]$ whose element a_{ij} are given by $a_{ij} = i + 2j$.

5. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$; find:

   (a) $A + B$ (b) $A - B$

   (c) $2A + B$ (d) $3A - 2B$.

6. (a) Find X, if $Y = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and $2X + Y = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$.

   (b) If $2A + 3B = \begin{pmatrix} 2 & 7 & 12 \\ 13 & 12 & 23 \end{pmatrix}$ and $A - 2B = \begin{pmatrix} 1 & 0 & -1 \\ -4 & -1 & -6 \end{pmatrix}$, find the matrices A and B.

   (c) Find the values of x, y and z if $\begin{pmatrix} x + y & z - x \\ y + 2z & x \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 8 & 1 \end{pmatrix}$.

7. (a) Given $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}$. Calculate AB and BA.

   (b) If $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$, show that $AB \neq BA$.

   (c) If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$, show that AB is a null matrix.

8. (a) If $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, show that $A^2 - 2A - 5I = O$ where I and O are 2×2 identity and null matrices respectively.

(b) If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, show that $A^2 - 4A + 3I = O$ where I and O are 2×2 identity and null matrix respectively.

9. (a) Given $A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}$, find AB and BA.

(b) Find 5 AB if $A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$.

10. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$,

verify that: A (BC) = (AB)C.

11. If $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$,

verify that A (BC) = (AB)C.

12. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$, find $A^2 - A - I$ where I be an identify matrix of order 3×3.

13. For the given matrices $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$, show that $(A + B)^T = A^T + B^T$.

14. If $A = \begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$ and $f(x) = x^2 - 5x + 3$, find $f(A)$.

15. Express $A = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 3 & 4 \\ 5 & 2 & 6 \end{pmatrix}$ as a sum of symmetric and skew symmetric matrix.

Answers

1. (a) 1×3 (b) 2×1 (c) 2×3
(d) 3×3

2. (a) 2, 1, 0 (b) 1, – 2, 1

3. (a) (5, 4, 6) (b) $\begin{pmatrix} 7 & 0 & 4 \\ 0 & 4 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 7 \\ -3 & 8 \\ 20 & 21 \end{pmatrix}$

4. (a) $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix}$

5. (a) $\begin{pmatrix} 1 & 3 & 5 \\ 5 & 5 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 5 & 8 \\ 7 & 8 & 13 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 4 & 5 \\ 0 & 5 & 2 \end{pmatrix}$

6. (a) $\begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$

(c) $x = 1, y = 2, z = 3$

7. (a) $\begin{pmatrix} 14 & 17 \\ 23 & 28 \end{pmatrix}, \begin{pmatrix} 14 & 23 \\ 17 & 28 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ 16 & 8 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix}$

9. (a) $AB = \begin{pmatrix} 3 & 7 \\ 26 & 14 \end{pmatrix}$ and $BA = \begin{pmatrix} 17 & -17 & 1 \\ -12 & -6 & 3 \\ 11 & -37 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 15 & 30 \\ 0 & -30 & -45 \end{pmatrix}$

12. $\begin{pmatrix} -1 & 0 & 0 \\ 14 & 1 & 6 \\ 2 & 0 & -1 \end{pmatrix}$ 14. $\begin{pmatrix} -13 & -6 \\ 8 & -15 \end{pmatrix}$

15. Symmetric matrix = $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{pmatrix}$

Skew-symmetric matrix = $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

Objective Questions

1. The size of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \end{bmatrix}$ is

(a) 2×3 (b) 3×2
(c) 3×3 (d) 2×2

2. If A is a square matrix then $A + A^T$ is a

(a) diagonal matrix (b) scalar matrix
(c) symmetric matrix (d) skew-symmetric matrix

3. A square matrix A is symmetric if

(a) $A^2 = A$ (b) $A^T = -A$
(c) $A^2 = -A$ (d) $A^T = A$

4. If $\begin{pmatrix} 0 & a+2 \\ 5 & 0 \end{pmatrix}$ is a skew-symmetric matrix then $a =$

(a) -7
(b) -5
(c) -3
(d) -2

5. Sum of diagonal elements of a skew-symmetric matrix is

(a) 0
(b) 1
(c) 2
(d) 100

6. If A is a matrix of order $m \times n$, B is a matrix of order $n \times p$, then the order of AB is

(a) $m \times n$
(b) $n \times p$
(c) $m \times p$
(d) $p \times m$

7. If A is a matrix of size $m \times n$ and B is a matrix of size $p \times q$ then BA is possible if

(a) $m = n$
(b) $p = q$
(c) $m = q$
(d) $n = p$

8. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $A^2 =$

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix}$
(c) $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$
(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then AB =

(a) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$
(d) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

10. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $AA^T =$

(a) 0
(b) I
(c) A
(d) A^T

Answer Sheet									
1	2	3	4	5	6	7	8	9	10
a	c	d	a	a	c	c	c	d	b

Determinants

4.8 Introduction

A determinant is defined as the square arrangement of numbers in rows and columns enclosed by vertical bars. It has a numerical value.

A determinant which has one row and one column is called determinant of order one. If A = [a] then | A | = a.

For examples

(a) $|5| = 5$ (b) $|-2| = -2$.

A determinant which has two rows and two columns is called the determinant of order two. If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ then $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

For examples

(a) $\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 4 \times 3 - 1 \times 2 = 12 - 2 = 10.$

(b) $\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 3 = 6 - 6 = 0.$

In a similar manner, a determinant which has three rows and three columns, is called determinant of order three.

$$\text{Let } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{Then } |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

This is called the expansion of the determinant along its first column. To obtain this expansion, we multiply each element of the first row by the determinant of the second order which is obtained by omitting the row and column passing through that element. Starting from the first element, the signs of the products are alternately positive and negative.

In the determinant of order three, we can find its value by expanding it along any of its rows or along any of its columns. In any of these expansions the element a_{ij} is multiplied by $(-1)^{i+j}$.

Example: Find the value of determinant by expanding second column

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

Solution

Let $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$

Expanding along C_2

$$= -2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$= -2\,(4 \times 9 - 7 \times 6) + 5\,(1 \times 9 - 7 \times 3) - 8\,(1 \times 6 - 4 \times 3)$

$= -2\,(36 - 42) + 5(9 - 21) - 8(6 - 12)$

$= -2\,(-6) + 5\,(-12) - 8\,(-6)$

$= 12 - 60 + 48$

$= 0.$

Example: Find the value of the following determinant $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}.$

Solution

Let $A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}.$

Expanding along R_1

$$= 1\begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix}$$

$= 1\,(-5 - 7) - 1\,(-2 - 14) + 1(2 - 10)$

$= -12 + 16 - 8$

$= -4.$

4.9 Minors and Cofactors

A determinant which is obtained by deleting the row and column passing through an element of a given matrix is called **minor** of that element.

For example

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then, $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ is the minor of a_{11}.

Similarly,

$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$ and $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ are the minors of the elements a_{21} and a_{32} respectively.

The minor multiplied by sign $(-1)^{i+j}$ is called the **cofactor** of that element.

For example

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$\text{Thus cofactor of } a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Cofactor of } a_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \text{ etc.}$$

Example: From the matrix $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, find out the minor and cofactor of 4, 5 and 9.

Solution

$$\text{Minor of 4} = \begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} = 18 - 24 = -6.$$

$$\text{Minor of 5} = \begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix} = 9 - 21 = -12.$$

$$\text{Minor of 9} = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = 5 - 8 = -3.$$

and

$$\text{Cofactor of } 4 = (-1)^{1+2}\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} = (-1)^3\,(18-24)$$

$$= (-1)\,(-6)$$
$$= 6.$$

$$\text{Cofactor of } 5 = (-1)^{2+2}\begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix} = (-1)^4\,(9-21)$$

$$= (1)\,(-12)$$
$$= -12.$$

$$\text{Cofactor of } 9 = (-1)^{3+3}\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = (-1)^6\,(5-8)$$

$$= (1)\,(-3)$$
$$= -3.$$

4.10 Sarrus Rule

Sarrus rule is the mathematical rule which is used to find out the values of three order determinant only.

$$\text{Let } |A| = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{array}{ccccc} x_1 & y_1 & z_1 & x_1 & y_1 \\ x_2 & y_2 & z_2 & x_2 & y_2 \\ x_3 & y_3 & z_3 & x_3 & y_3 \end{array}$$

$$= x_1 y_2 z_3 + y_1 z_2 x_3 + z_1 x_2 y_3 - x_3 y_2 z_1 - y_3 z_2 x_1 - z_3 x_2 y_1$$

Sarrus rule consists of the following steps:

(i) List the elements of the first three columns of the given determinants.

(ii) Repeat the first two columns.

(iii) Find the products of the elements lying on the diagonals from top to bottom containing three elements $x_1y_2z_3$, $x_3y_1z_2$, $x_2y_3z_1$.

(iv) Similarly, find the products of the elements lying on the off diagonals from bottom to top contianing three elements: $x_3y_2z_1$, $x_1y_3z_2$, $x_2y_1z_3$.

(v) The three products obtained in step (iii) is taken with positive signs and the three products obtained in step (iv) is taken with negative signs then the sum of six terms is the value of the determinant. Thus,

$$D = x_1y_2z_3 + x_3y_1z_2 + x_2y_3z_1 - x_3y_2z_1 - x_1y_3z_2 - x_2y_1z_3.$$

Example: Find the value of the following determinant by using Sarrus rule

$$\begin{vmatrix} 4 & 7 & 5 \\ 8 & 6 & 3 \\ 5 & 3 & 7 \end{vmatrix}.$$

Solution

$$\text{Let } |A| = \begin{vmatrix} 4 & 7 & 5 \\ 8 & 6 & 3 \\ 5 & 3 & 7 \end{vmatrix}$$

$$= \begin{array}{ccccc} 4 & 7 & 5 & 4 & 7 \\ 8 & 6 & 3 & 8 & 6 \\ 5 & 3 & 7 & 5 & 3 \end{array}$$

$$= (4\times6\times7) + (7\times3\times5) + (5\times8\times3) - (5\times6\times5) - (3\times3\times4) - (7\times8\times7)$$
$$= 168 + 105 + 120 - 150 - 36 - 392 = 185.$$

4.11 Properties of Determinants

The following properties of determinants hold good for determinants of any order

1. If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

$$\text{i.e.} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0, \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} = 0.$$

2. If all the elements of any row (or column) are zero, then the value of the determinant is also zero.

$$\text{i.e} \begin{vmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \qquad [\because R_1 = 0].$$

3. If all rows and columns of a determinant are interchanged, the value of the determinant remains unaltered i.e.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

4. If two adjacent rows (or columns) of a determinant are interchanged, the value of the determinant remains the same, but mathematical sign is changed.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \ R_1 \leftrightarrow R_2.$$

5. If all the elements of any one row (or column) are multiplied by the same constant, then the original determinant is also multiplied by that constant.

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

6. If any row (or column) is the sum of two or more elements then the determinant can be expressed as sum of two or more determinants.

$$\begin{vmatrix} a_1 & b_1 + k_1 & c_1 \\ a_2 & b_2 + k_2 & c_2 \\ a_3 & b_3 + k_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}.$$

7. If any row (or column) is added or subtracted k times the corresponding elements of another row (or column), the value of determinant remains unchanged.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_1 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ R_1 \rightarrow R_1 + k\,R_3.$$

Notation and Working Rule

To evaluate the value of a determinant, we first try to use the above properties whenever possible. We should try to bring as many zeros as possible in any row or column and then expand along that row or column. We use the notations R_1, R_2, R_3, … for the first, second and third … rows and C_1, C_2, C_3, … for the first, second, third … columns. If we change the first row by adding k times the second row, we write $R_1 \rightarrow R_1 + kR_2$. If we change the second column by subtracting from it k times the first column then we write $C_2 \rightarrow C_2 - kC_1$. It should be clear that in the first operation R_1 will change while R_2 will remain the same and in the second operation C_2 will change while C_1 will remain the same.

WORKED OUT EXAMPLES

Example 1. Find out the value of the following determinants by using their properties:

(a) $\begin{vmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ 13 & 16 & 19 \end{vmatrix}$ (b) $\begin{vmatrix} 15 & 1 & 2 \\ 25 & 4 & 3 \\ 46 & 4 & 6 \end{vmatrix}$.

Solution

(a)

$$\begin{vmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ 13 & 16 & 19 \end{vmatrix} = 0 \qquad [\because R_1 = R_2].$$

(b)

$$\begin{vmatrix} 15 & 1 & 2 \\ 25 & 4 & 3 \\ 46 & 4 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 14 & 1 & 2 \\ 21 & 4 & 3 \\ 42 & 4 & 6 \end{vmatrix} \qquad [C_1 \to C_1 - C_2]$$

$$= \begin{vmatrix} 7\times 2 & 1 & 2 \\ 7\times 3 & 4 & 3 \\ 7\times 6 & 4 & 6 \end{vmatrix}$$

$$= 7\begin{vmatrix} 2 & 1 & 2 \\ 3 & 4 & 3 \\ 6 & 4 & 6 \end{vmatrix}$$

$$= 7 \times 0$$

$$= 0 \qquad [\because C_1 = C_3].$$

Example 2. Evaluate without expanding:

(a) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (b) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

Solution

(a) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$= \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix}$ $C_2 \to C_2 + C_3$

$= (a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$ Taking common $(a+b+c)$ from C_2

$= (a+b+c) \times 0 \quad [\because C_1 = C_2]$.

$= 0$

(b) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$

$= 0 \quad [\because R_1 = 0]$.

Example 3. Prove that: $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

Solution

L.H.S. $= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$ $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$

Expanding along R_1

$= \begin{vmatrix} x-y & (y-z) \\ (x-y)(x+y) & (y-z)(y+z) \end{vmatrix}$

Taking common $(x-y)$ and $(y-z)$ from C_1 and C_2 respectively, we get.

$$= (x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix}$$

$$= (x-y)(y-z)(y+z-x-y)$$

$$= (x-y)(y-z)(z-x) = \text{R.H.S.}$$

Example 4. Prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$.

Solution

$$\text{LHS} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Applying. $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$$

Expanding along C_1

$$= 1\begin{vmatrix} b-a & c(a-b) \\ c-a & b(a-c) \end{vmatrix}$$

$$= b(b-a)(a-c) - c(a-b)(c-a)$$

$$= b(a-b)(c-a) - c(a-b)(c-a)$$

$$= (a-b)(c-a)(b-c)$$

$$= (a-b)(b-c)(c-a)$$

$$= \text{R.H.S.}$$

Example 5. Prove that: $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2(a+3)$.

Solution

$$\text{LHS} = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= \begin{vmatrix} 1+a+1+1 & 1 & 1 \\ 1+1+a+1 & 1+a & 1 \\ 1+1+1+a & 1 & 1+a \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= \begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix}$$

$$= (a+3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} \quad [\text{Taking common } (a+3) \text{ from } C_1]$$

$$= (a+3)\begin{vmatrix} 0 & 1 & 1 \\ -a & 1+a & 1 \\ 0 & 1 & 1+a \end{vmatrix} \quad [C_1 \to C_1 - C_2]$$

$$= (a+3)\left[0 - (-a)\begin{vmatrix} 1 & 1 \\ 1 & 1+a \end{vmatrix} + 0\right] \text{ [Expanding along } C_1]$$

$$= (a+3)\, a\,(1 + a - 1)$$

$$= (a+3)\, a(a)$$

$$= a^2\,(a+3)$$

$$= \text{R.H.S.}$$

Example 6. Prove that: $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$.

Solution

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

Dividing C_1 by x, C_2 by y and C_3 by Z

$$= xyz\begin{vmatrix} \frac{1}{x}+1 & \frac{1}{y} & \frac{1}{z} \\ \frac{1}{x} & \frac{1}{y}+1 & \frac{1}{z} \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z}+1 \end{vmatrix}$$

$$= xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{y} & \frac{1}{z} \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{y}+1 & \frac{1}{z} \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{y} & \frac{1}{z}+1 \end{vmatrix} \qquad [C_1 \rightarrow C_1 + C_2 + C_3]$$

Taking common $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)$ from C_1

$$= (xyz)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix} 1 & \frac{1}{y} & \frac{1}{z} \\ 1 & \frac{1}{y}+1 & \frac{1}{z} \\ 1 & \frac{1}{y} & \frac{1}{z}+1 \end{vmatrix}$$

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & \frac{1}{y} & \frac{1}{z}+1 \end{vmatrix} \quad [R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3]$$

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right) \times 1 \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} \quad [\text{Expanding along } C_1]$$

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right).$$

Example 7. Prove that:

$$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2).$$

Solution

$$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ 0 & 0 & -(ax^2+2bxy+cy^2) \end{vmatrix} \quad [R_3 \to R_3 - xR_1 - yR_2]$$

$$= -(ax^2+2bxy+cy^2)\begin{vmatrix} a & b \\ b & c \end{vmatrix} \quad [\text{Expanding along } R_3]$$

$$= (ax^2+2bxy+cy^2)(ac-b^2)$$

$$= (b^2-ac)(ax^2+2bxy+cy^2).$$

Example 8. Show that: $\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1+x^2+y^2+z^2.$

Solution

$$\text{L.H.S.} = \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$$

Applying $C_1 \to xC_1$, $C_2 \to yC_2$, $C_3 \to zC_3$

$$= \frac{1}{xyz}\begin{vmatrix} x(x^2+1) & xy^2 & xz^2 \\ x^2y & y(y^2+1) & yz^2 \\ x^2z & y^2z & z(z^2+1) \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \frac{1}{xyz}\begin{vmatrix} x(x^2+1+y^2+z^2) & xy^2 & xz^2 \\ y(x^2+y^2+1+z^2) & y(y^2+1) & yz^2 \\ z(x^2+y^2+z^2+1) & y^2z & z(z^2+1) \end{vmatrix}$$

Taking $1 + x^2 + y^2 + z^2$ common from C_1

$$= \frac{1}{xyz}(1+x^2+y^2+z^2)\begin{vmatrix} x & xy^2 & xz^2 \\ y & y(y^2+1) & yz^2 \\ z & y^2z & z(z^2+1) \end{vmatrix}$$

Taking x, y, z common from, R_1, R_2 and R_3 respectively

$$= \frac{1}{xyz}(1+x^2+y^2+z^2)(x{\cdot}y{\cdot}z)\begin{vmatrix} 1 & y^2 & z^2 \\ 1 & y^2+1 & z^2 \\ 1 & y^2 & z^2+1 \end{vmatrix}$$

$$= (1+x^2+y^2+z^2)\begin{vmatrix} 0 & -1 & 0 \\ 1 & y^2+1 & z^2 \\ 1 & y^2 & z^2+1 \end{vmatrix} \quad R_1 \to R_1 - R_2$$

Expanding along R_1

$$= (1+x^2+y^2+z^2)\begin{vmatrix} 1 & z^2 \\ 1 & z^2+1 \end{vmatrix}$$

$$= (1+x^2+y^2+z^2)(z^2+1-z^2)$$

$$= 1+x^2+y^2+z^2$$

= R.H.S. Proved.

Example 9. Solve the equation: $\begin{vmatrix} 2 & -3 & 4 \\ -5 & 6 & -7 \\ 8 & -9 & x \end{vmatrix} = 0.$

Solution

Here, we have to find the valve of x.

Now, $\begin{vmatrix} 2 & -3 & 4 \\ -5 & 6 & -7 \\ 8 & -9 & x \end{vmatrix} = 0$

or, $\begin{vmatrix} -3 & 3 & -3 \\ -5 & 6 & -7 \\ 8 & -9 & x \end{vmatrix} = 0 \qquad [R_1 \to R_1 + R_2]$

or, $\begin{vmatrix} 0 & 0 & -3 \\ 1 & -1 & -7 \\ -1 & -9+x & x \end{vmatrix} = 0 \qquad [C_1 \to C_1 + C_2,\ C_2 \to C_2 + C_3]$

or, $-3\begin{vmatrix} 1 & -1 \\ -1 & -9+x \end{vmatrix} = 0$ [Expanding along R_1]

or, $-3(-9+x-1) = 0$

$\therefore \quad x = 10.$

EXERCISE – 4 B

1. Evaluate the following determinants.

(a) $|-2|$ (b) $\begin{vmatrix} 5 & -3 \\ 2 & 5 \end{vmatrix}$ (c) $\begin{vmatrix} x+y & x \\ y & x-y \end{vmatrix}$.

2. Solve for x.

(a) $\begin{vmatrix} 1 & 3 \\ 2 & x \end{vmatrix} = 4$ (b) $\begin{vmatrix} x-3 & 4 \\ x-3 & 7 \end{vmatrix} = 0.$

3. Find the value of determinants by expanding along any row or column.

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$.

4. Find the value of determinants by using Sarrus rule.

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$.

5. Prove that the value of following determinants is zero by using their properties

(a) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 4 & 8 & 12 \end{vmatrix}$ (b) $\begin{vmatrix} 55 & 1 & 7 \\ 30 & 2 & 4 \\ 10 & 6 & 2 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

(d) $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$ (e) $\begin{vmatrix} b-c & b+c & b \\ c-a & c+a & c \\ a-b & a+b & a \end{vmatrix}$ (f) $\begin{vmatrix} a & a^2 & ab+ac \\ b & b^2 & bc+ab \\ c & c^2 & ca+cb \end{vmatrix}$.

6. Without expanding, prove that:

(a) $\begin{vmatrix} -2 & -7 & 4 \\ 3 & 13 & 2 \\ 4 & 6 & 11 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 11 \\ -2 & 3 & 4 \\ -7 & 13 & 6 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

(c) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

7. Prove that

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = (x-y)(y-z)(z-x)$

(c) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

(d) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(e) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

(f) $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$

(g) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(h) $\begin{vmatrix} a+b+2c & a & b \\ c & c+b+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

(i) $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

(j) $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$.

8. Solve the following equations

(a) $\begin{vmatrix} 1 & 4 & 4 \\ 1 & -2 & 1 \\ 1 & 2x & x^2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0.$

Answers

1. (a) -2 (b) 31 (c) $x^2 - y^2 - xy$
2. (a) $x = 10$ (b) $x = 3$
3. (a) -20 (b) 3
4. (a) 2 (b) 0
8. (a) $x = 2, -1$ (b) $x = -9, 1$

Objective Questions

1. If $A = [-2]$ then $|A| =$

(a) 2 (b) -2
(c) 0 (d) 1

2. The cofactor of 2 in the matrix $\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix}$ is

(a) -5 (b) 5
(c) 4 (d) 1

3. $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} =$

(a) 8 (b) 7
(c) 6 (d) 5

4. $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} =$

(a) 0 (b) 5
(c) 41 (d) 66

5. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} =$

(a) $a + b + c$ (b) $3abc$
(c) $1 + a + b + c$ (d) 0

6. $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} =$

(a) $a(a+3)$ (b) $a^2(a+3)$
(c) $3a(a+3)$ (d) $2a(a+3)$

7. If $\begin{vmatrix} -3-x & 4 \\ 2 & -1-x \end{vmatrix} = 0$ then $x =$

(a) 1, 5 (b) – 1, 5
(c) 1, – 5 (d) – 1, – 5

8. If $\begin{vmatrix} -4 & x-4 \\ -2 & x+1 \end{vmatrix} = 0$ then $x =$

(a) – 6 (b) 6
(c) 3 (d) – 3

9. If A is a square matrix then which of the following is always true?

(a) $|A| = 0$ (b) $|A| = |A^T|$
(c) $|A| = 1$ (d) $|A| = -|A^T|$

10. If A is a square matrix of order n and $A = kB$ where k is a scalar then $|A| =$

(a) $|B|$ (b) $k|B|$
(c) $n|B|$ (d) $k^n|B|$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	b	d	a	d	b	c	a	b	d

4.12 Singular and Non-singular Matrix

Definition

A square matrix A is said to be **singular** if $|A| = 0$ and **non-singular** if $|A| \neq 0$.

For example

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a singular matrix since $|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$ and

$B = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ is non-singular since $|B| = \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 2 + 3 = 5 \neq 0.$

4.13 Adjoint and Inverse of a Matrix

1. **Adjoint of a Matrix**

Let $A = (a_{ij})$ be any square matrix and A_{ij} be the cofactor of a_{ij}. Then, the adjoint of A is defined by adj. A = Transpose of its matrix of cofactors.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a given matrix.

Let A_{ij} be the cofactors of a_{ij}. Then,

$$\text{Adj. A} = \text{Transpose of} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}.$$

Examples

1. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

We have, $a_{11} = 1$, $a_{12} = 2$; $a_{21} = 4$; $a_{22} = 3$ then their cofactors are $A_{11} = 3$; $A_{12} = -4$; $A_{21} = -2$ and $A_{22} = 1$ then

$$\text{Adj. A} = \begin{pmatrix} 3 & -4 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

A_{11} = Cofactor of $1 = \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = 2 - 0 = 2$

A_{12} = Cofactor of $2 = -\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$

A_{13} = Cofactor of $-3 = \begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} = -2 - 0 = -2$

A_{21} = Cofactor of $-1 = -\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} = -(2 + 6) = -8$

A_{22} = Cofactor of $2 = \begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

A_{23} = Cofactor of $0 = -\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = -(2 - 0) = -2$

A_{31} = Cofactor of $0 = \begin{vmatrix} 2 & -3 \\ 2 & 0 \end{vmatrix} = 0 + 6 = 6$

A_{32} = Cofactor of $2 = -\begin{vmatrix} 1 & -3 \\ -1 & 0 \end{vmatrix} = -(0 - 3) = 3$

A_{33} = Cofactor of $1 = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 2 + 2 = 4$

$$\text{Thus, adj. A} = \begin{pmatrix} 2 & 1 & -2 \\ -8 & 1 & -2 \\ 6 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & -8 & 6 \\ 1 & 1 & 3 \\ -2 & -2 & 4 \end{pmatrix}.$$

2. Inverse of a Matrix

Let A and B be any square matrices such that $AB = BA = I$. Then B is called the inverse of A. We write, $B = A^{-1}$ i.e. $AA^{-1} = I$

Thus, if B is the inverse of A then A is also the inverse of B.

For example

Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$. Then,

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 6-5 & -2+2 \\ 15-15 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{and } BA = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 6-5 & 3-3 \\ -10+10 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore AB = BA = I$.

Hence, the two matrices are inverses to each other.

Example: If $A = \begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$, show that $A^{-1} = \frac{1}{19}A$.

Solution

Here, $A = \begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19 \neq 0$$

So, A^{-1} exists.

The cofactors are

$A_{11} = -2$, $A_{12} = -5$, $A_{21} = -3$, $A_{22} = 2$

$$\text{Adj. } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} -2 & -5 \\ -3 & 2 \end{pmatrix}^T$$

$$= \begin{pmatrix} -2 & -3 \\ -5 & 2 \end{pmatrix}.$$

We have,

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj. } A$$

$$= \frac{1}{-19}\begin{pmatrix} -2 & -3 \\ -5 & 2 \end{pmatrix} = \frac{1}{19}\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} = \frac{1}{19}A.$$

Some Useful Theorems on an Inverse and Adjoint Matrix

Theorem 1: If A is a square matrix of order n then $A \cdot (\text{adj. } A) = (\text{adj. } A)\, A = |A|\, I$.

Proof

Let $A = a_{ij}$, adj. $A = A_{ij}$ and $A\,(\text{adj. } A) = B_{ij}$ then

$$B_{ij} = a_{i1} A_{j1} + a_{i2} A_{j2} + a_{i3} A_{j3} = \begin{cases} |A| \text{ if } i = j \\ 0 \quad \text{if } i \neq j. \end{cases}$$

which shows that each diagonal element of $A \cdot (\text{adj. } A)$ is $|A|$ and non-diagonal elements are zeros.

Hence, $A \cdot (\text{Adj. } A) = |A|\, I$

Similarly, $(\text{adj. } A) \cdot A = |A|\, I$

$\therefore \quad A \cdot (\text{adj. } A) = (\text{adj. } A) \cdot A = |A|\, I.$

This theorem gives the result for an inverse matrix i.e.

$$A^{-1} = \frac{\text{adj. } A}{|A|}.$$

For example

Let $A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$. Find A^{-1}.

Solution

Here, $|A| = \begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix} = 3 + 8 = 11 \neq 0$

Now, $A_{11} = 3$; $A_{12} = -4$; $A_{21} = 2$; $A_{22} = 1$

Then, adj. $A = \begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{11}\begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 3/11 & 2/11 \\ -4/11 & 1/11 \end{pmatrix}$$

Theorem 2: The inverse of a square matrix A exists if and only if $|A| \neq 0$.

Proof

Let B be the inverse matrix of A then

$AB = BA = I$

or, $|A|\,|B| = |AB| = |I| = 1$

$\Rightarrow \quad |A| \neq 0.$

Conversely, let $|A| \neq 0$

Then we have:

$A \cdot (\text{adj. } A) = (\text{adj. } A) \cdot A = |A|\, I$

i.e. $A \cdot \left(\frac{\text{adj. } A}{|A|}\right) = \left(\frac{\text{adj. } A}{|A|}\right) \cdot A = I$

Thus, A^{-1} exists & $A^{-1} = \frac{\text{adj. } A}{|A|}$.

Theorem 3: Let A & B are two conformable matrices, the product AB and inverses of A & B exists then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

Similarly, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$.

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

For example

Find the inverse of $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.

Solution

Let $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} \begin{aligned} &= 1(16-9) - 3(4-3) + 3(3-4) \\ &= 7 - 3 - 3 = 1 \neq 0 \end{aligned}$$

Hence, A^{-1} exists.

Now,

$$A_{11} = \text{Cofactor of } 1 = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 16 - 9 = 7$$

$$A_{12} = \text{Cofactor of } 3 = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -(4-3) = -1$$

$$A_{13} = \text{Cofactor of } 3 = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = -1$$

$$A_{21} = \text{Cofactor of } 1 = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -(12-9) = -3$$

$$A_{22} = \text{Cofactor of } 4 = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 = 1$$

$$A_{23} = \text{Cofactor of } 3 = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = -(3-3) = 0$$

$$A_{31} = \text{Cofactor of } 1 = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 9 - 12 = -3$$

$$A_{32} = \text{Cofactor of } 3 = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = -(3-3) = 0$$

$$A_{33} = \text{Cofactor of } 4 = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 = 1$$

Then adj. A = Transpose of $\begin{pmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

$\therefore$ adj. A = $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

We have

$$A^{-1} = \frac{\text{adj. A}}{|A|}$$

$$= \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

WORKED OUT EXAMPLES

Example 1. Find the inverse of $\begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}$.

Solution

Let A = $\begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 3 & 6 \\ 7 & 2 \end{vmatrix} = 6 - 42 = -36 \neq 0$

$\therefore$ A^{-1} exists.

We find adj. A

$A_{11} = (-1)^{1+1} |2| = 2,$

$A_{12} = -7, A_{21} = -6, A_{22} = 3$

$\therefore$ adj. A = $\begin{pmatrix} 2 & -7 \\ -6 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 \\ -7 & 3 \end{pmatrix}$

$\therefore$ $A^{-1} = \frac{1}{-36}\begin{pmatrix} 2 & -6 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} \frac{-1}{18} & \frac{1}{16} \\ \frac{7}{36} & \frac{-1}{12} \end{pmatrix}.$

Example 2. If A = $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$, verify that $(A')^{-1} = (A^{-1})'$.

Solution

We have, A = $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$,

$|A| = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 10 - 3 = 7 \neq 0$

$\therefore$ A^{-1} exists.

We find adj. A

$$A_{11} = 5,\ A_{12} = -3,\ A_{21} = -1,\ A_{22} = 2$$

$$\therefore \quad \text{adj. } A = \begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{7}\begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{7} & \frac{-1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{pmatrix}$$

$$\therefore \quad \text{R.H.S.} = (A^{-1})' = \begin{pmatrix} \frac{5}{7} & \frac{-3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix}$$

For LHS $A' = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$

and $|A'| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 10 - 3 = 7 \neq 0$

$\therefore \quad (A')^{-1}$ exists and $(A')^{-1} = \dfrac{\text{adj. } A'}{|A'|}$

We find adj. A'

$$A_{11} = 5,\ A_{12} = -1,\ A_{21} = -3,\ A_{22} = 2$$

$$\therefore \quad \text{adj. } A' = \begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix}$$

Thus, $(A')^{-1} = \dfrac{\text{adj. } A'}{|A'|}$

$$= \frac{1}{7}\begin{pmatrix} 5 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{-3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix}$$

$\therefore \quad$ L.H.S. = R.H.S.

Example 3. Find the adjoint of $\begin{bmatrix} 1 & 4 & 3 \\ 3 & 9 & 0 \\ -5 & -6 & 2 \end{bmatrix}$.

Solution

The co–factors are:

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 9 & 0 \\ -6 & 2 \end{vmatrix} = 1(18) = 18$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 0 \\ -9 & 2 \end{vmatrix} = -1(6) = -6$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 9 \\ -5 & -6 \end{vmatrix} = 1(-18 + 45) = 27$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 3 \\ -6 & -2 \end{vmatrix} = -1(8+18) = -26$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -5 & -6 \end{vmatrix} = 2 + 15 = 17$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 4 \\ -5 & -6 \end{vmatrix} = -1(-6+20) = -14$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 3 \\ 9 & 0 \end{vmatrix} = 1(-27) = -27$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} = -1(-9) = 9$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 4 \\ 3 & 9 \end{vmatrix} = 1(9-12) = -3$$

Again,

$$\text{Adj. } A = \begin{bmatrix} 18 & -6 & 27 \\ -26 & 17 & -14 \\ -27 & 9 & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 18 & -26 & -27 \\ -6 & 17 & 9 \\ 27 & -14 & 9 \\ 27 & -14 & -3 \end{bmatrix}.$$

Example 4. Find the inverse of the matrix $\begin{bmatrix} 1 & 4 & 1 \\ 3 & 3 & -2 \\ 0 & -4 & 1 \end{bmatrix}$.

Solution

$$\text{Let, } A = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 3 & -2 \\ 0 & -4 & 1 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & 4 & 1 \\ 3 & 3 & -2 \\ 0 & -4 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -3 \\ 0 & -4 \end{vmatrix}$$

$$= 1\,(3-8) - 4\,(3) + 1(-12)$$

$$= -5 - 12 - 12 = -29 \neq 0$$

$\therefore$ A^{-1} exists.

The cofactors are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} = 3 - 8 = -5$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = -1(3) = -3$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & -3 \\ 0 & -4 \end{vmatrix} = 1(-12) = -12$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 1 \\ -4 & 1 \end{vmatrix} = -1(4+4) = -8$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 0 & -4 \end{vmatrix} = -1(-4) = 4$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = (-8-3) = -11$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -1(-2-3) = 5$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 4 \\ 3 & 3 \end{vmatrix} = (3-12) = -9$$

$$\text{Adj. } A = \begin{bmatrix} -5 & -3 & -12 \\ -8 & 1 & 4 \\ -11 & 5 & -9 \end{bmatrix}'$$

$$= \begin{bmatrix} -5 & -8 & -11 \\ -3 & 1 & 5 \\ -12 & 4 & -9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}.\text{ Adj.}A = \frac{1}{-29}\begin{bmatrix} -5 & -8 & -11 \\ -3 & 1 & 5 \\ -12 & 4 & -9 \end{bmatrix}.$$

EXERCISE – 4 C

1. Find the adjoint of the following matrices.

(a) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & -7 \\ -3 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -4 & 3 \\ 5 & 3 & 6 \\ 1 & 2 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$.

2. Find the inverse of the following matrices if possible.

(a) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 4 \\ -3 & 0 & 1 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$ be a matrix. Show that $AA^{-1} = A^{-1}A = I$.

4. Prove that the matrices $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$ are inverses of each other.

5. Given a matrix $\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$, find a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that they are inverse to each other.

6. If $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

7. If $A = \begin{pmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$, verify that: $A \cdot (\text{adj. } A) = (\text{adj. } A) \cdot A = |A|\ I$.

Answers

1. (a) $\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} -15 & 2 & -33 \\ 11 & -5 & 3 \\ 7 & -8 & 26 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$

2. (a) $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & -2 \\ \frac{3}{2} & \frac{5}{2} \end{pmatrix}$

(c) $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$ (d) $\begin{pmatrix} -\frac{1}{13} & \frac{1}{13} & -\frac{4}{13} \\ -\frac{14}{13} & \frac{1}{13} & -\frac{4}{13} \\ -\frac{3}{13} & \frac{3}{13} & \frac{1}{13} \end{pmatrix}$

5. $\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$

Objective Questions

1. Two matrices A and B are inverses of each other if and only if

(a) $AB = BA = 0$ (b) $AB = BA$
(c) $AB = I, BA = 0$ (d) $AB = BA = I$

2. A square matrix A is said to be singular if

(a) $|A| = 0$ (b) $|A| = 1$
(c) $|A| \neq 0$ (d) $|A| = 100$

3. If $A = \begin{pmatrix} 3 & -2 \\ 5 & 5 \end{pmatrix}$ then $A^{-1} =$

(a) $\frac{1}{5}\begin{pmatrix} 5 & 2 \\ -5 & 3 \end{pmatrix}$ (b) $\frac{1}{10}\begin{pmatrix} 5 & -2 \\ -5 & 3 \end{pmatrix}$

(c) $\frac{1}{25}\begin{pmatrix} 5 & 2 \\ -5 & 3 \end{pmatrix}$ (d) $\frac{1}{15}\begin{pmatrix} 3 & -2 \\ 5 & 5 \end{pmatrix}$

4. The adjoint of $\begin{pmatrix} 5 & -7 \\ -3 & 2 \end{pmatrix}$ is

(a) $\begin{pmatrix} 2 & 7 \\ -3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -7 \\ -3 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} -2 & 7 \\ -3 & -5 \end{pmatrix}$

5. If A is a square matrix then A · (adj. A) = (adj. A) · A =

(a) $|A|$ (b) $|A|I$

(c) $2|A|$ (d) 4A

6. If inverse of $\begin{bmatrix} 1 & 1 & 2 \\ x & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ doesn't exist then x =

(a) 0 (b) 2

(c) 3 (d) 1

Answer Sheet

1	2	3	4	5	6	7	8	9	10
d	a	c	b	b	d				

4.14 Linear Transformation

Transformation

A transformation (or mapping or function) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector **x** in $\mathbb{R}^n$ a vector T(**x**) in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain of T**, and $\mathbb{R}^m$ is called the **co-domain of T**. The set of all images T(**x**) is called the **range of T**. A vector is a physical quantity having both magnitude and direction.

Example: Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \\ 2 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, find $T(\mathbf{u})$.

Solution

$$T(u) = Au$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 \\ 3+10 \\ 2+14 \end{bmatrix} = \begin{bmatrix} -1 \\ 13 \\ 16 \end{bmatrix}.$$

Linear Transformation

A transformation (or mapping) T is called linear if

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T.

(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ where k is a scalar and $\mathbf{u}$ is in the domain of T.

In other words, we can say T is linear if $T(\mathbf{0}) = \mathbf{0}$.

$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v})$

for all scalars α & β and for vectors $\mathbf{u}$ & $\mathbf{v}$ in the domain of T.

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = 2\mathbf{x}$. Then show that T is linear.

Solution

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Consider the scalars α and β.

Now,

$$\begin{aligned} T(\alpha\mathbf{u} + \beta\mathbf{v}) &= 2(\alpha\mathbf{u} + \beta\mathbf{v}) \\ &= \alpha(2\mathbf{u}) + \beta(2\mathbf{v}) \\ &= \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \end{aligned}$$

Hence T is linear.

Theorem (Matrix of a linear transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, where $A = [T(\mathbf{e}_1) \dots T(\mathbf{e}_n)]$ where $\mathbf{e}_j$ is the j^{th} column of the identity matrix in $\mathbb{R}^n$.

Example: Find the standard matrix A for the transformation $T(\mathbf{x}) = 2\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^2$.

Solution

Here, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(\mathbf{e}_1) = 2\mathbf{e}_1 = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T(\mathbf{e}_2) = 2\mathbf{e}_2 = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The standard matrix is $[T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Orthogonal Matrix

A square matrix A is said to be orthogonal if $AA^T = AA^T = I$

i.e. $A^T = A^{-1}$. Also the determinant of orthogonal matrix is ± 1.

For example

The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal, since

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly, $$A^TA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore \quad AA^T = A^TA = I$

Thus, A is orthogonal matrix.

4.15 Orthogonal Transformations

A linear transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$ is said to be orthogonal transformation if matrix A is orthogonal.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if it preserves the length. That is length of $T(\mathbf{x})$ = length of $\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

For example

The transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal transformation since by above example we have seen that A is orthogonal matrix.

4.16 Rank of Matrices

An $m \times n$ matrix A is said to have a rank r if it has at least one square submatrix of order r which is non-singular and all submatrices of order greater than r are singular. It is denoted by $\rho(A)$ or rank A.

Note: (i) The rank r of an m × n matrix can at most be equal to the smaller of the numbers m and n.

(ii) An n × n matrix A has rank n iff $|A| \neq 0$.

(iii) An n × n matrix A has rank less than n iff $|A| = 0$.

Example: Find the rank of $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 1 \end{bmatrix}$.

Solution

Here, A is a matrix of size 3×2. Let us take a square submatrix of 2×2 namely $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$.

Now, $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$.

So, rank of A = 2.

Example: Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 0 & 5 & 7 \end{bmatrix}$.

Solution

Here, A is a 3×3 matrix.

So, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 0 & 5 & 7 \end{vmatrix}$

Expanding along C_1

$$1\begin{vmatrix} -1 & 2 \\ 5 & 7 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}$$

$= 1(-7-10) - 1(14-15)$

$= -17 + 1$

$= -16 \neq 0$

So, rank of A = 3.

WORKED OUT EXAMPLES

Example 1. Show that the transformation

$T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T : (x, y, z) = (x, y, z)$ is linear.

Solution

Let $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2) \in \mathbb{R}$ and α, β be scalars.

$$\begin{aligned}
\text{Now, } T(\alpha\mathbf{u} + \beta\mathbf{v}) &= T(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) \\
&= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \\
&= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \\
&= (\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2) \\
&= \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \\
&= \alpha\, T(x_1, y_1, z_1) + \beta\, T(x_2, y_2, z_2) \quad [\because T(x, y, z) = (x, y, z)] \\
&= \alpha\, T(\mathbf{u}) + \beta(\mathbf{v})
\end{aligned}$$

$\therefore$ T is linear.

Example 2. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \\
&= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= 1(45 - 48) - 2(36 - 42) + 3(32 - 35) \\
&= -3 + 12 - 9 \\
&= 0
\end{aligned}$$

So, rank of A is less than 3.

Now, consider a submatrix of order 2, for instance, take $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$.

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$$

So, rank of A = 2.

Example 3. Transform $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and check whether this transformation is linear? [TU BCA 2018]

Solution

Here, $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\text{Now, } Au = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0\times 1+(-1)\times(-1) \\ 1\times 1+0\times(-1) \end{bmatrix} = \begin{bmatrix} 0+1 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0\times(-2)+(-1)\times 3 \\ 1\times(-2)+0\times 3 \end{bmatrix} = \begin{bmatrix} 0-3 \\ -2+0 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

For linearity

Let $T(\mathbf{x}) = A\mathbf{x}$

Now, $T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v})$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}+\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0-2 \\ -1+0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad \dots \text{(i)}$$

And, $T(\mathbf{u}) + T(\mathbf{v}) = A\mathbf{u} + B\mathbf{v}$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}+\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1-3 \\ 1-2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad \dots \text{(ii)}$$

From (i) and (ii), we have,

$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

Again, $T(k\mathbf{u}) = A(k\mathbf{u})$ where k is a scalar. $[\because T(\mathbf{u}) = A\mathbf{u}]$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k \\ -k \end{bmatrix} = \begin{bmatrix} 0+k \\ k+0 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$= k\,A\mathbf{u} = k\,T(\mathbf{u})$

And, $T(kv)$ $= A(kv)$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2k \\ 3k \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 2k \\ 3k + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2k \\ 3k \end{bmatrix}$$

$$= k \begin{bmatrix} 2 \\ 3 \end{bmatrix} = k\,A\mathbf{v} = k\,T(\mathbf{v})$$

$\therefore$ T is linear.

EXERCISE – 4 D

1. Find the rank of the following matrices.

(a) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & 0 \\ -5 & 2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 7 & 2 \\ 4 & 8 & 12 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -5 \\ 7 & 0 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & -1 \\ 2 & 1 & 0 & 5 \end{bmatrix}$.

2. Given $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Transform $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

3. Transform $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

4. (a) Transform $\mathbf{u} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and check whether this transformation is linear or not?

(b) Transform $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and show that this transformation is linear.

5. Show that $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

(a) $T(x, y, z) = (x, y, 0)$ is linear.

(b) $T(x, y, z) = (0, y, z)$ is linear.

6. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

7. Find the standard matrix A for the transformation $T(\mathbf{x}) = 3\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^2$.

8. Prove that the matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal.

9. If a transformation T is defined by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that T is orthogonal transformation.

Answers

1. (a) 1 (b) 2 (c) 2 (d) 3 (e) 3

2. $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

4. (a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, linear. (b) $\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

7. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Objective Questions

1. The image of $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ by the matrix $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Then the image under T of $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is

(a) $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. A transformation T is linear if

(a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, where $\mathbf{u}$ and $\mathbf{v}$ are in domain of T.
(b) $T(k\mathbf{u}) = k\,T(\mathbf{u})$, where $\mathbf{u}$ is in domain of T and k is a scalar.
(c) both a and b.
(d) neither a nor b.

4. A matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ is orthogonal transformation if
   (a) $A^T = I$
   (b) $A^T = A$
   (c) $A^T = -A$
   (d) $A = I$

5. The rank of matrix $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ is
   (a) 0
   (b) 1
   (c) 2
   (d) 3

6. The rank of matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is
   (a) 0
   (b) 1
   (c) 2
   (d) 3

7. The rank of matrix $\begin{bmatrix} 10 & 2 & 5 \\ -1 & 3 & 1 \\ 2 & 4 & 5 \end{bmatrix}$ is
   (a) 0
   (b) 1
   (c) 2
   (d) 3

Answer Sheet

1	2	3	4	5	6	7	8	9	10
d	a	c	b	b	c	d			

OOO

5

UNIT

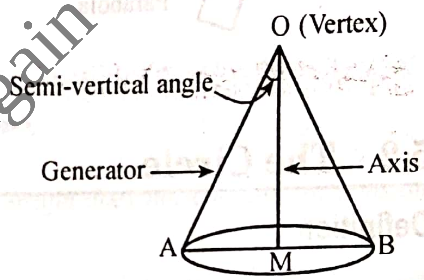
Analytical Geometry

Conic Section

5.1 Introduction

The curve obtained by the intersection of the cone and the plane is called a conic section. The nature of the curve depends upon how the plane cuts the cone.

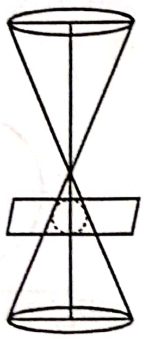
Let O be the fixed point and OM be the fixed line. The surface generated by rotating the line OA about the fixed line OM in such a way that ∡AOM is always constant is called the right circular cone.



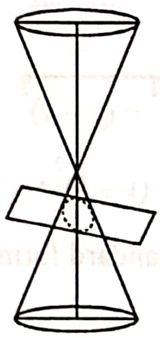
The point O is called vertex, OM is called the axis, OA is called generator and ∡AOM is called semi-vertical angle. If the cone symmetric to the cone OAB is added whose axis is opposite to OM, then it is double right cone.

(a) If a cutting plane cuts a cone parallel to the base, then the section is called a circle. It is shown in the figure.

(b) If a cutting plane cuts a cone neither parallel to base nor the side (generator) then the section is called an ellipse. It is shown in the figure.



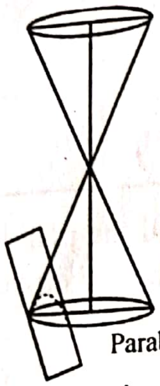
Circle

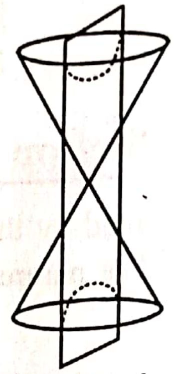


Ellipse

(c) If a cutting plane cuts the cone not passing through the vertex and is parallel to the generator of the cone, the section is called a parabola. It is shown in the figure.

(d) If a cutting plane cuts the double right cone in such a way that the semi-vertical angle is greater than the angle between the plane and axis then the section is called a hyperbola. It is shown in the figure.


Parabola


Hyperbola

5.2 The Circle

Definition

A circle is the locus of a point which moves in a plane so that it is always at a constant distance from a fixed point in the plane. The fixed point is called **centre** and the constant distance is called the **radius** of the circle.

Equation of the Circle

Central Form

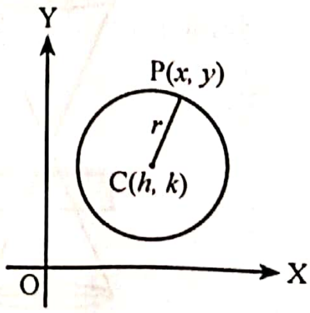
Let the centre of the circle be at a point C(h, k) and radius be equal to r. Then if P(x, y) be the moving point on the circle, we have by definition,

CP = r

i.e. $\sqrt{(x-h)^2+(y-k)^2} = r$

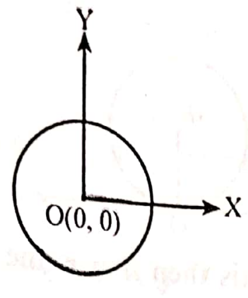
$\therefore\ (x-h)^2+(y-k)^2 = r^2.$

This is the standard form of the equation of the circle.

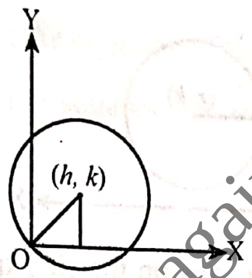


Particular Cases

(i) If the centre be at the origin and radius 'r' then $h = 0$, $k = 0$, so the equation of the circle is $x^2 + y^2 = r^2$.

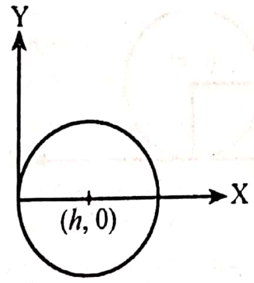


(ii) If the origin lies on the circle, then $h^2 + k^2 = r^2$ and so the equation of the circle in this case is $x^2 + y^2 - 2hx - 2ky = 0$.

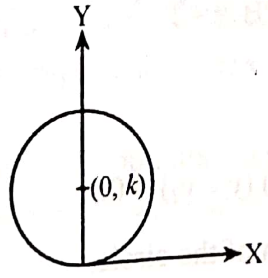


Note: The equation of a circle passing through the origin does not contain any constant term.

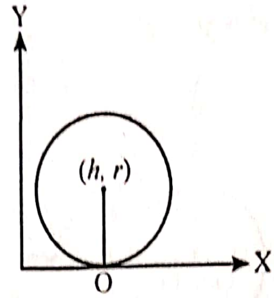
(iii) If the centre lies on the x-axis and the origin lies on the circle, then $k = 0$ and $h = r$ and so the equation of the circle is $x^2 + y^2 - 2rx = 0$.



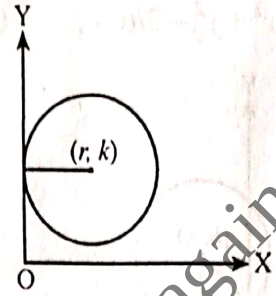
(iv) If the centre lies on the y-axis and the origin lies on the circle, the equation of the circle is $x^2 + y^2 - 2ry = 0$.



(v) If the circle touches the x-axis, then $k = r$, the equation of the circle in the case will be $(x-h)^2 + (y-r)^2 = r^2$.

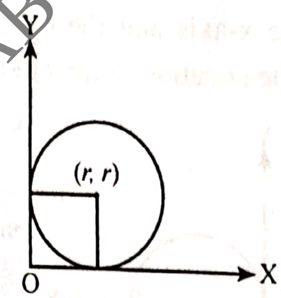


(vi) If the circle touches y-axis then $h = r$, the equation of circle becomes $(x-r)^2 + (y-k)^2 = r^2$.



(vii) If the circle touches both axes on the first quadrant, then $h = k = r$ and the equation of the circle is $(x-r)^2 + (y-r)^2 = r^2$,

i.e. $x^2 + y^2 - 2rx - 2ry + r^2 = 0$.



Diameter Form

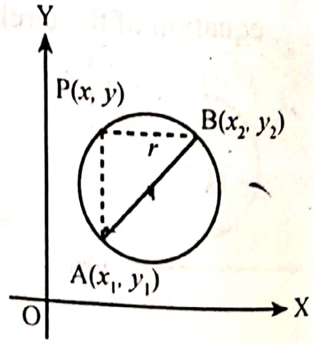
Let the ends of the diameter be $A(x_1, y_1)$ and $B(x_2, y_2)$. Join PA and PB. Then, by geometry $\measuredangle APB = 90^\circ$ i.e. $PA \perp PB$.

Then, slope of PA · slope of PB = – 1

or, $\left(\dfrac{y-y_1}{x-x_1}\right) \cdot \left(\dfrac{y-y_2}{x-x_2}\right) = -1$

$\therefore \quad (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0.$

which is the required equation of the circle.



Example: The equation of the circle described on the line joining the points (3, 4) and (2, – 7) as diameter is

$$(x-3)(x-2)+(y-4)(y+7)=0$$

i.e. $x^2+y^2-5x+3y-22=0$.

General Equation

The standard equation of the circle can be represented in another form.

$$(x-h)^2+(y-k)^2=r^2$$

or, $x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$

i.e. $x^2+y^2+2gx+2fy+c=0$ ...(i)

where $g=-h$ and $f=-k$ and $c=h^2+k^2-r^2$

This is called general form of the equation of the circle.

Conversely, by completing the squares, we may express it in the standard form.

Thus, we have $(x+g)^2+(y+f)^2=g^2+f^2-c$

i.e. $[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2$

Comparing it with the standard equation $(x-h)^2+(y-k)^2=c^2$, we can say that the general equation $x^2+y^2+2gx+2fy+c=0$ always represents a circle, its centre $=(-g,-f)$, radius $=\sqrt{g^2+f^2-c}$.

Note:

1. (i) *If $g^2+f^2-c>0$, the circle will be a real circle.*
   (ii) *If $g^2+f^2-c=0$, the circle will be a point circle.*
   (iii) *If $g^2+f^2-C<0$, the circle will be an imaginary circle and that circle cannot be sketch in a real plane.*
2. *The special features of the equation of the circle $x^2+y^2+2gx+2fy+c=0$ are*
   (i) *It is a second degree equation in both x and y.*
   (ii) *Coeff. of x^2 = Coeff of y^2.*
   (iii) *It does not contain any term involving the product xy.*

WORKED OUT EXAMPLES

Example 1. Find the center and radius of the circle $x^2+y^2-8x-10y+7=0$.

Solution

Given equation of circle is $x^2+y^2-8x-10y+7=0$ ... (i)

Comparing equation (i) with $x^2+y^2+2gx+2fy+c=0$, we get,

$g=-4, f=-5, c=7$

$\because$ Centre $=(-g,-f)=(4, 5)$

and radius $= \sqrt{g^2 + f^2 - c}$
$= \sqrt{4^2 + 5^2 - 7}$
$= \sqrt{16 + 25 - 7}$
$= \sqrt{34}$.

Example 2. Find the lengths of the intercepts of the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ on the axes of coordinates.

Solution

The equation of the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ ...(i)

Putting $y = 0$ in (i), we get,

$x^2 - 5x - 14 = 0$

or, $(x - 7)(x + 2) = 0$

$\therefore$ $x = 7, -2$

Hence, the circle cuts off an intercept of 7 units from the positive direction and 2 units from the negative direction of the axes.

$\therefore$ Length of the x-intercept = 9.

Now, putting $x = 0$ in equation (i), we get,

$y^2 - 13y - 14 = 0$

or, $(y - 14)(y + 1) = 0$

$y = 14, y = -1$

$\therefore$ Length of the y-intercept = 15.

Example 3. Find the equation of the circle which is concentric to the circle $x^2 + y^2 - 8x + 12y + 15 = 0$ and passes through (5, 4).

Solution

The equation of the circle concentric with

$x^2 + y^2 - 8x + 12y + 15 = 0$ is

$x^2 + y^2 - 8x + 12y + k = 0$ ... (i)

If (i) passes through (5, 4), then

$5^2 + 4^2 - 8 \times 5 + 12 \times 4 + k = 0$

$\therefore$ $k = -49$

Substituting the value of k in (i)

$x^2 + y^2 - 8x + 12y - 49 = 0$, which is the required equation of circle.

Example 4. Find the equation of a circle passing through the points (1, 2), (3, 1) and (–3, –1).

Solution

Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (i)

If the circle (i) passes through the points (1, 2), (3, 1) and (–3, –1), then

$1^2 + 2^2 + 2g \cdot 1 + 2f \cdot 2 + c = 0$

or, $2g + 4f + c = -5$ ... (ii)

Similarly, $3^2 + 1^2 + 2g \cdot 3 + 2f \cdot 1 + c = 0$

or, $6g + 2f + c = -10$ ... (iii)

And. $(-3)^2 + (-1)^2 + 2g\cdot(-3) + 2\cdot f(-1) + c = 0$

or, $-6g - 2f + c = -10$ ... (iv)

Adding (iii) and (iv), we get,

$$\begin{aligned} 6g + 2f + c &= -10 \\ -6g - 2f + c &= -10 \\ \hline 2c &= -20 \\ c &= -10 \end{aligned}$$

Putting the value of c in (ii) and (iii), we get,

$2g + 4f - 10 = -5$

$2g + 4f = 10 - 5$

$2g + 4f = 5$ ... (v)

And, $6g + 2f - 10 = -10$

$6g + 2f = 0$ ... (vi)

Multiplying equation (v) by 3 and subtracting (vi) from equation (v)

$$\begin{aligned} 6g + 12f &= 15 \\ 6g + 2f &= 0 \\ - \quad - \quad & \quad - \\ \hline 10f &= 15 \end{aligned}$$

$$f = \frac{15}{10} = \frac{3}{2}$$

Putting the value of f in (vi), we get,

$6g + 2 \times \frac{3}{2} = 0$

or, $6g + 3 = 0$

or, $6g = -3$

$g = -\frac{3}{6} = -\frac{1}{2}$

Now, putting the value of f, g & c in (i), we get,

$$x^2 + y^2 + 2\left(-\frac{1}{2}\right)x + 2\cdot\frac{3}{2}\cdot y + (-10) = 0$$

$$x^2 + y^2 - x + 3y - 10 = 0.$$

EXERCISE – 5 A

1. Find the equation of the circle with
   (a) centre (0, 0) and radius 5.
   (b) centre (4, 5) and radius 7.
   (c) centre (2, 3) and touches x-axis.
2. Find the equation of the circle
   (a) whose centre is (a, b) and which passes through the origin.

(b) whose centre is the point (2, 3) and which passes through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

3. Find the equation of the circle which has (1, 3) and (4, 5) as ends of a diameter.

4. Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the x-axis respectively.

5. One end of the diameter of the circle $x^2 + y^2 - 6x + 5y - 7 = 0$ is $(7, -8)$. Find the coordinates of other end.

6. Find the equation of circle concentric with $x^2 + y^2 + x + 2y + 3 = 0$ and through the point (1, 1).

7. Find the equation of circle passing through the following points.

   (a) (0, 0), (4, 0), (0, 3) (b) (5, 5), (6, 4), (–2, 4).

Answers

1. (a) $x^2 + y^2 = 25$ (b) $x^2 + y^2 - 8x - 10y - 8 = 0$

   (c) $x^2 + y^2 - 4x - 6y + 4 = 0$

2. (a) $x^2 + y^2 - 2ax - 2by = 0$ (b) $x^2 + y^2 - 4x - 6y - 12 = 0$

3. $x^2 + y^2 - 5x - 8y + 19 = 0$

4. $x^2 + y^2 - 3x - 4y = 0$

5. $(-1, 3)$

6. $x^2 + y^2 + x + 2y - 5 = 0$

7. (a) $x^2 + y^2 - 4x - 3y = 0$ (b) $x^2 + y^2 - 4x - 2y - 20 = 0$

Objective Questions

1. Which of the following can not be the equation of circle?

   (a) $x^2 + y^2 - 7 = 0$ (b) $x^2 + xy + y^2 = 5$

   (c) $x^2 + y^2 - 7x + 5y = 2$ (d) $x^2 + y^2 = 3x - 2y + 4$

2. The equation of circle in diameter form is

   (a) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = r$

   (b) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

   (c) $(x - x_1)(y - y_1) + (x - x_2)(y - y_2) = r$

   (d) $(x - x_1)(y - y_1) + (x - x_2)(y - y_2) = 0$

3. The equation of circle having centre (0, 0) and diameter 4 is

   (a) $x^2 + y^2 = 0$ (b) $x^2 + y^2 = 2$

   (c) $x^2 + y^2 = 1$ (d) $x^2 + y^2 = 4$

4. The centre of the circle $x^2 + y^2 + 4x - 6y + 4 = 0$ is

   (a) (2, 3) (b) (– 2, 3)

   (c) (– 2, – 3) (d) (– 2, – 3)

5. The radius of the circle $x^2 + y^2 + 4x + 6y + 4 = 0$ is
   (a) 1 (b) 2
   (c) 3 (d) 4
6. The equation of circle concentric with $x^2 + y^2 + 2x + 6y - 39 = 0$ and passing through the point (2, 0) is
   (a) $x^2 + y^2 + 2x + 6y - 8 = 0$ (b) $x^2 + y^2 + 2x + 6y - 12 = 0$
   (c) $x^2 + y^2 + 2x + 6y - 6 = 0$ (d) $x^2 + y^2 + 2x + 6y - 39 = 0$

Answer Sheet

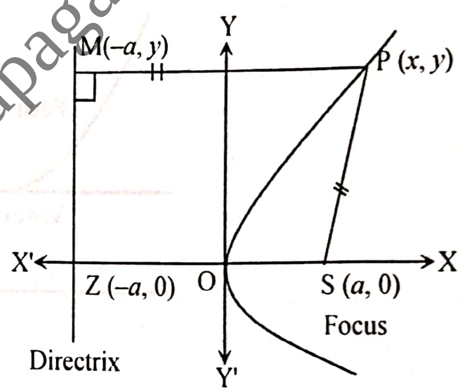
1	2	3	4	5	6	7	8	9	10
b	b	d	b	c	a				

5.3 Parabolas

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is called a parabola. The fixed point is called **focus** and fixed line is called **directrix** of the parabola.

Derivation of Equation of Parabola $y^2 = 4ax$

Let S be the focus and ZM be the directrix of the parabola. Let SZ be drawn perpendicular to ZM. Let O be the middle point of SZ i.e. SO = OZ. Then O is the vertex of the parabola and SZ is the axis of the parabola. Let OS = a. Then the coordinates of S, O and Z be $(a, 0)$, $(0, 0)$ and $(-a, 0)$ respectively.



Let P(x, y) be any point on the parabola. Join PS and draw PM perpendicular to ZM. Then by definition of parabola,

$$PS = PM$$

or, $PS^2 = PM^2$

or, $(x - a)^2 + (y - 0)^2 = (x + a)^2 + (y - y)^2$

or, $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

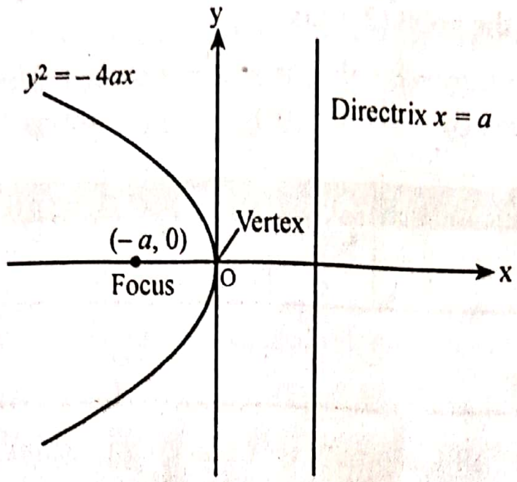
$\therefore$ $y^2 = 4ax$ which is the required equation of parabola in standard form.

The parabola $y^2 = 4ax$ is symmetric about x-axis. It is called the axis of the parabola.

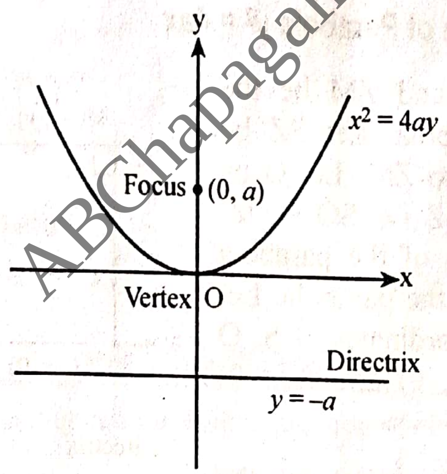
The point where a parabola crosses the axis is called the **vertex**. The vertex of the parabola $y^2 = 4ax$ lies at origin. The positive number a is called the parabola's **focal length**.

Latus rectum: The chord of the conic section through the focus and perpendicular to the axis is called the latus rectum of the conic section.

The Parabola $y^2 = -4ax$



The Parabola $x^2 = 4ay$



The Parabola $x^2 = -4ay$

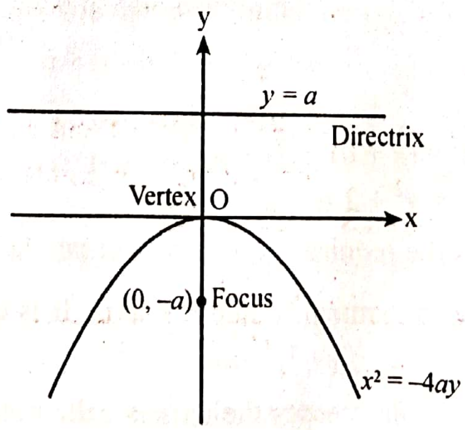


Table: Standard-form equations for parabolas with vertices at origin and $a > 0$.

Equation	Focus	Directrix	Axis	Opens
$y^2 = 4ax$	$(a, 0)$	$x = -a$	x-axis	right
$y^2 = -4ax$	$(-a, 0)$	$x = a$	x-axis	left
$x^2 = 4ay$	$(0, a)$	$y = -a$	y-axis	up
$x^2 = -4ay$	$(0, -a)$	$y = a$	y-axis	down

Example: Find the focus and directrix of the parabola $y^2 = 10x$.

Solution:

Comparing $y^2 = 10x$ with $y^2 = 4ax$, we get,

$$4a = 10$$

or, $a = \frac{5}{2}$

Focus: $(a, 0) = \left(\frac{5}{2}, 0\right)$

Directrix: $x = -a$

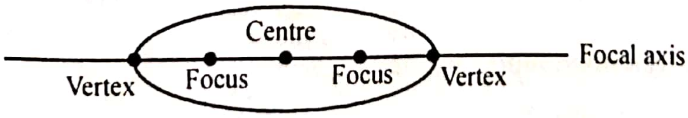
or, $x = \frac{-5}{2}$

$\therefore$ $2x + 5 = 0$.

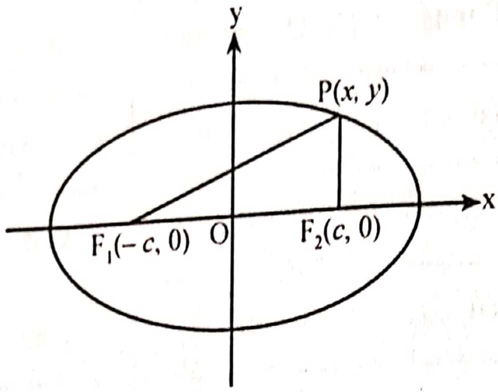
5.4 Ellipses

An ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The line through the foci of an ellipse is called **focal axis**. The point on the axis halfway between the foci is the **centre**. The points where the focal axis and ellipse cross are the ellipse's **vertices**.



Derivation of Equation of Ellipse in Standard Position



Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci. Let $P(x, y)$ be any point on the ellipse. Then by definition, $PF_1 + PF_2$ is constant.

Suppose $PF_1 + PF_2 = 2a$

or, $\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$

(Using distance formula)

or, $\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$

Squaring on both sides,

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2$$

or, $x^2 + 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2xc + c^2 + y^2$

or, $4xc = 4a^2 - 4a\sqrt{(x-c)^2+y^2}$

or, $a\sqrt{(x-c)^2+y^2} = a^2 - cx$

Again, squaring on both sides,

$$a^2\{(x-c)^2 + y^2\} = (a^2 - cx)^2$$

or, $a^2(x^2 - 2xc + c^2 + y^2) = a^4 - 2a^2cx + c^2x^2$

or, $a^2x^2 - 2a^2xc + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2$

or, $a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$

or, $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

Dividing both sides by $a^2(a^2 - c^2)$, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad \ldots \text{(i)}$$

From triangle inequality, we have,

$PF_1 + PF_2 > F_1F_2$

$\Rightarrow \quad 2a > 2c$

$\Rightarrow \quad a > c$

Hence, $a^2 - c^2$ is positive.

If $b = \sqrt{a^2 - c^2}$ then $a^2 - c^2 = b^2$

Then equation (i) can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The Major and Minor Axes of an Ellipse

Consider the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$.

Then the line segment of length $2a$ joining the points $(a, 0)$ & $(-a, 0)$ is called the **major axis**. The line segment of length $2b$ joining the point $(0, b)$ and $(0, -b)$ is called the **minor axis**. The number c, found from the relation $c = \sqrt{a^2 - b^2}$ is the centre to focus distance of the ellipse.

Example: Find the semimajor axis, semiminor axis, centre-to-focus distance, foci, vertices and centre of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. **(Major axis horizontal)**

Solution

Comparing $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

$a^2 = 16 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$

Semimajor axis $(a) = 4$

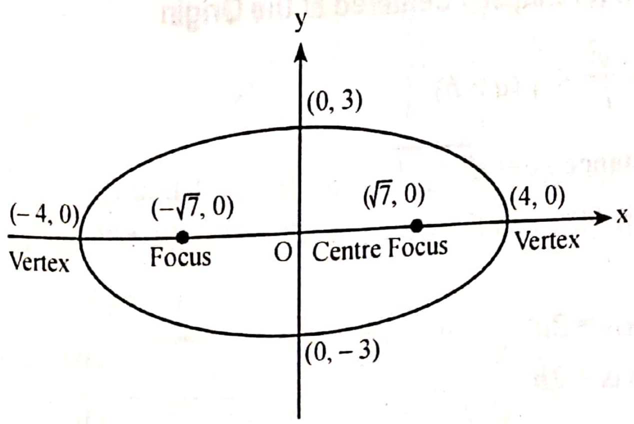
Semiminor axis $(b) = 3$

Centre-to-focus distance $(c) = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(\pm c, 0) = (\pm\sqrt{7}, 0)$

Vertices: $(\pm a, 0) = (\pm 4, 0)$

Centre: $(0, 0)$.



Example: Find the semimajor axis, semiminor axis, centre to focus distance, foci, vertices and centre of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$. **(Major axis vertical)**

Solution

Comparing $\frac{x^2}{9}+\frac{y^2}{16}=1$ with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get,

$a^2=9 \Rightarrow a=3$

$b^2=16 \Rightarrow b=4$

Semimajor axis $(b)=4$

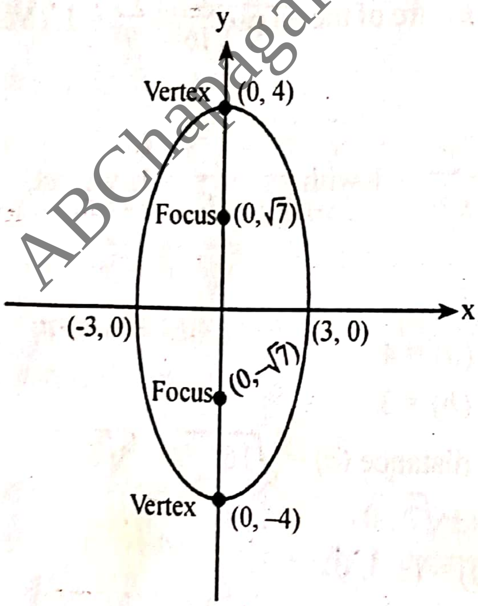
Semiminor axis $(a)=3$

Centre to focus distance $(c)=\sqrt{16-9}=\sqrt{7}$

Foci : $(0, \pm c)=(0, \pm\sqrt{7})$

Vertices: $(0, \pm b)=(0, \pm 4)$

Centre : $(0, 0)$.



Standard-Form Equation for Ellipses Centered at the Origin

Foci on the x-axis: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$

Centre to focus distance : $c=\sqrt{a^2-b^2}$

Foci: $(\pm c, 0)$

Vertices : $(\pm a, 0)$

Length of Major axis $=2a$

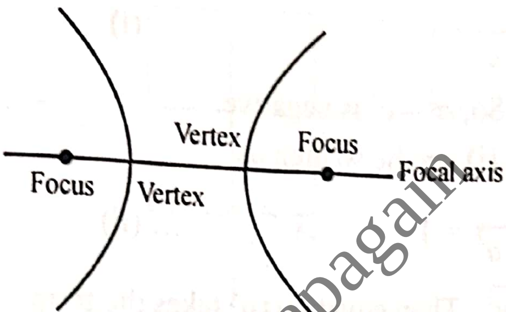
Length of Minor axis $=2b$

Foci on the y-axis: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (b>a)$

Centre to focus distance: $c=\sqrt{b^2-a^2}$
Foci: $(0, \pm c)$
Vertices: $(0, \pm b)$
Length of Major axis = $2b$
Length of Minor axis = $2a$.

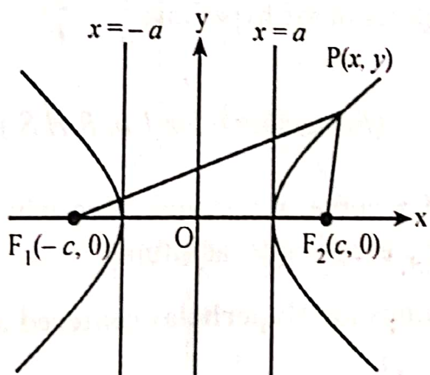
5.5 Hyperbolas

A hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the foci of the hyperbola.



The line through the foci of a hyperbola is called **focal axis.** The point where the focal axis and hyperbola cross are the **vertices.**

Derivation of Equation of Hyperbola in Standard Form



Let $F_1(-c, 0)$ and $F_2(c, 0)$ be two foci and the constant difference be $2a$. Then a point $P(x, y)$ lies on the hyperbola if and only if

$$PF_1 - PF_2 = 2a$$

or, $\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$

or, $\sqrt{(x+c)^2+y^2}=\sqrt{(x-c)^2+y^2}\pm 2a$

Squaring on both sides, we get,

$$(x + c)^2 + y^2 = (x - c)^2 + y^2 \pm 2a\sqrt{(x - c)^2 + y^2} + 4a^2$$

or, $x^2 + 2cx + c^2 + y^2 = x^2 - 2cx + c^2 + y^2 \pm 4a\sqrt{(x - c)^2 + y^2} + 4a^2$

or, $4cx - 4a^2 = \pm 4a\sqrt{(x - c)^2 + y^2}$

or, $cx - a^2 = \pm a\sqrt{(x - c)^2 + y^2}$

Again, squaring on both sides, we get,

$$c^2x^2 - 2a^2cx + a^4 = a^2 (x^2 - 2cx + c^2 + y^2)$$

or, $c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$

or, $a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$

or, $a^2(a^2 - c^2) = x^2 (a^2 - c^2) + a^2y^2$

Dividing both sides by $a^2(a^2 - c^2)$, we get,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad \ldots \text{(i)}$$

Here, $2a < 2c$. So, $a^2 - c^2$ is negative.

Now, equation (i) can be written as

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \qquad \ldots \text{(ii)}$$

Put $b = \sqrt{c^2 - a^2}$. Then equation (ii) takes the form,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Asymptotes of Hyperbolas

The equations of asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad \text{(Remember 0 for 1 in R.H.S.)}$$

Note that asymptote of a curve is a straight line which does not meet the curve but becomes very close to curve at infinity.

Standard - form equations for Hyperbolas centered at the origin.

Foci on the x-axis: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Centre-to-focus distance $(c) = \sqrt{a^2 + b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

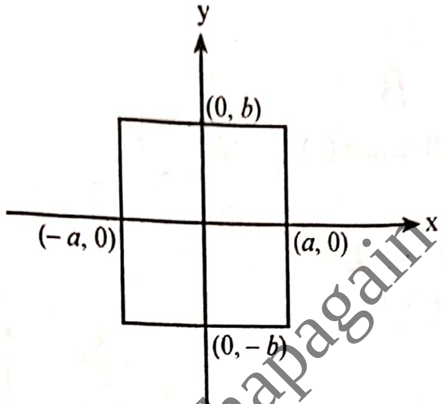
Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Foci on the y-axis: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

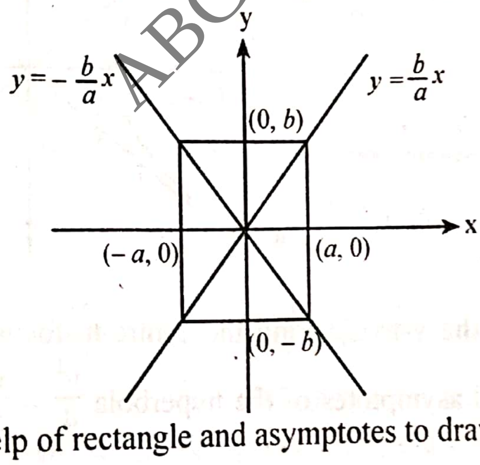
Centre-to-focus distance $(c) = \sqrt{a^2 + b^2}$
Foci: $(0, \pm c)$
Vertices: $(0, \pm a)$
Asymptotes: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$

Steps for drawing the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

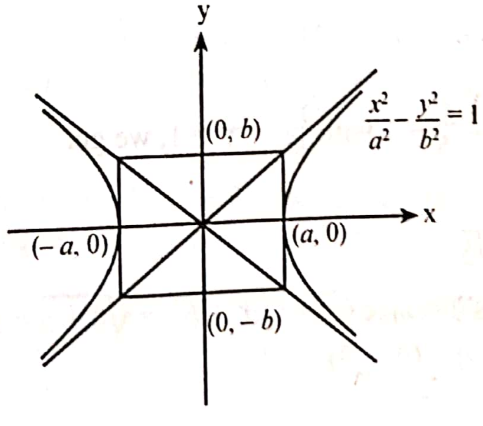
Step 1: Mark the points $(\pm a, 0)$ and $(0, \pm b)$ with the line segments and complete the rectangle.



Step 2: Sketch the asymptotes by extending the rectangle's diagonals.



Step 3: Take the help of rectangle and asymptotes to draw the hyperbola.



The line joining the points $(a, 0)$ and $(-a, 0)$ is called **transverse axis.** The length of transverse axis is $2a$. The line joining the points $(0, b)$ and $(0, -b)$ is called **conjugate axis.** The length of conjugate axis is $2b$.

Example (Foci on the x-axis): Find the centre-to-focus distance, vertices, foci, centre and asymptotes of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$. Also sketch the hyperbola.

Solution

Comparing $\frac{x^2}{4} - \frac{y^2}{5} = 1$ with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 4 \Rightarrow a = 2$

$b^2 = 5 \Rightarrow b = \sqrt{5}$

Centre-to-focus distance (c) $= \sqrt{a^2 + b^2}$

$= \sqrt{4 + 5} = 3$

Vertices: $(\pm a, 0) = (\pm 2, 0)$

Foci: $(\pm c, 0) = (\pm 3, 0)$

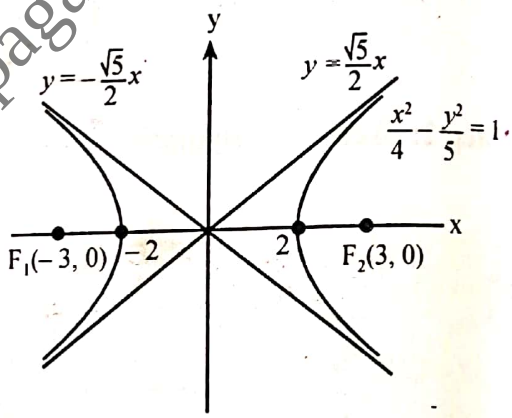
Centre: $(0, 0)$

Asymptotes: $\frac{x^2}{4} - \frac{y^2}{5} = 0$

or, $\frac{y^2}{5} = \frac{x^2}{4}$

or, $y^2 = \frac{5x^2}{4}$

$\therefore \quad y = \pm\frac{\sqrt{5}}{2}x$.



Example (Foci on the y-axis): Find the centre-to-focus distance, vertices, foci, centre and asymptotes of the hyperbola $\frac{y^2}{4} - \frac{x^2}{5} = 1$. Also sketch the hyperbola.

Solution

Comparing $\frac{y^2}{4} - \frac{x^2}{5} = 1$ with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get

$a^2 = 4 \Rightarrow a = 2$

$b^2 = 5 \Rightarrow b = \sqrt{5}$

Centre-to-focus distance $(c) = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$.

Vertices: $(0, \pm a) = (0, \pm 2)$

Foci: $(0, \pm c) = (0, \pm 3)$

Centre: $(0, 0)$

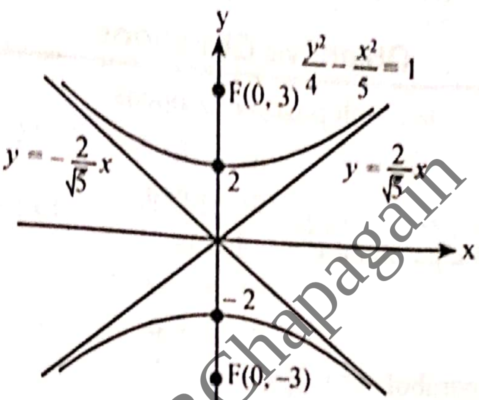
Asymptotes: $\frac{y^2}{4} - \frac{x^2}{5} = 0$

or, $\frac{y^2}{4} = \frac{x^2}{5}$

or, $y^2 = \frac{4x^2}{5}$

$\therefore \quad y = \pm \frac{2}{\sqrt{5}} x.$

Sketch



EXERCISE – 5 B

1. Find focus and directrix of following parabolas. Sketch the parabola including the focus and directrix.

   (a) $y^2 = 12x$ (b) $x = -3y^2$

   (c) $x^2 = 6y$ (d) $y = -8x^2$.

2. Find the vertices, centre-to-focus distance, foci of the ellipse and sketch the graph.

   (a) $16x^2 + 25y^2 = 400$ (b) $2x^2 + 3y^2 = 12$

   (c) $3x^2 + 2y^2 = 6$.

3. Find the centre-to-focus distance, vertices, foci and asymptotes of the following hyperbola. Sketch the graph.

   (a) $x^2 - y^2 = 1$ (b) $9x^2 - 16y^2 = 144$

   (c) $y^2 - x^2 = 4$ (d) $8y^2 - 2x^2 = 16$.

Answers

1. (a) $(3, 0), x = -3$ (b) $\left(-\frac{1}{12}, 0\right), x = \frac{1}{12}$
   (c) $\left(0, \frac{3}{2}\right), y = -\frac{3}{2}$ (d) $\left(0, -\frac{1}{32}\right), y = \frac{1}{32}$
2. (a) $(\pm 5, 0), 3, (\pm 3, 0)$ (b) $(\pm\sqrt{6}, 0), \sqrt{2}, (\pm\sqrt{2}, 0)$
   (c) $(0, \pm\sqrt{3}), 1, (0, \pm 1)$
3. (a) $\sqrt{2}, (\pm 1, 0), (\pm\sqrt{2}, 0), y = \pm x.$ (b) $5, (\pm 4, 0), (\pm 5, 0), y = \pm\frac{3}{4}x.$
   (c) $2\sqrt{2}, (0, \pm 2), (0, \pm 2\sqrt{2}), y = \pm x$
   (d) $\sqrt{10}, (0, \pm 2\sqrt{2}), (0, \pm\sqrt{10}), y = \pm\frac{x}{2}.$

Objective Questions

1. The parabola $y^2 = -4ax$ with positive 'a' opens
   (a) right (b) left
   (c) up (d) down
2. The directrix of the parabola $x^2 = -4ay$ is
   (a) $x = -a$ (b) $x = a$
   (c) $y = -a$ (d) $y = a$
3. The focus of the parabola $y^2 = 4ax$ is
   (a) $(a, 0)$ (b) $(-a, 0)$
   (c) $(0, a)$ (d) $(0, -a)$
4. Centre to focus distance of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$ is
   (a) $\sqrt{a^2 + b^2}$ (b) $\sqrt{a^2 - b^2}$
   (c) $\sqrt{b^2 - a^2}$ (d) a
5. The equations of asymptotes of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ are
   (a) $\frac{x^2}{4} - \frac{y^2}{9} = -1$ (b) $\frac{x^2}{4} - \frac{y^2}{9} = 0$
   (c) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (d) $\frac{x^2}{4} + \frac{y^2}{9} = 0$
6. The foci of the hyperbola $\frac{y^2}{4} - \frac{x^2}{5} = 1$ are
   (a) $(0, \pm 1)$ (b) $(0, \pm 2)$
   (c) $(0, \pm 3)$ (d) $(0, \pm 4)$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	d	a	b	b	c				

5.6 Classifying Conic Sections by Eccentricity

1. The eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ is given by

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

2. The eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

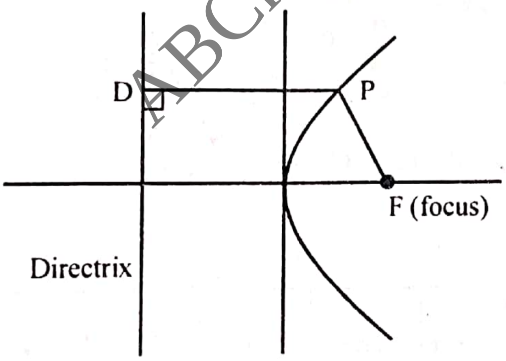
In both ellipse and hyperbola,

$$\text{the eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

3. The eccentricity of a parabola is $e = 1$.

In any conic section the ratio of $\frac{\text{PF}}{\text{PD}}$ is called eccentricity where F is focus, P is any point on the curve and PD is perpendicular to directrix.

Thus, $\frac{\text{PF}}{\text{PD}} = e$.



The conic section is

(a) a circle if $e = 0$.

(b) a parabola if $e = 1$.

(c) an ellipse if $e < 1$.

(d) a hyperbola if $e > 1$.

If the conic section has centre other than origin then we shift the origin to the point say (h, k).

We now give the formulae for the conic section whose centre is at (h, k) and axes parallel to the coordinate axes.

Parabola

Equation of parabola	Vertex	Focus	Equation of directrix	Axis	Length of latus rectum
1. $(y-k)^2 = 4a(x-h)$	(h, k)	$(h+a, k)$	$x = h - a$	$y = k$	$4a$
2. $(x-h)^2 = 4a(y-k)$	(h, k)	$(h, k+a)$	$y = k - a$	$x = h$	$4a$

Example: Find the coordinates of the vertex and the focus of the parabola whose equation is $y^2 = 6y - 12x + 45$.

Solution

Given equation of parabola is

$$y^2 = 6y - 12x + 45$$

or, $y^2 - 6y + 9 = -12x + 45 + 9$

or, $(y-3)^2 = -12x + 54$

or, $(y-3)^2 = -12\left(x - \dfrac{9}{2}\right)$ ... (i)

Comparing (i) with $(y-k)^2 = 4a(x-h)$, we get $h = \dfrac{9}{2}$, $k = 3$

$4a = -12$

$\therefore \quad a = -3$

Focus of the parabola $= (h + a, k)$

$$= \left(\frac{9}{2} - 3, 3\right) = \left(\frac{3}{2}, 3\right)$$

Vertex of the parabola $= (h, k) = \left(\dfrac{9}{2}, 3\right)$.

Ellipse

Equation	Centre	Vertex	Focus	Major axis	Minor axis	Eccentricity	Length of latus rectum	Equation of directrix
1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b > 0$	$(0, 0)$	$(\pm a, 0)$	$(\pm ae, 0)$	$2a$	$2b$	$\sqrt{1 - \dfrac{b^2}{a^2}}$	$\dfrac{2b^2}{a}$	$x = \pm \dfrac{a}{e}$
2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $b > a > 0$	$(0, 0)$	$(0, \pm b)$	$(0, \pm be)$	$2b$	$2a$	$\sqrt{1 - \dfrac{a^2}{b^2}}$	$\dfrac{2a^2}{b}$	$y = \pm \dfrac{b}{e}$
3. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, $a > b > 0$	(h, k)	$(h \pm a, k)$	$(h \pm ae, k)$	$2a$	$2b$	$\sqrt{1 - \dfrac{b^2}{a^2}}$	$\dfrac{2b^2}{a}$	$x = h \pm \dfrac{a}{e}$
4. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, $b > a > 0$	(h, k)	$(h, k \pm b)$	$(h, k \pm be)$	$2b$	$2a$	$\sqrt{1 - \dfrac{a^2}{b^2}}$	$\dfrac{2a^2}{b}$	$y = k \pm \dfrac{b}{e}$

Example: Find the foci of the ellipse $\frac{x^2}{8} + \frac{(y-2)^2}{12} = 1$.

Solution

Given equation of ellipse is $\frac{x^2}{8} + \frac{(y-2)^2}{12} = 1$ ... (i)

Comparing (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we have

$h = 0$, $k = 2$

$a^2 = 8$, $b^2 = 12$

$\Rightarrow a = 2\sqrt{2}$. $b = 2\sqrt{3}$

Since $b > a$, so the major axis is parallel to y-axis.

Eccentricity $(e) = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{8}{12}} = \frac{1}{\sqrt{3}}$

Foci $= (h, k \pm be) = \left(0, 2 \pm 2\sqrt{3} \cdot \frac{1}{\sqrt{3}}\right)$

$= (0, 2 \pm 2) = (0, 0)$ and $(0, 4)$.

Hyperbola

Equation	Centre	Vertex	Focus	Transverse axis	Conjuget axis	Eccentricity	Length of latus rectum	Equatio of directrix
1. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$(0, 0)$	$(\pm a, 0)$	$(\pm ae, 0)$	$2a$	$2b$	$\sqrt{1 + \frac{b^2}{a^2}}$	$\frac{2b^2}{a}$	$x = \pm \frac{a}{e}$
2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$	$(0, 0)$	$(0, \pm b)$	$(0, \pm be)$	$2b$	$2a$	$\sqrt{1 + \frac{a^2}{b^2}}$	$\frac{2a^2}{b}$	$y = \pm \frac{b}{e}$
3. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	(h, k)	$(h \pm a, k)$	$(h \pm ae, k)$	$2a$	$2b$	$\sqrt{1 + \frac{b^2}{a^2}}$	$\frac{2b^2}{a}$	$x = h \pm \frac{a}{e}$
4. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$	(h, k)	$(a, k \pm b)$	$(h, k \pm be)$	$2b$	$2a$	$\sqrt{1 + \frac{a^2}{b^2}}$	$\frac{2a^2}{b}$	$y = k \pm \frac{b}{e}$

Example: Find the eccentricity and the foci of the hyperbola: $3x^2 - 4y^2 = 36$.

Solution

Given equation of the hyperbola is

$$3x^2 - 4y^2 = 36$$

or, $\frac{x^2}{12} - \frac{y^2}{9} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$a^2 = 12$, $\quad b^2 = 9$

$\Rightarrow \quad a = 2\sqrt{3} \quad b = 3$

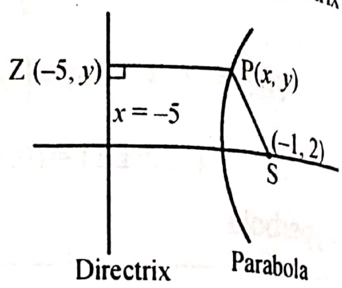
Eccentricity $(e) = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{12}} = \frac{\sqrt{7}}{2}$

Foci $= (\pm ae, 0) = \left(\pm 2\sqrt{3} \cdot \frac{\sqrt{7}}{2}, 0\right) = (\pm\sqrt{21}, 0)$.

WORKED OUT EXAMPLES

Example 1. Find the equation of the parabola with focus at (–1, 2) and directrix $x = -5$.

Solution



If P(x, y) be any point on the parabola then from figure, we have,

PS = PZ [Definition of parabola]

or, $(x + 1)^2 + (y - 2)^2 = (x + 5)^2 + (y - y)^2$

or, $x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 + 10x + 25$

$\therefore \quad y^2 - 4y - 8x - 20 = 0$.

Example 2. Find the eccentricity of the ellipse $16x^2 + 25y^2 = 400$.

Solution

$16x^2 + 25y^2 = 400$

or, $\frac{16x^2}{400} + \frac{25y^2}{400} = \frac{400}{400}$

or, $\frac{x^2}{25} + \frac{y^2}{16} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get,

$a^2 = 25$, $b^2 = 16$

Eccentricity $(e) = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{25 - 16}}{5} = \frac{\sqrt{9}}{5} = \frac{3}{5}$.

Example 3. Find the eccentricity and the foci of the ellipse $25x^2 + 4y^2 = 100$.

Solution

Given equation of ellipse is

$25x^2 + 4y^2 = 100$

or, $\frac{25x^2}{100} + \frac{4y^2}{100} = 1$

or, $\frac{x^2}{4} + \frac{y^2}{25} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 4,\ b^2 = 25$

$\Rightarrow\quad a = 2,\ b = 5$

Since $b > a$, so the major axis is along y-axis

$$\text{Eccentricity } (e) = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$$

$$\text{Foci} = (0, \pm be) = \left(0, 5 . \frac{\sqrt{21}}{5}\right) = (0, \pm\sqrt{21}).$$

Example 4. Show that: $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ represents the equation of an ellipse. Find its centre, vertex and focus.

Solution

Given equation is

$$9x^2 + 4y^2 - 18x - 16y - 11 = 0$$

or, $9(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 36$

or, $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$ ...(i)

Since equation (i) is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, it represents an equation of ellipse.

Here,

$h = 1,\quad k = 2$

$a^2 = 4,\quad b^2 = 9$

$\Rightarrow\quad a = 2,\quad b = 3$

Since $b > a$, so the major axis is parallel to y-axis.

Centre $= (h, k) = (1, 2)$

Vertex $= (h, k \pm b)$

$= (1, 2 \pm 3) = (1, 2 - 3)$ and $(1, 2 + 3)$

$= (1, -1)$ and $(1, 5)$

$$\text{Eccentricity } (e) = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$$

$$\text{Focus} = (h, k \pm be) = \left(1, 2 \pm 3\frac{\sqrt{5}}{3}\right) = (1, 2 \pm \sqrt{5}).$$

Example 5. Find the equation of the ellipse whose latus rectum is 3 and eccentricity is $\frac{1}{\sqrt{2}}$.

Solution

Latus rectum = 3

or, $\frac{2b^2}{a} = 3$

$\therefore \quad b^2 = \frac{3a}{2}$ ... (i)

Eccentricity $(e) = \frac{1}{\sqrt{2}}$

We have

$$e^2 = 1 - \frac{b^2}{a^2}$$

or, $\left(\frac{1}{\sqrt{2}}\right) = 1 - \frac{3\frac{a}{2}}{a^2}$

or, $\frac{1}{2} = 1 - \frac{3}{2a}$

or, $\frac{3}{2a} = \frac{1}{2}$

$\therefore \quad a = 3$

Now from (i)

$$b^2 = \frac{3 \times 3}{2} = \frac{9}{2}$$

The equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{9} + \frac{y^2}{9/2} = 1$

or, $\frac{x^2}{9} + \frac{2y^2}{9} = 1$

$\therefore \quad x^2 + 2y^2 = 9.$

Example 6. Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.

Solution

Given hyperbola is

$$9x^2 - 16y^2 = 144$$

or, $\frac{9x^2}{144} - \frac{16y^2}{144} = \frac{144}{144}$

or, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get,

$$a^2 = 16, b^2 = 9$$

We have,

$$c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$$

Eccentricity $(e) = \frac{c}{a} = \frac{5}{4}$.

Example 7. Find the equation of the hyperbola with focus at (–5, 0) and vertex at (2, 0).

Solution

Vertex $(a, 0) = (2, 0)$

$\therefore \quad a = 2$

Focus $(-ae, 0) = (-5, 0)$

or, $\quad ae = 5$

or, $\quad 2 \cdot e = 5 \qquad [\because a = 2]$

$\therefore \quad e = \frac{5}{2}$

We have,

$$e^2 = 1 + \frac{b^2}{a^2}$$

or, $\quad \left(\frac{5}{2}\right)^2 = 1 + \frac{b^2}{4}$

or, $\quad \frac{25}{4} = \frac{4 + b^2}{4}$

$\therefore \quad b^2 = 21$

Hence, the required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or, $\quad \frac{x^2}{4} - \frac{y^2}{21} = 1$

$\therefore \quad 21x^2 - 4y^2 = 84.$

Example 8. Find the vertices, centre, eccentricity and foci of the hyperbola $9(x - 1)^2 - 16(y + 2)^2 = 144$.

Solution

Given hyperbola is

$$9(x - 1)^2 - 16(y + 2)^2 = 144$$

Dividing both sides by 144, we get,

$$\frac{9(x - 1)^2}{144} - \frac{16(y + 2)^2}{144} = \frac{144}{144}$$

or, $\quad \frac{(x - 1)^2}{16} - \frac{(y + 2)^2}{9} = 1 \qquad \dots (1)$

Comparing (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get,

$$h = 1, k = -2$$
$$a^2 = 16, b^2 = 9$$
$$\therefore \quad a = 4, b = 3$$

Vertices $= (h \pm a, k)$
$= (1 \pm 4, -2)$
$= (5, -2)$ and $(-3, -2)$

Centre $= (h, k) = (1, -2)$

Eccentricity $(e) = \sqrt{1 + \frac{b^2}{a^2}}$
$= \sqrt{1 + \frac{9}{16}}$
$= \sqrt{\frac{16+9}{16}}$
$= \sqrt{\frac{25}{16}} = \frac{5}{4}$

Foci $= (h \pm ae, k)$
$= (1 \pm 4 \cdot \frac{5}{4}, -2)$
$= (1 \pm 5, -2)$
$= (1 + 5, -2)$ and $(1 - 5, -2)$
$= (6, -2)$ and $(-4, -2)$.

Example 9. Find equation for the hyperbola centered at the origin that has a focus at (3, 0) and the line $x = 1$ as the corresponding directrix.

Solution

Focus $(c, 0) = (3, 0)$
$\Rightarrow \quad c = 3$

The directrix is $x = \frac{a}{e}$

or, $\quad 1 = \frac{a}{e} \quad [\because x = 1]$

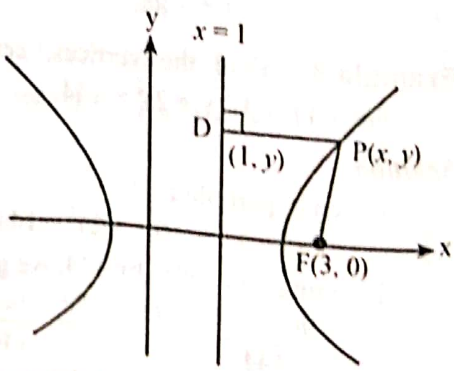
or, $\quad a = e$

Again,

$$e = \frac{c}{a} = \frac{3}{e}$$

or, $\quad e^2 = 3$

or, $\quad e = \sqrt{3}$.



The equation of hyperbola is

PF $= e \cdot$ PD. where P(x, y) is any point on the hyperbola.

or, $\quad \sqrt{(x-3)^2 + (y-0)^2} = \sqrt{3}\ \sqrt{(x-1)^2 + (y-y)^2}$

Squaring both sides, we get,

$$(x-3)^2 + y^2 = 3\,(x-1)^2$$

or, $x^2 - 6x + 9 + y^2 = 3x^2 - 6x + 3$

or, $2x^2 - y^2 = 6$

$\therefore \quad \dfrac{x^2}{3} - \dfrac{y^2}{6} = 1.$

EXERCISE – 5 C

1. Find eccentricity, foci and directrices of the following ellipses.
   (a) $6x^2 + 9y^2 = 54$ (b) $2x^2 + y^2 = 2$ (c) $3x^2 + 2y^2 = 6.$
2. Find the equation of ellipse in standard form:
   (a) Foci: $(\pm 8, 0)$
   Eccentricity: 0.2
   (b) Focus: $(\sqrt{5}, 0)$
   Directrix: $x = \dfrac{9}{\sqrt{5}}$
   (c) Foci : $(0, \pm 3)$
   Eccentricity: 0.5.
3. Find the eccentricity, foci and directrices of the hyperbolas.
   (a) $x^2 - y^2 = 1$ (b) $8x^2 - 2y^2 = 16$ (c) $y^2 - x^2 = 8.$
4. Find the equation of hyperbola in standard form.
   (a) Eccentricity: $\dfrac{5}{4}$
   Vertices: $(\pm 4, 0)$
   (b) Eccentricity: 3
   Vertices: $(0, \pm 1)$
   (c) Focus: $(-2, 0)$
   Directrix: $x = -\dfrac{1}{2}$.
5. Find the vertex, focus and directrix of the parabolas.
   (a) $(y+2)^2 = 8(x-1)$ (b) $(x-1)^2 = 4(y+2).$
6. Find the equation of parabola with focus at $(2, 1)$ and directrix $x = -1$.
7. Find the equation of the parabola in which the ends of the latus rectum have the coordinates $(-1, 5)$ and $(-1, -11)$ and the vertex is $(-5, -3)$.
8. Find centre, vertices, eccentricity and foci of the ellipses.
   (a) $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$ (b) $9x^2 + 6y^2 + 36y = 0.$
9. Find centre, vertices, eccentricity and foci of the hyperbolas.
   (a) $\dfrac{(x-2)^2}{16} - \dfrac{y^2}{9} = 1$ (b) $9(x-1)^2 - 16(y+2)^2 = 144.$

Answers

1. (a) $\frac{1}{\sqrt{3}}, (\pm\sqrt{3}, 0), x = \pm 3\sqrt{3}$ (b) $\frac{1}{\sqrt{2}}, (0, \pm 1), y = \pm 2.$

   (c) $\frac{1}{\sqrt{3}}, (0, \pm 1), y = \pm 3.$

2. (a) $\frac{x^2}{1600} + \frac{y^2}{1536} = 1$ (b) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (c) $\frac{x^2}{27} + \frac{y^2}{36} = 1.$

3. (a) $\sqrt{2}, (\pm\sqrt{2}, 0), x = \pm\frac{1}{\sqrt{2}}$ (b) $\sqrt{5}, (\pm\sqrt{10}, 0), x = \pm\sqrt{\frac{2}{5}}$

   (c) $\sqrt{2}, (0, \pm 4), y = \pm 2.$

4. (a) $9x^2 - 16y^2 = 144$ (b) $8y^2 - x^2 = 1$ (c) $3x^2 - y^2 = 3.$

5. (a) $(1, -2), (3, -2), x = -1$ (b) $(1, -2), (1, -1), y = -3$

6. $y^2 - 6x - 2y + 4 = 0$ 7. $y^2 + 6y - 16x - 71 = 0$

8. (a) $(-2, 1); (1, 1)$ and $(-5, 1); \frac{\sqrt{5}}{3}; (-2 \pm \sqrt{5}, 1).$

   (b) $(0, -3); (0, 0)$ and $(0, -6); \frac{1}{\sqrt{3}}; (0, -3 \pm \sqrt{2}).$

9. (a) $(2, 0); (6, 0)$ and $(-2, 0); \frac{5}{4}; (7, 0)$ and $(-3, 0).$

   (b) $(1, -2); (5, -2)$ and $(-3, -2); \frac{5}{4}; (6, -2)$ and $(-4, -2)$

Objective Questions

1. The conic section becomes a parabola if
   (a) $e = 0$ (b) $e = 1$
   (c) $e > 1$ (d) $e < 1$

2. The eccentricity of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is
   (a) $\frac{3}{5}$ (b) $\frac{3}{7}$
   (c) $\frac{4}{5}$ (d) $\frac{5}{7}$

3. The eccentricity of hyperbola $21x^2 - 4y^2 = 84$ is
   (a) $\frac{2}{5}$ (b) $\frac{5}{2}$
   (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

4. The equation of directrix of $\frac{x^2}{3} - \frac{y^2}{6} = 1$ is
   (a) $x = 4$ (b) $x = 3$
   (c) $x = 2$ (d) $x = 1$

5. The vertices of an ellipse whose foci are $(0, \pm 7)$ and eccentricity $\frac{4}{5}$ is

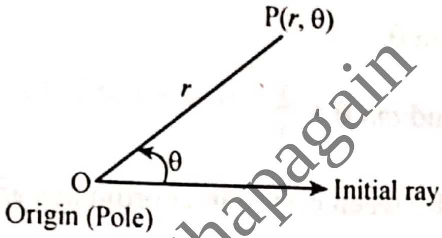
(a) $(0, \pm 4.75)$
(b) $(0, \pm 5.75)$
(c) $(0, \pm 7.75)$
(d) $(0, \pm 8.75)$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	a	b	d	d					

5.7 Polar Coordinates

Let us fix an origin O (called the pole) and an initial ray from O. Then each point P can be located by assigning to it a polar coordinate pair (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

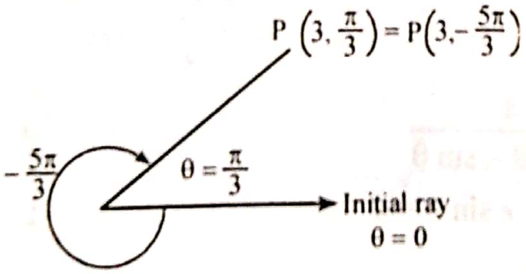


Note:

$P(r, \theta)$

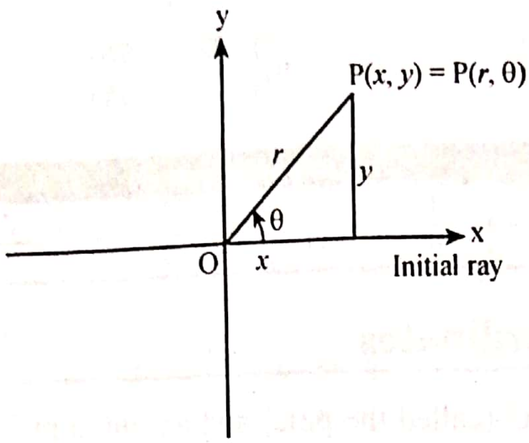
Directed distance from O to P. *Directed angle from initial ray to OP.*

Here θ is positive when measured anticlockwise and negative when measured clockwise. The angle associated with a given point is not unique. For example the point 3 units from the origin along the ray $\theta = \frac{\pi}{3}$ has polar coordinates $r = 3, \theta = \frac{\pi}{3}$. It also has polar coordinates $r = 3, \theta = -\frac{5\pi}{3}$.



Thus, polar coordinates are not unique.

Equations Relating Polar and Cartesian Coordinates



From right angled triangle,

$$\cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}$$

$\therefore$ $x = r\cos\theta,\ y = r\sin\theta$

Also, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$.

These are the relations between cartesian coordinates and polar coordinates.

Example: Find a polar equation for the curve $x^2 + (y-3)^2 = 9$.

Solution

$$x^2 + (y-3)^2 = 9$$

or, $x^2 + y^2 - 6y + 9 = 9$

or, $x^2 + y^2 - 6y = 0$

or, $x^2 + y^2 = 6y$

or, $r^2 = 6 \cdot r\sin\theta$

$\therefore$ $r = 6\sin\theta$.

Example: Change the polar equation $r = \dfrac{4}{2\cos\theta - \sin\theta}$ into cartesian form.

Solution

$$r = \frac{4}{2\cos\theta - \sin\theta}$$

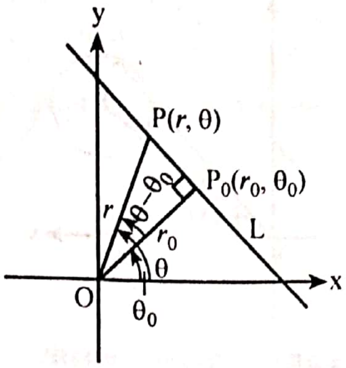
or, $2r\cos\theta - r\sin\theta = 4$

$\therefore$ $2x - y = 4$.

5.8 Polar Equations for Lines and Circles

Lines

Let the line be denoted by L. Suppose the perpendicular from origin to line L meets the line L at the points P_0 (r_0, θ_0), with $r_0 \geq 0$ as shown in the figure.



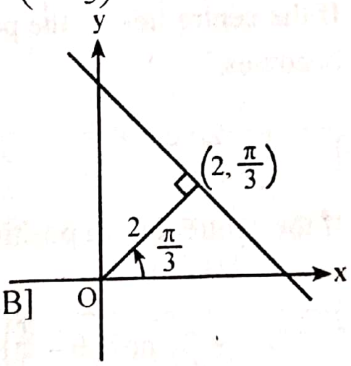
If $P(r, \theta)$ be any other point on the line L then Δ POP_0 is a right angled triangle.

Now, from right angled Δ POP_0, we have,

$$\cos(\theta - \theta_0) = \frac{\text{base}}{\text{hypotenuses}} = \frac{r_0}{r}$$

$\therefore$ $r\cos(\theta - \theta_0) = r_0$ which is the required equation of line L in polar form.

Example: Find a cartesian equation for the line $r\cos\left(\theta - \frac{\pi}{3}\right) = 2$.



Solution

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

or, $r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 2$

[$\because$ $\cos(A - B) = \cos A\cos B + \sin A\sin B$]

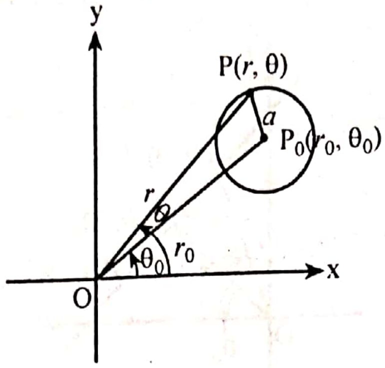
or, $r\cos\theta \cdot \frac{1}{2} + r\sin\theta \cdot \frac{\sqrt{3}}{2} = 2$

or, $x \cdot \frac{1}{2} + y \cdot \frac{\sqrt{3}}{2} = 2$

$\therefore$ $x + \sqrt{3}y = 4$.

Circles

Let centre of the circle be at $P_0(r_0, \theta_0)$ and radius be a. Let $P(r, \theta)$ be any point on the circle.



Applying cosine law to $\Delta\ OP_0P$, we have,

$$a^2 = r_0^2 + r^2 - 2r_0 r \cos(\theta - \theta_0) \quad \ldots \text{(i)}$$

If the circle passes through the origin, then $r_0 = a$. In this case, equation (i) becomes,

$$a^2 = a^2 + r^2 - 2ar \cos(\theta - \theta_0)$$

or, $r^2 = 2ar \cos(\theta - \theta_0)$

$$\therefore \quad r = 2a \cos(\theta - \theta_0) \quad \ldots \text{(ii)}$$

If the centre lies on the positive x-axis, then $\theta_0 = 0$. In this case, equation (ii) becomes,

$$r = 2a \cos\theta \quad \ldots \text{(iii)}$$

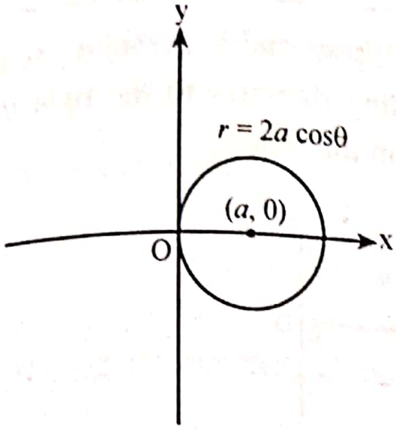
If the centre lies on positive y-axis, then $\theta_0 = \frac{\pi}{2}$. Then equation (ii) becomes

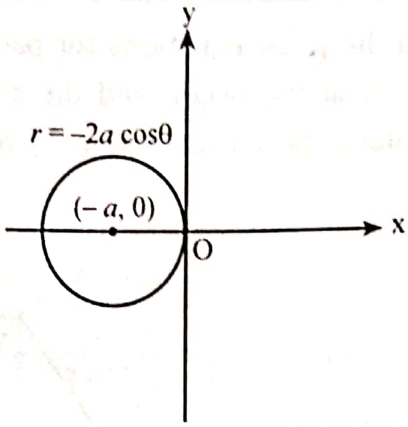
$$r = 2a \cos\left(\theta - \frac{\pi}{2}\right)$$

$$= 2a \cos\left(\frac{\pi}{2} - \theta\right) \quad [\because \cos(-\theta) = \cos\theta]$$

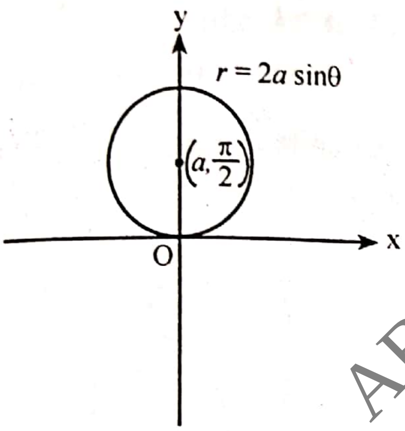
$$r = 2a \sin\theta.$$

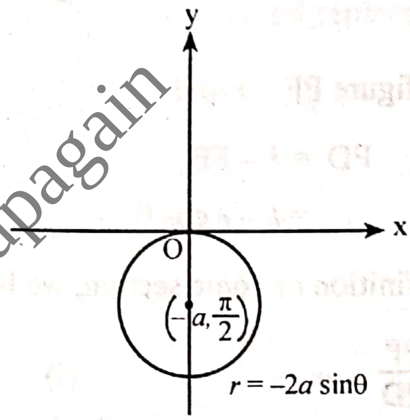
Polar equations for circles through origin centered on x-axis.





Polar equations for circles through origin centred on y-axis.



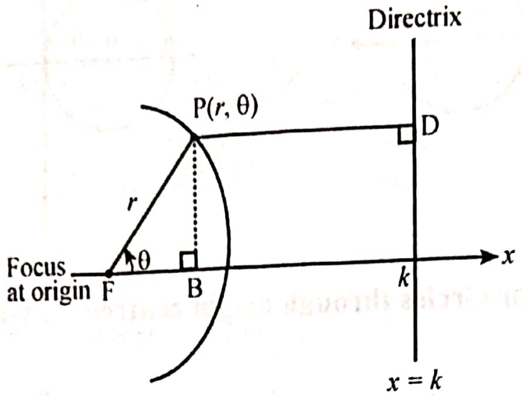


Example: Circle through origin

Radius	Polar coordinates	Polar equations
3	(3, 0)	$r = 6 \cos \theta$
2	$\left(2, \frac{\pi}{2}\right)$	$r = 4 \sin \theta$
$\frac{1}{2}$	$\left(-\frac{1}{2}, 0\right)$	$r = -\cos \theta$
1	$\left(-1, \frac{\pi}{2}\right)$	$r = -2 \sin \theta$

5.9 Ellipses, Parabolas and Hyperbolas in Polar Coordinates

To find the polar equations for parabolas, ellipses and hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$ as shown in the figure.



From figure PF = r and

$$PD = k - FB$$
$$= k - r\cos\theta$$

By definition of conic section, we have,

$$\frac{PF}{PD} = e \quad \text{... (i)}$$

Substituting the value of PF and PD in equation (i).

$$\frac{r}{k - r\cos\theta} = e$$

or, $r = ke - re\cos\theta$

or, $r + r\,e\cos\theta = ke$

or, $r(1 + e\cos\theta) = ke$

$\therefore \quad r = \dfrac{ke}{1 + e\cos\theta}.$

This equation represents an ellipse if $0 < e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$.

Example: (Typical Conics)

Polar equation for conic section is

$$r = \frac{ke}{1 + e\cos\theta} \qquad \dots \text{(i)}$$

If $e = \frac{1}{2}$, the equation of ellipse is $r = \frac{k}{2 + \cos\theta}$

If $e = 1$, the equation of parabola is $r = \frac{k}{1 + \cos\theta}$

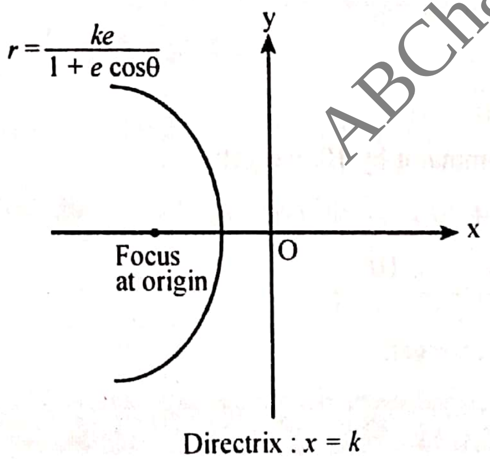
If $e = 2$, the equation of hyperbola is $r = \frac{2k}{1 + 2\cos\theta}$

If the directrix is the line $x = -k$ to the left of the origin, then the equation of the conic section is $r = \frac{ke}{1 - e\cos\theta}$.

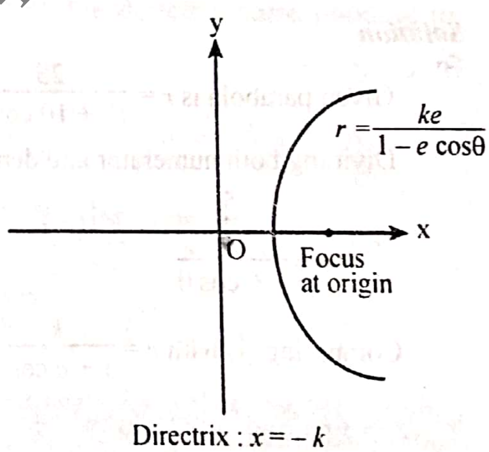
If the directrix is either of the lines $y = k$ or $y = -k$, we should replace cosine by sine in above formulae.

Equations for conic section ($e > 0$)

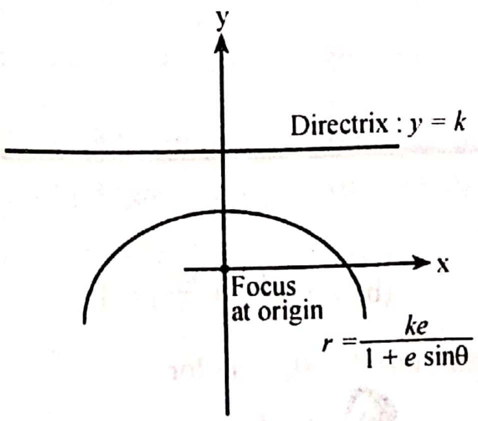
1.



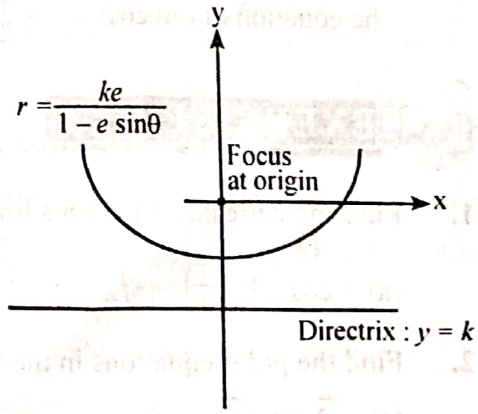
2.



3.



4.



WORKED OUT EXAMPLES

Example 1. Find a polar equation for the hyperbola with eccentricity $\frac{3}{2}$ and directrix $x = 2$.

Solution

Here,

$k = 2$ and $e = \frac{3}{2}$

We know the equation of hyperbola in polar form is

$$r = \frac{ke}{1 + e\cos\theta}$$

$$\text{or,}\quad r = \frac{2 \cdot \frac{3}{2}}{1 + \frac{3}{2}\cos\theta}$$

$$\therefore\quad r = \frac{6}{2 + 3\cos\theta}.$$

Example 2. Find the directrix of the parabola $r = \frac{25}{10 + 10\cos\theta}$.

Solution

Given parabola is $r = \frac{25}{10 + 10\cos\theta}$

Dividing both numerator and denominator by 10, we get,

$$r = \frac{\frac{5}{2}}{1 + \cos\theta} \qquad \ldots \text{(i)}$$

Comparing (i) with $r = \frac{ke}{1 + e\cos\theta}$, we get,

$k = \frac{5}{2}$ and $e = 1$.

The equation of directrix is $x = \frac{5}{2}$.

EXERCISE – 5 D

1. Find the cartesian equations for

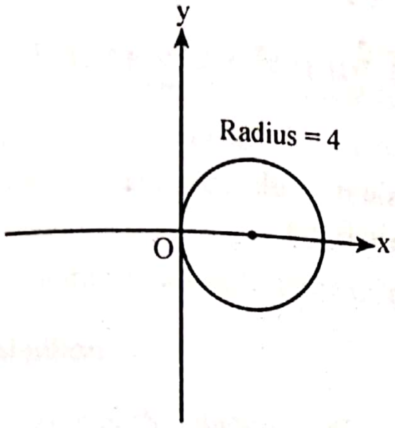
(a) $r\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}$ (b) $r\cos\left(\theta + \frac{2\pi}{3}\right) = 3$.

2. Find the polar equations in the form $r\cos(\theta - \theta_0) = r_0$ for

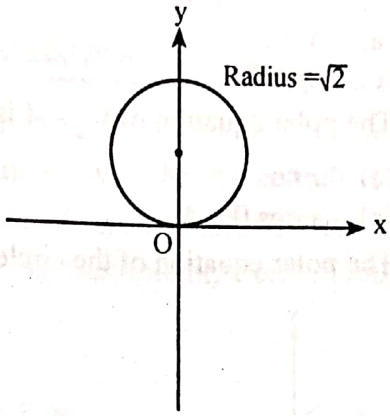
(a) $\sqrt{2}\,x + \sqrt{2}\,y = 6$ (b) $y = -5$.

3. Find polar equations for the following circles.

(a)



(b)



4. Sketch the circle.

(a) $r = 3\cos\theta$ (b) $r = -2\sin\theta$.

5. Find the polar equation for

(a) $(x-6)^2 + y^2 = 36$ (b) $x^2 + (y-5)^2 = 25$.

6. Find polar equation for each conic section, where the eccentricities of conic section with one focus at origin along with the directrix corresponding to that focus are given.

(a) $e = 1, x = 2$ (b) $e = \frac{1}{2}, x = 1$ (c) $e = 5, y = -6$.

7. Find the eccentricity and directrix of the following conic.

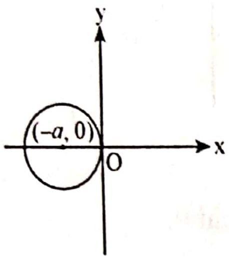
(a) $r = \frac{1}{1 + \cos\theta}$ (b) $r = \frac{25}{10 - 5\cos\theta}$ (c) $r = \frac{400}{16 + 8\sin\theta}$.

Answers

1. (a) $x + y = 2$ (b) $x + \sqrt{3}y + 6 = 0$
2. (a) $r\cos\left(\theta - \frac{\pi}{4}\right) = 3$ (b) $r\cos\left(\theta + \frac{\pi}{2}\right) = 5$
3. (a) $r = 8\cos\theta$ (b) $r = 2\sqrt{2}\sin\theta$
5. (a) $r = 12\cos\theta$ (b) $r = 10\sin\theta$
6. (a) $r = \frac{2}{1 + \cos\theta}$ (b) $r = \frac{1}{2 + \cos\theta}$ (c) $r = \frac{30}{1 - 5\sin\theta}$
7. (a) $e = 1, x = 1$ (b) $e = \frac{1}{2}, x = -5$ (c) $e = \frac{1}{2}, y = 50$

Objective Questions

1. The cartesian equation of $r \sin \theta = 2$ is
   - (a) $x = 2$
   - (b) $y = 2$
   - (c) $xy = 2$
   - (d) $y = x$
2. The polar equation of $x = -4$ is
   - (a) $r \cos \theta = -4$
   - (b) $r \sin \theta = -4$
   - (c) $r \cos \theta = 4$
   - (d) $r \sin \theta = 4$
3. The polar equation of the circle



is

   - (a) $r = 2a \cos \theta$
   - (b) $r = 2a \sin \theta$
   - (c) $r = -2a \cos \theta$
   - (d) $r = -2a \sin \theta$

4. The radius of the circle $r = 6 \cos \theta$ is
   - (a) 1
   - (b) 2
   - (c) 3
   - (d) 4
5. The equation $r = \dfrac{5}{1 + \cos \theta}$ represents
   - (a) circle
   - (b) ellipse
   - (c) hyperbola
   - (d) parabola
6. The equation $r = \dfrac{400}{16 + 8 \cos \theta}$ represents
   - (a) circle
   - (b) ellipse
   - (c) hyperbola
   - (d) parabola

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	a	c	c	d	b				

Vectors in Space

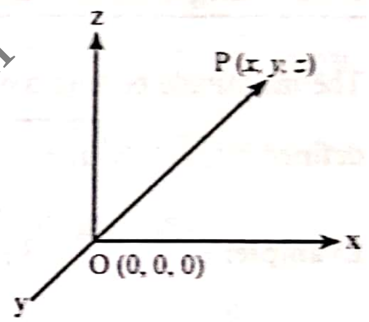
5.10 Introduction and Applications

Measurable quantities in science can be categorized in two types vectors and scalars. Quantities that have direction as well as magnitude are called vectors. For example force, velocity etc. A **scalar quantity** has magnitude only without any reference of direction. For example time and speed, etc.

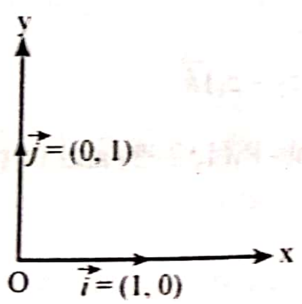
Definition

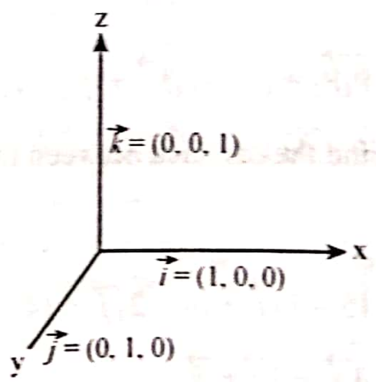
A vector in the plane is a directed line segment.

For the directed line $\overrightarrow{OP}$, the point O is called the initial point (origin) and P is called the terminal point (terminus). The arrow shows the direction of the vector and length of OP is the magnitude of vector $\overrightarrow{OP}$. In three dimensional space, if O is the initial point and P(x, y, z) is the terminal point then $\overrightarrow{OP} = (x, y, z)$. The $\overrightarrow{OP}$ vector is called the space vector.



Unit Vectors $\vec{i}, \vec{j}, \vec{k}$





The unit vectors $\vec{i}, \vec{j}, \vec{k}$ are mutually perpendicular to each other.

Notes: (a) Every plane vector can be written in the form $(a_1, a_2) = a_1\vec{i} + a_2\vec{j}$.

(b) Every space vector can be written in the form $(a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

5.11 Algebra of Vectors in Space

Let $\vec{v_1} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{v_2} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ be two vectors in space and k be a scalar.

1. Addition: $\vec{v_1} + \vec{v_2} = (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$.
2. Scalar multiplication: $k\vec{v_1} = k(a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$

$$= (ka_1)\vec{i} + (ka_2)\vec{j} + (ka_3)\vec{k}.$$

Equality

Two vectors are equal if they have the same magnitude (length) and direction.

5.12 Magnitude (Length) of a Vector

The magnitude or length of a vector $\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, denoted by $|\vec{v}|$, is defined by $|\vec{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Example: If $\vec{v} = 2\vec{i} + 3\vec{j} + 2\vec{k}$ then $|\vec{v}| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{18} = 3\sqrt{2}$.

5.13 Distance between Two Points

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2\ (x_2 + y_2, z_3)$ is denoted by $|\overrightarrow{P_1P_2}|$ and given by $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

where $\overrightarrow{P_1P_2} = (x_1 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$.

Example: Find the distance between the points $P_1(1, 2, 3)$ and $P_2\ (5, 4, 4)$.

Solution

$$\overrightarrow{P_1P_2} = (5 - 1)\vec{i} + (4 - 2)\vec{j} + (4 - 3)\vec{k}$$

$$= 4\vec{i} + 2\vec{j} + \vec{k}$$

So, $|\overrightarrow{P_1P_2}| = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$.

5.14 Unit Vector

Any vector having length 1 is called a unit vector. The vectors $\vec{i}, \vec{j}, \vec{k}$ are unit vectors. If $\vec{v} \neq 0$ then $\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the direction of $\vec{v}$. It is denoted by $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$.

Length vs Direction

If $\vec{v} \neq 0$ then,

(i) $\frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the director of $\vec{v}$.

(ii) The equation $\vec{v} = |\vec{v}| \frac{\vec{v}}{|\vec{v}|}$ expresses $\vec{v}$ in terms of its length (magnitude) and direction.

Example: Express $\vec{v} = 4\vec{i} + 3\vec{j}$ as a product of its length and direction.

Solution

Here, $\vec{v} = 4\vec{i} + 3\vec{j}$

Length of $\vec{v} = |\vec{v}| = \sqrt{4^2 + 3^2} = 5$

Direction of $\vec{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{4\vec{i} + 3\vec{j}}{5} = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$.

$$\therefore \quad \vec{v} = 4\vec{i} + 3\vec{j} = \underset{\text{Length}}{\overset{5}{\uparrow}} \quad \underset{\text{Direction}}{\left(\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}\right)}.$$

Example: Find a vector 7 units long in the direction of $\vec{u} = 2\vec{i} + \vec{j} + 2\vec{k}$.

Solution

$$\text{Required vector is } 7\frac{\vec{u}}{|\vec{u}|} = 7\left(\frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{2^2 + 1^2 + 2^2}}\right)$$

$$= \frac{14}{3}\vec{i} + \frac{7}{3}\vec{j} + \frac{14}{3}\vec{k}.$$

5.15 Zero (Null) Vector

A vector having length zero is called a zero vector.

Note that $\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is a zero vector if $a_1 = a_2 = a_3 = 0$.

Vector from a Point to another Point

The vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is $\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$ $+ \ (z_2 + z_1)\vec{k}$.

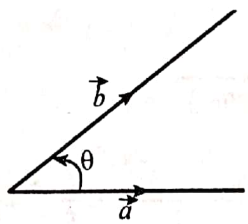
For example: The vector from $P_1(1, 2, 3)$ to $P_2(5, 4, 4)$ is

$$\begin{aligned}\overrightarrow{P_1P_2} &= (5-1)\vec{i} + (4-2)\vec{i} + (4-3)\vec{i} \\ &= 4\vec{i} + 2\vec{j} + \vec{k}.\end{aligned}$$

5.16 Dot (Scalar) Product and its Geometrical Interpretation

The scalar (dot) product of two vectors $\vec{a}$ and $\vec{b}$ denoted by

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.



Derivation of Dot Product $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

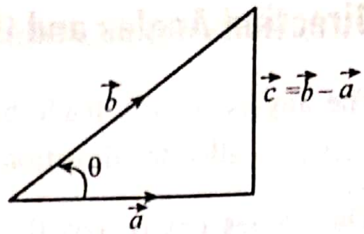
Put $\vec{c} = \vec{b} - \vec{a} = (b_1 - a_1)\vec{i} + (b_2 - a_2)\vec{j} + (b_3 - a_3)\vec{k}$.

Then, $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$,

$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$|\vec{b} - \vec{a}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

From cosine law, we have



$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}| \cos\theta$$

or, $$|\vec{a}|\,|\vec{b}| \cos\theta = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2}{2}$$

or, $$\vec{a}\cdot\vec{b} = \frac{a_1^2 + b_1^2 + c_1^2 + a_2^2 + b_2^2 + c_2^2 - \{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2\}}{2}$$

or, $$\vec{a}\cdot\vec{b} = \frac{a_1^2 + b_1^2 + c_1^2 + a_2^2 + b_2^2 + c_2^2 - b_1^2 + 2a_1b_1 - a_1^2 - b_2^2 + 2a_2b_2 - a_2^2 + b_3^2 + 2a_3b_3 - a_3^2}{2}$$

or, $$\vec{a}\cdot\vec{b} = \frac{2a_1b_1 + 2a_2b_2 + 2a_3b_3}{2}$$

or, $$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

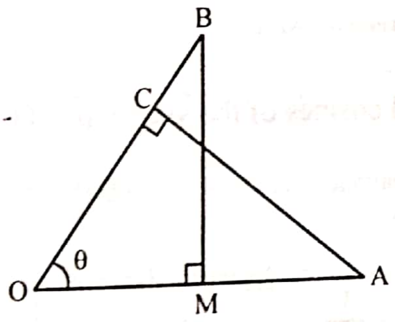
$\therefore$ $$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Geometrical Interpretation

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ be two vectors and θ be the angle between them.

Then, $|\overrightarrow{OA}| = |\vec{a}| = OA = a$, $|\overrightarrow{OB}| = |\vec{b}| = OB = b$

Draw AL ⊥ OB, BM ⊥ OA.



Now, $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$= (OA)\,(OB) \cos\theta$

$= (OA)\,(OM) = |\vec{a}|$ (Projection of $\vec{b}$ on $\vec{a}$)

Similarly $\vec{a}\cdot\vec{b} = |\vec{b}|$(Projection of $\vec{a}$ on $\vec{b}$)

Thus $\vec{a}\cdot\vec{b} = |\vec{a}|$ (Projection of $\vec{b}$ on $\vec{a}$) = $|\vec{b}|$ (Projection of $\vec{a}$ on $\vec{b}$).

Direction Angles and Direction Cosines

The angles α, β, γ made by the non zero vectors with positive coordinates axes are called the direction angles.

The values cos α, cos β and cos γ are called the direction cosines of the vector.

We have,

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|},$$

$$\cos\beta = \frac{\vec{a}\cdot\vec{j}}{|\vec{a}||\vec{j}|} = \frac{a_2}{|\vec{a}|}$$

and $\cos\gamma = \dfrac{\vec{a}\cdot\vec{k}}{|\vec{a}||\vec{k}|} = \dfrac{a_3}{|\vec{a}|}$. where $\vec{a} = (a_1, a_2, a_3)$

Note: *(i)* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(ii) The coefficient of $\vec{i}, \vec{j}$ *and* $\vec{k}$ *in the unit vectors are the direction cosines of the vectors.*

(iii) $\vec{a} = |\vec{a}|\,(\cos\alpha, \cos\beta, \cos\gamma)$

(iv) direction angles are usually in between 0 and π.

Example: Find the direction angles and cosines of the vector $\vec{a} = (1, 0, 1)$.

Solution

Here, $|\vec{a}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$.

So the direction cosines are $\cos\alpha = \dfrac{1}{\sqrt{2}}$, $\cos\beta = 0$, $\cos\gamma = \dfrac{1}{\sqrt{2}}$

Also, α = 45°, β = 90°, γ = 45° are the direction angles.

Note: *Scalar Projection of* $\vec{b}$ *on* $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Vector Projection of $\vec{b}$ *on* $\vec{a} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\left(\dfrac{\vec{a}}{|\vec{a}|}\right) = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$.

Example: Find the scalar projection and vector projection of $\vec{b} = (1, 1, 2)$ onto $\vec{a} = (-2, 3, 1)$.

Solution

$$|\vec{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$$

$$\vec{a}\cdot\vec{b} = (-2, 3, 1)\cdot(1, 1, 2)$$
$$= -2 + 3 + 2 = 3$$

The scalar projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$= \frac{3}{\sqrt{14}}$$

The vector projection of $\vec{b}$ on $\vec{a} = \dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\vec{a}$

$$= \frac{3}{(\sqrt{14})^2}(-2, 3, 1)$$

$$= \frac{3}{14}(-2, 3, 1)$$

$$= \left(-\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right).$$

Note: The angle between two non zero vectors $\vec{a}$ and $\vec{b}$ is $\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$.

Laws of Dot (Scalar) Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors.

(a) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative property)

(b) $(k\vec{a})\cdot\vec{b} = \vec{a}\cdot k\vec{b} = k(\vec{a}\cdot\vec{b})$ where k is a scalar

(c) $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (Left distributive)

(d) $(\vec{a} + \vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$ (Right distributive)

Some identities

(a) $(\vec{a} \pm \vec{b})^2 = a^2 \pm 2\vec{a}\cdot\vec{b} + b^2$

(b) $(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = a^2 - b^2$

Perpendicular (Orthogonal) Vectors

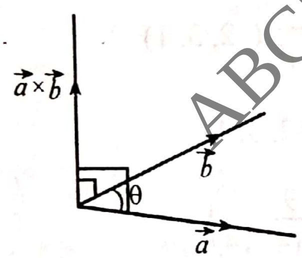
Non zero vectors $\vec{a}$ and $\vec{b}$ are perpendicular (orthogonal) if and only if $\vec{a}\cdot\vec{b} = 0$.

Example: $\vec{a} = 3\vec{i} + \vec{k}$ and $\vec{b} = -\vec{i} + \vec{j} + 3\vec{k}$ are orthogonal because

$\vec{a}\cdot\vec{b} = 3(-1) + 0(1) + 1(3) = 0.$

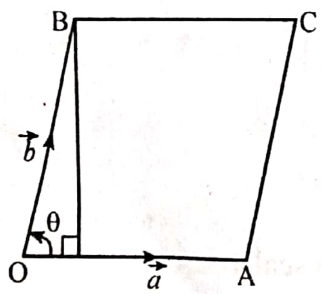
5.17 Cross (Vector) Product and its Geometrical Interpretation

The cross product of two non zero $\vec{a}$ and $\vec{b}$ is a vector defined by $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector in the direction of $\vec{a} \times \vec{b}$.



Note: The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Geometrical Interpretation



Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\measuredangle AOB = \theta$.

Draw a parallelogram OACB as shown in the figure. Also, draw BD perpendicular to OA.

$$\begin{aligned}\text{Area of parallelogram OACB} &= (\text{Base})\,(\text{height}) \\ &= (OA)\,(OD) \\ &= (OA)\,(OB \sin\theta) \\ &= ab\sin\theta \\ &= |\vec{a}\times\vec{b}|.\end{aligned}$$

Hence, $|\vec{a}\times\vec{b}|$ gives the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

Since the area of the triangle is half the area of parallelogram standing on the same base and between the same parallel lines, area of $\Delta OAB = \frac{1}{2}\,|\vec{a}\times\vec{b}|$.

Calculation for $\vec{a}\times\vec{b}$

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then

$$\begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$$

$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$.

Derivation of $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ be any two vectors.

Then,

$$\begin{aligned}|\vec{a}\times\vec{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^3 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2\,b_3^2 + a_1^2\,b_2^2 - 2a_1a_2b_1b_2 + a_2^2\,b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)\,(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \\ &= |\vec{a}|^2\,|\vec{b}|^2 - |\vec{a}|^2\,|\vec{b}|^2\cos^2\theta \\ &= |\vec{a}|^2\,|\vec{b}|^2\sin^2\theta\end{aligned}$$

or, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

$$\therefore \sin\theta = \frac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}|}.$$

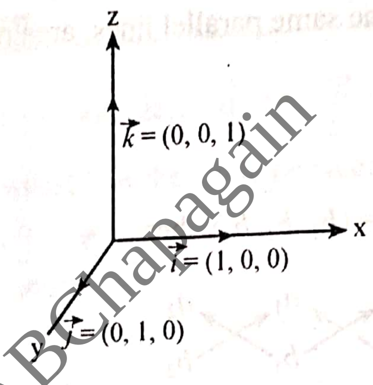
Parallel Vectors

Non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = 0$.

Cross Product of Unit Vectors

$$\vec{i} \times \vec{i} = 0, \vec{j} \times \vec{j} = 0, \vec{k} \times \vec{k} = 0$$

$$\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$$



But $\vec{j} \times \vec{i} = -\vec{k}, \vec{k} \times \vec{j} = -\vec{i}, \vec{i} \times \vec{k} = -\vec{j}$.

Associative law

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.$$

Commutative law

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \text{ but } \vec{a} \times \vec{b} = -(\vec{b} \times \vec{a}).$$

Distributive law

Let $\vec{a}, \vec{b}$ *and* $\vec{c}$ be three vectors.

(a) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (Left distributive law).

(b) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ (Right distributive law).

Also, $(-\vec{a}) \times \vec{b} = \vec{a} \times (-\vec{b}) = -a(\vec{a} \times \vec{b})$.

Determinant Formula for $\vec{a} \times \vec{b}$

Let, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$.

$$\begin{aligned}
\vec{a} \times \vec{b} &= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\
&= a_1b_1(\vec{i}\times\vec{i}) + a_1b_2(\vec{i}\times\vec{j}) + a_1b_3(\vec{i}\times\vec{k}) + a_2b_1(\vec{j}\times\vec{i}) + a_2b_2 \\
&\quad (\vec{j}\times\vec{j}) + a_2b_3(\vec{j}\times\vec{k}) + a_3b_1(\vec{k}\times\vec{i}) + a_3b_2(\vec{k}\times\vec{j}) + a_3b_3(\vec{k}\times\vec{k}) \\
&= 0 + a_1b_1\vec{k} + a_1b_3(-\vec{j}) + a_2b_1(-\vec{k}) + 0 + a_2b_3\vec{i} + a_3b_1\vec{j} + \\
&\quad a_3b_2(-\vec{j}) + 0 \\
&= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.
\end{aligned}$$

Example: Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ if $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 5\vec{k}$.

Solution

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ -2 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 1 \\ -2 & 5 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}\vec{k} \\
&= (10 - 1)\vec{i} - (5 + 2)\vec{j} + (1 + 4)\vec{k} \\
&= 9\vec{i} - 7\vec{j} + 5\vec{k} \\
\vec{b} \times \vec{a} &= -(\vec{a} \times \vec{b}) \\
&= -9\vec{i} + 7\vec{j} - 5\vec{k}.
\end{aligned}$$

Example: Find the area of a triangle with vertices are A(1, – 1, 2), B(2, 0, – 1) and C(0, 2, 1).

Solution

The area of the triangle is given by $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$\overrightarrow{AB} = 2 - 1)\vec{i} + (0 + 1)\vec{j} + (-1 - 2)\vec{k} = \vec{i} + \vec{j} - 3\vec{k}$

$\overrightarrow{AC} = (0 - 1)\vec{i} + (2 + 1)\vec{j} + (1 - 2)\vec{k} = -\vec{i} + 3\vec{j} - \vec{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 \\ 3 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} \vec{k}$$

$$= (-1 + 9)\vec{i} - (-1 - 3)\vec{j} + (3 + 1)\vec{k}$$

$$= 8\vec{i} + 4\vec{j} + 4\vec{k}.$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}\sqrt{8^2 + 4^2 + 4^2}$$

$$= \frac{1}{2}\sqrt{96}$$

$$= \frac{1}{2}(4\sqrt{6})$$

$$= 2\sqrt{6}.$$

5.18 Scalar and Vector Triple Product

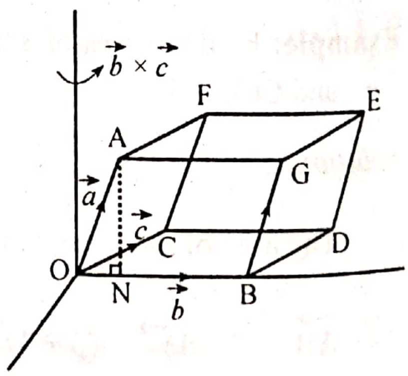
Scalar Triple Product

The scalar triple product of three vectors $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ and is given by

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Geometrical Interpretation of Scalar Triple Product

Let us consider a parallelopiped with three concurrent edges of OA, OB and OC where $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$.



Then, the magnitude of $\vec{b} \times \vec{c}$ is the area of parallelogram OBDC and the direction of $\vec{b} \times \vec{c}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

Now, $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

$= |\vec{b}\times\vec{c}|$ (projection of $\vec{a}$ on $\vec{b}\times\vec{c}$)

= (Area of parallelogram OBDC) (AN)

= (Area of parallelogram OBDC) (Height)

= Volume of parallelopiped.

Thus, the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ always represents the volume of parallelopiped forming by the sides which are represented by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Properties of Scalar Triple Product

1. The position of dot and cross can be interchanged without changing the value of scalar triple product. That is, if $\vec{a}, \vec{b}, \vec{c}$ be any three vectors then

   $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$.

2. The scalar triple product remains unaltered if the cyclic order of the vectors is maintained. That is, if $\vec{a}, \vec{b}, \vec{c}$ be any three vectors then

   $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$.

3. The conditions, when the value of the scalar triple product is zero are
   (a) when two of the vectors are equal.
   (b) when two of the vectors are parallel.
   (c) when the vectors are coplanar.

Example: For $\vec{a} = (1, 4, -7)$, $\vec{b} = (2, -1, 4)$ and $\vec{c} = (0, -9, 18)$; find $[\vec{a}\ \vec{b}\ \vec{c}]$.

Solution

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 2\begin{vmatrix} 4 & -7 \\ -9 & 18 \end{vmatrix}$$

$$= 1(-18 + 36) - 2(72 - 63)$$

$$= 18 - 18$$

$$= 0.$$

Vector Triple Product

The vector triple product of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ denoted by $\vec{a} \times (\vec{b} \times \vec{c})$ and is given by $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

Geometrical Interpretation of Vector Triple Product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar vectors, then $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$. Hence their dot product is zero.

i.e. $\quad \vec{a} \times (\vec{b} \times \vec{c}) \cdot (\vec{b} \times \vec{c}) = 0$

or, $\quad \vec{u} \cdot (\vec{b} \times \vec{c}) = 0 \qquad \ldots$ (i) $\quad$ where, $\vec{u} = \vec{a} \times (\vec{b} \times \vec{c})$.

We know that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ if $\vec{a}$ is coplanar with $\vec{b}$ and $\vec{c}$. So, from (i), we can say that $\vec{u}$ is coplanar with $\vec{b}$ and $\vec{c}$.

Hence, $\vec{a} \times (\vec{b} \times \vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$.

Example: For $\vec{a} = (1, 4, -7)$, $\vec{b} = (2, -1, 4)$ and $\vec{c} = (0, -9, 18)$; find $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution

$$\begin{aligned}\vec{a} \cdot \vec{c} &= (1, 4, -7) \cdot (0, -9, 18)\\ &= -36 - 126\\ &= -162\end{aligned}$$

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (1, 4, -7) \cdot (2, -1, 4)\\ &= 2 - 4 - 28\\ &= -30\end{aligned}$$

and $$\begin{aligned}\vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}\\ &= -162\,(2, -1, 4) + 30\,(0, -9, 18)\\ &= (-324, -108, -108).\end{aligned}$$

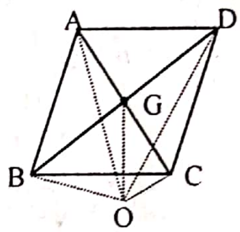
WORKED OUT EXAMPLES

Example 1. ABCD is a parallelogram. G is the point of intersection of its diagonals and if O is any point, show that: $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 4\,\vec{OG}$.

Solution

Given, ABCD is a parallelogram. Also G is the point of intersection of its diagonals and O is any point.

Now,



$$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}$$

$$= (\vec{OG} + \vec{GA}) + (\vec{OG} + \vec{GB}) + (\vec{OG} + \vec{GC}) + (\vec{OG} + \vec{GD})$$

$$= 4\vec{OG} + (\vec{GA} + \vec{GC}) + (\vec{GD} + \vec{GB})$$

$$= 4\vec{OG} + (\vec{GA} - \vec{GA}) + (\vec{GD} - \vec{GD})$$

($\because$ The diagonals of parallelogram bisect each other)

$$= 4\vec{OG}.$$

Example 2. Find a vector perpendicular to the plane of P(1, – 1. 0). Q(2, 1, – 1) and R (–1, 1, 2).

Solution

The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to the both vectors.

$$\vec{PQ} = (2-1)\vec{i} + (1+1)\vec{j} + (-1-0)\vec{k} = \vec{i} + 2\vec{j} - \vec{k}$$

$$\vec{PR} = (-1-1)\vec{i} + (1+1)\vec{j} + (2-0)\vec{k} = -2\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\vec{k}$$

$= 6\vec{i} + 6\vec{k}$ which is the required vector.

Example 3. Find a unit vector $\vec{u}$ in the direction of the vector from P(1, 0, 0) to Q(3, 2, 1).

Solution

$$\vec{PQ} = (3-1)\vec{i} + (2-0)\vec{j} + (1-0)\vec{k}$$

$$= 2\vec{i} + 2\vec{j} + \vec{k}$$

$$|\vec{PQ}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\therefore \quad \vec{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

$$= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}.$$

Example 4. If $\vec{a} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{b} = \vec{i} + \vec{j} + 2\vec{k}$ then find $\vec{a}\cdot\vec{b}$ and the angle between $\vec{a}$ and $\vec{b}$.

Solution

$$\vec{a}\cdot\vec{b} = (2\vec{i} + \vec{j} + \vec{k})\cdot(\vec{i} + \vec{j} + 2\vec{k})$$
$$= (2)(1) + (1)(1) + (1)(2)$$
$$= 2 + 1 + 2 = 5$$

$$|\vec{a}| = \sqrt{2^2 + 1^1 + 1^2} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

We know,

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{5}{6}\right)$$

Example 5. Show that the vectors $2\vec{i} + 3\vec{j} - 8\vec{k}$ and $2\vec{i} + 4\vec{j} + 2\vec{k}$ are orthogonal.

Solution

Let $\vec{a} = 2\vec{i} + 3\vec{j} - 8\vec{k}$

$\vec{b} = 2\vec{i} + 4\vec{j} + 2\vec{k}$

Now,

$$\vec{a}\cdot\vec{b} = (2\vec{i} + 3\vec{j} - 8\vec{k})\cdot(2\vec{i} + 4\vec{j} + 2\vec{k})$$
$$= 4 + 12 - 16 = 0$$

This shows that $\vec{a}$ and $\vec{b}$ are orthogonal.

Example 6. Find the angle between two vectors:

$\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$.

Solution

Here, $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$

$\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$

$$|\vec{a}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$$

$$\vec{a} \cdot \vec{b} = (\vec{i} + \vec{j} - 2\vec{k}) \cdot (2\vec{i} - \vec{j} - \vec{k})$$
$$= 2 - 1 + 2 = 3$$

If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{3}{(\sqrt{6})(\sqrt{6})} = \frac{3}{6} = \frac{1}{2} = \cos 60°$$

$\therefore \quad \theta = 60°.$

Example 7. Show that vector product $\vec{a} \times \vec{b}$ is perpendicular to both vectors $\vec{a}$ and $\vec{b}$.

Solution

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

Then,

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}$$

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

Now, $(\vec{a} \times \vec{b}) \cdot \vec{a}$

$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot (a_1, a_2, a_3)$

$= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$

$= a_1a_2b_3 - a_1b_2a_3 + b_1a_2a_3 - a_1a_2b_3 + a_1b_2a_3 - b_1a_2a_3$

$= 0$

$\therefore \quad \vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Similarly, $(\vec{a} \times \vec{b}) \cdot \vec{b}$

$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot (b_1, b_2, b_3)$

$= b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1)$

$= 0$

$\therefore \quad \vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

Example 8. If $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$, prove that $|\vec{a}| = |\vec{b}|$.

Solution

Here, $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

or, $\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 0$

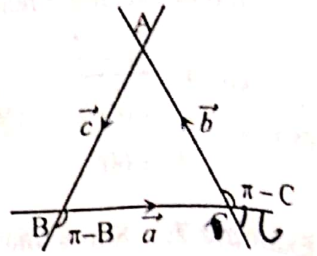
or, $|\vec{a}|^2 - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - |\vec{b}|^2 = 0 \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$

or, $|\vec{a}|^2 = |\vec{b}|^2$

$\therefore \quad |\vec{a}| = |\vec{b}|.$

Example 9. Using vector method, prove in any triangle ABC that:
$a = b\cos C + c\cos B$.

Solution



In ΔABC, let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

By definition of vector addition, we have

$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$$

or, $\vec{a} = -\vec{c} - \vec{b}$

Multiplying both sides scalarly by $\vec{a}$, we have,

$$\vec{a}\cdot\vec{a} = -\vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{b}$$

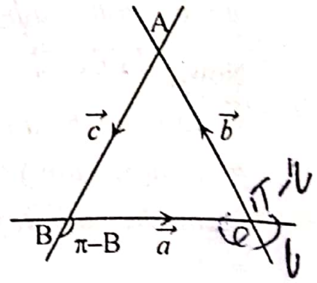
or, $a^2 = -ac\cos(\pi - B) - ab\cos(\pi - C)$

or, $a^2 = ac\cos B + ab\cos C$

$\therefore$ $a = c\cos B + b\cos C$.

Example 10. Using vector method, prove in any triangle ABC, that:
$b^2 = c^2 + a^2 - 2ac\cos B$.

Solution



In ΔABC, let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$

By definition of vector addition, we have

$$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$$

or, $\vec{b} = -\vec{a} - \vec{c}$

or, $b^2 = (-\vec{a} - \vec{c})^2 = (\vec{a} + \vec{c})^2$

$= a^2 + 2\vec{a}\cdot\vec{c} + c^2$

$= a^2 + c^2 + 2ac\cos(\pi - B)$

$\therefore$ $b^2 = a^2 + c^2 - 2ac\cos B$.

Example 11. If $\vec{a}$ and $\vec{b}$ are two vectors of unit length and θ is the angle between them. Show that $\frac{1}{2}|\vec{a} - \vec{b}| = \sin\frac{\theta}{2}$.

Solution

$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})^2$

$= a^2 - 2\vec{a}\cdot\vec{b} + b^2$

$= 1 - 2\vec{a}\cdot\vec{b} + 1 \quad (\because |\vec{a}| = |\vec{b}| = 1)$

$= 2 - 2|\vec{a}||\vec{b}|\cos\theta \qquad [\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$

$= 2 - 2\cos\theta = 2(1 - \cos\theta) = 2\left(2\sin^2\frac{\theta}{2}\right)$

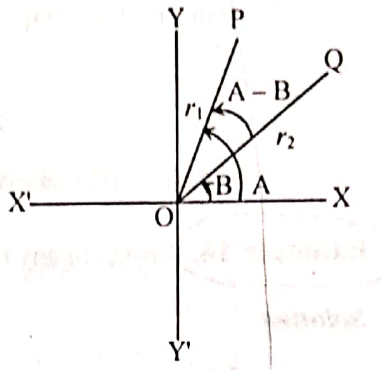
or. $|\vec{a}-\vec{b}|^2 = \left(2\sin^2\frac{\theta}{2}\right)$

or. $|\vec{a}-\vec{b}| = 2\sin\frac{\theta}{2}$

$\therefore \quad \frac{1}{2}|\vec{a}-\vec{b}| = \sin\frac{\theta}{2}.$

Example 12. Prove by vector method: cos (A – B) = cos A cos B + sin A sin B.

Solution



Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let $\measuredangle XOQ = B$ and $\measuredangle XOP = A$ so that $\measuredangle QOP = A - B$. Again, let $OP = r_1$ and $OQ = r_2$. Then the coordinates of P and Q are $(r_1 \cos A, r_1 \sin A)$ and $(r_2 \cos B, r_2 \sin B)$ respectively. So,

$\vec{OP} = (r_1 \cos A, r_1 \sin A)$

$\vec{OQ} = (r_2 \cos B, r_2 \sin B)$

Now,

$$\begin{aligned}\vec{OP}\cdot\vec{OQ} &= (r_1 \cos A, r_1 \sin A)\cdot(r_2 \cos B, r_2 \sin B)\\ &= r_1 r_2 \cos A \cos B + r_1 r_2 \sin A \sin B\\ &= r_1 r_2 (\cos A \cos B + \sin A \sin B)\end{aligned}$$

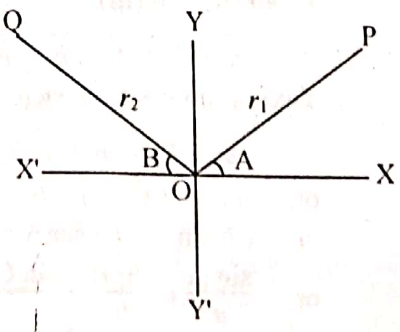
Here, (A – B) is the angle between $\vec{OQ}$ and $\vec{OP}$.

$$\begin{aligned}\therefore \quad \cos(A-B) &= \frac{\vec{OQ}\cdot\vec{OP}}{|\vec{OQ}|\,|\vec{OP}|}\\ &= \frac{r_1 r_2 (\cos A \cos B + \sin A \sin B)}{r_1 r_2}\\ &= \cos A \cos B + \sin A \sin B.\end{aligned}$$

Example 13. Prove by vector method: sin (A + B) = sin A cos B + cos A sin B.

Solution

Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let $\measuredangle XOP = A$ and $\measuredangle QOX' = B$ so that $\measuredangle POQ = \pi - (A + B)$. Also, let $OP = r_1$ and $OQ = r_2$.



Then the coordinates of P and Q are $(r_1 \cos A, r_1 \sin A)$ and $(r_2 \cos(\pi - B), r_2 \sin(\pi - B)) = (-r_2 \cos B, r_2 \sin B)$.

So, $\vec{OP} = (r_1 \cos A, r_1 \sin A)$
$= (r_1 \cos A, r_1 \sin A, 0)$

and $\vec{OQ} = (-r_2 \cos B, r_2 \sin B) = (-r_2 \cos B, r_2 \sin B, 0)$.

Now,

$$\begin{array}{cccccc} r_1 \cos A & r_1 \sin A & 0 & r_1 \cos A & r_1 \sin A \\ -r_2 \cos B & r_2 \sin B & 0 & -r_2 \cos B & r_2 \sin B \end{array}$$

$$\vec{OP} \times \vec{OQ} = (0, 0, r_1 r_2 \cos A \sin B + r_1 r_2 \sin A \cos B)$$

$$|\vec{OP} \times \vec{OQ}| = r_1 r_2(\sin A \cos B + \cos A \sin B)$$

Since $\pi - (A + B)$ is the angle between OP and OQ, so

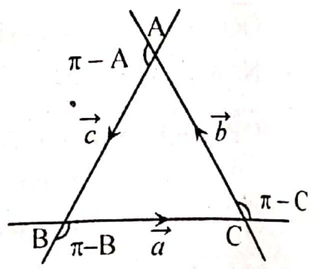
$$\sin [\pi - (A + B)] = \frac{|\vec{OP} \times \vec{OQ}|}{|\vec{OP}|\,|\vec{OQ}|}$$

$$= \frac{r_1 r_2 (\sin A \cos B + \cos A \sin B)}{r_1 r_2}$$

$\therefore \quad \sin (A + B) = \sin A \cos B + \cos A \sin B.$

Example 14. Prove, in any triangle, by vector method that: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Solution



In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, and $\vec{AB} = \vec{c}$

By definition of vector addition,

$$\vec{AB} = \vec{AC} + \vec{CB}$$

or, $\quad \vec{c} = -\vec{b} - \vec{a}$

or, $\quad \vec{a} + \vec{b} + \vec{c} = 0 \quad \ldots (i)$

From (i)

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$

or, $\quad 0 + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c} \qquad (\because \vec{a} \times \vec{a} = 0)$

or, $\quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad \ldots (ii)$

Again, from (i)

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{b} + \vec{c} \times \vec{b} = 0$$

or, $\quad \vec{a} \times \vec{b} + 0 = -\vec{c} \times \vec{b}$

or, $\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} \quad \ldots (iii)$

From (ii) and (iii),

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

Taking modulus on each side, we have

$$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

or, $\quad ab \sin (\pi - C) = bc \sin (\pi - A) = ca \sin (\pi - B)$

or, $\quad ab \sin c = bc \sin A = ca \sin B$

or, $\quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Example 15. If $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + \vec{k}$, find $[\vec{a}\ \vec{b}\ \vec{c}]$.

Solution

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(1 - 0) - 2(2 - 1) + 3(0 - 1)$$

$$= 1 - 2 - 3$$

$$= -4.$$

Example 16. Find the volume of parallelopiped whose concurrent edges are determined by the vectors $\vec{a} = (1, -2, 3)$, $\vec{b} = (2, 1, -1)$ and $\vec{c} = (0, 1, 1)$.

Solution

$$\text{Volume of parallelopiped} = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 1(1 + 1) - 2(-2 - 3)$$

$$= 2 + 10$$

$$= 12.$$

Example 17. If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ then find $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (-1 - 2)\vec{i} - (-2 - 1)\vec{j} + (4 - 1)\vec{k}$$

$$= -3\vec{i} + 3\vec{j} + 3\vec{k}.$$

$$\text{Now, } \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -2 & 1 \\ 3 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix}$$

$$= (-6 - 3)\vec{i} - (3 + 3)\vec{j} + (3 - 6)\vec{k}$$

$$= -9\vec{i} - 6\vec{j} - 3\vec{k}.$$

EXERCISE – 5 E

1. Find a vector with representation given by the directed line segment $\overrightarrow{AB}$.

   (a) A(–1, 1), B(3, 2) (b) A(–1, 3, 4), B(2, 2, 2).

2. Find $\vec{a} + \vec{b}$, $2\vec{a} + 3\vec{b}$, $|\vec{a}|$ and $|\vec{a} - \vec{b}|$

   (a) $\vec{a} = (5, -12)$, $\vec{b} = (-3, -6)$

   (b) $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{b} = -2\vec{i} - \vec{j} + 5\vec{k}$.

3. Find a unit vector that has the same direction as the given vector.

   (a) $-3\vec{i} + 7\vec{j}$ (b) $8\vec{i} - \vec{j} + 4\vec{k}$.

4. If $\vec{v}$ lies in the first quadrant and makes an angle $\frac{\pi}{3}$ with the positive x-axis and $|\vec{v}| = 4$, find v in component form.

5. Find scalar product of the following pair of vectors.

   (a) (1, 2) and (3, 4) (b) (1, 2, 0) and (3, 2, 1)

   (c) $3\vec{i} - 7\vec{j} + \vec{k}$ and $-\vec{i} + \vec{j} - 2\vec{k}$.

6. Find the angle between the following vectors.

   (a) $\vec{a} = \vec{i} - 2\vec{j} - 2\vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

   (b) $\vec{a} = 2\vec{j} + 4\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$.

7. Find the direction cosines of the following vectors.

   (a) (2, 1, 2) (b) (c, c, c), where $c > 0$.

8. Find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$.

   (a) $\vec{a} = (-5, 12)$, $\vec{b} = (4, 6)$ (b) $\vec{a} = (3, 6, -2)$, $\vec{b} = (1, 2, 3)$

9. Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

   (a) $\vec{a} = (6, 0, -2)$, $\vec{b} = (0, 8, 0)$

   (b) $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{b} = -\vec{i} + 5\vec{k}$

   (c) $\vec{a} = \vec{i} - \vec{j} - \vec{k}$, $\vec{b} = \frac{1}{2}\vec{i} + \vec{j} + \frac{1}{2}\vec{k}$.

$|\vec{a} \times \vec{b}|$

10. Find a unit vector orthogonal to both (3, 2, 1) and (–1, 1, 0).

11. Find the area of the parallelogram with vertices A(–2, 1), B(0, 4), C(4, 2) and D(2, –1).

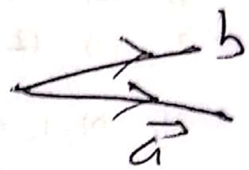
12. Find a nonzero vector orthogonal to the plane through the points P(1, 0, 1), Q(–2, 1, 3), and R(4, 2, 5), and also find the area of triangle PQR.

13. Determine whether the given vectors are orthogonal, parallel or neither.

(a) $\vec{a} = (4, 6), \vec{b} = (-3, 2)$

(b) $\vec{a} = (-5, 3, 7), \vec{b} = (6, -8, 2)$

(c) $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}, \vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}.$



14. (a) If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, prove that $\vec{a}$ is perpendicular to $\vec{b}$.

(b) Show that: $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$.

15. If $\vec{a} \cdot \vec{b} = \sqrt{3}$ and $\vec{a} \times \vec{b} = (1, 2, 2)$, find the angle between $\vec{a}$ and $\vec{b}$.

16. $\vec{a} + \vec{b} + \vec{c} = 0$, show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

17. By vector method, prove in any ΔABC that

(a) $c = a \cos B + b \cos A$ (b) $c^2 = a^2 + b^2 - 2ab \cos C$

18. Using vector method, prove that

(a) cos (A + B) = cos A cos B – sin A sin B

(b) sin (A – B) = sin A cos B – cos A sin B.

19. Find $[\vec{a}\ \vec{b}\ \vec{c}]$.

(a) $\vec{a} = 3\vec{i} + \vec{j} + \vec{k}, \vec{b} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = 2\vec{j} - 3\vec{k}$.

(b) $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}, \vec{b} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + \vec{j} + \vec{k}$.

20. $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}, \vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$, find $\vec{a} \times (\vec{b} \times \vec{c})$ and verify that find $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

21. If $\vec{a} = (1, 0, 1), \vec{b} = (2, 1, -1)$, and $\vec{c} = (0, 1, 3)$, show that

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

22. Find the volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ where $\vec{a} = (6, 3, -1)$, $\vec{b} = (0, 1, 2)$, $\vec{c} = (4, -2, 5)$.

23. Find the volume of the parallelepiped with adjacent edges PQ, PR and PS, where P(– 2, 1, 0), Q(2, 3, 4), R(1, 4, –1), S(3, 6, 1).

Answers

1. (a) $4\vec{i} + \vec{j}$ (b) $3\vec{i} - \vec{j} - 2\vec{k}$
2. (a) (2, – 18), (1, – 42), 13, 10
   (b) $-\vec{i} + \vec{j} + 2\vec{k}, -4\vec{i} + \vec{j} + 9\vec{k}, \sqrt{14}, \sqrt{6}$
3. (a) $\frac{-3}{\sqrt{58}}\vec{i} + \frac{7}{\sqrt{58}}\vec{j}$ (b) $\frac{8}{9}\vec{i} - \frac{1}{9}\vec{j} + \frac{4}{9}\vec{k}$
4. $2\vec{i} + 2\sqrt{3}\,\vec{j}$
5. (a) 11 (b) 7 (c) – 12
6. (a) $\cos^{-1}\left(-\frac{4}{21}\right)$ (b) $\frac{\pi}{2}$
7. (a) $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ (b) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
8. (a) $4, \left(-\frac{20}{13}, \frac{48}{13}\right)$ (b) $\frac{9}{7}, \left(\frac{27}{49}, \frac{54}{49}, -\frac{18}{49}\right)$
9. (a) (16, 0, 48) (b) $15\vec{i} - 3\vec{j} + 3\vec{k}$
   (c) $\frac{1}{2}\vec{i} - \vec{j} + \frac{3}{2}\vec{k}$
10. $\left(-\frac{1}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}\right)$
11. 16 square units
12. $18\vec{j} - 9\vec{k}, \frac{1}{2}\sqrt{405}$ square units
13. (a) perpendicular (b) neither (c) parallel
15. 60°
19. (a) – 10 (b) – 4
20. $-9\vec{i} - 6\vec{j} - 3\vec{k}$
22. 82 cubic units
23. 16 cubic units

Objective Questions

1. If $\vec{a} = 3\vec{i} + 2\vec{j} + 2\vec{k}$ then $|\vec{a}| =$

(a) 1 (b) $2\sqrt{2}$

(c) 3 (d) $3\sqrt{2}$

2. If $\vec{a}$ is a non-zero vector then unit vector in the direction of $\vec{a}$ is

(a) $\vec{a}$ (b) $\dfrac{\vec{a}}{|\vec{a}|}$

(c) $\vec{a}\,|\vec{a}|$ (d) $|\vec{a}|$

3. Scalar projection of $\vec{b}$ on $\vec{a}$ =

(a) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ (b) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

(c) $\dfrac{|\vec{a}|}{\vec{a}\cdot\vec{b}}$ (d) $\dfrac{|\vec{b}|}{\vec{a}\cdot\vec{b}}$

4. Vector projection of (1, 1, 2) onto (–2, 3, 1) is

(a) $\left(\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right)$ (b) $\left(-\frac{3}{7}, -\frac{9}{14}, \frac{3}{14}\right)$

(c) $\left(-\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right)$ (d) $\left(-\frac{3}{7}, -\frac{9}{14}, -\frac{3}{14}\right)$

5. If $\vec{a} = 3\vec{i} + \vec{k}$ and $\vec{b} = \lambda\vec{i} + \vec{j} + 3\vec{k}$ are orthogonal then $\lambda =$

(a) 0 (b) 1

(c) – 1 (d) 2

6. If $\vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 2\vec{i} + 4\vec{j} - 4\vec{k}$ then $\vec{a}\cdot\vec{b} =$

(a) 14 (b) 18

(c) 10 (d) 6

7. If $\vec{a} = \vec{i} + \vec{j} - 3\vec{k}$ and $\vec{b} = -\vec{i} + 3\vec{j} - \vec{k}$ then $\vec{a} \times \vec{b} =$

(a) $\vec{i} - 4\vec{j} - 4\vec{k}$ (b) $-8\vec{i} - 4\vec{j} - 4\vec{k}$

(c) $8\vec{i} + 4\vec{j} + 4\vec{k}$ (d) $8\vec{i} - 4\vec{j} + 4\vec{k}$

8. If $\vec{a} = (1, 2, 3)$, $\vec{b} = (5, -2, 3)$ and $\vec{c} = (2, 4, 6)$ then $[\vec{a}\ \vec{b}\ \vec{c}\,] =$

(a) -4
(b) -2
(c) 3
(d) 0

9. The vector triple product $\vec{a} \times (\vec{b} \times \vec{c}\,)$ is given by

(a) $(\vec{a} \cdot \vec{c}\,)\vec{b} + (\vec{a} \cdot \vec{b}\,)\,\vec{c}$
(b) $(\vec{a} \cdot \vec{c}\,)\vec{b} - (\vec{a} \cdot \vec{b}\,)\,\vec{c}$
(c) $(\vec{a} \times \vec{c}\,) \cdot \vec{b} + (\vec{a} \times \vec{b}\,) \cdot \vec{c}$
(d) $(\vec{a} \times \vec{c}\,) \cdot \vec{b} - (\vec{a} \times \vec{b}\,) \cdot \vec{c}$

Answer Sheet

1	2	3	4	5	6	7	8	9	10
d	b	a	c	c	b	c	d	b	

○○○

6

UNIT

Permutation and Combination

6.1 Introduction

The concept of permutation and combination are very important in Mathematics. Before, we discuss about permutation and combination, it is necessary to introduce the basic principle of counting. These principles help to understand permutation and combination.

6.2 Basic Principle of Counting

There are two fundamental principles of counting. These are as follows:

1. **Multiplication Principle of Counting**

   If an event can occur in m_1 different ways and a second event can occur in m_2 different ways then the two events in succession can occur in m_1m_2 different ways.

Example: In a college, there are 3 entrance doors and 4 exit doors. In how many ways can a student enter the college and exit?

Solution

A student can enter the college through any of the 3 entrance doors, so there are 3 ways of entering the college. After entering the college, the student can come out through any of the 4 exit doors, so there are 4 ways of coming out. Hence the number of ways in which a student can enter and exit the college = 3 × 4 = 12.

2. **Additional Principle of Counting**

   If two events can be performed independently in 'm_1' ways and 'm_2' ways respectively. Then either of two events can be performed in $(m_1 + m_2)$ ways.

Example: There are 5 bus routes and 6 train routes between the two cities X and Y. In how many ways can a person go from X to Y?

Solution

A person can go from X to Y by any one of the 5 bus routes or any one of the 6 train routes. Thus, there are $5 + 6 = 11$ ways by which the person can travel from X to Y.

6.3 The Factorial

The continued product of first n natural numbers is called n factorial or factorial n. It is denoted by $n!$ or $\underline{|n}$.

Thus, $n! = 1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n$.

Note : *(i)* $n! = n \cdot (n-1)!$
(ii) According to the definition 0! makes no sense. However, we define 0! = 1.
(iii) When n is a negative integer or a fraction, n! is not defined. Thus, n! is defined only for natural numbers.

Example: Compute (i) $\dfrac{9!}{6!}$ (ii) $\dfrac{12!}{9!3!}$.

Solution

(i) $\dfrac{9!}{6!} = \dfrac{9 \times 8 \times 7 \times 6!}{6!} = 9 \times 8 \times 7 = 504.$

(ii) $\dfrac{12!}{9!3!} = \dfrac{12 \times 11 \times 10 \times 9!}{9! \times 3 \times 2 \times 1} = 2 \times 11 \times 10 = 220.$

Example: Convert $4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$ into factorial form.

Solution

$$4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3} = \frac{9!}{3!}.$$

6.4 Permutations

The arrangements of objects taken some or all at a time in some order is called permutation. For example, 548 and 845 are two numbers formed from the digits 4, 5 and 8. Since they are arranged in different order, they are different permutations. In permutation, the order is essential.

6.5 Set of Objects All Different

Theorem: The number of permutation of n different things taken r at a time is given by,

$$^{n}P_r = P(n, r) = \frac{n!}{(n-r)!}, \ (n \geq r).$$

Proof

The number of permutations of a set of n objects taken r at a time is same as the number of ways in which r places can be filled up by n objects. Now, there are n choices to fill up the first place. When first place has been filled up, there will be left $n-1$ objects to fill up the second place. So, there are $(n-1)$ choices to fill up the second place. Similarly, there are $(n-2)$ choices to fill up the third place and so on. Finally, to fill up the r^{th} place, there are $n-(r-1) = n-r+1$ choices. Then by basic principle of counting,

total number of ways $= n(n-1)\cdot(n-2) \ldots (n-r+1)$

$$\therefore \quad P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 3\cdot 2\cdot 1}{(n-r)\ldots 3\cdot 2\cdot 1}$$

$$= \frac{n!}{(n-r)!}, \; n \geq r.$$

Summary

Place :	1	2	3	...	r
	$\downarrow$	$\downarrow$	$\downarrow$	...	$\downarrow$
Ways:	n	$n-1$	$n-2$	...	$n-(r-1)$

$${}^nP_r = n\cdot(n-1)\cdot(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Corollary:

$${}^nP_n = n!$$

Proof

We have, ${}^nP_r = \dfrac{n!}{(n-r)!}$

When, $r = n$, ${}^nP_n = \dfrac{n!}{(n-n)!}$

$$= \frac{n!}{0!}$$

$$= n! \; [\because 0! = 1].$$

6.6 Permutation of Objects Not at All Different

In this section, we will find the number of permutations of things taking all at a time which are not all different.

Theorem: The number of permutation of 'n' objects taken all at a time, when 'p' objects are of first kind, 'q' objects are of second kind and 'r' objects are of third kind is given by, $\dfrac{n!}{p!\,q!\,r!}$.

Proof

Let the required number of permutations be x. First we fix one permutation among x permutations.

Now, suppose p alike objects are replaced by p different objects. These p different objects can be arranged among themselves in $p!$ ways.

Similarly, suppose q alike objects are replaced by q different objects. These q different objects can be arranged among themselves in $q!$ ways.

Again, suppose r alike objects are replaced by r different objects. These r different objects can be arranged among themselves in $r!$ ways.

Thus, if three of these replacements are done simultaneously, then each one of x permutations give rise to $p!q!r!$ permutations.

$\therefore$ x permutations gives rise to $(p!q!r!)x$ permutations.

Now, each of these $(p!q!r!)x$ is a permutation of n different objects taken all at a time.

Hence, $p!q!r!\,x = n!$

$$\therefore \quad x = \frac{n!}{p!q!r!}.$$

Example: How many permutations are there in the letters of the word 'SAARC'?

Solution

There are 5 letters in the word SAARC. Also there are two A's and the rest are single.

$\therefore \quad n = 5, p = 2$

$$\text{Total number of arrangements} = \frac{n!}{p!} = \frac{5!}{2!}$$

$$= \frac{5 \times 4 \times 3 \times 2!}{2!}$$

$$= 60.$$

Example: How many permutations are there of the letters of the word 'MATHEMATICS' taken all together?

Solution

There are 11 letters in the word 'MATHEMATICS' in which 'M' comes twice, 'A' comes twice, 'T' comes twice and the rest are single. So

$n = 11, p = 2, q = 2, r = 2$

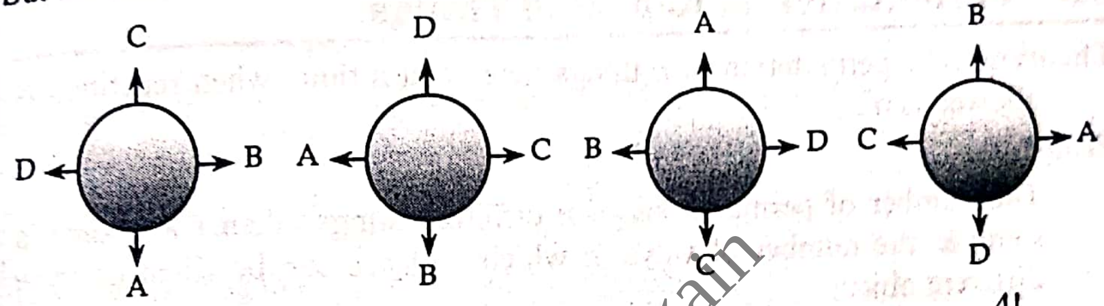
$$\text{Total number of arrangements} = \frac{n!}{p!\,q!\,r!}$$

$$= \frac{11!}{2!\,2!\,2!} = 4989600.$$

6.7 Circular Permutations

In the previous article, we have considered permutations in row (line). These permutations are called linear permutations. The permutations (arrangements) in a circle are called circular permutation. For example, suppose 4 persons are to be seated at a round table. Let 4 persons be represented by A, B, C, D. If A, B, C, D are seated in a row, four permutations ABCD, BCDA, CDAB, DABC are all different from each other. But if they are seated at a round table, all the above four permutations are the same. It can be seen by the following figures. Actually, 4 linear permutations correspond to one circular permutation in this case.

But the total number of linear permutations of A, B, C, D $= {}^4P_4 = 4!$



Hence, the number of circular permutations of 4 letters A, B, C, D $= \frac{4!}{4} = 3!$.

Theorem: Circular Permutations of n different objects $= (n-1)!$.

Proof

Let x be the total number of circular permutations. To each one of these x circular permutations, there are n linear permutations. Thus, all circular permutations give rise to $x \cdot n$ linear permutations.

Hence, $x \cdot n = n!$

$$\therefore \quad x = \frac{n!}{n} = (n-1)!.$$

Example: In how many ways can 7 girls be arranged in a (i) circle (ii) line.

Solution

(i) The required number of ways in which 7 girls can be arranged in a circle $= (7-1)! = 6! = 6 \times 5 \times 4 \times 3 \times 2 = 720$.

(ii) The required number of ways in which 7 girls can be arranged in a line $= {}^7P_7 = \frac{7!}{(7-7)} = \frac{7!}{0!} = 7! = 5040$.

Note: *(i) In circular permutations, the permutations are always read in anticlockwise direction.*

(ii) When there is no difference between anticlockwise and clockwise permutations, then the circular permutations of n things is $\frac{(n-1)!}{2}$ because in this case, if we turn around an anticlockwise circular permutations, we get the corresponding clockwise circular permutations. Basically, this case occurs in forming bracelet, necklace, garland etc.

Example: In how many ways can 8 beads of different colours form a necklace?

Solution

Here, $n = 8$

Number of ways of forming a necklace $= \dfrac{(n-1)!}{2}$

$$= \frac{(8-1)!}{2} = \frac{7!}{2}$$

$$= \frac{5040}{2} = 2520.$$

6.8 Permutation of Repeated Things

Theorem: The permutation of n things taken r at a time, when repetition is allowed is n^r.

Proof

The number of permutations of n different things taken r at a time is same as the number of ways in which r places can be filled up by n different objects.

Now, no. of ways by which first place can be filled $= n$

no. of ways by which second place can be filled $= n$

[$\because$ repetition is allowed]

no. of ways by which third place can be filled $= n$

... ...

no. of ways by which r^{th} place can be filled $= n$

By basic principles of counting, total no. of ways $= n \cdot n \cdot n \ldots n$ (r factors)

$= n^r$.

Summary

Place :	1	2	3	...	r
	↓	↓	↓	...	↓
Ways:	n	n	n	...	n

$^nP_r = n^r$

Example: How many 3-digit numbers can be formed by using the digits 2, 3, 4, 5, when the repetitions of digits is allowed?

Solution

Here, $n = 4, r = 3$

Since repetition of digits is allowed, so total no. of permutations $= n^r$

$= 4^3$

$= 64.$

WORKED OUT EXAMPLES

Example 1. Find the value of $^9P_6$.

Solution

We have,

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\therefore \quad {}^9P_5 = \frac{9!}{(9-5)!}$$

$$= \frac{9!}{4!}$$

$$= 9 \times 8 \times 7 \times 6 \times 5$$

$$= 15120.$$

Example 2. If $^nP_4 = 12\,{}^nP_2$, find n.

Solution

$$^nP_4 = \frac{n!}{(n-4)!} \text{ and } {}^nP_2 = \frac{n!}{(n-2)!}$$

According to the given question

$$\frac{n!}{(n-4)!} = \frac{12\,n!}{(n-2)!}$$

or, $12\,(n-4)! = (n-2)!$

or, $12\,(n-4)! = (n-2)\,(n-3)\,(n-4)!$

or, $12 = (n-2)\,(n-3)$

or, $12 = n^2 - 5n + 6$

or, $n^2 - 5n - 6 = 0$

or, $(n-6)\,(n+1) = 0$

Either, $n = 6$ or, $n = -1$

Since n is a positive integer, we reject the second value of $n = -1$.

Hence $n = 6$.

Example 3. If $^nP_5 : {}^nP_3 = 2 : 1$, find the value of n.

Solution

We have,

$$\frac{^nP_5}{^nP_3} = \frac{2}{1}$$

or, $^nP_5 = 2 \cdot {}^nP_3$

or, $\dfrac{n!}{(n-5)!} = 2\dfrac{n!}{(n-3)!}$

or, $\dfrac{1}{(n-5)!} = \dfrac{2}{(n-3)\,(n-4)\,(n-5)!}$

or, $1 = \frac{2}{(n-3)(n-4)}$

or, $n^2 - 7n + 12 = 2$

or, $n^2 - 7n + 10 = 0$

or, $(n-5)(n-2) = 0$

Either, $n = 5$ or, $n = 2$

Since n cannot be equal to 0, 1, 2, 3, 4 because $n \geq r$, therefore $n = 5$.

Example 4. In how many ways can 6 passengers sit in a bus having 16 vacant seats?

Solution

Here, $n = 16$, $r = 6$

Required number of ways $= {}^{16}P_6$

$= \frac{16!}{(16-6)!}$

$= \frac{16!}{10!}$

$= 16 \times 15 \times 14 \times 13 \times 12 \times 11$

$= 5765760.$

Example 5. How many numbers of four digits can be formed with the digits 2, 3, 4, 5, 6, 7, none of the digits being repeated in any of the numbers so formed?

Solution

To form the numbers of 4 digits, we have to arrange the given digits 2, 3, 4, 5, 6, 7 taking four at a time. 6 digits can be arranged taken 4 at a time in ${}^6P_4$ ways

${}^6P_4 = \frac{6!}{(6-4)!}$

$= \frac{6!}{2!}$

$= \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!}$

$= 6 \times 5 \times 4 \times 3 = 360$ ways.

Example 6. There are 20 micro buses running between Ratnapark and Balaju. In how many ways can Mr. John go from Ratnapark to Balaju and return by a different microbus?

Solution

While going $n = 20$, $r = 1$

${}^{20}P_1 = \frac{20!}{(20-1)!}$

$= \frac{20 \times 19!}{19!}$

$= 20$

While returning $n = 19$ and $r = 1$

$$^{19}P_1 = \frac{19!}{(19-1)!} = 19$$

$\therefore$ Total number of ways $= 20 \times 19 = 380.$

Example 7. In how many ways can the letters of the word DAUGHTER be arranged so that the vowels may never be separated?

Solution

There are 3 vowels (A, U, E) and 5 consonants (D, G, H, T, R) in the given word. Considering the 3 vowels as one letter, we have only 6 different letters namely D, G, H, T, R, (A, U, E) to arrange. These 6 letters can be arranged taking all at a time in $^6P_6$ ways $= \frac{6!}{(6-6)!} = \frac{6!}{0!} = 720$

In each of these 720 ways, 3 vowels (A, U E) can be arranged among themselves in $^3P_3$ ways $= \frac{3!}{0!} = 6$

Hence, the required no. of ways $= 720 \times 6 = 4320.$

Example 8. 4 men and 3 women are to be seated for a dinner such that none of the 2 women sit together nor the 2 men. Find the number of ways in which this can be arranged?

Solution

3 women can be placed in 3 even places in $^3P_3$ ways $= 3 \times 2 \times 1 = 6$. In between and at the end of 3 women, there are 4 odd places in which 4 men can be placed in $^4P_4$ ways $= 4 \times 3 \times 2 \times 1 = 24$

Hence, the required no. of ways $= 6 \times 24 = 144$

Example 9. A family of 4 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated if all the sisters sit together?

Solution

Taking 3 sisters as 1, we have to arrange 4 + 1 i.e. 5 different persons taking 5 at a time in row. This can be done in $^5P_5$ ways $= \frac{5!}{0!} = 120$ ways. Also 3 sister can be arranged among themselves in $^3P_3$ ways $= 6$ ways

$\therefore$ Hence, the required no. of ways $= 120 \times 6 = 720$

Example 10. Find the number of ways in 6 books can be arranged on a self so that two particular books are not together?

Solution

Without any restriction 6 books can be arranged on a self in $^6P_6$ ways

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Now, let us find the number of ways in which two particular books always come together. Considering the two particular books as 1, we have to arrange $4 + 1 = 5$ different books and this can be done in ${}^5P_5$ ways $= 5 \times 4 \times 3 \times 2 \times 1 = 120$.

But in each of these 120 ways, 2 particular books which are always together, can be arranged among themselves in ${}^2P_2 = 2 \times 1 = 2$.

The number of arrangements in which two particular books always come together $= 120 \times 2 = 240$.

Hence, the required number of arrangements in which the two particular books never come together $= 720 - 240 = 480$.

Example 11. How many numbers of four different digits each greater than 5000 can be formed from the digit 2, 4, 5, 7, 8, 0?

Solution

Number of four digits greater than 5000 must begin with either 5 or 7 or 8. Now, number of 4 different digit numbers beginning with $5 = {}^5P_3 = 5 \times 4 \times 3 = 60$

[$\therefore$ the remaining 5 digits 2, 4, 7, 8, 0 can be arranged by taking 3 at a time in ${}^5P_3$ ways]

Similarly,

The number of 4 different digit numbers beginning with $7 = {}^5P_3 = 60$

Again, number of 4 different digit numbers beginning with $8 = {}^5p_3 = 60$

Hence, the required numbers $= 60 + 60 + 60 = 180$.

Example 12. In how many ways can the letters of the word 'STRANGE' be arranged so that the vowels may appear in the odd places?

Solution

There are 5 consonants and 2 vowels in the word STRANGE. There are 7 places for the 7 letters. Also 4 places are odd and 3 places are even.

Two vowels can be arranged in 4 odd places in ${}^4P_2$ ways $= 4 \times 3 = 12$

Five consonants can be arranged in the remaining 5 places in ${}^5P_5$ ways

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

Hence, the required number of ways $= 12 \times 120 = 1440$.

Example 13. In how many of the permutations of 10 different things, taken 4 at a time, will one particular thing (i) never occur (ii) always occur?

Solution

(i) Leaving aside the particular things which will never occur, we have 9 things to arrange taking 4 at a time and number of permutations of 9 things taken 4 at a time is ${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$.

(ii) The number of permutations of 10 things taken 4 at a time is ${}^{10}P_4 = 10 \times 9 \times 8 \times 7$

$$= 5040$$

Hence, the required no. of permutation of 10 different things taken 4 at a time in which one particular thing always occurs $= 5040 - 3024 = 2016$.

Example 14. In how many ways can the letters of the word 'PETROL' be arranged? How many of these do not begin with P?

Solution

There are 6 different letters in the word PETROL and 6 letters are to be arranged taking all at a time.

Here, $n = 6, r = 6$

$$\begin{aligned} {}^nP_r &= {}^6P_6 \\ &= \frac{6!}{(6-6)!} \\ &= 6! \\ &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \text{ ways} \end{aligned}$$

Again, if words begin with P, then P is fixed at first place and we can arrange the letters E, T, R, O, L taking 5 at a time in ${}^5P_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Hence, the required no. of ways which do not begin with P = 720 – 120 = 600.

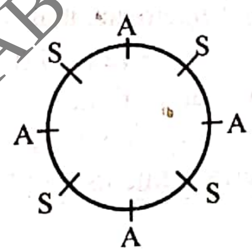
Example 15. In how many ways the numbers on the clock face be arranged?

Solution

There are 12 numbers can on the clock face. So, the numbers can be arranged in $(12-1)!$ ways = $11!$ ways.

Example 16. In how many ways can 4 Art students and 4 Science students be arranged alternately at a round table?

Solution



4 Art students at a round table can be arranged in $(4-1)! = 3!$ ways
$= 3 \times 2 \times 1$
$= 6$ ways.

Since Art and Science students must be arranged alternatively, so 4 Science students in 4 seats can be arranged in P(4, 4) ways $= 4!$ ways
$= 4 \times 3 \times 2 \times 1$
$= 24$ ways

Total number of arrangements $= 6 \times 24 = 144$.

Example 17. Show that the number of ways in which the letters of the word 'ARRANGE' can be arranged so that no two R's come together is 900.

Solution

Given word is 'ARRANGE'

Total number of letters $(n) = 7$

Number of A's $(p) = 2$

Number of R's $(q) = 2$

And the rest are single.

Total number of arrangements $= \dfrac{n!}{p!\,q!} = \dfrac{7!}{2!\,2!}$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2! \times 2 \times 1} = 1260$$

To find the number of arrangements in which no two 'R' come together, we first find the number of arrangements in whioh two R's come together. For this, consider two R's as a single letter, then the number of letters will be 6.

Total number of arrangements in which two R's come together

$$= \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$$

$\therefore$ Required number of arrangements in which no two 'R' come together

$$= 1260 - 360 = 900.$$

Example 18. In how many ways can the letters of the word "MONDAY" be arranged? How many of these arrangements do not begin with M? How many begin with M and don't end with Y?

Solution

Total number of letters in the word 'MONDAY' is 6.

$\therefore$ Number of arrangements $= 6!$

$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

To find the number of arrangements that do not begin with M, firstly we find the number of arrangements that begin with 'M'. For this fix 'M' at first place, then remaining 5 letters can be arranged in P (5, 5) ways

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120 \text{ ways}$$

$\therefore$ Required number of arrangements that do not begin with 'M'

$$= 720 - 120 = 600$$

Again, fix 'M' at first place and 'Y' at last, then remaining 4 letters can be arranged in P(4, 4) ways $= 4 \times 3 \times 2 \times 1 = 24$

$\therefore$ Required number of arrangements that begin with 'M' and do not end with 'Y' $= 120 - 24 = 96$.

Example 19. In how many ways can the letters of the word LOGIC be arranged so that

(i) Vowels may occupy odd position?

(ii) No vowels are together?

Solution

There are 5 letters in the word LOGIC. There are 3 consonants and 2 vowels.

(i) If the vowels may occupy odd position then 2 vowels in 3 odd positions can be arranged in ${}^3P_2$ ways $= \dfrac{3!}{(3-2)!}$

$$= \frac{3 \times 2 \times 1}{1} = 6.$$

Also, 3 consonants in remaining 3 positions can be arranged in ${}^3P_3$ ways

$$= 3! \text{ ways}$$

$$= 3 \times 2 \times 1 \text{ ways}$$

$$= 6 \text{ ways}$$

Total no. of arrangements $= 6 \times 6 = 36$.

(ii) Total no. of arrangements of the letter of the word LOGIC = 5!

$= 5\times4\times3\times2\times1$

$= 120$

Next, consider 2 vowels as a single letters then there will be 4 letters namely L, G, C, (I, O). Then 4 letters can be arranged in 4! ways $= 4 \times 3 \times 2 \times 1$ ways

$= 24$ ways

Also, 2 vowels among themselves can be arranged $= 2!$ ways

$= 2 \times 1$

$= 2$ ways

Total no. of arrangements in which the vowels are always together $= 24 \times 2$

$= 48$

Required no. of arrangements in which no vowels are together

= Total arrangements – no. of arrangements in which all vowels are together.

$= 120 - 48$

$= 72.$

EXERCISE– 6 A

1. Evaluate the following:

   (a) $\dfrac{10! - 9!}{9!}$ (b) ${}^{8}P_{8}$ (c) ${}^{14}P_{4} \div {}^{12}P_{2}$

2. (a) If ${}^{n}P_{2} = 20$, find the value of n.

   (b) If ${}^{n}P_{5} : {}^{n}P_{3} = 2 : 1$, find the value of n.

3. (a) There are 7 entrance doors and 4 exit doors in a stadium. In how many ways can a person enter the stadium and exit?

   (b) There are 8 local taxies plying between Dhulikhel and Banepa. In how many ways can Mr. Dinesh go from Dhulikhel to Banepa and return by a different taxies?

4. (a) How many ways can 6 passengers sit in a compartment having 10 vacant seats?

   (b) If 4 people enter a bus in which there are 7 vacant seats, in how many ways can they take their seats?

5. (a) How many numbers of plates of vehicles consisting of 4 different digits be made out of integers 4, 5, 6, 7, 8, 9?

   (b) How many different numbers of five digits can be formed with the digits 0, 1, 2, 3, 4 ?

   (c) How many numbers between 4000 and 5000 can be formed with the digits 2, 3, 4, 5, 6, 7; if no digits being repeated? How many of these numbers are divisible by 5?

6. In how many ways can 5 girls and 4 boys be arranged in a row of 9 seats if
   (a) they may sit anywhere
   (b) the girls and boys must sit alternatively
   (c) all the boys are together.
7. In how many ways can 6 people be seated in a row of 6 seats so that two particular persons always come together?
8. In how many ways can 5 girls and 3 boys stand in a row so that all the boys stand together?
9. Find the number of arrangements that can be made out of the letters of the following words.
   (a) EXAMINATION
   (b) KATHMANDU
   (c) NEPAL
   (d) HUMANITIES
   (e) MATHEMATICS
10. There are 5 red balls, 4 white balls and 3 yellow balls. In how many ways can they be arranged in a row?
11. (a) In how many ways can 10 girls be arranged at a round table?
    (b) In how many ways can 7 boys be arranged at a round table so that two particular boys can be together?
    (c) In how many ways 4 BCA students and 4 BBA students be arranged alternatively at a round table?

12. In how many ways can 10 different beads form a bracelet?
13. (a) In how many ways can 3 letters be posted in 5 letter boxes?
    (b) How many 3 digits numbers can be formed by using the digits 3, 4, 5, 6 when each digits may be repeated any number of times?
    (c) There are 3 prizes to be distributed among 5 BCA students. In how many ways can it be done if:
       (i) no students gets more than one prize?
       (ii) a student may get any number of prizes?
       (iii) no students get all the prizes?
14. In how many ways can the letters of the word ARRANGE be arranged so that no two R's come together?
15. In how many ways can the letter of the word 'SUNDAY' be arranged? How many of these arrangements do not begin with S? How many begin with S and do not end with Y?
16. In how many ways can the letters of the word "TUESDAY" be arranged? How many of these arrangements do not begin with T? How many begin with T and do not end with Y?

17. In how many ways can the letters of the word "COMPUTER" be arranged so that:
    (a) all vowels are always together?
    (b) the relative positions of the vowels and consonants are not changed?
    (c) the vowels may occupy only the odd positions?

Answers

1.	(a) 9	(b) 40320	(c) 182
2.	(a) 5	(b) 5	
3.	(a) 25	(b) 56	
4.	(a) 151200	(b) 840	
5.	(a) 360	(b) 96	(c) 60, 12
6.	(a) 362880	(b) 2880	(c) 17280
7.	240	8. 4320	
9.	(a) 4989600	(b) 181440	(c) 120
	(d) 181440	(e) 4989600	
10.	27720		
11.	(a) 362880	(b) 240	(c) 144
12.	181440		
13.	(a) 125	(b) 64	
	(c) (i) 60	(ii) 125	(iii) 120
14.	900		
15.	720, 600, 96	16. 5040, 4320, 600	
17.	(a) 4320	(b) 720	(c) 2880

Objective Questions

1. $\dfrac{100!}{98!\,2!} =$
   (a) 9900 (b) 4950
   (c) 3300 (d) 2475

2. $\dfrac{11! - 10!}{10} =$
   (a) 10 (b) 10!
   (c) 11 (d) 11!

3. Which of the following is true?
   (a) $(2 + 5)! = 2! + 5!$ (b) $(5 - 3)! = 5! - 3!$
   (c) $(5 \times 3)! = 5! \times 3!$ (d) none

4. How many different numbers of five digits can be formed with the digits 0, 1, 2, 3, 4?
   (a) 120 (b) 114
   (c) 96 (d) 60

5. The no. of ways in which 4 boys and 3 girls can be seated in a row if they sit alternatively?

   (a) 12! (b) 7!

   (c) $4! \times 3!$ (d) $4! + 3!$

6. The number of ways can the letters of the word ELEMENT be arranged?

   (a) 420 (b) 840

   (c) 210 (d) 1680

7. The number of 6 digit numbers that can be formed form 2, 2, 2, 5, 5, 0?

   (a) 30 (b) 40

   (c) 60 (d) 50

8. In how many ways can 6 BCA students be seated in a round table?

   (a) 720 (b) 360

   (c) 120 (d) 60

9. The number of ways in which 6 different beads be strung on a necklace?

   (a) 120 (b) 60

   (c) 30 (d) 15

10. In how many ways can 7 letters be posted in 3 letter boxes?

   (a) 21 (b) 3^7

   (c) 7^3 (d) 42

Answer Sheet

1	2	3	4	5	6	7	8	9	10
b	b	d	c	c	b	d	c	b	b

6.9 Combination

The collection or selection of objects taken all or some at a time without any order is called combination. In combination, we select the objects irrespective of their order but in permutation the order is essential.

Theorem: The number of combinations of n different things taken r at a time is given by $C(n, r) = {}^nC_r = \dfrac{n!}{(n-r)!\, r!}$, $(n \geq r)$.

Proof

Consider any one of the $C(n, r)$ combinations. This combination contains r objects. These r objects among themselves can be arranged in $r!$ different ways. So, for each combination, there are $r!$ permutations. Hence, for the $C(n, r)$ combinations, there are $C(n, r) \cdot r!$ different permutations. Since these are all possible permutations of n objects taken r at a time. So, we have

$$C(n, r) \cdot r! = P(n, r)$$

$$\text{or,} \quad C(n, r) \cdot r! = \frac{n!}{(n-r)!}$$

$$\therefore \quad C(n, r) = \frac{n!}{(n-r)!\, r!}.$$

Note: The relationship between permutation and combination of n objects taken r at a time is $\boxed{{}^nP_r = r! \cdot {}^nC_r}$.

6.10 Properties of Combination

Some properties of combinations are as follows:

(i) ${}^nC_r = {}^nC_{n-r}$ (Complementary combination)

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) ${}^nC_n = 1$

(iv) ${}^nC_1 = n$

(v) ${}^nC_0 = 1$

(vi) If ${}^nC_x = {}^nC_y$ then, either $x = y$ or $x + y = n$

Proofs

(i)
$$
\begin{aligned}
{}^nC_{n-r} &= \frac{n!}{(n-(n-r))!\,(n-r)!} \\
&= \frac{n!}{(n-n+r)!\,(n-r)!} \\
&= \frac{n!}{r!\,(n-r)!} \\
&= \frac{n!}{(n-r)!\,r!} \\
&= {}^nC_r.
\end{aligned}
$$

(ii)
$$
\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\cdot r\cdot(r-1)!} + \frac{n!}{(n-r+1)\cdot(n-r)!\cdot(r-1)!} \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left(\frac{1}{r} + \frac{1}{n-r+1}\right) \\
&= \frac{n!}{(n-r)!\,(r-1)!}\cdot\frac{(n-r+1+r)}{r(n-r+1)} \\
&= \frac{(n+1)\cdot n!}{(n-r+1)\cdot(n-r)!\cdot r\cdot(r-1)!} \\
&= \frac{(n+1)!}{(n+1-r)!\,r!} \\
&= {}^{n+1}C_r.
\end{aligned}
$$

(iii) $$^nC_n = \frac{n!}{(n-n)!n!}$$

$$= \frac{n!}{0!n!}$$

$$= \frac{n!}{n!} \qquad [\because 0! = 1]$$

$$= 1.$$

(iv) $$^nC_1 = \frac{n!}{(n-1)!\ 1!}$$

$$= \frac{n\cdot(n-1)!}{(n-1)!}$$

$$= n.$$

(v) $$^nC_0 = \frac{n!}{(n-0)!\ 0!}$$

$$= \frac{n!}{n!} \qquad [\because 0! = 1]$$

$$= 1.$$

(vi) If $^nC_x = {}^nC_y$

Either $x = y$

or, $^nC_x = {}^nC_{n-y}$

or, $x = n - y$

$\therefore$ $n = x + y$.

WORKED OUT EXAMPLES

Example 1. Find the value of (i) $^8C_5$ (ii) $^{10}C_6 + {}^{10}C_5$.

Solution

(i) $$^8C_5 = \frac{8!}{(8-5)!5!}$$

$$= \frac{8!}{3!5!}$$

$$= \frac{8\times7\times6\times5!}{5!\times3\times2\times1}$$

$$= \frac{8\times7\times6}{3\times2\times1}$$

$$= 56$$

(ii) $^{10}C_6 + {}^{10}C_5 = \frac{10!}{(10-6)!6!} + \frac{10!}{(10-5)!5!}$

$= \frac{10!}{4!6!} + \frac{10!}{5!5!}$

$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} + \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$

$= 210 + 252$

$= 462.$

Example 2.

(i) If $^{21}C_x = {}^{21}C_y$ and $x \neq y$, what is the value of $x + y$.

(ii) If $^{16}C_r = {}^{n}C_{r+4}$ find r and $^{r}C_3$.

Solution

(i) We have, $^{n}C_x = {}^{n}C_y$, then either $x = y$ or $x + y = n$

So, $^{21}C_x = {}^{21}C_y$, then either $x = y$ or $x + y = 21$

Then, $x + y = 21$. $[\because x \neq y]$

(ii) We have, $^{n}C_x = {}^{n}C_y$ then either $x = y$ or $x + y = n$

We have, $^{16}C_r = {}^{16}C_{r+4}$, then either $r = r + 4$ or $r + (r + 4) = 16$

If $r = r + 4$ then $0 = 2$ which is false.

Hence,

$r + (r + 4) = 16$

or, $2r + 4 = 16$

or, $2r = 16 - 4$

or, $2r = 12$

$\therefore$ $r = 6$

Now,

$^{r}C_3 = {}^{6}C_3$

$= \frac{6!}{(6-3)!3!}$

$= \frac{6!}{3!3!}$

$= \frac{6 \times 5 \times 4}{3 \times 2 \times 1}$

$= 20.$

Example 3. If ${}^nC_r = 56$ and ${}^nP_r = 336$, find n and r.

Solution

$$\frac{{}^nP_r}{{}^nC_r} = \frac{336}{56}$$

or, $\dfrac{n!}{(n-r)!} \times \dfrac{(n-r)!r!}{n!} = 6$

or, $r! = 6$

or, $r! = 3 \times 2 \times 1$

or, $r! = 3!$

$\therefore$ $r = 3$

Then,

${}^nP_r = 366$

or, ${}^nP_3 = 336$

or, $\dfrac{n!}{(n-3)!} = 336$

or, $\dfrac{n(n-1)(n-2)(n-3)}{(n-3)!} = 336$

or, $n(n-1)(n-2) = 8 \times 7 \times 6$

$\therefore$ $n = 8$

Hence, $n = 8$ and $r = 3$.

Example 4. In an examination paper on Mathematics, 18 questions are set. In how many different ways can you choose 15 questions to answer?

Solution

Here, $n = 18$, $r = 15$

No. of ways of selecting 15 questions out of 18

$= {}^{18}C_{15}$

$= \dfrac{18!}{(18-15)!\ 15!}$

$= \dfrac{18 \times 17 \times 16 \times 15!}{3 \times 2 \times 1 \times 15!}$

$= 816.$

Example 5. How many different committees of 6 members may be formed with 7 boys and 5 girls?

Solution

We have to select 6 members from 7 + 5 = 12 persons.

Here, $n = 12$, $r = 6$

Total number of ways $= {}^nC_r = {}^{12}C_6$

$= \dfrac{12!}{(12-6)!6!}$

$= \dfrac{12}{6!\ 6!}$

$= \dfrac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= 924$ ways

Hence, the required number of committees = 924.

Example 6. Find the number of ways in which a cricket team consisting of 11 players be selected out of 14 players. In how many ways can it be done if one particular player is always included?

Solution

First case

$n = 14;\ r = 11$

Total number of selections $= {}^{n}C_{r} = {}^{14}C_{11}$

$$= \frac{14!}{(14-11)!11!}$$

$$= \frac{14!}{3!\,11!}$$

$$= \frac{14 \times 13 \times 12}{3 \times 2 \times 1}$$

$= 364$ ways.

Second case

Since a particular player is always included in the team, we have to select $11 - 1 = 10$ players from the remaining $14 - 1 = 13$ players.

$n = 13;\ r = 10$

Total no. of selections $= {}^{n}C_{r} = {}^{13}C_{10}$

$$= \frac{13!}{(13-10)!10!}$$

$$= \frac{13!}{3!10!}$$

$$= \frac{13 \times 12 \times 11}{3 \times 2 \times 1}$$

$= 286$ ways.

Example 7. A box contains 7 red, 6 white and 4 blue balls. How many selections of three balls can be made so that (i) all three are red balls (ii) no red ball (iii) one ball of each colour.

Solution

(i) All the three balls will be red if 3 balls are drawn out of 7 red balls. So, 3 red balls can be drawn in ${}^{n}C_{r} = {}^{7}C_{3}$ ways

$$= \frac{7!}{(7-3)!\,3!}$$

$$= \frac{7!}{4!3!}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}$$

$= 35$ ways.

(ii) $n = 6 + 4 = 10$

$r = 3$

Total no. of selections $= {}^{n}C_r = {}^{10}C_3$

$$= \frac{10!}{(10-3)!\,3!}$$

$$= \frac{10!}{7!3!}$$

$$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$$

$= 12$ ways.

(iii) 1 red ball can be selected out of $7 = {}^{7}C_1$

1 white ball can be selected out of $6 = {}^{6}C_1$

1 blue ball can be selected out of $4 = {}^{4}C_1$

Required no. of selections $= {}^{7}C_1 \times {}^{6}C_1 \times {}^{4}C_1 = 168$ ways.

Example 8. A person has got 12 acquaintances of whom 8 are relatives. In how many ways can he invite 7 guests so that 5 of them may be relatives?

Solution.

Total number of acquaintances = 12

Number of relatives = 8

Number of non-relatives = 12 – 8 = 4

Relatives (8)	Non-relatives (4)	Selection
5	$7 - 5 = 2$	${}^{8}C_5 \times {}^{4}C_2$

Total number of selections

$= {}^{8}C_5 \times {}^{4}C_2$

$$= \frac{8!}{3!\,5!} \times \frac{4!}{2!\,2!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} = 56 \times 6$$

$= 336$

Example 9. Mr. John has 7 friends. In how many ways can he invite one or more of his friends to a dinner?

Solution

Mr. John may invite one or more friends by selecting either 1 friend, or 2 friends, or 3 friends or 4 friends or 5 friends, or 6 friends or 7 friends out of 7. Then the required no. of ways in which he may invite one or more of them to a dinner.

$$= {}^{7}C_1 + {}^{7}C_2 + {}^{7}C_3 + {}^{7}C_4 + {}^{7}C_5 + {}^{7}C_6 + {}^{7}C_7$$

$$= \frac{7!}{6!1!} + \frac{7!}{5!2!} + \frac{7!}{4!3!} + \frac{7!}{3!4!} + \frac{7!}{2!5!} + \frac{7!}{1!6!} + \frac{7!}{0!7!}$$

$= 7 + 21 + 35 + 35 + 21 + 7 + 1 = 127$ ways.

Example 10. In an examination paper on Mathematics 10 questions are set. In how many different ways can you choose 6 questions to answer? However, if question no. 2 is made compulsory, in how many ways can you select to answer 6 questions in all?

Solution

Case (i)

Here, $n = 10; r = 6$

Required no. of ways $= {}^nC_r = {}^{10}C_6 = \frac{10!}{4!6!}$

$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$ ways.

Case (ii)

If question number 2 is made compulsory, we can select 5 questions from the remaining 9 questions.

$n = 10 - 1 = 9$

$r = 6 - 1 = 5$

$\therefore$ Required number of ways $= {}^9C_5 = \frac{9!}{4!5!}$

$= \frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2 \times 1} = 126$ ways.

Example 11. In how many ways can a volleyball team of 6 persons be chosen out of 11 players if

(i) there are no restrictions on the selection.

(ii) a particular player is always chosen.

(iii) a particular player is never chosen.

Solution

(i) 6 players can be chosen out of 11 in ${}^{11}C_6$ ways.

$= \frac{11!}{(11-6)!6!}$

$= \frac{11!}{5!6!}$

$= \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1}$

$= 462$ ways.

(ii) When a particular player is always chosen, then

$n = 11 - 1 = 10$

$r = 6 - 1 = 5$

Total no. of selections $= {}^{10}C_5$

$= \frac{10!}{(10-5)!5!}$

$= \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$

$= 252$ ways.

(iii) When a particular player is never chosen, then

$n = 11 - 1 = 10$

$r = 6$

Total no. of selections $= {}^{10}C_6$

$$= \frac{10!}{(10-6)!\,6!}$$

$= 210$ ways.

Example 12. From 10 football players in how many ways can a selection of a 4 be made (i) when one particular player is always included (ii) when two particular players are always excluded?

Solution

Total number of players = 10

Number of players to be selected = 4

When one particular player is always included, then we have to select $4 - 1 = 3$ players out of $10 - 1 = 9$

Total number of selections $= C(9, 3) = \frac{9!}{6!\,3!}$

$$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$$

When two particular players are excluded, then we have to select 4 players out of $10 - 2 = 8$

Required number of selections $= C(8, 4) = \frac{8!}{4!\,4!}$

$$= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70.$$

Example 13. There are 7 men and 3 ladies. Find the number of ways in which a committee of 6 persons can be formed if the committee is to have at least one lady.

Solution

The selections of the members in the committee can be made as follows:

3 Ladies	7 Men	Selection
1	5	${}^3C_1 \times {}^7C_5$
2	4	${}^3C_2 \times {}^7C_4$
3	3	${}^3C_3 \times {}^7C_3$

Total no. of committees of 6 members with at least one lady.

$= {}^3C_1 \times {}^7C_5 + {}^3C_2 \times {}^7C_4 + {}^3C_3 \times {}^7C_3$

$= \frac{3!}{2!1!} \times \frac{7!}{2!5!} + \frac{3!}{1!2!} \times \frac{7!}{3!4!} + \frac{3!}{0!3!} \times \frac{7!}{4!3!}$

$= 63 + 105 + 35 = 203$ ways.

Example 14. A committee of 5 is to be constituted from 6 boys and 5 girls. In how many ways can this be done so as to include at least a boy and a girl?

Solution

The selection of the members in the committee can be made as follows

Boys (6)	Girls (5)	Selection
1	4	C(6, 1) × C(5, 4)
2	3	C(6, 2) × C(5, 3)
3	2	C(6, 3) × C(5, 2)
4	1	C(6, 4) × C(5, 1)

Required number of selections

$= C(6, 1) \times C(5, 4) + C(6, 2) \times C(5, 3) + C(6, 3) \times C(5, 2) + C(6, 4) \times C(5,1)$

$= \frac{6!}{5!\,1!} \times \frac{5!}{1!\,4!} + \frac{6!}{4!\,2!} \times \frac{5!}{2!\,3!} + \frac{6!}{3!\,3!} \times \frac{5!}{3!\,2!} + \frac{6!}{2!\,4!} \times \frac{5!}{4!\,1!}$

$= 6 \times 5 + 15 \times 10 + 20 \times 10 + 15 \times 5$

$= 30 + 150 + 200 + 75 = 455.$

Example 15. A candidate is required to answer 6 out of 10 questions which are divided into two groups each containing 5 questions and he is not permitted to attempt more than 4 from any group. In how many different ways can he make up his choice?

Solution

The selection of questions can be made as follows

1st group (5)	2nd group (5)	Selection
4	2	C(5, 4) × C(5, 2)
3	3	C(5, 3) × C(5, 3)
2	4	C(5, 2) × C(5, 4)

Total number of selections

$= C(5, 4) \times C(5, 2) + C(5, 3) \times C(5, 3) + C(5, 2) \times C(5, 4)$

$= \frac{5!}{1!\,4!} \times \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} \times \frac{5!}{2!\,3!} + \frac{5!}{3!\,2!} \times \frac{5!}{1!\,4!}$

$= 5 \times 10 + 10 \times 10 + 10 \times 5 = 200$

Example 16. If $C(n, r - 1) = 36$, $C(n, r) = 84$ and $C(n, r + 1) = 126$, find the value of r and n.

Solution

Given,

$C(n, r - 1) = 36$ ... (i)

$C(n, r) = 84$ ... (ii)

$C(n, r + 1) = 126$ ... (iii)

Dividing equation (ii) by (i),

$$\frac{C(n, r)}{C(n, r-1)} = \frac{84}{36}$$

or, $$\frac{\frac{n!}{(n-r)!\,r!}}{\frac{n!}{(n-r+1)!\,(r-1)!}} = \frac{7}{3}$$

or, $$\frac{(n-r+1)!\,(r-1)!}{(n-r)!\,r!} = \frac{7}{3}$$

or, $$\frac{(n-r+1)\cdot(n-r)!\,(r-1)!}{(n-r)!\cdot r\cdot(r-1)!} = \frac{7}{3}$$

or, $$\frac{n-r+1}{r} = \frac{7}{3}$$

or, $3n - 3r + 3 = 7r$

or, $3n - 10r = -3$ ... (iv)

Again, dividing equation (iii) by (ii)

$$\frac{C(n, r+1)}{C(n, r)} = \frac{126}{84}$$

or, $$\frac{\frac{n!}{(n-r-1)!\,(r+1)!}}{\frac{n!}{(n-r)!\,r!}} = \frac{3}{2}$$

or, $$\frac{(n-r)!\,r!}{(n-r-1)!\,(r+1)!} = \frac{3}{2}$$

or, $$\frac{(n-r)\cdot(n-r-1)!\,r!}{(n-r-1)!\,(r+1)\cdot r!} = \frac{3}{2}$$

or, $$\frac{n-r}{r+1} = \frac{3}{2}$$

or, $2n - 2r = 3r + 3$

or, $2n - 5r = 3$ ...(v)

Multiplying (v) by 2 and subtracting from (iv)

$$\begin{array}{r} 3n - 10r = -3 \\ \underset{-}{4n} \underset{+}{-} 10r = \underset{-}{} 6 \\ \hline -n = -9 \end{array}$$

$\therefore$ $n = 9$

Putting the value of n in equation (v), we get,

$2 \times 9 - 5r = 3$

or, $18 - 3 = 5r$

or, $r = 3$

$\therefore$ $r = 3, n = 9.$

EXERCISE – 6 B

1. Find the value of:

   (a) ${}^{7}C_{5}$ (b) ${}^{100}C_{2}$ (c) ${}^{10}C_{4} + {}^{10}C_{3}$

2. State the meaning of ${}^{n}P_{r}$ and ${}^{n}C_{r}$. Give the relation between them. Find the value of ${}^{3}P_{2} + {}^{7}C_{2}$.

3. (a) If ${}^{18}C_r = {}^{18}C_{r+2}$ find the value of r.

(b) If ${}^{n}P_r = 336$ and ${}^{n}C_r = 56$, find n and r.

4. (a) In the question paper of an examination, 12 questions were asked and the students had to give the answer of only 10 questions. In how many ways can the students select the questions?

(b) In an entrance test, 22 questions are set. In how many ways can you choose 18 questions to answer?

(c) How many different committees of 7 members may be formed from 8 Nepalese and 4 Chinese?

5. (a) A committee is to be chosen from 12 men and 8 women and is to consist of 3 men and 2 women. How many committees can be formed?

(b) In how many ways can a committee of 5 members be selected from 6 men and 5 women consisting of 3 men and 2 women?

(c) A bag contains 8 red balls and 5 blue balls. In how many ways can 3 red balls and 4 blue balls be drawn?

6. Ram has got 15 friends of whom 10 are relatives. In how many ways can he invite 9 guests so that 7 of them be his relatives?

7. From 6 men and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include 2 ladies?

8. Mr. A has five friends. In how many ways can he invite one or more of them to a dinner?

9. In an examination, a candidate has to pass in each of the five subjects. In how many ways can the candidate fail?

10. In an examination question paper 9 questions were asked. In how many different ways can a student choose 5 questions to answer? If question number 1 is made compulsory, in how many ways can the student select to answer 5 questions in all?

11. In how many ways can 4 students be selected out of 12 students if (a) 2 particular students are excluded? (b) 2 particular students are included?

12. In how many ways a committee of 8 members be selected from 8 men and 6 ladies, if the committee is to include not more than three ladies?

13. From 3 men and 7 women a committee of 5 is to be formed. In how many ways can this be done so as to include at least one man?

14. An examination paper consisting of 10 questions is divided into two groups A and B. Group A contains 6 questions. In how many ways can an examinee attempt 7 questions selecting at least two questions from each group?

Answers

1. (a) 21 (b) 4,950 (c) 330
2. 27
3. (a) 8; 165 (b) $n = 8, r = 3$
4. (a) 66 (b) 7315 (c) 792
5. (a) 6160 (b) 200 (c) 280

6. 1200 7. 120 8. 31
9. 31 10. 126, 70
11. (a) 210 (b) 45
12. 1589 13. 231 14. 116

Objective Questions

1. Which of the following is not true?
   (a) ${}^nC_r = r! \cdot {}^nP_r$
   (b) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
   (c) ${}^nC_0 = 1$
   (d) ${}^nC_n = n$
2. If ${}^nP_r = 336$ and ${}^nC_r = 56$ then $r =$
   (a) 2
   (b) 3
   (c) 4
   (d) 5
3. If there are 10 persons in a party and each two of them shakes hands with each other, how many hand shakes happen in the party?
   (a) 90
   (b) 45
   (c) 30
   (d) 15
4. A man has 4 friends. In how many ways can he invite one or more of them to a dinner?
   (a) 16
   (b) 15
   (c) 12
   (d) 4
5. If ${}^9C_{2r} = {}^9C_{3r-1}$ then $r =$.
   (a) 2, 4
   (b) 1, 3
   (c) 2, 3
   (d) 1, 2
6. From 10 persons, in how many ways can a committee of 4 be made when one particular person is always included?
   (a) 72
   (b) 84
   (c) 92
   (d) 100
7. How many different sums of money can be made from 4 coins of different denominations?
   (a) 4
   (b) 8
   (c) 15
   (d) 16
8. In an examination, a candidate has to pass in each of the 5 subjects. In how many ways can the candidate fail?
   (a) 31
   (b) 30
   (c) 32
   (d) 1

Answer Sheet

1	2	3	4	5	6	7	8	9	10
d	b	b	b	d	b	c	a		

Bibliography

Bajracharya, D.R., Shrestha, R.M. & et al, *Basic Mathematics I, II*. Kathmandu: Sukunda Pustak Bhawan.

Brown, J.W., Churchill, R. V. (2009), *Complex Variables and Applications*. McGraw Hill Higher Education.

Budnick, F.S., *Applied Mathematics for Business, Economics and the Social Sciences*. McGraw-Hill Ryerson Limited.

Cheney, W., Kincaid, D., *Linear Algebra, Theory and Applications*. 2nd Edition, Jones & Bartlett Learning.

Hall, H.S., Knight, S.R., *Higher Algebra*. New Delhi: Book Palace.

Monga, G.S., *Mathematics for Management and Economics*. New Delhi: Vikas Publishing House Pvt. Ltd.

Paudel, K.C., GC. F.B., and et. al, *Higher Secondary Mathematics*. Kathmandu: Asmita Books Publishers & Distributors Pvt. Ltd.

Stewart, J., *Calculus*. 7th Edition, New York.

Thomas, G.B., Finney, R.L., *Calculus and Analytic Geometry*. 9th Edition, Addison–Wesley.

Upadhayay, H.P., Paudel, K.C. & et al. *Elements of Business Mathematics*. Kathmandu: Pinnacle Publication Pvt. Ltd.

William, L. S., (1965), *Basic Concepts of Elementary Mathematics*. John Wiley and Sons.

Yamane, T., *Mathematics for Economist*. Prentice-hall of India.

Appendix

Mathematical Software

(Mathematica and MATLAB)

The most popular mathematical software are MATLAB, Mathematica and Maple. The websites for these software systems are MATLAB (www.mathworks.com), Mathematica (www.wolfram.com), and Maple (www.maplesoft.com). There are many other mathematical software systems. But each of these systems has its own syntax. As per the demand of course, two software systems MATLAB and Mathematica are introduced.

According to Wikipedia, the free encyclopedia, "Wolfram Mathematica (usually termed Mathematica) is a modern technical computing system spanning most areas of technical computing inlucding neural networks, machine learning, image processing, geometry, data science, visualizations, and others. The system is used in many technical, scientific, engineering, mathematical, and computing fields. It was conceived by Stephen Wolfram and is developed by Wolfram Research of Champaign, Illinois. The Wolfram Language is the programming language used in Mathamatica."

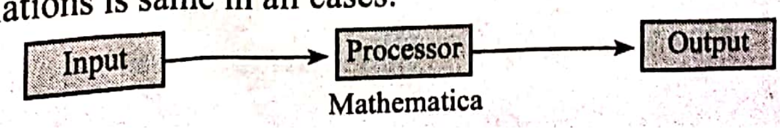
According to Wikipedia, the free encyclopedia "MATLAB (matrix laboratory) is a multi-paradigm numerical computing environment and proprietary programming language developed by Mathworks. MATLAB allows matrix manipulations, plotting of functions and data, implementations of algorithms, creation of user interfaces, and interfacing with programs written in other languages, including C, C++, C#, Java, Fortran and Python.

Although MATLAB is intended primarily for numerical computing, an optional toolbox uses the MuPAD symbolic engine, allowing access to symbolic computing abilities. An additional package, Simulink, adds graphical multi-domain simulation and model-based design for dynamic and embedded systems.

MATLAB has more than 3 millions users worldwide. MATLAB users come from various backgrounds of engineering, science, and economics."

Unit 2 : Relation, Functions and Graphs [Mathematica]

First of all, install and run Mathematica in your computer. The details of running Mathematica differ from one compute to another computer, but the calculations is same in all cases.

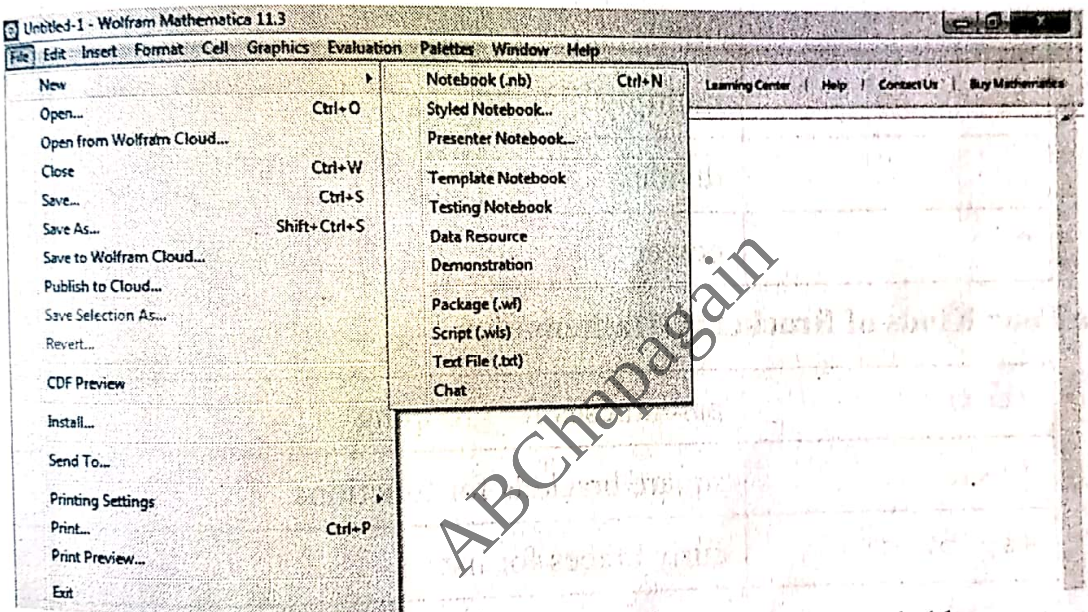


Mathematica is the creation of Stephen Wolfram, a theoretical physicist who made the important contribution to Mathematics. Wolfram describes Mathematica as "the world's only fully integrated environment for technical computing." It is a software package that does algebraic manipulations as well as the graphics. It is the world's fully integrated environment for technical computing.

Mathematica Notebook

When we start Mathematica, we get a menu bar at the top of the screen. To get started computing with Mathematica, we need to create a Notebook. For this, from the menu bar select

File $\Rightarrow$ New $\Rightarrow$ Notebook (.nb).



When we enter a command then the Mathematica Kernel (the part of the software that actually does the computation) executes it and returns the result. For example

```
In[1]:= 1 + 3
Out [1]= 4
```

The input cell (labeled by `In[1]:=`) contains the expression 1 + 3, which Mathematica evaluates and gives the result 4 in the output cell (labeled by `Out[1]=`).

Here, we have to type only '1 + 3'; then Mathematica automatically supplied the label "`In[1]:=`" and same for output. If we see far right of the document, we see the brackets that indicate the grouping of the material into cells.

To Run Mathematica

1. When Mathematica starts up, it gives a blank notebook.
2. Enter the input into the notebook.
3. Type Shift + Enter for output [To type Shift + Enter, hold down the shift key, then press Enter]

When we send Mathematica input from notebook, it will label input with `In[n]:=` and corresponding output with `Out[n]=`. To exist Mathematica, we can choose the Quit menu item in the notebook interface.

Note: (i) *The arguments of Mathematica functions should be enclosed by square [ ] brackets.*

(ii) *The built-in functions should begin with CAPITAL LETTERS.*

Arithmetic Operations in Mathematica

Symbols	Operations
+	add
–	minus
`space or *`	multiply
/	divide
^	power

The Four Kinds of Bracket in Mathematica

`(term)`	parentheses for grouping
`f[x]`	square brackets for functions
`{a, b, c}`	curly braces for list
`a[[i]]`	double brackets for indexing (Part [*a*, *i*])

Using Mathematica, we can calculate exact value of expression which has more than 10 (in general electronic calculator) places after decimal. Mathematica can handle numbers of any size.

End you input with // N.

It gives approximate value.

For Examples

In[2]:= `2^50`

Out[2]= 1 125 899 906 842 624

In[3]:= `2^50 // N`

Out[3]= 1.1259×10^{15}

Some Mathematical Functions

`Sqrt[x]`	$\sqrt{x}$
`Exp[x]`	e^x
`Log[x]`	$\log_e x$
`Log[a, x]`	$\log_a x$
`Sin[x], Cos[x], Tan[x]`	$\sin x$, $\cos x$, $\tan x$
`Abs[x]`	$\lvert x \rvert$, absolute value
`Mod[m, n]`	m modulo n
`Pi`	π
`E`	e

For Examples

In[4]:= `Sqrt[81]`

Out[4]= 9

In[5]:= `Sqrt[2]`

Out[5]= $\sqrt{2}$

In[6]:= `Sqrt[2] // N`

Out[6]= 1.41421

In[7]:= `Sin[Pi / 6]`

Out[7]= $\frac{1}{2}$

In[8]:= `Log[3, 27]`

Out[8]= 3

For Previous Results

`%` the last result

`% %` the next to last result

`% % ... %` (n times) the n^{th} previous result.

Note: *x = value assign a value to the variable x.*

x = . or clear [x] remove any value assigned to x.

Some examples of finding values, expanding, factorizing and solving are as follows:

```
In[1]:= 3 x + 5 /. x → 1
Out[1]= 8

In[2]:= Expand[(1 + x) ^2]
Out[2]= 1 + 2 x + x^2

In[3]:= Factor[%]
Out[3]= (1 + x)^2

In[4]:= Factor[x^33 + y^33]
```

Out[4]= $(x+y)\,(x^2 - x y + y^2)\,(x^{10} - x^9 y + x^8 y^2 - x^7 y^3 + x^6 y^4 - x^5 y^5 + x^4 y^6 - x^3 y^7 + x^2 y^8 - x y^9 + y^{10})$
$(x^{20} + x^{19} y - x^{17} y^3 - x^{16} y^4 + x^{14} y^6 + x^{13} y^7 - x^{11} y^9 - x^{10} y^{10} - x^9 y^{11} + x^7 y^{13} + x^6 y^{14} - x^4 y^{16} - x^3 y^{17} + x y^{19} + y^{20})$

```
In[5]:= 2 < 1
Out[5]= False

In[6]:= Solve[x^2 - 5 x + 4 == 0, x]
Out[6]= {{x → 1}, {x → 4}}
```


Relational and Logical Operations

`x = = y`	equal
`x! = y or x ≠ y`	unequal
`x > y`	greater than
`x > = y or x ≥ y`	greater than or equal to
`x < y`	less than
`x < = y or x ≤`	less than or equal to

Graphing

To plot function *f* from $x_{\min}$ to $x_{\max}$, the basic syntax is

```
Plot [f[x], {x, xmin, xmax}]
```

Redrawing and Combining Plots

`Show[Plot]`	redraw a plot
`Show[Plot, Option → Value]`	redraw with options changed
`Show[Plot1, Plot2, …]`	combine several plots

Plotting List of Data

`ListPlot [{y1, y2, ...}]` plot $y_1, y_2, \ldots$ at $x = 1, 2, \ldots$

`ListPlot [{x1, y1} {x2, y2}, ...]` plot points $(x_1, y_1) \ldots$

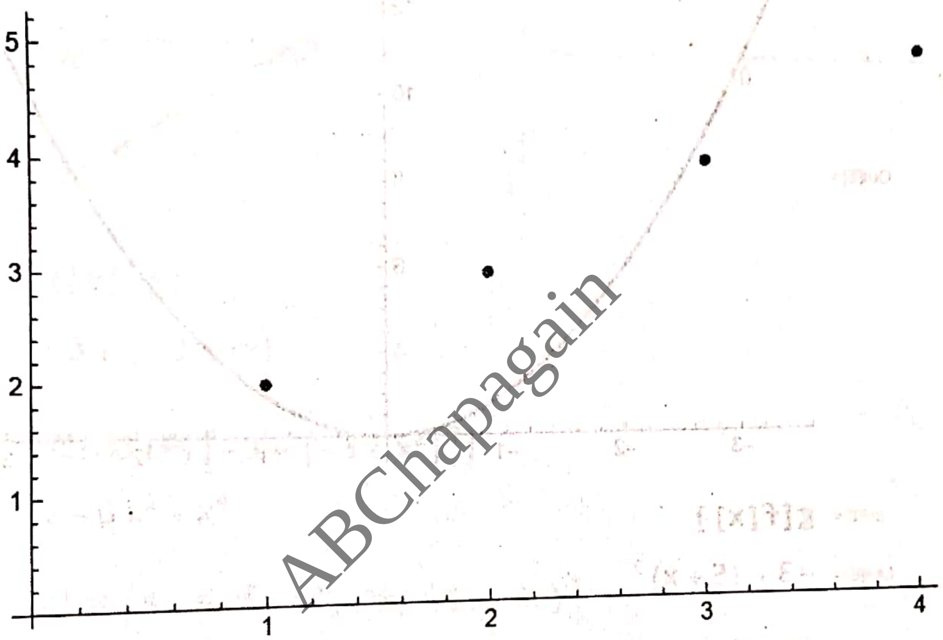
`ListPlot [List, PlotJoined → True]` join the points with lines.

For Example

```
In[7]:= x = Table[x + 1, {x, 4}]
Out[7]= {2, 3, 4, 5}

In[8]:= ListPlot[x]
```

Out[8]=



Applying Functions Repeatedly

`Nest [f, x, n]` apply the function *f* nested *n* times to *x*.

For example

```
In[1] : = Nest [f, x, 3]
Out[1] = f[f[f[x]]]
```

Applying Functions to Parts of Expressions

```
In[1] : = Map [f, {x, y, z}]
Out[1] = {f[x], f[y], f[z]}.
```

Inverse

`InverseFunction[f]` the inverse of *f*.

Composite

`Composition[f, g, ...]` the composition of *f*, *g*, …

Example: If $f(x) = x + 5$ and $g(x) = x^2 - 3$ then find $(fog)(x)$, $(gof)(x)$, $(fof)(x)$ and $(gof)(x)$ with graph using Mathematica.

Solution

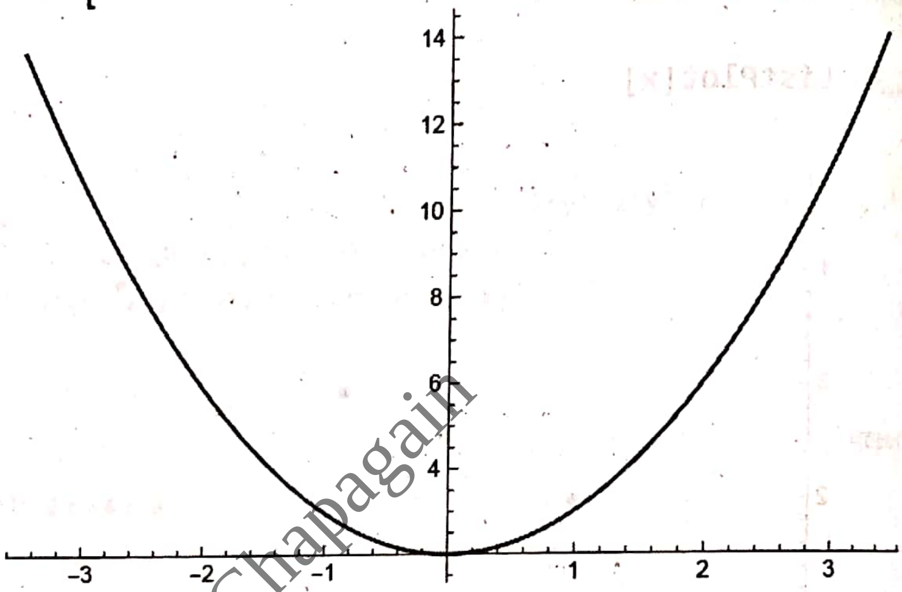
```
In[2]:= f[x_] := x + 5
        g[x_] := x^2 - 3

In[4]:= f[g[x]]

Out[4]= 2 + x^2

In[5]:= Plot[2 + x^2, {x, -3.4641, 3.4641}]
```

Out[5]=

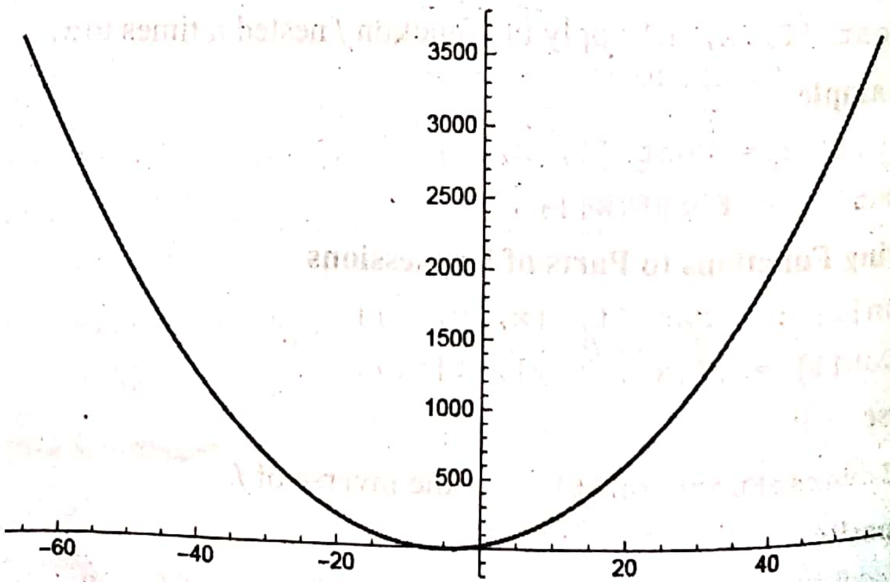


```
In[6]:= g[f[x]]

Out[6]= -3 + (5 + x)^2

In[7]:= Plot[-3 + (5 + x)^2, {x, -65., 55.}]
```

Out[7]=

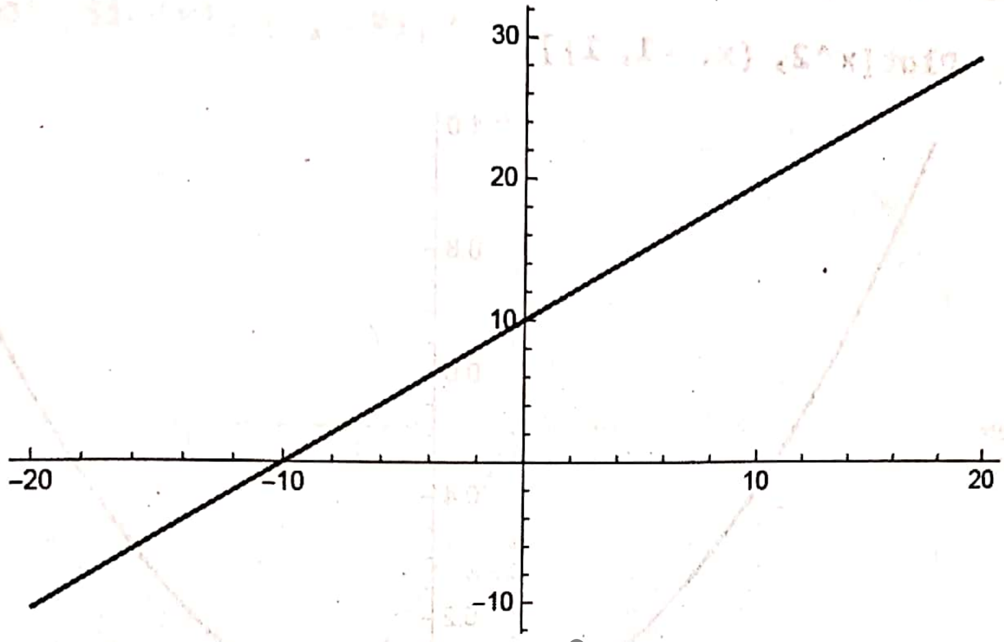


```
In[8]:= f[f[x]]

Out[8]= 10 + x
```

In[9]:= `Plot[10 + x, {x, -20, 20}]`

Out[9]=



In[10]:= `g[g[x]]`

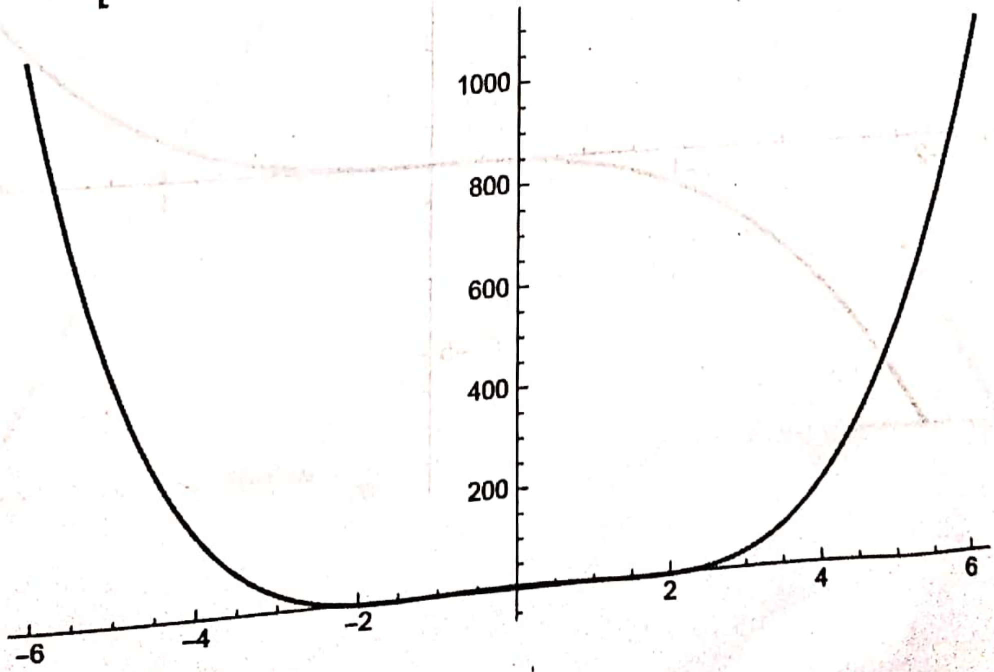
Out[10]= $-3 + \left(-3 + x^2\right)^2$

In[11]:= `Simplify`$\left[-3 + \left(-3 + x^2\right)^2\right]$

Out[11]= $6 - 6x^2 + x^4$

In[12]:= `Plot`$\left[6 - 6x^2 + x^4, \{x, -6., 6.\}\right]$

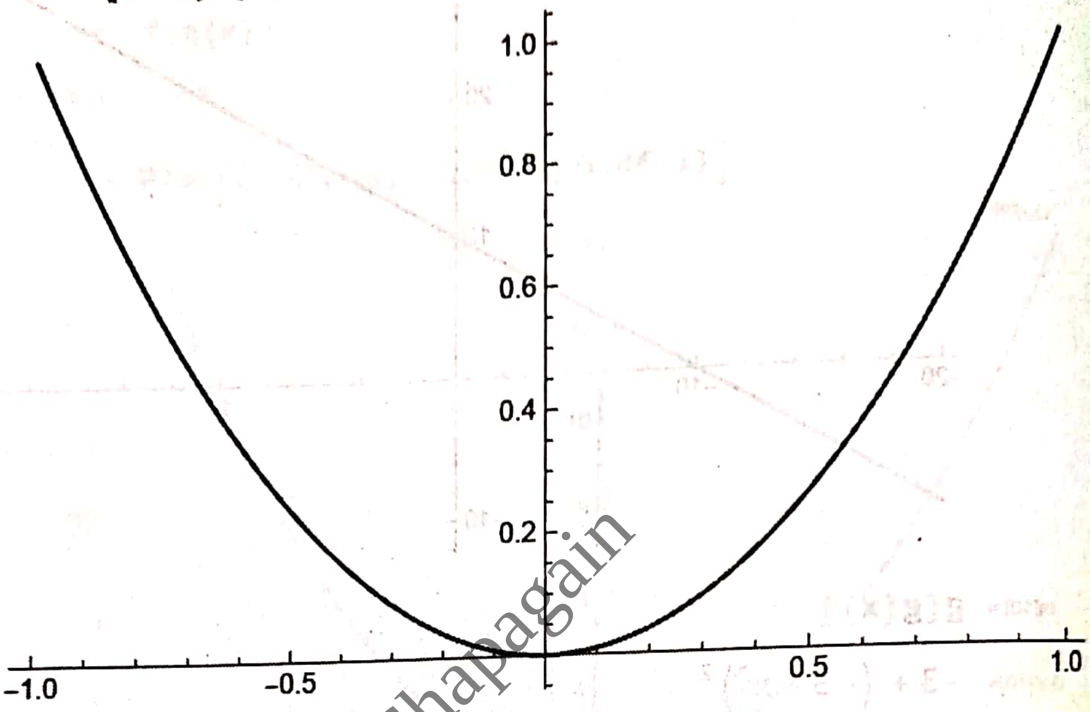
Out[12]=



Graph of x^2 and x^3

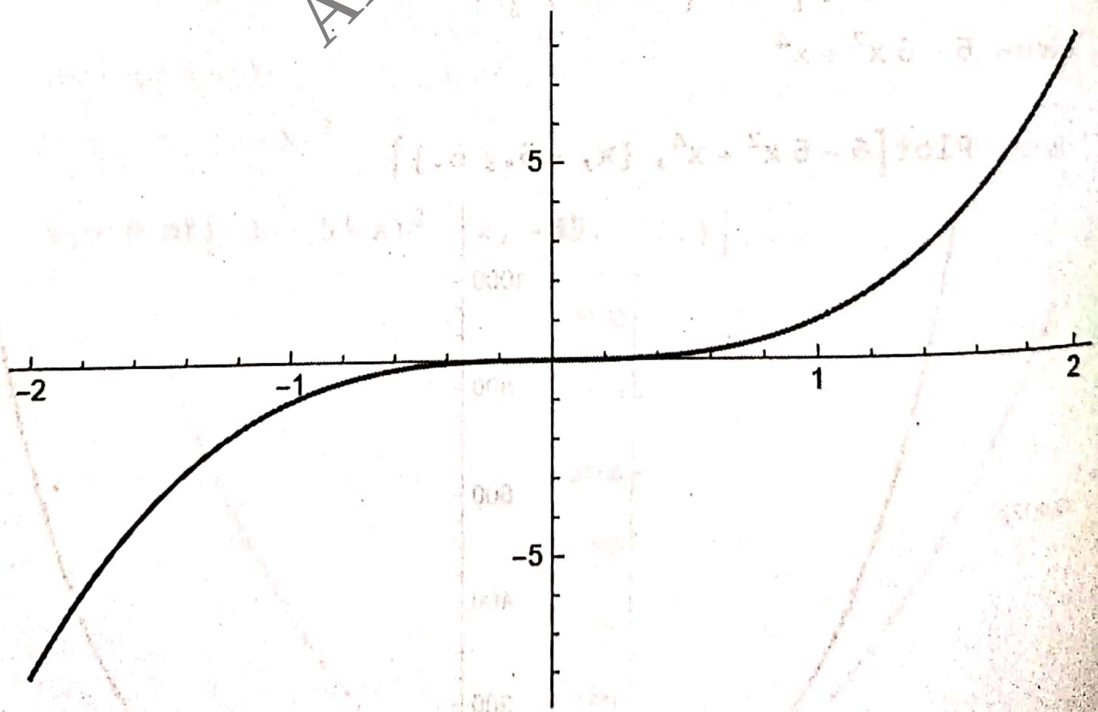
In[3]:= `Plot[x^2, {x, -1, 1}]`

Out[3]=



In[4]:= `Plot[x^3, {x, -2, 2}]`

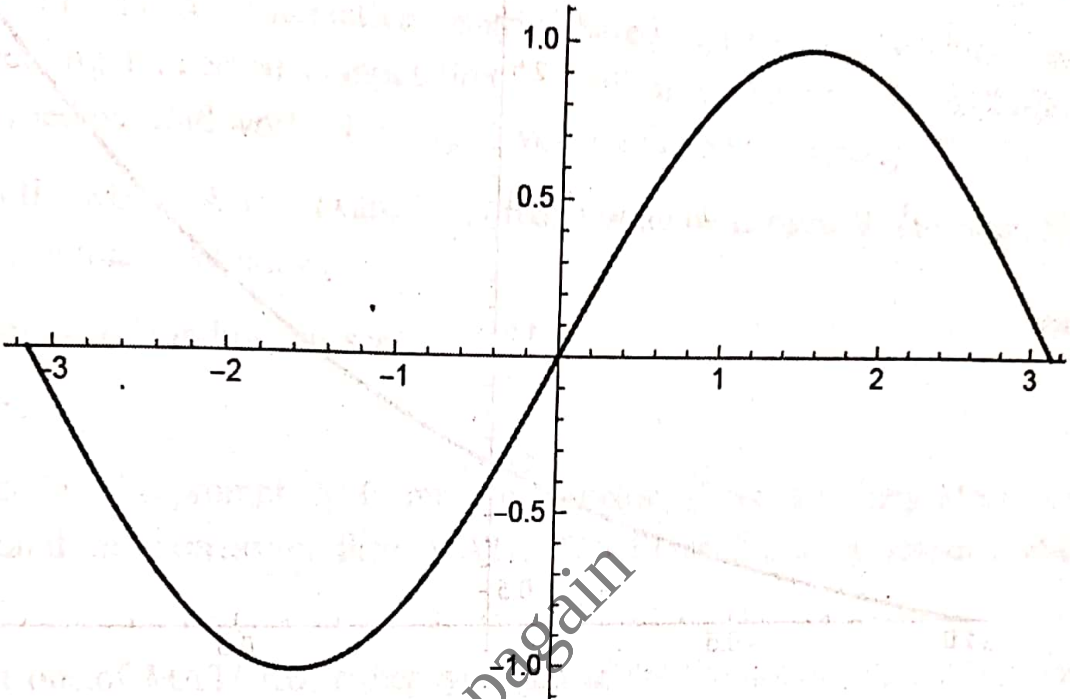
Out[4]=



Graph of Some Trigonometric Functions

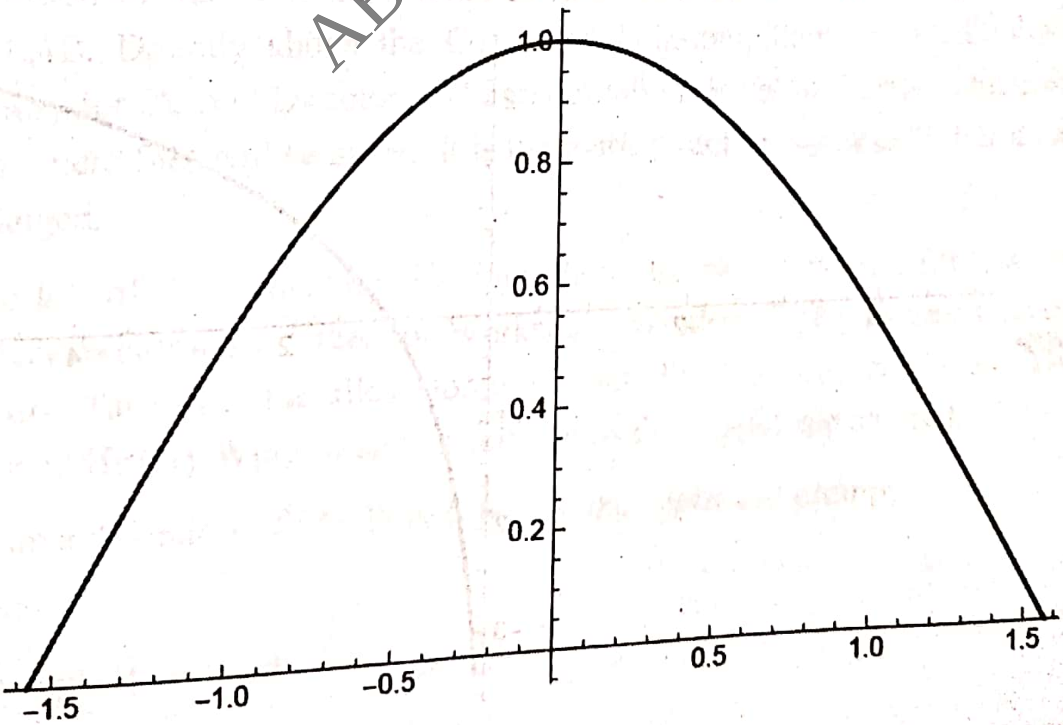
In[5]:= `Plot [Sin[x], {x, -Pi, Pi}]`

Out[5]=



In[7]:= `Plot[Cos[x], {x, -Pi/2, Pi/2}]`

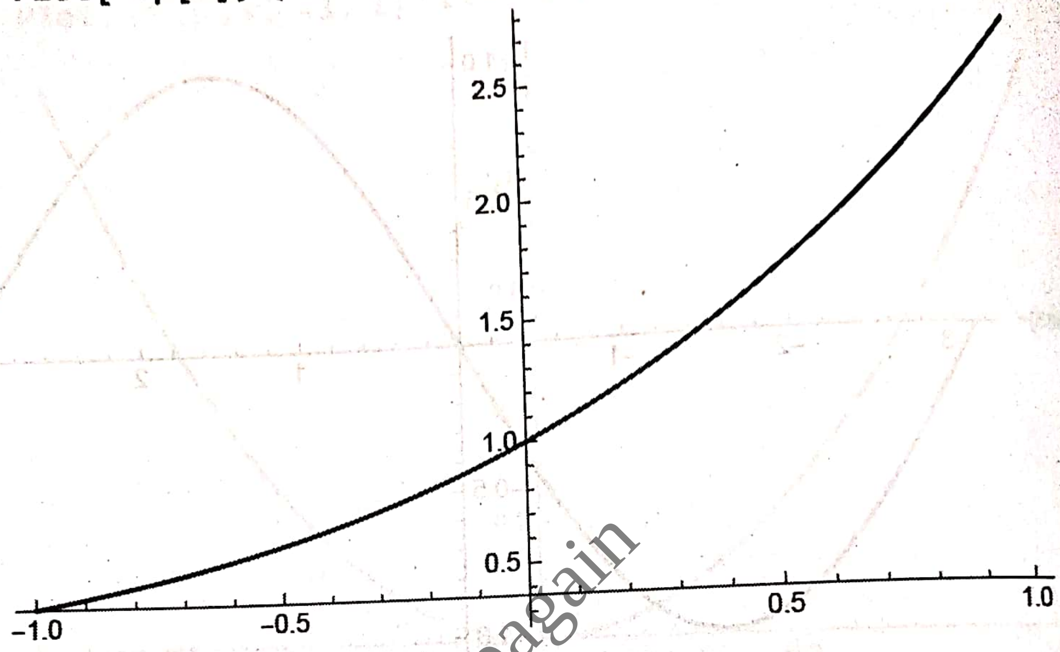
Out[7]=



Graph of Exponential Function

In[9]:= `Plot[Exp[x], {x, -1, 1}]`

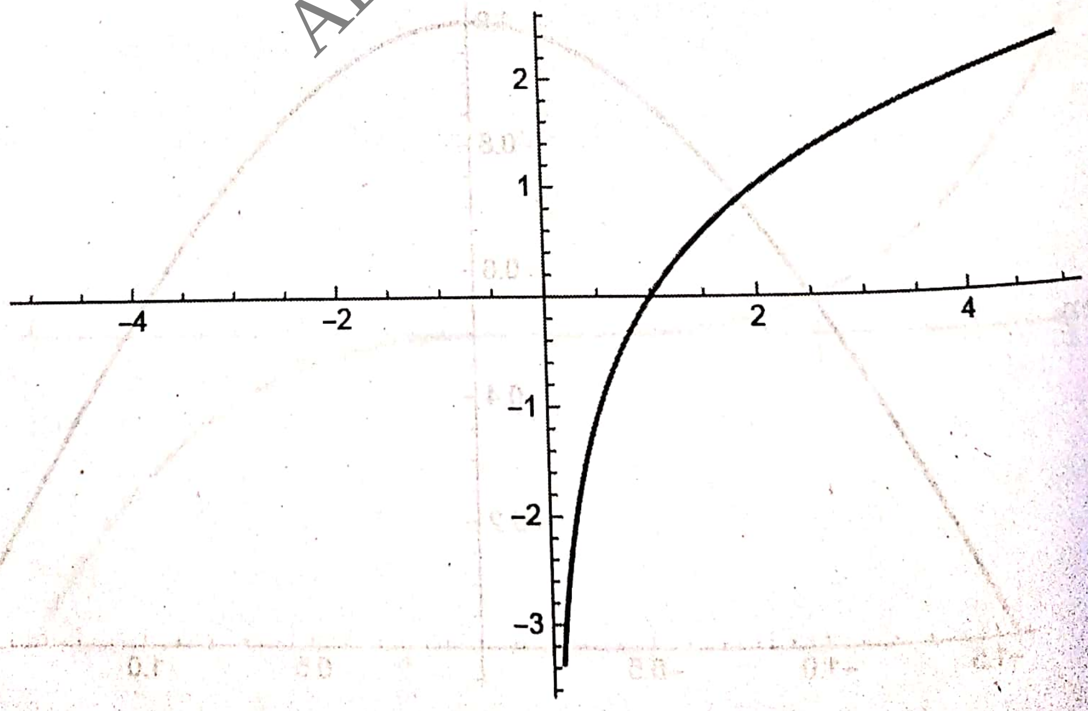
Out[9]=



Graph of Logarithmic Function

In[1]:= `Plot[Log[2, x], {x, -5, 5}]`

Out[1]=



Unit 4 : Matrices and Determinants (MATLAB)

MATLAB is an interactive matrix based system for scientific and engineering numerical computation as well as visualization. MATLAB is developed by Mathworks. It is high level programming language.

When the MATLAB software is started, a window is opened. The main part is the Command Window.

In Command Window, we see

```
>>
```

It is called the prompt. In Command Window, if we enter any MATLAB command or expression then MATLAB will immediately respond with result.

To get out of MATLAB, either type quit at the prompt or choose file then Exit MATLAB from the menu.

In addition to the Command Window, there are several other windows in MATLAB. Directly above the Command Window, there is a pull down Directory for Current Directory. The folder which is set as Current Directory is that where files will be saved. It is the work directory by default, but it can be changed.

To the left of the Command Window, there are two tabs; one for Current Directory Window and other for Workspace Window. If we choose Current Directory Tab, then the files stored in that directory are displayed. The Command History Window shows commands that have been entered.

In command window of MATLAB, we see the command prompt

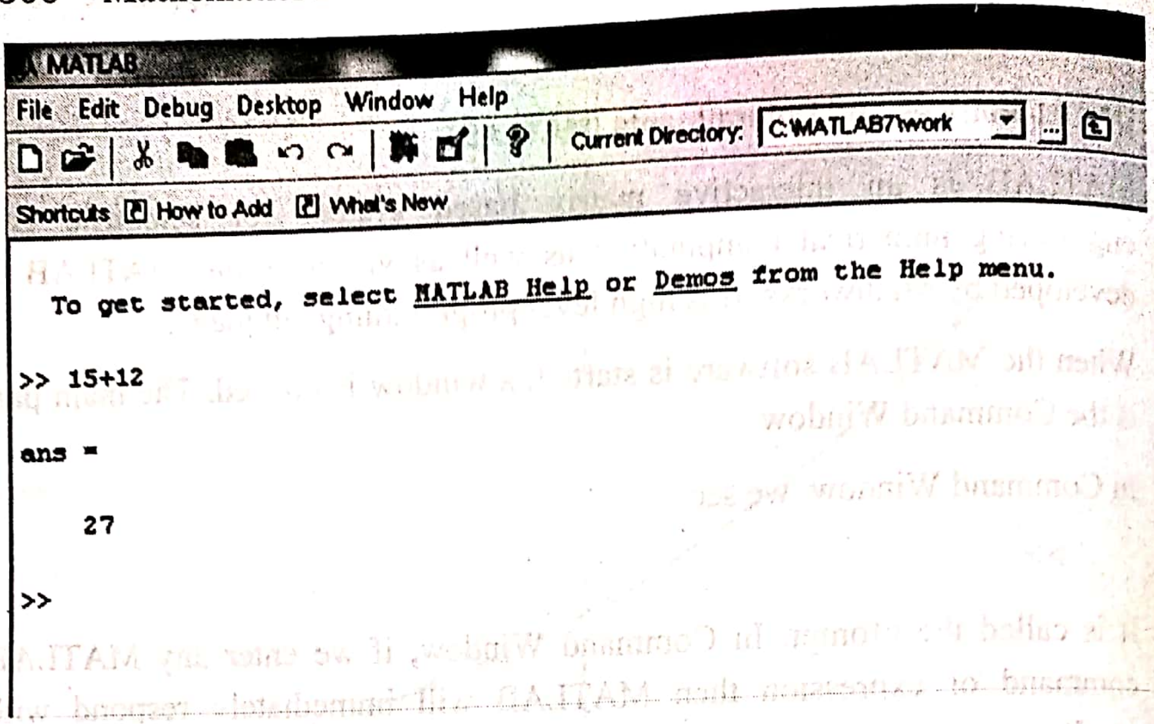
```
>>
```

For example if we type

```
>> 15 + 12
```

MATLAB will display (Press enter for result)

```
ans =
      27
```



Next,

```
>> ans - 27
```

gives the

```
ans =
      50
```

Assignment

It refers to assigning values to the variable.

1. Assigning Scalars

Type

```
>> x = 4
```

It will display

```
x =
   4
```

2. Assigning Arrays, Vectors and Matrices

An array is a collection of values by a single variable. One dimensional arrays are called vectors and two dimensional arrays are called matrices.

For example

a.
```
>> a = [3 4 5 6]
a =
   3 4 5 6
```

b.
```
>> b = [3; 4; 5; 6]
```

or by transposing

The result is

```
b =
     3
     4
     5
     6
```

A matrix can be displayed as follows:

```
>> A = [3 4 5; 5 6 7; 7 8 9]
A =
     3    4    5
     5    6    7
     7    8    9
```

Also, we can seperate the rows by the help of Enter key

```
>> A = [3     4     5
        5     6     7
        7     8     9]
```

A(m, n) selects the element in m^{th} row and n^{th} column. For example

```
>> A(1, 2)
   ans =
        4
```

Note: There are several built in functions to create matrices.

Arithmetic operation

Operations	Meaning
+	addition
–	subtraction
*	multiplication
\	left division
/	right division
^	power
'	transpose

*Note: In MATLAB A * A is same as A^2 but it is different from A .^2. The last gives the square of every numbers.*

The difference between left division (\) and right division (/) is as follows:

8 / 4 = 2 [It means $\frac{8}{4} = 2$]

& 8 \ 4 = 0.5000 [It means $\frac{4}{8} = 0.5000$]

Built in function

Command	Function
`abs ( )`	absolute value
`exp ( )`	exponential function
`sqrt ( )`	square root
`log ( )`	logarithmic function

Note: By default MATLAB produces 4 digits after decimal places .

If we want move digits after the decimal places, then we should >> format long.

If you need, any help, take the help of >> help format.

Trigonometric function

Command	Function
`sin ( )`	sine
`cos ( )`	cosine
`tan ( )`	tangent
`cot ( )`	cotangent
`sec ( )`	secant
`csc ( )`	cosecant

Examples

1. To create 3 × 3 matrix of zeros, we should do

```
>> zeros (3, 3)
ans =
        0    0    0
        0    0    0
        0    0    0
```

2. To create 2 × 3 matrix of ones, we should do

```
>> ones (2, 3)
ans =
        1    1    1
        1    1    1
```

Matrix Functions

`eye`	identity matrix
`zeros`	matrix of zeros
`ones`	matrix of ones
`diag`	create diagonal matrix
`triu`	upper triangular part of a matrix
`tril`	lower triangular part of a matrix
`rand`	randomly generated matrix

Example

```
eye (5, 5)
ans =
    1    0    0    0    0
    0    1    0    0    0
    0    0    1    0    0
    0    0    0    1    0
    0    0    0    0    1
```

Some other Commands of Matrix Functions

`size`	size of a matrix
`det`	determinant of a square matrix
`inv`	inverse of a matrix
`rank`	rank of a matrix
`det*inv`	adjoint of a matrix

Example

```
>> A = [1, 2, 5; 0, 7, 1; 2, 3, -5]
>> size (A)
ans =
    3    3
>> det (A)
ans =
    -104
```

Since the determinant is not zero, the matrix is invertible.

```
>> inv (A)
ans =
    0.3654     -0.2404     0.3173
    -0.0192    0.1442      0.0096
    0.1346     -0.0096     -0.0673
```

We can check the result by verifying

$AA^{-1} = A^{-1}A = I$

```
>> A * inv (A)
ans =
    1    0    0
    0    1    0
    0    0    1
>> inv (A) * A
ans =
    1.0000    0.0000    0
    0         1.0000    0
    0         0         1.0000
>> rank (A)
    ans =
        3
```

Some Examples

1. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$; find:

   (a) $A + B$ (b) $A - B$

Solution

In MATLAB, we can solve this in the following way

```
>> A = [1, 2, 3; 2, 3, 4]
A =
      1      2      3
      2      3      4
>> B = [0, 1, 2; 3, 2, 5]
B =
      0      1      2
      3      2      5
>> A + B
ans =
      1      3      5
      5      5      9
>> A - B
ans =
      1      1      1
     -1      1     -1
```


2. Given $A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{pmatrix}$, find AB and BA.

Solution

In MATLAB, we do the following steps for the solution.

```
>> A = [4, 2, -1; 3, -7, 1]
A =
      4      2     -1
      3     -7      1
>> B = [2, 3; -3, 0; -1, 5]
B =
      2      3
     -3      0
     -1      5
```

```
>> A * B
ans =
     3      7
     26     14
>> B * A
ans =
     17     -17     1
     -12    -6      3
     11     -37     6
```

3. Find the value of determinants by expanding along any row or column.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$$

Solution

Using MATLAB

```
>> A = [1, 2, 3; 2, 4, 1; 3, 2, 9]
A =
     1      2      3
     2      4      1
     3      2      9
>> det (A)
ans =
     - 20
```


4. Find the adjoint of the following matrices. $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

Solution

Using MATLAB

```
>> A = [1, 2, -2; -1, 3, 0; 0, -2, 1]
A =
     1      2      -2
     -1     3      0
     0      -2     1
>> det (A) * inv (A)
ans =
     3.0000      2.0000      6.0000
     1.0000      1.0000      2.0000
     2.0000      2.0000      5.0000
```

5. Find the inverse of the following matrices if possible.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Solution

Using MATLAB

```
>> A = [1, 1, 1; 1, -1, 1; 1, 1, -1]
A =
      1      1      1
      1     -1      1
      1      1     -1
>> inv (A)
ans =
         0       0.5000      0.5000
    0.5000      -0.5000           0
    0.5000            0     -0.5000
```

Matrix Equation

Let $Ax = b$ be a system of equation in matrix form.

Then the solution is

```
x = A \ b.
```

For example:

Solve

$x + y + z = 9$; $2x + 5y + 7z = 52$, $2x + y - z = 0$

In MATLAB,

```
>> A = [1   1   1;   2   5   7;   2   1   -1];
>> b = [ 9   52   0]';
>> X = A \ b
```

Enter

```
x =
     1
     3
     5
```

6. Find the rank of the following matrices. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & -1 \\ 2 & 1 & 0 & 5 \end{bmatrix}$

Solution

Using MATLAB

```
>> A = [1, 2, 3, 1; 2, 5, 7, -1; 2, 1, 0, 5]
A =
      1      2      3      1
      2      5      7     -1
      2      1      0      5
>> rank (A)
   ans =
        3
```

Solving Algebraic Equations in MATLAB

We solve the algebraic equations as follows:

Example (1) Solve $x + 3 = 0$

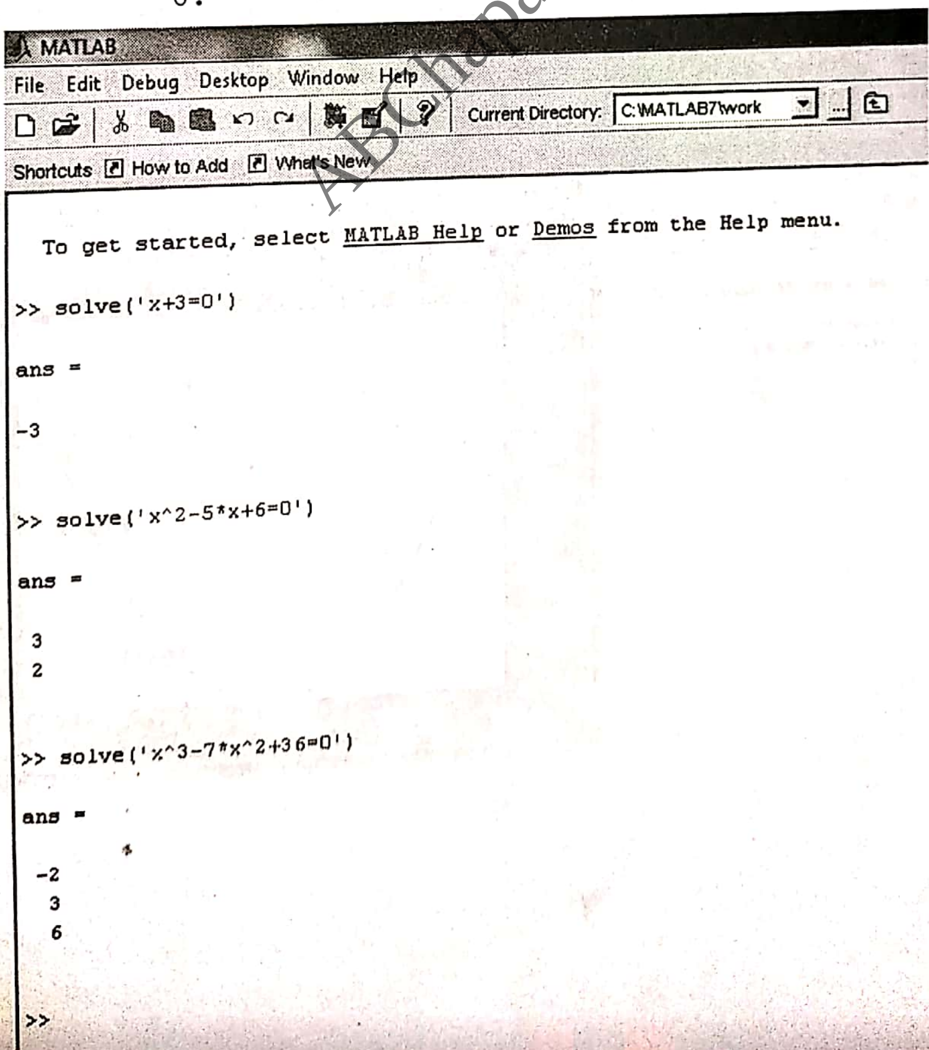
```
>>  solve ('x + 3 = 0')
ans =
          -3.
```

(2) Solve $x^2 - 5x + 6 = 0$

```
>>  solve ('x^2 - 5x + 6 = 0')
ans =
          3
          2.
```

(3) Solve $x^3 - 7x^2 + 36 = 0$

```
>>  solve ('x^3 - 7*x^2 + 36 = 0')
ans =
          -2
           3
           6.
```

Plotting

The plot in MATLAB appears in the Graphic / Figure Window. MATLAB provides facilities for 2 – D and 3 – D graphics. The most command for 2 – D and 3 – D plot are plot and plot 3 command respectively.

2–D plotting

Syntax: plot (x data, y data)

Here, y data are the value of y variable corresponding to x variable which are given at x data, such as $y = f(x)$.

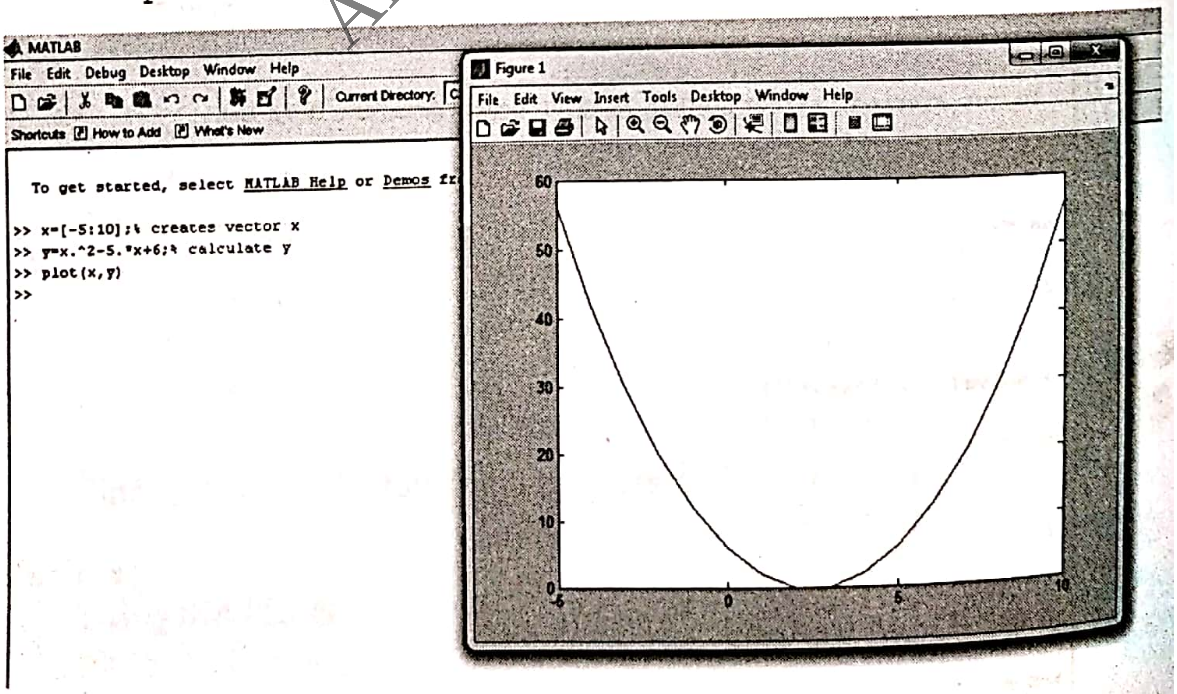
This command plots x data along horizontal axis and y data along vertical axis.

Example:

Plot the function $y = x^2 - 5x + 6$, where $x \in [-5, 10]$

Solution

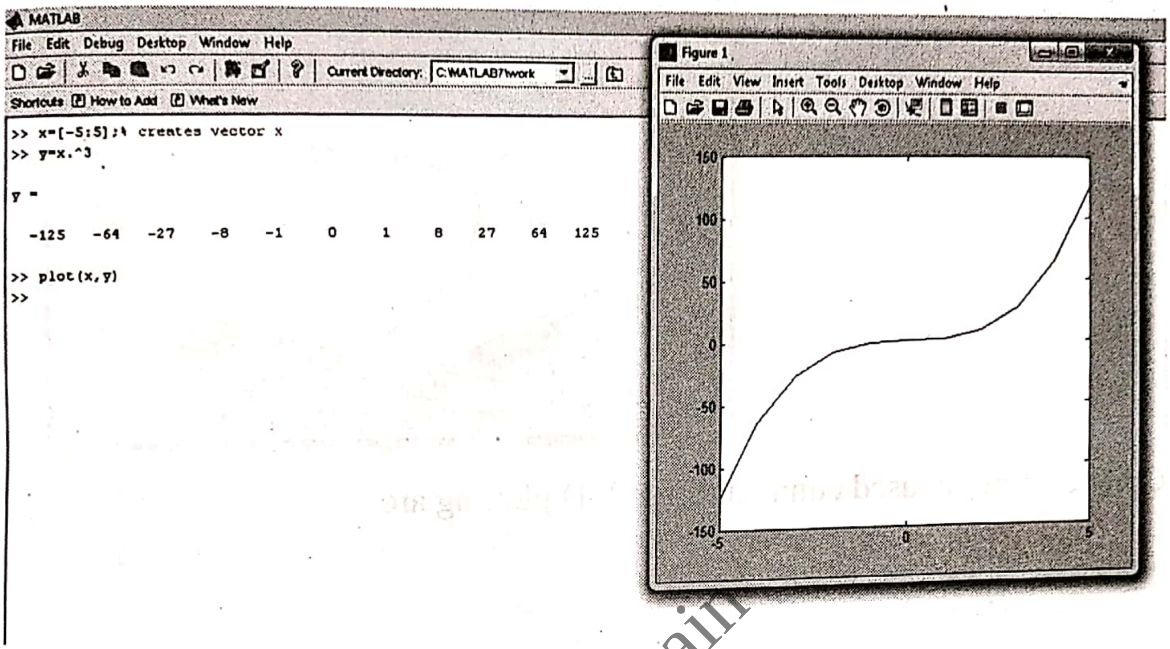
```
>> x = [-5:10]; % creates vector x
>> y = x.^2 - 5.*x + 6; % calculate y
>> plot (x, y)
```

Example:

Plot the function $y = x^3$ in $[-5, 5]$

Solution



Arrays

Variable = initial value: step size: final value

Here, initial value is the first element of the row, final value is the last element of the row and step size is the difference between the consecutive elements of that row.

For example

```
>> x = 2:1:10
   x =
          2   4   6   8   10
```

3–D plotting

Syntax: plot 3(x, y, z)

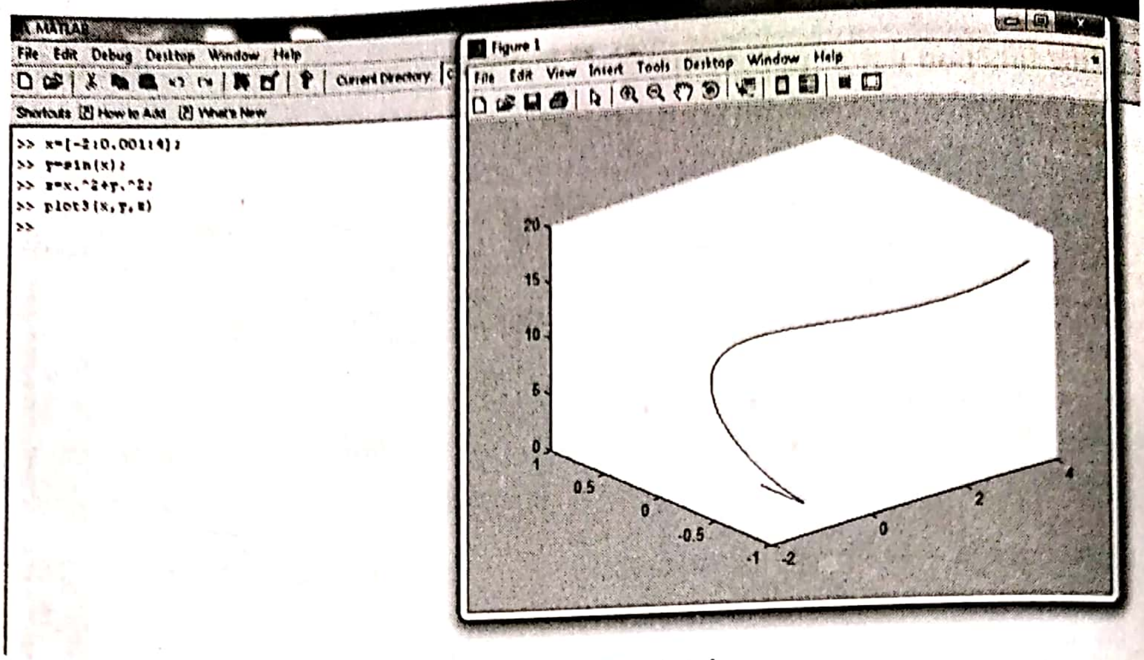
Example:

$x = [-2 : 0.001 : 4]$;

$y = \sin(x)$;

$z = x.\hat{}2 + y.\hat{}2$;

plot 3(x, y, z).



Other commonly used commands for 3–D plotting are

plot 3(x, y, z)

mesh (x, y, z)

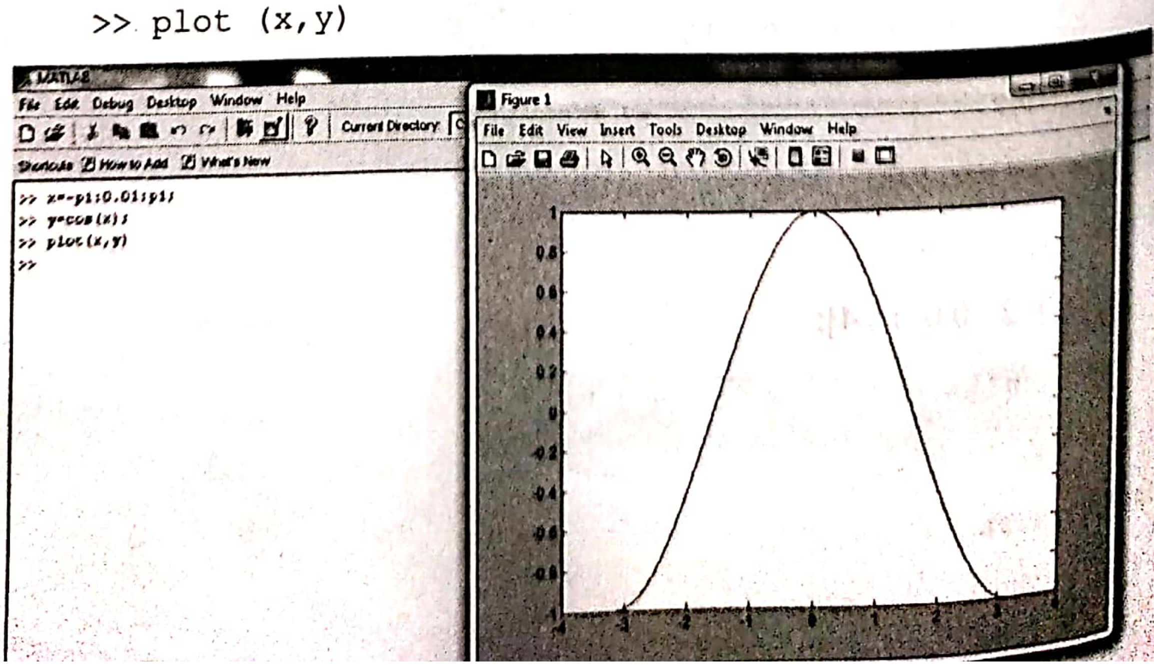
surf (x, y , z)

contour (x, y , z)

Example

To obtain the graph of y = cos (x) from $-\pi$ to π, we first define the vector with components equally spaced numbers between $-\pi$ and π, with increment 0.01 (say).

```
>> x = -pi:0.01:pi;
>> y = cos(x);
>> plot (x,y)
```



In order to label the axis on the graph, we have to do followings.

```
>> xlabel ('x')

>> ylabel ('y = cos(x)')
```

We can put a title on the top using

```
>> title ('Graph of cosine from -\pi to \pi').
```

For different line types, plot symbol and colours, we have to use the followings.

y	yellow
k	black
w	white
g	green
b	blue
r	red
.	point
*	star
–	solid
--	dashed

For example, to obtain the graph in red, we type

```
>> plot (x, y, 'r')
```

Command	**Explanation**
who	Gives a list of the variables in use
whos	Gives a list of the variables in use as well as some extra information
clear	Removes all variables
clc	Clear all
clear x y	Removes the variables x and y

Unit 5

Conic Section (Mathematica)

To draw circle using Mathematica software, the command is

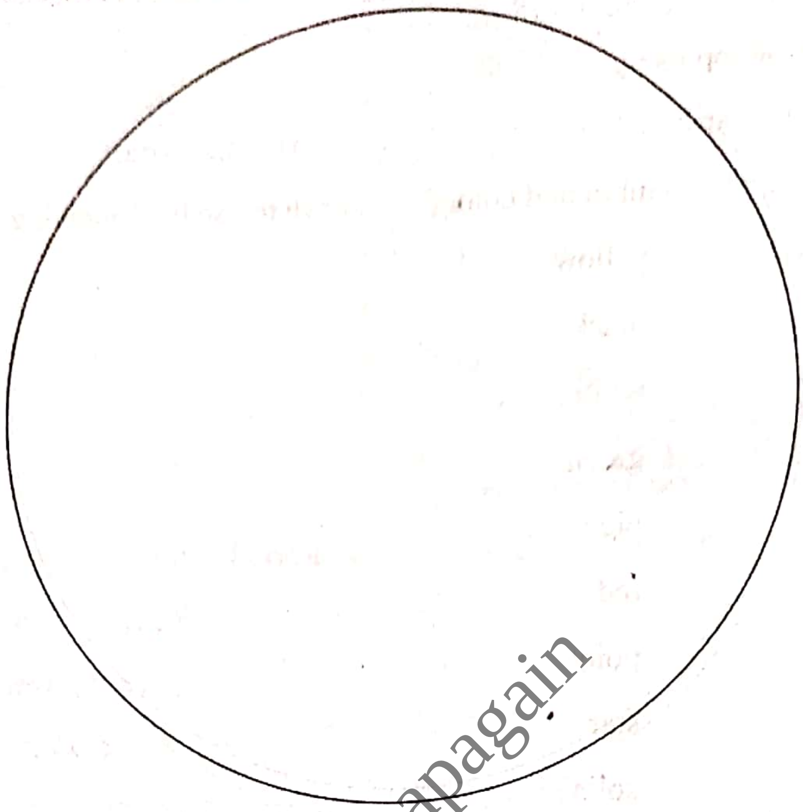
```
Graphics [{Circle [{h, k}, r]}]
```

where (h, k) is the centre and r is the radius.

For Examples

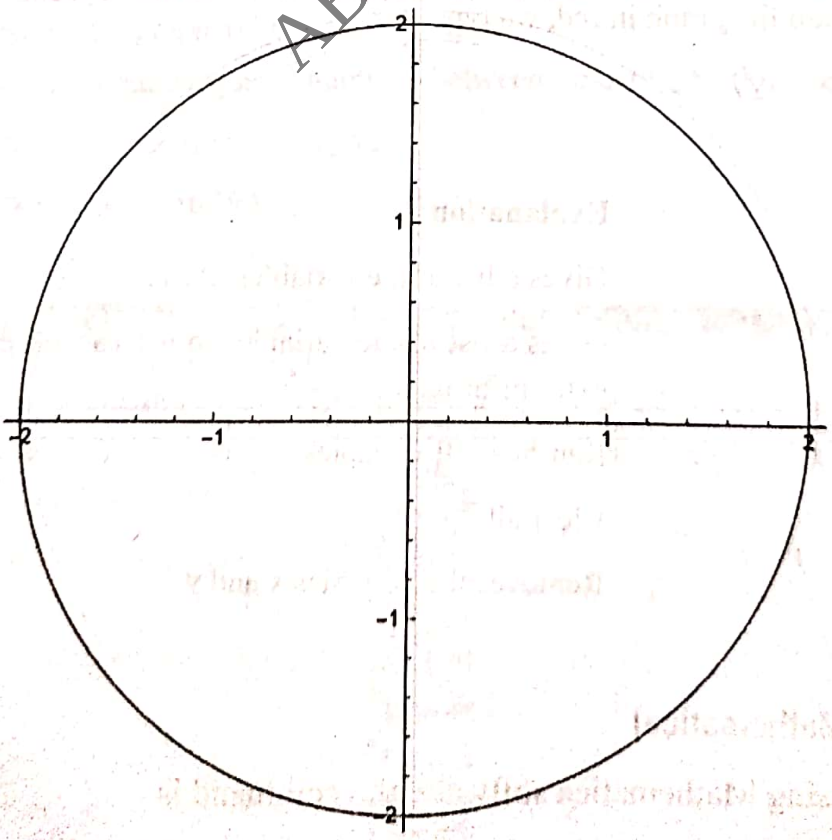
```
In[1]:= Graphics[{Circle[{0, 0}, 2]}]
```

Out[1]=



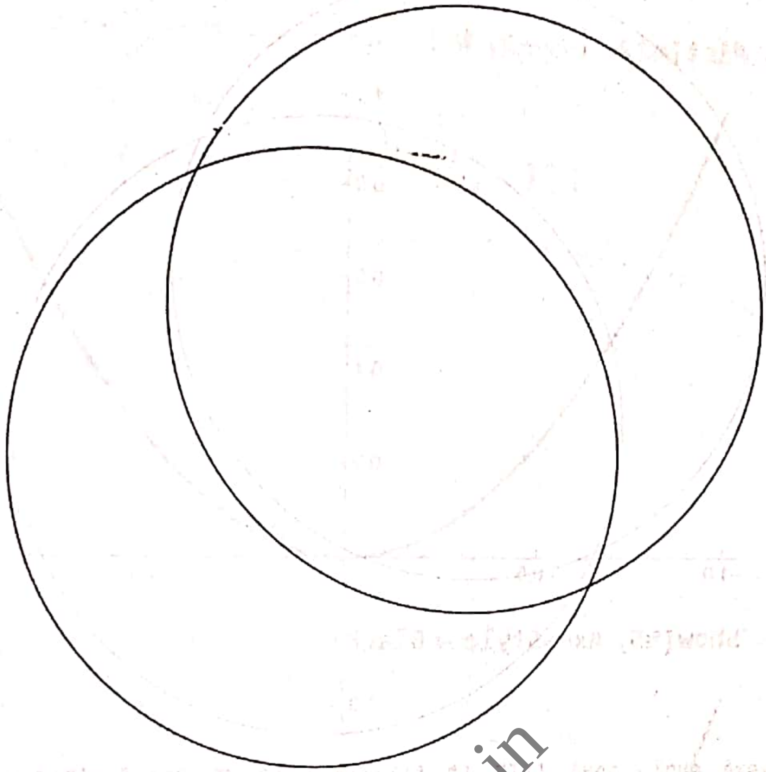
```
In[2]:= Show[Graphics[{Circle[{0, 0}, 2]}], Axes → True, AxesStyle → Black]
```

Out[2]=



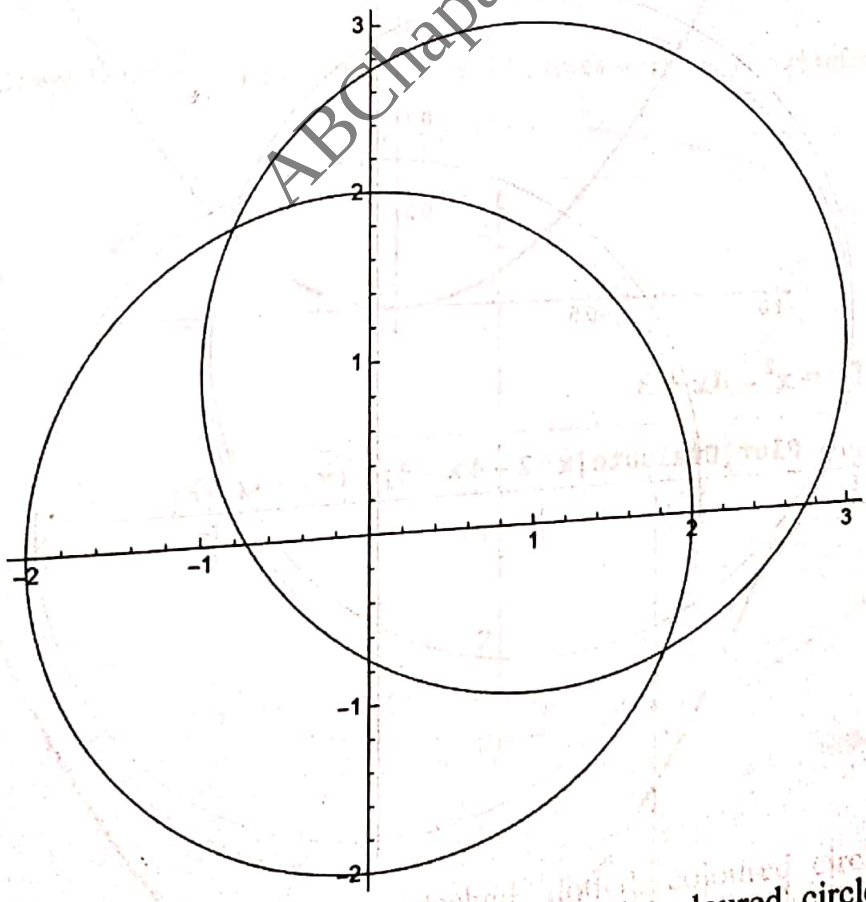
```
In[3]:= Graphics[{Circle[{0, 0}, 2], Circle[{1, 1}, 2]}]
```

Out[3]=



```
In[4]:= Show[Graphics[{Circle[{0, 0}, 2], Circle[{1, 1}, 2]}], Axes → True, AxesStyle → Black]
```

Out[4]=

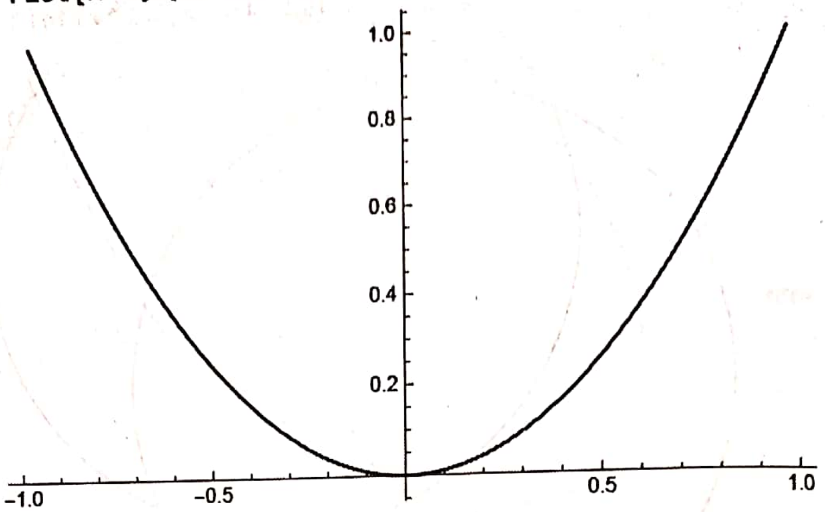


We can draw different thickness, dashed, dotted, coloured circles using Mathematica.

Parabola

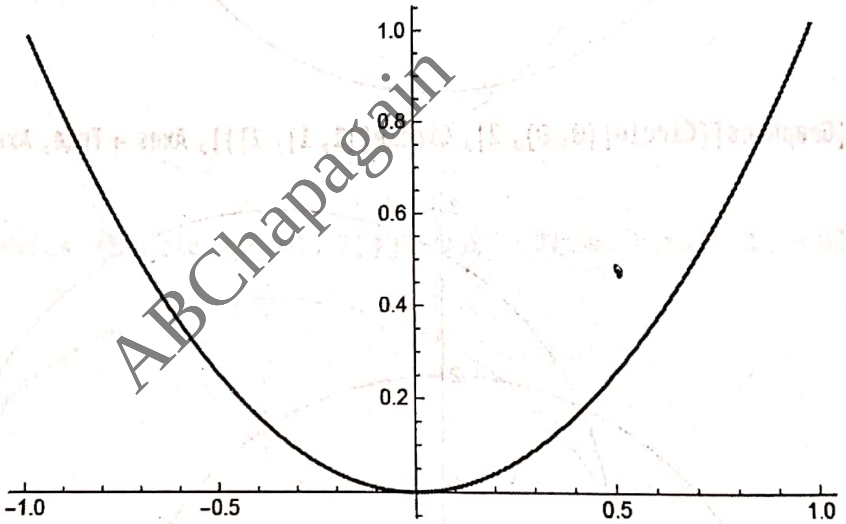
In[5]:= `Plot[x^2, {x, -1, 1}]`

Out[5]=



In[6]:= `Show[%5, AxesStyle → Black]`

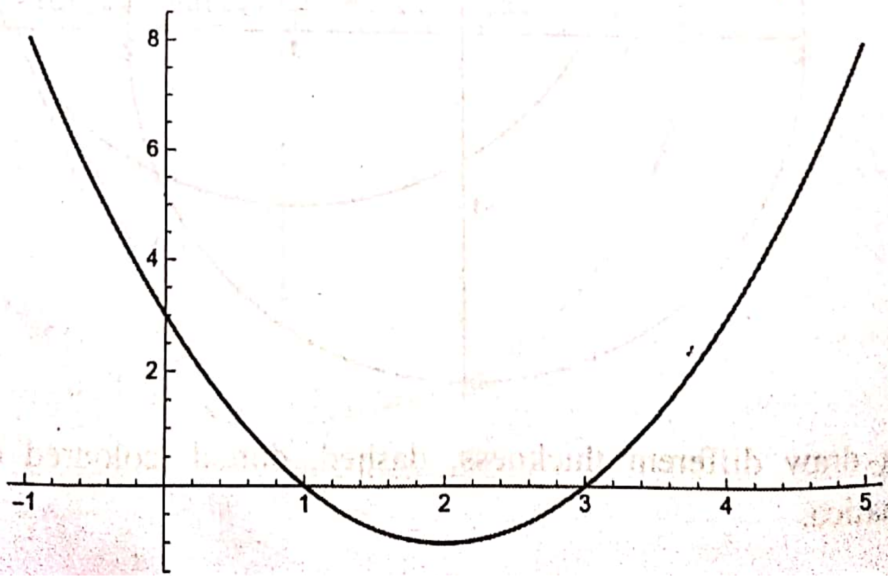
Out[6]=



Graph of $y = x^2 - 4x + 3$

In[19]:= `Plot[Evaluate[x^2 - 4 x + 3], {x, -1, 5}]`

Out[19]=



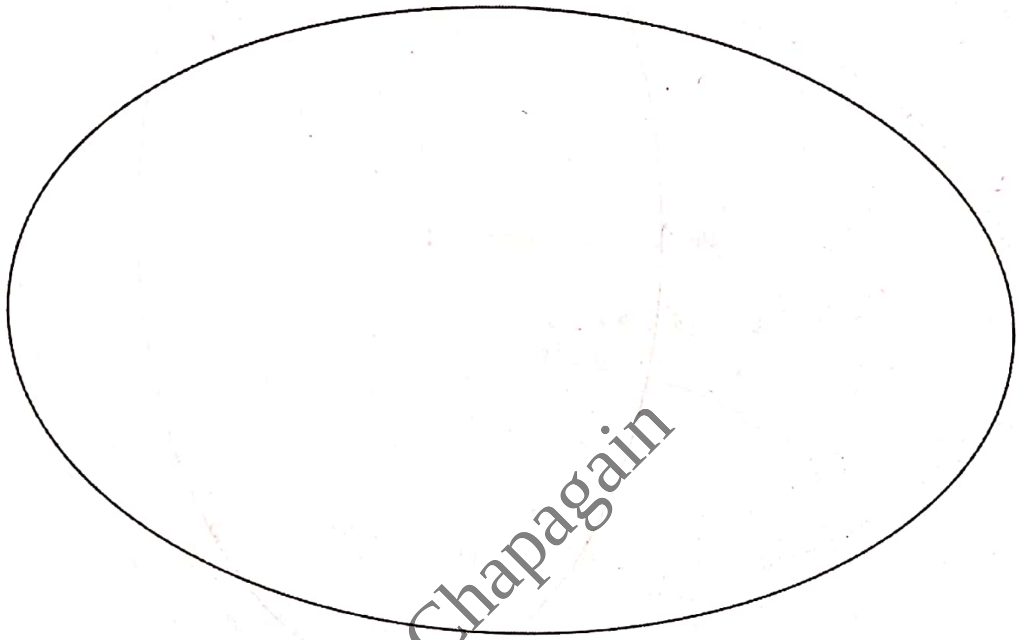
Ellipses

To display an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b$ in Mathematica, we use the command

```
Graphics [Circle [{0, 0}, {a, b}]
```

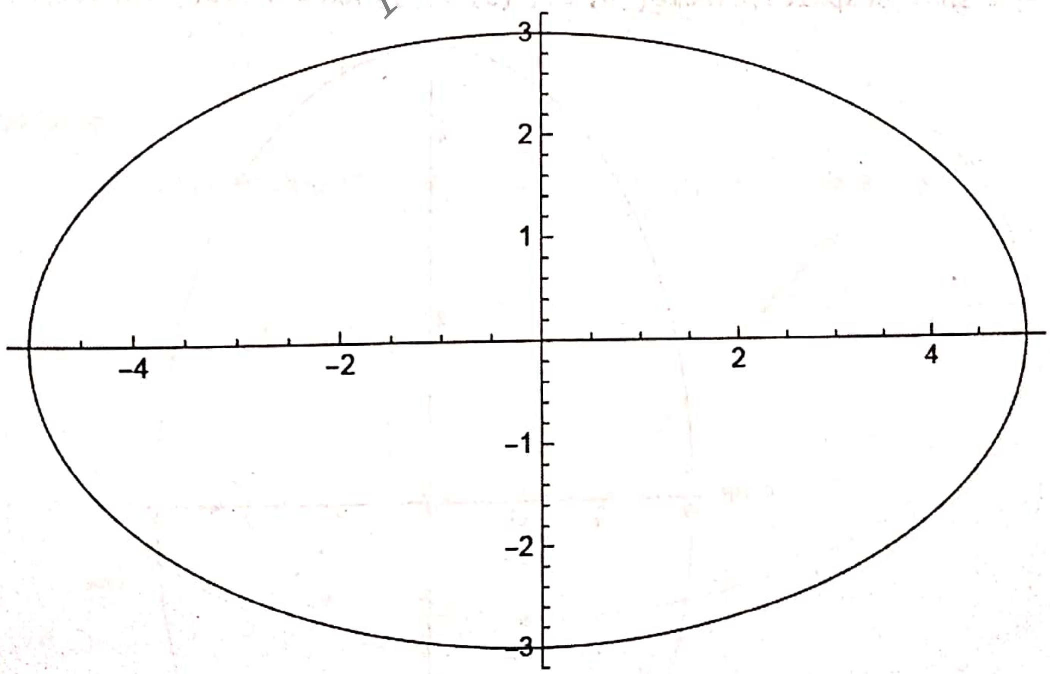
```
In[1]:= Graphics[Circle[{0, 0}, {5, 3}]]
```

Out[1]=



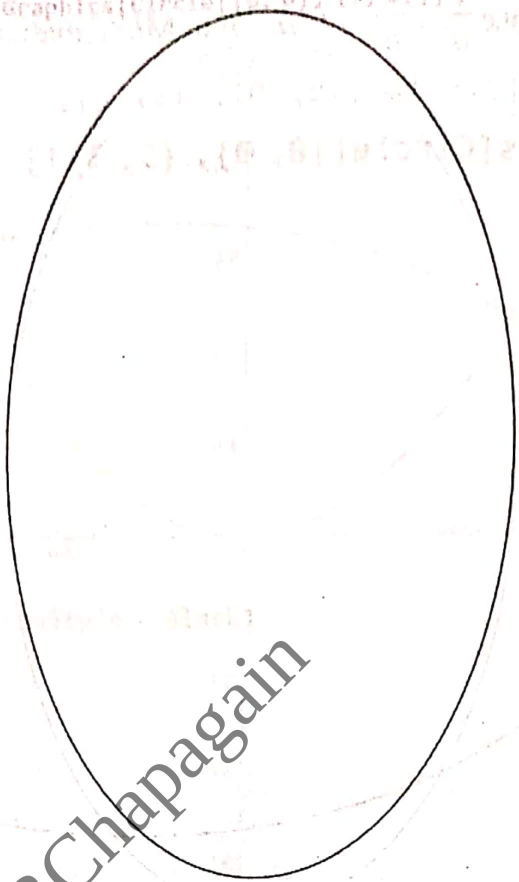
```
In[2]:= Show[Graphics[Circle[{0, 0}, {5, 3}]], Axes → True, AxesStyle → Black]
```

Out[2]=



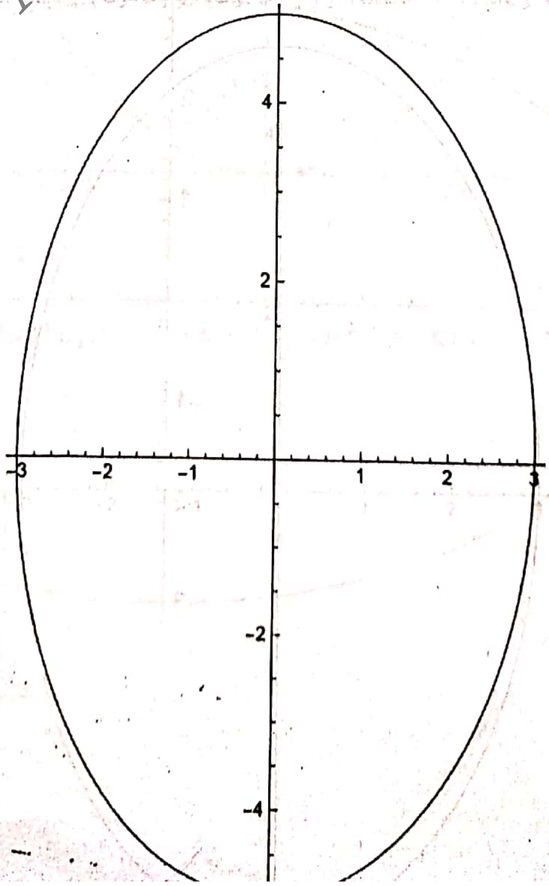
In[3]:= Graphics[Circle[{0, 0}, {3, 5}]]

Out[3]=



In[4]:= Show[Graphics[Circle[{0, 0}, {3, 5}]], Axes → True, AxesStyle → Black]

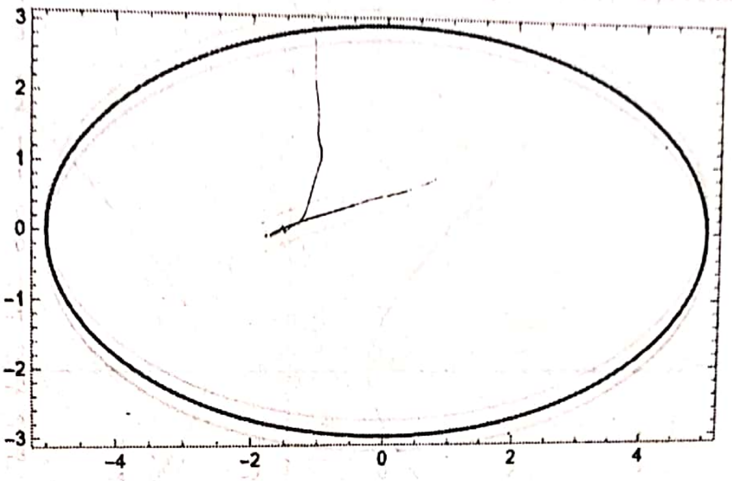
Out[4]=



The ellipses can also be drawn in the following command:

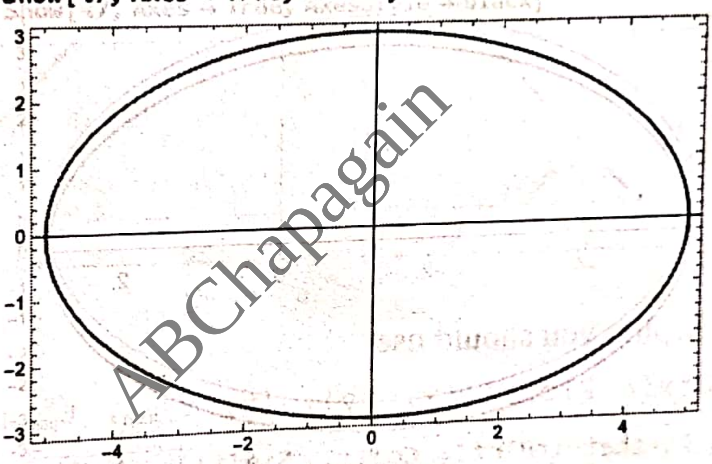
```
In[7]:= ContourPlot[(x / 5) ^2 + (y / 3) ^2 == 1, {x, -5, 5}, {y, -3, 3}, AspectRatio → Automatic]
```

Out[7]=




```
In[8]:= Show[%7, Axes → True, AxesStyle → Black]
```

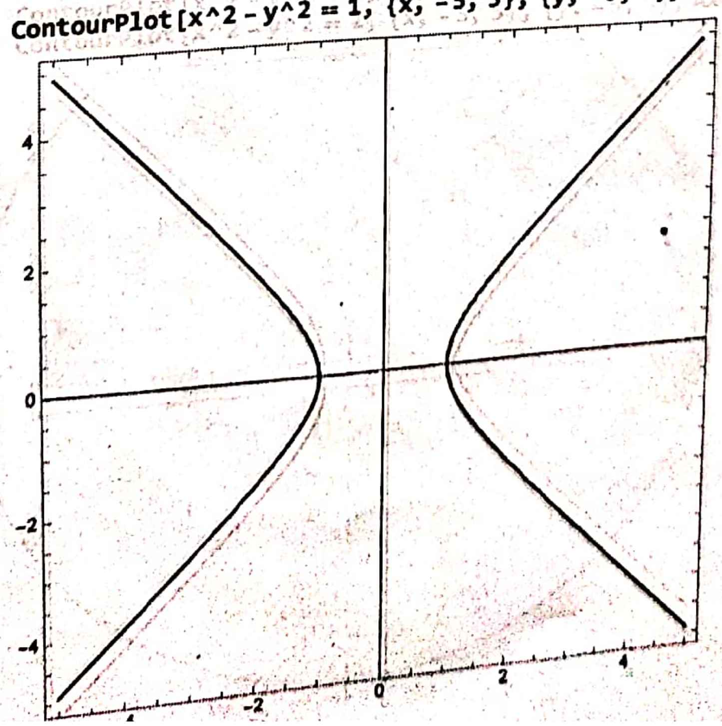
Out[8]=



Hyperbola

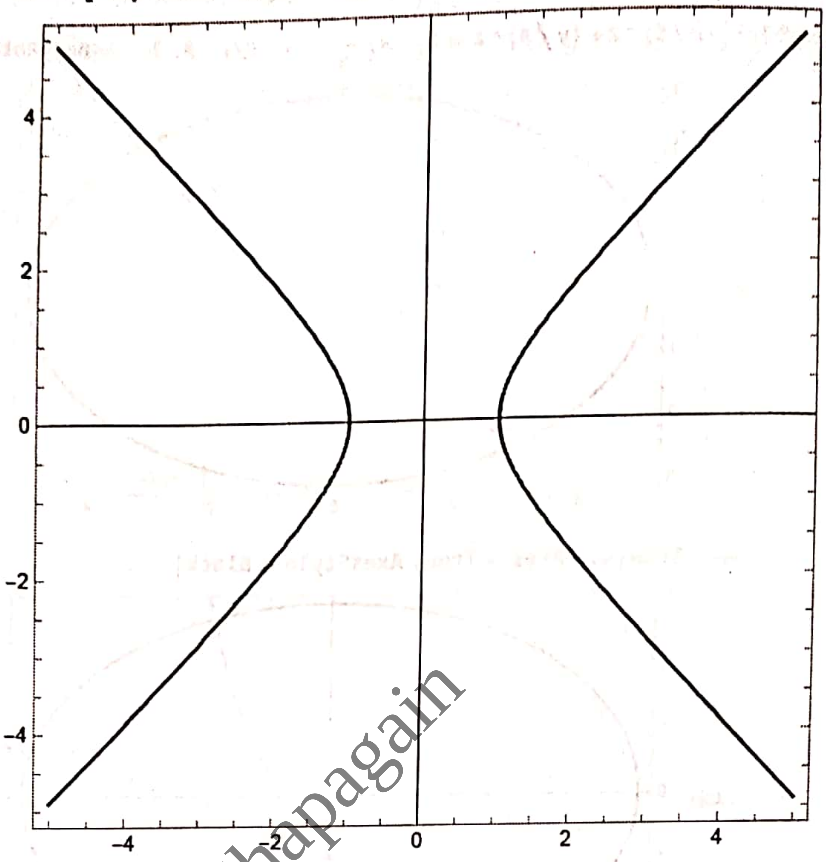
```
In[5]:= ContourPlot[x^2 - y^2 == 1, {x, -5, 5}, {y, -5, 5}, Axes → True]
```

Out[5]=



```
In[6]:= Show[%5, AxesStyle → Black]
```

Out[6]=

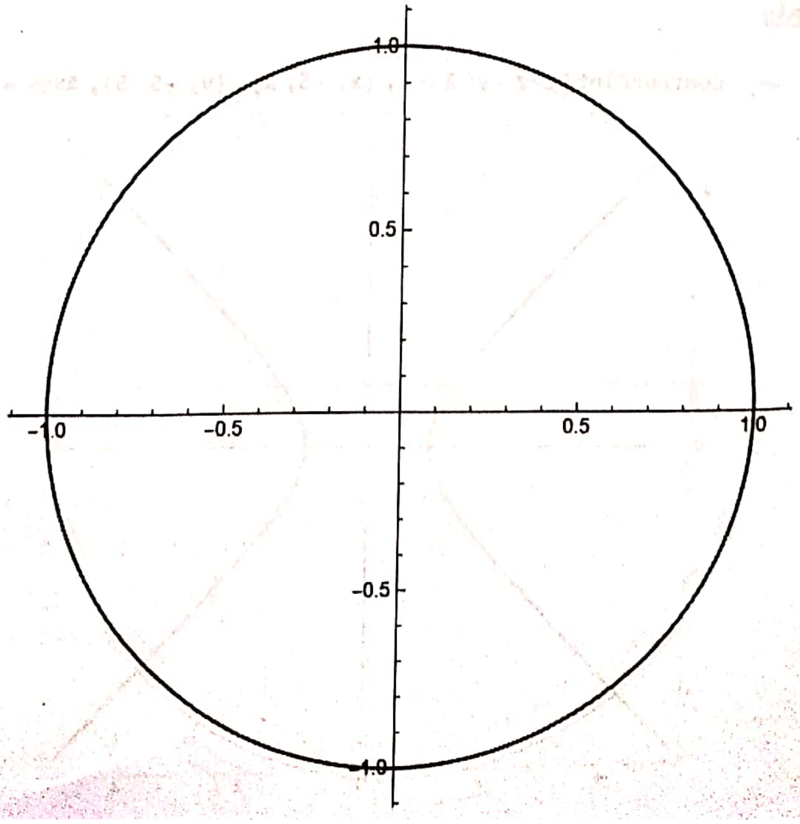


For parametric plot, you should use

```
Parametric Plot [{fx, fy}, {t, t,min, tmax}]
```

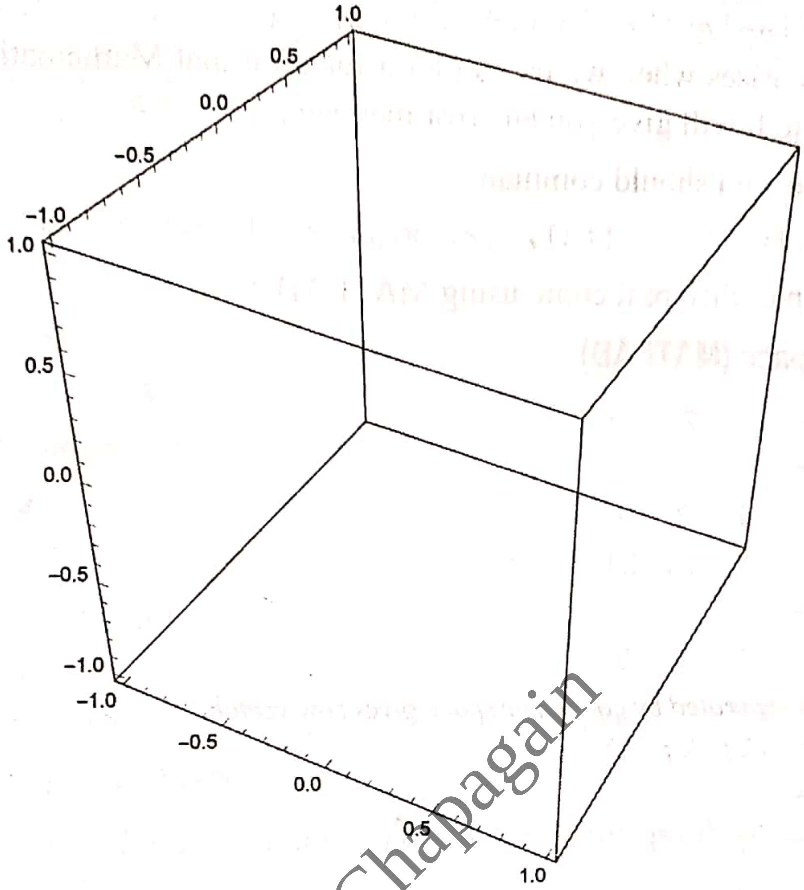
```
In[9]:= ParametricPlot[{ Cos[t], Sin[t]}, {t, 0, 2 Pi}]
```

Out[9]=



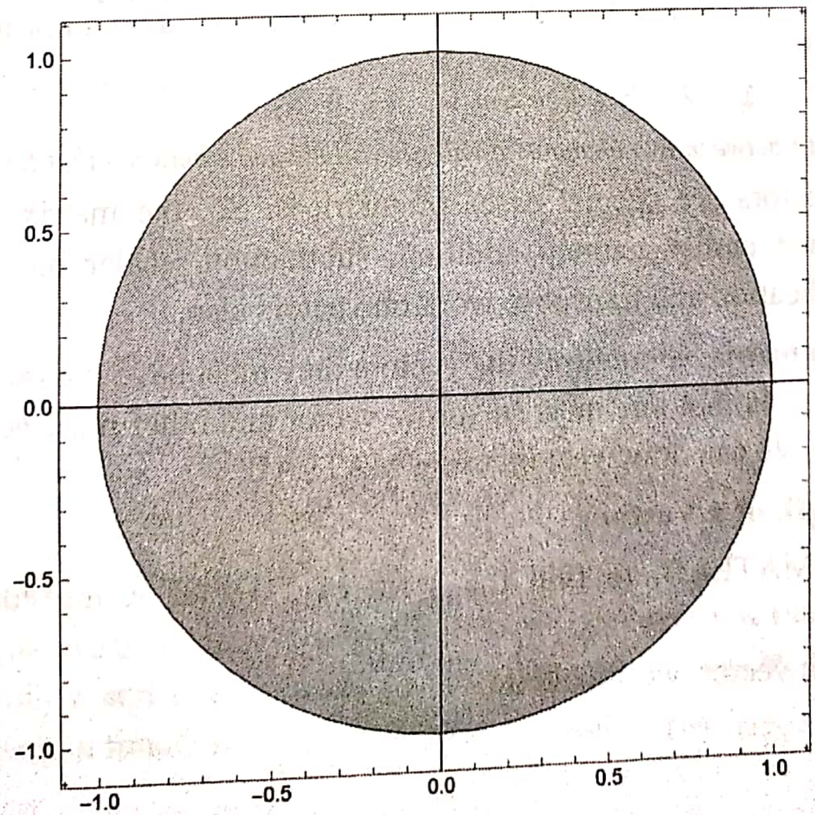
```
In[10]:= ParametricPlot3D[{ Cos[t], Sin[t]}, {t, 0, 2 Pi}]
```

Out[10]=




```
In[14]:= ParametricPlot[{r Cos[t], r Sin[t]}, {r, 0, 1}, {t, 0, 2 Pi}]
```

Out[14]=



Plotting is easy in Mathematica, the basic syntax is

```
Plot [f(x), {x, xmin, xmax}]
```

One problem arises when we try to plot a function that Mathematica has not first evaluated. It will give you an error message.

In such a case, you should command

```
Plot [Evaluate [fx], {x, xmin, xmax}]
```

We can also plot different conic using MATLAB.

Vectors in Space (MATLAB)

```
>> v = [1  2  3]
   v =
        1   2   3
>> v = [1, 2, 3]
   v =
        1   2   3
```

Note: Elements seperated by comma or space gives row vector.

```
>> v = [1; 2; 3]
   v =
        1
        2
        3
```

Note: Elements seperated by semi-colon gives column vector.

```
>> v = [1 : 8]
   v =
        1   2   3   4   5   6   7   8
```

Note: If we put : between two elements it will give all elements between that two.

The vectors are special cases of matrices. So, the matrix operations described earlier namely addition, subtraction, scalar multiplication, multiplication and transpose work on vectors also.

Like in matrix, to multiply the vectors they must have the same number of elements, but one must be a row vector and other must be a column vector. We can draw vectors using MATLAB.

Norm (Length of a Vector)

Using MATLAB, to find the norm of a vector v, the command is `norm (v)`.

For unit vector, we should do

```
v/norm (v)
```

Examples

1. Find a unit vector that has the same direction as the given vector.

$8\vec{i} - \vec{j} + 4\vec{k}$

Solution

Using MATLAB

```
>> v = [8, -1, 4]
   v =
         8   -1   4
>> norm (v)
ans =
    9
>> v/norm(v)
ans =
    0.8889      -0.1111       0.4444
```

Dot (Scalar Product)

In MATLAB, the dot Product or inner product is like a matrix multiplication.

That is, we multiply a row vector by a column vector. Note here that the number of elements must be same in both vectors. We can do this product by using the dot function in MATLAB.

```
>> a = [3, 2, 5, -1]
   a =
         3   2   5   -1
>> b = [1, 2, 0, 5]
   b =
         1   2   0   5
>> dot (a  b)
   ans =
         2
```

Cross (Vector Product)

The cross or outer product $a \times b$ of two vectors is defined only when both a and b are vectors in three dimensional space. MATLAB has a built in function 'cross' for cross product.

2. If $\vec{a} = (6, 0, -2)$, $\vec{b} = (0, 8, 0)$ then $\vec{a} \times \vec{b}$?

Solution

Using MATLAB

```
>> a = [6, 0, -2]
   a =
          6   0   -2
>> b = [0, 8, 0]
   b =
          0   8   0
>> cross(a, b)
   ans =
          16   0   48
```

Command

`dot (a, b)`	dot product of *a* and *b*
`cross (a, b)`	cross product of *a* and *b*
`dot (a, cross (b, c))`	scalar triple product of *a, b, c*
`cross (a, cross (b, c))`	vector product of *a, b, c*

3. Find $[\vec{a}\ \vec{b}\ \vec{c}]$ and $\vec{a} \times (\vec{b} \times \vec{c})$:

$\vec{a} = 3\vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = 2\vec{j} - 3\vec{k}$.

Solution

Using MATLAB

```
>> a = [3, 1, 1]
   a =
          3   1   1
>> b = [1, 1, 1]
   b =
          1   1   1
>> c = [0  2  -3]
   c =
          0   2   -3
>> dot (a, cross (b, c))
ans =
     -10
>> cross (a, cross (b, c))
ans =
     -1    -11    14
```

Tribhuvan University
Faculty of Humanities & Social Sciences
OFFICE OF THE DEAN
2018

Bachelor in Computer Applications

Course Title: Mathematics — **Full Marks: 60**

Code No.: CAMT 104 — **Pass Marks: 24**

Semester: 1st — **Time: 3 hours**

Candidates are required to answer the questions in their own words as far as possible.

Group A

Attempt all the questions. **[10×1=10]**

In this group, there will be 10 objectives questions.

This paper is collected at the time of examination.

Group 'B'

Attempt any SIX questions: **6×5=30**

11. 32 students play basketball and 25 students play volleyball. It is found that 20 students play both the games. Find the number of students playing at least one game. Also, find total number of students if 13 students play none of these games.

12. Let $f: \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 2x$ for all $x \in \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. Show that f is one-one but not onto function.

13. If the three consecutive term of a geometric series be increased by their middle term, then prove that the resulting terms will be in harmonic progression. (H.S.)

14. Find the adjoin of the matrix: $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$.

15. Prove that: $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)$.

16. Find the equation of parabola with focus $(-1, 2)$ and directrix $x = -5$.

17. Transform $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and check whether this transformation is linear.

Group 'C'

Attempt any TWO questions: **2×10=20**

18. Define permutation and combination. Try to establish relationship between them with the help of formulae. In how many ways can the letters of the word "LOGIC" be a arranged so that:

(i) Vowels may occupy odd position?

(ii) No vowels are together?

19. Define scalar and vector product in three dimensional space with their geometrical interpretation and prove the formula

sin (A + B) = sin A cos B + cos A sin B by using vector method.

20. Define the logarithmic function, state its properties and if

$f(x) = \log \frac{1+x}{1-x} (-1 < x < 1)$, show that

$f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right) (|a| < 1, |b| < 1).$

◆◆◆

Tribhuvan University

Faculty of Humanities & Social Sciences

OFFICE OF THE DEAN

2019

Bachelor in Computer Applications — **Full Marks: 60**

Course Title: Mathematics — **Pass Marks: 24**

Code No.: CAMT 104 — **Time: 3 hours**

Semester: 1st

Candidates are required to answer the questions in their own words as far as possible.

Group A

Attempt all the questions. **[10×1=10]**

In this group, there will be 10 objectives questions.

This paper is collected at the time of examination.

Group 'B'

Attempt any SIX questions: **6×5=30**

2. In a class of 100 students, 40 students failed in Mathematics, 70 failed in English and 20 failed in both subjects. Find

a. How many students passed in both subjects?

b. How many students passed in Mathematics only?

c. How many students failed in Mathematics only?

3. Find the domain and range of the function $f(x) = \dfrac{2x+1}{3-x}$.

4. Find the Maclaurin series of the function $f(x) = \sin x$.

5. Prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

6. Find a unit vector perpendicular to the plane containing points P(1, – 1, 0), Q(2, 1, – 1) and R(– 1, 1, 2).

7. In how many ways can be letters of words "Sunday" be arranged? How many of these arrangement begin with S? How many begin with S and don't end with y?

8. If $x + iy = \sqrt{\dfrac{1+i}{1-i}}$ then show that $x^2 + y^2 = 1$.

Group 'C'

Attempt any TWO questions: **2×10=20**

9. a. Define conic section. Find the coordinates of vertices, eccentricity and foci of the ellipse $9x^2 + 4y^2 - 18x - 16y - 11 = 0$. [1+5]

b. If $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1 + x_2, x_2, x_1)$ be the linear transformation, then find matrix associated with linear map T. [4]

10. a Define irrational number: Prove that $\sqrt{2}$ is an irrational number . [1+4]

b. If functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 - 2$. Find the formulae for composite functions fog and gof and also verify that $fog \neq gof$. [4 + 1]

11. a. If arithmetic mean, geometric mean and harmonic mean between two unequal positive numbers are A, G, H respectively, then prove that $A > G > H$. [4]

b. What is the relation between permutation and combination of n objects taken r at a time? A committee of 5 is to be constituted from 6 boys and 5 girls. In how many ways can this be done so as to include at least a boy and a girl? [1+5]

◆◆◆