Gaussian Integration

Gaussian integration is based on the concept that the accuracy of numerical integration can be improved by choosing the sampling point wisely rather than on the basis of equal spacing. For example consider the following figure.

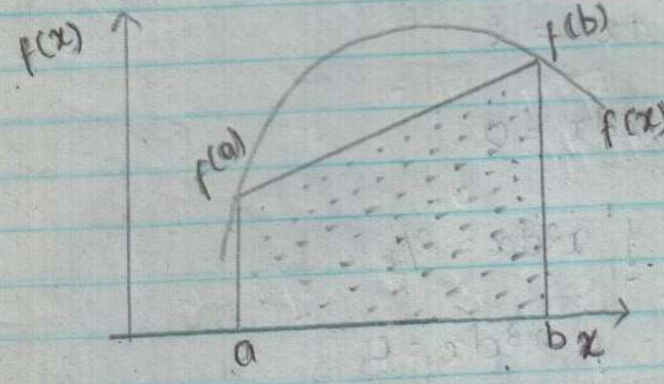


fig1: Trapezoidal rule

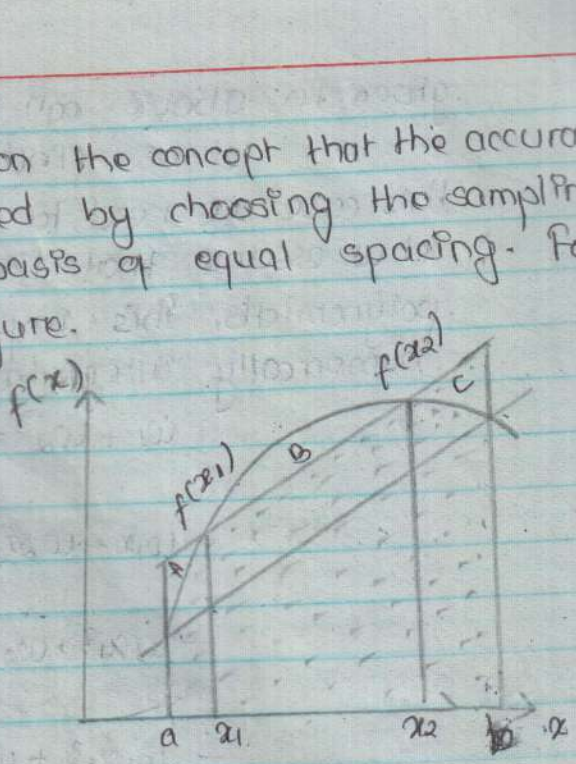


fig2: Gaussian rule

in figure 1 for trapezoidal rule the end point of integral lies in the function curve, whereas in figure 2 the straight line has been moved up such that area B = A + C. In this case the function values at the end points are not used in computation whereas the function values $f(x_1)$ & $f(x_2)$ are used to compute the shaded area. Here, the problem is to compute the value of x_1 & x_2 given the values of a & b & choose appropriate weights [w_1 & w_2]

Hence the method of implementing the strategy of finding appropriate values of x_i & w_i & obtaining the integral of $f(x)$ is called gaussian integration or quadrature

Gaussian integration assumes the approximation of the form,

$$I_g = \int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i) \quad \text{------ (1)}$$

This eq^n contains 2n unknowns to be determine these unknowns can be determined using the condition given in above eqn ①

let us find a Gaussian quadrature formula for $n=2$. In this case we need to find the values of w_1, w_2, x_1 & x_2. Also, let us assume that the integral will be exact upto cubic polynomials, this implies that the function $1, x, x^2, x^3$ can be numerically integrated to obtain exact result.

$$w_1 + w_2 = \int_{-1}^{1} dx = 2$$

$$w_1x_1 + w_2x_2 = \int_{-1}^{1} x\,dx = 0$$

$$w_1x_1^2 + w_2x_2^2 = \int_{-1}^{1} x^2dx = 2/3$$

$$w_1x_1^3 + w_2x_2^3 = \int_{-1}^{1} x^3dx = 0$$

Solving these simultaneous eqn, we obtain,

$$w_1 = w_2 = 1$$
$$x_1 = -1/\sqrt{3}$$
$$x_2 = 1/\sqrt{3}$$

Thus we have gaussian quadrature formula for $n=2$ as,

$$\int_{-1}^{1} f(x)dx = f\left[\frac{-1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right]$$

This formula gives correct values for integral of $f(x)$ in the range $(-1, 1)$ for any function upto 3rd order. This eqn is known as Gauss legendre formula.

Q) Compute $\int_{-1}^{1} e^x dx$ using two point gauss legendre formula.

soln,

we know that, for two point

$$x_1 = -1/\sqrt{3}$$
$$x_2 = 1/\sqrt{3}$$

then $f(x_1) = e^{-1/\sqrt{3}} = 0.56138$

$f(x_2) = e^{1/\sqrt{3}} = 1.78131.$

Then we know

$$\int_{-1}^{1} f(x)\,dx = f(x_1) + f(x_2)$$

$$= 0.56138 + 1.78131$$

$$= 2.34269$$

for $n = 3$:

$i =$	1	2	3
$w_i =$	0.55556	0.88889	0.55566
$x_i =$	-0.7746	0.0000	0.7746.

Romberg Integration Formula

This method can often only used to improve the approximate results obtained by the finite difference method. We consider the definite integral

$$I = \int_a^b f(x)\,dx$$

and evaluate it by trapezoidal rule with different sub intervals as shown below.

let us consider a Romberg integration formula.

$$R_{ij} = \frac{4^j R_{i,j-1} - R_{i-1,j-1}}{4^j - 1} \quad \text{-----} \ (1)$$

when this equation is expanded it will form a lower diagonal matrix where the elements of matrix R are computed Row by Row In The Order Indicated In The Figure Below.

$$\begin{tikzcd}
R(0,0) \arrow[d, "1"] \arrow[dr] & & & \\
R(1,0) \arrow[r, "2"] \arrow[d, "3"] \arrow[dr] & R(1,1) \arrow[dr] & & \\
R(2,0) \arrow[r, "4"] \arrow[d, "6"] \arrow[dr] & R(2,1) \arrow[r, "5"] \arrow[dr] & R(2,2) \arrow[dr] & \\
R(3,0) \arrow[r, "7"] & R(3,1) \arrow[r, "8"] & R(3,2) \arrow[r, "9"] & R(3,3)
\end{tikzcd}$$

Elements in the 1st column represent trapezoidal rule at $h, h/2, h/4$ etc. They can be evaluated recursively as follows,

$$h = (b-a)$$

$$R(0,0) = \frac{h}{2}(f(a) + f(b))$$

$$R(i,0) = \frac{R(i-1,0)}{2} + h_i \sum_{k=1}^{2^{i-1}} f(x_{2k-1}) \quad (2) \text{ for } i = 1,2 \ldots$$

$$\text{where } h_i = \frac{b-a}{2^i}$$

$$x_k = a + k h_i$$

Q. Compute Romberg estimate R_{22} for $\int_1^2 1/x \, dx$

Soln,

Given = $a = 1$

$b = 2$

$h = b-a = 2-1 = 1$

Then $f(a) = f(1) = 1/1 = 1$

$f(b) = f(2) = 1/2 = 0.5$

So,

$$R(0,0) = \frac{h}{2}[f(a) + f(b)]$$

$$= \frac{1}{2}[1 + 0.5]$$

$$= 3/4$$

&

$$R(1,0) = \frac{R(0,0)}{2} + h_1 \sum_{k=1}^{1} f(x_1)$$

$$\text{So, } h_1 = \frac{b-a}{2^i} = \frac{2-1}{2^1} = 1/2$$

$$x_i = a + 1 \cdot h_1$$

$$= 1 + 1 \times 1/2$$

$$= 3/2$$

etc

$$f(x_1) = f(3/2) = 2/3$$

So we have:

$$R(1,0) = \frac{R(0,0)}{2} + h_1 \sum_{k=1}^{1} f(x_1)$$

$$= \left(\frac{3}{4}\right)\Big/2 + \frac{1}{2} \times \frac{2}{3}$$

$$= \frac{3}{8} + \frac{1}{3}$$

$$= \frac{17}{24}$$ //

Unit. 5

Solution of Ordinary Differential Equation.

mathematical models for different laws in scientific work are often expressed in terms of not only certain system parameters but also in terms of their derivatives. Such mathematical models which use differential calculus to express relationship between variables are known as differential variables.

For example

① law of cooling

Newton's law of cooling.

$$\frac{dT(t)}{dt} = k\left[T_s - T(t)\right] \quad \text{------ ①}$$

where T_s = temp^r of surrounding
$T(t)$ = temp^r of liquid at time t
k = constant of ~~proportonility~~ proportionality

Radioactive decay

The radioactive decay of an element is given by,

$$\frac{dm}{dt} - km = 0 \quad \text{---- ②}$$

where,
m = mass
t = time
k = constant rate of decay.

Simple harmonic motion

$$\frac{md^2y}{dt^2} + a\frac{dy}{dt} + ky = 0 \quad \text{---- ③}$$

where

y = displacement
m = mass
d^2y/dt^2 = acceleration
dy/dt = velocity of moving particle

If there is only one independent variable, the eq^n is called an ordinary differential equation,

Example, above eq^n (1) (2) (3)

If there is two or more independent variables, the derivatives will be partial therefore the eq^n is called partial differential equation,

example, Heat flow in a rectangular plate is given by,

$$\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = f(x,y) \quad --- (4)$$

where $u(x,y)$ denotes the temperature at point (x,y) &

$f(x,y)$ = head source.

Order of differential Eq^n.

The order of differential equation is the highest derivative that appears in the eq^n. for example.

$$\frac{dT(t)}{dt} = k\,[T_s - T(t)] \quad ------ \text{ First ordered.}$$

$$\frac{m\,d^2y}{dt^2} + a\frac{dy}{dt} + ky = 0 \quad --- \text{ Second ordered,}$$

A first ordered eq^n can be expressed on the form $\frac{dy}{dx} = f(x)$. A second ordered eq^n can be expressed on the form $\frac{d^2y}{dx^2} = y'' = f(x, y, y')$

where y'' = Second derivative,

y' = first derivative.

Higher order eq^n can be reduced to a set of first ordered equations by suitable transformation. For example,

$$y'' = f(x, y, y') \text{ can be equivalently represented by,}$$

$$u' = f(x, y, y') \quad \text{where } y' = u.$$

Degree of equation

The degree of a differential eq^n is the power of the highest order derivative. for example,

$$xy'' + y^2 y' = 2y + 3 \rightarrow \text{first degree / second order}$$

$$(y''')^2 + 5y' = 0 \rightarrow \text{second degree / third order}$$

Initial value problem.

It is well known that a differential eq^n of n^{th} order will have n arbitrary constant in its general solution. In order to compute the numerical soln of such an eq^n we therefore need n conditions. Problems in which all the initial conditions are specified at the initial point only are called initial value problem.

On the other hand, in problems involving second & higher order differential eq^{n}s we may prescribe the conditions at two points or more. Such problems are called boundary value problems.

Consider the soln $y = ae^x$ to the eq^n $y' = y$

If we are given the value of y for some x, the constant a can be determined.

Suppose $y = 1$ at $x = 0$ then,

$$y(0) = ae^0 = 1$$

$$\therefore a = 1$$

$$y(x) = y(x_0) + y'(x_0)(x-x_0) + \frac{y''(x_0)(x-x_0)^2}{2!}$$

& the particular solⁿ is

$$y = e^{x}.$$

If the order of equation is n we will have to obtain n constants & therefore we need n conditions, in order to obtain a unique solⁿ.

~~Tyalor~~ Taylor series method

A function $y(x)$ is extended about a point $x = x_0$ using Taylor's theorem of expansion as,

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \frac{y'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{y^n(x_0)}{n!}(x-x_0)^n \quad \cdots \text{(1)}$$

where,

$y^i(x_0)$ = ith derivative of $y(x)$ evaluated at $x = x_0$.

The value of y can be obtained if we know the value of its derivatives, This implies that if we are given the eqⁿ

$$y' = f(x,y) \quad \cdots \text{(2)}$$

we must then repeatitly differentiate $f(x,y)$ empliciltly with respect to x and evaluate them at x_0.

Example

If $\frac{dy}{dx} = y' = f(x,y)$ then,

$$y'' = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}[f(x,y)]$$

$$= \frac{\partial}{\partial x}[f(x,y)] + \frac{\partial}{\partial y}(f(x,y)).\frac{dy}{dx}$$

$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x-x_0) + y''(x_0)(x-x_0)$

$x = x_0$ $\quad y(x) = y(x_0) + y'(x_0)(x-x_0) + \frac{y''(x-x_0)^2}{2!}$

$= \frac{df}{dx} + \frac{df}{dy} \cdot f$

$= f_x + f_y \cdot f$ where

f = function ~~of~~ $f(x,y)$ & f_x & f_y denote the partial derivative of the function $f(x,y)$ with respect to x & y respectively.

Similarly we can obtained

$$y''' = f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$$

let us ellustrate with example.

consider the eq^n.

$$y' = x^2 + y^2$$

under the condition,

$$y(x) = 1 \text{ where } x = 0.$$

Given,

$$y' = x^2 + y^2$$

$$y'' = \frac{d}{dx}(x^2+y^2)$$

$$= \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{dy}(x^2+y^2)\frac{dy}{dx}$$

$$= 2x + 2y \cdot y'$$

$$y''' = \cancel{2x} + 2yy'' + 2(y')^2$$

at $x=0$, $y(0) = 1$. & therefore

$y'(0) = 1$

$y''(0) = 2$

$y'''(0) = 8$

finally substituting the values, taylor series become

$$y(x) = 1 + x + x^2 + \frac{8}{3!} x^3 + \dots \quad (8)$$

The number of terms to be used depends on the accuracy of the soln needed.

Q// Use the taylor method to solve the eqn,
$y' = x^2 + y^2$ for $x = 0.25$ & $x = 0.5$
given $y(0) = 1$

soln
we have,
given eqn as,
$y' = x^2 + y^2$
The taylor series for above eqn is,
$$y(x) = 1 + x + x^2 + \frac{8}{3!} x^3 + \dots$$

Now for $x = 0.25$
$$y[0.25] = 1 + 0.25 + (0.25)^2 + \frac{8}{3!}(0.25)^3$$
$$= 1.025 + 0.0625 + 1.33 \times 0.015625$$
$$= 1.0875 + 0.02078125$$
$$= 1.1083$$

& for $x = 0.5$,
$$y[0.5] = 1 + 0.5 + 0.5^2 + \frac{8}{3!}(0.5)^3$$
$$= 1.5 + 0.25 + 1.33 \times 0.125$$
$$= 1.75 + 0.16625$$
$$= 1.91625$$

One major problem with taylor series method is the evaluation of higher order derivative. This method is therefore generally impractical from a computational point of view.

The error in Taylor method is in the order of $(x-x_0)^{n+1}$. If $|(x-x_0)|$ is large the error can also become large, therefore the result is often found unsatisfactory.

Picard's method:

Consider the first ordered eqn $dy/dx = f(x,y)$ ---- ①

It is necessary to find that particular soln of eqn ① which assums the value y_0 when $x = x_0$. Integrating eqn ① between limits we get,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\text{or,}\quad y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\therefore\quad y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \quad ---- ②$$

This is an integral eqn equivalent to eqn ①, for it contains the unknown y under the integral sign.

As a first approximation y_1 to the soln, we put $y = y_0$ in $f(x,y)$ & integrate eqn ② giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

For a second approximation y_2 we put $y = y_1$ in $f(x,y)$ & integrate eqn ② giving,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

Simillarly 3rd approximation is given by

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

continuing this process we obtain $y_4, y_5 \cdots\cdots y_n$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

Hence this method gives a sequence of approximations $y_1,$ $y_3 \cdots$ each giving a better result than preeceding one

Q. Using pecard's process of successive approximation, obtain a solution upto 5th approximation of the equation $\frac{dy}{dx} = y + x$ such that $y_0 = 1$ when $x_0 = 0$

Soln,

we have,

$$y = 1 + \int_0^x (y + x)\, dx$$

for the first approximation, put $y = 1$ in $y + x$ then,

$$y_1 = 1 + \int_0^x (1 + x)\, dx$$

$$= 1 + \left[x + \frac{x^2}{2}\right]_0^x$$

$$= 1 + x + \frac{x^2}{2}$$

for the second approximation put

$$y = 1 + x + \frac{x^2}{2}$$

Then

$$y_2 = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} + x\right) dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2}$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

for third approximation put $y = 1 + x + x^2 + x^3/6$

Then,

$$y_3 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{6} + x\right) dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$ //

for the fourth approximation put $y = 1 + x + x^2 + x^3/3 + x^4/24$

Then,

$$y_4 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + x\right) dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right] dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

for 5th approximation put $y = 1+x+x^2+x^3/3+x^4/12+x^5/120$ we get

$$y_5 = 1+\int_0^x (1+x+x^2+x^3/3+x^4/12+x^5/120+x)\,dx$$

$$= 1+x+x^2+x^3/3+x^4/12+x^5/60+x^6/720.$$

Q. Solve the following eqn by picard's method

(i) $y'(x) = x^2+y^2$, $y(0)=0$

(ii) $y'(x) = xe^y$ $y(0)=0$

& estimate $y(0.1)$, $y(0.2)$ & $y(1)$.

Soln,

Given,

$$y'(x) = x^2+y^2$$

$$y_0=0,\ x_0=0$$

then,

$$y_1 = y_0 + \int_0^x (x^2+y^2)\,dx$$

$$= 0 + \left[\cancel{2x}\ \frac{x^3}{3} + xy^2\right]_0^x$$

$$= \frac{x^3}{3} + xy^2$$

$$y_2 = y_0 + \int_0^x (x_0^2+y_1^2]\,dx$$

$$= 0 + \int_0^x \left(x^2+\left(\frac{x^3}{3}+xy^2\right]^2\right)dx$$

$$= 0 + \int_0^x \left(x^2+\frac{x^6}{9}+\frac{2x^4y^2}{3}+x^2y^4\right)dx$$

$$= 0 + \left[\frac{x^3}{3}+\frac{x^7}{63}+\frac{2x^5y^2}{15}+\frac{x^3}{3}y^4\right]_0^x$$

$$= \frac{x^3}{3}+\frac{x^7}{63}+\frac{2x^5y^2}{15}+\frac{x^3}{3}y^4$$

Since picard's method involves actual integration, sometimes it maynot be possible to carry out the integratio. Also this method is not convenient for computer based soln. like taylor series this is also semi numeric method.

Euler's method:

It is the simplest one step method & has a limited application because of its low accuracy.

Consider the first two terms of the expansion of taylor series,

$$y(x) = y(x_0) + y'(x_0)(x - x_0) \quad --- (1)$$

given the differential eqn,

$$y'(x) = f(x, y) \text{ with } y(x_0) = y_0$$

Then we have,

$$y'(x_0) = f(x_0, y_0)$$

and therefore,

$$y(x) = y(x_0) + (x - x_0) f(x_0, y_0)$$

Then the value of $y(x)$ at $x = x_1$, is given by,

$$y(x_1) = y(x_0) + (x_1 - x_0) f(x_0, y_0)$$

let $h = x_1 - x_0$, then,

$$y_1 = y_0 + h f(x_0, y_0)$$

Similarly, y_x at point $x = x_2$ is given by

$$y_2 = y_1 + h f(x_1, y_1)$$

In general we obtain a recursive relation,

$$y_{i+1} = y_i + h f(x_i, y_i)$$

This formula is known as Euler's method, and can be used recursively to evaluate y_1, y_2, y_3 ------ ~~g of y_n~~

of $y(x_1); y(x_2); y(x_3)$ .... starting from the initial condition $y_0 = y(x_0)$.

Here a new value of y is estimated using the previous value of y as the initial condition. Note that the term $hf(x_i, y_i)$ represents the incremental value of y & $f(x_i, y_i)$ is the slope of $y(x)$ at point (x_i, y_i) i.e the new value is obtained by extrapolating linearly over the step size h using the slope ~~at~~ its previous value, i.e

new value = old value + stepsize × slope.

This is illustrated in the figure below.

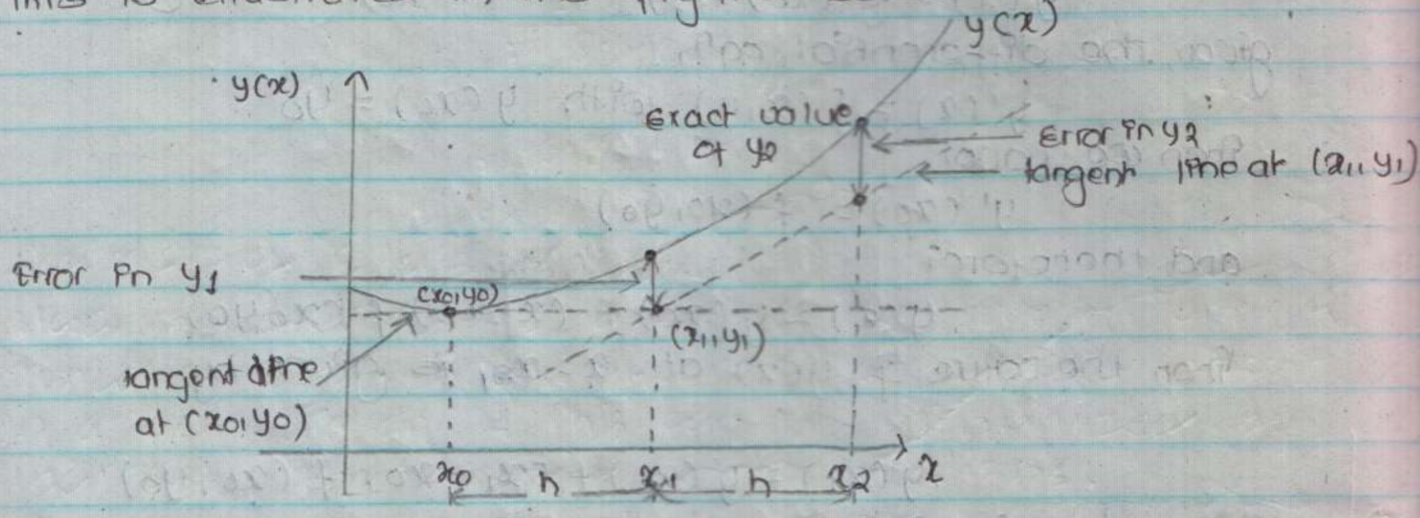


fig - Illustration of Euler's method for two steps.

$\sqrt{\text{Imp}}$ 2/11 Given the equation $dy/dx = 3x^2 + 1$ with $y(1) = 2$ estimate $y(2)$ by Euler's method using

(i) $h = 0.5$
(ii) $h = 0.25$.

Soln

We have

$$h = 0.5$$

& the given condition is

$$y(1) = 2$$

$x_0 = 1$

$y_0 = 2$

$f(x_0) = 3x^2 + 1$

Now,

$y(1 + 0.5) = y_0 + hf(x_0, y_0)$

or $y(1.5) = 2 + 0.5 \times (3 \times 1^2 + 1)$

$= 2 + 0.5 \times 4$

$= 2 + 0.2$

$= 4$ //

Again,

$y(1.5 + 0.5) = y_{1.5} + 0.5 \times f(x_{1.5}, y_{1.5})$

$= 4 + 0.5 \times (3 \times 1.5^2 + 1)$

$= 7.875$ //

for $h = 0.25$

$y(1) = 2$

$x_0 = 1$

$y_0 = 2$

$f(x) = 3x^2 + 1$

Now,

$y(1 + 0.25) = y_0 + hf(x_0, y_0)$

$= 2 + 0.25 \times 4$

$= 3$

Again,

$y(1.25 + 0.25) = y_{1.25} + 0.25 \times f(x_{1.25}, y_{1.25})$

$= 3 + 0.25 \times (5.6875)$

$= 4.421875$

CA → 23
NM → 20
POM → 14

Now,

$$y(1.5 + 0.25) = y_{1.5} + hf(x_{1.5}, y_{1.5})$$

$$= 4.421875 + 0.25 \times 7.75$$

$$= 6.359375$$

Again

$$y(1.75 + 0.25) = y_{1.75} + hf(x_{1.75}, y_{1.75})$$

$$= 6.359375 + 0.25 \times 10.1875$$

$$= 8.90625$$

Accuracy of Euler's method.

- Here, the accuracy is affected by roundoff error & truncation error. Roundoff error is always present in a computation & this can be minimized by increasing the precision of calculations.
- The major cause of laws of accuracy is truncation error because of the use of truncated taylor series.
- Also, the total truncation error in any iteration state will consist of two components.
  - (i) The propagated truncation error
  - (ii) The truncation error introduced by the step itself
- The truncation introduced by the step itself is known as local truncation error & the sum of propagated error & the local error is called the global truncation error.

Recall the Taylor series expansion used in Euler's method for estimating the values of y_{i+1} as,

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 + \frac{y_i'''}{3!} h^3 + \cdots$$

Since only the first two terms are used in Euler's Formula, The local truncation error is given by,

$$E_{t,i+1} = \frac{y_i'' h^2}{2!} + \frac{y_i''' h^3}{3!} + \frac{y_i^{(4)} h^4}{4!} + \cdots$$

If the step size h is very small, the higher order terms can be neglated & therefore

$$E_{t,i+1} = \frac{y_i'' h^2}{2!}$$

This shows that the local truncation error of Euler's method is of the order h^2. If the final estimation requires n steps, the total (global) truncation error at the target point say b will be,

$$|E_{tg}| = \sum_{i=1}^{n} c_i h^2 = (c_1 + c_2 + c_3 \cdots c_n) h^2$$

$$= nch^2 \quad \text{where } c = \frac{c_1 + c_2 \cdots c_n}{n}$$

Since $n = \dfrac{b - x_0}{h}$

$$|E_{tg}| = \frac{(b - x_0)\, ch^{\cancel{2}}}{\cancel{h}}$$

$$= (b - x_0)\, ch.$$

Q. Compute the error in the previous example when $h = 0.5$

Soln,

Step 1, $x_0 = 1$ $y_0 = 2$

$y_1 = y(1.5) = 4$

$$Et_1 = \frac{y_0''}{2} h^2 + \frac{y_0'''}{6} h^3$$

$$= \frac{6x}{2} (0.5)^2 + \frac{(6) \cdot (0.5)^3}{6}$$

$$= \frac{6 \cdot 1}{2} (0.5)^2 + 0.5^3$$

$$= 0.75 + 0.125$$

$$= 0.875$$

Heun's method.

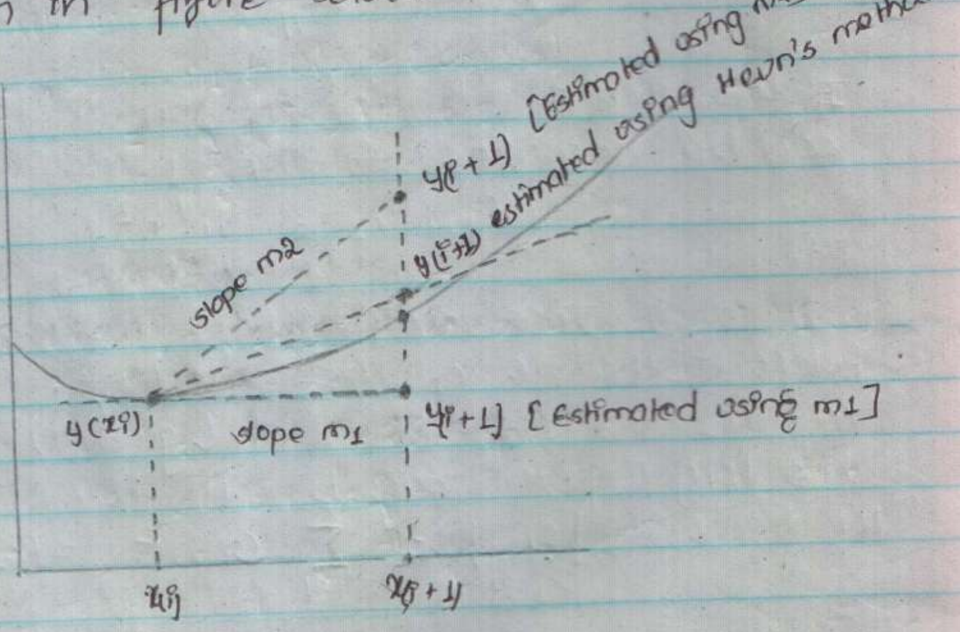
It is the improvement to Euler's method, so it is also called modified Euler's method.

In Euler's method the slope at the begining of the interval is used to extrapolate y_i to y_{i+1}. Thus

$$y_{i+1} = y_i + m_1 * h$$

where m_1 is the slope at (x_i, y_i)

In Heun's method, we use the line which is parallel to the tangent at point (x_{i+1}, y_{i+1}) to extrapolate from y_i to y_{i+1} as shown in figure below.



i.e.,

$$y_{i+1} = y_i + m_2 * h.$$

where m_2 = slope at point (x_{i+1}, y_{i+1})

The next step is to use a line whose slope is the average of the slopes at the end points of the interval then,

$$y_{i+1} = y_i + \frac{m_1 + m_2}{2} * h. \quad \text{---} ①$$

This gives better approximation to y_{i+1}. This approach is known as Heun's method.

Formula for implementing Heun's method is constructed as follows,

given the eqn,

$$y'(x) = f[x,y]$$

we obtain m_1, m_2

$$m_1 = y'(x_i) = f(x_i, y_i) \qquad y_{i+1} = y_i + hf(x_i, y_i)$$

$$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1})$$

and therefore,

$$m = \frac{m_1 + m_2}{2} = \frac{f(x_i, y_i)}{2} + \frac{f(x_{i+1}, y_{i+1})}{2}$$

so,

eqn ① becomes,

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right] \quad \text{--- ②}$$

Note that the term y_{i+1} of RHS of the eqn is predicted using Euler's formula as,

$$y_{P+1} = y_i + hf(x_i, y_i]. \quad \text{--- ③}$$

Then Heun's formula becomes,

$$y_{P+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y^{e}_{i+1})\right) \quad \text{--- ④}$$

This is an improved version of Eulers method also called one step predictor corrector method. Hence substituting eqn ③ in eqn ④ we get,

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i)\right]$$

Q// Given the equation $y'(x) = 2y/x$ with $y(1) = 2$ estimate $y(2)$ using (i) Euler's method.
(ii) Heun's method using $h = 0.25$ & compare the result with exact answer.

Soln,

Given,

$$y'(x) = 2y/x = f(x,y)$$

$$x_0 = 1, \quad y_0 = 2, \quad h = 0.25$$

(i) Using Eulers method.

$$y(1+0.25) = y_0 + 0.25 \, f(x_0, y_0)$$
$$= 2 + 0.25 \times 4$$
$$= 3$$

$$y(1.25+0.25) = y_{1.25} + 0.25 \, f(x_{1.25}, y_{1.25})$$

$$= 3 + 0.25 \times 4.8$$
$$= 3 + 1.2$$
$$= \cancel{3.5} \; 4.2$$

$$y(1.5+0.25) = y_{1.5} + 0.25 \times f(x_{1.5}, y_{1.5})$$

$$= 4.2 + 0.25 \times 5.6$$

$$= 4.2 + 1.4$$
$$= 5.6$$

$$y(1.75+0.25) = y_{1.75} + 0.25 \times f(x_{1.75}, y_{1.75})$$
$$= 5.6 + 0.25 \times 6.4$$
$$= 5.6 + 1.6$$
$$= 7.2$$

(ii) ~~Using Heun's method,~~

~~$y(1+0.25) = y_1 + \frac{0.25}{2}[4+3]$~~

~~$= 2 + 0.125 \times 7$~~

~~$= 2.875$~~

(ii) Using Heun's method,

$m_1 = f(x_0, y_0) = 4$

$y^e(1.25) = y_0 + hm$

$= 2 + 0.25 \times 4$

$= 3$

$m_2 = f(x_1, y^e)$

$= \frac{2 \times 3}{1.25}$

$= 4.8$

then,

$y(1.25) = y_0 + \frac{h}{2}(m_1 + m_2)$

$= 2 + \frac{0.25}{2}(4 + 4.8)$

$= 3.1$//

Again,

$m_1 = f(x_{1.25}, y_{1.25})$

$= 4.96$//

$y^e(1.5) = y_{1.25} + hm$

$= 3.1 + 0.25 \times 4.96$

$= 4.34$//

$m_2 = f(x_{1.25}, y^e)$

$= \frac{2 \times 4.34}{1.5}$

$= 5.787$

~~Again~~, then

$\cancel{m_1 = f(x_{1.5}, y_{1.5})}$

$y(1.5) = y_{1.25} + \frac{h}{2}(m_1 + m_2)$

$= 3.1 + \frac{0.25}{2}(4.096 + 5.787)$

$= 4.4434.$

Again, $m_1 = f(x_{1.5}, y_{1.5})$

$= \frac{2 \times 4.4434}{1.5}$

$= 5.92$

$y^e(1.75) = y_{1.5} + hm$

$= 4.4434 + 0.25 \times 5.92$

$= 5.9234$

$m_2 = f(x_1, y^e)$

$= \frac{2 \times 5.9234}{1.75}$

$= 6.07696$

Then,

$$y(1.75) = y_{1.5} + \frac{h}{2}(m_1 + m_2)$$

$$= 4.4434 + \frac{0.25}{2}(5.92 + 6.07696)$$

$$= 6.0296$$

Again,

$$m_1 = f(x_{1.75}, y_{1.5})$$

$$= \frac{2 \times 6.0296}{1.75}$$

$$= 6.89$$

$$y^e(2.0) = y_{1.75} + hm$$

$$= 6.0296 + 0.25 \times 6.89$$

$$= 7.07521$$

$$m_2 = f(x_i, y^e)$$

$$= \frac{2 \times 7.07521}{2}$$

$$= 7.07521$$

Then,

$$y(2) = y_{1.75} + \frac{h}{2}(m_1 + m_2)$$

$$= 6.0296 + \frac{0.25}{2}[6.89 + 7.07521]$$

$$= 7.86$$

Runge- Kutta method.

This method refers to the family of one step method, used for the solution of initial value problems. They are all based on the general form of the extrapolation equation,

$$y_{i+1} = y_i + \text{slope} \times \text{interval size}.$$

$$= y_i + mh.$$

where m = slope weighted averages of the slopes at various point in the interval h. h = interval size

Runge - Kutta (RK) methods are known by their order. For instance an RK method is called the r-order Runge--Kutta method when slope at r-points are used to construct the weighted average slope m. Euler's method uses only one slope at (x_i, y_i) to estimate y_{i+1}. And therefore Eulers method is the first order Runge-Kutta method. Similarly Heun's method is a 2nd order Runge-Kutta method because it employes slopes at two end points of the interval.

From Euler's & Heuns method it is seen that higher the order better would be the accuracy of estimates.

It is clear from above discussion that it is possible to construct RK method of different orders. However, the commonly used ones is the 4th order method. Although there are different versions of 4th order RK method, the most popular method is, the classical 4th order RK method, which is given below,

$$m_1 = f(x_i, y_i)$$
$$m_2 = f\left[x_i + h/2,\ y_i + \frac{m_1 h}{2}\right)$$

$$m_3 = f(x_i + h/2,\ y_i + m_2 h/2)$$

$$m_4 = f(x_i + h,\ y_i + m_3 h)$$

$$y_{i+1} = y_i + \left(\frac{m_1 + 2m_2 + 2m_3 + m_4}{6}\right) * h$$

Q. Use the classical RK method to estimate $y(0.4)$ when $y'(x) = x^2 + y^2$ with $y(0) = 0$. Assume $h = 0.2$
$y(0.4)$ when $h = 0.1$

Soln,

Given $y'(x) = x^2 + y^2 = f(x, y)$

$y_0 = 0,\ x_0 = 0$

$h = 0.2$

then iteration 1,

$m_1 = f(x_0, y_0)$

$= 0^2 + 0^2$

$= 0$

$m_2 = f\left(x_i + h/2,\ y_i + \frac{m_1 h}{2}\right)$

$= f(0.1, 0)$ 　　　 $|0.1|$,

$= 0.1^2 + 0^2$

$= 0.01$

$m_3 = f(x_i + h/2,\ y_i + m_2 h/2)$

$= f(0 + 0.1,\ 0 + 0.01 \times 0.2/2)$

$= f(0.1,\ 0.001)$

$= 0.01,$

$m_4 = f(x_i + h, \ y_i + m_3 h)$

$= f(0 + 0.2, \ 0 + 0.01 \times 0.2)$

$= f(0.2, \ 0.002)$

$= 0.04$

then,

$$y_{0.2} = y_0 + \frac{(m_1 + 2m_2 + 2m_3 + m_4)}{6} \times h$$

$$= 0 + \frac{(0 + 2 \times 0.01 + 2 \times 0.01 + 0.04)}{6} \times 0.2$$

$= \cancel{0.008}$

$= 0.00267$

for 2nd iteration, we have,

$m_1 = f(x_{0.2}, \ y_{0.2})$

$= 0.2^2 + 0.00267^2$

$= 0.04$

$m_2 = f(x_{0.2} + h/2 + y_{0.2} + m_1 h/2)$

$= f[0.2 + 0.1 + 0.00267 + 0.04 \times 0.1]$

$= f(0.3, \ 0.00307)$

$= 0.3^2 + 0.00667^2$

$= 0.09004$

$m_3 = f(x_{0.2} + h/2 \,,\ y_{0.2} + m_2\, h/2)$

$= f[0.2 + 0.1 \,,\ 0.00267 + 0.09004 \times 0.1]$

$= f(0.3, 0.01167)$

$= 0.3^2 + 0.01167^2$

$= 0.09014$ //

$m_4 = f(x_{0.2} + h \,,\ y_{0.2} + m_3 h)$

$= f(0.2 + 0.2 \,,\ 0.00267 + 0.09014 \times 0.2)$

$= f(0.4, 0.0207)$

$= 0.4^2 + 0.0207^2$

$= 0.16043$

Now,

$$y(0.4) = y_{0.2} + \left(\frac{m_1 + 2m_2 + 2m_3 + m_4}{6}\right) \times h$$

$$= 0.00267 + \left(\frac{0.04 + 2 \times 0.09004 + 2 \times 0.09014 + 0.16043}{6}\right) \times 0.2$$

$= 0.00267 + 0.09352 \times 0.2$

$= 0.02137$ //

Soln,

Given $f(x,y) = x^2 + y^2$

$x_0 = 0,\ y_0 = 0,\ h = 0.1 \quad h/2 = 0.05$

Then in First iteration,

$$m_1 = f(x_0, y_0)$$
$$= 0^2 + 0^2$$
$$= 0$$

$$m_2 = f(x_i + h/2,\ y_i + m_1 h/2)$$
$$= f(0 + 0.05,\ 0 + 0 \times 0.05)$$
$$= f(0.05, 0)$$
$$= 0.05^2 + 0^2$$
$$= 0.0025$$

$$m_3 = f(x_i + h/2,\ y_i + m_2 h/2)$$
$$= f(0 + 0.05,\ 0 + 0.0025 \times 0.05)$$
$$= f(0.05,\ 0.000125)$$
$$= 0.05^2 + 0.000125^2$$
$$= 0.0025$$

$$m_4 = f(x_i + h,\ y_i + m_3 h)$$
$$= f(0.1,\ 0.0025 \times 0.1)$$
$$= f(0.1,\ 0.00025)$$
$$= 0.1^2 + 0.00025^2$$
$$= 0.01$$

Then,

$$y_{0.1} = y_0 + \frac{(m_1 + 2m_2 + 2m_3 + m_4)}{6} * h$$

$$= 0 + \frac{[0 + 2 \times 0.0025 + 2 \times 0.0025 + 0.01)}{6} * 0.1$$

$$= 0.000333$$

For 2nd iteration we have

$x_i = 0.1, \; y_i = 0.000333$

Then, $m_1 = f(0.1, 0.000333)$

$= 0.01$

$m_2 = f(x_i + h/2, \; y_i + m_1 * h/2)$

$= f(0.15, \; 0.000833)$

$= 0.023$

$m_3 = f(x_i + h/2, \; y_i + m_2 * h/2)$

$= f(0.15, \; 0.001483$

$= 0.023$

$m_4 = f(x_i + h, \; y_i + m_3 * h)$

$= f(0.2, \; 0.002633)$

$= 0.04$

Then,

$$y_{0.2} = y_i + \frac{(m_1 + 2m_2 + 2m_3 + m_4)}{6} * h$$

$$= 0.000333 + \frac{(0.01 + 2 \times 0.023 + 2 \times 0.023 + 0.04)}{6} * 0.1$$

$= 0.002699667$

For 3rd iteration we have

$x_i = 0.2, \; y_i = 0.002699667$

Then,

$m_1 = f(0.2, \; 0.002699667)$

$= 0.04$

$m_2 = f(x_i + h/2, \; y_i + m_1 * h/2)$

$= f(0.21, \; 0.003099667)$

$= 0.0441$

$$m_3 = f(x_i + h/2,\ y_i + m_2 * h/2)$$
$$= f(0.21,\ 0.00314)$$
$$= 0.0441$$

$$m_4 = f(x_i + h,\ y_i + m_3 h)$$
$$= f(0.3,\ 0.0071)$$
$$= 0.09$$

Then, $y_{0.3} = y_i + \frac{(m_1 + 2m_2 + 2m_3 + m_4) * h}{6}$

$$= 0.0049$$

Now for fourth iteration we have

$x_i = 0.3,\ y_i = 0.0049$

Then,

$$m_1 = f(x_i, y_i)$$
$$= 0.09$$

$$m_2 = f(x_i + h/2,\ y_i + m_1 * h/2)$$
$$= 0.096$$

$$m_3 = f(x_i + h/2,\ y_i + m_2 * h/2)$$
$$= 0.096$$

$$m_4 = f(x_i + h,\ y_i + m_3 * h)$$
$$= 0.16$$

Then,

$$y_{0.4} = y_i + \frac{(m_1 + 2m_2 + 2m_3 + m_4) * h}{6}$$

$$= 0.0049 + 0.0105667$$

$$= 0.0154667$$

System of Differential Equations

mathematical model of many application involve a system of 1st order differential eq^n which may be represented as follows,

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2 \ldots y_m) \qquad y_1(x_0) = y_{10}$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2 \ldots y_m) \qquad y_2(x_0) = y_{20}$$

$$\vdots$$

$$\frac{dy_m}{dx} = f_m(x, y_1, y_2 \ldots y_m) \qquad y_m(x_0) = y_{m0}$$

These eq^{n}s can be solved by any of the method studied before for $y_1(x)$, $y_2(x)$ ----- $y_m(x)$ over the interval (a, b). At each stage, all the equations are solved before preceding to the next stage. For examp if $h = 0.5$ & $a = x_0 = 0$ then we must evaluate $y_1(0.5)$, $y_2(0.5)$ ----- $y_m(0.5)$ before preceding to the next stage $h = 1$.

let us consider a system of two equations for the purpose of illustration.

$$y_1'(x) = f_1(x, y_1, y_2), \quad y_1(x_0) = y_{10}$$
$$y_2'(x) = f_2(x, y_1, y_2), \quad y_2(x_0) = y_{20}$$

If we use Heun's method, The 1st stage would involve the following calculations.

$$m_1(1) = f_1(x_0, y_{10}, y_{20})$$
$$m_1(2) = f_2(x_0, y_{10}, y_{20})$$

$$m_2(1) = f_1 [x_0 + h,\ y_{10} + h\, m_1(1),\ y_{20} + h\, m_1(2)]$$

$$m_2(2) = f_2 [x_0 + h,\ y_{10} + h\, m_1(1),\ y_{20} + h\, m_1(2)]$$

Now,

$$m(1) = \frac{m_1(1) + m_2(1)}{2}$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2}$$

$$y_1(x_1) = y_1(x_0) + y_1\, x_0 + m(1) \times h$$
$$= y_{10} + m(1) h$$

$$y_2(x_1) = y_2(1) = y_2(x_0) + m(2) h$$
$$= y_{20} + m(2) h$$

The next stage uses $y_1(1)$ & $y_2(1)$ as initial values & by following similar procedure, $y_1(2)$ & $y_2(2)$ can be obtained.

Q. Given the equation $dy_1/dx = x + y_1 + y_2$, $y_1(0) = 1$
$dy_2/dx = 1 + y_1 + y_2$, $y_2(0) = -1$
estimate the values of $y_1(0.1)$ & $y_2(0.1)$ using Heun's method.

Soln,

Given $x_0 = 0$, $y_{10} = 1$
$y_{20} = -1$

Then,

$$m_1(1) = f_1(x_0, y_{10}, y_{20})$$
$$= f(0, 1, -1)$$
$$= 0$$

$m_1(2) = f_2(x_0, y_{10}, y_{20})$
$= f_2(0, 1, -1)$
$= 1$ //

$m_2(1) = f_1 [x_0 + h, y_{10} + h m_1(1), y_{20} + h m_1(2)]$
$= f_1 [0 + 0.1, 1 + 0.1 * 0 + (-1) + 0.1 * 1]$
$= f_1 (0.1, 1, -0.9)$
$= 0.2$ //

$m_2(2) = f_2 [x_0 + h, y_{10} + h m_1(1), y_{20} + h m_1(2)]$
$= f_2 (0 + 0.1, 1 + 0.1 \times 0, -1 + 0.1 * 1)$
$= f_2 (0.1, 1, -0.9)$
$= 1.1$

Now,

$m(1) = \frac{m_1(1) + m_2(1)}{2}$

$= \frac{0 + 0.2}{2}$

$= 0.1$

$m(2) = \frac{m_1(2) + m_2(2)}{2}$

$= \frac{1 + 1.1}{2}$

$= 1.05$

Then,

$y_1(x_1) = y_1 x_0 + m(1) h$
$= y_{10} + m(1) h$
$= 1 + 0.1 \times 0.1$
$= 1.01$ //

$$y_2(x_1) = y_{20} + m(2)h$$
$$= -1 + 1.05 \times 0.1$$
$$= -0.895$$

Higher order difference equation.

It is in the form $d^m y / dx^m = f(x, y, dy/dx, d^2y/dx^2$ ---- $\frac{d^{m}y'}{dx^{m-1}})$ ------ (1)

with m initial conditions given as,

$$y(x_0) = a_1$$
$$y'(x_0) = a_2$$
$$y^{m-1}(x_0) = a_m$$

we replace eqn (1) by a system of 1^{st} order equations as follows.

let us denote $y = y_1$, $\frac{dy}{dx} = y_2$, $\frac{d^2y}{dx^2} = y_3$,

---------- $\frac{d^{m-1}y}{dx^{m-1}} = y_m$

Then,

$$\frac{dy_1}{dx} = y_2, \; y_1(x_0) = y_{10} = a_1$$

$$\frac{dy_2}{dx} = y_3, \; y_2(x_0) = y_{20} = a_2$$

$$\frac{dy_{m-1}}{dx} = y_m, \; y_{m-1}(x_0) = y_{m-1,0} = a_{m-1}$$

$$\frac{dy_m}{dx} = f(x, y, y_2 \text{ ---- } y_m)$$

d^2y/dx^2

Q Solve the following eqⁿ for $y(0.2)$ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$

given $y(0) = 0$, $y'(0) = 1$

Solⁿ,

$$\frac{d^2y}{dx^2} = 6x + 3y - 2\frac{dy}{dx}$$

$$y = y_1, \quad \frac{dy}{dx} = y_2$$

then $\frac{dy_1}{dx} = y_2$, $y_{10} = 0$

$$\frac{dy_2}{dx} = 6x + 3y - 2y_2, \quad y_{20} = 1$$

let $h = 0.2$,

$$m_1(1) = y_{20} = 1$$

$$m_1(2) = 6x_0 + 3y_{10} - 2y_{20}$$
$$= 6\cdot 0 + 3\times 0 - 2\times 1$$
$$= -2$$

$$m_2(1) = y_{20} + hm_1(2)$$
$$= 1 + 0.2\times -2$$
$$= 0.6$$

$y_{10} + 0.2\times 1$

$(x_0 + h, y_{10} + h\times m_1(1), y_{20} + h\times m_1(2))$

$$m_2(2) = \cancel{6x_0}\ f(0.2, 0.2, 0.6)$$
$$= 0.6$$

Then,

$$m(1) = \frac{m_1(1) + m_2(1)}{2}$$

$$= \frac{1 + 0.6}{2}$$

$$= \cancel{0.5}$$

$$= 0.8$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2}$$

$$= \frac{-2+0.6}{2}$$

$$= -0.7$$

Boundary value problems.

It is seen previously that to solve m order differential equation, we require m conditions to be specified when all m conditions were specified at one point, $x = x_0$, and, therefore, we called this poblem as an initial value problem.

It is not always necessary to specify a condition at one point of the independent variable. They may be specified at different points in the interval (a, b) and, therefore, such problems are called boundary value problems.

In solving initial value problem, we move in steps from the given initial value of x to the point where solⁿ is required.

In case of boundary value problem we seek solutions at specified points within the domain of given boundries. For instance,

$$\frac{d^2y}{dx} = f(x, y, y') \quad , y(a) = y_a, \; y(b) = y_b$$

we are interested in finding the value of y in the range $a \leq x \leq b$. There are two popular methods available for solving the boundary value problem.

(a) shooting method.

(b) Finite difference method.

(a) shooting method.

In this method, the given boundary value problem is first transformed to an initial value problem, then these initial value problem is solved by taylor series method or Runge kutta method etc. Finally the given boundary value problem is solved.

consider the equation,

$$y'' = f(x, y, y') \qquad y(a) = A \;,\; y(b) = B$$

by letting $y' = z$ we obtain the following set of two eq^ns.

$$y' = z$$
$$z' = f(x, y, z)$$

In order to solve this set as an initial value problem we need two condition at $x = a$. We have one condition

i.e $y(a) = A$

and, therefore, required another condition for z at $x = a$. let us assume that,

$$z(a) = m_1,$$

where m_1 is a guess.

Note that m_1 represents the slope.

$$y'(x) \text{ at } x = a$$

Thus the problem is reduced to a system two first order equations, with the initial condition

$$y' = z \qquad y(a) = A,$$
$$z' = f(x, y, z) \qquad z(a) = m_1, \; (= y'(a)) \;\text{---}\; (1)$$

The above eqn (1) can be solved for y, and z using any one step method, using step of h, $O(h^{?})$ the solution at $x = b$ is reached.

let the estimated value of $y(x)$ at $x = b$ be B_1. If $B_1 = B$ then we have obtained the required soln.

If $B_1 \neq B$, then we obtain the soln with another guess say $z(a) = m_2$. let the new estimate of $y(x)$ at $x = b$ be B_2.

If $B_2 \neq B$, then the process may be continued until we obtain the correct estimate of $y(b)$. However The procedure can be accelerated by using an improved guess $z(a)$, after the estimates of B_1 & B_2 are obtained.

let us assume that $z(a) = m_3$ leads to the value of $y(b) = B$. If we assume that the value of m and B are linearly related then,

$$\frac{m_3 - m_2}{B - B_2} = \frac{m_2 - m_1}{B_2 - B_1}$$

$$\text{so } m_3 = m_2 + \frac{B - B_2}{B_2 - B_1} \times (m_2 - m_1)$$

$$= m_2 - \frac{B_2 - B}{B_2 - B_1} * (m_2 - m_1) \quad \text{---} (2)$$

Now with $z(a) = m_3$ we can again obtain the soln $y(x)$

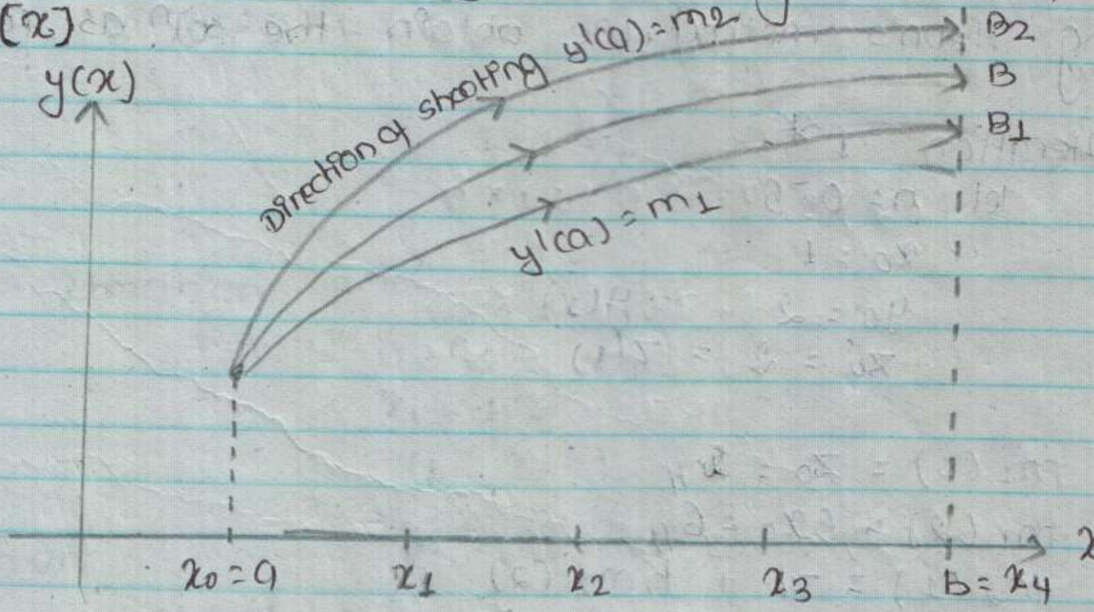
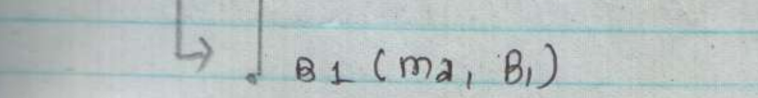


fig: Illustration of shooting method.



(9.75) 3.25

Q/ using shooting method solve the equation $\frac{d^2y}{dx^2} = 6x$
$y(1) = 2$; $y(2) = 9$

Soln,

By transformation we obtain the following,

$f_1 = \frac{dy}{dx} = z$; $y(1) = 2$ ||

$f_2 = \frac{dz}{dx} = 6x$

let us assume that $z(1) = y'(1) = 2$ $(= m_1)$

Applying Heun's method we obtain the soln as following,

Iteration 1st.

let $h = 0.5$

$x_0 = 1$

$y_0 = 2 = y(1)$

$z_0 = 2 = z(1)$

$m_1(1) = z_0 = 2$ ||

$m_1(2) = 6x_0 = 6$ ||

$m_2(1) = z_0 + hm_1(2)$

$= 2 + 0.5 * 6$

$= 5$

$m_2(2) = 6(x_0 + h)$

$= 6(1 + 0.5)$

$= 9$

$m(1) = \frac{m_1(1) + m_2(1)}{2}$

$= \frac{2 + 5}{2}$

$= 3.5$

$$m(2) = \frac{m_1(2) + m_2(2)}{2}$$
$$= \frac{6 + 9}{2}$$
$$= 7.5$$

Now,

$$y(1.5) = y_0 + m(1) * h$$
$$= 2 + 3.5 * 0.5$$
$$= 3.75 = y_1$$

$$z(1.5) = z(1) + m_2 * h$$
$$= 2 + 7.5 * 0.5$$
$$= 5.75$$

Iteration 2nd:

$$h = 0.5$$
$$x_1 = 1.5$$
$$y_1 = 3.75$$
$$z_1 = 5.75$$

Then

$$m_1(1) = z_1 = 5.75$$
$$m_1(2) = 6x_1 = 6 * 1.5 = 9$$
$$m_2(1) = z_1 + h m_1(2)$$
$$= 5.75 + 0.5 * 9$$
$$= 10.25$$
$$m_2(2) = 6(x_1 + h)$$
$$= 6(1.5 + 0.5)$$
$$= 12$$

Now,

$$m(1) = \frac{m_1(1) + m_2(1)}{2}$$

$$= \frac{5.75 + 10.25}{2}$$

$$= 8$$

$$m(2) = \frac{m_1(2) + m_2(2)}{2}$$

$$= \frac{9 + 12}{2}$$

$$= 10.5$$

Now,

$$y(2) = y_1 + m(1) \times h$$

$$= 3.75 + 8 \times 0.5$$

$$= 7.75$$

$$z(2) = z_1 + m(2) \times h$$

$$= 5.75 + 10.5 \times 0.5$$

$$= 11$$

This gives $B_1 = 7.75$ which is less than $B = 9$
Now, let us assume,

$$z(1) = y'(1) = 4 \; (= m_2)$$

Applying Heun's method we obtain the soln as following.

Iteration 1st,

let $h = 0.5$, $x_0 = 1$, $y_0 = 2$, $z_0 = 4$

then,

$m_1(1) = z_0 = 4$

$m_1(2) = 6x_0 = 6 \times 1 = 6$

$m_2(1) = z_0 + hm_1(2)$

$= 4 + 0.5 * 6$

$= 7$

$m_2(2) = 6(x_0 + h)$

$= 6(1 + 0.5)$

$= 9$

Then $m_1 = \dfrac{m_1(1) + m_2(1)}{2}$

$= \dfrac{4 + 7}{2}$

$= 5.5$

$m_2 = \dfrac{m_1(2) + m_2(2)}{2}$

$= \dfrac{6 + 9}{2}$

$= 7.5$

Now

$y(1.5) = y_0 + m(1) * h$

$= 2 + 5.5 * 0.5$

$= 4.75$

$z(1.5) = z_0 + m(2) * h$

$= 4 + 7.5 * 0.5$

$= 7.75$

Iteration 2nd,

$h = 0.5, \quad x_1 = 1.5, \quad y_1 = 4.75, \quad z_1 = 7.75$

then

$m_1(1) = z_q = 7.75$

$m_1(2) = 6x_1 = 6 \times 1.5$
$= 9$

$m_2(1) = z_q + h\, m_1(2)$
$= 7.75 + 0.5 * 9$
$= 12.25$

$m_2(2) = 6(x_q + h)$
$= 6(1.5 + 0.5)$
$= 12$ //

then

$$m(1) = \frac{m_1(1) + m_2(1)}{2}$$

$$= \frac{7.75 + 12.25}{2}$$

$= 10$

$$m(2) = \frac{m_1(2) + m_2(2)}{2}$$

$$= \frac{9 + 12}{2}$$

$= 10.5$

then

$y(1.5) = y_{1.5} + m(1)h$
$= 4.75 + 10 \times 0.5$
$= 9.75$

Now we have, $B_1 = 7.75 \qquad m_1 = 2$

$B_2 = 9.75 \qquad m_2 = 4$

$B = 9$

then

$$m_3 = m_2 + \frac{B - B_2}{B_2 - B_1} * m_2 - m_1$$

$$= 4 + \frac{9 - 9.75}{9.75 - 7.75} (4-2)$$

$$= 4 + \left[\frac{-0.75}{2}\right] \times 2$$

$$= 4 + (-0.375) * 2$$

$$= 3.25$$

Now,

Iteration 3rd

$h = 0.5 \quad x_0 = 1 \quad y_0 = 2, \; z_0 = 3.25$

then,

$$m_1(1) = z_0 = 3.25$$

$$m_1(2) = 6x_0 = 6 * 1 = 6$$

$$m_2(1) = z_0 + h m_1(2)$$

$$= 3.25 + 0.5 * 6$$

$$= 6.25$$

$$m_2(2) = 6(x_0 + h)$$

$$= 6(1 + 0.5)$$

$$= 9$$

Then

$$m(1) = \frac{m_1(1) + m_2(1)}{2}$$

$$= \frac{3.25 + 6.25}{2}$$

$$= 4.75$$

$$m(2) = \frac{m_2(2) + m_2(2)}{2}$$

$$= \frac{6 + 9}{2}$$

$$= 7.5$$

Now,

$$y(x) = y_0 + m(1)\,h$$

$$= 2 + 4.75 * 0.5$$

$$= 4.375$$

&

$$z(x) = z_0 + m(2) * h$$

$$= 3.25 + 7.5 * 0.5$$

$$= 7 //$$

Algorithm for shooting method.

1. convert the given problem into an initial value problem.
2. initialize the variables including two gueses at the initial slope.
3. solve the equations with this guesses using either a one step or a multi step method.
4. Interpolate from these results to find an improved value of the slope obtained.
5. Repeate the process until a specified accuracy in the final function value is obtained (or until a limit to the number of iteration is reached.

Eigenvalue & Eigenvector Problem.

Boundary value problems such as study of vibrating systems, structure analysis, electric circuit system analysis, reduce to a system of eq^n of the form

$$Ax = \lambda x \quad \text{-----} \quad (1)$$

such problems are called eigen value problem.

The sol^n vector x corresponding to eigen values λ are called eigenvectors.

we need to find the value of λ and vector x which satisfies the eq^n (1). we have two simple methods available to solve this type of problem.

(i) Polynomial method.

~~ii~~ (ii) Power method.

Power method

It is a 'single value' method used for determine the 'dominant' eigen value of a matrix. It is an iterative method implemented using an initial starting vector X. It is implemented as,

$$Y = AX \quad --- (2)$$
$$X = \frac{1}{K} Y \quad --- (3)$$

The new value of X obtained from eq^n (3) is used in eq^n (2) to obtain a new value of y. and the process is repeated until the desired level of accuracy is obtained.

The parameter K is known as the scaling factor & is the element of Y with the largest magnitude.

let us assume that the eigen values are $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$

and the corresponding eigen vectors $x_1, x_2 \ldots x_n$

after repetate application of eq^n (2) & (3) the vector X converges to x_1 and K converges to λ_1.

Q, find the largest eigen value λ_1 & corresponding eigen vector v_1 of the matrix
$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
using power method.

Soln,

let us assume the starting vector as.
$$X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eq^n (2) & (3) are repetatly used as follows, Iteration (1)

$$y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Now,

$$x = \frac{1}{k} y$$

$$= \frac{1}{2} \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

Iteration (2)

$$y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix}$$

$$x = \frac{1}{k} y$$

$$= \frac{1}{2.5} \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

Iteration ③

$$y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix}$$

&, $x = \frac{1}{2.8} \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix}$$

Iteration ④

$$y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix}$$

&

$$x = \frac{1}{2.93} \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}$$

Iteration ⑤

$$y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 2.96 \\ 0 \end{bmatrix}$$

& $x = \frac{1}{2.98}\begin{bmatrix} 2.98 \\ 2.96 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix}$

Iteration (6)

$y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 2.99 \\ 0 \end{bmatrix}$

&

$x = \frac{1}{2.99}\begin{bmatrix} 2.98 \\ 2.99 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Iteration (7)

$y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

& $x = \frac{1}{3} \times \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

The final answer shows that $\lambda_1 = 3$ (element with largest magnitude) & the corresponding eigen vector is the last x

i.e $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ...

Solution of Partial Differential Equation.

It is a differential eqn involving more than one independent variable. mathematical model of many physical phenomenan in applied science & engeneering fall into a catagory of systems known as partial differential Equations

For example.

(i) Study of displament of vibrating string.
(ii) Heat flow problem.
(iii) Fluid flow analysis.

Most of these problems can be formulated as 2nd order partial differential eqns (with the highest order of derivative being the 2nd).

If we represent the dependent variable as f, & the two independent variable as x & y, then we will have three possible 2nd order partial derivatives.

$$\frac{d^2f}{dx^2}, \quad \frac{d^2f}{dxdy}, \quad \frac{d^2f}{dy^2}$$

In addition to two first order partial derivatives,

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

we can write a 2nd order eqn involving two independent variables in general form

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x\partial y} + c\frac{\partial^2 f}{\partial y^2} = f\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \quad \text{------ (1)}$$

where a, b & c may be constants or functions of x & y. Depending on the value of these coefficients there are three types of eqn's.

If,

$b^2 - 4ac < 0$ (elliptic curve)

$b^2 - 4ac = 0$ (parabolic curve)

$b^2 - 4ac > 0$ (Hyperbolic curve)

Since the application of analytical method becomes more complex, we seek the help of numerical technique to solve partial differential equation.

There are two types of numerical technique,

(I) finite difference method.

(II) finite element method

Deriving difference Equation

while deriving difference eqⁿ, we will discuss two dimensional problems only, consider a two dimensional solⁿ domain as shown in figure below.

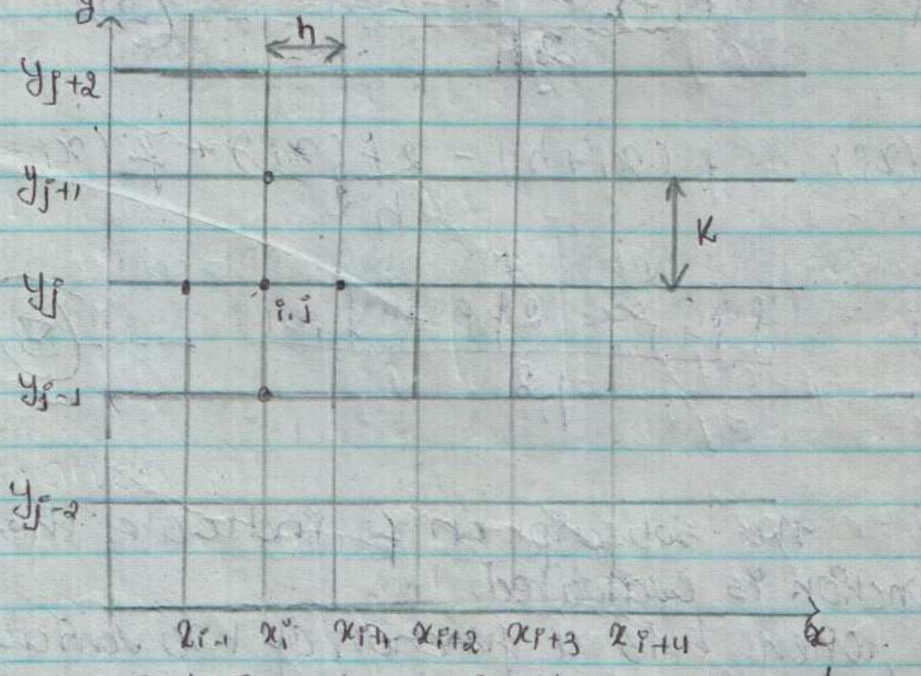


fig: Two dimensional finite difference grid.

Here, the domain is split into regular rectangular grids of width h & height k. The pivotal value at the point of intersection (known as grid point or node) are denoted by f_{ij} which is a function of two space variables x & y.

In finite difference method, we write the difference eqn corresponding to each "grid point" (where derivative is required) using functions values at the surrounding grid points. Solving these eqn's simultaneously gives values for the functions at each grid points.

If the function $f(x)$ has a continuous 4^{th} derivative then its 1^{st} & 2^{nd} derivatives are given by the following central difference approximation-

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h}$$

$$\text{or,} \quad f'_i = \frac{f_{i+1} - f_{i-1}}{2h} \quad \text{---} (2)$$

$$f''(x_i) = \frac{f(x_i+h) - 2f(x_i) + f(x_i-h)}{h^2}$$

$$\text{or,} \quad f''_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \quad \text{---} (3)$$

The subscript on f indicates the x value where the function is evaluated,

When f is a function of two variables x & y, the partial derivative of f with respect to x (or y) are the ordinary derivative of f with respect to x (or y) when y (or x) doesnot change.

We use eqn (2) & (3) in x direction to determine derivatives with respect to x & in y direction to determine derivatives with respect to y. Thus we have,

$$\frac{\partial f(x_i, y_j)}{\partial x} = f_x(x_i, y_j) = \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{2h}$$

$$\frac{\partial f(x_i, y_j)}{\partial y} = f_y(x_i, y_j) = \frac{f(x_i, y_{j+1}) - f(x_i, y_{j-1})}{2k}.$$

$$\frac{\partial^2 f(x_i, y_j)}{\partial x^2} = f_{xx}(x_i, y_j) = \frac{f(x_{i+1}, y_j) - 2f(x_i, y_j) + f(x_{i-1}, y_j)}{h^2}$$

$$\frac{\partial^2 f(x_i, y_j)}{\partial y^2} = f_{yy}(x_i, y_j) = \frac{f(x_i, y_{j+1}) - 2f(x_i y_j) + f(x_i, y_{j-1})}{k^2}$$

$$\frac{d^2 f(x_i y_j)}{dx dy} = f_{xy}(x_i, y_j) = \frac{f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_{j-1}) - f(x_{i-1}, y_{j+1}) + f(x_{i-1}, y_{j-1})}{4hk}$$

It is convinent to use double subscript i, j on f to indicate x & y values. Then the above eq^n becomes,

$$f_{x, ij} = \frac{f_{i+1, j} - f_{i-1, j}}{2h} \quad \text{------ (4)}$$

$$f_{y, ij} = \frac{f_{i, j+1} - f_{i, j-1}}{2k} \quad \text{---- (5)}$$

$$f_{xx, ij} = \frac{f_{i+1, j} - 2f_{i, j} + f_{i-1, j}}{h^2} \quad \text{--- (6)}$$

$$f_{yy, ij} = \frac{f_{i, j+1} - 2f_{i, j} + f_{i, j-1}}{k^2} \quad \text{--- (7)}$$

$$f_{xy, ij} = \frac{f_{i+1, j+1} - f_{i+1, j-1} - f_{i-1, j+1} + f_{i-1, j-1}}{4hk} \quad \text{--- (8)}$$

we will use those above formula to construct various types of differential Equations.

Elliptic Equations:

The two most commonly used elliptic eq^{n}s are,

(a) laplace's Equation.
(b) Poisson's equation.

① laplace's Equation

The equation

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = f\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right),$$

when,

$a=1$, $b=0$ & $c=1$ and,

$f\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0$ becomes.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 = \nabla^2 f \quad \text{--- } ⑨$$

The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the laplacian operator; and the eq^n ⑨ is called laplace equation. (sometimes u may be written in place of f). To solve laplace equation on a region in xy plane we divide the region as shown in the above figure. (ODE)

let us consider the portion of the region near (x_i, y_j). We have to approximate

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

replacing the 2nd order derivatives by their finite difference equivalence (from eqn 6 & 7) at the point (x_i, y_j), we obtain

$$\nabla^2 f_{ij} = \frac{f_{i+1,j} - 2f_{ij} + f_{i-1,j}}{h^2} + \frac{f_{i,j+1} - 2f_{ij} + f_{i,j-1}}{k^2} = 0$$

let us assume $h = k$ for simplicity,
we get,

$$\nabla^2 f_{ij} = \frac{1}{h^2}\left(f_{i+1,j} + f_{i-1,j} - 4f_{ij} + f_{i,j+1} + f_{i,j-1}\right) = 0 \quad \text{----} \quad (10.)$$

Note that eqn (10) contains four neighbouring points around the central points (x_i, y_j) (on all four sides) as shown in the figure below:

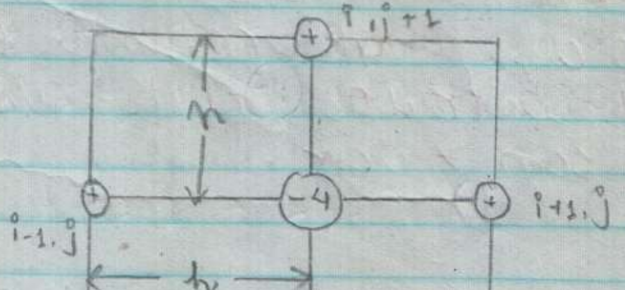


fig - Grid for laplac equation.

The relationship of pivotal values are represented as,

$$\nabla^2 f_{ij} = \frac{1}{h^2}\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix} f_{ij} = 0 \quad \text{---} \quad (11)$$

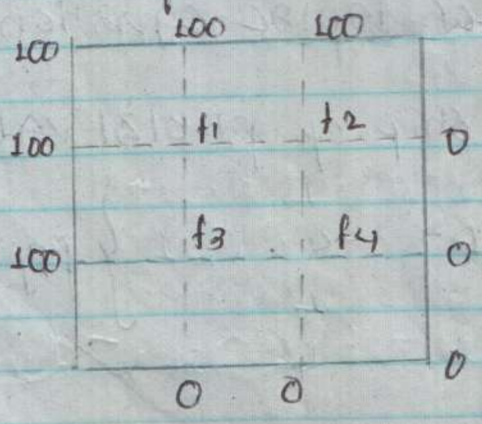
From eqⁿ (11) we can show that, the function value at the grid point (x_i, y_j) is the average of the values at the four adjoining point, i.e.,

$$f_{ij} = \frac{1}{4}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}\right) \quad \text{--- (12)}$$

To evaluate numerically the soln of laplace eqn at the grid points, we can apply eqn (12) at the grid points where f_{ij} is required. Thus obtaining a system of linear eqn in the pivotal values f_{ij}. The system of linear eqn may be solved using either direct method or iterative method.

Q. Consider a steel plate of size 15cm * 15 cm. If the two sides are held at 100°C & other two sides are held at 0°C, what are the steady state temperature at interior points, assuming a grid size of 5cm × 5cm.
2017

A problem with the values known on each boundary is said to have dirichlet boundary condition. The problem is illustrated as follows.



The system of eq^n is as follows,

at point 1 (f_1)

$$f_2 + f_3 - 4f_1 + 100 + 100 = 0$$

at point 2 (f_2)

$$f_1 + f_4 - 4f_2 + 100 + 0 = 0$$

$f_1 = 75$
$f_2 = 50$
$f_3 = 50$
$f_4 = 25$

at point 3 (f_3)

$$f_1 + f_4 - 4f_3 + 100 + 0 = 0$$

at point 4 (f_4)

$$f_2 + f_3 - 4f_4 + 0 + 0 = 0$$

Note,

By symmetry we can say that $f_2 = f_3$ so the system of eq^n reduced to,

$$f_2 + f_2 - 4f_1 + 200 = 0$$

i.e. $$2f_2 - 4f_1 + 200 = 0$$

f_1

Poisson's Equation:-

The equation

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = F\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

when $a=1$
$b=0$
$c=1$ & $F\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = g(x,y)$

becomes,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = g(x,y)$$

or,

$$\nabla^2 f = g(x,y) \quad \text{----} \quad (13)$$

then this eq^n (13) is called poisson's eq^n.

using the notation

$$g_{i,j} = g(x_i, y_j)$$

& using eq^n (6) & (7) at point (x_i, y_j) we obtain,

$$f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j} = h^2 g_{i,j} \quad \text{------} \quad (14)$$

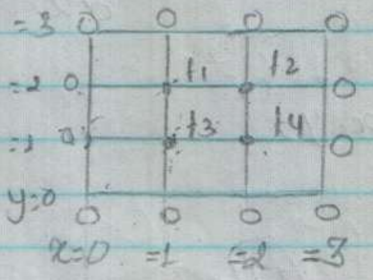
By applying the replacement formula to each grid point, in the domain of consideration, we will get a system of linear equations of ~~fi~~ f_{ij}.

these equations may be solved either by any of the elimination method or by any iterative method as done in solving laplace eq^n.

Q. Solve the poisson's Eqn
$\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3$ & $0 \le y \le 3$ with $f = 0$ on the boundary & $h = 1$.

Soln

The domain is divided into squares of 1 unit size as illustrated below.



at point 1 (f_1)

$$0 + 0 + f_2 + f_3 - 4f_1 = 1^2 \cdot 2x^2y^2$$

or $f_1 + f_3 - 4f_1 = 2 \cdot 1 \cdot 2^2$

or $f_1 + f_3 - 4f_1 = 8$

at point 2 (f_2)

or $0 + 0 + f_1 + f_4 - 4f_2 = 1^2\, 2x^2y^2$

or $f_1 + f_4 - 4f_2 = 2 \cdot 2^2 \cdot 2^2$

or, $f_1 + f_4 - 4f_2 = 32$

at point 3 f_3.

$$0 + 0 + f_1 + f_4 - 4f_3 = 1^2\, 2x^2y^2$$

or, $f_1 + f_4 - 4f_3 = 1 \cdot 2 \cdot 1 \cdot 1$

or $f_1 + f_4 - 4f_3 = 2$

$f_1 = -22/4$
$f_2 = -43/4$
$f_3 = -13/4$
$f_4 = -22/4$

at point f_4,

$$0 + 0 + f_2 + f_3 - 4f_4 = 2x^2y^2$$

or $f_2 + f_3 - 4f_4 = 2 \cdot 2^2 \cdot 1$

or, $f_2 + f_3 - 4f_4 = 8$ //