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2

Relation, Function and Graph

Learning objectives/outcomes:

After completion of this unit, the students will be enable to

i) Define ordered pair, cartesian product, relation, domain and range of relation, inverse relation and solve the related problems.
ii) Define function, domain and range of a function, one to one function (injective funtion), onto function (surjective function), bijective function (both injective and surjective), inverse of a function and composite function of given functions.
iii) Find domain and range of a function.
iv) Find inverse function of given invertible(bijective) function.
v) Calculate composite function of given functions.
vi) Check whether the given function is one to one or onto or both.

1.3.1 Introduction

In mathematics, function plays very crucial role in all the brances of mathematics, not only in the mathematics but also other fields of studies such as physics, engineering, medicine, biology, business, computer science and industry has been used widely. The notion of the function are used day to day life as well.

1.3.2 Ordered pair

Before defining an ordered pair we need to define what is a pair? A set with two elements is called a pair. For examples:- a couple of crows, a pair of sandals, a pair of bags and a pair of natural number and so on. But orders are not mentioned in each of the above examples. Infact, **a pair which are kept(arranged) in an order is**

called an ordered pair. Since, (a, b) and (b, a) are two ordered pairs. In first ordered pair a is the first element and b is the second element. But, in second ordered pair, b is the first element and a is second element although their elements are same. The ordered pair are written within a parenthesis bracket i.e. (,) but {} or curly brackets are used to denote a set. The co-ordinates on a graph are represented by an ordered pair, x and y. Where first element is x and second element is y. Two ordered pairs are said to be equal orif the corresponding elements are same which is also called an equality of an ordered pair. But (a, b) and (b, a) are not equal ordered pair because their corresponding elements are different i.e. $a \neq b$ and $b \neq a$.

1.3.3 Cartesian product

Let X and Y be two non-empty sets. Then, the Cartesian product X × Y between two sets X and Y is the **set of all possible ordered pairs** with first element from X and second element from Y.

X × Y = {(x, y) : x ∈ X and y ∈ Y}.

Where X is the set of points on the x – axis, Y is the set of points on the y-axis, and X×Y is the xy – plane. Since, the Cartesian product of X and Y is denoted by X × Y and read as "**X cross Y**".

Examples:

1. If P = {2, 4, 6} and Q = {8, 9}, find P×Q and Q× P.

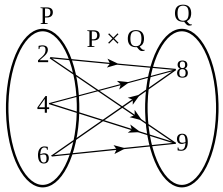
Solution:Here,

P = {2, 4, 6}

Q = {8, 9}

∴ P × Q = {2, 4, 6} × {8, 9}

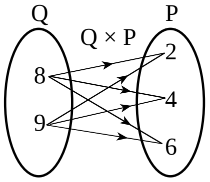
= {(2, 8), (2, 9), (4, 8), (4, 9), (6, 8), (6, 9)}



And,

∴ Q × P = {8, 9} × {2, 4, 6}

= {(8, 2), (8, 4), (8, 6), (9, 2), (9, 4), (9, 6)}



Hence, P × Q ≠ Q × P

2. Find A = {2, 8, 9} find the Cartesian product with itself.

Sol^{n}: Here,

A = {2, 8, 9}

A × A = {2, 8, 9} × {2, 8, 9}

= {(2, 2), (2, 8), (2, 9), (8, 2), (8, 8), (8, 9), (9, 2), (9, 8), (9, 9)}

Representation of Cartesian product: The Cartesian product can be represented in a mapping diagram, in a tabular form and Graphical form.

Example: If A = {2, 4, 6} and B = {a, b, c} find (A × B) in a mapping diagram, in a tabular form and a graphical form.

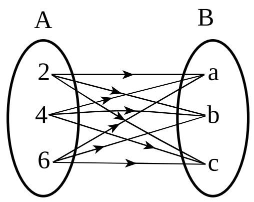
Here,

A = {2, 4, 6} and B = {a, b, c}

A × B = {2, 4, 6} × {a, b, c}

= {(2, a), (2, b), (2, c), (4, a), (4, b), (4, c), (6, a), (6, b), (6, c)}

i) In a mapping diagram



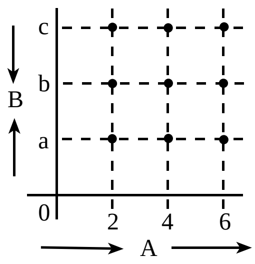
ii) In a tabular form

B

A

A × B	**a**	**b**	**c**
2	(2, a)	(2, b)	(2, c)
4	(4, a)	(4, b)	(4, c)
6	(6, a)	(6, b)	(6, c)

iii) In a graphical method:



1.3.4 Relation

If A and B be two non-empty sets, then the relation $\mathfrak{R}$ from set A to set B is **a subset of the Cartesian product** A and B i.e. A × B such that $\mathfrak{R} \subseteq (A \times B)$. If $(x, y) \in \mathfrak{R}$,then we write x $\mathfrak{R}$ y and read as x is related to y. A relation from set A to itself is called a relation on A.

In other word, **an association/mapping between the elements of first set A to the elements of second set B based on some properties(conditions)** possesses by them is called a relation. It is denoted by $\mathfrak{R}$. Indeed, a relation is the relationship between two or more sets of values. Let us assume that X and Y be two sets. Since, set X has relation with set Y, then the values of x are called domain where the values of set Y are called range.

For example:

$\mathfrak{R} = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$

Domain = $D(f) = \{1, 3, 5, 7\}$ [Input]

Range = $R(f) = \{2, 4, 6, 8\}$ [Output]

Types of Relations:

There are various types of relations which are described below:

(i) **Empty Relation:** A relation which has no relation between any elements of set is called empty relation or **void** relation. For example: If A = {1, 2}. Then one of the void relation can be $\mathfrak{R} = \{ x, y\}$ where $| x - y| = 6$

(ii) **Identity Relation**: A relation in which every element of a set is related to itself only.
If A = {a, b} then its identity relation i.e.
$A \times A = \{a, b\} \times \{a, b\} = \{(a, a), (a, b), (b, a), (b, b)\}$
$\therefore$ It's identity Relation = I = {(a, a), (b, b)}

(iii) **Inverse Relation :** An inverse relation is the set of ordered pairs which is obtained by interchanging the first and second elements of each pair. It is denoted by R^{-1}. For example:

$\mathfrak{R} = \{(2, 3), (4, 5), (6, 7), (8, 9)\}$

$\therefore \quad \Re^{-1} = \{(3, 2), (5, 4), (7, 6), (9, 8)\}$

(iv) Reflexive Relation: A relation $\Re$ in a set A is said to be reflexive if and only if for each $a \in A : (a, a) \in \Re$ or $a \Re a$.
For example: Let $\Re$ be a relation is defined on a set A = {1, 2, 3}.
$\Re = \{(1, 1), (2, 2), (3, 3)\}$

(v) Symmetric Relation: A relation $\Re$ on set A is said to be a symmetric relation iff (if and only if) $(a, b) \in \Re \Leftrightarrow (b, a) \in \Re$ or $a \Re b \Leftrightarrow b \Re a$
Thus, a relation is a symmetric if $\Re = \Re^{-1}$.
For example:
Let A = {1, 2, 3} and $\Re$ be a relation defined on set A as:
$\Re = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ then
$\Re^{-1} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
Since, $\Re = \Re^{-1}$, it is symmetric Relation.
Note:
a) ϕ is symmetric relation b) $A \times A$ is a symmetric relation.

(vi) Transitive relation: Let A be any set. A relation $\Re$ on A is said to be a transitive relation if $(a, b) \in \Re$ and $(b, c) \in \Re \Rightarrow (a, c) \in \Re$ for all $a, b, c \in A$.
For example: If A = {1, 2, 3}. Then, the relation R = {(1, 2), (2, 3), (1, 3)} is a transitive because ${}^{1}\Re_{2}$ and ${}^{2}\Re_{3} \Rightarrow {}^{1}\Re_{3}$.

(vii) Equivalence Relation: A relation on a set A is called an equivalence relation on A if it is reflexive, symmetric and transitive. It is generally denoted by the symbol $\sim$.

Moreover,

(viii) Anti-Symmetric Relation: A relation $\Re$ is said to be anti-symmetric on a set A, If $x \Re y$ and $y \Re x$ hold when $x \neq y$ or it can be defined as; relation R is anti-symmetric if either $(x, y) \notin \Re$ or $(y, x) \notin \Re$ whenever $x = y$. For example:

a) If $\Re = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$

Here, $\Re$ is not anti-symmetric because of $(1, 2) \in \Re$ and $(2, 1) \in \Re$, but $1 \neq 2$. Also, $(1, 4) \in \Re$ and $(4, 1) \in \Re$ but $1 \neq 4$.

b) If A = {1, 2, 3, 4} then, $\Re = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ is anti-symmetric relation on A.

(ix) Asymmetric Relation: A relation $\Re$ on a set A is said to be Asymmetric relation if $(a, b) \in \Re \Rightarrow (b, a) \notin \Re$.
For example: If A = {1, 2, 3, 4}
$\Re_1 = \{(1, 2), (2, 3), (3, 1), (4, 1)\} \rightarrow$ is an Asymmetric

$\Re_2 = \{(1, 2), (2, 1), (2, 3), (4, 3)\} \rightarrow$ It is not Asymmetric

Worked out examples:

Example:1

If (2x + y, 3) and (2, x – y) are equal ordered pairs, find the values of x and y.

Sol^n: Given,

$(2x + y, 3) = (2, x - y)$

Equating the corresponding elements, we get,

$2x + y = 2$ ..... (i) and $x - y = 3$ ..... (ii)

Adding eq^n (i) and eq^n (ii), we get,

$3x = 5$ or, $x = \frac{5}{3}$

Substituting the value of x in eq^n (i).

$2 \times \frac{5}{3} + y = 2$

or, $y = 2 - \frac{10}{3} = \frac{6-10}{3} = \frac{-4}{3}$

Hence, the required values of x is $\frac{5}{3}$ and y is $\frac{-4}{3}$

Example:2

If A = {1, 2}, B = {2, 3} and C = {1, 3, 4} find (i) $(A \cap B) \times C$ (ii) $(A - B) \times C$ (iii) $(A \Delta B) \times C$,

Note: $A\Delta B = (A - B)\cup(B - A)$, where, Δ - Symmetric Difference between sets A and B.

Solution: Here,

$A = \{1, 2\}$

$B = \{2, 3\}$

$C = \{1, 3, 4\}$

(i) $(A \cap B) \times C$

$A \cap B = \{1, 2\} \cap \{2, 3\} = \{2\}$

$\therefore (A \cap B) \times C = \{2\} \times \{1, 3, 4\} = \{(2, 1), (2, 3), (2, 4)\}$

(ii) $(A - B) \times C$

$(A - B) = \{1, 2\} - \{2, 3\} = \{1\}$

$\therefore \quad (A - B) \times C = \{1\} \times \{1, 3, 4\} = \{(1, 1), (1, 3), (1, 4)\}$

(iii) $(A \Delta B) \times C$

$A \Delta B = (A - B) \cup (B - A) = \{1\} \cup \{3\} = \{1, 3\}$

$(A \Delta B) \times C = \{1, 3\} \times \{1, 3, 4\} = \{(1, 1), (1, 3), (1, 4), (3, 1), (3, 3), (3, 4)\}$

Example:3

If $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$. Find the relation $\mathfrak{R}$ from set A to set B determined by the relation $x > y$ where, $x \in A$ and $y \in B$.

Solution: Here,

$A = \{1, 2, 3, 4\}$

$B = \{1, 3, 5\}$

$A \times B = \{1, 2, 3, 4\} \times \{1, 3, 5\}$

$= \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5), (4, 1), (4, 3), (4, 5)\}$

Under the relation $\mathfrak{R}_{x>y}$ is $\{(2, 1), (3, 1), (4, 1), (4, 3)\}$

Example:4

If $A = \{3, 4, 5, 6\}$, find the relation $\mathfrak{R}$ in $A \times A$ satisfying the condition $x + y = 9$; $x \in A$, $y \in A$.

Solution: Here,

$A = \{3, 4, 5, 6\}$

$A \times A = \{3, 4, 5, 6\} \times \{3, 4, 5, 6\}$

$= \{(3, 3), (3, 4), (3, 5), (3, 6), (4, 3), (4, 4), (4, 5), (4, 6), (5, 3), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Under the relation $\mathfrak{R}_{x+y=9}$ i.e.

$(A \times A)_{x+y=9}$ is $\{(3, 6), (4, 5), (5, 4), (6, 3)\}$

Example:5

If $\mathfrak{R} = \{(a, b), (c, d), (e, f), (g, h)\}$. Find $\mathfrak{R}^{1}$.

Solution: Here

$\mathfrak{R} = \{(a, b), (c, d), (e, f), (g, h)\}$

$\therefore \quad \mathfrak{R}^{-1} = \{(b, a), (d, c), (f, e), (h, g)\}$

Exercise

1. Find the values of a and b if

a) $(a + b, 2) = (3, a - b)$

b) $(a - 1, b + 2) = (b - 2, 2a + 1)$

2) If P = {2, 3}, Q = {1, 2} and R = {2, 4, 6}. Find

a) $P \times Q$ b) $Q \times P$ c) $(P \cup Q) \times R$

d) $(P \cap Q) \times R$ e) $(P - Q) \times R$ f) $(P \Delta Q) \times R$

3) If A = {2, 4, 6} and B = {a, b, c}, find $A \times B$ and $B \times A$. Are they equal?

4) If P = {1, 2, 3}, Q = {4, 5, 6} and R = {c, d, e} verify that:

a) $P \times (Q \cup R) = (P \times Q) \cup (P \times R)$

b) $(P \times Q) \cap (P \times R) = P \times (Q \cap R)$

5) Find the domain, range and inverse of the following relations.

a) $\mathfrak{R}_1 = \{(4, 5), (5, 6), (6, 7), (7, 8)\}$

b) $\mathfrak{R}_2 = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$

c) $\mathfrak{R}_3 = \{(1, 4), (3, 4), (5, 4), (7, 4)\}$

d) $\mathfrak{R}_4 = \{(1, 4), (1, 4), (2, 1), (4, 3), (4, 5)\}$

6. If P = {2, 4, 6} and Q = {2, 3, 6,8}. Find the relation from set P to set Q determined by the condition that x divides y.Also,find the domain and range of the relation.

7. Let P= { 1,2,3,4} and the relation is defined as $\mathfrak{R} = \{(x, y): x, y \in P \text{ and } x + y \leq 4\}$. Express $\mathfrak{R}$ as a set of ordered pairs. Find the domain, Range and $\mathfrak{R}^{-1}$

Answers

1.(a) $a = \frac{5}{2}$ and $b = \frac{1}{2}$ b) a = 2 and b=3.

2. (a) $P \times Q = \{(2,1),(2,2),(3,1),(3,2)\}$

(b)$Q \times P = \{(1,2),(1,3),(2,2),(2,3)\}$

(c) $(P \cup Q) \times R = \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6),(3,2),(3,4),(3,6)\}$.

(d)$(P \cap Q) \times R = \{(2,2),(2,4),(2,6)\}$.

(e) $(P - Q) \times R = \{(3,2),(3,4),(3,6)\}$.

(f) $(P \Delta Q) \times R = \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6)\}$.

3.$A \times B = \{(2,a),(2,b),(2,c),(4,a),(4,b),(4,c),(6,a),(6,b),(6,c)\}$

$B \times A = \{(a,2),(b,2),(c,2),(a,4),(b,4),(c,4),(a,6),(b,6),(c,6)\}$

$\therefore (A \times B) \neq (B \times A)$.

5. a) Domain ={4,5,6,7}, Range ={ 5,6,7,8} and inverse relation i.e. $R_1^{-1} = \{(5, 4), (6, 5), (7, 6), (8,)\}$

b) Domain ={1,2,3,4}, Range = {3,5,7,9} and inverse relation i.e$R_2^{-1} = \{(3, 1), (5, 2), (7, 3), (9, 4)\}$

c) Domain ={ 1,3,5,7},Range ={4} and inverse relation i.e. R_3^{-1} = {(4, 1), (4, 3), (4, 5), (4, 7)}

d) Domain ={ 1,2,4},Range ={ 1,3,5}and inverse relation i.e. R_4^{-1} = {(4, 1), (4, 1), (1, 2), (3, 4), (5, 4)}

6. R={ (2,2),(2,6),(2,8),(4,8),(6,6)}, Domain = { 2,4,6} and Range ={ 2,6,8}.

7. R = {(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)},Domain ={1,2,3} and Range ={ 1,2,3}

1.3.5 Functions

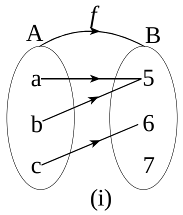
The concept of a function was developed after the development of calculus in the 17th century. Functions are everywhere where the technologies have been used widely. A refinement of the concept of relation provides is the important definitions of function. In information technology, function is regarded as **processing**.

Definition: Let A and B be two non-empty sets. A function from set A to set B is a relation/association/mapping/rule from A to B such that for all $x \in A$ there is one and only one (**unique**) element $y \in B$ with $(x, y) \in f$. We write $f : A \to B$ to mean f is a function from set A to set B.

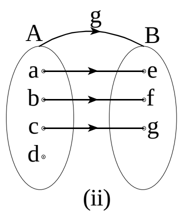
If a variable y is so related to a variable x that whenever a numerical value is assigned to x, there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x. Commonly, the relation in Cartesian form is symbolized as **$y = f(x)$**. In addition to $f(x)$ other symbols also is used such as g(x), h(x) and p(x) etc.

The above definition can be rephrase as given below:

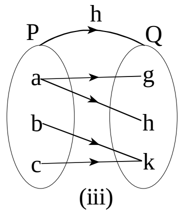
A function from set A to set B is a relation or rule which associates each element of set A with the unique element of set B. Symbolically, we write $f : A \to B$ which means f is the function from set A to set B. The equation $y = f(x)$, read, 'y equals f of x', is said to define the function f from A to B. Here, $f(x)$ is known as the image of f at x. The following mapping diagram clearly define which one is a function or not.



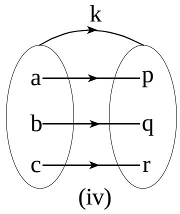
(i)



(ii)



(iii)



(iv)

From the above mapping diagrams (i) and (iv) are functions because every elements in a domain has unique image in the Co-domain. But, fig (ii) and fig (iii) are not functions because in fig (iii) element 'a' has two images which disobey the definition of a function and in fig (ii) there is one-element in a set A which doesnot associate in set B.

Domain, Co-domain and Range of the function

Domain and Co-domain: If $f : A \to B$ be a function then the **first** set A or set of input is the domain and **second** set B is called the Co-domain of the function f.

Range : The range is the set of possible output (images) values which are shown on the y-axis of the graph. In shorten: the set of all values of f is called the range of f. The range of f is denoted by $\boldsymbol{f(x)}$. Or, Set of images is Range.

Example: Let P = {– 3, – 1, 0, 2, 4} and a function is defined by $f(x) = |x| - 4$. Find the values of $f(-3)$, $f(-1)$, $f(0)$, $f(2)$ and $f(4)$.

Solution: Given,

$f(x) = |x| - 4$

$f(-3) = |-3| - 4 = 3 - 4 = -1$

$f(-1) = |-1| - 4 = 1 - 4 = -3$

$f(0) = |0| - 4 = 0 - 4 = -4$

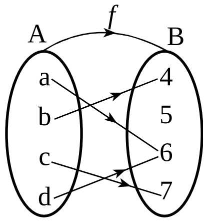
$f(2) = |2| - 4 = 2 - 4 = -2$

$f(4) = |4| - 4 = 4 - 4 = 0$

$\therefore$ Domain of f = P = {– 3, – 1, 0, 2, 4}

$\therefore$ Range of f = {– 1, – 3, – 4, – 2, 0}

Image and Pre-imageof a function: If a function $f : A \rightarrow B$ where $x \in A$ and $y \in B$ then $y = f(x)$ is said to be image of f at x or value of f at x. Then, x is said to be a Pre-image of y under f. For eg:-



Since, image of a is 6, image of an element b is 4, image of an element c is 7, image of an element d is 6 and,

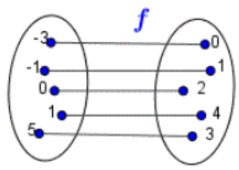
Pre-image of 6 is a and Pre-image of 4 is b.

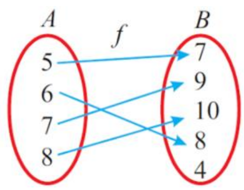
Pre-image of 7 is c and Pre-image of 6 is d.

Note: The set of all images is called range of the function.

1.3.6 Types of function

(i) **One to one function (injective function):** A function $f : A \rightarrow B$ is said to be injective or one to one function if it maps distinct elements of its domain to distinct elements of its Co-domain.





Thus, in this case, for $x_1 \neq x_2$, we must have $f(x_1) \neq f(x_2)$. [distinct object $\Rightarrow$ distinct image]

Equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. [equal image $\Rightarrow$ equal object]

Counter Example:-

a) The function $f : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $f(x) = x^2$ where $\mathbb{N}$ is the set of natural numbers.

Solution:-

Given, $f : \mathbb{N} \rightarrow \mathbb{N}$ where, $\mathbb{N} = \{1, 2, 3, 4 .....\}$

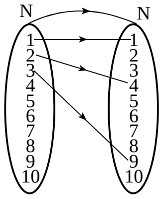
Now,

$2, 3 \in \mathbb{N} \Rightarrow f(2) = 2^2 = 4$

$\Rightarrow f(3) = 3^2 = 9$

Then, $2 \neq 3 \quad \Rightarrow f(2) \neq f(3)$

Hence, f is one to one function.



b) If $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 4x - 3$.

Solution:- For any $x_1, x_2 \in \mathbb{R}$

let $f(x_1) = f(x_2) \Rightarrow 4x_1 - 3 = 4x_2 - 3$

$\Rightarrow 4x_1 = 4x_2$

$\Rightarrow x_1 = x_2$

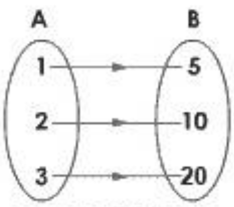
$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Hence, f is one to one or injective function.

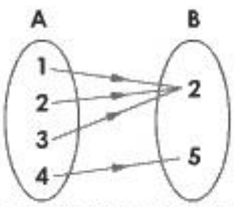
(ii) **Onto function (or Surjective function):** A function $f : A \to B$ is said to be onto or surjective if every element of B has at least one Pre-image in set A.

OR,

This means that $f : A \to B$ is onto function if range of f is equal to Co-domain then $\boldsymbol{f(A) = B}$ **or** $\mathbf{f(x) = y}$, Where $x \in A$ and $y \in B$.



one-one and onto



many-one and onto

Counter Examples:

a) If $\mathbb{R}$ be the set of real numbers, show that the function defined by $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = 4x - 7$, $x \in \mathbb{R}$ onto function.

Solution:-

Let $y = f(x)$, then

$y = 4x - 7$

or, $y + 7 = 4x$

or, $x = \dfrac{y+7}{4}$

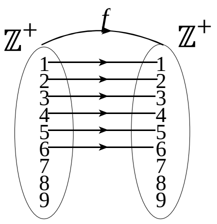
Given, $f(x) = 4x - 7$

$\therefore\ f\left(\dfrac{y+7}{4}\right) = \cancel{4}\left(\dfrac{y+7}{\cancel{4}}\right) - 7 = y + 7 - 7 = y$

$\Rightarrow f(x) = y$

So, f is onto function.

b) If $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $f(x) = x$, where $\mathbb{Z}^+$ is the set of positive integers. It can be shown as;



Since, Range is equal to Co-domains. It is onto function.

Or,

Let $y = f(x)$

$\Rightarrow y = x$

$\Rightarrow x = y$

$\Rightarrow$ for every $y \in \mathbb{Z}^+$, $x \in \mathbb{Z}^+$. Hence, given function is onto.

c) If $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $f(x) = 2x$, where $\mathbb{Z}^+$ is the set of positive integers.

Solution:-

Let $y = f(x)$

$\Rightarrow y = 2x$

$\Rightarrow x = \frac{y}{2}$

$\Rightarrow$ for every $y \in \mathbb{Z}^+$, $x \notin \mathbb{Z}^+$. Hence, given function is not onto.

d) Check whether $f : [-2, 3] \to \mathbb{R}$ given $f(x) = x^3$ onto function.

Solution: Here,

$f(x) = x^3$

And, $f : [-2, 3] \to \mathbb{R}$

$f(-2) = (-2)^3 = 8$

$f(3) = 3^3 = 27$

$\therefore \quad -2 \neq 3 \Rightarrow 8 \neq 27$

So, f is one to one function

Again, For onto function,

Range of f = Co-domain of f

But, Range of $f = [f(-2), f(3)]$

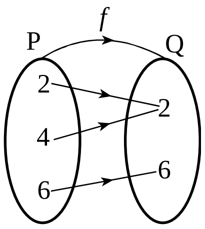
$= [-8, 27]$

Here, Co-domain of f = R

but, $[-8, 27] \neq \mathbb{R}$

Hence, f is not onto function.

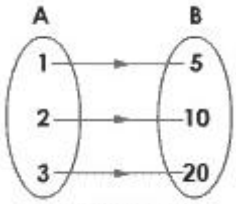
e) Let P = {2, 4, 6} and Q = {2, 6}. If a function $f : P \to Q$ is onto function.



Here, Range of f = Co-domain of f

Hence, f is onto function.

(iii) Bijective function (One to one onto): A function $f : A \to B$ is said to be bijective function if it is both injective (one to one) and surjective (onto).



one-one and onto

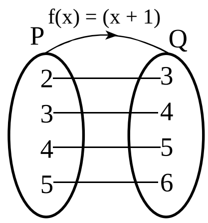
Counter Examples:

a) Let P = {2, 3, 4, 5} and Q = {3, 4, 5, 6} and a function $f : P \to Q$ is defined by $f(x) = x + 1$. Is it a bijective function ?

Here,

P = {2, 3, 4, 5}

Q = {3, 4, 5, 6}



Since, every elements of domain have unique image in co-domain. So, it is one to one function. And, range is equal to co-domain. It is also onto function.

$\therefore$ yes, it is a bijective function because it is both one to one onto

b) If a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ for all $x \in R$ is a bijective.

Here,

$f(x) = 2x + 3$

Let, $x_1, x_2 \in R$ then,

$f(x_1) = 2x_1 + 3$

$f(x_2) = 2x_2 + 3$

$\therefore \quad f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

So, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

So, it is injective function.

Again, for onto

$f(x) = 2x + 3$

or, $y = 2x + 3$

or, $y - 3 = 2x$

or, $x = \frac{(y-3)}{2}$

Now,

$$f\left(\frac{y-3}{2}\right) = 2.\frac{(y-3)}{2} + 3 = y - 3 + 3 = y$$

or, $f\left(\frac{y-3}{2}\right) = y$

$\Rightarrow f(x) = y$

$\therefore$ It is surjective function as the range is equal to Co-domain.

Hence, the given function is bijective as it is injective and surjective both.

c) Let $f : \mathbb{N} \to \{2\}$ defined by $f(x) = 2$ for all $x \in \mathbb{N}$, obviously, f is onto function but is not one to one function. Hence, it is not bijective function.

d) The identity function is always a bijective as it is one to one both.

Worked out Examples:

Example:1

A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 4x - 2 & \text{for } x \geq 1 \\ 2x & \text{for } x < 1 \end{cases}$$

Find (i) $f(2)$ (ii) $f(1)$ (iii) $f(-1)$ (iv) $\frac{f(h) - f(1)}{h}$ for $1 \leq h$.

Solution: Here,

$$f(x) = \begin{cases} 4x - 2 & \text{for } x \geq 1 \\ 2x & \text{for } x < 1 \end{cases}$$

(i) $f(2) = 4 \times 2 - 2 = 8 - 2 = 6$

(ii) $f(1) = 4 \times 1 - 2 = 4 - 2 = 2$

(iii) $f(-1) = 2 \times (-1) = -2$

(iv) For $\frac{f(h) - f(1)}{h}$

$f(h) = 4h - 2$ as $1 \leq h$

$$\therefore \frac{f(h) - f(1)}{h} = \frac{4h - 2 - 2}{h} = \frac{4h - 4}{h} = \frac{4(h-1)}{h}.$$

Example:2

Let A = {0, 1, 2, 3, 4, 5, 6} and a function $f : A \to \mathbb{Q}$ is defined by $f(x) = \frac{x}{2}$. Find the range of f, where $\mathbb{Q}$ is the set of rational numbers.

Solution: Here,

$f : A \to Q$ be defined by $f(x) = \frac{x}{2}$,

where A = {0, 1, 2, 3, 4, 5, 6}

when x = 0, $f(0) = \frac{0}{2} = 0$

when x = 1, $f(1) = \frac{1}{2}$

when x = 2, $f(2) = \frac{2}{2} = 1$

when x = 3, $f(3) = \frac{3}{2}$

when x = 4, $f(4) = \frac{4}{2} = 2$.

when x = 5, $f(5) = \frac{5}{2}$

when x = 6, $f(6) = \frac{6}{2} = 3$

$\therefore$ Range of $f = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$

Example:3

If a function $f(x)$ is defined by $f(x) = 4x - 3$ on the interval $[-2, 2]$, find the value of the following: a) f(0) b) f(3) c) f(2) d) $f\left(\frac{1}{2}\right)$

Solution: Here,

$f(x) = 4x - 3$

a) When x = 0, $f(0) = 4 \times 0 - 3 = -3$ [$\therefore x \in [-2, 2] \Rightarrow -2 \le x \le 2$]

b) When x = 3, $f(3)$ is not defined because $3 \notin [-2, 2]$.

c) When x = 2, $f(2) = 4 \times 2 - 3 = 8 - 3 = 5$

d) When $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = 4 \times \frac{1}{2} - 3 = 2 - 3 = -1$

Example:4

If $f : \mathbb{N} \to \mathbb{N}$ is a functions defined by $f(x) = 4x - 3$. Find

(i) $f(2)$, $f(3)$ and $f(4)$

(ii) The domain and range of f

(iii) Is this function bijective. Justify your answer.

Solution: Here,

$f(x) = 4x - 3$

For (i) $f(2) = 4 \times 2 - 3 = 8 - 3 = 5$

$f(3) = 4 \times 3 - 3 = 12 - 3 = 9$

$f(4) = 4 \times 4 - 3 = 16 - 3 = 13$

For (ii) Domain of function $f = \{2, 3, 4\}$ and the range of function $f = \{5, 9, 13\}$.

For (iii) Let, $x_1, x_2 \in N$ then,

$f(x_1) = 4.x_1 - 3$

$f(x_2) = 4.x_2 - 3$

Now, $f(x_1) = f(x_2) \Rightarrow 4x_1 - 3 = 4x_2 - 3$

$\Rightarrow 4x_1 = 4x_2$

$\Rightarrow x_1 = x_2$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

So, f is one to one

For onto

$f(x) = 4x - 3$

or, $y = 4x - 3$

or, $4x = y + 3$

or, $x = \dfrac{(y+3)}{4} \notin \mathbb{N}$ which is not natural number for each $y \in \mathbb{N}$.

Hence, it is not onto function

OR,

Range of $f \neq \mathbb{N}$. So, f is not onto function

$\therefore$ f is not bijective function.

Example:5

If $P = \{-3, -1, 0, 1, 3\}$ and $Q = \{9, 0, 1\}$. A function $f : P \to Q$ is defined by.
$\left.\begin{matrix} f(-3) \\ f(3) \end{matrix}\right\} = 9$, $f(0) = 0$, $\left.\begin{matrix} f(-1) \\ f(1) \end{matrix}\right\} = 1$

Solution: Here,

$\left.\begin{matrix} f(-3) \\ f(3) \end{matrix}\right\} = 9$, $f(0) = 0$, $\left.\begin{matrix} f(-1) \\ f(1) \end{matrix}\right\} = 1$. Is this function bijetive?

The elements – 3 and 3 have same image i.e. 9. So the given function is not one to one. But, it is onto function as

Range = {9, 0, 1}

Co-domain i.e. Q = {9, 0, 1}

$\therefore$ Range = Co-domain. It is an onto function. So, the given function is not a bijective as it is not one to one onto both.

Example:6

Check whether the function $f : [-2, 2] \to \mathbb{R}$ is defined by $f(x) = x^2$ is one to one, onto or both.

Solution: Here,

$f(x) = x^2$

$f(-2) = (-2)^2 = 4$

$f(2) = (2)^2 = 4$

$\therefore \quad -2 \neq 2 \Rightarrow f(-2) = f(2)$

Which contradicts to our definition of one to one function

i.e. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. So, it is not one to one function.

And, Range of a function is 4. i.e. Range ≠ Real number (R).

$\therefore$ The given function is not a bijective function.

Example:7

Let Q be the set of all rational numbers. Show that the function $f : Q \to Q$ such that $f(x) = 3x + 5$ for all $x \in Q$ is one to one and onto.

Solution: Here,

$f(x) = 3x + 4$

(i) For one to one function:

Let, $x_1, x_2 \in Q$. Then

$f(x_1) = 3x_1 + 5$

$f(x_2) = 3x_2 + 5$.

Now,

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow 3x_1 + 5 = 3x_2 + 5 \\ &\Rightarrow 3x_1 = 3x_2 \\ &\Rightarrow x_1 = x_2 \end{aligned}$$

Since, for $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. So, f is one to one function.

ii) For onto function:

$f(x) = 3x + 5$

or, $y = 3x + 5$

or, $3x = y - 5$

or, $x = \dfrac{(y-5)}{3} \in Q$.

Thus, for every $y \in Q$, there exists $\dfrac{(y-5)}{3} \in Q$ such that

$f\left(\dfrac{y-5}{3}\right) = 3.\left(\dfrac{y-5}{3}\right) + 5 = y - 5 + 5 = y$. So, f is onto function.

Exercise-3.2

1. If P = {1, 3, 5, 7} and Q = {a, b, c, d}. Determine which of the following relations between P and Q are functions. Find the domain and range if it is a function.
   a) {(1, a), (3, b), (5, c), (7, d)}
   b) {(1, a), (1, b), (3, b), (5, c), (7, d)}
   c) {(1, d), (3, a), (5, b), (7, c), (3, c)}
   d) {(1, a), (1, b), (1, c), (1, d)}
2. a) If f $(x, 2x^2 - 4x + 1)$. Find the values of $f(0)$, $f(3)$, $f(-2)$, $f(P-1)$ and $\dfrac{f(a+h) - f(a)}{h}$

   b) If $h(x) = \dfrac{(5-x)}{(x-2)}$; find $h(5)$, $h(-3)$, $h\{1, -2\}$ and $h\left(\dfrac{3+a}{2}\right)$.

   c) If $f(x) = \begin{Bmatrix} 3 - x^2 & \text{for} & x > 1 \\ 4x & \text{for} & x = 1 \\ 7 + x & \text{for} & x < 1 \end{Bmatrix}$ find the value of

   i) $f(-2)$ ii) $f\left(\dfrac{3}{2}\right)$ iii) $f(-4.5)$ iv) $f(1)$
3. If a function h(x) is defined by $h(x) = 3x^2 - 2$ on an interval $-1 \le x \le 4$. Find the values of the following:

   i) $h(-1)$ ii) $h(0)$ iii) $h\left(\dfrac{1}{2}\right)$ iv) $h\left(\dfrac{-3}{2}\right)$ v) $h(4)$
4. Let A = {– 1, 0, 2, 4, 6, 8} and $f : A \to \mathbb{R}$ is defined by $f(x) = \dfrac{x}{x+2}$. Find range of f.
5. Check whether the following functions are one to one, onto or neither if
   a) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = 3x$
   b) $g : \mathbb{Q} \to \mathbb{Q}$ given by $g(x) = 2x + 3$
   c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 1$

d) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$

e) $f : [-2, 3] \to \mathbb{R}$ defined by $f(x) = x^2$

f) $k : [0, 3] \to \mathbb{R}$ defined by $k(x) = x^2$

g) $h : [0, \infty) \to \mathbb{R}$ defined by $k(x) = x^2$

h) $f: [1,4] \to \mathbb{R}$ defined by $f(x) = x^2$

6. Let function $f : A \to B$ be defined by $f(x) = \frac{x+1}{2x-1}$ .Find the range of f.Is the function f one to one onto both? If not ,how can the function be made one to one and onto both?

7. Let function $f : A \to B$ be defined by $f(x)= \frac{x-1}{x+2}$ with A ={ -1,0,1,2,3,4} and

B= $\{-2,1,-\frac{1}{2},0,\frac{1}{2},\frac{1}{4},\frac{2}{5}\}$ Find the range of f. Is the function f one to one onto both?

If not ,how can the function be made one to one and onto both?

Answers

1. a, c and d defined a function a function.

For (a) Domain ={1,3,5,7} and Range ={a,b,c,d}

(c) Domain ={1,3,5,7} and Range= {a,b,c,d}

(d) Domain = {1} and Range = {a,b,c,d}

2. (a) $f(0)= 1$, $f(3) = 7$, $f(-2) = 17$ and $f(p-1) = 2p^2 - 8p + 7$.

(b) $h(5)= 0$, $h(-3)= \frac{-8}{5}$, $h\{1, -2\}= \{-4, \frac{-7}{4}\}$ and $h\left(\frac{3+a}{2}\right) = \frac{7-a}{a-1}$.

(c) i) 5 ii) $\frac{3}{4}$ iii) 2.5 iv) *4*

3.(i) 1 ii) -2 iii) $\frac{-5}{4}$ iv) $\frac{19}{4}$ v) 46

4. $\{-1,0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5}\}$

5.(a) One to one function . (b) one to one onto (c) Neither (d)Neither (e) one to one (f) one to one onto

g) one to one but not onto. h) one to one but not onto.

6. Range of f = $\mathbb{R} - \{\frac{1}{2}\}$ 7.Range of f = $\mathbb{R} - \{1\}$.

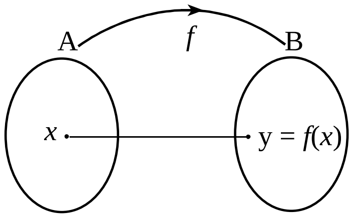
1.3.7 (a) Inverse image of an element:

Let A and B be sets and let f: A $\rightarrow$ B. Let x$\subseteq$ A. Then we define f(x) by

$f(x) = \{y\in B: \exists\, x\in x : y = f(x)\} = \{f(x)\in B: x\in x\}$.

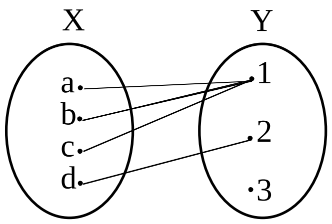
We say that f(x) as "the image of x". Which is the image of a set. Then, for inverse image of an element.

Let A and B be sets and f: A $\rightarrow$ B. Let Y $\subseteq$ B then we define f^{-1} (y) be f^{-1} (y) = { x $\in$ A: there exists y $\in$ Y such that f^{-1} (y) is the inverse of y:



Example:

1. Let f: X $\rightarrow$ Y be defined by the function shown in an arrow diagram. Then,



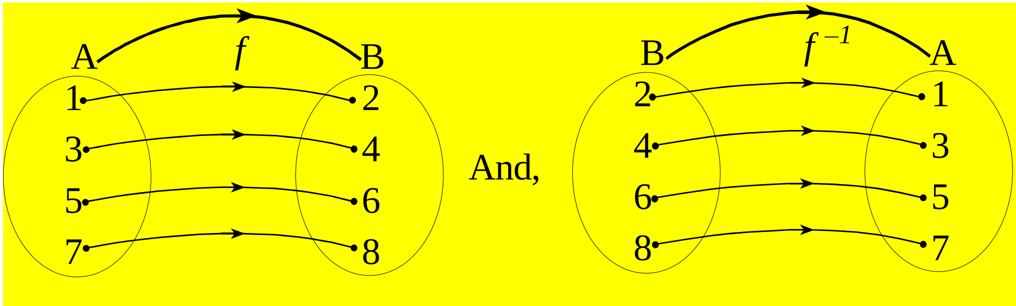
2. If h : Z$\rightarrow$ Z defined by the function $f(x) = x^2$, then
   $f^{-1}(9) = \{3,-3\}$, $f^{-1}(4) = \{2, -2\}$, $f^{-1}(0) = 0$,
   $f^{-1}(-2) = \phi$ and so on.

b) **Inverse function:** Let f: A $\rightarrow$ B be a **bijective** function (i.e. one to one and onto). Then, there exists an another function f^{-1}: B $\rightarrow$ A, which associate every elements of set B to the unique element of set A. Such function is called the inverse function of f and it is denoted by f^{-1}.

In other word "If f is a function then the set of ordered pair obtained by interchange the first and second co-ordinates of each ordered pair in f is called the inverse of *f*. It is denoted by f^{-1}. For example:

If G = {(1, 2), (3,4), (5, 6), (7,8)}

$\therefore$ G^{-1} = {(2, 1) (4,3), (6,5), (8,7)}



Notes:

i) To find inverse function, first of all, need to check whether it is bijective or not.

ii) When a function is a bijective, then f^{-1} exist.

iii) $f^{-1}(x) \neq \frac{1}{f(x)}$. It is very important not to confuse function notation, with negative exponents.

Required steps to find the inverse function:

Sept 1: Check whether the given function is a bijective or not. If it is not then write inverse does not exist. If it exists then go to step 2.

Step 2: Change f(x) to y.

Step 3: Interchange x and y.

Step 4: Solve for y.

Step 5: Change y back to $f^{-1}(x)$.

Counter Example:-

If a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x - 7$, $x \in \mathbb{R}$. Find $f^{-1}(x)$.

Solution:-

let $y = f(x)$

Given, $f(x) = 4x - 7$.

Or, $y = 4x - 7$

Interchanging x and y. We get

$x = 4y - 7$

Or, $4y = x + 7$

Or, $y = \frac{x + 7}{4}$

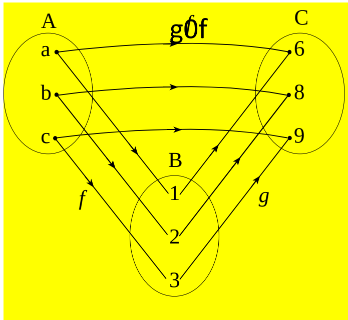
Or, $f^{-1}(x) = y = \frac{(x + 7)}{4}$

$\therefore$ $f^{-1}(x) = \frac{(x + 7)}{4}$.

c) **Composition of functions:** A composite function is a function that depends on another function. A composite function is created when one function is substituted into another function. For example, f (g (*x*)) is the composite function that is formed when g(x) is substituted for x in *f*(*x*). *f*(g(*x*)) is read as "f of g of *x*".

f(g(*x*)) can also be written as (fog) (*x*) or fg(x). In the composition (fog)(x), then domain of *f* becomes g(x).

Let $f: A \to B$ and $g: B \to C$ be any two functions defined from A to B and B to C. Then, the new function defined from A to C is called composite function(**function of function**) of f and g. It is denoted by gof or gf. For example:



From the above mapping diagram, gof = {(a,6), (b,8), (c, 9)}.

Example:1

If f = {(a, 1), (b, 2), (c, 3) (d, 4)} and g = {1, 4), (2, 5), (3, 6), (4, 8)}. Find the value of gf (a), gf(c) and gf(d).

Solution: Here,

f = {(a, 1), (b, 2), (c, 3), (d, 4)}.

g = {(1, 4), (2, 5), (3, 6), (4, 8)}.

$\therefore$ gf (a) = g[f(a)] = g(1) = 4 i.e (a, 4).

$\therefore$ gf (c) = g[f(c)] = g(3) = 6 i.e. (c, 6).

$\therefore$ gf (d) = g[f(d)] = g(4) = 8 i.e (d, 8).

Example:2

If A, B and C be the sets of real number. Such that $f: A \to B$ and $g: B \to C$ are defined by $f(x) = x^2 + 6$ and $g(x) = 2x - 1$, find (a) (fog) (x) (b) (gof) (x).

Solution: Here,

$f(x) = x^2 + 6$.

$g(x) = 2x - 1$.

(a) (fog) (x) = f[g(x)]

= f(2x – 1) = $(2x – 1)^2 + 6$.

= $4x^2 – 4x + 1 + 6$. = $4x^2 – 4x + 7$.

(b) (gof) (x) = g[f(x)]

= g $(x^2 + 6)$ = 2 $(x^2 + 6) – 1$

= $2x^2 + 12 – 1$ = $2x^2 + 11$.

Properties of composite functions:

1. Composite functions are not commutative i.e. fog ≠ gof.
2. Composite function are associative i.e. (fog)oh = fo(goh).
3. A function f : A → B and g : B → C are one to one functions.
   Then gof : A → C is also one to one.
4. A function f: A → B and g: B → C are onto function then gof: A → C is also onto. Indeed, two functions f and g can be combined in various ways to form a new function, such as:
   i) (f ± g) (x) = f(x) ± g(x)
   ii) (fg) (x) = f(x). g(x)
   iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$; g (x) ≠ 0.

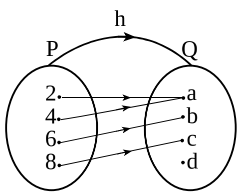
Worked out examples:

Example:1

By studying the adjoining diagram, find the values of the following:

i) h^{-1} (d) ii) h^{-1} (a)

iv) h^{-1}(a, b, c) iv) h(p)



Solution:

Here,

From the given mapping diagram

i) h^{-1}(d) = ϕ ii) h^{-1} (a) = {2, 4}

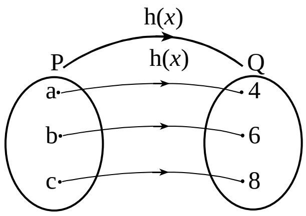
iii) h^{-1}(a, b, c) = {2, 4, 6} iv) h(p) = {a, b, c}

Example:2

If h(a) = 4, h(b) = 6 and h(c) = 8 where P = {a, b, c} and Q = {4, 6 8}. Write down h^{-1}: Q → P as a set of ordered pairs

Solution: Here,

h(a) = 4
h(b) = 6
h(c) = 8



It is a bijective function because distinct elements of set P has distinct image in set Q i.e. it is one to one function. Also, it is onto function as well because range i.e. {4, 6 8} is equal to co-domain.

$\therefore$ Function h(x) is a bijective so that $h^{-1}(x)$ exists.

Now, $h(a) = 4 \Rightarrow a = h^{-1}(4)$ or, $h^{-1}(4) = a$

$h(b) = 6 \Rightarrow b = h^{-1}(6)$ or, $h^{-1}(6) = b$

$h(c) = 8 \Rightarrow c = h^{-1}(8)$ or, $h^{-1}(8) = c$

$\therefore h^{-1} : Q \to P = \{(4, a), (6, b), (8, c)\}$

Example:3

If a function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 4x - 7$, $x \in \mathbb{R}$. Check whether g is one to one or onto or both. Find $g^{-1}(x)$.

Solution:

Here,

Since, $g : \mathbb{R} \to \mathbb{R}$ defined by

$g(x) = 4x - 7$, $x \in \mathbb{R}$.

At first, we need to check whether a given function is a bijective or not.

For this, Let $x_1, x_2 \in \mathbb{R}$ (domain).

$g(x_1) = 4x_1 - 7$ and $g(x_2) = 4x_2 - 7$.

Now, $g(x_1) = g(x_2) \Rightarrow 4x_1 - 7 = 4x_2 - 7$

$\Rightarrow \quad 4x_1 = 4x_2$

$\Rightarrow \quad x_1 = x_2$

i.e. $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$.

$\therefore$ g is one to one function.

For onto function: Let, $y \in \mathbb{R}$. Then,

$g(x) = 4x - 7$, $x \in \mathbb{R}$.

Or, $y = 4x - 7$

Or, $y + 7 = 4x$

Or, $x = \frac{(y+7)}{4} \in R$ (Which is a real number).

Or, $g\left(\frac{y+7}{4}\right) = 4.\left(\frac{y+7}{4}\right) - 7$

$= y + 7 - 7 \quad = y$

$\therefore \quad g\left(\frac{y+7}{4}\right) = y$

It is onto function because range is equal to co-domain.

Hence, function g(x) is one to one onto so the inverse of a function g(x) i.e. $g^{-1}(x)$ exists.

Now, To find $g^{-1}(x)$,

$g(x) = 4x - 7$.

Or, $y = 4x - 7$

Interchanging x and y. We get

$x = 4y - 7$

Or, $4y = x + 7$

Or, $y = \frac{x+7}{4}$

Or, $g^{-1}(x) = y = \frac{(x+7)}{4}$

$\therefore \quad g^{-1}(x) = \frac{(x+7)}{4}$

Example:4

Let $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = 3x + 2$ and $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3$. Then find:

a) goh (x) b) hog (x) c) gog (– 3) d) $goh^{-1}(x)$.

Solution:

Here,

$h(x) = 3x + 2$ and $g(x) = x^3$.

Since, the function h(x) is a linear. So, it is always a bijective. Therefore, $h^{-1}(x)$ exists.

Then, $h(x) = 3x + 2$

Or, $y = 3x + 2$

Interchanging x and y, we get.

$x = 3y + 2$.

Or, $x - 2 = 3y$

Or, $y = \frac{(x-2)}{3}$

Or, $h^{-1}(x) = y = \dfrac{(x-2)}{3}$.

a) $goh(x) = g[h(x)] = g(3x+2)$
$= (3x+2)^3 = (3x)^3 + 3.(3x)^2.\ 2 + 3\ .3x.2^2 + 2^3$
$= 27x^3 + 54\ x^2 + 36x + 8$

b) $hog(x) = h[g(x)] = h(x^3)$
$= 3.x^3 + 2 = 3x^3 + 2.$

c) $gog(-3) = g[g(-3)] = g\{(-3)^3\}$
$= g(-27) = (-27)^3 = -19683.$

d) $goh^{-1}(x) = g\left(\dfrac{x-2}{3}\right) = \left(\dfrac{x-2}{3}\right)^3 = \dfrac{(x-2)^3}{27} = \dfrac{x^3 - 6x^2 + 12x - 8}{27}$.

Example:5

Find the domain and range of the following function defined on the real valued functions.

a) $y = 3x + 1$ b) $y = -x^2 + 4x - 3$ c) $y = \dfrac{1}{x+1}$

d) $y = \sqrt{x^2 - 3x - 4}$ e) $\dfrac{|x-2|}{x-2}$

a) Solution: Here,

$y = 3x + 1.$

For domain,

For all $x \in \mathbb{R}$, y is defined.

$\therefore$ Domain of f = D(f) = $(-\infty, \infty) = \mathbb{R}$.

For range;

$y = 3x + 1.$

Or, $3x = (y - 1)$

Or, $x = \dfrac{y-1}{3} \in \mathbb{R}$.

For all $x \in \mathbb{R}$, $y \in \mathbb{R}$

$\therefore$ Range of f = $\mathbb{R} = (-\infty, \infty)$.

b) Solution: Here,

$y = -x^2 + 4x - 3.$

For domain,

For all $x \in \mathbb{R}$, y is defined.

$\therefore$ Domain of f = D (f) = $\mathbb{R} = (-\infty, \infty)$.

For range:

$y = -x^2 + 4x - 3$

Or, $-y = x^2 - 2.x.2 + 2^2 - 2^2 + 3.$

Or, $-y = (x-2)^2 - 4 + 3.$

Or, $-y = (x-2)^2 - 1$

Or, $1 - y = (x-2)^2$

As, $(x-2)^2 \geq 0.$

So, $(1-y) \geq 0.$

Or, $1 \geq y.$

Or, $y \in (-\infty, 1]$

$\therefore$ Range of f = R(f) = $(-\infty, 1)$

c) Solution: Here,

$$y = \frac{1}{(x+1)}$$

For domain:

y is defined for all x except at x = – 1

$\therefore$ Domain of f = y = $\mathbb{R} - \{-1\}$.

For range;

$$y = \frac{1}{x+1}$$

Or, $xy + y = 1.$

Or, $xy = 1 - y$

Or, $x = \frac{1-y}{y}$; Provided $y \neq 0$.

$\therefore$ Range of f = $\mathbb{R} - \{0\}$.

d) Solution:

Here,

$$y = \sqrt{x^2 - 3x - 4}$$

Or, $y = \sqrt{x^2 - 2.x.\frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 4}$

Or, $y = \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{25}{4}}$

Or, $y = \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2}$

The function is defined only where

$$\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right\} \geq 0.$$

Or, $\left(x - \frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2 \geq 0.$

Or, $\left(x - \frac{3}{2}\right)^2 \geq \left(\frac{5}{2}\right)^2.$

Or, $\left(x - \frac{3}{2}\right) \geq \pm\frac{5}{2}$

For + ve sign,

$x - \frac{3}{2} \geq \frac{5}{2}$

Or, $x \geq \frac{8}{2}$

Or, $x \geq 4$

For – ve sign,

$x - \frac{3}{2} \geq -\frac{5}{2}$

Or, $x \geq \frac{3}{2} - \frac{5}{2}$

Or, $x \geq \frac{3-5}{2}$

Or, $x \geq -1$

Domain of f = $[4, \infty) \cup [-1, \infty)$

For range;

$y = \sqrt{x^2 - 3x - 4}$

Or, $y^2 = x^2 - 3x - 4$

Or, $y^2 + 4 = x^2 - 3x.$

Or, $y^2 + 4 = x^2 - 2.x.\frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2.$

Or, $y^2 + 4 + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2.$

Or, $y^2 = \left(x - \frac{3}{2}\right)^2 - \frac{25}{4}$

Or, $y = \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2}$

If $\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right\} \geq 0$ the range is only defined. So,

Range of f = R (f) = $[0, \infty)$.

e) $\frac{|x-2|}{x-2}$

Solution: Let y = e) $\frac{|x-2|}{x-2} = \begin{cases} \frac{(x-2)}{x-2} = 1 & if\ x-2 \geq 0 \\ -\frac{(x-2)}{x-2} = -1 & if\ x-2 < 0 \end{cases}$

So, Domain = $\mathbb{R}$ -{2} and Range = {1, 1}

Exercise-3.3

1. a) Let f: P $\rightarrow$ Q be defined by f = {(a, b), (b, c), (c, d), (d, e) be a function. If f^{-1} exists, find f^{-1} as a set of ordered pairs.
   b) Draw a mapping diagram of the inverse of the function f.
   c) Let P = {1, 2, 3, 4}, Q = {4, 5, 6, 7} and f: P $\rightarrow$ Q is a function such that f(1) = 4, f(2) = 5, f(3) = 6 and f(4) = 7. Find f^{-1} as a set of ordered pairs.
2. If f: $\mathbb{R} \rightarrow \mathbb{R}$ defined by the following functions. Find the inverse of each of the given functions.
   a) $f(x) = x^3 + 5$
   b) $f(x) = 4x - 7, x \in \mathbb{R}$
   c) $f(x) = cx + d$, where $c \neq 0$.
3. a) If f = {(1, 2), (3, 4), (5, 6)} and g = {(2, 3), (4, 1), (6, 5)}. Find fog and gof. Are they equal?
   b) Find fog and gof of the following:
      i) $f(x) = 2x + 3, g(x) = x^2$, ii) $h(x) = 2x - 1, k(x) = x^2 - 2$.
      iii) $k(x) = 2x + 1, g(x) = \dfrac{x+2}{3}$ iv) $p(x) = x^3 - 1$ and $q(x) = x^2$.
4. Let $\mathbb{Q}$ be the set of all rational number. Show that the function f: $\mathbb{Q} \rightarrow \mathbb{Q}$ Such that f(x) = 3x – 5 for all $x \in \mathbb{Q}$. Find $f^{-1}(x)$.
5. State the condition for a function to be bijective, $f(x) = x^3 + 5$, $x \in R$, find f^{-1}.
6. If f: $\mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2 - 3$, find $f^{-1}(x)$. Also, determine whether $fof^{-1}(x) = f^{-1}of(x)$.
7. Show that f : $\mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{0\}$ given by $f(x) = \dfrac{1}{(x-2)}$ is bijective. Also, find f^{-1}.
8. If f : $\mathbb{R} \rightarrow \mathbb{R}$ defined by f(x) = 2x + 3 is bijective. Also, find $f^{-1}(2)$.
9. Find the domain and range of the following real valued functions.
   a) $y = 4x - 3$ b) $y = x^2 - 1$ c) $y = x^3$.
   d) $y = \dfrac{1}{x+5}$ e) $y = \sqrt{x}$ f) $y = -x^2 + 4x - 3$
   g) $y = 5 - (x+3)^2$ h) $y = \dfrac{x}{|x|}$ i) $y = \sqrt{x-2}$
   j) $y = \dfrac{x^2 - 16}{x - 4}$ k) $y = \sqrt{x^2 - 2x - 8}$ l) $y = \sqrt{21 - 4x - x^2}$
   m) $y = \dfrac{|x-1|}{x-1}$ n) $y = \dfrac{1}{\sqrt{x^2 + 6x + 8}}$ o) $y = \dfrac{1}{\sqrt{4 + 3x - x^2}}$

10. If $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ defined by g = {(a, d), (b, e), (c, f)} and hog = {(a, a), (b, e), (c, i)}. Find h. Present hog in arrow diagram. Can goh be defined? Given reason.

Answers

1. (a) f^{-1} = { (b,a),(c,b),(d,c),(e,d)} (c) f^{-1} = { (4,1),(5,2),(6,3),(7,4)}

2. (a) $f^{-1}(x) = y = \sqrt[3]{x-5}$ (b) $f^{-1}(x) = \frac{x+7}{4}$ (c) $f^{-1}(x) = \frac{x-d}{c}$

3 (a) $f_o g$={ (1,3),(3,1),(5,5)} (b) $g_o f$={ (3,1),(1,3),(5,5)}.Yes, they are equal
(b) (i) $g_o f = 4x^2 + 12x + 9$ (ii) $h_o k = 2x^2 - 5$
(iii) $k_o h = 4x^2 - 4x - 1$ (iv) $p_o q = x^6 - 1$
(v) $q_o p = x^6 - 2x^3 + 1$

4. $f^{-1}(x) = \frac{x+5}{3}$ (5) $f^{-1}(x) = \sqrt[3]{x-5}$ (6) $f^{-1}(x) = \sqrt{x+3}$

7. $f^{-1}(x) = \frac{1}{x} + 2$ (8) $f^{-1}(x) = -\frac{1}{2}$

9. (a) D $=(-\infty, \infty)$ and R $= (-\infty, \infty)$ (b) D $= (-\infty, \infty)$ and R $= [-1, \infty)$
(c) D $=(-\infty, \infty)$ and R $= (-\infty, \infty)$ (d) D $= \mathbb{R} - \{-5\}$ and R $= (0, 1]$
(e) D= $[0,\infty)$ and R = $[0,\infty)$ (f) D= $(-\infty, \infty)$ and R = $(-\infty$,
(g) D = $(-\infty, \infty)$ and R = $(-\infty, 5]$ (h) D = $\mathbb{R} - \{0\}$ and R = {1,–1}
(i) D = $[2,\infty)$ and R = $[0,\infty)$ (j) D = $\mathbb{R} - \{4\}$ and R = $(-\infty, \infty)$
(k) D = $(-\infty,-2] \cup [4,\infty)$ and R = $[0,\infty)$ (L) D = [–7,3] and R = $[0,\infty)$
(m) D = $\mathbb{R}$ –(1} and R ={–1,1} (n) ......, R = (0,1)
(o) D = (–1,4) and R = (0,1)

1.3.8 Real valued function:

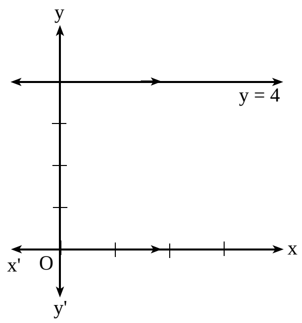
A real valued function is a function whose value are real number. In other words; A function $f : A \to B$ is said to be a real valued function whose range is R or some subset of R. There are two types of real valued functions;

- Algebraic function.
- Transcendental functions.

Algebraic functions: An algebraic function is a function that can be defined as the root of a polynomials. An algebraic functions are algebraic expressions using a finite number of terms, involving only the algebraic operators such as addition, subtraction, multiplication division or raising the power of the independent variable *x* is called algebraic functions such as constant function, identity function, linear function, quadratic function, cubic function, polynomial function, Rational function, polynomial function, Absolute value function and greatest integer function are the algebraic functions.

a) **Constant function:** A function f : A → B which is expressed in the f(x) = C for all $x \in$ A and for some C∈ B is called a constant function. Example: The function y = 4 is a constant function where C = 4. The domain is the sets of all real number ℝ. The co-domain is just 4.

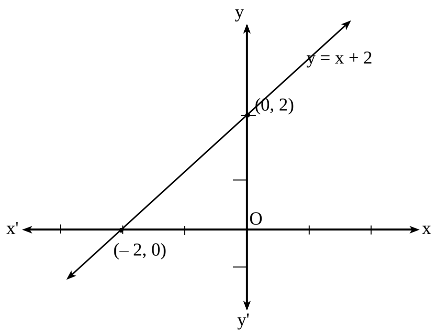
The constant function is an even function. So, y-axis is the line of symmetry.



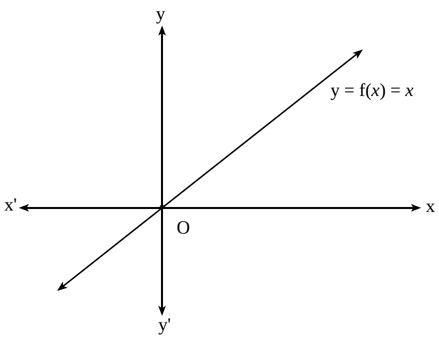
b) Linear function: A function f: A → B which is expressed in the form

y = mx + c for all x ∈ A; m and c are constants is called a linear function. The graph of a linear function is shown in the adjoining figure:

If A = B = ℝ, the set of real numbers, the function defined by f(x) = y = x + 2

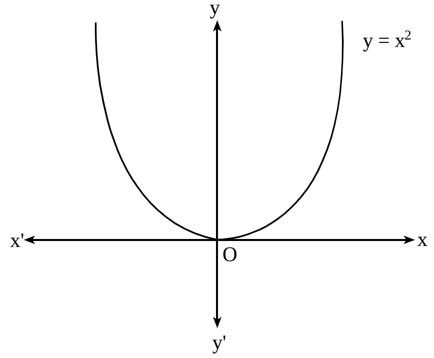


c) **Identity function:** A function f: A→A is said to be an identity function, if it is expressed in the form y = f(x) = x for all x ∈ ℝ. It is usually denoted by I_A. The graph of an identity function is shown in the adjoining diagram.

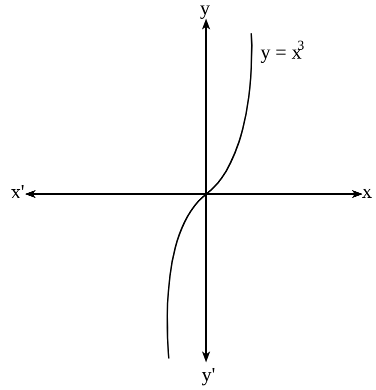


d) Quadratic function: A function f: A→ B defined by y =f (x) = ax^2 + bx + c, a ≠ 0 for x ∈A, where a, b and c are constants, is called a quadratic function.

If A = B = ℝ, the set of real numbers, the function defined by y = f(x)= x^2 is a quadratic function. The graph of a quadratic function is shown in the adjoining diagram.



e) Cubic function: A function f: A→B defined by y = f(x) = $ax^3 + bx^2 + cx + d$, a ≠ 0, where a, b, c and d are constants and x ∈ ℝ is called a cubic function. If A = B = ℝ, the set of real numbers, the function defined by y = f(x) = x^3 is a cubic function. Its graph is shown alongside.



f) Greatest integer function: The greatest integer function is denoted by y = [x]. For all real number x. the greatest integer function returns the largest integer less than or equal to x. Since, it can be written to the nearest integer. For example:

[1] = 1, [1.5] = 1, [3.7] = 4, [4.3] = 4

Beware;

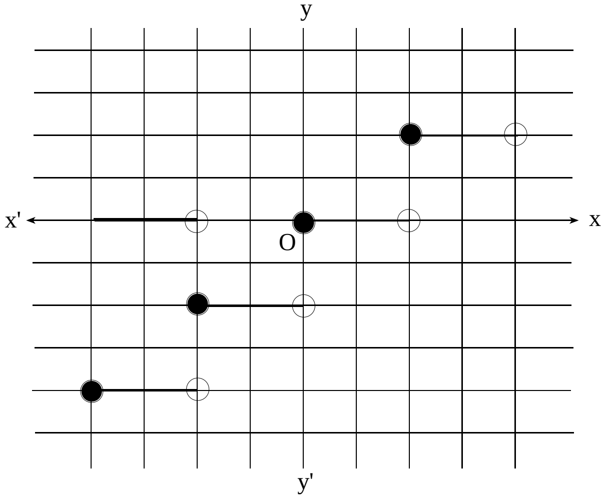
[– 2] = – 2 [– 1.6] = – 2, [– 3.2] = – 3.

For example; if we have f(x) = [– 1.6], the two closest integer are – 1 and – 2. For the greatest integer value, we always choose the small integer, this means that [– 1.6] = – 2.

Also, if the number inside the brackets is an integer, we return to the original number such as [2] = 2, [– 5] = – 5 and [0] = 0

Let us see the graph of the greatest integer function as:

x	$-4 \le x < -2$	$-2 \le x < 0$	$0 \le x < 2$	$2 \le x < 4$
Y	– 4	– 2	0	2



g) Rational function: A function f defined by $f(x) = \frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial in x and q(x) ≠ 0 is known as a rational function for example:

$f(x) = \frac{4x^3 - 3x^2}{x^2 + 8x - 7}$ is a rational function.

h) Absolute value function: If x is a real number, then absolute value or modulus of x is a non- negative real number denoted by |x| is defined by;

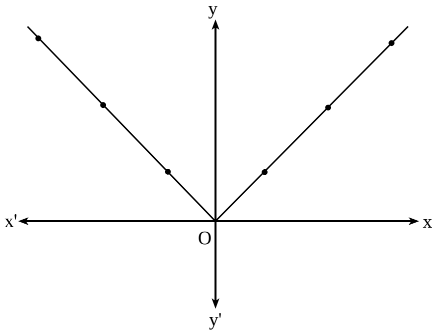
$$|x| = \begin{cases} x & if\ x \ge 0 \\ -x & if\ x < 0 \end{cases}$$

Since, the domain of the absolute value is the set of real number and range is non-negative real number. Thus,

Domain = D(f) = ℝ

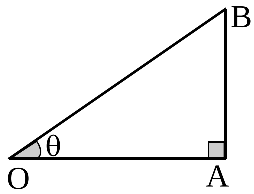
Range = R(f) = [0, ∞)

The graph of the absolute value function f(x) = |x| is shown in the adjoining figure,



- **Transcendental functions:** A transcendental function is an analytic function that does not satisfy a polynomial equation. For examples; trigonometric functions, exponential functions, logarithmic functions and hyperbolic functions.

a) **Trigonometric function:** A function which is defined as the function of an angle of a triangle. Indeed, the relationship between the angles and sides of a triangle are given by trigonometric function. Such as sine, cosine, tangent, cotangent, secant and cosecant are six simple trigonometric functions.



Let OA be an initial line and AB be a revolving line which maker an angle 'O' with the positive direction of OA. Such that $\angle$ AOB = θ. In right angled triangle, OAB, $\sin\theta = \frac{AB}{OB}$, $\text{Cos } \theta = \frac{OA}{OB}$, $\tan\theta = \frac{AB}{OA}$, $\text{cosec}\theta = \frac{OB}{AB}$, $\text{Sec}\theta = \frac{OB}{OA}$ and $\text{Cot}\theta = \frac{OA}{AB}$ are Sin functions defined above are known as trigonometric functions or trigonometric ratios.

i) **Sine function:** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be sine function if it is expressed as f(x) = y = sin x.

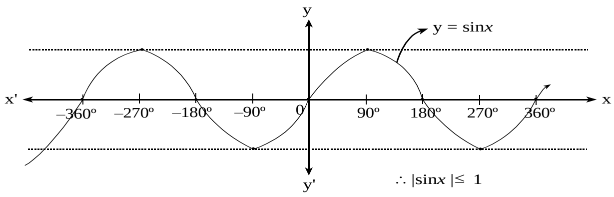
Domain = D(f) = $\mathbb{R}$

Range = [– 1, 1]

Table of since function:

θ	–360°	– 270°	–180°	– 90°	0	90°	180°	270°	360°
y = sinθ	0	–1	0	–1	0	1	0	–1	0

Graph of sine function:

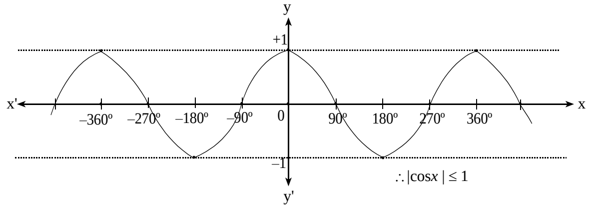


ii) Cosine function: A function f: $\mathbb{R} \to \mathbb{R}$ is said to be cosine function if it is expressed as $f(x) = y = \cos x$.

Domain = D (f) = $\mathbb{R}$.

Range = [– 1, 1]

Table of cosine function:



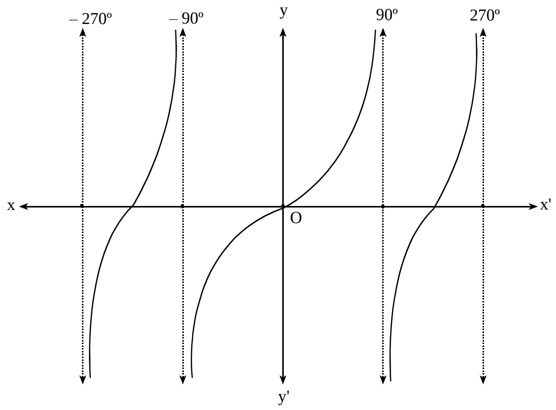
iii) Tangent function: A function f: $\mathbb{R} - \{ (2n + 1).\frac{\pi}{2}, n \in \mathbb{Z}\} \to \mathbb{R}$ is said to be tangent function if it is expressed as $y = f(x) = \tan x$.

Domain = D (f) = $\mathbb{R} - \{ (2n + 1).\frac{\pi}{2}, n \in \mathbb{Z} \}$.

Range = $\mathbb{R}$

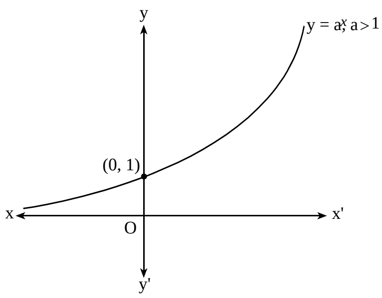
Table of tangent function:

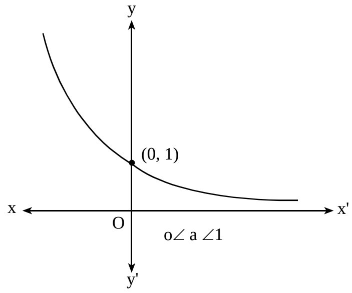
θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
Y= tanθ	0	0.58	1	0.73	∞	–1.73	– 1	–0.58	0	0.58	1	1.73	∞	–1.73	–1	–0.58	0



By the above method, the graph of cosecθ, secθ and cotθ can be shown.

b) Exponential function: A function f: $\mathbb{R} \to (0, \infty)$ defined by $y = f(x) = a^x$, $x \in \mathbb{R}$ where $a > 0$, $a \neq 1$ is called an exponential function of base a. It is bijective e^x is a typical exponential function. Where, $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{h \to \infty} (1+h)^{1/h}$.



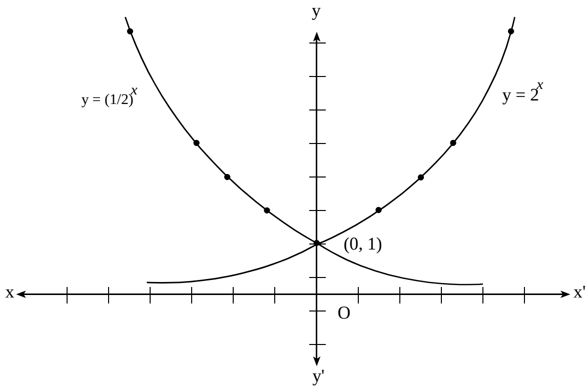


y = f(x) > 0, domain = ℝ and Range = (0, ∞)

Since,E is an irrational number whose values lies between 2, 3 i.e. 2 < e < 3.

The graph of simple exponential function $y = 2^x$.

x	–3	–2	–1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



And, Graph of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$intersect at the point (0, 1).

c) Logarithmic function: Let a and b are two positive real number and a ≠ 1. Such that a^x= b then x is called logarithmic of b to the base a . i.e. $a^x = b$ x ⇔ $\log_a b$.

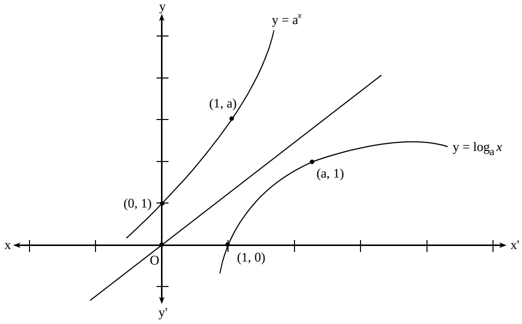
The domain of the logarithmic function is the set of positive real numbers $\mathbb{R}^+$ and the range is ℝ.

Remarks

i) The logarithm of a number with base 10 is called common logarithm and the logarithm with base e is called natural logarithm.

ii) The natural logarithm i.e. $log_e x$ is simply written as $l$nx.

The graph of $y = log_e x = log x = l nx$.



Properties of logarithms function:

Theorem- 1: For any number x, y and a; $\log_a (x, y) = \log_a x + \log_a y$

Proof: For any positive number x, y and a;

$\log_a x = b$ and $\log_a y = c$

Or, $x = a^b$ or, $y = a^{c.}$

So, $x.\ y = a^b.\ a^c = a^{b + c}$

Or, $a^{(b + c)} = x.\ y$

By the definition of logarithm.

Hence, $\log_a(x.y) = b + c$

Or, $\log_a(x.y) = \log_a x + \log_a y$

$\therefore$ $\log_a (x.y) = \log_a x + \log_a y$

Theorem-2: For any positive number a and x,

$\log_a x^m = m \log_a x$. Where m is any real number.

Proof: Here,

For any positive number a and x,

Put $\log_a x = b \Rightarrow x = a^b$

Or, $x^m = (a^b)^m = a^{mb}$

Taking $\log$ both sides

Hence, $\log_a x^m = mb = m\log_a x$.

$\therefore$ $\log_a x^m = m \log_a x$.

Theorem-3: For all positive real numbers x, y and a.

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

Proof: Here,

For any positive numbers x, y and a

Put, $\log_a x = p \Rightarrow x = a^p$

$\log_a y = q \Rightarrow y = a^q$.

Now, $\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$.

Or, $(p - q) = \log_a\left(\frac{x}{y}\right)$

Or, $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$

$\therefore$ $\log_a\left(\frac{x}{y}\right) = (\log_a x - \log_a y)$

Theorem-4: For any positive number x, y and a.

$\log_a x = \log_a b. \log_b x$ (change base)

Proof:

Let, $\log_b x = P \Rightarrow x = b^p$

Hence, $\log_a x = \log_a b^p = p\log_a b = \log_a b. \log_b x$

Theorem -5: For any positive number x, n and a, $\log_{a^n} a = \frac{1}{n} \log_a x$; $n \neq 0$.

Proof:

$\log_{a^n} x = \log_{a^n} a. \log_a x$ …… (i)

Put, $\log_{a^n} a = y \Rightarrow a = (a^n)y$

$\Rightarrow$ $a = a^{ny}$

$\Rightarrow$ $ny = 1$

$\Rightarrow y = \frac{1}{n}$

$\therefore$ $\log_{a^n} a = \frac{1}{n}$

Then,

$\log_{a^n} x = \log_{a^n} a. \log_a x$.

$\therefore$ $\log_{a^n} x = \frac{1}{n} . \log_a x$.

Theorem -6: Prove that $\log_a 1 = 0$

Proof: Here,

$a^{\circ} = 1$

Or, $0 = \log_a 1$

Or, $\log_a 1 = 0$

Theorem-7: For any positive integer a, Prove that $\log_a a = 1$.

Proof: Here,

$a^1 = a$

Or, $\log_a a = 1$ [$\because 2^3 = 8 \Rightarrow \log_2 8 = 3$]

Theorem-8: For all positive real numbers a and b, prove that: $\log_a b.\ \log_b a = 1$.

Proof: Here,

$\log_a a = \log_a b.\ \log_b a.$

Or, $1 = \log_a b.\ \log_b a.$

$\therefore \log_a b.\ \log_b a = 1.$

Worked out examples:

Example:1

Prove that: $\log_a x^2 - 2\log_a\sqrt{x} = \log_a x$.

Solution: Here,

L.H.S $= \log_a x^2 - 2\log_a\sqrt{x}$

$= \log_a x^2 - \log_a(\sqrt{x})^2. \quad = \log_a x^2 - \log_a(x)^{\frac{1}{2}\times 2} \quad = \log_a x^2 - \log_a x$

$= \log_a \dfrac{x^2}{x} \quad = \log_a x$

R.H.S Proved ***Ans***

Example:2

If $x^2 + y^2 = 7xy$, prove that: $\log\dfrac{(x + y)}{3} = \dfrac{1}{2}(\log x + \log y)$

Solution: Here,

$x^2 + y^2 = 7xy$

Or, $(x + y)^2 - 2xy = 7xy$

Or, $(x + y)^2 = 2xy + 7xy$

Or, $(x + y)^2 = 9xy$

Or, $\dfrac{(x + y)^2}{9} = xy$

Taking 'log' both sides

$\log\left(\dfrac{x + y}{3}\right)^2 = l\log(xy)$

$2\log\left(\dfrac{x + y}{3}\right) = \log x + \log y$

$\log\left(\dfrac{x + y}{3}\right) = \left(\dfrac{\log x + \log y}{2}\right)$

$\therefore \quad \log\left(\dfrac{x + y}{3}\right) = \dfrac{1}{2}\ (\log x + \log y).$

Example:3

Prove that: $x^{\log y - \log z} . y^{\log z - \log x} . z^{\log x - \log y} = 1.$

Solution: Here,

Put, $k = x^{\log y - \log z} . y^{\log z - \log x} . z^{\log x - \log y}$

Taking 'log' both sides.

$\log k = \log (x^{\log y - \log z} . y^{\log z - \log x} . z^{\log x - \log y})$

Or, $\log k = \log x^{\log y - \log z} + y^{\log z - \log x} + z^{\log x - \log y}$

Or, $\log k = (\log y - \log z). \log x + (\log z - \log x). \log y + (\log x - \log y). \log z$

$= \log x. \log y - \log x. \log z + \log y. \log z - \log x. \log y + \log x. \log z - \log y. \log z$

Or, $\log k = 0$

Or, $\log k = \log 1$

Or, $k = 1.$

$\therefore$ $x^{\log y - \log z} . y^{\log z - \log x} . z^{\log x - \log y} = 1.$

R.H.S. = R.H.S

Example:4

If $x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$, Prove that: $xyz + 1 = 2yz$.

Solution: Here,

$x = \log_{2a} a$, $y = \log_{3a} 2a$ and $z = \log_{4a} 3a$.

L.H.S.

$xyz + 1$

=	$\log_{2a} a. \log_{3a} 2a. z + 1$	=	$\log_{3a} 2a. \log_{2a} a.z + 1$	=	$\log_{3a} a.z + 1$
=	$\log_{3a} a. \log_{4a} 3a + 1$	=	$\log_{4a} 3a. \log_{3a} a + 1$	=	$\log_{4a} a + 1$
=	$\log_{4a} a + \log_{4a} 4a$	=	$\log_{4a} (a. 4a)$	=	$\log_{4a} (2a)^2$
=	$2\log_{4a} 2a.$	=	$2. \log_{4a} 3a. \log_{3a} 2a$	=	$2.z.y$
=	$2yz$ = R.H.S				

Example:5

If $\log_4 5 = a$ and $\log_5 6 = b$, then prove that: $l\log_2 3 = \dfrac{1}{2ab - 1}$

Solution: Here,

$a = \log_4 5$ and $b = \log_5 6$.

$a.b = \log_4 5. \log_5 6$

Or, $2ab = 2\log_4 6.$

Or, $2ab - 1 = 2\dfrac{\log 6}{\log 4} - 1$

Or, $2ab - 1 = 2.\dfrac{\log(2 \times 3)}{\log 2^2} - 1.$

Or, $2ab - 1 = 2.\frac{\log 2 + \log 3}{2\log 2} - 1$

Or, $2ab - 1 = \frac{\log 2}{\log 2} + \frac{\log 3}{\log 2} - 1$

Or, $2ab - 1 = 1 + \frac{\log 3}{\log 2} - 1.$

Or, $2ab - 1 = \frac{\log 3}{\log 2}$

Or, $\frac{1}{2ab - 1} = \frac{\log 2}{\log 3}$

$\therefore\ \log_3 2 = \frac{1}{2ab - 1}$ [$\because\ \log_a b = \frac{\log b}{\log a}$]

Note: [Let, $\log_a b = p \Rightarrow b = a^p \Rightarrow \log b = \log a^p \Rightarrow \log b = p\log a \Rightarrow \frac{\log b}{\log a} = P \Rightarrow \frac{\log b}{\log a} = \log_a b$*]*

Exercise - 3.5

1 (a) Prove that: $\cosh 2x = 2\cosh^2 x - 1$

b) Prove that: $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$

2. Find the value of x if

a) $\log_5 x = 5$ b) $\log_5 x = \frac{-1}{2}$

3. Prove that:

a) $\log_a\left(\frac{pq^3}{r^2}\right) = \log_a p + 3\log_a q - 2\log_a r.$ b) $\log_a x = \log_a x^2 - 2\log_a\sqrt{x}$

c) $b^{\log x} = x.$ d) $\log(4 + 5 + 6) = \log_4 + \log_5 + \log_6$

e) $\log_a\sqrt{a\sqrt{a\sqrt{a}}} = \frac{7}{8}$

4. Prove that:

a) $x^{\log(y/z)}.\ y^{\log(z/x)}.\ z^{\log(x/y)} = 1.$

b) $(yz)^{\log y - \log z}.\ (zx)^{\log z - \log x}.\ (xy)^{\log x - \log y} = 1.$

5. If $a^2 + b^2 = 7ab$, then prove that: $\log_{10}\frac{a + b}{3} = \frac{1}{2}(\log_{10} a + \log_{10} b)$

6. If $h(x) = \log\left(\frac{1 + x}{1 - x}\right)$ $(-1 < x < 1)$, show that;

$h(a) + h(b) = h\left(\frac{a + b}{1 + ab}\right)$ $(|a| < 1, |b| < 1)$

7. If $\frac{\log x}{y - z} = \frac{\log y}{z - x} = \frac{\log z}{x - y}$, prove that $x^x . y^y . z^z = 1$.

8. Show that: $\log_v \sqrt[4]{4^3} \times \log_w v^5 \times \log_4 \sqrt[5]{w4} = 3$

9. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$ Prove that: $\frac{1}{x + 1} + \frac{1}{y + 1} + \frac{1}{z + 1} = 1$

Answer

2(a) 3125 b) $\frac{1}{\sqrt{5}}$

♦●♦●♦