Convergence of Secant method

- Let the iterative formula for secant method is:

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{——— eq}^n \text{ (1)}$$

Let x_r be the root of function $f(x) = 0$ then

$$e_i = x_i - x_r \quad \Rightarrow x_i = e_i + x_r$$

$$e_{i+1} = x_{i+1} - x_r \quad \Rightarrow x_{i+1} = e_{i+1} + x_r$$

Substituting these values in eq^n (1), we get,

$$e_{i+1} + \cancel{x_r} = e_i + \cancel{x_r} - \frac{f(x_i)\,(e_i + x_r - e_{i-1} - x_r)}{f(x_i) - f(x_{i-1})}$$

or, $$e_{i+1} = e_i - \frac{f(x_i)\,(e_i - e_{i-1})}{f(x_i) - f(x_{i-1})}$$

or, $$e_{i+1} = \frac{e_i \cancel{f(x_i)} - e_i f(x_{i-1}) - e_i \cancel{f(x_i)} + e_{i-1} f(x_i)}{f(x_i) - f(x_{i-1})}$$

or, $$e_{i+1} = \frac{e_{i-1} f(x_i) - e_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{——— eq}^n \text{ (2)}$$

According to mean value theorem, there exist at least one point $x_i = R_i$ in the interval $[x_i, x_r]$ such that

$$f'(R_i) = \frac{f(x_i) - f(x_r)}{x_i - x_r}$$

or, $$f'(R_i) = \frac{f(x_i) - 0}{x_i - x_r} \quad [\because \text{ since } x_r \text{ is root}]$$

or, $$f(x_i) = e_i\, f'(R_i) \quad [\because e_i = x_i - x_r]$$

Similarly,

$$f(x_{i-1}) = e_{i-1} f'(R_{i-1})$$

Substituting the values of $f(x_i)$ & $f(x_{i-1})$ in numerator of eq^n (2)

$$e_{i+1} = \frac{e_{i-1} \cdot e_i f'(R_i) - e_i \cdot e_{i-1} f'(R_{i-1})}{f(x_i) - f(x_{i-1})}$$

or, $e_{i+1} = \dfrac{e_i e_{i-1} [f'(x_i) - f'(x_{i-1})]}{f(x_i) - f(x_{i-1})}$ —— eqⁿ (3)

Here,

$e_{i+1} \propto e_i \cdot e_{i-1}$

In order to find the order of convergence, it is necessary to find a formula of type

$e_i \propto e_{i-1}^{p}$ & [order of Convergence of iteration process is p].

$e_{i+1} \propto e_i^{p}$

Now,

$e_{i+1} \propto e_i \cdot e_{i-1}$ becomes

or, $e_i^{p} \propto e_{i-1}^{p} \cdot e_{i-1}$

or, $e_i^{p} \propto e_{i-1}^{p+1}$

or, $e_i \propto e_{i-1}^{(p+1)/p}$

Comparing $e_i \propto e_{i-1}^{p}$ and $e_i \propto e_{i-1}^{(p+1)/p}$, we get,

$p = (p+1)/p$

or, $p^2 - p - 1 = 0$

$\therefore p = \dfrac{1 \pm \sqrt{1+4 \cdot 1 \cdot 1}}{2 \cdot 1 \cdot 1} = \dfrac{1 \pm \sqrt{5}}{2} = \pm 1.618$

Since p is always positive, we have $p = 1.618$.

It follows that the order of convergence of secant method is 1.618 and the convergence is referred to as super linear convergence.