UNIT 5

Vectors

Learning Outcomes:

At the end of this chapter, students will be enable to

- Define vector product of two vectors and, interpret vector product geometrically.
- Solve the problems using properties of vector product.
- Apply vector product in plane trigonometry and geometry.

5.3

Product of Vectos

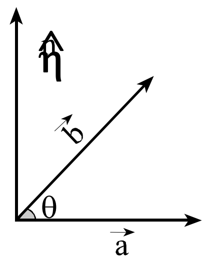
Vector product of two vectors

There are two types of vector products: Scalar (dot) product and vector (cross) product. We have already discussed about scalr (dot) product in grade-XI. Here, we discuss only about vector (cross) product of two vectors.

1. Vector (or cross) product:-

Let $\vec{a}$, $\vec{b}$ be two non-zero non-parallel vectors. Then the vector product $\vec{a} \times \vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}$, $\vec{b}$ and this direction constitute a right handed system.

In other words, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{\eta}$ form a right handed system.



The right handed system means if we rotate vector $\vec{a}$ in to the vector $\vec{b}$, then $\hat{\eta}$ will point in the direction perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in which a right handed screw will move if it is turned in the same manner.

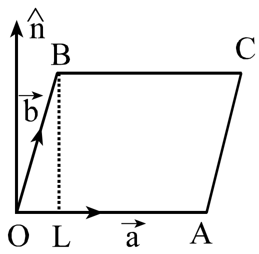
Note- 1: If one of $\vec{a}$ or $\vec{b}$ or both is $\vec{0}$, then θ is not defined as $\vec{0}$ has no direction and so $\hat{\eta}$ is not defined. In this case, we define $\vec{a} \times \vec{b} = \vec{0}$.

Note 2: If $\vec{a}$ and $\vec{b}$ are collinear i.e. if $\theta = 0$ or π, then the direction of $\hat{\eta}$ is not well defined. So, in this case also we define $\vec{a} \times \vec{b} = \vec{0}$.

Note 3: $\vec{a} \times \vec{b}$ is read as $\vec{a}$ cross $\vec{b}$ and is called cross product. Since the resulting quantity is a vector, it is known as a vector product.

Geometrical Interpretation of Vector Product of Two Vectors

Let $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and let θ be the angle between them, complete the parallelogram OACB and draw BL ⊥ OA at L.



In Δ OBL, $\sin\theta = \frac{BL}{OB}$

$\Rightarrow BL = OB \sin\theta = |\vec{a}| \sin\theta$

Now,

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta}$

$=$ (OA) (BL) $\hat{\eta}$

$=$ (Base × height) $\hat{\eta}$

$=$ (Area of parallelogram OACB) $\hat{\eta}$

$=$ Vector area of the parallelogram OACB,

Thus, $\vec{a} \times \vec{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its adjacent sides and whose direction $\hat{\eta}$ is perpendicular to the palne of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system.

In other words $\vec{a} \times \vec{b}$ represents the vector area of the parallelogram having adjacent sides along $\vec{a}$ and $\vec{b}$.

Thus, area of parallelogram OACB = $|\vec{a} \times \vec{b}|$

Also,

Area of $\Delta OAB = \frac{1}{2}$ area of parallelogram OACB

$= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}|$.

Note: By the term vector area of a plane figure we mean that a vector of magnitude equal to the area of the figure and direction normal to the plane of the figure in the sense of right handed rotation.

Note:

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta}$

Taking modulus on both sides,

$|\vec{a} \times \vec{b}| = |\,|\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta}|$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|\,|\hat{\eta}|$ [Since, $|xy| = |x|\,|y|$]

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta\,.1$

$$\Rightarrow \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| \times |\vec{b}|}$$

Cross product in component form

Let $\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = (b_1, b_2, b_3) = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any two vectors. Then, their cross product $\vec{a} \times \vec{b}$ is defined as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

Note: We define the cross product of three dimensional vectors only.

Properties of vector product

Property- I :- Vector product is not commutative in general i.e. if $\vec{a}$ and $\vec{b}$ are any two vectors, then $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

Property-II: If $\vec{a}$, and $\vec{b}$ are two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and let θ be the angle between them. Then $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$

Property -III: If $\vec{a}, \vec{b}$ are two vectors and m,n are scalars, then

$m\vec{a} \times n\vec{b} = mn\,(\vec{a} \times \vec{b})$

Property-III: Distributivity of vectors product over vector addition:

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors. Then

i) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (Left Distributivity)

ii) $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$ (Right Distributivity)

Property IV: The vector product of two non-zero vectors is zero vector iff they are parallel (collinear) i.e. $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b}$, $\vec{a}, \vec{b}$ are non-zero vectors.

Note -1: It follows from the above property that $\vec{a} \times \vec{a} = 0$ for every non-zero vector $\vec{a}$ which in turn implies that $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

Proof:

For $\hat{i}$ and $\hat{i}$, $\boldsymbol{\theta = 0}$

We have,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta$$

$\Rightarrow \hat{i} \times \hat{i} = |\hat{i}|\,|\hat{i}| \sin 0$

$\Rightarrow \hat{i} \times \hat{i} = 1.\ 1\ .\ 0$

$\Rightarrow \hat{i} \times \hat{i} = 0$. Similarly other.

Note-2: $\hat{i} \times \hat{j} = \hat{j} \times \hat{k} = \hat{k} \times \hat{i} = 1$

Proof:

For $\hat{i}$ and $\hat{j}$, $\theta = 90^0$

We have,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta$$

$\Rightarrow \hat{i} \times \hat{j} = |\hat{i}|\,|\hat{j}| \sin 90^0$

$\Rightarrow \hat{i} \times \hat{j} = 1.\,1\,.\,1$

$\Rightarrow \hat{i} \times \hat{j} = 1$. Similarly other.

Note-3: Vector product of orthogonal (orthonormal also) triad of unit vectors $\hat{i}, \hat{j}, \hat{k}$ is given by

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$$

Vectors Normal/Perpendiculr/Orthogonal to the plane of two given vectors

Let $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors.

The vector product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Also, let θ be the angle between $\vec{a}$ and $\vec{b}$ then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta}$

Where, $\hat{\eta}$ is a unit vector perpendicular ot the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ from a right-handed system.

$\therefore \quad (\vec{a} \times \vec{b}) = |\vec{a} \times \vec{b}|\,\hat{\eta}$

$\Rightarrow \quad \hat{\eta} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Thus, $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Note 1: $-\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ **is also a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.**

Note-2: Vector of magnitude 'k' normal to the plane $\vec{a}$ and $\vec{b}$ is given by

$$\pm \frac{k(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$$

Note-3: **Prove that the vector product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.**

Proof: Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ Then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

Now, $(\vec{a} \times \vec{b}).\ \vec{a} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1).(a_1,a_2,a_3)$

$= \quad (a_2b_3 - a_3b_2)a_1 + (a_3 b_1 - a_1 b_3).a_2 + (a_1 b_2 - a_2b_1)\, a_3$

$= \quad 0$

$\Rightarrow \quad \vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Similarly, $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

Note-4: Expression for sinθ

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = ab\cos\theta.$$

Where θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $(\vec{a} \times \vec{b})^2 = (\vec{a} \times \vec{b}).(\vec{a} \times \vec{b})$

$= \quad (a_2 b_3 - a_3 b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2 b_1)^2$

$= \quad (a_1^2 + a_2^2 + a_3^2)(a_1^2 + a_2^2 + a_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$

$= \quad a^2 b^2 - (\vec{a}\,.\,\vec{b})^2$

$\Rightarrow \quad |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - |\vec{a}|^2\,|\vec{b}|^2 \cos^2 \theta.$

$= \quad |\vec{a}|^2\,|\vec{b}|^2\,(1 - \cos^2 \theta)$

$= \quad |\vec{a}|^2\,|\vec{b}|^2 \sin^2 \theta$

$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$

$$\Rightarrow \quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| \times |\vec{b}|}$$

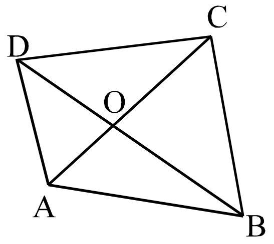
Note- 1: The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.

Note- 2: The area of a traiangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Note-3: The area of a triangle ABC is $\frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}|$ or $\frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|$

Note -4: The area of a parallelogram with diagonals $\vec{d}_1$ and $\vec{d}_2$ is $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

Note-5: The area of a plane quadrilateral ABCD is $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$. Whene AC and BD are its diagonals.



i.e. Area of quadrilateral ABCD $= \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$

Examples 1: If $\vec{a}, \vec{b}, \vec{c}$ are the position vector of vertices A,B,C of a triangle ABC, Show that the area of triangle ABC is $\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$. Also, deduce the condition for points $\vec{a}, \vec{b}, \vec{c}$ to be a collinear,

Solution:- We have, $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$

Area of Δ ABC $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}|$

$$= \frac{1}{2}|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{b})|$$

$$= \frac{1}{2}|\vec{b} \times \vec{c} - \vec{b} \times \vec{b} - \vec{a} \times \vec{c} + \vec{a} \times \vec{b})|$$

$$= \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

If the points, A, B,C are collinear, then

Area of Δ ABC = 0

$\Rightarrow \quad \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} \times \vec{a}| = 0$

$\Rightarrow \quad |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| = 0$

$\Rightarrow \quad \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

which is the required condition of collinearity of three points $\vec{a}, \vec{b}, \vec{c}$.

2. Prove that the points A, B and C with position vectors $\vec{a}$, $\vec{b}$ are $\vec{c}$ respectively are collinear if and only if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

Solution:

The point A, B and C are collinear

$\Leftrightarrow \quad \overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors.

$\Leftrightarrow \quad \overrightarrow{AB} \times \overrightarrow{BC} = \vec{0}$

$\Leftrightarrow \quad (\vec{b} - \vec{a}) \times (\vec{c} - \vec{b}) = \vec{0}$

$\Leftrightarrow \quad (\vec{b} - \vec{a}) \times \vec{c} - (\vec{b} - \vec{a}) \times \vec{b} = 0$

$\Leftrightarrow \quad (\vec{b} \times \vec{c} - \vec{a} \times \vec{c}) - (\vec{b} \times \vec{b} - \vec{a} \times \vec{b}) = 0$

$\Leftrightarrow \quad (\vec{b} \times \vec{c} + \vec{c} \times \vec{a}) - (0 - \vec{a} \times \vec{b}) = 0 \; (\because -(\vec{a} \times \vec{c}) = \vec{c} \times \vec{a})$

$\Leftrightarrow \quad \vec{a} \times b + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

3. For any three vectors $\vec{a}, \vec{b}, \vec{c}$, show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$

Solution:

L.H.S $= \vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$ [Distribuive law]

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{c}$

$= \vec{0}$

4. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Solution:-

Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$ [Taking cross-production with $\vec{a}$]

$\Rightarrow \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$ [Using distributive law]

$\Rightarrow \vec{0} + \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ……………….. (i)

Similarly, $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ ……………(ii)

(i) and (ii) gives the result

5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are the non-zero vectors prove that: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ Solution:-

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$ be three non-zero vectors. Then

$\vec{b} + \vec{c} = (b_1, b_2, b_3) + (c_1, c_2, c_3)$

Now,

$$\vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{a}\times\vec{b} + \vec{a}\times\vec{c} \text{ Proved}$$

<u>Application of Cross Product in Geometrical Problem</u>

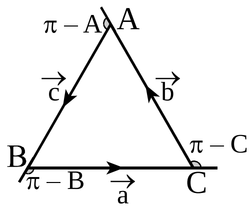
1. Sine Law:-

In a triangle ABC, prove by vectors method that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Solution

Let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$.

Then $|\vec{a}| = a$, $|\vec{b}| = b$, $|\vec{c}| = c$



We have,

$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

$\Rightarrow \vec{a}+\vec{b} = -\vec{c}$

$\Rightarrow \vec{a}+\vec{b} + \vec{c} = \vec{o}$

$\Rightarrow \vec{a} \times (\vec{a}+\vec{b}+\vec{c}) = \vec{a} \times \vec{0}$

$\Rightarrow \vec{a}\times\vec{a} + \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = \vec{0}$

$\Rightarrow \vec{0} + \vec{a}\times\vec{b} - \vec{c}\times\vec{a}$ ………. (i)

Again,

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0}$

$\Rightarrow \vec{b}\times\vec{a} + \vec{b}\times\vec{b} + \vec{b}\times\vec{c} = \vec{0}$

$\Rightarrow -(\vec{a}\times\vec{b}) + \vec{0} + \vec{b}\times\vec{c} = \vec{0}$

$\Rightarrow \vec{b}\times\vec{c} = \vec{a}\times\vec{b}$ ………….. (ii)

From (i) and (ii)

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

$\Rightarrow \quad ab \sin (\pi - C) = bc \sin (\pi - A) = ca \sin (\pi - B)$

$\Rightarrow \quad ab \sin C = bc \sin A = ca \sin B$

$$\Rightarrow \quad \frac{ab \sin C}{abc} = \frac{bc \sin A}{abc} = \frac{ca \sin B}{abc}$$

$$\Rightarrow \quad \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

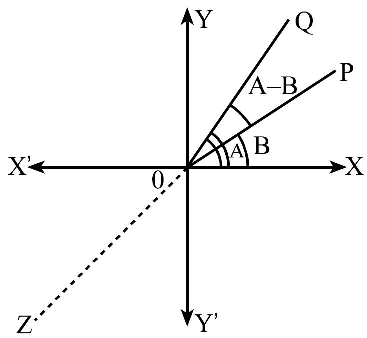
2. Prove by vector method that

(i) $\sin (A - B) = \sin A \cos B - \cos A \sin B$

(ii) $\sin (A + B) = \sin A \cos B + \cos A \sin B$

Soltion:-

(i) Let XOX' and YOY' be the mutually perpendicular lines taken as axes, and let $\rightarrow XOQ = A$, $\rightarrow XOP = B$



So, $\rightarrow POQ = A - B$.

Let, $OP = r_1$, and $OQ = r_2$

Now, co-ordinates of P and Q are

$(r_1 \cos B, r_1, \sin B)$ and $(r_2 \cos A, r_2 \sin A)$

Then, $\overrightarrow{OP} = (r_1 \cos B, r_1, \sin B, 0)$

and $\overrightarrow{OQ} = (r_2 \cos A, r_2 \sin A, 0)$

$$\therefore \quad \overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_1 \cos B & r_1 \sin B & 0 \\ r_2 \cos A & r_2 \sin A & 0 \end{vmatrix} = (0, 0, r_1, r_2 \sin A \cos B - r_1 r_2 \cos A \sin B)$$

$\Rightarrow \quad |\overrightarrow{OP} \times \overrightarrow{OQ}| = r_1 r_2 (\sin A \cos B - \cos A \sin B)$

We have,

$$\sin(A - B) = \frac{|\overrightarrow{OP} \times \overrightarrow{OQ}|}{OP.OQ} = \frac{r_1 r_2 (\sin A \cos B - \cos A \sin B)}{r_1 r_2}$$

$\therefore \quad \sin(A - B) = \sin A \cos B - \cos A \sin B$

ii) Let XOX' and YOY' be two mutually perpendicular lines taken as axes and $\rightarrow$ XOP = A, $\rightarrow$ X' OQ = B, the $\rightarrow$POQ = π – (A + B)

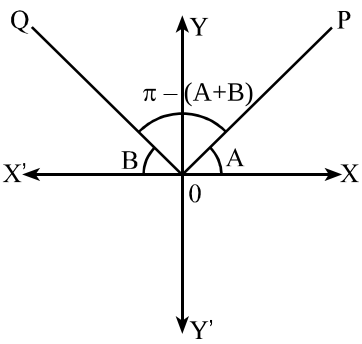
Let OP = r_1, and OQ = r_2

Now co-ordiantes of P and Q are (r_1 cosA, r_1, sinA) and (– r_2 cosB, r_2 sinB) respectively.

Then, $\overrightarrow{OP}$ = (r_1, cosA, r_1, sinA, 0)

And $\overrightarrow{OQ}$ = (– r_2 cosB, r_2 sinB, 0)

$$\therefore \quad \overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_1\cos A & r_1\sin A & 0 \\ r_2\cos B & r_2\sin B & 0 \end{vmatrix} = (0, 0, r_1 r_2 \sin A \cos B + r_1 r_2 \cos A \sin B)$$



$$\Rightarrow \quad |\overrightarrow{OP} \times \overrightarrow{OQ}| = r_1 r_2 (\sin A \cos B + \cos A \sin B)$$

We have,

$$\sin\{\pi - (A + B)\} = \frac{|\overrightarrow{OP} \times \overrightarrow{OQ}|}{OP.OQ}$$

$$\Rightarrow \quad \sin (A + B) = \frac{r_1 r_2 (\sin A \cos B + \cos A \sin B)}{r_1 r_2}$$

$$\therefore \quad \sin (A + B) = \sin A \cos B + \cos A \sin B$$

Worked out Examples

Example-1: If $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} - 2\vec{j} + 4\vec{k}$ are two vectors then find

(i) $\vec{a} \times \vec{b}$ (ii) unit vector perpendicular to $\vec{a}$ and $\vec{b}$

(iii) sine of an angle between $\vec{a}$ and $\vec{b}$

Solution:

Given, $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} - 2\vec{j} + 4\vec{k}$

$$\text{i)} \quad \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$$

$$= \quad 8\vec{i} - 8\vec{j} - 8\vec{j}$$

(ii) $|\vec{a} \times \vec{b}| = \sqrt{(8)^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$

We have,

Unit vector perpendicular to $\vec{a}$ and $\vec{b} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$= \frac{8\vec{i} - 8\vec{j} - 8\vec{k}}{8\sqrt{3}} = \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})$$

iii) let θ be angle between $\vec{a}$ and $\vec{b}$.

Then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{a\,b} = \dfrac{8\sqrt{3}}{\sqrt{(3)^2 + (1)^2 + (2)^2}\sqrt{(2)^2 + (-2)^2 + (4)^2}}$

$$= \frac{8\sqrt{3}}{\sqrt{14}\sqrt{24}} = \frac{2}{\sqrt{7}}$$

Exampl-2: Find the area of the triangle determined by the

(i) Vectors: $3\vec{i} + 4\vec{j} + \vec{k}$ and $-5\vec{i} + 7\vec{j}$

(ii) Vertices: P (1, 2, 3), Q (3, 4, 5) and R (2, 4, 7).

Solution:

(i) Let $\vec{a} = 3\vec{i} + 4\vec{j} + \vec{k}$ and $\vec{b} = -5\vec{i} + 7\vec{j}$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 1 \\ -5 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 7 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 1 \\ -5 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 4 \\ -5 & 7 \end{vmatrix}\vec{k}$$

$$= -7\vec{i} + 5\vec{j} + 41\vec{k}$$

And $|\vec{a} \times \vec{b}| = \sqrt{(-7)^2 + 5^2 + 41^2} = \sqrt{1755}$

We have,

Area of a triangle determined by $\vec{a}$ and $\vec{b} = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{1755} = 20.95$ sq. unit

(ii) Let p(1,2,3), Q (3, 4, 5) and R (1, 4, 7) be three vertices of ΔPQR and O be the origin. Then, $\overrightarrow{OP} = (1, 2, 3)$, $\overrightarrow{OQ} = (3,4,5)$, $\overrightarrow{OR} = (1, 4, 7)$

Now, $\overrightarrow{OQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (3, 4, 5) - (1, 2, 3) = (2, 2, 2)$

and $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (1, 4, 7) - (1, 2, 3) = (0, 2, 4)$

Now,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 2 \\ 0 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 2 \\ -0 & 2 \end{vmatrix} \vec{k}$$

$= \quad 4\vec{i} - 8\vec{j} + 4\vec{k} = (4, -8, 4)$

and $|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{4^2\, 7(-8)^2 + 4^2} = \sqrt{96} = 4\sqrt{6}$

We have,

Area of $\Delta PQR = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}4\sqrt{6} = 2\sqrt{6}$ sq. unit

Example-5: If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $c = \vec{j} - \vec{k}$ are given vectors then find a vector $\vec{b}$ satisfying the equation $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a}.\vec{b} = 3$

Solution:

Let $\vec{b} = x\vec{i} + y\vec{j} + z\vec{k}$. Then

$$\vec{a} \times \vec{b} = \vec{c} \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \vec{j} - \vec{k}$$

$\Rightarrow \quad (z - y)\vec{i} - (z - x)\vec{j} + (y - x)\vec{k} = \vec{j} - \vec{k}$

Equating,

$z - y = 0, -(z - x) = 1, y - x = -1$

$\Rightarrow \quad y = z, x - z = 1, x - y = 1$

$\Rightarrow \quad x - z = 1, x - y = 1$ [These two equation are equivalent to $y = z$]

$\Rightarrow \quad z = x - 1$ and $y = x - 1$ …………..(i)

Also,

$\vec{a}.\vec{b} = 3 \Rightarrow (\vec{i} + \vec{j} + \vec{k}).(x\vec{i} + y\vec{j} + z\vec{k}) = 3$

$\Rightarrow \quad x + y + z = 3 \Rightarrow x + x - 1 + x - 1 = 3$ [uning (i)]

$\Rightarrow \quad 3x = 5 \Rightarrow x = \frac{5}{3}$

From (i) $y = x - 1 \Rightarrow y = \frac{5}{3} - 1 = \frac{2}{3}$

Also, $y = z \Rightarrow z = \frac{2}{3}$

Hence, $\vec{b} = \frac{5}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$

Exercise

1. Define a vector product of two vectors. Interprect it geometrically. Find a unit vector perpendicular to the plane of $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + \vec{k}$. Also compute the sine of the angle between them.
2. Find the unit vectors perpendicular to the given vectors.

   (i) $\vec{i} + 3\vec{j} - 4\vec{k}$ and $2\vec{i} + \vec{j} - \vec{k}$

   (ii) $4\vec{i} - 2\vec{j} + 3\vec{k}$ and $5\vec{i} + \vec{j} - 4\vec{k}$

   (iii) $2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{i} + \vec{j} - 2\vec{k}$
3. Find the area of the triangle determined by the following vectors.

   (i) $3\vec{i} + 4\vec{j}$ and $-5\vec{i} + 7\vec{j}$

   (ii) $-2\vec{i} - 5\vec{k}$ and $-10\vec{i} - 7\vec{j} + 4\vec{k}$
4. Find the area oif the parallelogram determined by the following vectors.

   (i) $\vec{i} + \vec{j} - 3\vec{k}$ and $-\vec{i} - 2\vec{j} - 3\vec{k}$

   (ii) $\vec{i} + 2\vec{j} + 3\vec{k}$ and $3\vec{i} - 2\vec{j} + \vec{k}$
5. Find the area of the parallelogram having the diagonals $4\vec{i} - \vec{j} - 3\vec{k}$ and $-2\vec{i} + \vec{j} - 2\vec{k}$
6. If $\vec{a} = 6\vec{i} + 3\vec{j} - 5\vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$,

   Show that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ [Hint : Show $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$]
7. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, find $\vec{a} \cdot \vec{b}$.

[Hint: First the sinθ using $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$, then $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$ as $\theta = |\vec{a}||\vec{b}|$ $\sqrt{1 - \sin^2\theta}$]

8) Let $\vec{a} = \vec{i} - \vec{j}$, $\vec{b} = 3\vec{j} - \vec{k}$ and $\vec{c} = 7\vec{i} - \vec{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d} = 1$

[Hint: Let $\vec{a} = \lambda(\vec{a} \times \vec{b})$, find λ using $\vec{c}.\vec{d} = 1$]

Answer:

1. $\dfrac{1}{\sqrt{3}}(-\vec{i} - \vec{j} - \vec{k}), \dfrac{\sqrt{3}}{2}$

2. (i) $\dfrac{1}{\sqrt{75}}$ (1, – 7, – 5) (ii) $\dfrac{1}{\sqrt{1182}}$ (5, – 3, 14) (iii) $\dfrac{1}{\sqrt{35}}$ (– 5,3,– 1)

3. (i) 20.5 sq. unit (ii) $\dfrac{1}{2}\sqrt{165}$ sq. unit

4. (i) $\sqrt{118}$ sq. unit (ii) $8\sqrt{3}$ sq. unit

5. 7.5 sq. unit

6. …..?

7. 6

8. $\dfrac{1}{4}(\vec{i} + \vec{j} + 3\vec{k})$

Product of Three Vectors

1. Introduction

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors. Consider the product $(\vec{a}.\vec{b})\ \vec{c}$. Since $\vec{a}.\vec{b}$ is a scalar quantity and the dot product is defined between two vector quantity $(\vec{a}.\vec{b}).\vec{c}$ is not meaningful. But $(\vec{a}\times\vec{b}).\vec{c}$ is meaningful and scalar quantity. This product in known as the scalar triple product of $\vec{a},\vec{b}$ and $\vec{c}$. The product $(\vec{a}\times\vec{b})\times\vec{c}$ is also meaningful and ….. vector triple product of $\vec{a}$, $\vec{b}$, $\vec{c}$.

Scalar Triple Product

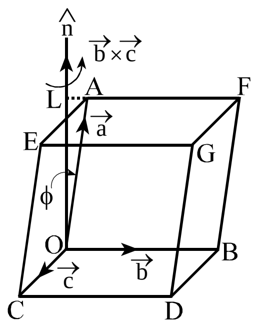
Definition: Let $\vec{a},\vec{b}$ and $\vec{c}$ be three vector. Then the scalar $\vec{a}.(\vec{b}\times\vec{c})$ is called the scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$. Thus, $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}.(\vec{b}\times\vec{c})$. Similarly, $(\vec{a}\times\vec{b}).\vec{c}$, $(\vec{b}\times\vec{c}).\ \vec{a}$ and $(\vec{c}\times\vec{a}).\vec{b}$ are scalar triple products.

<u>Geometrical Interpretation</u>

Let, $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors. Comsider a parallelopriped having co-terminous edges OA, OB and OC such that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC}$ = c. The dirction of perpendicular to the plance of $\vec{b}$ and $\vec{c}$.

Let θ be the angle between $\vec{a}$ and $\vec{b}\times\vec{c}$. If $\hat{\eta}$ is a unit vector along $\vec{b}\times\vec{c}$, then θ is also the angle between $\hat{\eta}$ and $\vec{a}$.

Now, $[\vec{a}\ \vec{b}\ \vec{c}]$



= $\vec{a}\,.\,(\vec{b}\times\vec{c}) = \vec{a}\,.\,$(Area of parallelogram OBDC) $\hat{\eta}$

= (Area of parallelogram OBDC) $(\vec{a}\,.\hat{\eta})$

= (Area of parallelogram OBDC) $(|\vec{a}|\cos\theta)$ $[\because |\hat{\eta}| = 1]$

= Area of parallelogram OBDC) (OL) $[\because OC\cos\theta = OL]$

= (Area o the base of the parallelopied) × height

= Volume of the parallelepiped with conterminous edges $\vec{a}, \vec{b}, \vec{c}$

Thus, the scalar triple product $[\vec{a}, \vec{b}, \vec{c}]$ represents the volume of the parallelepiped whose co-terminuous edges $\vec{a}, \vec{b}, \vec{c}$ form a right handed system of vectors.

<u>Properties of scalar Triple Product</u>

Property-I: If $\vec{a}, \vec{b}$ and $\vec{c}$ are cyclically permuted the value of scalar triple product remains same (unaltered).

i.e. $(\vec{a}\times\vec{b})\,.\,\vec{c} = (\vec{b}\times\vec{c}).\,\vec{a} = (\vec{c}\times\vec{a}).\,\vec{b}$

$\Rightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{b}\ \vec{c}\ \vec{a}) = [\vec{c}, \vec{a}, \vec{b}]$

Property-II: The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.

i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\ \vec{b})\,.\,\vec{c} = -(\vec{b}\times\vec{a}).\,\vec{c} = -\{(\vec{b}\times\vec{a})\}.\,\vec{c} = -[\vec{b}\ \vec{a}\ \vec{c}]$

Property -III : In scalar triple product the positions of dot and cross can be intercharged provided that the cyclic order of the vectors remains same.

i.e. $(\vec{a}\times\vec{b}).\,\vec{c} = \vec{a}\,.(\vec{b}\times\vec{c})$

Property -IV: The scalar triple product of three vectors is zero if any two of them are equal proof:- let $\vec{a}\ \vec{b}$ and $\vec{c}$ be threreetors.

Property-V: For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar λ,

we have $[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda\,[\vec{a}\,\vec{b}\ \vec{c}]$

Property VI:- The scalar triple product of their three vectors is zero if two of them are parallel or collinear.

Property-VII: If $\vec{a}\ \vec{b}\ \vec{c}$ and $\vec{d}$ ar four vectors, then

$[\vec{a}+\vec{b}\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

Property-VIII : The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar is $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

i.e. $\vec{a}, \vec{b}, \vec{c}$ are coplanner $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

Scalar Triple product in terms of component (In determinant form)

Theorem:- Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three vectors. Then,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof:- We have, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (a_3 b_3 - a_3 b_2)\ \hat{i} - (a_1, b_3 - a_3 b_1)\ \hat{j} + (a_1b_2 - a_2b_1)\ \hat{k}$

Now, $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}).\ \vec{c} = [(a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\ \hat{j} + (a_1, b_2 - a_2 b_1)\ \hat{k}].\ (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$= (a_3 b_3 - a_3 b_2)\, c_1 - (a_1 b_3 - a_3 b_1)\, c_2 + (a_1 b_2 - a_2 b_1)\, c_3$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Distributivity of cross product over vector Addition

Theorem: For any three vector $\vec{a}$, $\vec{b}$ and $\vec{c}$, $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Worked out Examples

1. Prove that $[\vec{i}\ \vec{j}\ \vec{k}] + [\vec{i}\ \vec{k}\ \vec{j}] = 0$

Sol^n $[\vec{i}\ \vec{j}\ \vec{k}] + [\vec{i}\ \vec{k}\ \vec{j}]$

$= \vec{i}.(\vec{j} \times \vec{k}) + \vec{i}.(\vec{k} \times \vec{j}) \qquad = \vec{i}.\vec{i} + \vec{i}.(-\vec{i})$

$= \quad \vec{i}\ \vec{i} - \vec{i}\,.\,\vec{i} = 0$

2) Find $[\vec{a}\ \vec{b}\ \vec{c}]$ when $\vec{a} = (2, -3, 4)$ $\vec{b} = (1, 2, -1)$ and $\vec{c} = (3, -1, 2)$.

Solution:-

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ -3 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -1 \\ -3 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} = -7$$

3) Find the volume of a parallelepiped whose sides are given by $-3\vec{i} + 7\vec{j} + 5\vec{k}$, $-5\vec{i} + 7\vec{j} - 3\vec{k}$ and $7\vec{i} - 5\vec{j} - 3\vec{k}$.

Sol^n:

Let $\vec{a} = (-3, 7, 5)$, $\vec{b} = (-5, 7, -3)$ and $\vec{c} = (7, -5, -3)$ be three adjacent edges of a parallelopied. Then its volume is equal to $[\vec{a}\ \vec{b}\ \vec{c}]$.

$$\text{Now, } [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$= \quad -3\begin{vmatrix} 7 & -3 \\ -5 & -3 \end{vmatrix} - 7\begin{vmatrix} -5 & -3 \\ 7 & -3 \end{vmatrix} + 5\begin{vmatrix} -5 & 7 \\ 7 & -5 \end{vmatrix} = -264$$

$\therefore$ Required volume of the parallelepiped $= |[\vec{a}\ \vec{b}\ \vec{c}]| = |-264| = 264$ cubic unit.

4) Show that the vectors $\vec{a} = (-2, -2, 4)$ $\vec{b} = (-2, 4, -2)$, $\vec{c} = (4, -2, -2)$ are coplanar.

Solution:- $\vec{a}, \vec{b}, \vec{c}$ are coplanar iff $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

$$\text{So, } [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = 0 \text{ (on solving)}$$

Hence, $\vec{a}\,\vec{b}\,\vec{c}$ are coplanar.

5) Find the value of λ so that $\vec{a} = (2, -1, 1)$, $\vec{b} = (1, 2, -3)$ and $\vec{c} = (3, \lambda, k)$ are coplanar.

Sol^n:

$\vec{a}, \vec{b}, \vec{c}$ are coplarar iff $[\vec{a}, \vec{b}\,\vec{c}] = 0$

Since, $\vec{a}, \vec{b}, \vec{c}$] are coplarar $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

$$\Rightarrow \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2\begin{vmatrix} 2 & -3 \\ \lambda & 5 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(10 + 3\lambda) + 1\,(5 + 9) + (\lambda - 6) = 0$$

$$\Rightarrow \quad 7\lambda + 28 = 0 \Rightarrow -4$$

Vector Triple Product

Definition: Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vector. Then the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is called the vector triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is given by

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\,.\,\vec{c})\vec{b} - (\vec{a}\,.\,\vec{b})\vec{c}$$

Geometrical Interpretatio of Vector Triple Product

If $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero and non-coplanar vectors. Then the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to both the vectors $\vec{a}$ and $\vec{b} \times \vec{c}$.

Then dot product of $\vec{a}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ is zero.

Similarly,

dot product of $\vec{b} \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ is zero.

Example:-

20. $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$, find $\vec{a} \times (\vec{b} \times \vec{c})$ and verify that find $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

Solution

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (-1-2)\vec{i} - (-2-1)\vec{j} + (4-1)\vec{k} = -3\vec{i} + 3\vec{j} + 3\vec{k}$$

$$\therefore \quad \vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 1 \\ 3 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix}$$

$$= (-6-3)\vec{i} - (3+3)\vec{j} + (3-6)\vec{k}$$

$$= -9\vec{i} - 6\vec{j} - 3\vec{k} \qquad \dots \text{(i)}$$

Again, $\vec{a} \cdot \vec{c} = (\vec{i} - 2\vec{j} + \vec{k}) \cdot (\vec{i} + 2\vec{j} - \vec{k}) = 1 - 4 - 1 = -4$

$\vec{a} \cdot \vec{b} = (\vec{i} - 2\vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k}) = 2 - 2 + 1 = 1$

$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = -4(2\vec{i} + \vec{j} + \vec{k}) - 1(\vec{i} + 2\vec{j} - \vec{k})$

$= -8\vec{i} - 4\vec{j} - 4\vec{k} - \vec{i} - 2\vec{j} + \vec{k} = -9\vec{i} - 6\vec{j} - 3\vec{k}$

Hence, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ verified

21. If $\vec{a} = (1, 0, 1)$, $\vec{b} = (2, 1, -1)$, and $\vec{c} = (0, 1, 3)$, show that $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

Solution

$$\vec{a} = (1, 0, 1) = \vec{i} + \vec{k}$$

$$\vec{b} = (2, 1, -1) = 2\vec{i} + \vec{j} - \vec{k}$$

$$\vec{c} = (0, 1, 3) = \vec{j} + 3\vec{k}$$

Now, $$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (3 + 1)\vec{i} - (6 - 0)\vec{j} + (2 - 0)\vec{k} = 4\vec{i} - 6\vec{j} + 2\vec{k}.$$

Again, $$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 4 & -6 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ -6 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 4 & -6 \end{vmatrix}$$

$$= (0 + 6)\vec{i} - (2 - 4)\vec{j} + (-6 - 0)\vec{k}$$

$$= 6\vec{i} + 2\vec{j} - 6\vec{k} \qquad \text{... (i)}$$

Next, $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= (0 - 1)\vec{i} - (-1 - 2)\vec{j} + (1 - 0)\vec{k} = -\vec{i} + 3\vec{j} + \vec{k}$$

$\therefore$ $$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= (9 - 1)\vec{i} - (-3 - 0)\vec{j} + (-1 - 0)\vec{k}$$

$$= 8\vec{i} + 3\vec{j} - \vec{k} \qquad \text{... (ii)}$$

From (i) and (ii), we get that

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Exercise

1. Prove that $[\vec{i}\,\vec{j}\,\vec{k}] + [\vec{j}\,\vec{k}\,\vec{i}] + [\vec{k}\,\vec{i}\,\vec{j}] = 3$

2. Find $[\vec{a}\,\vec{b}\,\vec{c}]$ when $\vec{a} = (2, -3, 0)$, $\vec{b} = (1, 1, -1)$ and c = (3, 0, – 1)

3. Find the volume of parallelepiped whose co-terminom edges are represented by $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

4. Show that the triad of vectors $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{b} = 3\vec{i} + 2\vec{j} + 7\vec{k}$, $\vec{c} = 5\vec{i} + 6\vec{j} + 5\vec{k}$ are coplanar.

5. Find the value of λ if the vectors $\vec{a} = (1, -1, 1)$, $\vec{b} = (2, 1, -1)$ and $\vec{c} = (\lambda, -1, \lambda)$ are coplanar

6. If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ then find the value of $\vec{a} \times (\vec{b} \times \vec{c})$.

Answers:

2. 4 3. 37 5. 1 6. $-9\vec{i} - 6\vec{j} - 3\vec{k}$

Multiple Choice Questions

Choose the best option.

1. A unit vector perpendicular to the plane $\vec{a} = 2\vec{i} - 6\vec{j} - 3\vec{k}$, $\vec{b} = 4\vec{i} + 3\vec{j} - \vec{k}$ is

   a) $\dfrac{4\vec{i} + 3\vec{j} - \vec{k}}{\sqrt{26}}$ b) $\dfrac{2\vec{i} - 6\vec{j} - 3\vec{k}}{7}$ c) $\dfrac{3\vec{i} - 2\vec{j} + 6\vec{k}}{7}$ d) $\dfrac{2\vec{i} - 3\vec{j} - 6\vec{k}}{7}$

2. For non-zero vectors $\vec{a}, \vec{b}, \vec{c}$, $(\vec{a} \times \vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|$ holds if

   a) $\vec{a}.\vec{b} = 0$, $\vec{b}.\vec{c} = 0$, $\vec{c}.\vec{c} \neq 0$ (b) $\vec{b}.\vec{c} = 0$, $\vec{c}.\vec{a} = 0$, $\vec{a}.\vec{b} \neq 0$

   c) $\vec{c}.\vec{c} = 0$, $\vec{a}.\vec{b} = 0$, $\vec{b}.\vec{c} \neq 0$ d) $\vec{a}.\vec{b} = 0$, $\vec{b}.\vec{c} = 0$, $\vec{c}.\vec{a} = 0$

[Hint:- $(\vec{a} \times \vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|$ if the parallelepiped is rectangular. i.e. $\vec{a}.\vec{b} = 0$, $\vec{c}.\vec{a} = 0$]

3. if $\vec{a}\,.\,\vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$ then

a) $\vec{a}$ is parallel to $\vec{b}$

(b) $\vec{a}$ is perpendicular to $\vec{b}$

c) either $\vec{a}$ or $\vec{b}$ is a non-zero vectors (d) non of above

[Hint: $\vec{a}$ and $\vec{b}$ cannot be parallel and perpendicular both at the same time].

4. The area of the triangle having vertices $\vec{i} - 2\vec{j} + 3\vec{k}, -2\vec{i} + 3\vec{j} - \vec{k}, 4\vec{i} - 7\vec{j} + 7\vec{k}$ is

a) 36 sq. unit b) 0 sq. unit c) 39 sq. unit d) 11 sq. unit

5. Area of a parallelogram wgose adjacent sides are represented by the vectors $3\vec{i} - \vec{k}$ and $\vec{i} + 2\vec{j}$ is a) $\frac{\sqrt{17}}{2}$ b) $\frac{\sqrt{14}}{2}$ c) $\sqrt{41}$ d) $\frac{\sqrt{7}}{2}$

6. If $\vec{a}$ and $\vec{b}$ are two vectors, then $(\vec{a} \times \vec{b})^2$ equals

a) $\begin{vmatrix} \vec{a}.\vec{b} & \vec{a}.\vec{a} \\ \vec{b}.\vec{b} & \vec{b}.\vec{a} \end{vmatrix}$ b) $\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$ c) $\begin{vmatrix} \vec{a}.\vec{b} \\ \vec{b}.\vec{a} \end{vmatrix}$ d) None

7) The geometrical meausing of cross product of two vectors $\vec{a}$ and $\vec{b}$ is

a) Vectors area of parallelogram whose adjacent sides are represented by $\vec{a}$ on $\vec{b}$

b) Projection of $\vec{a}$ on $\vec{b}$

c) Projection of $\vec{b}$ on $\vec{a}$

d) Vector area of a triangle whose two sides are represented by $\vec{a}$ and $\vec{b}$

8. $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ is equal to

a) $2(\vec{b} \times \vec{a})$ b) $\vec{a}^2 - \vec{b}^2$ (c) $2(\vec{a} \times \vec{b})$ (d) $\vec{a}^2 + \vec{b}$

9. Geometrical meaning of scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is

a) $|\vec{a}|$ (projection of $\vec{b}$ on $\vec{a}$ and $\vec{c}$)

b) $|\vec{b}|$ (projection of $\vec{b}$ on $\vec{a}$ and $\vec{c}$) c) $|\vec{a}||\vec{b}||\vec{c}|$

d) Volume of parallelepiped formed by $\vec{a}$, $\vec{b}$, $\vec{c}$.

10. The condition, when the value of the scaler triple product is zero are

a) When two of the vectors are equal

b) When two of the vectors are parallel

c) When the vectors are coplanar.

d) All of above

Answers

1) c 2) d 3) c 4) c 5) c 6) b 7) a 8) a 9) d 10) d

Project work

Prepare a project report using scalar triple product to find the volume of any three parallelepiped which you frequently use.

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