Numerical Method

CSC - 204 (BSc CSIT)

MTH - 317.3 (P.U)

P.U

	Theory	Practical	Total
Sessional	30	20	50
Final	50	—	50
Total	80	20	100

Course Contents:

1. Solution of Nonlinear Equations (10hrs)
   - → Review of calculus & Taylor's theorem
   - → Errors in numerical calculation
   - → Trial & Error Method
   - → Half interval method
   - → Secant Method
   - → Newton Method
   - → Fixed-point iteration
   - → Newton's Method of polynomials

2. Interpolation and Approximation
   - → Interpolation
   - → Lagrang's interpolation
   - → Newton's "
   - → Cubic Spline Interpolation
   - → Least Squares Approximations

3. Numerical Differentiations and Integration (5hrs) — Gaussian, Romberg, Newtons

4. Solution of Linear Algebric Equations (10hrs)
   - → Review
   - → Gaussian elimination method and algorithm, pivoting, ill-conditioning, Gauss-Jordan method & algorithm, matrix inversion.
   - → Matrix factorization
   - → Iterative methods
   - → Eigen values & eigen vectors problems, solving eigen value problem using power method.

5. Solution of ordinary Differential Equations (7hrs)
   - → Review
   - → Taylor's method, Picard's Method, Euler's method, Heun's method, Runge-kutta Method (4th order)
   - → Solution of higher-order equations.
   - → Boundary Value Problems, shooting method

6. Solution of Partial Differential Equations (5hrs)
   - → Review
   - → Deriving difference eqn
   - → Laplacian eqn.
   - → Poisson's "

Books
- → E Balagurusamy
- → Grewal
- → E. Ward Cheney

Errors in Numerical Calculations:

1) Inherent Error: Those which are already present in the statement of problem itself.

a. Data Errors:

Data error (also known as empirical error) arises when data for a problem are obtained by some experimental means and are therefore of limited accuracy and precision. This may be due to some limitations in instrumentation and reading, and therefore may be unavoidable. Such as distance, a voltage, or a time period, weighing machine etc.

b. Conversion Errors:

Conversion error (also known as representation errors) arise due to the limitation of the computer to store the data exactly. We know that the floating point representation retains only a specified number of digits. The digits that are not retained constitute the roundoff error.

2) Numerical Error: Occurs due to the ~~limitation~~ implementation of numerical methods to solve certain problems.

a. Roundoff Error:

Roundoff errors occur when a fixed number of digits are used to represent exact number. Since the numbers are stored at every stage of computation, roundoff error is introduced at the end of every arithmatic operation.

Rounding a number can be done in two way. One is known as chopping and the other is known as symmetric rounding.

Eg: 24.457 → 24.46

b. Truncation Errors:

Truncation errors arises from ~~the~~ using an approximation in place of an exact mathematical procedure. Typically, it is the error resulting from the truncation of the numerical process. We often use some finite numbers of terms to estimate the sum of an infinite series. For example,

$$S = \sum_{i=0}^{\infty} a_i x^i \text{ is replaced by finite sum}$$

$$S = \sum_{i=0}^{n} a_i x^i$$

The series has been truncated.

c. Blunders:

Blunders are errors caused by human imperfection. Such error may cause a very serious disaster in the result. Such errors can be avoided by acquiring a sound knowledge of all aspects of the problem as well as the numerical process. Such errors arise due to lack of understanding of the problem, wrong assumption etc.

Error representation:

Let suppose the true value of a data item is denoted by x_t and its approximate value is denoted by x_a.

Then error is given by,

$$\boxed{Error = x_t - x_a}$$

The error may be both negative or positive. Therefore absolute error is denoted by

$$\boxed{e_a = |x_t - x_a|}$$

and relative error is denoted by

$$e_r = \frac{\text{absolute error}}{\text{true value}} = \frac{|x_t - x_a|}{x_t}$$

Half-Interval (Bisection Method):

The bisection method is one of the simplest and most reliable of iterative methods for the solution of non linear equations. This method, also known as binary chopping or half-interval method, relies that if $f(x)$ is real and continuous in the interval $a < x < b$, i.e. (a,b) and $f(a)$ and $f(b)$ are of opposite signs.

i.e. $f(a)f(b) < 0$

then there is at least one real root between the interval a and b.

Let $x_1 = a$ and $x_2 = b$ let us also define another midpoint x_0 between a and b.

i.e. $x_0 = \frac{x_1 + x_2}{2}$

Now there exists the following conditions:

1. if $f(x_0) = 0$, we have a root at x_0
2. if $f(x_0)f(x_1) < 0$, there is a root between x_0 & x_1
3. if $f(x_0)f(x_2) < 0$, " " " " " x_0 & x_2

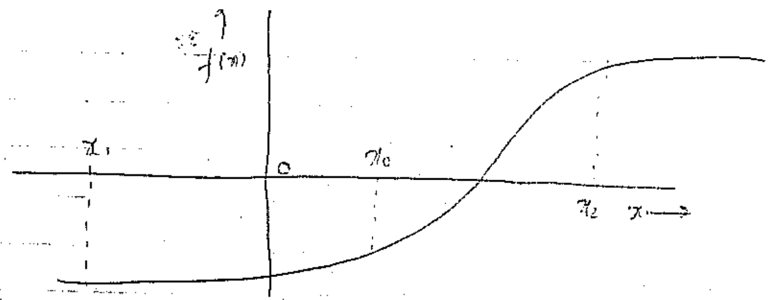


fig: illustration of bisection method

(2) $|x_{max}| \leq 1 + \frac{1}{|a_n|} \times \max \{ |a_{n-1}|, |a_{n-2}|, \ldots \}$

Algorithm

1) Start
2) Input initial guess x_1 & x_2 and initialize error ϵ
3) Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$
4) If $f_1 \times f_2 > 0$, x_1 and x_2 do not bracket the root, goto step 2
5) Compute $x_0 = (x_1 + x_2)/2$ and $f_0 = f(x_0)$
6) If $f_1 \times f_0 < 0$
 set, $x_2 = x_0$, $f_2 = f_0$
 else, set $x_1 = x_0$, $f_1 = f_0$
7) If $|(x_2 - x_1)/x_2| < \epsilon$, then root $= x_0$ else goto step 5
8) Stop

Q.1 Find the root of the equation
$x^2 - 4x - 10 = 0$

Soln

we have, $f(x) = x^2 - 4x - 10$
$f(-2) = 2$ and $f(-1) = -5$
Since $f(-2) \cdot f(-1) < 0$, so let the initial guess be $x_1 = -2$ & $x_2 = -1$
then $x_0 = \frac{x_1 + x_2}{2} = \frac{-2-1}{2} = -1.5$

iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_0)$
1	-2	-1	-1.5	2	-5	-1.75
2	-2	-1.5	-1.75	2	-1.75	0.0625
3	-1.75	-1.5	-1.625	0.0625	-1.75	-0.859
4	-1.75	-1.625	-1.6875	0.0625	-0.859	-0.4
5	-1.75	-1.6875	-1.72	0.0625	-0.4	-0.1616
6	-1.75	-1.72	-1.735	0.0625	-0.1616	-0.05
7	-1.75	-1.735	-1.7425	0.0625	-0.05	0.006
8	-1.7425	-1.735	-1.73875	0.0065	-0.05	-6.06
9	-1.742	-1.749	-1.741	0.0065	-6.07	-13.93
10	-1742	-1.741	-1.741	0.0065	-13.93	-13.93

Here 9th & 10th iteration value upto 3 decimal is same so we stop the iteration.

$\boxed{\therefore \text{root} = \cancel{0.0065}}$ -1.741

Q.2 Find the root of $\cos x - 3x + 1 = 0$ upto 3 decimal places.

Soln let $f(x) = \cos x - 3x + 1$
$f(0) = 2$, $f(1) = -1.4597$
Then, let the initial guess: $x_1 = 0$, $x_2 = 1$
The tabular form is,

iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_0)$
1	0	1	0.5	2 +ve	-1 -ve	0.5 +ve
2	0.5	1	0.75	0.5 +ve	-1 -ve	-0.25 -ve
3	0.5	0.75	0.625	0.5 +ve	-0.25 -ve	+0.125 -ve
4	0.625	0.75	0.6875	0.125	-0.25	-0.0626
5	0.625	0.6875	0.656	0.125	-0.626	0.032
6	0.656	0.6875	0.672	0.032	-0.626	-0.02
7	0.656	0.672	0.664	0.032	-0.02	0.007
8	0.664	0.672	0.668	0.007	-0.02	-0.004
9	0.664	0.668	0.666	0.007	-0.04	0.002
10	0.664	0.666	0.665	0.007	0.002	0.005
11	0.664	0.665	0.6645	0.007	0.005	0.002
12	0.6645	0.665	0.6645	0.002	0.005	0.007

$\therefore$ Root $= 0.664$

iteration	x_1	x_2	x_0	$f(x_1)$	$f(x_2)$	$f(x_0)$
1	0	1	0.5	+ve	-ve	+ve
2	0.5	1	0.75	+ve	-ve	-ve
3	0.5	0.75	0.625	+ve	-ve	-ve
4	0.5	0.625	0.5625	+ve	-ve	+ve
5	0.5625	0.625	0.59375	+ve	-ve	+ve
6	0.59375	0.625	0.609375	+ve	-ve	-ve
7)	0.59375	0.609375	0.6016	+ve	-ve	+ve
7	0.6016	0.609375	0.60855	+ve	-ve	+ve
8	0.60655	0.609375	0.6079	+ve	-ve	+ve
9	0.60655	0.6079	0.6072	+ve	-ve	-ve
10	0.60655	0.6072	0.606875	+ve	-ve	+ve
11	0.606875	0.6072	0.60708	+ve	-ve	+ve
12	0.60708	0.6072	0.6071	+ve	-ve	+ve
13						

Convergence of Bisection method:

In the bisection method, we choose a midpoint x_0 in the interval between x_1 & x_2. Depending upon the sign of functions $f(x_0)$, $f(x_1)$ and $f(x_2)$, x_1 or x_2 is set equal to x_0 such that new interval contains the root. In either case, the interval containing the root is reduced by a factor 2. If the procedure is repeated n times, then the interval containing the root is reduced to the size

$$\frac{x_2 - x_1}{2^n} = \frac{\Delta x}{2^n}$$

After n iteration, the root lies within $\pm \Delta x / 2^n$. This means the error bound at nth iteration is

$$E_n = \left| \frac{\Delta x}{2^n} \right|$$

Similarly,

$$E_{n+1} = \left| \frac{\Delta x}{2^{n+1}} \right| = \frac{E_n}{2}$$

i.e the error decreases linearly with each step by factor 0.5. The bisection method is, therefore linearly convergent. ~~To achieve~~ Since the convergence is slow to achieve a high degree of accuracy, a large number of iteration may be needed.

SECANT METHOD:

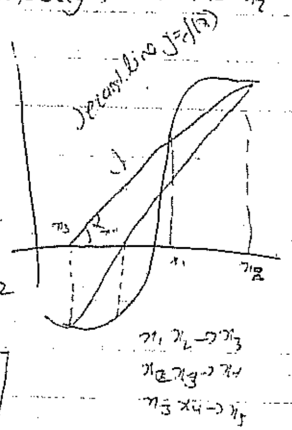
Secant method is like bisection methods, uses two ~~roots~~ initial estimates but does not require that they must be bracket the root.

Let the initial guess be x_1 & x_2. Then let $f(x_1) = f_1$ and $f(x_2) = f_2$.

Now,

slope of the secant line passing through x_1 and x_2 is given by.

$$\frac{f_1}{x_1 - x_3} = \frac{f_2}{x_2 - x_3}$$



or, $f_1(x_2 - x_3) = f_2(x_1 - x_3)$

or, $x_3(f_2 - f_1) = f_2 x_1 - f_1 x_2$

or

$$\boxed{x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1}}$$

This equation is known as secant formula.

By adding & subtracting $f(x_2)\, x_2$ to the numerator.

In general,

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} = x_2 - \frac{f_2(x_2 - x_1)}{f_2 - f_1}$$

$$x_4 = x_3 - \frac{f_3(x_3 - x_1)}{f_3 - f_2}$$

i.e $x_{i+1} = x_i - \dfrac{f_i\,(x_i - x_{i-1})}{f_i - f_{i-1}}$

Secant Algorithm

1. Start
2. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$
3. Compute

$$x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1}$$

4. Test for the accuracy of x_3

if $\left| \dfrac{x_3 - x_2}{x_3} \right| > E$ then

set $x_1 = x_2$ & $f_1 = f_2$

set $x_2 = x_3$ & $f_2 = f(x_3)$

otherwise

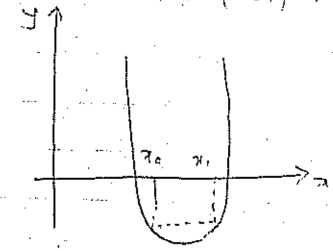
set root $= x_3$

print result

5. Stop.

Drawback

① When secant line become parallel to x-axis (i.e. $f(x_0) = f(x_1)$)



Q. Use the secant method to estimate

$x^2 - 4x - 10$

with the initial estimates of $x_1 = 4$ & $x_2 = 2$

Soln

Given $x_1 = 4$ & $x_2 = 2$

$$\left.\begin{aligned} f(x_1) &= f(4) = -10 \\ f(x_2) &= f(2) = -14 \end{aligned}\right\} \text{do not bracket root.}$$

We know secant formula

$$x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1}$$

Now,

iteration	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$
1	4	2	-10	-14	9
2	2	9	-14	35	4
3	9	4	35	-10	5.1111
4	4	5.1111	-10	-4.321	5.9565
5	5.1111	5.9565	-4.321	1.6538	5.7225
6	5.9565	5.7225	1.6538	-0.143	5.7411
7	5.7225	5.7411	-0.143	-4.17×10^{-3}	5.7417
8	5.7411	5.7417	-4.17×10^{-3}	3.19×10^{-7}	5.7417

$\therefore$ Root = 5.7417

Q. Calculate the root of non-linear equation $f(x) = \sin x - 2x + 1$ using secant method. The absolute error of functional value should be less than 10^{-3}.

Soln

Given,

$f(x) = \sin x - 2x + 1$

$f(0) = 1$ & $f(1) = -0.1586$

Now,

Let initial guess be $x_1 = 0$ & $x_2 = 1$

The value in tabular form:

iteration	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1}$
	0	1	1	-0.1586	0.8613
	1	0.8613	-0.1585	0.036	0.887
	0.8613	0.887	0.036	0.0012	0.8878
	0.887	0.8878	0.0012	0.000085	0.8878

$\therefore$ Root = 0.8878

Q. Find the root of eqn (i) $xe^x = \cos x$

(ii) $3x = \cos x + 1$

Rate of convergence in secant Method:

We know secant method formula as,

$$x_3 = \frac{f(x_2)\,x_1 - f(x_1)\,x_2}{f(x_2) - f(x_1)}$$

By adding & subtracting $f(x_2)\,x_2$ to the numerator we get

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

In general,

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \cdot f(x_k)$$

Let us assume that, 'α' is a simple root of the eq^n $f(x) = 0$. Then $f(\alpha) = 0$.

Substituting, $x_k = \alpha + \varepsilon_k$ in the secant method we obtain,

$$\alpha + \varepsilon_{k+1} = (\alpha + \varepsilon_k) - \frac{(\alpha + \varepsilon_k) - (\alpha + \varepsilon_{k-1})}{f(\alpha + \varepsilon_k) - f(\alpha + \varepsilon_{k-1})} \cdot f(\alpha + \varepsilon_k)$$

or,

$$\varepsilon_{k+1} = \varepsilon_{k-1} - \frac{\varepsilon_k - \varepsilon_{k-1}}{f(\alpha + \varepsilon_k) - f(\alpha + \varepsilon_{k-1})} \cdot f(\alpha + \varepsilon_k)$$

Expanding $f(\alpha + \varepsilon_k)$ & $f(\alpha + \varepsilon_{k-1})$ using taylor series.

$$\varepsilon_{k+1} = \varepsilon_{k-1} - [\varepsilon_k - \varepsilon_{k-1}] \left[\frac{f(\alpha) + \varepsilon_k f'(\alpha) + \frac{\varepsilon_k^2}{2!} f''(\alpha) + \dots}{\{f(\alpha) + \varepsilon_k f'(\alpha) + \frac{\varepsilon_k^2}{2!} f''(\alpha) + \dots\} - \{f(\alpha) + \varepsilon_{k-1} f'(\alpha) + \frac{\varepsilon_{k-1}^2}{2!} f''(\alpha) + \dots\}}\right]$$

On solving we get,

$$\varepsilon_{k+1} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} \varepsilon_k \cdot \varepsilon_{k-1} + O$$

i.e. $\varepsilon_{k+1} \simeq c\,\varepsilon_k \varepsilon_{k-1}$ ——— (1)

where, $c = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$

The relation (1) is called error eq^n.

According to the def^n of rate of convergence, we see the relation in the form.

$\varepsilon_{k+1} = A\,\varepsilon_k^{\,p}$ ——— (2)

where A & p are constants to be determined. From (2)

$$\varepsilon_k = A\,\varepsilon_{k-1}^{\,p}$$

i.e. $\varepsilon_{k-1} = A^{-1/p}\,\varepsilon_k^{\,1/p}$ ——— (3)

From, (1), (2) & (3) we get,

$$\varepsilon_{k+1} = C\,\varepsilon_k \cdot A^{-1/p}\,\varepsilon_k^{\,1/p} = D\,\varepsilon_k^{\,p}$$

or $\varepsilon_k^{\,p} = C A^{-(1+1/p)} \cdot \varepsilon_k^{\,(1+1/p)}$

Equating the power of ε_k on both sides

$$p = 1 + 1/p$$

$$\Rightarrow p^2 - p - 1 = 0$$

$$\therefore p = \frac{1 \pm \sqrt{(1)^2 - 4\cdot(1)\cdot(-1)}}{2\cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

Neglecting the -ve sign we find that the rate of convergence for the secant method is,

$$p = \left(\frac{1 + \sqrt{5}}{2}\right) = 1.618$$

Therefore, the secant method of convergence has the order of 1.618 also called as superlinear convergence.

NEWTON-RAPHSON METHOD :-

Consider a graph of $f(x)$ as shown in figure. Let us assume that x_1 is an approximate root of $f(x)=0$. Draw a tangent at the curve $f(x)$ at $x=x_1$ as shown in fig. The point of intersection of this tangent with the x-axis gives the second approximation to the root. Let the point of intersection be x_2.

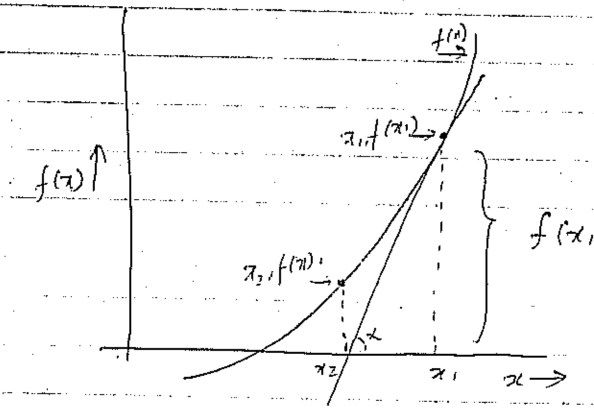


fig: Newton-Raphson Method

The slope of the tangent is given by,

$$\tan\alpha = \frac{f(x_1)}{x_1 - x_2} = f'(x_1)$$

or, $$x_1 - x_2 = \frac{f(x_1)}{f'(x_1)}$$

or $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Advantage
→ Only one initial guess is required
→ open bracket method
→ faster method.

The next approximation would be,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method of successive approximation is called Newton-Raphson method.

Limitation
→ If $f'(x_i)$ is zero or near to zero, division by zero occurs
→ If initial guess is too far from desired root, process may converge to ~~root of function~~ some other root.

Algorithm
1. Start
2. Input ~~two~~ initial guess x_0 and initialize error E
3. Compute $f_0 = f(x_0)$ and $g_0 = f'(x_0)$
4. Compute $x_1 = x_0 - f_0/g_0$
5. If $|(x_1 - x_0)/x_1| < E$, then root $= x_1$
else $x_0 = x_1$ and go to step 3
6) Stop.

Q Find the root of the equation $f(x) = x^2 - 3x + 2$. Using NR

Sol^n

Given $f(x) = x^2 - 3x + 2$

thus $f'(x) = 2x - 3$

Let initial guess be $x_1 = 5$

We know NR formula,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In tabular form,

Iteration	x_1	$f(x)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	5	12	7	3.2857
2	3.2857	2.9387	3.5714	2.4629
3	2.4629	0.6772	1.9258	2.1113
4	2.1113	0.1237	1.2226	2.0101
5	2.0101	0.0102	1.0202	2.0001
6	2.0001	0.0001	1.0002	2.0000

$\therefore$ Root = 2

Q Find the root of the equation $xe^x - 2 = 0$.

Sol^n

Given, $f(x) = xe^x - 2 = 0$

$f'(x) = xe^x + e^x = e^x(x+1)$

Let initial guess is $x_1 = 0$

The tabular form is,

No of iteration	x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	0	−2	1	2
2	2	12.7781	22.1672	1.4236
3	1.4236	3.9108	10.0629	1.0350
4	1.0350	0.9137	5.7288	0.8755
5	0.8755	0.1013	4.5014	0.8530
6	0.8530	0.0017	4.3484	0.8526
7	0.8526	−0.000024	4.3457	0.8526

$\therefore$ Root = 0.8526

Q $3x - e^{-x} = 0$; guess $x_1 = 0$, Ans: 0.2576

Q $x^3 - x^2 + x + 7$; guess $x_1 = 0$, Ans = −1.4883

Q $3x - \cos x - 1 = 0$; guess $x_1 = 0$; Ans = 0.6071

Fixed Point Iteration Method : (Method of successive approximation).

Any function in the form of

$$f(x) = 0 \quad \text{---} \quad (1)$$

Can be manipulated such that x is on the left hand side of the eqn as

$$x = g(x) \quad \text{---} \quad (2)$$

Eqn (1) & (2) are equivalent and they have the same root. The root of eqn (2) is given by the intersection of two curves $y = x$ & $y = g(x)$

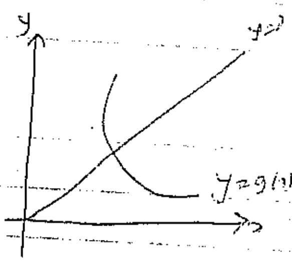
The intersection of the curves is called as fixed point of $g(x)$. Transformation can be done either by algebraic manipulation or adding 'x' on both sides

Example

$$x^2 + x - 2 = 0$$

Can be algebrically manipulate as.

$x = 2 - x^2$ or $x^2 = 2 - x$ or $x = \sqrt{2-x}$



Adding 'x' on both sides are only done when the eqn cannot be manipulated algebrically such as.

$$\tan x = 0$$

i.e $\quad x = \tan x + x$

Q.1 Find the root of eqn $\cos x - 2x + 3 = 0$ correct upto 3 decimal places.

$$\Rightarrow x = \frac{\cos x + 3}{2}$$

i.e $x = g(x)$

let $x_0 = 0$, $x_1 = g(x_0) = 2$

$x_2 = g(x_1) = 1.292$

$x_3 = 1.638$

$x_4 = 1.467$

$x_5 = 1.552$

$x_6 = 1.509$

$x_7 = 1.531$

$x_8 = 1.520$

$x_9 = 1.525$

$x_{10} = 1.523$

$x_{11} = 1.524$

$x_{12} = 1.524$

$\therefore$ Root $= 1.524$

Q.2

$x^2 + x - 2$

$\Rightarrow x = 2 - x^2 \Rightarrow x = g(x)$

let $x_0 = 0$

$x_1 = 2$

$x_2 = -2$

$x_3 = -2$

$\therefore$ Root $= -2$

Q.3 Evaluate the square root of 5 using the eq.

$$x^2 - 5 = 0$$

Sol

Let us reorganise the function as follow

$$x^2 = 5$$

or $x = 5/x$

and assume $x_0 = 1$. Then

$x_1 = 5$

$x_2 = 1$

$x_3 = 5$

$x_4 = 1$

$\Big\}$ oscillatory divergence

The process does not converge to the solution. This type of divergence is known as oscillatory divergence.

Now, let us consider another form of $g(x)$ as shown below.

$$x = x^2 + x - 5$$

let $x_0 = 0$

$x_1 = -5$

$x_2 = 15$

$x_3 = 235$

$x_4 = 55455$

Again it doesnot converge. Rather it diverges rapidly. This type of divergence is called Monotone divergence.

Again considering another form of $g(x)$

$$x = 5/x$$

$$x + x = 5/x + x$$

$$\Rightarrow 2x = x + 5/x$$

$$\Rightarrow x = \frac{x + 5/x}{2}$$

let $x_0 = 1$

$x_1 = 3$

$x_2 = 2.3333$

$x_3 = 2.2381$

$x_4 = 2.2361$

$x_5 = 2.2361$

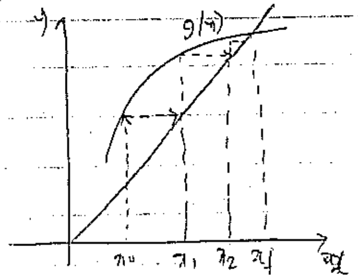
$\therefore$ This ~~the~~ time, the process converges rapidly - the solution. The square root of 5 is 2.2361

⑧

Convergence and divergence of Fixed Point Iteration:

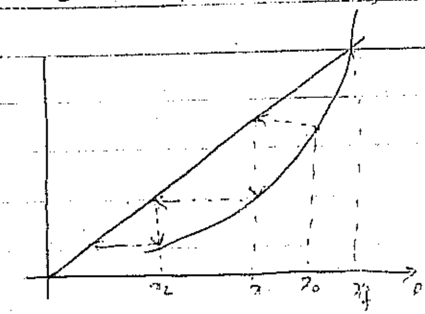
- Convergence & divergence depends on slope of $g(x)$

1. Monotone Convergence
⇒ If slope of $g(x)$ is +ve & $0 < g'(x) < 1$



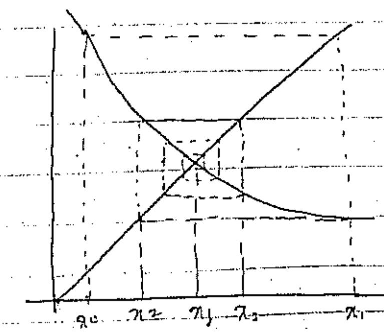
a) Monotone Convergence

2. Monotone Divergence
⇒ If slope of $g(x)$ is +ve & $g'(x) > 1$

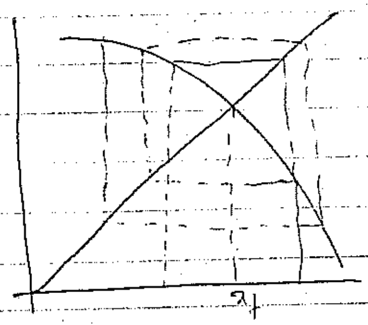


(b) Monotone divergence

3. Spiral / Oscillating Convergence
⇒ If slope of $g(x)$ is -ve and $-1 < g'(x) < 0$



4. Spiral Divergence
⇒ If slope of $g(x)$ is -ve & $g(x)$ is < -1



SYNTHETIC DIVISION:

Polynomial of degree n can be expressed as.

$$P(x) = (x - x_r)\, q(x).$$

where x_r is the root of the polynomial $p(x)$ and $q(x)$ is the quotient polynomial of degree $n-1$. Once a root is found, we can use this fact to obtain a lower degree polynomial $q(x)$ by dividing $p(x)$ by $(x - x_r)$. This process is known as synthetic division. This division is continued until the degree is reduced to one. The activity of reducing degree is ~~refer~~ referred as deflation.

Let, $\quad p(x) = \sum_{i=0}^{n} a_i x^i$

and $\quad q(x) = \sum_{i=0}^{n-1} b_i x_i$

If, $\quad P(x) = (x - x_r) q(x)$ then.

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = (x - x_r)(b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \cdots b_1 x + b_0) \quad \text{—(1)}$$

Comparing the corresponding coefficient we get,

$a_n = b_{n-1}$

$a_{n-1} = b_{n-2} - x_r b_{n-1}$

⋮

$a_1 = b_0 - x_r b_1$

$a_0 = -x_r b_0$

i.e. $\quad a_i = b_{i-1} - x_r b_i \quad , \quad i = n, n-1, \cdots 0$

where $b_n = 0$

∴ $\boxed{b_{i-1} = a_i + x_r b_i}$ $\quad , i = n \cdots 1$, $b_n = 0$

Q The polynomial equation

$$P(x) = x^3 - 7x^2 + 15x - 9 = 0$$

has a root at $x=3$. Find the quotient polynomial $q(x)$ such that

$$P(x) = (x-3)\, q(x)$$

Soln

Here we have,

$a_3 = 1,\ a_2 = -7,\ a_1 = 15$ & $a_0 = -9$

$b_3 = 0$

$$b_{i-1} = a_i + x_r b_i$$

So,

$$b_2 = a_3 + b_3 \times x_r = 1 + 0 \times 3 = 1$$

$$b_1 = a_2 + b_2 \times 3 = -7 + 1 \times 3 = -4$$

$$b_0 = a_1 + b_1 \times 3 = 15 + (-12) = 3$$

Thus the polynomial eqn $q(x)$ is.

$$q(x) = b_2 x^2 + b_1 x + b_0 = 0$$

i.e $\quad x^2 - 4x + 3 = 0$

Horner's Method:

The polynomial eqn is given by

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_0 + \sum_{i=1}^{n} a_i x_i \quad \text{——} \ (1)$$

The polynomial can also be evaluated as.

$$f(x) = ((\cdots((a_n x + a_{n-1})x + a_{n-2})x \cdots + a_1)x + a_0)$$

Here the innermost expression $a_n x + a_{n-1}$ is evaluated first. The resulting value constitutes a multiplicand for the expression at the next level. This approach needs a total of n additions and n multiplications.

~~This rule is called Horner's rule or nested multipli~~

Horner's rule, also known as nested multiplication, is implemented using following process

$$P_n = a_n$$

$$P_{n-1} = P_n x + a_{n-1}$$

$$\vdots$$

$$P_j = P_{j+1} x + a_j$$

$$\vdots$$

$$P_1 = P_2 x + a_1$$

$$\therefore f(x) = P_0 = P_1 x + a_0$$

Q. Evaluate the polynomial

$$f(x) = x^3 - 4x^2 + x + 6$$

using Horner's rule at $x=2$

Soln

Here, $n=3$, $a_3 = 1$, $a_2 = -4$, $a_1 = 1$ & $a_0 = 6$

$\therefore P_3 = a_3 = 1$

$P_2 = P_3 \times x + a_2 = 1 \times 2 + (-4) = -2$

$P_1 = P_2 \times x + a_1 = -2 \times 2 + 1 = -3$

$P_0 = P_1 \times x + a_0 = -3 \times 2 + 6 = 0$

$\therefore f(2) = 0$

2. INTERPOLATION AND APPROXIMATION

Polynomial Interpolation

x	x_0	x_1	......	x_n
y	y_0	y_1		y_n

$\Rightarrow f(x_i)$

⇒ A polynomial $p(x_i) = y$, when $0 \le i \le n$ is said to be interpolate the table.

⇒ The points x_i are called nodes.

Types of Interpolation

1. Lagrange's Interpolation

Given a set of $(n+1)$ tabulated values (x_0, y_0) (x_1, y_1), (x_2, y_2) ---- (x_n, y_n) of a function $y = f(x)$ whose expressive nature is unknown.

Corresponding to above given set of $(n+1)$ tabulated values, we can construct a polynomial $P_n(x)$ of degree 'n' to interpolate the function. So we can now consider $P_n(x)$ as.

$$P_n(x_k) = f_k(x_k) \quad \text{for } k = 0, 1, 2, \ldots, n$$

This is called an Interpolation function.

Let us Consider a second-order polynomial of the form,

$$P_2(x) = b_1 (x - x_0)(x - x_1) + b_2 (x - x_1)(x - x_2) + b_3 (x - x_0)(x - x_2)$$

where b_1, b_2, b_3 are constants to be determined.

at $x = x_0$ i.e (x_0, y_0)

$$P_2(x_0) = b_2 (x_0 - x_1)(x_0 - x_2) = y_0$$

$$\therefore b_2 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)}$$

at $x = x_1$ i.e (x_1, y_1)

$$P_2(x_1) = b_3 (x_1 - x_2)(x_1 - x_0) = y_1$$

$$\therefore b_3 = \frac{y_1}{(x_1 - x_2)(x_1 - x_0)}$$

at (x_2, y_2)

$$P_2(x_2) = b_1 (x_2 - x_0)(x_2 - x_1) = y_2$$

$$\therefore b_1 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

Substituting the value of b_1, b_2 & b_3 in eq^n ①, we get

$$P_2(x) = y_0 \cdot \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \cdot \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \cdot \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \quad \text{--- } ②$$

Eq^n ② can be represented as,

$$P_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$$

$$= \sum_{i=0}^{2} y_i l_i(x) \quad \text{where}$$

$$l_i(x) = \prod_{j=0, j \neq i}^{2} \frac{(x - x_j)}{(x_i - x_j)}$$

In general form,

$$P_n(x) = \sum_{i=0}^{n} y_i l_i(x) \quad \text{where } l_i(x) = \prod_{j=0, j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

or,

$$\boxed{P_n(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}}$$

is called Lagrange Interpolation polynomial.

Q Find the Lagrange's interpolation polynomial as

x_i	0	1	2	3
$f(x_i)$	0	1.783	6.3891	19.0855

where, $f(x_i) = e^{x}-1$

Use the polynomial to get $e^{1.5}$

Soln

Here given

i	0	1	2	3
x_i	0	1	2	3
$f(x_i)$	0	1.7183	6.3891	19.0855

i.e 4 points ⇒ 3^{rd} order i.e $n=3$

we know Lagrange's interpolation is

$$P_n(x) = \sum_{i=0}^{n} y_i\, l_i(x) \quad \text{where } l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Here $n=3$,

$$P_3(x) = \sum_{i=0}^{3} y_i\, l_i(x)$$

$$= y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x) \quad \text{—(1)}$$

Now,

$$l_0(x) = \cancel{\frac{x-x_0}{x_0-x_0}}^{1} \times \frac{x-x_1}{x_0-x_1} \times \frac{x-x_2}{x_0-x_2} \times \frac{x-x_3}{x_0-x_3}$$

$$= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = \frac{(x^2-3x+2)(x-3)}{-6}$$

$$= \frac{x^3-3x^2+2x-3x^2+9x-6}{-6} = \frac{x^3-6x^2+11x-6}{-6}$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} \times \cancel{\frac{x-x_1}{x_1-x_1}}^{1} \times \frac{x-x_2}{x_1-x_2} \times \frac{x-x_3}{x_1-x_3}$$

$$= \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{(x^2-2x)(x-3)}{2}$$

$$= \frac{x^3-3x^2-2x^2+6x}{2} = \frac{x^3-5x^2+6x}{2}$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0} \times \frac{x-x_1}{x_2-x_1} \times \cancel{\frac{x-x_2}{x_2-x_2}}^{1} \times \frac{x-x_3}{x_2-x_3}$$

$$= \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = \frac{(x^2-x)(x-3)}{-2}$$

$$= \frac{x^3-3x^2-x^2+3x}{-2} = \frac{x^3-4x^2+3x}{-2}$$

$$\cancel{l_4(x)=}\ l_3(x) = \frac{x-x_0}{x_3-x_0} \times \frac{x-x_1}{x_3-x_1} \times \frac{x-x_2}{x_3-x_2} \times \cancel{\frac{x-x_3}{x_3-x_3}}^{1}$$

$$= \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{(x^2-x)(x-2)}{6} = \frac{x^3-3x^2+2x}{6}$$

Now, substituting the values in eqn (1)

$$P_3(x) = 0 \times \frac{(x^3-6x^2+11x-6)}{-6} + (1.7183) \times \frac{x^3-5x^2+6x}{2}$$

$$+ 6.3891 \times \frac{(x^3-4x^2+3x)}{-2} + 19.0855 \times \frac{x^3-3x^2+2x}{6}$$

$$= 0 + \frac{1.7183x^3 - 8.591x^2 + 10.3098x}{2} +$$

$$\frac{6.3891x^3 - 25.5564x^2 - 19.1673x}{-2} + \frac{19.0855x^3 - 57.2569x^2 + 38.171x}{-6}$$

$$= 5.1849x^3 + 25.7745x^2 + 30.9294x + 19.3391x^3 + \frac{76.669x^2 - 57.5019x + 19.0855x^3 - 57.256x^2 + 38.171x}{6}$$

$$= \frac{4.8513x^3 - 60.3619x^2 + 11.5985x}{6}$$

$$\therefore P_3(x) = 0.8085x^3 - 1.0603x^2 + 1.93308x$$

Again,

For 2nd part,

$$P_3(1.5) = 2.7289 - 2.3857 + 2.8996$$
$$= 3.2428$$

$$P_3(1.5) = 3.2428$$

Now,

$$e^{1.5} - 1 = 3.2428$$

$$\boxed{e^{1.5} = 4.2428}$$

Ans

Q.2 Find the polynomial $f(x)$ by using Lagrange's formula & hence find $f(3)$ for,

x:	0	1	2	5
$f(x)$:	2	3	12	147

Sol^n

4th points $\Rightarrow$ 3rd order

$$P_3(x) = \sum_{i=0}^{3} y_i l_i(x) \quad \text{where,} \quad l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{3} \frac{x - x_i}{x_i - x_j}$$

or $P_3(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x)$ ——— (1)

Now,

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}$$

$$= \frac{(x^2-3x+2)(x-5)}{-10} = \frac{x^3-5x^2-3x^2+15x+2x-10}{-10}$$

$$= \frac{x^3-8x^2+17x-10}{-10}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}$$

$$= \frac{(x^2-2x)(x-5)}{4} = \frac{x^3-5x^2-2x^2+10x}{4} = \frac{x^3-7x^2+10x}{4}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}$$

$$= \frac{(x^2-x)(x-5)}{-6} = \frac{x^3-6x^2+5x}{-6}$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}$$

$$= \frac{(x^2-x)(x-2)}{60} = \frac{x^3-3x^2+2x}{60}$$

Now, Substituting all these values in eq ①,

$$P_3(x) = 2x\left(\frac{x^3-8x^2+17x-10}{-10}\right) + 3x\left(\frac{x^3-7x^2+10x}{4}\right) + 12x\left(\frac{x^3-6x^2+5x}{-6}\right)$$

$$+ 147x\left(\frac{x^3-3x^2+2x}{60}\right)$$

$$= \frac{2x^3-16x^2+34x-20}{-10} + \frac{3x^3-21x^2+30x}{4} + \frac{12x^3-72x^2+60x}{-6}$$

$$+ \frac{147x^3-441x^2+294x}{60}$$

$$= -0.2x^3+1.6x^2-3.4x+2+0.75x^3-5.25x^2+7.5x-2x^3+12x^2-10x$$
$$+2.45x^3-7.35x^2+4.9x$$

$\therefore P_3(x) = x^3+x^2-x+2$

& $f(3) = 3^3+3^2-3+2 = 35$

Ans

Q. Find the missing term in the following table using interpolation.

x:	0	1	2	3	4
y:	1	3	9	----	91

Ans: $f(x) = 2x^3-4x^2+4x+1$

at $x=3, y=31$

Notes:

1) More arithmatic operation are required.
2) With addition or elimination of one or more points from the table, requires solve of the same problem from the begining.

2. NEWTON'S INTERPOLATION.

Two types: → 1. Divided Difference
→ 2. Simple Difference

1. Divided Difference:

Consider a set of $(n+1)$ tabulated values (x_0, y_0), (x_1, y_1) ---- (x_n, y_n) of a function $y = f(x)$ whose explicit nature is not known, and.

Let $P_n(x)$ be an interpolating polynomial of the function $y = f(x)$ such that $P_n(x)$ agree at the given tabulated values. Then $P_n(x)$ can be written as,

$$P_n(x) = a_0 + (x-x_0)a_1 + (x-x_0)(x-x_1)a_2 + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})a_n \quad \text{—— (1)}$$

where $a_0, a_1, \dots a_n$ are constants to be determined.

Substituting successively we get

at $(x_0, y_0) \Rightarrow P_n(x_0) = a_0 + 0 = y_0 = f(x_0)$

at $(x_1, y_1) \Rightarrow P_n(x_1) = a_0 + a_1(x_1 - x_0) + 0 = y_1$

$$\therefore a_1 = \frac{y_1 - y_0}{x_1 - x_0} = f[x_0, x_1]$$

at (x_2, y_2)

$$\Rightarrow P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) + 0 = y_2$$

$$\therefore a_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} = f[x_0, x_1, x_2]$$

Similarly $a_n = f[x_0, x_1, x_2, \dots, x_n]$

Here, a_1 represents ~~first~~ first divided difference, a_2 the second divided difference and so on.

Substituting the value of a_i in eq^n (1), we get

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots + f[x_0, x_1, \dots x_n](x - x_0)(x - x_1)\dots(x - x_{n-1})$$

This can be written as

$$\boxed{P_n(x) = \sum_{i=1}^{n} f[x_0, \dots x_i] \prod_{j=0}^{n-1} (x - x_j)}$$

this is called Newton's divided difference interpolation polynomial.

Divided difference table. (for 4th order)

x	y	1st d.d Δy	2nd d.d $\Delta^2 y$	3rd d.d $\Delta^3 y$	4th d.d $\Delta^4 y$
x_0	y_0				
		$\frac{y_1 - y_0}{x_1 - x_0} = \Delta y_0$			
x_1	y_1		$\frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \Delta^2 y_0$		
		$\frac{y_2 - y_1}{x_2 - x_1} = \Delta y_1$		$\frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0} = \Delta^3 y_0$	
x_2	y_2		$\frac{\Delta y_2 - \Delta y_1}{x_3 - x_1} = \Delta^2 y_1$		$\frac{\Delta^3 y_1 - \Delta^3 y_0}{x_4 - x_0} = \Delta^4 y_0$
		$\frac{y_3 - y_2}{x_3 - x_2} = \Delta y_2$		$\frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4 - x_1} = \Delta^3 y_1$	
x_3	y_3		$\frac{\Delta y_3 - \Delta y_2}{x_4 - x_2} = \Delta^2 y_2$		
		$\frac{y_4 - y_3}{x_4 - x_3} = \Delta y_3$			
x_4	y_4				

Q.1 Given the values

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

evaluate $f(9)$ using Newton's divided difference formula.

Sol^n

The divided difference table is.

x	y (a_0)	Δ (a_1)	Δ^2 (a_2)	Δ^3 (a_3)	Δ^4 (a_4)
5	150				
		$\frac{392-150}{7-5} = 121$			
7	392		$\frac{265-121}{11-5} = 24$		
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$	
11	1452		$\frac{457-265}{13-7} = 32$		$\frac{1-1}{17-5} = 0$
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$	
13	2366		$\frac{709-457}{17-11} = 42$		
		$\frac{5202-2366}{17-13} = 709$			
17	5202				

i.e $f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$
$+ a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)$

at $x = 9$

$f(9) = 150 + 121(9-5) + 24(9-5)(9-7) + 1(9-5)(9-7)(9-11)$
$+ 0(9-5)(9-7)(9-11)(9-13)$

$= 810$

Q.2 Using Newton's divided difference formula, find the missing value from the table

x:	1	2	4	5	6
y:	14	15	5	...	9

Sol^n

x	y (a_0)	Δ (a_1)	Δ^2 (a_2)	Δ^3 (a_3)
1	14			
		$\frac{15-14}{2-1} = 1$		
2	15		$\frac{-5-1}{4-1} = -2$	
		$\frac{5-15}{4-2} = -5$		$\frac{7/4+2}{6-1} = 3/4$
4	5		$\frac{2+5}{6-2} = 7/4$	
		$\frac{9-5}{6-4} = 2$		
6	9			

Now, Applying Newton's divided difference formula,

$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$
$= 14 + 1(x-1) + (-2)(x-1)(x-2) + \frac{3}{4}(x-1)(x-2)(x-4)$

Now, at $x=5$, we get

$f(5) = 14 + 1(5-1) - 2(5-1)(5-2) + \frac{3}{4}(5-1)(5-2)(5-4)$

$= 3$

Hence the missing value is 3.

Extra Questions

Q.1) Using Newton's divided difference formula, evaluate $f(8)$ & $f(15)$. Given

x:	4	5	7	10	11	13
y:	48	100	294	900	1210	2028

Ans: $f(8) = 448$ & $f(15) = 3150$

Q.2) ~~Usi~~ Determine $f(x)$ as a polynomial in x for the following data:

x:	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

Ans. $f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$

Q.3) Estimate $\log 2.5$ using 2nd order newton's polynomial

x:	1	2	3	4
$\log x$:	0	0.3010	0.4771	0.6021

Ans $\log(2.5) = f(2.5) = 0.4047$

2. Simple Difference:
- → Forward Difference
- → Backward "

2.1) Newton Forward Difference:

Here the function are tabulated at equal interval i.e $x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = h$ a constant with tabulation at equal intervals. A difference table for n points is expressed as,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$y_1 - y_0 = \Delta y_0$			
x_0+h	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$		
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$	
x_0+2h	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$		$\Delta^3 y_1 - \Delta^3 y_0 =$
		$y_3 - y_2 = \Delta y_2$		$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$	
x_0+3h	y_3		$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$		
		$y_4 - y_3 = \Delta y_3$			
x_0+4h	y_4				

$\therefore \boxed{\Delta y_i = y_{i+1} - y_i}$

– first forward difference

Similarly,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - y_1 - y_1 + y_0$$
$$= y_2 - 2y_1 + y_0$$

$\therefore \boxed{\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i}$

"Second ~~order~~ forward difference"

→ The polynomial that passes through a group of equidistance points, i.e Newton's Gregory forward polynomial, we can write in terms of index S, such that

$$S = \frac{x - x_0}{h} \qquad x = x_0 + S.h$$

$$\therefore \boxed{P_n(x) = y_0 + \frac{S\Delta y_0}{1!} + \frac{S(S-1)}{2!}\Delta^2 y_0 + \frac{S(S-1)(S-2)}{3!}\Delta^3 y_0 + \cdots + \frac{S(S-1)(S-2)(S-3)\cdots(S-(n-1))}{n!}\Delta^n y_0}$$

This relation is known as "Newton's Gregory Forward Interpolation" formula.

→ It is applied within when the required point is close to the start of table.

2) Newton's Backward Difference.

$$P_n(x) = y_n + S.\nabla y_n + \frac{S(S+1)}{2!}\nabla^2 y_n + \frac{S(S+1)(S+2)}{3!}\nabla^3 y_n \cdots$$

where $S = \frac{x - x_n}{h}$

→ Simple-difference table

x	y	1st.S.d. Δy	2nd.S.d. $\Delta^2 y$	3rd.S.d. $\Delta^3 y$
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_3	y_3			

Forward difference

Backward difference

$y_n \quad \nabla y_n \quad \nabla^2 y_n \quad \nabla^3 y_n$

Note:
- By default use forward difference if not clarified in question.

Q.1) Estimate the value of function $x=0.16$ from following tabulated function

x	0.1	0.2	0.3	0.4
y	1.005	1.020	1.045	1.081

Soln

Here, $h = 0.2-0.1 = 0.1$

$$\& \; s = \frac{x-x_0}{h} = \frac{0.16-0.1}{0.1} = 0.6$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.005			
		0.015		
0.2	1.020		0.010	
		0.025		0.001
0.3	1.045		0.011	
		0.036		
0.4	1.081			

Now,

$$P_3(0.16) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0$$

$$= 1.005 + 0.6\times 0.015 + \frac{0.6(0.6-1)}{2}\times 0.010$$

$$+ \frac{0.6(0.6-1)(0.6-2)}{6}\times 0.001$$

$$= 1.0128$$

Q.2) The ~~table~~ gives the distance in miles of the visible ~~earth's surf~~ horizon for the given heights in feet above the earth surface:

Forward ... Backward

x=height:	100	150	200	250	300	350	400
y=distance:	10.63	13.03	15.04	16.81	18.42	19.90	21.27

find the values of y when (i) $x = 218$ ft, (ii) $x = 410$ ft

Sol: The difference ~~table~~ table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
100	10.63						
		2.4					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

i) $x = 218$ ft

Soln $x = 218$ ft lies at back in the table so we use Newton's forward difference formula,

$x_0 = 200$ then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$,

$\Delta^3 y_0 = 0.03$ & $\Delta^4 y_0 = -0.01$

Since, $x = 218$ & $h = 50$

$\therefore s = \dfrac{x - x_0}{h} = \dfrac{218 - 200}{50} = 0.36$

$\therefore$ Using Newton's forward difference formula, we get

$$y(218) = y_0 + s\,\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 y_0$$

$$\therefore y(218) = 15.04 + 0.36(1.77) + \frac{0.36(0.36-1)}{2}(-0.16)$$

$$+ \frac{0.36(0.36-1)(0.36-2)}{6} \times (0.03) + \frac{0.36(0.36-1)(0.36-2)(0.36-3)}{24} \times (-0.01)$$

$= 15.04 + 0.637 + 0.019 + 0.002 + 0.0004$

$= 15.698$ miles

ii) $x = 410$ ft

Soln Since, $x = 410$ ft lies near the end of table, we use Newton's backward difference formula.

$\therefore$ Taking $x_n = 400$, $s = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

$y_n = 21.27$, $\nabla y_n = 1.37$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$ & $\nabla^4 y_n = -0.07$

$\nabla^5 y_n = 0.04$, $\nabla^6 y_n = 0.02$

Using Newton's backward difference formula gives,

$$f(x) = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 y_n$$

$$+ \frac{s(s+1)(s+2)(s+3)(s+4)}{5!}\nabla^5 y_n + \frac{s(s+1)(s+2)(s+3)(s+4)(s+5)}{6!}\nabla^6 y_n$$

$$\& \; f(410) = 21.27 + 0.2 \times (0.37) + \frac{0.2(0.2+1)}{2} \times (-0.11)$$

$$+ \frac{0.2(0.2+1)(0.2+2)}{6} \times (0.02) + \frac{0.2(0.2+1)(0.2+2)(0.2+3)}{24} \times (-0.07)$$

$$+ \frac{0.2(0.2+1)(0.2+2)(0.2+3)(0.2+4)}{120} \times (0.04)$$

$$+ \frac{0.2(0.2+1)(0.2+2)(0.2+3)(0.2+4)(0.2+5)}{720} \times (0.02)$$

$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007 + 0.002 + 0.0001$

$= 21.5340$ miles

Extra questions

Q.1) From the following table, estimate the number of ~~stds~~ students who obtained marks between 40 & 50

Marks:	30-40	40-50	50-60	60-70	70-80
No of students:	31	42	57	35	31

clue ⇒ cumulative freq. table

x :	40	50	60	70	80
y :	31	73	124	159	190

$y(45) = 47.87 \approx 48$

std having marks < 40 = 31

∴ 40-45 = 48 - 31 = 17

Q.2) Find the cubic polynomial which takes following values & evaluate $f(4)$

x:	0	1	2	3
f(x):	1	2	1	10

→ forward $f(x) = 2x^3 - 7x^2 + 6x + 1$

→ backward $f(4) = 41$ which is equal to f(x) when x = 4

Q.3) Find $\sin\theta$ at $\theta = 25°$ & $55°$

θ :	10	20	30	40	50
sinθ :	0.1736	0.3420	0.5000	0.6428	0.7660

Ans : $\sin 25° = 0.42259$

$\sin 55° = 0.8189$

3. CUBIC SPLINE INTERPOLATION:

Formula of cubic spline interpolation (s):

1) $h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1} a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$ — (1)

where, $a_0 = a_n = 0$

and, $h_i = x_i - x_{i-1}$

2) $S_i(x) = \frac{a_{i-1}}{6h_i}\left(h_i^2 u_i - u_i^3\right) + \frac{a_i}{6h_i}\left(u_{i-1}^3 - h_i^2 \cdot u_{i-1}\right)$

$+ \frac{1}{h_i}\left(f_i u_{i-1} - f_{i-1} u_i\right)$ — (2)

Where $u_i = x - x_i$

Q Given the data points:

x:	4	9	16
y:	2	3	4

Estimate the functional value f at x=7 using cubic spline

Sol^n $h_i = x_i - x_{i-1}$ & $u_i = x - x_i$

Here, there are 3 points i.e. a_0, a_1, a_2

where $a_0 = a_2 = 0$ [1st & last are zero] for formula 1

① $\Big|_{i=1} \Rightarrow h_1 \cancel{a_0} + 2a_1(h_1+h_2) + \cancel{h_2 a_2} = 6\left[\frac{f_2-f_1}{h_2} - \frac{f_1-f_0}{h_1}\right]$

$h_1 = x_1 - x_0$ & $h_2 = x_2 - x_1$

$= 9-4 = 5$ $\quad = 16-9 = 7$

$\Rightarrow 2a_1(5+7) = 6\left[\frac{4-3}{7} - \frac{3-2}{5}\right]$

$\Rightarrow 24a_1 = 6\left[\frac{1}{7} - \frac{1}{5}\right]$

$\therefore a_1 = -0.0143$

for next point

② $\Big|_{i=1} \Rightarrow S_1(x) = \frac{\cancel{a_0}}{6h_1}(h_1^2 u_1 - u_1^3) + \frac{a_1}{6h_1}(u_0^3 - h_1^2 u_0)$

$+\frac{1}{h_1}(f_1 u_0 - f_0 u_1)$

$u_0 = x - x_0$ & $u_1 = x - x_1$

$= x-4$ $\quad = x-9$

~~at x=7~~

$\therefore S_1(x) = \frac{-0.0143}{6\times5}\left[(x-4)^3 - 5^2(x-4)\right]$

$+\frac{1}{5}\left[3(x-4) - 2(x-9)\right]$

at $x = 7$

$S_1(7) = \frac{-0.0143}{6\times5}\left[27-75\right] + \frac{1}{6}\left[9+4\right]$

$= 2.6228$

4. LEAST SQUARE APPROXIMATION:

Mathematical eqn of st. line is

$$y = a + bx = f(x)$$

where a & b are constants to be determined.

Thus

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

& $$a = \frac{\sum y_i - b \sum x_i}{n}$$

Q Fit a st. line to the following set of data,

x	1	2	3	4	5
y	3	4	5	6	8

Soln

x_i	y_i	x_i^2	$x_i y_i$
1	3	1	3
2	4	4	8
3	5	9	15
4	6	16	24
5	8	25	40
$\sum x_i = 15$	$\sum y_i = 26$	$\sum x_i^2 = 55$	$\sum x_i y_i = 90$

Now,

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$= \frac{5 \times 90 - 15 \times 26}{5 \times 55 - (15)^2}$$

$$= 1.2$$

& $$a = \frac{\sum y_i - b \sum x_i}{n}$$

$$= \frac{26 - 1.2 \times 15}{5}$$

$$= 1.6$$

Therefore, the linear eqn is,

$$y = a + bx$$
$$= 1.6 + 1.2x$$

3. NUMERICAL DIFF. AND INTEG.

1. MAXIMA AND MINIMA:

⇒ Given $(n+1)$ points

⇒ objective to find x at which f is max^m or min^m

⇒ Use Newtons simple forward diff.

Q. From the table below, for what value of x, y is minimum? Also find the value of y.

x:	3	4	5	6	7	8
y:	0.205	0.240	0.259	0.262	0.250	0.224

Soln

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3	0.205					
		0.035				
4	0.240		-0.016			
		0.019		0.0		
5	0.259		-0.016		0.001	
		0.003		0.001		-0.001
6	0.262		-0.015		0	
		-0.012		0.001		
7	0.250		-0.014			
		-0.026				
8	0.224					

Now,

Newtons forward diff. formula is,

$$y = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots$$

~~at $x = x_0$ we get~~

or, $y = 0.205 + s(0.035) + \frac{s(s-1)}{2}(-0.016) + \cdots$

diff w.r.t s we get,

$$\frac{dy}{ds} = 0 + 0.035 + \left(\frac{2s-1}{2}\right)(0.016)$$

for y to be minimum, $\frac{dy}{ds} = 0$

i.e. $0.035 + \frac{(2s-1)}{2}(-0.016) = 0$

or $(2s-1) = \frac{0.035 \times 2}{0.016}$

$\therefore s = 2.6875$

$\therefore x = x_0 + sh$

$= 3 + 2.6875 \times 1$

$= 5.6875$

Hence y is ~~maxm~~ min^m at $x = 5.6875$.

putting $s = 2.6875$ in eqn ① we get the minimum value of y

$$y = 0.205 + 2.6875 \times 0.035 + \frac{1}{2}(2.6875 \times 1.6875) \times (-0.016)$$

$$= 0.2623$$

9

1st

Simple difference: Forward

$$P_n(x) = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\frac{d}{dx}[P_n(x)] = \frac{d[P_n(x)]}{ds} \cdot \frac{ds}{dx}$$

$x = x_0 + s \cdot h$

$\therefore dx = 0 + ds \cdot h$

$\therefore \frac{ds}{dx} = \frac{1}{h}$

$$= \frac{1}{h}\left[0 + \Delta y_0 + \frac{2s-1}{2}\Delta^2 y_0 + \frac{3s^2-6s+2}{6}\Delta^3 y_0 + \cdots\right]$$

$$\& \frac{d^2[P_n(x)]}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}P_n(x)\right] \cdot \frac{ds}{dx}$$

$$= \frac{1}{h^2}\left[0 + 0 + \Delta^2 y_0 + \frac{6s-6}{6}\Delta^3 y_0 + \cdots\right]$$

Q. Find the max^m & min^m value of y from the following data

x:	-2	-1	0	1	2	3	4
y:	2	-0.25	0	-0.25	2	15.75	56

Sol^n

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
-2	2						
		-2.25					
-1	-0.25		2.5				
		0.25		-3			
0	0		-0.5		6		
		-0.25		3		0	
1	-0.25		2.5				0
		2.25		9	6	0	
2	2		11.5				
		13.75		15	6		
3	15.75		26.5				
		40.25					
4	56						

Sol^n

~~takin~~ Newtons forward diff.

$$y = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots$$

taking derivative w.r.t. x

$$\frac{dy}{dx} = \frac{dy}{ds}\cdot\frac{ds}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2s-1}{2!}\Delta^2 y_0 + \frac{3s^2-6s+2}{3!}\Delta^3 y_0 + \frac{4s^3-18s^2+22s-6}{24}\Delta^4 y_0\right]$$

at $x_0 = 0$ we get $x = x_0 + sh$

$\therefore x = sh$

$x = s \times 1$

$\therefore s = x$

So,

$$\frac{dy}{dx} = \frac{1}{1}\left[-0.25 + \frac{2x-1}{2}\times(2.5) + \frac{3x^2-6x+2}{6}\times 9 \cdots\right.$$

$$\left. + \frac{4x^3-18x^2+22x-6}{24}\times 6\right]$$

$$= \left[-0.25 + 2.5x - 1.25 + \cancel{4.5x^2} - 9x + 3 + x^3 - \cancel{4.5x^2} + 5.5x - 1.5\right]$$

$$= x^3 - x$$

For y to be max^m or min^m $\frac{dy}{dx} = 0$

i.e. $x^3 - x = 0$

$\Rightarrow x = 0, 1$ & -1

Now,

$$\frac{d^2y}{dx^2} = 3x-1 = \begin{array}{l} -ve \text{ for } x=0 \\ +ve \text{ for } x=1 \\ \& +ve \text{ for } x=1 \end{array}$$

Since,

$$y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \cdots$$

$$y(0) = 0 + 0 + \frac{0(0-1)}{2}\times(2\cdot5)$$

$$= 0$$

$$\therefore y(0) = 0$$

thus y is max^m for $x=0$ & max^m ~~value~~ value of y at 0 is $y(0)=0$.

Again,

$$y(1) = 0 + 1\times(-0.25) + \frac{1(1-1)}{2!}\times2\cdot5 + \cdots$$

$$= -0.25$$

i.e. y is minimum for $x=1$ & minimum value of $y(1) = -0.25$

2. NEWTON-COTES METHOD:

Let the interval $[a,b]$ be divided into n sub intervals each of equal length h such that,

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Then clearly, $x_n = x_0 + sh$, where $s = 0,1,2,\ldots,n$

Then the integral,

$$I = \int_a^b f(x)dx, \text{ becomes as}$$

$$I = \int_{x_0}^{x_n} f(x)dx \quad \text{——} \quad (1)$$

Applying $y = f(x)$ by Newton's forward difference interpolation formula, we have

$$I = \int_{x_0}^{x_n} \left[y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \right] dx \quad \text{——} \quad (2)$$

when $x = x_0 + sh$ —— (3)

$dx = hds$

Now eq^n (2) becomes,

$$I = h\int_0^n \left[y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \cdots \right] ds$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \text{—} (4)$$

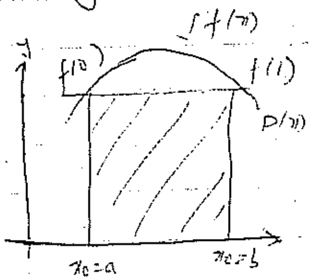
The relation (4) is the general formula to compute numerical integration based on Newton's forward difference interpolation formula.

Cases:

1) For $n=1$, we find
Trapezoidal rule for numerical integration q (6)

2) For $n=2$ i.e for two points we find,
Simpson's 1/3 rule for numerical integration q (7)

3) For $n=3$, we find (8)
Simpson's 3/8 rule for numerical integration q (7)



A) Trapezoidal Rule:

$$\begin{matrix} x_0 & x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & & \Downarrow & \Downarrow \\ y_0 & y_1 & y_2 & y_3 & \cdots & y_n & y_n \end{matrix}$$

Here, we take $n=1$ in the general formula (4)

Then all difference higher than the first will become zero and we obtain.

$$\int_{x_0}^{x_1} y\,dx = 1\cdot h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

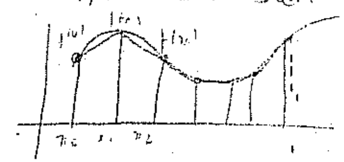
$$= \frac{h}{2}\left[y_0 + y_1\right]$$

Combining all these,

$$I = \int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \int_{x_2}^{x_3} y\,dx + \cdots \int_{x_{n-1}}^{x_n} y\,dx$$

$$= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n\right] \quad \text{---} (5)$$

The relation (5) is known as Trapezoidal rule with n-sub interval.



B) Simpson's 1/3 Rule:

It is obtained by taking $n=2$ in general formula (4). Then all the differences higher than 1st and 2nd will become zero. Then

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2(4-3)}{12}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}\left[(y_2 - y_1) - (y_1 - y_0)\right]\right]$$

$$= 2h\left[y_0 + y_1 - y_0 + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly for the interval $[x_2, x_4]$ we have.

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

For interval $[x_4, x_6]$ we have,

$$\int_{x_4}^{x_6} = \frac{h}{3}\left[y_4 + 4y_5 + y_6\right]$$

Proceeding similarly we obtain,

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Combining all these we obtain,

1/3rd

$$I = \int_{x_0}^{x_n} y\,dx = \frac{h}{3}\Big[y_0 + 4(y_1 + y_3 + \cdots y_{n-1})$$
$$+ 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n\Big] \quad \text{— (6)}$$

The relation (6) is known as Simpson's 1/3 rule. This rule required the division of whole range into an even no. of width n. The rule gives the correct value of the interval only if F(x) is a quadratic function.

c) Simpson's 3/8 Rule:

Taking n=3 in Newton-Cotes formula. Then the higher order differences other than upto 3rd differen becomes zero and we obtain,

$$\int_{x_1}^{x_3} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(6-3)}{12}\Delta^2 y_0 + \frac{3(3-2)^2}{24}\Delta^3 y_0\right.$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0)\right.$$
$$\left. + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3}{8}h\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Proceeding similarly we obtain

$$\int_{x_3}^{x_6} = \frac{3}{8}h\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$\int_{x_6}^{x_9} = \frac{3}{8}h\left[y_6 + 3y_7 + 3y_8 + y_9\right]$$

$$\vdots$$

$$\int_{x_{n-3}}^{x_n} = \frac{3}{8}h\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Combining all these we obtain,

$$I = \int_{x_0}^{x_n} y\,dx = \frac{3}{8}h\Big[(y_0+3y_1+3y_2+y_3)+(y_3+3y_4+3y_5+3y_6)$$

$$+(y_6+3y_7+3y_8+y_9)+\cdots+(y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)\Big]$$

$$=\frac{3}{8}h\Big[y_0+3(y_1+y_2+y_4+y_5+\cdots y_{n-1})+2(y_3+y_6+y_9+\cdots y_{n-3})$$

$$+y_n\Big] \quad \text{--- (7)}$$

This relation (7) is known as simpson's 3/8 rule.

Q Find the area bounded by the functions, x-axis from, $x = 7.47$ to $x = 7.52$ (using Trapezoidal Rule)

x:	7.47	7.48	7.49	7.50	7.51	7.52
y:	1.93	1.95	1.98	2.01	2.03	2.06
	y_0	y_1	y_2	y_3	y_4	y_5

Sol.

$h = 0.01$

$$\int_{7.47}^{7.52} = \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3+y_4)+y_5\right]$$

$$= \frac{0.01}{2}\left[1.93+2(1.95+1.98+2.01+2.03)+2.06\right]$$

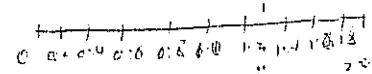
$$= \cancel{0.09985}\ 0.09965$$

Q Use the Trapezoidal rule to estimate the integral $\int_0^2 e^{x^2}dx$ taking the number 10 intervals

Sol.

let $y = e^{x^2}$

$$h = \frac{b-a}{n} = \frac{2-0}{10} = 0.2$$



By Trapezoidal rule, we have

$$\int_0^1 e^{x^2}dx = \frac{h}{2}\left[y_0+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)+y_{10}\right]$$

$$=\frac{0.2}{2}\Big[1+2(1.0408+1.1735+1.4333+1.8964+2.1782+4.2206+7.0993+12.9358+25.5337)+54.5981\Big]$$

$$\therefore \int_0^1 e^{x^2}dx = 17.6021$$

Q. Evaluate using Simpson's $\frac{1}{3}$ rule

$$\int_0^{\pi/2} \sqrt{\sin x}\, dx \text{ with } n=4$$

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4$

$0 \quad \frac{\pi}{8} \quad \frac{\pi}{4} \quad \frac{3\pi}{8} \quad \frac{\pi}{2}$

Soln

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{4} = \pi/8$$

$$y = \sqrt{\sin x}$$

Now

$$I = \int_0^{\pi/2} \sqrt{\sin x} = \frac{h}{3}\left[y_0 + 4(y_1+y_3) + 2(y_2) + y_4\right]$$

$$= \frac{\pi}{24}\left[0 + 4\left(\cancel{0.6326} + \cancel{0.9238} \atop 0.6186 + 0.9612\right) + 2\times\left(\cancel{0.7070} \atop 0.8409\right) + 1\right]$$

$$= \cancel{1.1783} \quad 1.1783$$

Q. Evaluate the integral $\int_0^1 \frac{x^2}{1+x^3}$ using Simpson's $\frac{1}{3}$rd.

let,

$$y = \frac{x^2}{1+x^3}$$

assume $n=4$

$0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

By Simpson's $\frac{1}{3}$rd rule, we have,

$$\int_0^1 \frac{x^2}{1+x^3}\, dx = \frac{h}{3}\left[y_0 + 4(y_1+y_3) + 2(y_2) + y_4\right]$$

$$= \frac{0.25}{3}\left[0 + 4(0.0616 + \cancel{0.397}\ 0.3956) + 2\times(0.2223) + 0.5\right]$$

$$= 0.2312$$

Q. Use Simpson's $3/8$ rule to evaluate

$$\int_0^{\pi/2} \sqrt{\sin(x)}\, dx \quad \text{assume } n=3$$

Sol^n

$$h = \frac{\pi/2 - 0}{3} = \pi/6$$

$0 \quad \pi/6 \quad \pi/3 \quad \pi/2$

Now, By Simpson's $3/8$ rule

$$I = \int_0^{\pi/2} \sqrt{\sin(x)}\, dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2) + y_3\right]$$

$$= \frac{3\pi}{48}\left[0 + 3(0.7072 + 0.9306) + 1\right]$$

$$= 1.61093$$

Q Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ Using simpson's 3/8th rule. ($n = 6$)

Sol^n

let, $y = \sin x - \log x + e^x$

$h = \frac{b-a}{n} = \frac{1.4-0.2}{6} = 0.2$

y_0	y_1	y_2	y_3	y_4	y_5	y_6
0.2	0.4	0.6	0.8	1.0	1.2	1.4

Now By Simpson's 3/8 rule

$$I = \int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) + y_6\right]$$

$$= \frac{3 \times 0.2}{8}\left[2.119 + 3(2.2792 + 2.6086 + 3.5598 + 4.1730) + 2 \times 3.0398 + 4.8945\right)$$

$$= 3.8217$$

Tutorial

Q.1) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Trapezoidal rule ($n=6$) Ans: 1.4108

Q.2) Evaluate the following integrals using Simpson's 1/3.

a) $\int_{-1}^{1} e^x dx$; $n=2$ b) $\int_0^{\pi/2} \sqrt{\sin x}\, dx$; $n=6$

Ans: 2.3621   Ans: 1.1783

Q.3) The velocity V of a particle at a distance S from a point on its path is given in the table below:

S(ft):	0	10	20	30	40	50	60
V(ft/sec):	47	58	64	65	61	52	38

Estimate the time taken to travel a distance of 60 ft by using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule

Ans: 1.06355 & 1.06445

$\int \frac{1}{V}$, $V = \frac{ds}{dt}$

$\frac{dt}{ds} = \frac{1}{V} = y$ (say)

ROMBERG INTEGRATION:

Error in quadrature formula, $E = \int_a^b y\,dx - \int_a^b P(x)\,dx$

where $P(x)$ is the polynomial representing function $y = f(x)$ in interval $[a, b]$

Error in Trapezoidal rule,

$$E = \frac{-(b-a)h^2}{12} f''(x) = Ch^2$$

where,

$$C = -\frac{(b-a) f''(x)}{12} \quad \& \quad f''(x) \text{ is max of } f''(x_0),$$

$f''(x_1), \ldots, f''(x_{n-1})$

Romberg method is done for better approximations.

Given,

$$I = \int_a^b f(x)\,dx$$

Using Trapezoidal rule, we calculate I_1 & I_2 for sub intervals of width h_1 & h_2 respectively then.

$$I = I_1 + Ch_1^2 \quad \text{——— (1)}$$

$$\& \quad I = I_1 + Ch_2^2 \quad \text{——— (2)}$$

Equating both eqns,

$$I_1 + Ch_1^2 = I_2 + Ch_2^2$$

$$\Rightarrow C = \frac{I_1 - I_2}{(h_2^2 - h_1^2)}$$

Substituting the values of 'C' on eqn (1) we get,

$$I = I_1 + \frac{(I_1 - I_2)}{(h_2^2 - h_1^2)} h_1^2 = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$$

which is better approximation of I

for, $h_1 = h$ & $h_2 = h/2$

$$I = \frac{I_1 \frac{h^2}{4} - I_2 h^2}{\frac{h^2}{4} - h^2} = \frac{I_1/4 - I_2}{-3/4}$$

$$= \frac{4I_2 - I_1}{3}$$

i.e. $I(h, h/2) = \dfrac{4I(h/2) - I(h)}{3}$

Using trapezoidal rule several times successively having h & applying in eqn (1) to each pair of values we get

$I(h)$			
	$I(h, h/2)$		
$I(h/2)$		$I(h, h/2, h/4)$	
	$I(h/2, h/4)$		$I(h, h/2, h/4, h/8)$
$I(h/4)$		$I(h/2, h/4, h/8)$	
	$I(h/4, h/8)$		
$I(h/8)$			

Q) Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to 3 decimal places.

Sol:

let $h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

(i) When $h = 0.5$, the values of $y = (1+x)^{-1}$ are.

x :	0	0.5	1
y :	1	0.6667	0.5

$\therefore I(h) = \frac{h}{2}[y_0 + 2(y_1) + y_2]$

$= \frac{0.5}{2}[1 + 2\times 0.6667 + 0.5]$

$= 0.70835$

(ii) when $h = 0.25$, the values of $y = (1+x)^{-1}$ are.

x :	0	0.25	0.5	0.75	1
y :	1	0.8	0.6667	0.5715	0.5

$\therefore I(h/2) = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4]$

$= \frac{0.25}{2}[1 + 2(0.8 + 0.6667 + 0.5715) + 0.5]$

$= 0.6971$

(iii) when $h = \frac{0.25}{2} = 0.125$

x:	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y:	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5715	0.5334	0.5

$\therefore I(h/4) = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$

$= \frac{0.125}{2}[1 + 2(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5715 + 0.5334) + 0.5]$

$= 0.6942$

Now, using Romberg formula.

$I(h, h/2) = \frac{4I(h/2) - I(h)}{3}$

$I(h) = 0.70835$

$I(h/2) = 0.6971$

$I(h/4) = 0.6942$

$I(h, h/2) = 0.6934$

$I(h/2, h/4) = 0.6932$

$I(h, h/2, h/4) = 0.6931$

$\therefore$ The value of integral $\int_0^1 \frac{dx}{1+x} = 0.693$

Q. Use Romberg Integration method, evaluate the integral $\int_1^2 \frac{dx}{x}$ correct upto 3 decimalplaces taking the initial subinterval size as $h = (b-a)/2$.

Soln: $h = \frac{b-a}{2} = \frac{2-1}{2} = 0.5$

i) when $h = 0.5$, the value of $y = x^{-1}$ are

x:	1	1.5	2
y:	1	0.6667	0.5

$\therefore I(h) = \frac{h}{2}[y_0 + 2y_1 + y_2]$

$= \frac{0.5}{2}[1 + 2\times 0.6667 + 0.5]$

$= 0.70835$

ii) When $h = \frac{0.5}{2} = 0.25$, the value of $y = x^{-1}$ are:

x:	1	1.25	1.5	1.75	2
y:	1	0.8	0.6667	0.5714	0.5

$\therefore I\left(\frac{h}{2}\right) = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4]$

$= \frac{0.25}{2}[1 + 2(0.8 + 0.6667 + 0.5714) + 0.5]$

$= 0.697$

iii) when $h = \frac{0.25}{2} = 0.125$

x:	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y:	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

$\therefore I\left(\frac{h}{4}\right) = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$

$= \frac{0.125}{2}[1 + 2(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333) + 0.5]$

$= \cancel{0.3939}\ 0.6941$

Now, using Romberg formula,

$I(h, h/2) = \frac{4I(h/2) - I(h)}{3}$

$I(h) = 0.70835$	$I(h, h/2) = \cancel{0.712133}$ 0.69322	$\cancel{0.15946}$ 0.69306
$I(h/2) = 0.697$	$I(h/2, h/4) = \cancel{0.2929}$ $= 0.6931$	
$I(h/4) = \cancel{0.3939}$ 0.6941		

$\therefore I = \int_1^2 \frac{1}{x}\,dx = 0.69306$

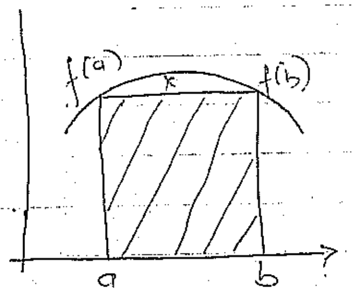
Correct ans $= \ln(2) = 0.69314$

GAUSSIAN INTEGRAL:

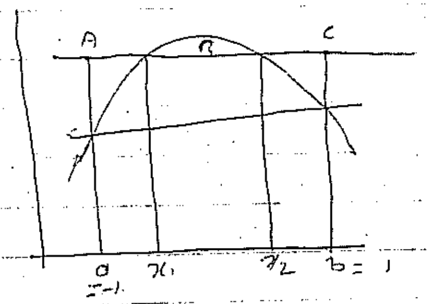
In Newton's cotes formula, the integration is performed by taking the functional values at equal interval of space & fitting polynomial through these function.

In case of Gaussian integration, the integration is performed by computing the functional values at non equal interval of x.

The domain of integration for such a rule is conventionally taken as $(-1,1)$



Trapezoidal



Gaussian Integration.

Gaussian integration formula is expressed as,

$$I_g = \int_{-1}^{1} f(x)\,dx = \omega_1 f_1(x) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n)$$

$$= \sum_{i=1}^{n} \omega_i f(x_i)$$

where ω_i = weights, x_i = abscissae

for $n = 2$,

There being $2n$ unknown parameters is given by

$$2n = 4 = (\omega_1, \omega_2, x_1, x_2)$$

Degree of polynomial $= 2n-1 = 3$

Assume that the integral will be exact upto a cubic polynomial. This implies that the function $1, x, x^2 \,\&\, x^3$ can be numerically integrated to obtain exact result.

$$f(x) = 1;\ f(x) = x;\ f(x) = x^2;\ f(x) = x^3$$

Now,

$$\sum_{i=1}^{n} \omega_i f(x_i) = \int_{-1}^{1} f(x)\,dx$$

$$\omega_1 f(x_1) + \omega_2 f(x_2) = \int_{-1}^{1} 1 \cdot dx$$

$$\omega_1 + \omega_2 = 2 \quad \text{———} \quad (i)$$

then $f(x) = x$

$$\omega_1 f(x_1) + \omega_2 f(x_2) = \int_{-1}^{1} x\,dx$$

$$\omega_1 x_1 + \omega_2 x_2 = \left.\frac{x^2}{2}\right|_{-1}^{1} = 0 \quad \text{———} \quad (ii)$$

Similarly,

$f(x) = x^2$ & $f(x) = x^3$

$$\omega_1 x_1^2 + \omega_2 x_2^2 = 2/3 \quad \text{———} \quad (iii)$$

$$\& \quad \omega_1 x_1^3 + \omega_2 x_2^3 = 0 \quad \text{———} \quad (iv)$$

on solving we get,

$\omega_1 = \omega_2 = 1$

$x_1 = -\frac{1}{\sqrt{3}}$

& $x_2 = \frac{1}{\sqrt{3}}$

$\int_{-1}^{1}$

Now the integral becomes,

$$I = \int_{-1}^{1} f(x)dx = f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

also known as Gauss-Legendre formula,

When $n=3$ (i.e Gaussian 3 point formula)

$$I_g = \int_{-1}^{1} f(x)dx = \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3)$$

where,

$x_1 = -0.77460 (-\sqrt{3/5})$, $\omega_1 = 0.5555$

$x_2 = 0.0$, $\omega_2 = 0.88889$

$x_3 = 0.77460 (\sqrt{3/5})$, $\omega_3 = 0.5555$

Table

n	weight (ω_i)	x_i
2	$\omega_1 = \omega_2 = 1$	$x_1 = -1/\sqrt{3}$ $x_2 = 1/\sqrt{3}$
3	$\omega_1 = 5/9$ $\omega_2 = 8/9$ $\omega_3 = 5/9$	$x_1 = -\sqrt{3/5}$ $x_2 = 0$ $x_3 = \sqrt{3/5}$
4	$\omega_1 = 0.34785$ $\omega_2 = 0.65214$ $\omega_2 = 0.65214$ $\omega_4 = 0.34785 = \omega_1$	$x_1 = -0.86114$ $x_2 = -0.33998$ $x_3 = 0.33998$ $x_4 = 0.86114$

Q Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using two point Gauss legendre formula.

Soln we know,

$n = 2$

$\omega_1 = \omega_2 = 1$, $x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$

$$I_g = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{dx}{1+x^2}$$

$$f(x) = \frac{1}{1+x^2}$$

$$\therefore I_g = f(-1/\sqrt{3}) + f(1/\sqrt{3}) = 0.75 + 0.75 = 1.5$$

Changing the limit of integration (Interval transformation)

The limit of the integration $\int_a^b f(x)dx$ are changed to -1 to 1 by means of the transformation,

$$\int_a^b f(x)dx = c\int_{-1}^{1} g(u)du$$

where,

$$c = \frac{b-a}{2}$$

$$\boxed{x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)}$$

$$\therefore \boxed{I_g = \left(\frac{b-a}{2}\right)\sum_{i=1}^{n} \omega_i g(u_i)}$$

(Q.1 पर्यो)

Q.2 Using three point Gaussian quadrature formula evaluate,

$$\int_0^1 \frac{dx}{1+x}$$

Soln

We first change the limit from (0,1) to (-1,1)

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

$$x = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0)$$

$$= \frac{1}{2}u + \frac{1}{2}$$

$$x = \frac{1}{2}(u+1)$$

$f(x) = \frac{1}{1+x}$

& $g(u) = \frac{1}{1+\frac{1}{2}(u+1)}$

Now, $\frac{dx}{du} = \frac{1}{2}\frac{d(u+1)}{du}$

$$\frac{dx}{du} = \frac{1}{2}$$

$$\therefore dx = \frac{1}{2}du$$

Now,

$$I_g = c\sum_{i=1}^{n} \omega_i g(u_i)$$

$$= \frac{1}{2}\left[\omega_1 g(u_1) + \omega_2 g(u_2) + \omega_3 g(u_3)\right]$$

$$= \frac{1}{2}\left[\frac{5}{9}\left(\frac{1}{1+\frac{1}{2}(-\sqrt{3/5}+1)}\right) + \frac{8}{9}\left(\frac{1}{1+\frac{1}{2}(0+1)}\right) + \frac{5}{9}\left(\frac{1}{1+\frac{1}{2}(\sqrt{3/5}+1)}\right)\right]$$

$$= \cancel{0.69315}\ 0.721256\ (0.6931)$$

Q.1 Using Gaussian 2 point formula, compute

$$\int_{-2}^{2} e^{-x/2}\,dx$$

Soln: we first change the limits from $(-2, 2)$ to $(-1, 1)$ by using transformation rule.

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

$$= \frac{1}{2}(2+2)u + \frac{1}{2}(2-2)$$

$$x = 2u$$

$$\therefore dx = 2du$$

Now,

$$I_g = \int_{-2}^{2} e^{-x/2}\,dx = 2\int_{-1}^{1} e^{-u}\,du = 2\sum_{i=1}^{2} \omega_i f(u_i)$$

$$= 2\left[\omega_1 f(u_1) + \omega_2 f(u_2)\right]$$

$$= 2\left[1 \cdot f\left(\frac{-1}{\sqrt{3}}\right) + 1 \cdot f\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= 2\left(e^{1/\sqrt{3}} + e^{-1/\sqrt{3}}\right)$$

$$= \underline{\underline{4.6854}}$$

Assignment

Q.1 $\int_{-2}^{2} e^{-x/2}\,dx = 4.70$

Q.2 $\int_{0}^{\pi/2} \sin x\,dx = 0.99847 \simeq \underline{1}$

4. Solution Of Linear Algebric Equations

Linear Algebric Equation:

→ Equation of the form $ax+by=c$ is called linear equation, whose x & y are variables and a, b, c are real numbers.

→ Generally linear eqn with 'n' variables has form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

→ Soln of x is infinite. There is no unique soln. For the soln of eqn with 'n' variables (unknown) we need a set of 'n' such eqn. This set of eqn is called "system of linear eqn".

→ A system of 'n' linear eqn can be expressed as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

In matrix form,

$$AX = B$$

A = $n \times n$ matrix of co-efficient of x

X = vector of n unknown

B = " " n constant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \text{ \& } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

→ There are two different approach for the soln of linear eqn.

1) Direct method (Elimination Method)
   a) Gauss elimination Method
   b) Gauss elimination with pivoting
   c) Gauss Jordan Method
   d) Factorization Method.
2) Indirect Method (iterative method)
   a) Gauss Seidel
   b) Gauss Jacobi

1. Gauss Elimination Method:

This method uses two different phases to find the soln of the system of linear eqn.

a) Forward Elimination:

In this phase, the co-efficient matrix is converted into upper triangular matrix

b) Backward Substitution:

The values of unknowns are determined by substituting the values backward from the reduced upper triangular matrix.

Consider $AX=B$ and eqn defined as.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{—} (1)$$

$$\therefore A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

→ Consider ~~eqn~~ augmented matrix as

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \quad \text{—} (2)$$

→ Reducing eqn (2) to upper triangular matrix, let $a_1 \neq 0$ & applying following row operation in eqn (2)

$$R_2 \to R_2 - \frac{a_2}{a_1} R_1 \quad ; \quad R_3 \to R_3 - \frac{a_3}{a_1} R_1$$

$$\therefore [A|B] = \begin{bmatrix} a_1 & b_1 & c_1 & : d_1 \\ 0 & b_2' & c_2' & : d_2' \\ 0 & b_3' & c_3' & : d_3' \end{bmatrix} \quad (3)$$

→ Take $b_2' \neq 0$ as the pivot element and applying row operation as.

$$R_3 \to R_3 \to \frac{b_3'}{b_2'} R_2$$

$$[A|B] = \begin{bmatrix} a_1 & b_1 & c_1 & : d_1 \\ 0 & b_2' & c_2' & : d_2' \\ 0 & 0 & c_3'' & : d_3'' \end{bmatrix} \quad \text{—} (4)$$

If $c_3'' \neq 0$, then eqn can be written as

$$\left.\begin{aligned} a_1x_1 + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \quad \text{—} (5)$$

Similarly,

Backward elimination & forward substitution.

$$\left[\begin{array}{ccc:c} a_1'' & 0 & 0 & d_1'' \\ a_2' & b_2' & 0 & d_2' \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Q. Solve the following set of eqns by Gauss elimination with forward elimination method.

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + y + 3z = 14$$

Soln

Consider $Ax = B$ where,

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 18 \\ 13 \\ 14 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc:c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right]$$

$R_2 \leftarrow R_2 - \frac{1}{2}R_1$; $R_3 \leftarrow R_3 - \frac{3}{2}R_1$

$$[A|B] = \left[\begin{array}{ccc:c} 2 & 2 & 4 & 18 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -3 & -13 \end{array}\right]$$

$R_3 \leftarrow R_3 + R_2$

$$[A|B] = \left[\begin{array}{ccc:c} 2 & 2 & 4 & 18 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

So,

$-3z = -9$

$\therefore z = 3$

$2y + 0.z = 4$

$y = 2$

and $2x + 2y + 4z = 18$

or $2x + 2(2) + 4(3) = 18$

or $2x = 2$

$x = 1$

$\therefore x = 1$

$y = 2$

& $z = 3$

Q2 Apply Gauss Elimination method to solve eqs

$3x_1 + 2x_2 + x_3 = 10$; $2x_1 + 3x_2 + 2x_3 = 14$; $x_1 + 2x_2 + 3x_3 = 14$

Sol^n

The given eq^s is written as,

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Consider $AX = B$ where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Now

$$[A|B] = \begin{bmatrix} 3 & 2 & 1 & : 10 \\ 2 & 3 & 2 & : 14 \\ 1 & 2 & 3 & : 14 \end{bmatrix}$$

$R_2 \leftarrow R_2 - \frac{2}{3}R_1$; $R_3 \leftarrow R_3 - \frac{1}{3}R_1$

$$[A|B] = \begin{bmatrix} 3 & 2 & 1 & : 10 \\ 0 & 5/3 & 4/3 & : 22/3 \\ 0 & 4/3 & 8/3 & : 32/3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & : 10 \\ 0 & 5 & 4 & : 22 \\ 0 & 4 & 8 & : 32 \end{bmatrix}$$

$R_3 \leftarrow R_3 - \frac{4}{5}R_2$

$$[A|B] = \begin{bmatrix} 3 & 2 & 1 & : 10 \\ 0 & 5 & 4 & : 22 \\ 0 & 0 & \frac{24}{5} & : \frac{72}{5} \end{bmatrix}$$

Now,

$$\frac{24}{5} x_3 = \frac{72}{5}$$
$$x_3 = 3$$

$$5x_2 + 4x_3 = 22$$
$$5x_2 + 4 \times 3 = 22$$
$$x_2 = 2$$

$$3x_1 + 2x_2 + x_3 = 10$$
$$3x_1 + 2 \times 2 + 3 = 10$$
$$x_1 = 1$$

$\therefore$ $x_1 = 1$

$x_2 = 2$

& $x_3 = 3$

Tutorial

1. Solve

$x+2y+3z-\cancel{4u}=10;\ 2x+3y-3z-u=1;\ 2x-y+2z+3u=7\cancel{4}$
$3x+2y-4z+3u=2$ Ans $[1, 2, 2, 1]$

2. $x+4y-z=-5;\ x+y-6z=-12,\ 3x-y-z=4$
$[1.6479,\ -1.1409,\ 2.0845]$

3. $10x-7y+3z+5u=6;\ -6x+8y-z-4u=5;$
$3x+y+4z+11u=2;\ 5x-9y-2z+4u=7$
$[5,\ 4,\ -7,\ 1]$

B) Gauss Elimination with Pivoting:

In the elimination phase, in case of Gauss elimination method, each row is normalized by dividing the coefficient of that row by its pivot element (a_{ii}). Normalization cannot be done when the pivot element is zero. This problem can be overcome by interchanging the rows such that pivot elements are non-zero.

Besides making the pivot element non-zero, rearranging can be done to minimize the error resulting from round off by making the pivot element larger. So row with zero pivot element should be interchanged with the row having the largest (absolute value) coefficient in that position. Rearranging can be done to improve accuracy, even if the pivot element is non-zero.

1. Partial Pivoting:

Eg: For system of eqs below.

$$13x_1+5x_2+3x_3=10$$
$$20x_1+10x_2+5x_3=15$$
$$10x_1-40x_2-16x_3=9$$

Sol

$$[A|B]=\left[\begin{array}{ccc|c} 13 & 5 & 3 & 10 \\ 20 & 10 & 5 & 15 \\ 10 & -40 & -16 & 9 \end{array}\right]$$

Steps

(i) Choose the greatest element in 1st column.

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 13 & 5 & 3 & 10 \\ 10 & -40 & -6 & 9 \end{array}\right]$$

$R_2 \leftarrow R_2 - \frac{13}{20}R_1$; $R_3 \leftarrow R_3 - \frac{10}{20}R_1$

Use Gauss Elimination metho

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 0 & -1.5 & -0.25 & 0.25 \\ 0 & -45 & -18.5 & 1.5 \end{array}\right]$$

Check
$|a_{22}| < |a_{32}|$
$|a_{32}| > |a_{22}|$
apply $R_2 \leftrightarrow R_3$

$R_2 \leftrightarrow R_3$

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 0 & -45 & -18.5 & 1.5 \\ 0 & -1.5 & -0.25 & 0.25 \end{array}\right]$$

$$R_3 \leftarrow R_3 - \frac{1.5}{45}R_2$$

$$[A|B] = \left[\begin{array}{ccc|c} 20 & 10 & 5 & 15 \\ 0 & -45 & -18.5 & 1.5 \\ 0 & 0 & 0.3667 & 0.2 \end{array}\right]$$

$$\therefore z = \frac{0.2}{0.3667} = 0.545$$

$$-45y - 18.5 \times 0.545 = 1.5$$
$$y = -0.257$$

& $20x + 10 \times (-0.257) + 5 \times 0.547 = 15$

$x = \cancel{0.485}\ 0.742$

2) Complete Pivoting:

→ Interchange of both row & coloumn.
→ In whole given eqn which has greatest value of given coefficient, ~~th~~ is kept in th top of the left side.

Q.1) Solve:

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 + x_2 + x_3 = 0$$

Soln

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$C_1 \leftrightarrow C_2$

$$[A|B] = \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - \frac{2}{4}R_1$ & $R_3 \leftarrow R_3 - \frac{1}{4}R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & 1 & -0.5 & 4 \\ 0 & 0.5 & 0.25 & -1 \end{array}\right]$$

$R_3 \leftarrow R_3 - 0.5 R_2$

$$[A|B] = \begin{bmatrix} 4 & 2 & 3 & : & 4 \\ 0 & 1 & -0.5 & : & 4 \\ 0 & 0 & 0.5 & : & -3 \end{bmatrix}$$

Now,

$0.5x_3 = -3$

$\therefore x_3 = -6$

$x_2 - 0.5x_3 = 4$

$x_2 = 4 + 0.5\times(-6) = 1$

also, $4x_1 + 2\times1 + 3\times(-6) = 4$

$\therefore x_1 = 5$

The values are,

$x_1 = 5,\ x_2 = 1\ \&\ x_3 = -6$

Q.2 Solve,

$13x + 5y + 3z = 10$

$20x + 10y + 5z = 15$

$10x - 40y - 16z = 9$

Soln

$$[A|B] = \begin{bmatrix} 13 & 5 & 3 & : & 10 \\ 20 & 10 & 5 & : & 15 \\ 10 & -40 & -16 & : & 9 \end{bmatrix}$$

$C_1 \leftrightarrow C_2$

$$\begin{array}{cccccc} & y & x & z & & \\ [A|B] = & 5 & 13 & 3 & : & 10 \\ & 10 & 20 & 5 & : & 15 \\ & -40 & 10 & -16 & : & 9 \end{array}$$

$R_1 \leftrightarrow R_3$

$$[A|B] = \begin{bmatrix} -40 & 10 & -16 & : & 9 \\ 10 & 20 & 5 & : & 15 \\ 5 & 13 & 3 & : & 10 \end{bmatrix}$$

$R_2 \leftarrow R_2 + \frac{10}{40} R_1 \ : \ R_3 \leftarrow R_3 + \frac{5}{40} R_1$

$$= [A|B] = \begin{bmatrix} -40 & 10 & -16 & : & 9 \\ 0 & 22.5 & 1 & : & 17.25 \\ 0 & 14.25 & 1 & : & 11.725 \end{bmatrix}$$

$\therefore R_3 \leftarrow R_3 - \frac{14.25}{22.5} R_2$

$$[A|B] = \begin{bmatrix} -40 & 10 & -16 & : & 9 \\ 0 & 22.5 & 1 & : & 17.25 \\ 0 & 0 & 11.733 & : & 5.425 \end{bmatrix}$$

Now,

$11.7333z = 5.425$

$\therefore z = 0.487$ (0.545) correct ans

$\therefore 22.5x + 0.487 = 17.25$

$x = 0.745$

$\& -40y + 10\times0.745 - 16\times0.487 = 9$

$y = -0.236$

C) Gauss Jordan Method:

1) Simple Gauss Jordan Method:

→ In this method, matrix obtained from the system of eqn is converted into identity matrix by row operation.

→ The conversion process is implemented using following procedure.

i) The element in 1st row, 1st column is made 1 & the first element of all rows below it are reduced to zero.

ii) Element in 2nd colm and 2nd row is made 1 & element in 2nd coloumn of all rows above & below are reduced to zero. The process is continued until identity matrix is obtained.

Q) Solve: $10x + y + z = 12$
$2x + 10y + z = 13$
$x + y + 5z = 7$ using Gauss Jordan Method.

Soln

Augmented matrix

$$[A|B] = \left[\begin{array}{ccc:c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

$R_1 \leftarrow R_1 - 9R_3$

$$[A|B] = \left[\begin{array}{ccc:c} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$; $R_3 \leftarrow R_3 - R_1$

$$= \left[\begin{array}{ccc:c} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

$R_2 \leftarrow 3R_3 - R_2$

$$= \left[\begin{array}{ccc:c} 1 & -8 & -44 & -51 \\ 0 & 1 & 58 & 59 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

$R_1 \leftarrow R_1 + 8R_2$; $R_3 \leftarrow R_3 - 9R_2$

$$= \left[\begin{array}{ccc:c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & -473 & -473 \end{array}\right]$$

$R_3 \leftarrow R_3 / -473$

$$= \left[\begin{array}{ccc:c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 - 420R_3$; $R_2 \leftarrow R_2 - 58R_3$

$$= \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore x = 1, y = 1 \; \& \; z = 1$ //

Q.2) $2x + y + z = 10$

$3x + 2y + 3z = 18$

$x + 4y + 9z = 16$

Soln

The augmented matrix is,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_1 \leftarrow R_1/2$

$$= \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 5 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$

$R_3 \leftarrow R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 5 \\ 0 & 0.5 & 1.5 & 3 \\ 0 & 3.5 & 8.5 & 11 \end{array}\right]$$

$R_2 \leftarrow 2R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 5 \\ 0 & 1 & 3 & 6 \\ 0 & 3.5 & 8.5 & 11 \end{array}\right]$$

$R_1 \leftarrow R_1 - 0.5R_2$

$R_3 \leftarrow R_3 - 3.5R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$R_3 \leftarrow R_3/-2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_3$

$R_2 \leftarrow R_2 - 3R_3$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\therefore x = 7,\ y = -9\ \&\ z = 5$

Tutorial

Q) Solve: $10x - 7y + 3z + 5u = 6$

$-6x + 8y - z - 4u = 5$

$3x + y + 4z + 11u = 2$

$5x - 9y - 2z + 4u = 7$

Ans. $\begin{bmatrix} 5 \\ 4 \\ -7 \\ 1 \end{bmatrix}$

2) Inverse Gauss Jordan Method (Inverse Elimination):

$AX = B$

$\therefore \boxed{X = A^{-1}B}$

Q.1) Solve using ~~Gauss J~~ Inverse Gauss Jordan Method.

$2x + y + z = 10$
$3x + 2y + 3z = 18$
$x + 4y + 9z = 16$

Soln :
The augmented matrix is,

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftarrow \frac{R_1}{2}$

$$\left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$
$R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1.5 & -1.5 & 1 & 0 \\ 0 & 3.5 & 8.5 & -0.5 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow 2 * R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 3.5 & 8.5 & -0.5 & 0 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 - 0.5R_2$
$R_3 \leftarrow R_3 - 3.5R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right]$$

$R_3 \leftarrow -R_3/2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & 1 & -5 & 3.5 & -0.5 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_3$
$R_2 \leftarrow R_2 - 3R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2.5 & -0.5 \\ 0 & 1 & 0 & 12 & -8.5 & 1.5 \\ 0 & 0 & 1 & -5 & 3.5 & -0.5 \end{array}\right]$$

Now,
The inverse of given matrix is

$$A^{-1} = \begin{bmatrix} -3 & 2.5 & -0.5 \\ 12 & -8.5 & 1.5 \\ -5 & 3.5 & -0.5 \end{bmatrix}$$

Here,

$AX = B$

$X = A^{-1}B$

$$= \begin{bmatrix} -3 & 2.5 & -0.5 \\ 12 & -8.5 & 1.5 \\ -5 & 3.5 & -0.5 \end{bmatrix} \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} -3\times10 + 2.5\times18 - 0.5\times16 \\ 12\times10 - 8.5\times18 + 1.5\times16 \\ -5\times10 + 3.5\times18 - 0.5\times16 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \\ 5 \end{bmatrix}$$

$\therefore$ $x = 7$, $y = -9$ & $z = 5$

Q.7 Solve,

$10x - 7y + 3z + 5u = 6$

$-6x + 8y - z - 4u = 5$

$3x + y + 4z + 11u = 2$

$5x - 9y - 2z + 4u = 7$

Ans $\begin{bmatrix} 5 \\ 4 \\ -7 \\ 1 \end{bmatrix}$

D) Factorization Method:

The coefficient matrix A of the system of linear equation can be factorized (or decompared) into two triangular matrices L and U such that,

$$A = LU \quad \text{——— (1)}$$

where,

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

L is known as lower triangular matrix and U is known as upper triangular matrix. Hence this method is also known as LU factorization or Triangular method or Decomposition method.

Once A is factorized to L and U, the system of eqns.

$$AX = B$$

can be expressed as,

$$(LU)X = B \quad \text{—— (2)}$$

or, $L(UX)=B$ —— (2)

Let us assume that,

$$UX = Z \quad \text{—— (3)}$$

where Z is an unknown vector. Substituting eqn (3) in eqn (2), we get

$$LZ = B \quad \text{—— (4)}$$

Now we can solve the system

$$AX = B$$

in two stages

1) Solve the eqn

$$LZ = B$$

for Z by forward substitution

2) Solve the eqn

$$UX = Z$$

for x using Z.

The elements of L and U can be determined comparing the elements of the product of L and U with those of A.

This is done by assuming the diagonal elements of L or U to be unity. The decomposition with L having unit diagonal values is called Dolittle LU decomposition while with U having unit diagonal elements is called the Crout LU decomposition.

Q) Solve the eqn by Decomposition method

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Soln

Here,

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

we know $A = LU$

$$\therefore LU = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

by using dolittle decomposition

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = A$$

or,

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} =$$

$\therefore U_{11} = 3, \quad U_{12} = 2, \quad U_{13} = 1$

Solving second row.

$L_{21}U_{11} = 2$	$L_{21} * U_{12} + U_{22} = 3$	$L_{21}U_{13} + U_{23} = 2$
$\Rightarrow L_{21} = \frac{2}{3}$	or, $\frac{2}{3} \times 2 + U_{22} = 3$	$\frac{2}{3} \times 1 + U_{23} = 2$
	$\Rightarrow U_{22} = 5/3$	$\Rightarrow U_{23} = 4/3$

solving 3rd row,

$L_{31}U_{11} = 1$	$L_{31}U_{12} + L_{32}U_{22} = 2$	$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 3$
$L_{31} \times 3 = 1$	$\Rightarrow \frac{1}{3} \times 2 + L_{32} \times \frac{5}{3} = 2$	$\Rightarrow \frac{1}{3} \times 1 + \frac{4}{5} \times \frac{4}{3} + U_{33} = 3$
$\Rightarrow L_{31} = 1/3$	$\Rightarrow L_{32} = \frac{4}{5}$	$\Rightarrow U_{33} = \frac{8}{5}$

Now,

$LZ = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

on solving,

$\Rightarrow Z_1 = 10$

$\Rightarrow \frac{2}{3} Z_1 + Z_2 = 14 \Rightarrow Z_2 = \frac{22}{3}$

and,

$\frac{1}{3} Z_1 + \frac{4}{5} Z_2 + Z_3 = 14$

or $\frac{1}{3} \times 10 + \frac{4}{5} \times \frac{22}{3} + Z_3 = 14$

$\Rightarrow Z_3 = \frac{24}{5}$

Now again,

$UX = Z$

i.e $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$

on solving,

i) $\frac{8}{5} x_3 = \frac{24}{5} \Rightarrow x_3 = 3$

ii) $\frac{5}{3} x_2 + \frac{4}{3} \times 3 = \frac{22}{3} \Rightarrow x_2 = 2$

& iii) $3x_1 + 2 \times 2 + 1 \times 3 = 10 \Rightarrow x_1 = 1$

$\therefore x_1 = 1,\ x_2 = 2$ & $x_3 = 3$

Cholesky's Factorization / Method of Square Root:

In case A is symmetric, the LU decomposition can be modified so that the upper factor is the transpose of the lower one (or vice versa).

i.e $A = LL^T$ or $A = U^T U$

$$U_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} U_{ki}^2}, \quad i = 1 \text{ to } n$$

$$U_{ij} = \frac{1}{U_{ii}} \left[a_{ij} - \sum_{k=1}^{i-1} U_{ki} U_{kj} \right] \text{ for } j > i$$

This decomposition is called Cholesky's factorized or the method of square root.

Here, $A = LL^T$ or UU^T

Then $AX = B$

or, $(LL^T)X = B$

or $L(L^T X) = B$

or $L Z = B$ where $Z = L^T X$

Steps

1. solve for Z using $LZ = B$
2. solve for X using $Z = L^T X$

Q. Solve by Cholesky method.

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Soln

$A = LL^T$

or, $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix}$

$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} L_{11}^2 & \cancel{L_{11}L_{21}}\ L_{11}L_{21} & L_{11}L_{31} \\ L_{11}L_{21} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{11}L_{31} & L_{31}L_{21} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix}$

solving 1st row

(i) $L_{11}^2 = 3 \Rightarrow L_{11} = \sqrt{3}$

(ii) $L_{11} \cdot L_{21} = 2 \Rightarrow \sqrt{3} \cdot L_{21} = 2 \Rightarrow L_{21} = 2/\sqrt{3}$

iii) $L_{11} \cdot L_{31} = 1 \Rightarrow \sqrt{3} \cdot L_{31} = 1 \Rightarrow L_{31} = 1/\sqrt{3}$

Solving 2nd row,

(i) $\cancel{L_{11}L_{21} = 2}$ $L_{21}^2 + L_{22}^2 = 3 \Rightarrow (2/\sqrt{3})^2 + L_{22}^2 = 3$

$\Rightarrow \frac{4}{3} + L_{22}^2 = 3 \Rightarrow L_{22} = \sqrt{5/3}$

ii) $L_{21}L_{31} + L_{22}L_{32} = 2$

or, $\frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}} + \sqrt{\frac{5}{3}} \cdot L_{32} = 2$

or, $\frac{2}{3} + \sqrt{5/3}\ L_{32} = 2$

$(\sqrt{5/3}\ L_{32})^2 = \left(\frac{4}{3}\right)^2 \Rightarrow L_{32} = \frac{4}{\sqrt{15}}$

Solving 3rd row

$L_{31}^2 + L_{32}^2 + L_{33}^2 = 3$

or $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{4}{\sqrt{15}}\right)^2 + L_{33}^2 = 3$

or $L_{33}^2 = \frac{8}{5}$

$\Rightarrow L_{33} = \sqrt{\frac{8}{5}}$

i.e $L = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & \sqrt{5/3} & 0 \\ 1/\sqrt{3} & 4/\sqrt{15} & \sqrt{8/5} \end{bmatrix}$

Now,

$LZ = B$

$$\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & \sqrt{5/3} & 0 \\ 1/\sqrt{3} & 4/\sqrt{15} & \sqrt{8/5} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

on solving we get,

(i) $\sqrt{3}\, z_1 = 10$

$\therefore z_1 = 10/\sqrt{3}$

(ii) $\frac{2}{\sqrt{3}} z_1 + \sqrt{\frac{5}{3}} z_2 = 14$

or $\frac{2}{\sqrt{3}} \times \frac{10}{\sqrt{3}} + \sqrt{\frac{5}{3}} z_2 = 14$

$\frac{20}{3} + \sqrt{5/3}\, z_2 = 14$

$\frac{5}{3} z_2^2 = \frac{484}{9}$

$\Rightarrow z_2 = 22/\sqrt{15}$

iii) $\frac{1}{\sqrt{3}} \times \frac{10}{\sqrt{3}} + \frac{4}{\sqrt{15}} \times \frac{22}{\sqrt{15}} + \sqrt{\frac{8}{5}} \times z_3 = 14$

$\Rightarrow z_3 = \frac{24}{\sqrt{40}}$

Finally $z = L^T . x$

$$\begin{bmatrix} 10/\sqrt{3} \\ 22/\sqrt{15} \\ 24/\sqrt{40} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{5/3} & 4/\sqrt{15} \\ 0 & 0 & \sqrt{8/5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

on solving,

i) $\sqrt{\frac{8}{5}}\, x_3 = \frac{24}{\sqrt{40}} \Rightarrow x_3 = 3$

ii) $\sqrt{\frac{5}{3}}\, x_2 + \frac{4}{\sqrt{15}} x_3 = \frac{22}{\sqrt{15}}$

$\Rightarrow x_2 = 2$

iii) $\sqrt{3}\, x_1 + \frac{2}{\sqrt{3}} \times 2 + \frac{1}{\sqrt{3}} \times 3 = \frac{10}{\sqrt{3}}$

$\therefore x_1 = 1$

$x_1 = 1, \; x_2 = 2, \; x_3 = 3$

2) Indirect Method (Iterative Method)

a) Gauss Jacobi Method

=> Based on fixed point iteration method.

i.e any function $f(x)=0$ can be manipulated as.

$$x=g(x)$$

=> Take initial guess : $x_1, x_2, x_3, \ldots x_n$

=> To get better values, we use the values in next iteration.

Q Solve by Jacobi's iteration method, the eqn.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Soln

we write the above eqn in the form

$$x = (17 - y + 2z)/20$$
$$y = (-18 - 3x + z)/20$$
$$z = (25 - 2x + 3y)/20$$

Take initial guess: $x=0, y=0$ & $z=0$

Iteration 1

$$x = (17-0+0)/20 = 0.85$$
$$y = (-18-0+0)/20 = -0.9$$
$$z = 1.25$$

Iteration 2

$$x = 1.02$$
$$y = -0.965$$
$$z = 1.03$$

3rd iteration

$$x = 1.00125$$
$$y = -1.0015$$
$$z = 1.00325$$

4th iteration

$$x = 1.004$$
$$y = -1.000025$$
$$z = 0.99965$$

5th iteration

$$x = 0.99996625$$
$$y = -1.0000775$$
$$z = 0.9999565$$

Use complete solution

6th iteration

$$x = 0.999965 = 1$$
$$y = -0.999997 = -1$$
$$z = 0.99999175 = 1$$

$\therefore x = 1, y = -1$ & $z = 1$

Q Solve by Jacobi's iteration method, the eqn
$10x+y-z=11.19$, $x+10y+z=28.08$, $-x+y+10z=35.61$
correct upto two decimal places

Soln

$$x = (11.19 - y + z)/10$$
$$y = (28.08 - x - z)/10$$
$$z = (35.61 + x - y)/10$$

let $x=0$, $y=0$ & $z=0$

Itr 1

$x = 1.119$
$y = 2.808$
$z = 3.561$

Itr 2

$x = 1.194$
$y = 2.340$
$z = 3.392$

Itr 3

$x = 1.224$
$y = 2.349$
$z = 3.446$

Itr 4

$x = 1.228$
$y = 2.341$
$z = 3.448$

$\therefore$ $x = 1.22$
$y = 2.34$
& $z = 3.44$ //

B) Gauss Siedal Method:

This method is same as Gauss Jacobi's method but the only different is we use most recent values.

Q) Apply Gauss Siedal Iteration method to solve the eqs

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Sol: we write the eqn in the form

$$x = (85 - 6y + z)/27$$
$$y = (72 - 6x - 2z)/15$$
$$z = (110 - x - y)/54$$

let initial guess be $x=0$, $y=0$ & $z=0$

Itr 1

$x = (85 - 6 \cdot 0 + 0)/27 = 3.148$
$y = (72 - 6 \times 3.148 - 2 \times 0)/15 = 3.541$
$z = (110 - 3.148 - 3.541)/54 = 1.914$

Itr 2

$x = (85 - 6 \times 3.541 + 1.914)/27 = 2.432$
$y = (72 - 6 \times \overset{2.432}{\cancel{3.148}} - 2 \times 1.914)/15 = 3.572$
$z = (110 - 2.432 - 3.572)/54 = 1.926$

Itr 3

$x = (85 - 6 \times 3.572 + 1.926)/27 = 2.426$
$y = (72 - 6 \times 2.426 - 2 \times 1.926)/15 = 3.572$
$z = (110 - 2.426 - 3.572)/54 = 1.926$

Itr 4

$x = (85 - 6 \times 3.572 + 1.926)/27 = 2.426$
$y = (72 - 6 \times 2.426 - 2 \times 1.926)/15 = 3.572$
$z = (110 - 2.426 - 3.572)/54 = 1.926$ //

Powers Method:

Power method is a single value method used for determining the 'dominant' eigen value of a matrix. It is an iterative method implemented using an initial starting vector X. The starting vector can be arbitrary if no suitable ~~exa~~ approximation is available. Powers method ~~is~~ implemented as follows.

$$Y = AX$$
$$X = \frac{1}{k} Y$$

Q. Find the largest eigen and corresponding eigen vector using Powers method.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Soln

Let $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Itr 1

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

Itr 2

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix} = 2.5 \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

Itr 3

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix} = 2.8 \begin{bmatrix} 0.9 \\ \\ 0 \end{bmatrix}$$

Itr 4

$$Y = AY = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix} = 2.93 \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}$$

Itr 5

$$Y = AY = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 2.96 \\ 0 \end{bmatrix} = 2.98 \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix}$$

Itr 6

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 2.99 \\ 0 \end{bmatrix} = 2.99 \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix}$$

Itr 7

$$Y = AY = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 2.98 \\ 0 \end{bmatrix} = 2.99 \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix}$$

Now,

largest Eigen value $= 2.99 \simeq 3$

and corresponding eigen vector $= \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Q.2) Find the largest eigen value & corresponding eigen vector of the matrix $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ using Power Method.

Sol: let initial guess be $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now,

Itr 1

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

Itr 2

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

Itr 3

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix}$$

Itr 4

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.43 \\ 1.86 \end{bmatrix} = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix}$$

Itr 5

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 2.74 \\ -3.41 \\ 2.08 \end{bmatrix} = 3.41\begin{bmatrix} 0.8 \\ -1 \\ 0.61 \end{bmatrix}$$

Itr 6

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.8 \\ -1 \\ 0.61 \end{bmatrix} = \begin{bmatrix} 2.6 \\ -3.41 \\ 2.22 \end{bmatrix} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix}$$

Itr 7

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \begin{bmatrix} 2.52 \\ -3.41 \\ 2.3 \end{bmatrix} = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix}$$

Itr 8

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 2.48 \\ -3.41 \\ 2.34 \end{bmatrix} = 3.41\begin{bmatrix} 0.73 \\ -1 \\ 0.67 \end{bmatrix}$$

Itr 9

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.73 \\ -1 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 2.46 \\ -3.4 \\ 2.34 \end{bmatrix} = 3.4\begin{bmatrix} 0.72 \\ -1 \\ 0.69 \end{bmatrix}$$

Itr 10

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.72 \\ -1 \\ 0.69 \end{bmatrix} = \begin{bmatrix} 2.44 \\ -3.41 \\ 2.38 \end{bmatrix} = 3.41\begin{bmatrix} 0.72 \\ -1 \\ 0.70 \end{bmatrix}$$

Itr 11

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.72 \\ -1 \\ 0.70 \end{bmatrix} = \begin{bmatrix} 2.44 \\ -3.42 \\ 2.4 \end{bmatrix} = 3.42 \begin{bmatrix} 0.71 \\ -1 \\ 0.70 \end{bmatrix}$$

Itr 12

$$Y = AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.71 \\ -1 \\ 0.70 \end{bmatrix} = \begin{bmatrix} 2.42 \\ -3.41 \\ 2.4 \end{bmatrix} = 3.41 \begin{bmatrix} 0.71 \\ -1 \\ 0.70 \end{bmatrix}$$

$\therefore$ largest eigen value $= 3.41$

& Eigen vector $= \begin{bmatrix} 0.71 \\ -1 \\ 0.70 \end{bmatrix}$

5. Solution of Ordinary Differential Equations

Initial Value Problem:

Let us consider the eqn

$$y = ae^{x} \quad \text{——} \ (1)$$

where a is constant to be determined.

Suppose $y = 1$ at $x = 0$ then,

$$y(0) = ae^{0} = 1$$

$$\therefore \ a = 1$$

and the particular solution is

$$\boxed{y = e^{x}}$$

If the order of the eqn is n, we will have to obtain n constants and therefore, we need n conditions in order to obtain unique solution. When all the conditions are specified at a particular value of the independent variable x, then the problem is called an initial value problems. If the condition are specified at two or more points then it is called boundary value problem.

This problem can be solved by many methods. They can be categorized in two methods

1) Single step method or Pt. wise method: A series of y in terms of powers of x from which the value of y can be obtained by direct substitution. In this method, y is approximated by truncated series each term is a function of x. The information about the curve at one pt. is used & soln is not iterated. Taylor's series & Picard method belong to this method.

2) Step by Step Method: In this method, the values of y are computed by short step ahead for equal interval of h of independent variable. The values are iterated till we get desired accuracy. Euler's method, Heun's Method, Runge kutta Method belongs to this method.

1. Taylor Series Method:

We can expand a function $y(x)$ about a point $x = x_0$ using Taylor's theorem of expansion

$$y(x) = y(x_0) + (x-x_0)\frac{y'(x_0)}{1!} + (x-x_0)^2 \frac{y''(x_0)}{2!} + \dots + (x-x_0)^n \frac{y^n(x_0)}{n!} \quad \text{--- (1)}$$

Where $y^n(x_0)$ is the nth derivative of $y(x)$, evaluated at $x = x_0$. The value of $y(x)$ can be obtained if we know the values of derivatives. This implies that if we are given the equation

$$y' = f(x,y)$$

we must repeatedly differentiate $f(x,y)$ implicitly with respect to x and evaluate them at x_0.

Let us consider a equation

$$y' = x^2 + y^2$$

under the condition $y(x) = 1$ when $x = 0$

$$y' = x^2 + y^2$$
$$y'' = 2x + 2yy'$$
$$y''' = 2 + 2yy'' + 2(y')^2$$

at $x=0$, $y(0)=1$. Therefore,

$$y'(0) = 1$$
$$y''(0) = 2$$
$$y'''(0) = 2 + 2\times1\times2 + 2\times1^2 = 8$$

Substituting these value, the taylor series becomes.

$\cancel{y(x) = 1 + x + x^2 + 8}$

$$y(x) = y(0) + (x-0)y'(0) + (x-0)^2\frac{y''(0)}{2!} + \frac{(x-x_0)^3 y'''(0)}{3!} + \dots$$
$$= 1 + x\cdot1 + x^2\cdot\frac{2}{2!} + x^3\cdot\frac{8}{3!} + \dots$$
$$= 1 + x + x^2 + \frac{8}{3!}x^3 + \dots$$

The number of terms depends upon the accuracy of the solution needed.

Q.) Use Taylor Method to solve the equation

$$y' = x^2 + y^2$$

for $x = 0.25$ and $x = 0.5$ given $y(0)=1$

Soln

The solution of $y' = x^2 + y^2$ is given by

$$y(x) = 1 + x + x^2 + \frac{8x^3}{3!} + \dots$$

$\therefore$ at $x = 0.25$

$$y(0.25) = 1 + 0.25 + 0.25^2 + \frac{8\times0.25^3}{6} + \dots$$
$$= 1.33333$$

& at $x = 0.5$

$$y(0.5) = 1 + 0.5 + 0.5^2 + \frac{8\times0.5^3}{6} + \dots$$
$$= 1.9167$$

Q.2) Use Taylor Method recursively to solve the eq².

$$y' = x^2 + y^2, \quad y(0) = 0$$

for the interval (0, 0.4) using sub intervals of size 0.2

Sol

The derivatives of y are given by

$$y' = x^2 + y^2$$
$$y'' = 2x + 2yy'$$
$$y''' = 2 + 2yy'' + 2(y')^2$$
$$\cancel{y^4 = 2yy''' + 2(y'')^2 + 4y''}$$
$$y^4 = 2yy''' + 2y'y'' + 4y''$$

The taylor series becomes,

$$y(x) = y_0 + \frac{y_0'}{1!}h + \frac{y_0''}{2!}h^2 + \frac{y'''}{3!}h^3 + \frac{y^4}{4!}h^4 + \cdots \quad \text{—— (1)}$$

Itr 1,

$$h = 0.2, \quad y_0 = y(0) = 0$$
$$y_0' = y'(0) = 0 + y(0)^2 = 0$$
$$y_0''(0) = 2\times0 + 2\times0 + 0 = 0$$
$$y_0'''(0) = 2 + 2\times0\times0 + 2\times0^2 = 2$$
$$y_0^4(0) = 2\times0\times2 + 2\times0\times0 + 4\times0 = 0$$

∴ eqn (1) becomes

$$y(0.2) = 0 + 0 + 0 + \frac{2}{3!}(0.2)^3 + 0$$
$$= 0.002667$$

Itr 2

$$x_1 = 0.2$$
$$y_1' = x_1^2 + y_1^2 = 0.2^2 + (0.002667)^2 = 0.04$$
$$y_1'' = 2x_1 + 2y_1y_1'$$
$$= 2(0.2) + 2(0.002667)(0.04)$$
$$= 0.400213$$
$$y_1''' = 2 + 2y_1y_1'' + 2(y_1')^2$$
$$= 2 + 2(0.002667)(0.400213) + 2(0.04)^2$$
$$= 2.005335$$

Now,

$$y(0.4) = y_1 + \frac{y_1'h}{1!} + \frac{y_1''}{2!}h^2 + \frac{y_1'''}{3!}h^3 + \cdots$$

$$\cancel{= 0.04 +}$$
$$= 0.002667 + 0.04\times0.2 + \frac{0.400213}{2}\times0.2^2$$
$$+ \frac{2.005335}{6}\times0.2^3$$

$$= 0.02135$$

2) Picard Method:

Consider the differential equation.

$$\frac{dy}{dx} = f(x,y)$$

we can integrate this to obtain the solution in the interval (x_0, x).

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

or, $$y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx$$

we can use this as a first approximation to the solution and the result can be used on the right-hand side to obtain the next approximation. The iterative equation is written as.

$$y^{i+1} = y_0 + \int_{x_0}^{x} f(x, y^i)\,dx \quad — ①$$

Eqn ① is known as Picard's Method.

Disadvantage:

→ Since it involves actual integration, thus sometimes it may not be possible to carry out integration.

Q1) Solve the eqn using Picard's Method:

i) $y'(x) = x^2 + y^2$, $y(0) = 0$ estimate $y(0.1)$, $y(0.2)$, $y(1)$

Here,

$y_0 = 0$ & $x_0 = 0$

$$y^1 = y_0 + \int_{x_0}^{x} (x^2 + (y^0)^2)\,dx$$

$$= 0 + \int_0^x x^2\,dx = \frac{x^3}{3}$$

$$y^2 = y_0 + \int_{x_0}^{x} (x^2 + (y^1)^2)\,dx$$

$$= 0 + \int_0^x \left(x^2 + \frac{x^6}{9}\right)dx = \frac{x^3}{3} + \frac{x^7}{63}$$

This process can be continued further although it may be a difficult task. If we stop at y^2 then.

$$y(x) = \frac{x^3}{3} + \frac{x^7}{63}$$

Now, $y(0.1) = 0.000333$

$y(0.2) = 0.002667$

& $y(1) = 0.34921$

Q) Solve, $y'(x) = xe^{y}$, $y(0) = 0$ & estimate $y(0.1)$, $y(0.2)$, $y(1)$

Soln

$y_0 = 0$ & $x_0 = 0$

$y' = \cancel{y_0 + \int_{x_0}^{x} (x + (y^0)^2)\,dx}$

$\cancel{= 0 + \int_0^x (x^2}\ x$

$$y^1 = y_0 + \int_{x_0}^{x} x e^{y^0}\, dx = 0 + \int_0^x x e^0\, dx$$

$$= \frac{x^2}{2}$$

$$y^2 = y_0 + \int_{x_0}^{x} x e^{y^1}\, dx = 0 + \int_0^x x e^{(x^2/2)}\, dx$$

$$= e^{x^2/2} - 1$$

Now,

$$y(x) = e^{x^2/2} - 1$$

$$y(0.1) = 0.0050125$$

$$y(0.2) = 0.0202$$

$$y(1) = 0.6487$$

$$\boxed{\int x e^{cx^2} = \frac{1}{2c} e^{cx^2}}$$

$$\int_0^x x e^{(x^2/2)} = \frac{1}{2 \cdot \frac{1}{2}} e^{x^2/2} \Big|_0^x$$

$$= e^{x^2/2} \Big|_0^x$$

$$= e^{x^2/2} - e^0 = e^{x^2/2} - 1$$

3) EULER'S METHOD.

Eulers method is the simplest one-step meth and has limited approach because of its low accur

Consider the first two terms of the expansion.

$$y(x) = y(x_0) + y'(x_0)(x - x_0)$$

Given the differential equation

$$y'(x) = f(x, y) \text{ with } y(x_0) = y_0$$

we have,

$$y'(x_0) = f(x_0, y_0)$$

so,

$$y(x) = y(x_0) + (x - x_0) f(x_0, y_0)$$

Then the values of $y(x)$ at $x = x_1$ is given by

$$y(x_1) = y(x_0) + (x_1 - x_0) f(x_0, y_0)$$

let $h = x_1 - x_0$. so.

$$y_1 = y_0 + h f(x_0, y_0)$$

Similarly at $x = 2$

$$y_2 = y_1 + h f(x_1, y_1)$$

$m = f(x_0$

In general

$$\boxed{y_{i+1} = y_i + h f(x_i, y_i)} \quad \text{or} \quad y_{i+1} = y_i + m l$$

This formula is known as Eulers method.

Q> Given the eqn

$$\frac{dy}{dx} = 3x^2+1 \quad \text{with } y(1)=2$$

estimate $y(2)$ by Euler's method using (i) $h=0.5$ and (ii) $h=0.25$

Soln

given

$$y' = \frac{dy}{dx} = 3x^2+1 = m$$

(1) for $h=0.5$

1st iteration

$y(1) = 2$

$m = f(x_0,y_0) = f(1,2) = 3(1)^2+\cancel{1} = 4$

$y(1.5) = y_0+mh = 2+4\times0.5 = 4$

2nd iteration

$y(1.5) = 4$

$m = f(x_0,y_0) = f(1.5,4) = 3(1.5)^2+\cancel{1} = 7.75$

$\therefore y(2) = y_0+mh = 4+7.75\times0.5 = 7.875$

(2) for $h=0.25$

1st iteration

$y(1) = 2$

$m = f(x_0,y_0) = f(1,2) = 3(1)^2+1 = 4$

$y(1.25) = y_0+mh = 2+4\times0.25 = 3$

2nd iteration

$y(1.25) = 3$

$m = f(x_0,y_0) = f(1.25,3) = 3(1.25)^2+1 = 5.6875$

$y(1.5) = y_0+mh = 3+5.6875\times0.25 = 4.421875$

3rd iteration

$y(1.5) = 4.421875$

$m = f(x_0,y_0) = f(1.5, 4.421875) = 3(1.5)^2+1 = 7.75$

$y(1.75) = y_0+mh = 4.421875+7.75\times0.25 = 6.35938$

4th iteration

$y(1.75) = 6.35938$

$m = f(x_0,y_0) = f(1.75, \text{[illegible]}) = 3(1.75)^2+1 = 10.1875$

$y(2) = y_0+mh = 6.35938+10.1875\times0.25 = 8.90625$

Heun's method (Modified Euler's Method)

→ It is an improvement on Euler's Method.

→ As we know in Euler's method, the slope at the begining of the interval is used to extrapolate y_i to y_{i+1} over the entire interval

ie $y_{i+1} = y_i + mh$

when m_i = slope at (x_i, y_i)

→ The Heun's method is implement as

Given the equation

$$\frac{dy}{dx} = y'(x) = f(x, y)$$

$$m_1 = y'(x_i) = f(x_i, y_i)$$
$$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1}) = f(x_0+h, y_0+m_1h)$$

$$m = \frac{m_1+m_2}{2}$$

$$\therefore y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right]$$

i.e $\boxed{y_{i+1} = y_i + mh}$

Q. Given the eqn $y'(x) = 2y/x$ with $y(1) = 2$, $y(2) = ?$ with $h = 0.25$.

sol: Given, $y'(x) = \frac{2y}{x} = m$

1st iteration

$y_0 = y(1) = 2$

$m_1 = f(x_0, y_0) = f(1, 2) = \frac{2y}{x} = \frac{2\times 2}{1} = 4$

$m_2 = f(x_0+h, y_0+m_1h) = f(1.25, 3) = \frac{2\times 3}{1.25} = 4.8$

$m = \frac{m_1+m_2}{2} = \frac{4+4.8}{2} = 4.4$

$\therefore y(1.25) = y_0 + mh = 2 + 4.4\times 0.25 = 3.1$

2nd iteration

$y(1.25) = 3.1$

$m_1 = f(x_0, y_0) = f(1.25, 3.10) = \frac{2\times 1.25}{3.1} = \frac{2\times 3.1}{1.25} = 4$

$m_2 = f(x_0+h, y_0+m_1h) = f(1.5, 4.34) = 5.7867$

$m = \frac{m_1+m_2}{2} = \frac{4.96+5.7867}{2} = 5.3733$

$\therefore y(1.5) = y_0 + mh = 3.1 + 5.3733\times 0.25 = 4.443$

3rd iteration

$y(1.5) = 4.4433$

$m_1 = f(x_0, y_0) = \cancel{f(4.4433, 1.5)}\ f(1.5, 4.4433) = 5.92$

$m_2 = f(x_0+h, y_0+m_1h) = (1.75, 5.9244) = 6.7708$

$m = \frac{m_1+m_2}{2} = \frac{5.92+6.7708}{2} = 6.3454$

$\therefore y(1.75) = y_0 + mh = 4.4434 + 6.3454\times 0.25 = 6.0296$

4th iteration

$y(1.75) = 6.0296$

$m_1 = f(x_0, y_0) = f(1.75, 6.0296) = 6.891$

$m_2 = f(x_0+h, y_0+m_1h) = f(2, 7.7524) = 7.7524$

$m = \frac{m_1+m_2}{2} = 7.3217$

$\therefore y(2) = y_0 + mh = 6.0296 + 7.3217 \times 0.25$

$= 7.860025$

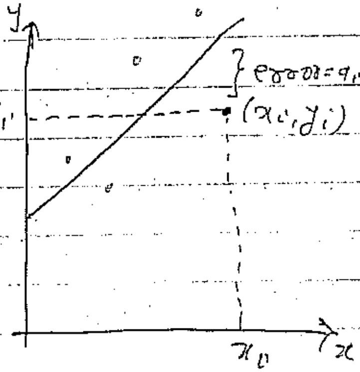
Fitting Linear Equation:

Fitting a linear line is simplest approach of regression analysis.

Let $f(x) = y = a + bx$ is the eqn of a straight line ~~wheer~~ where,

a = Intercept of line

b = slope



→ Let (x_i, y_i) be the point as shown in fig. The vertical distance of this point from the line $f(x) = a + bx$ is the error a_i

$$a_i = y_i - f(x_i)$$
$$= y_i - (a + bx_i)$$

for fitting a best line passing through data by means of minization of error, the various approach ~~at~~ are

1) Minimize the sum of error

$$\sum a_i = \sum (y_i - a - bx_i)$$

2) Minimize the sum of absolute values of error

$$\sum |a_i| = \sum |y_i - a - bx_i|$$

3) Minimize the sum of square of error.

$$\sum a_i^2 = \sum (y_i - a - bx_i)^2$$

Least Square Regression:

It is the technique to minimize the sum of square of error of individual error

$$Q = \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

in this method, we choose 'a' & 'b' such that Q is minimize

∴ Necessary condition for Q to be minimum is

$$\frac{\partial Q}{\partial a} = 0 \quad \& \quad \frac{\partial Q}{\partial b} = 0$$

$$\therefore \frac{\partial Q}{\partial a} = -2 \sum_{i=1}^{n} (y_i - a - bx_i) = 0$$

$$\therefore \sum y_i = an + b \sum x_i \quad \text{—— (I)}$$

$$\frac{\partial Q}{\partial b} = -2 \sum_{i=1}^{n} x_i (y_i - a - bx_i) = 0$$

or, $$\sum x_i y_i = a \sum x_i + b \sum x_i^2 \quad \text{—— (II)}$$

From eqn (I) & (II) in matrix form,

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\therefore \boxed{\begin{aligned} b &= \frac{n \sum (x_i y_i) - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \\ a &= \frac{\sum y_i - b \sum x_i}{n} \end{aligned}}$$

Quadratic Regression 2nd order Polynomial.

→ A quadratic eqn is given by,

$$y = a + bx + cx^2$$

$$S = \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} \left(y_i - a - bx_i - (x_i^2)\right)^2$$

$$\begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

Q) Fit a straight line ($y = a + bx$) to the following set of data

x:	1	2	3	4	5
y:	3	4	5	6	8

Soln we know

$$\begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

x_i	x_i^2	y_i	$x_i y_i$
1	1	3	3
2	4	4	8
3	9	5	15
4	16	6	24
5	25	8	40
Σ 15	55	26	90

$$\begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 26 \\ 90 \end{bmatrix}$$

$5a + 15b = 26$

$15a + 55b = 90$

$\therefore a = 1.\bar{6}$

$b = 1.2$

$\therefore y = 1.6 + 1.2x$

Q)

x:	1	2	3	5
y:	−1	2	9	35

Soln

$$\begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

	x_i	x_i^2	x_i^3	x_i^4	y_i	$x_i y_i$	$x_i^2 y_i$
	1	1	1	1	−1	−1	−1
	2	4	8	16	2	4	8
	3	9	27	81	9	27	81
	5	25	125	625	35	175	875
Σ	11	39	161	723	45	205	963

$$\begin{bmatrix} 4 & 11 & 39 \\ 11 & 39 & 161 \\ 39 & 161 & 723 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 45 \\ 205 \\ 963 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 4 & 11 & 39 \\ 11 & 39 & 161 \\ 39 & 161 & 723 \end{vmatrix} = 440$$

$$\Delta_1 = \begin{vmatrix} 45 & 11 & 39 \\ 205 & 39 & 161 \\ 963 & 161 & 723 \end{vmatrix} = 0$$

$$\Delta_2 = \begin{vmatrix} 4 & 45 & 39 \\ 11 & 205 & 161 \\ 39 & 963 & 723 \end{vmatrix} = -1320, \quad \Delta_3 = \begin{vmatrix} 4 & 11 & 45 \\ 11 & 39 & 205 \\ 39 & 161 & 963 \end{vmatrix} = 880$$

$c = \frac{\Delta_1}{\Delta} = 0, \quad b = \frac{\Delta_2}{\Delta} = -3, \quad a = \frac{\Delta_3}{\Delta} = 2$

$\therefore y = ax^2 + bx + c$

i.e. $y = 2x^2 - 3x$ //

Q. Fit the following set of data to estimate the coefficient 'a' and 'b' for the function $y = e^{ax+b}$.

x	-3	-1.5	-1	0	1	2.5	4.0
y	5.77	2.77	2.22	1.33	0.77	0.33	0.11

Soln: $y = e^{ax+b}$

Taking ln on both side

$\ln y = ax+b$

$\Rightarrow Y = ax+b$

x_i	x_i^2	y_i	$Y_i = \ln y_i$	$x_i Y_i$
-3	9	5.77	1.7527	-5.2581
-1.5	2.25	2.77	1.0188	-1.5282
-1	1	2.22	0.7975	-0.7975
0	0	1.33	0.2852	0
1	1	0.77	-0.2614	-0.2614
2.5	6.25	0.33	-1.1087	-2.7718
4.0	16	0.11	-2.2073	-8.8292
Σ 2	35.5		0.2768	-19.4462

$$\begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} \Sigma Y_i \\ \Sigma x_i Y_i \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 2 & 35.5 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0.2768 \\ -19.4462 \end{bmatrix}$$

$b = 0.199$

$a = -0.559$ $\quad \therefore y = e^{-0.559x + 0.199}$

Q) $y = ax^b$

$\ln y = \ln(ax^b) = \ln a + \ln x^b$

or, $\ln y = \ln a + b \ln x$

$\Rightarrow Y = A + bX$

Q) $P = P_0 e^{kt}$

$\ln P = \ln(P_0 e^{kt}) = \ln P + \ln e^{kt}$

$\Rightarrow \ln P = \ln P_0 + kt$

$Y = A + BX$

$y = ax^2 + b\log x$

$\frac{y}{\log x} = \frac{ax^2}{\log x} + b$

$Y = aX + b$

$T = be^{t/4} + a$

$Y = bX + a$

$y = \frac{1}{a+bx}$

$\frac{1}{y} = a+bx$

$Y = ax+b$

Runge-kutta Method (R-K Method):

Taylor Series method of solving differential equations numerically handicapped by the problems of finding higher order derivatives to find the accurate value. Euler's method is less efficient in practical problems since it required interval h should be small for obtaining reasonable accuracy.

Runge-kutta method do not required higher order derivatives and they are designed to give greater accuracy with the advantages of requiring functional values at some selected points on the sub-interval. These method are agreed with Taylor Series method upto term h^r where r is the order of R-K method.

Order of R-K Method:

1. Runge kutta 1st order (Euler's Method):

$m_1 = f(x_0, y_0)$

$y_1 = y_0 + m_1 h$ $\quad \left[y_{i+1} = y_i + \text{slope} \times \text{interval} \right]$

2. R-K 2nd order (Heun's Method):

$m_1 = f(x_0, y_0)$

$y_e = y_0 + m_1 h$

$m_2 = f(x_0+h, y_e) = f(x_0+h, y_0+m_1 h)$

$m = \frac{m_1+m_2}{2}$

$\therefore y_1 = y_0 + mh$

③ Rk 3rd order:

$m_1 = f(x_0, y_0)$

$m_2 = f\left(x_0 + h/2,\ y_0 + \frac{m_1 h}{2}\right)$

$m_3 = f\left(x_0 + h,\ y_0 + m_2 h\right)$

$m = \frac{1}{6}\left(m_1 + 4m_2 + m_3\right)$

$y_1 = y_0 + mh$

④ Rk 4th order:

$m_1 = f(x_0, y_0)$

$m_2 = f\left(x_0 + \frac{h}{2},\ y_0 + \frac{m_1 h}{2}\right)$

$m_3 = f\left(x_0 + \frac{h}{2},\ y_0 + \frac{m_2 h}{2}\right)$

$m_4 = f\left(x_0 + h,\ y_0 + m_3 h\right)$

$m = \frac{1}{6}\left(m_1 + 2m_2 + 2m_3 + m_4\right)$

$\therefore y_1 = y_0 + mh$

Q $\frac{dy}{dx} = 3x^2+1$ with $y(1) = 2$, estimate $y(2)$ using

sol: ⓘ $h = 0.5$ ⓘⓘ $h = 0.25$

Given,

$\frac{dy}{dx} = 3x^2+1 = m$

① for $h = 0.5$

1st iteration

$y(1) = 2$

$m = f(x_0, y_0) = f(1, 2) = 3(1)^2 + 1 = 4$

$y(1.5) = y_0 + mh = 2 + 4 \times 0.5 = 4$

2nd iteration

$y(1.5) = 4$

$m = f(x_0, y_0) = f(1.5, 4) = 3(1.5)^2 + 1 = 7.75$

$\therefore y(2.0) = y_0 + mh = 4 + 7.75 \times 0.5 = 7.575$

② for $h = 0.25$

1st iteration

$y(1) = 2$

$m_1 = f(x_0, y_0) = f(1, 2) = 3(1)^2 + 1 = 4$

$y(1.25) = y_0 + mh = 2 + 4 \times 0.25 = 3$

2nd iteration

$y(1.25) = 3$

$m_1 = f(x_0, y_0) = f(1.25, 3) = 3(1.25)^2 + 1 = 5.6875$

$y(1.5) = y_0 + mh = 3 + 5.6875 \times 0.5 = 4.421875$

3rd iteration.

$y(1.5) = 4.421875$

$m = f(x_0, y_0) = f(\cancel{4.421875}, f(1.5, 4.421875) = 7.75$

$y(1.75) = y_0 + mh = 6.35938$

4th iteration.

$y(1.75) = 6.35938$

$m = f(x_0, y_0) = f(1.75, 6.35938) = 10.1875$

$y(2.0) = 6.35938 + 10.1875 \times 0.25$

$= 8.906255$

Q Given the eqn $y'(x) = 2y/x$ with $y(1) = 2$, $y(2) = ?$ with $h = 0.25$

Soln Given $y'(x) = \frac{2y}{x} = m$

Here,

1st iteration.

$y_0 = y(1) = 2$

$m_1 = f(x_0, y_0) = f(1, 2) = \frac{2y}{x} = \frac{2 \times 2}{1} = 4$

$y_e = y_1 + m_1 h = 2 + 4 \times 0.25 = 3$

$m_2 = f(x_0 + h, y_e) = f(1 + 0.25, 3) = f(1.25, 3)$

$= \frac{2 \times 3}{1.25} = 4.8$

$m = \frac{m_1 + m_2}{2} = \frac{4 + 4.8}{2} = 4.4$

$\therefore y(1.25) = y_0 + mh = 2 + 4.4 \times 0.25 = 3.1$

2nd iteration

$y(1.25) = 3.1$

$m_1 = f(x_0, y_0) = f(1.25, 3.1) = 4.96$

$y_e = y_0 + m_1 h = 3.1 + 4.96 \times 0.25 = 4.34$

$m_2 = f(x_0 + h, y_e)$

$= 5.7867$

$m = \frac{m_1 + m_2}{2} = \frac{4.96 + 5.7867}{2} = 5.3733$

$\therefore y(1.5) = y_0 + mh = 4.4433$

3rd iteration

$y(1.5) = 4.4433$

$m_1 = f(x_0, y_0) = 5.92$

$y_e = y_0 + m_1 h = 5.9244$

$m_2 = f(x_0 + h, y_e) = 6.7708$

$m = \frac{m_1 + m_2}{2} = 6.3454$

$y(1.75) = y_0 + mh = 6.0296$

4th iteration

$y(1.75) = 6.0296$

$m_1 = f(x_0, y_0) = 6.891$

$y_e = y_0 + mh = 7.7584$

$m_2 = f(x_0 + h, y_e) = 7.7524$

$m = \frac{m_1 + m_2}{2} = 7.3217$

$y(2.0) = y_0 + mh = 7.8713$

Q $y'(x) = x^2 + y^2$ using $y(0) = 0$, $h = 0.2$ estimate $y(0.4)$

Soln Here,

$y'(x) = \frac{dy}{dx} = x^2 + y^2 = m$

& $y(0) = 0$

7:00 - 8:30 wednesday C(101) → BCE - 48 & 12:15 - 1:45 (cc) BEZ1
7:00 - 8:30 A (501) BCE 11 (C204)
E21
7:00 - 6:30
BEZ1 (C106)

~~2591/069~~
2256/0695-00
2253 x /0695-0

1st iteration

$y(0) = 0$

$m_1 = f(x_0, y_0) = f(0,0) = 0^2 + 0^2 = 0$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right)$

$= f\left(0 + 0.1, 0 + \frac{0 \times 0.2}{2}\right)$

$= f(0.1, 0)$

$= 0.1^2 + 0^2 = 0.01$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right)$

$= f\left(0.1, 0 + \frac{0.01 \times 0.2}{2}\right)$

$= f(0.1, 0.001)$

$= 0.01$

$m_4 = f(x_0 + h, y_0 + m_3 h)$

$= f(0 + 0.2, 0 + 0.01 \times 0.2)$

$= f(0.2, 0.002)$

$= 0.04$

$m = \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$= \frac{1}{6}(0 + 2 \times 0.01 + 2 \times 0.01 + 0.04)$

$= 0.0133$

$y(0.2) = y_0 + mh$

$= 0 + 0.0133 \times 0.2$

$= 0.00267$

2nd iteration A (501) BCE

$y(0.2) = 0.00267$

$m_1 = f(x_0, y_0) = 0.2^2 + 0.00267^2 = 0.04$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right)$

$= f\left(0.2 + \frac{0.2}{2}, 0.00267 + \frac{0.04 \times 0.2}{2}\right)$

$= f(0.3, 0.00667) = 0.09$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right)$

$= f\left(0.3, 0.00267 + \frac{0.09 \times 0.2}{2}\right)$

$= f(0.3, 0.01167) = 0.09.$

$m_4 = f(x_0 + h, y_0 + m_3 h)$

$= f(0.2 + 0.2, 0.00267 + 0.09 \times 0.2)$

$= f(0.4, 0.02067) = 0.160428$

$m = \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$

$= \frac{1}{6}(0.04 + 2 \times 0.09 + 2 \times 0.09 + 0.160428)$

$= 0.0934$

$y(2.0) = y_0 + mh$

$= 0.00267 + 0.0934 \times 0.2$

$\boxed{y(2.0) = 0.02135}$

11 may