1.3

Complex Numbers

Learning outcomes or objectives:
On the completion of this chapter, the students will be enable to
(i) Define a complex number.
(ii) Solve the problems related to algebra of complex numbers.
(iii) Represent complex numbers geometrically
(iv) Find conjugate and absolute value (modulus) of a complex numbers and verify their properties.
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1.7.1 Introduction

The equation of the form
$x^2 + 1 = 0$ or, $x^2 = -1$ and
$x^2 + 4 = 0$ or, $x^2 = -4$ etc. are not solvable in $\mathbb{R}$
because the square of a real number is never negative.

Euler was the first mathematician to introduce the symbol **i** (iota) for the square root of -1 with property $\mathbf{i^2 = -1}$. The symbol i is also called imaginary unit.
By introducing i, the set of real number is extended to new system of numbers, known as **complex numbers system**.

The set of all complex numbers is denoted by $\mathbb{C}$.

Complex Number

An ordered pair (a, b) of two real numbers a and b can be considered as a complex number. A complex number is usually denoted by z or w.

The ordered pair (a, b) is expressed as a + ib, where $i = \sqrt{-1}$.

Also, i = 0 + 1. i = (0,1)

If z = (a, b) = a + ib is a complex number, then **a** is known as its real part and **b** is known as the imaginary part of the complex number z. They are denoted by Re(z) and Im(z) i.e. Re(z) = a and Im(z) = b but a and b both are real numbers.

Note: Any complex number z is said to be purely real if Im(z) = 0 and purely imaginary if Re(z) = 0

Question:- Is a real number a complex number?

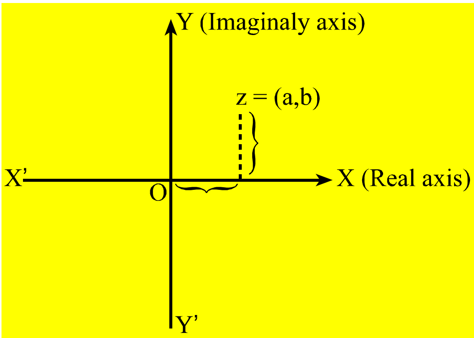
Eg:- Real number = 2 = 2 + 0.i = (2, 0)= Complex number

Henc, a real number is also a complex number.

Geometrical Representation

As we have already known in real number system that any ordered pair of real numbers can be represented as a point on carterian plane with **a** as abscissa and **b** as ordinate. Similarly, a complex number a + ib can be represented in the cartesian plane as a point.

The plane in which a complex number is ploted is known as Argand plane or complex plane. In the plane, *x*-axis is taken as real axis and y-axis is taken as imaginary axis as shown in the figure.



Equality of Complex Numbers:

Two complex numbers are said to be equal if and only if their corresponding parts are equal. i.e. a + ib = c + id if only if a = c and b = d.

Example: Find x and y if x + iy = 5 – 3i

Solution: Equating the corresponding real and imaginary parts, we get x = 5 and y = – 3.

Integral Power of i:

Any integral power of i can be equal to only one of the four quantities 1, – 1, i, – i.

Example: Compute (a) i^3 (b) i^{20} (c) i^{25} (d) i^{-13}

Solution:

(a) $i^3 = i^2 .i = -1. i = -i$ (b) $i^{20} = (i^2)^{10} = (-1)^{10} = 1$

(c) $i^{25} = (i^2)^{12}.i = (-1)^{12}. i = i$ (d) $i^{-13} = \frac{1}{i^{13}} = \frac{1}{(i^2)^6 \cdot i} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = -i$

1.7.2 Algebra of Complex Numbers

Addition of complex numbers

Let z = a + ib and w = c + id be two complex numbers. Then their sum z + w is defined as the complex number (a + c) + i(b + d).

i.e. z + w = (a + c) + i(b + d)

Example: If z = 2 + 3i and w = 1 – 4i then z + w = (2 + 1) + (3 – 4)i = 3 – i.

Properties of Addition of Complex Numbers:

(i) **Commutative**: For any two complex number z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

Proof: Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, $a_1, b_2, a_2, b_2 \in \mathbb{R}$.

$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

= $(a_2 + a_1) + i(b_2 + b_1)$ ($\because$ Addition is commutative in $\mathbb{R}$)

= $z_2 + z_1$

(ii) **Associative**: For any complex numbers z_1, z_2, z_3.

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

Proof: Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$, $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$.

$(z_1 + z_2) + z_3 = [(a_1 + a_2) + i(b_1 + b_2)] + (a_3 + ib_3)$

$= [(a_1 + a_2) + a_3] + i[(b_1 + b_2) + b_3]$

$= [a_1 + (a_2 + a_3)] + i[b_1 + (b_2 + b_3)]$ ($\because$ Addition is associate in $\mathbb{R}$)

$= (a_1 + ib_1) + [(a_2 + a_3) + i(b_2 + b_3)]$

$= z_1 + (z_2 + z_3)$

Hence, addition of complex numbers is associate.

(iii) **Existence of additive identity**:

The complex number $0 = 0 + i.\,0$ is the additive identity element i.e. for any complex number z, $0 + z = z = z + 0$.

Proof: Let $z = a + ib$

$0 + z = (0 + a) + i(0 + b) = a + ib = z$

and $z + 0 = (a + 0) + i(b + 0) = a + ib = z$

(iv) **Additive inverse**: $-z$ is the additive inverse of any complex number z

i.e. $z + (-z) = 0 = (-z) + z$

Proof: Let $z = a + ib$ then $-z = -(a + ib) = -a - ib$

So, $z + (-z) = (a - a) + i(b - b) = 0 + i.0 = 0$

And $(-z) + z = (-a + a) + i(-b + b) = 0 + i.0 = 0$

Multiplication of complex numbers:

Let $z = a + ib$ and $w = c + id$ be any two complex numbers. The product of z and w is denoted by zw and defined by $zw = (ac - bd) + i(bc + ad)$.

Example: if $z = 2 + 3i$ and $w = 5 - 3i$ the

$zw = (2 + 3i)(5 - 3i) = \{2\cdot5 - 3\cdot(-3)\} + i\{3\cdot5 + 2\cdot(-3)\} = 19 + 9i$

Next Method:-

$zw = (2 + 3i)(5 - 3i) = 2\cdot5 - 2\cdot3i + 3i\cdot5 - 3i\cdot3i = 10 - 6i + 15i - 9i^2$

$= 10 + 9i - 9\cdot(-1) = 10 + 9i + 9 = 19 + 9i$

Note: The product zw can actually be computed as below:

$zw = (a + ib)(c + id)$

$= a(c + id) + ib(c + id)$

$= ac + iad + ibc + i^2 bd$

$= (ac - bd) + i(ad + bc)$ ($\because i^2 = -1$)

Properties of Multiplication

i) **Commutative**: For any complex numbers z_1 and z_2, $z_1 z_2 = z_2 z_1$

Proof: Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Then,

$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$

$= (a_2 a_1 - b_2 b_1) + i(b_2 a_1 + a_2 b_1)$ ($\because$ Multiplication is commutative in $\mathbb{R}$)

$= z_2 z_1$.

ii) **Associative**: For any complex numbers z_1 z_2 and z_3.

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Proof: Let $z_1 = a_1 + ib_1$, $z_2 = a_2\ ib_z$ and $a_3 = a_3 + ib_3$, $a_1, b_1, a_2, b_3, a_3\ b_3 \in \mathbb{R}$

$(z_1 z_2) z_3 = [(a_1 + ib_1)(a_2 + ib_2)](a_3 + ib_3)$

$= [(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)](a_3 + ib_3)$

$= [(a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + a_2 b_1) b_3] + i[(a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + a_2 b_1) a_3]$

$= [a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_2 b_3 + a_3 b_2)] + i[b_1 (a_2 a_3 + a_2 b_3) + a_1 (a_3 b_2 + a_2 b_3)]$

$= (a_1 + ib_1)[(a_2 a_3 - b_2 b_3) + i(a_2 b_3 + a_3 b_2)]$

$= z_1 (z_2 z_3)$

(iii) **Existence of identity element**:

The complex number $1 = 1 + i \cdot 0$ is the multiplicative identity element. i.e. for any complex number a, $1.\ z = z = z.1$

Proof: Let $z = a + ib$ be a complex number.

$1.z = (1 + i \cdot 0)(a + ib) = (1 \cdot a - 0.b) + i(0 \cdot a + 1.b) = a + ib = z$

and $a.1 = (a + ib)(1 + i \cdot 0) = (a \cdot 1 - b \cdot 0) + i(a \cdot 0 + b \cdot 1) = a + ib = z$

iv) **Multiplicative inverse**:

$\frac{1}{z}$ or z^{-1} is the multiplicative inverse of any non-zero complex number z.

i.e. $z \cdot z^{-1} = z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z$

If $z = a + ib$ and $\frac{1}{z} = x + iy$

Then $z \cdot \frac{1}{z} = 1$

$\Rightarrow (a + ib)(x + iy) = 1 + i \cdot 0$

$\Rightarrow (ax - by) + i(bx + ay) = 1 + i \cdot 0$

Equating the real and imaginary parts, we get

$ax - by = 1$ ………... (i)

$bx + ay = 0$ ……….. (ii)

Solving equation (i) and (ii) we get

$$x = \frac{a}{a^2 + b^2},\ y = \frac{-b}{a^2 + b^2}$$

Thus, the multiplicative inverse of a non-zero complex number $z = a + ib$ is

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - i\,\frac{b}{a^2 + b^2}$$

Note: Multiplicative inverse of a complex number is denoted by z^{-1}. i.e. $z^{-1} = \frac{1}{z}$.

Example: The multiplicative inverse of $z = 3 + 4i$ is

$$z^{-1} = \frac{3}{3^2 + 4^2} - i\,\frac{4}{3^2 + 4^2} = \frac{3}{25} - i\,\frac{4}{25}$$

Next Method:-

$$z^{-1} = \frac{1}{z} = \frac{1}{3 + 4i} = \frac{1}{3 + 4i} \times \frac{3-4i}{3-4i} = \frac{3-4i}{25} = \frac{3}{25} - i\,\frac{4}{25}$$

(v) Multiplicaion is **distributive** over addition:

For any complex number z_1, z_2, z_3.

(a) Left distribution: $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) Right distribution: $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$

Proof:

(a) Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$, $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$.

$z_1 (z_2 + z_3) = (a_1 + ib_1) [(a_2 + ib_2) + (a_3 + ib)]$

$= (a_1 + ib_1) [(a_2 + a_3) + i (b_2 + b_3)]$

$= [a_1(a_2 + a_3) - b_1 (b_2 + b_3)] + i [a_1 (b_2 + b_3) + b_1 (a_2 + a_3)]$

$= [a_1 a_2 + a_1 a_3 - b_1 b_2 - b_1 b_3) + i (a_1 b_2 + a_1 b_3 + b_1 a_2 + b_1 a_3)$

$= [(a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1)] + [(a_1 a_3 - b_1 b_3) + i(a_1 b_3 + a_3 b_1)]$

$= z_1 z_2 + z_1 z_3$

b) Similar as (a)

Worked out examples

Example 1: Evaluate

(a) $(1,0)^5$ (b) $(0, 1)^7$ (c) $i^{100} + i^{101} + i^{102} + i^{103}$

Solution:

(a) $(1, 0)^5 = (1 + i.0)^5 = 1^5 = 1$

(b) $(0, 1)^7 = (0 + i.1)^7 = i^7 = (i^2)^3.i = (-1)^3.\ i = -1$

(c) $i^{100} + i^{101} + i^{102} + i^{103} = i^{100}\,(1 + i + i^2 + i^3)$

$= i^{100}\,(1 + i - 1 + i^2) \qquad = i^{100}\,(1 + i - 1 + i^2)$

$= i^{100}\,(1 + I - 1 - i) \qquad = i^{100} \times 0$

$= 0$

Example-2: Find the value of x and y if $(x + 2) + iy = (3 + i)(1 - 2i)$

Solution: $(x + 2) + iy = (3 + i)(1 - 2i)$

$= (3{\cdot}1 + 1{\cdot}2) + i(1{\cdot}1 - 3{\cdot}2) \qquad = 5 - 5i$

Equating the real and imaginary parts, we get

$x + 2 = 5$ and $y = -5$

i.e. $x = 3$ and $y = -5$

Example-3: Express the following complex numbers in the form of a + ib (a) $\dfrac{1}{1-i}$

(b) $(1 - i)^9 \left(1 - \dfrac{1}{i^3}\right)$

Solution:

(a) $\dfrac{1}{1-i} = \dfrac{1}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{1+i}{1^2 - i^2} \qquad = \dfrac{1+i}{2}$

$= \dfrac{1}{2} + \dfrac{1}{2}i = a + ib$ where $a = \dfrac{1}{2},\quad b = \dfrac{1}{2}$

(b) $(1 - i)^9\left(1 - \dfrac{1}{i^3}\right)^9 = (1 - i)^9\left(1 - \dfrac{1}{i^2.i}\right)^9$

$= (1 - i)^9\left(1 + \dfrac{1}{i}\right)^9 \ (i^2 = -1) \qquad = (1 - i^9)\,\dfrac{(1+i)^9}{i^9}$

$= \dfrac{(1 - i^2)^9}{(i^2)^4.i} \qquad = \dfrac{(1+1)^9}{(-1)^4\,i}$

$= \dfrac{512}{i} \times \dfrac{i}{i} \qquad = -512i$

$= 0 - 512i = a + ib$, where $a = 0$, $b = -512$.

Example-4: If $\sqrt{x - iy}$ = a – ib, prove that $\sqrt{x + iy}$ = a + ib.

Solution: Given, $\sqrt{x - iy}$ = a – ib

Squaring both sides, $x - iy = (a - ib)^2 = a^2 - 2ab\,i + i^2b^2$

$= (a^2 - b^2) - 2ab\,i$ [since i^2=-1]

Equating the real and imaginary parts,

$$x = (a^2 - b^2) \text{ and } y = 2ab$$

Then, $x + iy = (a^2 - b^2) + 2ab\,i = a^2 + 2ab\,i + i^2b^2$

$\Rightarrow x + iy = (a + ib)^2$

$\Rightarrow \sqrt{x + iy}$ = a + ib

Example-5 Find the additive inverse of 2+3i **.**

Solution: Let z = 2 + 3i

Since, additive inverse of z is –z then

–z = –(2+3i) = –2–3i .

Example-6 Find the multiplicative inverse of 2+3i **.**

Solution: Let z = 2 + 3i

Since, multiplicative inverse of z is z^{-1} or $\frac{1}{z}$ then

$$\frac{1}{z} = \frac{1}{2+3i} = \frac{1}{2+3i} \times \frac{2-3i}{2-3i} = \frac{2-3i}{4-9i^2} = \frac{2-3i}{4+9} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i\,.$$

Exercise

1. Evaluate:

   (a) $(1, 0)^7$ (b) $(0, 1)^{99}$ (c) i^{49} (d) $1 + i^{10} + i^{20} + i^{30}$

2. Compute the following:

   (a) $\sqrt{-49}$ (b) $\sqrt{-4} \times \sqrt{-9}$ (c) $\sqrt{-25} + \sqrt{-4} + 2\sqrt{-9}$

3. Show that $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$

4. Simplify:

   (a) $\frac{1}{i} + \frac{1}{i^2} + \frac{1}{i^3} + \frac{1}{i^4}$ (b) $(1 + i)^4\left(1 + \frac{1}{i}\right)^4$

5. Express the following complex numbers in the form of a + ib:

   (a) (1 + i) (4 – 3i) (b) $\frac{3-2i}{3+2i}$ (c) $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$ d) $\frac{3-\sqrt{-16}}{1-\sqrt{-9}}$

e) $\dfrac{1}{(1+i)^3}$ (f) $\sqrt{\dfrac{1+i}{1-i}}$ (g) $(1-2i)^{-3}$ (h) $(1-i)^9\left(1-\dfrac{1}{i}\right)^9$

6. Find the real number x and y if

(i) $(x+3)+(1-y)i = 7-4i$ (ii) $(x-1)i+(y+1) = (1+i)(4-3i)$

(iii) $x+iy = (2-3i)(3-2i)$ (iv) $\dfrac{x-1}{3+i}+\dfrac{y-1}{3-i} = i$

7. Find the multiplication inverse of the following complex numbers:

(a) $3-2i$ (b) $\dfrac{4+5i}{2+3i}$ (c) $(1+i\sqrt{3})^2$

8. Prove that $\dfrac{3+2i}{2-3i}+\dfrac{3-2i}{2+3i}$ is purely real.

Answer:

1. (a) 1 (b) $-i$ (c) i (d) 0
2. (a) $7i$ (b) -6 (c) $13i$
4. (a) 0 (b) 16
5. (a) $7+i$ (b) $\dfrac{5}{13}-\dfrac{12}{13}i$ (c) $\dfrac{22}{17}+\dfrac{3}{17}i$ (d) $\dfrac{3}{2}+\dfrac{1}{2}i$

   (e) $-\dfrac{1}{4}-\dfrac{1}{4}i$ (f) $\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i$ (g) $-\dfrac{11}{125}-\dfrac{2}{125}i$ (h) $-512i$
6. (i) $x = 4, y = 5$ (ii) $x = 2, y = 6$, (iii) $x = 3, y = -5$ (iv) $x = -4, y = 6$.
7. (a) $\dfrac{3}{13}+\dfrac{2}{13}i$ (b) $\dfrac{23}{41}+\dfrac{2}{41}i$ (c) $-\dfrac{1}{8}-i\dfrac{\sqrt{3}}{8}$

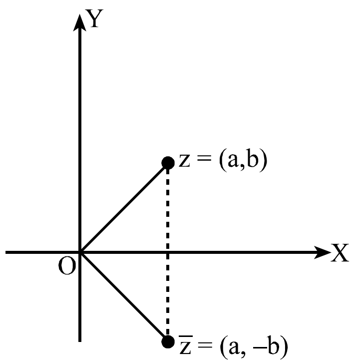
1.7.3 Conjngate of a Complex Number

The conjugate of a complex number $z = a + ib$ is denoted by $\overline{z}$ and defined by $\overline{z} = a - ib$.

The conjngate of a complex number is a complex number obtained by replacing i by $-i$.

Geometrically, it represents the reflection of the point represented by a complex number about real axis (x-axis).

Example: If $z = 3 + 2i$ then its conjngate is $\overline{z} = 3 - 2i$.



Properties of conjugate

For any complex number z, z_1 and z_2,

(i) $z + \bar{z} = 2\text{Re}(z)$

(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $\bar{\bar{z}} = z$

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(vi) $\left(\frac{\bar{z}_1}{\bar{z}_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

(vii) $\overline{z^2} = (\bar{z})^2$

Proofs:

Let $z = a + ib$, $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$,

(i) $z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\text{Re}(z)$

(ii) $z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2i\,\text{Im}(z)$

(iii) $z = a + ib \Rightarrow \bar{z} = a - ib \Rightarrow \bar{\bar{z}} = a + ib = z$

$\therefore$ $\bar{\bar{z}} = z$

(iv) $\overline{z_1 + z_2} = \overline{(a_1 + a_2) + i(b_1 + b_2)} = (a_1 + a_2) - i(b_1 + b_2) = (a_1 - ib_1) + (a_2 - ib_2)$

$= \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 z_2} = \overline{(a_1 + ib_1)(a_2 + ib_2)}$

$= \overline{(a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)}$

$= (a_1a_2 - b_1b_2) - i(a_1b_2 + b_1a_2)$

$= a_1a_2 + i^2 b_1b_2 - ia_1b_2 - ib_1a_2$

$= a_1(a_2 - ib_2) - ib_1(a_2 - ib_2)$

$= (a_1 - ib_1)(a_2 - ib_2)$

$= \bar{z}_1\bar{z}_2$

(vi) $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}$

$$= \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$$

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\,\frac{(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} - i\,\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}$$

Also,

$$\frac{\overline{z}_1}{\overline{z}_2} = \frac{a_1 - ib_1}{a_2 - ib_2}$$

$$= \frac{a_1 - ib_1}{a_2 - ib_2} \times \frac{a_2 + ib_2}{a_2 + ib_2}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} - i\,\frac{b_1 a_2 - b_2 a_1}{a_2^2 + b_2^2}$$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$$

(vii) $\overline{z}^2 = \overline{(a + ib)^2}$

$$= \overline{a^2 + 2aib + (ib)^2}$$

$$= \overline{a^2 - b^2 + 2abi}$$

$$= a^2 - b^2 - 2abi$$

$$= a^2 - 2aib + (ib)^2$$

$$= (a - ib)^2 = \overline{(a + ib)}^2 = (\overline{z})^2$$

1.7.4 Modulus (Absolute value) of a complex number:

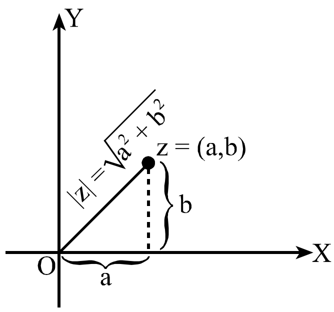
The modulus or absolute value of a complex number $z = a + ib$ is denoted by $|z|$ and defined by $|z| = \sqrt{a^2 + b^2}$. Clearly, $|z|$ is non-negative real number.

Geometrically, the absolute value of a complex number represents the length from the origin to the point represented by the complex number on the plane.

Example: If $z = 3 + 4i$ then $|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

<u>Properties of Absolute values:</u>

For any complex numbers z, z_1, z_2,



(i) $|z| = 0$ if and only if $z = 0$

(ii) $|z| = |\overline{z}|$, i.e. modulus of z and $\overline{z}$ are equal.

(iii) $\text{Re}(z) \le |z|$, $\text{Im}(z) \le |z|$

(iv) $z\,\overline{z} = |z|^2$

(v) $|z_1 z_2| = |z_1|\,|z_2|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $(z_2 \ne 0)$

(vii) $|z_1 + z_2| \le |z_1| + |z_2|$ (Triangle inequality)

Proofs:

(i) Let $z = a + ib$, $a, b \in \mathbb{R}$

$|z| = 0 \Leftrightarrow \sqrt{a^2 + b^2} = 0$

$\Leftrightarrow$ $a^2 + b^2 = 0$

$\Leftrightarrow$ $a = 0$ and $b = 0$

$\Leftrightarrow$ $z = a + ib = 0 + 0i$

$\Leftrightarrow$ $z = 0$

$\therefore$ $|z| = 0$ if and only if $z = 0$.

(ii) Let $z = a + ib$ then $\overline{z} = a - ib$.

$\therefore$ $|z| = \sqrt{a^2 + b^2}$

And $|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$

$\therefore$ $|z| = |\overline{z}|$.

iii) Let $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$

Since $a^2 \le a^2 + b^2$, $a \le \sqrt{a^2 + b^2}$

$\Rightarrow$ $\text{Re}(z) \le |z|$

Also, $b^2 \le a^2 + b^2 \Rightarrow b \le \sqrt{a^2 + b^2}$

$\Rightarrow$ $\text{Im}(z) \le |z|$.

$\therefore$ $\text{Re}(z) \le |z|$ and $\text{Im}(z) \le |z|$.

(iv) Let $z = a + ib$, then $|z| = \sqrt{a^2 + b^2}$

Now, $z\,\overline{z} = (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 + b^2 = (\sqrt{a^2 + b^2})^2 = |z|^2$

$\therefore$ $z\,\overline{z} = |z|^2$

(v) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_z$, $a_1, b_1, a_2 \in \mathbb{R}$, then $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$

Now, $|z_1 z_2| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + b_1 a_2)^2}$

$= \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + b_1^2 a_2^2)}$

$= \sqrt{a_1^2 (a_2^2 + b_2^2) + b_1^2 (a_2^2 + b_2^2)}$

$= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$

$= |z_1| |z_2|$

$\therefore$ $|z_1 z_2| = |z_1| |z_2|$

We have,

$\frac{Z_1}{Z_2} . Z_2 = Z_1$

$\Rightarrow \left|\frac{z_1}{z_2}.z_2\right| = |z_1|$

$\Rightarrow \left|\frac{z_1}{z_2}\right| |z_2| = |z_1|$

$\Rightarrow \left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|.$

(vi) we have to show $|z_1 + z_2| \le |z_1| + |z_2|$

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, $a_1, b_1, a_2, b_2 \in \mathbb{R}$,

Then $|z_1| = \sqrt{a_1^2 + b_1^2}$, $|z_2| = \sqrt{a_2^2 + b_2^2}$

and $|z_1 + z_2| = |(a_1 + a_2) + i(b_1 + b_2)| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$

Now, $|z_1 + z_2| \le |z_1| + |z_2|$ will be true

if $\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} \le \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2}$

i.e. $a_1^2 + 2a_1 a_2 + a_2^2 + b_1^2 + 2b_1 b_2 + b_2^2 \le a_1^2 + b_1^2 + 2\sqrt{a_1^2 + b_1^2}.\sqrt{a_2^2 + b_2^2} + a_2^2 + b_2^2$

i.e. $a_1 a_2 + b_1 b_2 \le \sqrt{a_1^2 + b_1^2}.\sqrt{a_2^2 + b_2^2}$

i.e. $a_1^2 a_2^2 + 2a_1 a_2 b_1 b_2 + b_1^2 b_1^2 \le a_1^2 a_2^2 + a_1^2 b_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2$

i.e. $2a_1 a_2 b_1 b_2 \le (a_1^2 b_1^2)^2 + (b_1 a_2)^2$

i.e. $(a_1 b_2 - b_1 a_2)^2 \ge 0$

This is always true for all $a_1, b_1, a_2, b_2 \in \mathbb{R}$

$\therefore$ $|z_1 + z_2| \le |z_1| + |z_2|$

Alternative method:

Since, $|z_1 + z_1|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$ ($\because |z|^2 = z\overline{z}$)

$= \quad (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= \quad z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\,\bar{z}_2$

$= \quad |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \ (\because z = \bar{\bar{z}})$

$= \quad |z_1|^2 + |z_2|^2 + z_1\,\bar{z}_2 + \overline{z_1\,\bar{z}_2} \ (\because \bar{z}_1\,\bar{z}_2 = \overline{z_1\,z_2}\ )$

$= \quad |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(z\,\bar{z}_2) \ (\because z + \bar{z} = 2\,\mathrm{Re}\,(z)]$

$\leq \quad |z_1|^2 + |z_2|^2 + 2\,|z_1\,\bar{z}_2|$

$= \quad |z_1|^2 + |z_2|^2 + 2|z_1|\,|\bar{z}_2|$

$= \quad |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \ (\because |z| = |\bar{z}|)$

$= \quad (|z_1| + z_2|)^2$

$\because \quad |z_1 + z_2| \leq |z_1| + |z_2|.$

Worked examples

Example 1: Find the conjugate of the complex number $\frac{5-7i}{5+8i}$.

Solution: $\frac{5-7i}{5+8i} = \frac{5-7i}{5+8i} \times \frac{5-8i}{5-8i}$

$= \quad \frac{(5.5-7.8)+i(5(-8)-7.5)}{5^2+8^2}$

$= \quad \frac{-31-75i}{89}$

$= \quad \frac{-31}{89} - \frac{75}{89}$, it is in a+ib form.

$\therefore$ Conjugate of $\frac{5-7i}{5+8i}$ is $\frac{-31}{89} + \frac{75}{89}i$

Example 2: If $x + iy = \frac{a+ib}{c+id}$, prove that $x - iy = \frac{a-ib}{c-id}$

Solution:

We have,

$x + iy = \frac{a+ib}{c+id}$

Taking conjugate, we get

$$\overline{x + iy} = \overline{\left(\frac{a + ib}{c + id}\right)}$$

$$\Rightarrow \quad x - iy = \frac{\overline{a + ib}}{\overline{c + id}} \quad \left[\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\right]$$

$$\Rightarrow \quad x - iy = \frac{a - ib}{c - id}$$

Example 3: If $x + iy = \frac{a + ib}{a - ib}$, prove that $x^2 + y^2 = 1$.

Solution:

Here, $x + iy = \frac{a + ib}{a - ib}$

Taking modulus, we get

$$|x + iy| = \left|\frac{a + ib}{a - ib}\right|$$

Or, $\sqrt{x^2 + y^2} = \frac{|a + ib|}{|a - ib|} \quad \left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$

Or, $\sqrt{x^2 + y^2} = \frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2}}$

Or, $\sqrt{x^2 + y^2} = 1$

Squarig, we get , $x^2 + y^2 = 1$.

Example 4: Find the modulus of $\frac{1}{2+3i}$.

Solution:- $\frac{1}{2+3i} = \frac{1}{2+3i} \times \frac{2-3i}{2-3i} = \frac{2-3i}{4-9i^2} = \frac{2-3i}{4+9} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$

Now, Required modulus = $\left|\frac{2}{13} - \frac{3}{13}i\right| = \sqrt{\left(\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{1}{\sqrt{13}}$

Next Method:- $\left|\frac{1}{2+3i}\right| = \frac{|1|}{|2+3i|} = \frac{1}{\sqrt{(2)^2+(3)^2}} = \frac{1}{\sqrt{13}}$

Exercise

1. Find the conjugate of the following complex numbers.

(a) $(3 + 5i)(7 - 5i)$ (b) $\frac{1}{1 + i}$ (c) $\frac{3 + 4i}{3 - 4i}$ (d) $\frac{(3 - i)^2}{2 + i}$

2. Find the modulus of the following complex numbers.

(a) $(3 + 4i)(5 + 12i)$ (b) $\frac{1}{3 + 5i}$ (c) $\frac{2 + 3i}{2 - 3i}$ (d) $\frac{(1 + i)(1 + \sqrt{3}i)}{1 - i}$

3. If $z = 2 + 3i$ and $w = 1 - i$, verify that

(a) $\overline{z\,w} = \overline{z}.\,\overline{w}$ (b) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ (c) $|zw| = |z|\,|w|$

(d) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ (e) $|z + w| \le |z| + |w|$

4 (a) If $x - iy = \frac{3 - 2i}{3 + 21}$. Prove that $x^2 + y^2 = 1$.

(b) If $\sqrt{x + iy} = a + ib$, prove that $\sqrt{x - iy} = a - ib$.

(c) If $(3 - 4i)(a + ib) = 3\sqrt{5}$, prove that $a^2 + b^2 = \frac{9}{5}$.

Solution:- Given, $(3 - 4i)(a + ib) = 3\sqrt{5}$

Taking mosulus on both sides,

$$|(3 - 4i)(a + ib)| = \left|3\sqrt{5}\right|$$

$$\Rightarrow |(3 - 4i)||a + ib| = 3\sqrt{5}$$

$\Rightarrow \sqrt{3^2 + (-4)^2}\,\sqrt{a^2 + b^2} = 3\sqrt{5}$

$\Rightarrow 5\,\sqrt{a^2 + b^2} = 3\sqrt{5}$

$\Rightarrow \sqrt{a^2 + b^2} = \frac{3\text{Error ! Bookmark not defined .}}{5}$

Squaring both sides, we get

$a^2 + b^2 = \frac{9}{5}$

(d) If $x - iy = \sqrt{\frac{1 - i}{1 + i}}$, prove that $x^2 + y^2 = 1$

Answers

1.	(a) $46 - 20i$	(b) $\frac{1}{2} + \frac{1}{2} i$	(c) $\frac{-7}{25} - \frac{24}{25} i$	(d) $2 + 4i$
2	(a) 65	(b) $\frac{1}{\sqrt{34}}$	(c) 1	(d) 2

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