Solving polynomial

Newton's method / Horner's Rule:

Synthetic Division

⇒ A polynomial of degree 'n' can be expressed as,

$$p(x) = (x - x_r)\, q(x)$$

Where, x_r is root of the polynomial $p(x)$

$q(x)$ is the quotient polynomial of degree $n-1$.

Once a root is found, we can use this root to find a lower degree polynomial $q(x)$ by dividing $p(x)$ by $(x - x_r)$ using a process known as synthetic division. Here, the activity of reducing the degree of polynomial is known as deflection.

Again, the quotient polynomial $q(x)$ can be used to determine the other roots of $p(x)$. A further deflection can be performed and the process can be continue until the degree is reduced to 1.

Synthetic division of a polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$ by $(x - \alpha)$ is done as follows.

α	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
		αb_0	αb_1	$\cdots$	αb_{n-2}	αb_{n-1}
	a_0 $(= b_0)$	$a_1 + \alpha b_0$ $(= b_1)$	$a_2 + \alpha b_1$ $(= b_2)$	$\cdots$	$a_{n-1} + \alpha b_{n-2}$ $(= b_{n-1})$	$a_n + \alpha b_{n-1}$ $(= R)$

Therefore, quotient $= b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1}$

Qn ⊗ Divide the polynomial $f(x) = x^3 + x^2 - 3x - 3$ by $(x-2)$ using synthetic division and apply Newton Raphson method.

Soln

Here, the given function is,

$$f(x) = x^3 + x^2 - 3x - 3$$

Now, Iteration 1:

2	1	1	-3	-3
	↓	2	6	6
	1	3	3	3 = R = $f(x_1)$
	↓	2	10	
	1	5	13 = $f'(x_1)$	

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{3}{13} = 1.769231$$

Iteration 2.

1.769231	1	1	-3	-3
	↓	1.769231	4.89941	3.36049
	1	2.769231	1.89941	0.360493 = R = $f(x_2)$
	↓	1.769231	8.02959	
	1	4.53846	9.928997 = $f'(x_2)$	

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.769231 - \frac{0.360493}{9.928997}$$

$$= 1.73292.$$

Iteration: 3

1.73292	1	1	-3	-3
	↓	1.73292	4.73593	3.00823
	1	2.73292	1.73593	0.00823 $= f(x_3)$
	↓	1.73292	7.73834	
	1	4.46584	9.47487 $= f'(x_3)$	

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.73292 - \frac{0.00823}{9.47487}$$

$$= 1.73379$$

Iteration: 4

1.73379	1	1	-3	-3
	↓	1.73379	4.73982	3.01648
	1	2.73379	1.73982	0.01648 $= f(x_4)$
	↓	1.73379	7.74585	
	1	4.46758	9.48567 $= f'(x_4)$	

$$\therefore x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.73379 - \frac{0.01648}{9.48567}$$

$$= 1.73205$$

Iteration: 5

1.73205	1	1	-3	-3
	↓	1.73205	4.73205	2.99999
	1	2.73205	1.73205	-0.00001 $= f(x_5)$
	↓	1.73205	7.73204	
	1	4.4641	9.46409 $= f'(x_5)$	

$$x_6 = 1.73205 - \frac{(-0.00001)}{9.46409}$$

$$= 1.732051$$

Hence, the required root after fifth iteration is 1.732051.

Horner's Rule

→ Let us consider the evaluation of a polynomial by using Horner's rule as follows

$$f(x) = ((\cdots((a_n x + a_{n-1})x + a_{n-2})x + \cdots + a_1)x + a_0) \quad \text{— eq}^n(1)$$

Here, the innermost expression $a_n x + a_{n-1}$ is evaluated first.

Horner's method is also known as nested multiplication and is implemented using the following algorithm

$$p_n = a_n$$
$$p_{n-1} = p_n x + a_{n-1}$$
$$\cdots$$
$$p_j = p_{j+1} x + a_j$$
$$\cdots$$
$$p_1 = p_2 x + a_1$$
$$f(x) = p_0 = p_1 x + a_0$$

(*) Evaluate the polynomial $f(x) = x^3 - 4x^2 + x + 6$ using Horner's rule at $x = 2$.

Soln. Here, $n = 3$. (degree of polynomial)

$a_3 = 1,\ a_2 = -4,\ a_1 = 1,\ a_0 = 6$

Then,

$$p_3 = a_3 = 1$$
$$p_2 = p_3 x + a_2 = 1*2 + (-4) = 2 - 4 = -2$$
$$p_1 = p_2 x + a_1 = -2*2 + 1 = -4 + 1 = -3$$
$$p_0 = f(2) = p_1 x + a_0 = -3*2 + 6 = -6 + 6 = 0$$

Hence, $f(2) = 0$. Ans