4

Matrices and Determinants

Learning outcomes or objectives:

On the completion of this chapter, the students will be enable to

(i) know the meaning of matrix, types, equality of matrices, algebra of matrices and solve the related problems.

(ii) know the meaning of determinant, singular and non-singular matrix,

(iii) obtain transpose of a matrix and know its properties,

(iv) calculate minors, cofactors, adjoint, determinant and inverse of a square matrix,

(v) say the properties of determinant(without proof) and solve the problems using properties of determinants,

(vi) say the meaning of transformation, linear transformation, orthogonal transformation, rank of matrices and solve the related problems.

Introduction:

The matrix has the long history of application. The term matrix was introduced by English mathematician Arthur Cayley (1821-1895) and developed the theory of matrices in the connection with the linear transformation in 1857.Matrix has the wide application in the engineering, physics, economist and statistics as well as various branches of mathematics. In mathematics a matrix is rectangular array or table of a numbers, symbols or expressions. Matrices are often denoted using capital letters such as A,B,C,D…..X,Y,Z but its members are denoted by small letters, numbers and objects etc and it is enclosed round bracket ( )

or [ ].But, the theory of determinants is said to have originated with Leibniz in 1693 in connection with the system of linear equations .Here we used matrix and determinant to solve linear equations.

Definition: A matrix is the rectangular array of the numbers/elements which are arranged into different rows and columns and it is enclosed by round (.) or square [ ] brackets. Mtrices are denoted by A, B, C, ..., X, Y, Z.

Examples

1. The linear systems 3x –4y = 2 and 3x +7y = 1can be represented as below as the matrix form as $\begin{bmatrix} 3 & -4 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2. The information below are the stationary obtained by the three students.

Students	**Items**		
	Pencil	eraser	Books
Ayush	2	3	4
Yumon	3	4	5
prayush	1	3	3

can be presented in the rectangular form as

$$\begin{array}{cl}
\begin{array}{ccc} 1^{st}\text{ Column} & 2^{nd}\text{Column} & 3^{rd}\text{Column} \\ C_1 & C_2 & C_3 \\ \uparrow & \uparrow & \uparrow \end{array} & \\
\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 1 & 3 & 3 \end{bmatrix} & \begin{array}{l} \rightarrow 1^{st}\text{ Row; } R_1 \\ \rightarrow 2^{nd}\text{ Row; } R_2 \\ \rightarrow 3^{rd}\text{ Row; } R_3 \end{array}
\end{array}$$

1.6.1 Notation

A matrix is usually denoted by a capital letters which is printed in the boldface font (e.g. A, B, C, D……..X,Y,Z). The elements of the matrix are printed in the lower cade letters with double subscript (e.g. a_{ij}, b_{ij}, x_{ij}etc).The rows of the matrices are denoted by R_1,R_2,R_3……………and columns of the matrices are denoted by C_1, C_2, C_3…………..etc. In the matrix A, a_{13} is the element in the first row and third column. For examples: The general form for 2× 2 and 3×3 matrices are;

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{and} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

1.6.2 order or size of a matrix:

The number of rows and the number of columns contains in a matrix is called size or order of a matrix. If a matrix A has m rows and n columns is called m × n matrix and can be written as;

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Since, the expression m × n read as m by n. An element of a matrix is denoted by $\mathbf{a_{ij}}$, where i means row and j means column.

Example:1 If $A = \begin{bmatrix} 3 & -5 \\ 4 & 0 \\ -1 & 7 \end{bmatrix}$. Then find the value of the following:

(a) The order of A (b) a_{22}, a_{31} and a_{12} (c) i for $a_{i1}=3$.

Solution: Here,

(a) The number of rows of matrix A =3.

The number of columns of matrix A =2.

∴The order of matrix A = 3 × 2.

(b) a_{22} = the element (entry) in second row and second column of matrix A i.e.0

∴ $a_{22} = 0$

a_{31} = the element in third rows and first column of matrix A i.e.0

∴ $a_{31} = -1$

a_{12} = First row and second column of matrix A i.e.–5

∴ $a_{12} = -5$

(c) since, the element 3 is the position of first row and first column i.e. $a_{11} = 3$

∴ i = 1.

Example:2 Construct a 2 × 2 matrix whose a_{ij} is given by $a_{ij} = 3i - 2j$.

Solution: let a 2 × 2 matrix be $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Given, $\mathbf{a_{ij} = 3i - 2j}$

$a_{11} = 3 \times 1 - 2 \times 1 = 3 - 2 = 1.$

$a_{12} = 3 \times 1 - 2 \times 2 = 3 - 4 = -1.$

$a_{21} = 3 \times 2 - 2\times1 = 6 - 2 = 4.$

$a_{22} = 3 \times 2 - 2\times2 = 6 - 4 = 2.$

∴ Required 2 × 2 matrix is $A= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 4 \end{bmatrix}$

1.6.3 Types of Matrices

1. Row matrix:

A matrix having only one row is called a row matrix or a row vector.

For example,

A = [-3 5]is a row matrix of order 1 × 2.

B= [a b c d ………………n] is a row matrix of order 1×n

2. Column matrix

A matrix having only one column is called a column matrix or a column vector.

For examples,

[a], $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} a \\ b \\ c \\ d \\ . \\ . \\ . \\ n \end{bmatrix}$ are also column matrices of order 1×1,2×1 and n×1 respectively.

3. Null or zero matrix

A matrix whose each entries is zero is called a null matrix or a zero matrix. A null matrix of order m × n is denoted by A_{mn} or simply by A.

For example,

$A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix of order 2 × 2.

Which is actually denoted by $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. Rectangular matrix

A matrix which is not a square matrix is called a rectangular matrix. The number of rows is not equal to the number of columns in a rectangular matrix.

For example, $\begin{bmatrix} 1 & -2 & 5 \\ 3 & 6 & 9 \end{bmatrix}$ is a rectangular matrix of order 2 × 3 because number of rows and number of columns are not equal to each other.

5. **Square matrix:**

A matrix having equal number of rows and columns is called a square matrix. A matrix of order n × n is called a square matrix of order n.

For example, the matrix

$A = \begin{bmatrix} 2 & 3 & 4 \\ -4 & 5 & 6 \\ 1 & -3 & 7 \end{bmatrix}$ is a square matrix of order 3×3 because it has equal number of rows and columns.

6. **Diagonal matrix**

A square matrix whose all the non-diagonal elements are zero is called a diagonal matrix.

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix of order 3×3.

$B = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is a diagonal matrix of order 2×2

7. **Scalar matrix**

A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix. Thus a scalar matrix is a square matrix with all the diagonal entries equal and zero elsewhere.

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ are scalar matrices of order 2×2 and 3 × 3 respectively.

8. **Unit or Identity matrix**

A diagonal matrix in which each element in the principal diagonal is unity is called a unit matrix or an identity matrix. An identity matrix of order n is denoted by I_n or simply by **I**.

For examples,

$I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the unit matrices of order 1×1, 2×2 and 3×3 respectively.

9. Triangular matrix

A square matrix in which all the elements below or above the principal (main) diagonal are zero is called a triangular matrix. Thus there are two types of triangular matrices.

(a) Upper triangular matrix

A square matrix having all the elements below the principal diagonal are zero is called an upper triangular matrix.

For example,

$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & f \end{bmatrix}$ are upper triangular matrices.

(b)Lower triangular matrix

A square matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix.

For example,

$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$, $B = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ are lower triangular matrices.

Note: A diagonal matrix is both an upper triangular and a lower triangular matrix.

10. Symmetric Matrix or Even Matrix.

A square matrix $A = (a_{ij})$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all i and j.

For example; $P = \begin{bmatrix} 2 & 5 & 7 \\ 5 & 8 & 6 \\ 7 & 6 & 9 \end{bmatrix}$ is a symmetric matrix.

Note: In a symmetric matrix, the elements placed symmetrically about the principal diagonal are equal.

11. Skew - Symmetric matrix (Anti - symmetric or odd matrix)

A square matrix $A = (a_{ij})$ is called a skew - symmetric matrix if $a_{ij} = -a_{ji}$, for all i and j.

Thus, the diagonal elements of a skew-symmetric matrix are zero.

For example, $A = \begin{bmatrix} 0 & 5 & -4 \\ -5 & 0 & 3 \\ 4 & -3 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Note:- Since, $a_{ij} = -a_{ji}$. For, i= j, $a_{ii} = -a_{ii} \Rightarrow a_{ii} + a_{ii} = 0 \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$.

1.6.4 Equality of Matrices

Definition: Two matrices A and B of the **same order** are said to be equal matrices if their **corresponding elements are same** i.e. Matrix A = Matrix B.

Example: Let $P = \begin{bmatrix} a & e & b \\ c & d & e \\ x & y & z \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 7 & 8 \\ 1 & 5 & 6 \\ 0 & 3 & 8 \end{bmatrix}$

Solution: Here,

Matrix p = Matrix Q .

Or, $\begin{bmatrix} a & e & b \\ c & d & e \\ x & y & z \end{bmatrix} = \begin{bmatrix} 3 & 7 & 8 \\ 1 & 5 & 6 \\ 0 & 3 & 8 \end{bmatrix}$

Comparing the corresponding elements, we get.

a=3,e=7,b=8

c=1,d=5,e =6

x=0,y=3,z=8

1.6.5 Operation on Matrices

The operation of the matrices is given below:

i. Addition of matrices
ii. Multiplication of a matrix by a scalar (real constant).
iii. Difference of a matrix from a matrix.
iv. Multiplication of matrices.

(i) Addition of matrices

Definition: If Aand B are two matrices of **equal order** then the sum of A and B, denoted by A + B, is the matrix which is obtained by adding corresponding elements of AandB.

Example 1

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 4 & 7 \\ 8 & 2 & 5 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 3 \\ 1 & 4 & 7 \\ 8 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 7 \\ 1 & 5 & 13 \\ 9 & 5 & 7 \end{bmatrix}$

(ii) Multiplication of a matrix by a scalar

Definition: Let P be a matrix and $k \neq 0$. Then the matrix kP obtained by multiplying each element of P by k is called the scalar multiple of P by k.

Example

If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $2P = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$ is the scalar multiple of P by 2 and $(-1)\,P = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is the scalar multiple of P by –1.

(ii) The scalar multiple of a matrix A by –1,denoted by –A, is called additive inverse of A or negative of A.

(iii) Difference of a matrix from a matrix

Definition: Let P and Q be any two matrices of same order. Then the difference of matrix Q from P denoted by P – Q is the matrix given by P + (–1) Q.

i.e. P – Q = P + (–1)Q.

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -4 \\ -5 & -3 \end{bmatrix}$.Then

A – B = A + (–1) B

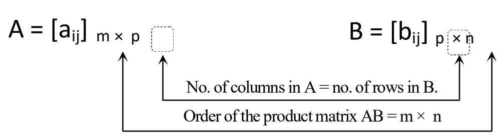
$$= \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} + (-1)\begin{bmatrix} 7 & -4 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -5 \\ -1 & -4 \end{bmatrix}$$

(iv) Multiplication of Matrices

Two matrices A and B are said to be conformable or computable for the product AB if the **number of columns of Matrix A is equal to the number of rows of Matrix B.**

The following diagram may give the idea for the multiplication of two matrices.



Note: If the product AB exists, then it is not necessary that the product BA also exists.

Examples

1.Let $A = \begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix}_{3 \times 2}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{2 \times 2}$

Then, AB = $\begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix}_{3\times 2} \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{2\times 2}$

= $\begin{bmatrix} ae+pg & af+ph \\ be+qg & bf+qh \\ ce+rg & cf+rh \end{bmatrix}$ it order is 3×2

But BA doesn't exist since the number of columns in B is not equal to the number of rows in A.

BA = $\begin{bmatrix} e & f \\ g & h \end{bmatrix}_{2\times 2} \begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix}_{3\times 2}$ is impossible.

2 .Let P = $\begin{bmatrix} 1 & 5 & 6 \\ 1 & 2 & 9 \end{bmatrix}$ and Q = $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Find PQ if possible.

Solution: Here,

The order of Matrix P = $2\times$ **3** and order of Matrix Q = **3** $\times 1$

Since, P and Q are computable as the columns of matrix p is equal to rows of Matrix Q. Then

PQ = $\begin{bmatrix} 1 & 5 & 6 \\ 1 & 2 & 9 \end{bmatrix} . \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 1\times 2 & +5\times 3 & +6\times 4 \\ 1\times 2 & +2\times 3 & +9\times 4 \end{bmatrix} = \begin{bmatrix} 2 & +15 & +24 \\ 2 & +6 & +36 \end{bmatrix} = \begin{bmatrix} 41 \\ 44 \end{bmatrix}$, its order is 2×1

1.6.6 Properties of Matrix Multiplication

The basic properties of addition for real numbers also hold true for matrices.

Let A, B and C be m × n matrices, **I** is the n × n identity matrix and O is the n × n zero matrix.

(i) The commutative property of multiplication does not hold i.e. $AB \neq BA$ for $A \neq B$ and $A \neq B^{-1}$ or $B \neq A^{-1}$.

(ii) Associative property of multiplication: (AB)C = A(BC)

(iii) Distributive properties: A(B+C) = AB + AC and (B + C)A= BA + CA.

(iv) Multiplication identity property: IA = A and AI = A.

(v) Multiplication property of zero: 0 A = 0 and A 0 = 0..

1.6.7 Transpose of a matrix

Definition: A matrix which is obtained by switching(interchanging) its rows with/and its columns is called a transpose matrix. If P be a given matrix of order m × n. Then the matrix $\mathbf{P^T}$ or $\mathbf{P'}$ or $\mathbf{P^t}$ or $\overline{\mathbf{P}}$ of order n × m obtained by **interchanging the rows and columns** of P is called the transpose of P.

Symbolically, if P = $[a_{ij}]_{m \times n}$, then $P^T = [a_{ji}]_{n \times m}$

For example,

If $P = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 5 & 0 \end{bmatrix}_{2 \times 3}$. Then, $P^T = \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 1 & 0 \end{bmatrix}_{3 \times 2}$

Here, we see that, $(i, j)^{th}$ element of P = $(j, i)^{th}$ element of P^T.

1.6.8 Properties of Transpose of Matrices:

If A and B be two matrices of same orders. Then,

(i) The transpose of trapnspoe of a matrix is equal to the given matrices i.e. $(A^T)^T = A$.

(ii) The transpose of the sum is the sum of their transposes. $(A + B)^T = A^T + B^T$ (i.e. the transpose of the sum is the sum of their transposes).

(iii) The transpose of a scalar multiple is the scalar multiple of its transpose) $(kA)^T = kA^T$

(iv) The transpose of the product equals to the product of their transposes but in reverse order.

i. e. $(AB)^T = B^T \cdot A^T$.

Now, let us verify the above properties with examples,

(i) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Then, $A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$(A^T)^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = P.$

$\therefore (A^T)^T = P.$

(ii) Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 7 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 7 & 8 \\ 0 & 4 & 2 \end{bmatrix}$. Then

$A + B = \begin{bmatrix} 3 & 2 & 5 \\ 7 & -1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 7 & 8 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 13 \\ 7 & 3 & 6 \end{bmatrix}$

$\therefore (A + B)^T = \begin{bmatrix} 2 & 7 \\ 9 & 3 \\ 13 & 6 \end{bmatrix}$

and $A^T + B^T = \begin{bmatrix} 3 & 7 \\ 2 & -1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 7 & 4 \\ 8 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 7 \\ 9 & 3 \\ 13 & 6 \end{bmatrix}$$

Thus $(A + B)^T = A^T + B^T$

(iii) Let $A = \begin{bmatrix} a \\ b \end{bmatrix}$ and k = 3. Then

$$kA = 3\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}$$

$\therefore$ $(kA)^T = [3a \quad 3b]$

And, $kA^T = 3\begin{bmatrix} a \\ b \end{bmatrix}^T = 3[a \ \ b] = [3a \ \ b]$.

Thus $(kA)^T = k.A^T$.

Example 1

If $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$, then show that $(AB)^T = A^T B^T$.

Solution:

Here, $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

$\therefore$ $A^T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $B^T = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2.(-2) + 0.3 & 2.1 + 0.2 \\ 1.(-2) + 3.3 & 1.1 + 3.2 \end{bmatrix}$$

$$AB = \begin{bmatrix} -4 & 2 \\ 7 & 7 \end{bmatrix}$$

$\therefore$ $(AB)^T = \begin{bmatrix} -4 & 7 \\ 2 & 7 \end{bmatrix}$ .....(i)

Next, $B^T. A^T = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -2 \times 2 + 3 \times 0 & -2 \times 1 + 3 \times 3 \\ 1 \times 2 + 2 \times 0 & 1 \times 1 + 2 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 + 9 \\ 2 + 0 & 1 + 6 \end{bmatrix}$$

$\therefore$ $B^T. A^T = \begin{bmatrix} -4 & 7 \\ 2 & 7 \end{bmatrix}$ ....(ii)

Hence, from (i) and (ii) $(PQ)^T = Q^T . P^T$.Proved

Example 2

If $P = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 2 & -3 & -4 \end{bmatrix}$, then show that

$P + P^T$ is a symmetric and $P - P^T$ is a skew- symmetric matrix.

Solution:

Here, $P = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 2 & -3 & -4 \end{bmatrix}$

$\therefore P^T = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 5 & -3 \\ 4 & 7 & -4 \end{bmatrix}$

Now, $P + P^T = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 2 & -3 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 2 \\ 3 & 5 & -3 \\ 4 & 7 & -4 \end{bmatrix}$

$= \begin{bmatrix} 2+2 & 3+1 & 4+2 \\ 1+3 & 5+5 & 7-3 \\ 2+4 & -3+7 & -4-4 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 6 \\ 4 & 10 & 4 \\ 6 & 4 & -8 \end{bmatrix}$

which is a symmetric matrix.

Next, $P - P^T = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 2 & -3 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 2 \\ 3 & 5 & -3 \\ 4 & 7 & -4 \end{bmatrix} = \begin{bmatrix} 2-2 & 3-1 & 4-2 \\ 1-3 & 5-5 & 7+3 \\ 2-4 & -3-7 & -4+4 \end{bmatrix}$

$= \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 10 \\ -2 & -10 & 0 \end{bmatrix}$

which is a skew-symmetric matrix.

Example 3

Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & -3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric matrix and a skew -symmetric matrix.

Solution: Here,

$B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ and $B^T = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & -4 & -3 \end{bmatrix}$

Now, Symetirc Mtrix + Skew-symetric Matrix = $(B + B^T) + (B - B^T) = B + B^T + B - B^T = 2B$

So, $2B = (B + B^T) + (B - B^T)$

Or, $B = \frac{1}{2}[(B + B^T) + (B - B^T)]$

i.e. $B = \frac{1}{2}(B + B^T) + \frac{1}{2}(B - B^T)$ ……………..(i)

Now,

$$(B + B^T) = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & -2-1 & -4+1 \\ -1-2 & 3+3 & 4-2 \\ 1-4 & -2+4 & -3-3 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix}$$

And $$(B - B^T) = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & -2+1 & -4-1 \\ -1+2 & 3-3 & 4+2 \\ 1+4 & -2-4 & -3+3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$$

Hence, from equation (i), $B = \frac{1}{2}(B + B^T) + \frac{1}{2}(B - B^T)$

$$= \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4+0 & -3-1 & -3-5 \\ -3+1 & 6+0 & 2+6 \\ -3+5 & 2-6 & -6+0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & -4 & -8 \\ -2 & 6 & 8 \\ 2 & -4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -2 \end{bmatrix}$$

Exercise

1. If $P = \begin{bmatrix} 4 & -5 \\ 3 & 6 \end{bmatrix}_{2\times 2}$ $Q = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}_{2\times 2}$ and k = 3,

then verify that

(a) $(P^T)^T = P$ (b) $(P + Q)^T = P^T + Q^T$

(c) $(kP)^T = kP^T$ (d) $(PQ)^T = Q^T P^T$

2. If P = $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}_{2\times 2}$ and Q = $\begin{bmatrix} -2 & -9 \\ -1 & -5 \end{bmatrix}$, then show that $(PQ)^T = Q^T P^T$.

3. If P = $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then show that

   (a) $P + P^T$ is a symmetric matrix; and

   (b) $P – P^T$ is a skew-symmetric matrix.

4. Express the following matrices as the sum of a symmetric and a skew-symmetric matrix.

   (a) $\begin{bmatrix} 2 & 3 & -4 \\ -3 & 5 & 1 \\ 4 & 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

5. For a matrix P = $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$, verify that PP^T and P^TP both are symmetric matrices.

6. If P = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ and $g(y) = y^2 -5y + 7$, prove that g(P) = 0.

   [Note:- 7 = 7I, I is a 2 x 2 identity matrix]

7. If A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then show that $A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$.

1.6.9 Determinants

The determinant of a matrix is a number that is specially defined only for square matrices. If A be a square matrix of order n × n then the determinant of matrix A is denoted by det. A or |A|.

Determinant of a square matrix of order 1.

Let A = [a] be a square matrix of order 1×1. Then the determinant of the matrix A is defined to be equal to a.

Examples:

(i) If A = [–3], then |P| = |– 3| = – 3.

(ii) If A = [9], then |P| = | 9 | = 9.

(iii) If P = [0], then |P| = |0| = 0.

Determinant of a square matrix of order 2 × 2.

Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ be a square matrix of order 2 × 2.

Then the determinant of the matrix A is A = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad – bc.

(i) If A = $\begin{bmatrix} -2 & 5 \\ -6 & 0 \end{bmatrix}$, then |A|= $\begin{vmatrix} -2 & 5 \\ -6 & 0 \end{vmatrix}$ = – 2 × 0 – 6 × 5 = 0 – 30 = – 30.

(ii) If A = $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2$.

Minor and cofactors: Minors and cofactors are most crucial parts in finding the adjoint and the inverse of a matrix. To find the determinants of a large square matrix more than 2 orders, we use minor and cofactors to find the inverse of a matrix.

Definition: If $P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$ be a 3 × 3 matrix. The determinant of 2× 2 matrix formed by omitting the i^{th} row and j^{th}column of P is called the minor of the element $p_{ij}$and is denoted by number M_{ij}. Thus

M_{11} = minor of $a_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

M_{12} = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

M_{13} = minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

And so on.

Definition:The cofactor $A_{ij}$of the i^{th} row and j^{th} column element a_{ij} of the 3 ×3 matrix A is the number $A_{ij} = (-1)^{i+j}M_{ij}$,where i=1,2,3 and j= 1,2,3.

i.e. cofactor of $a_{ij} = A_{ij} = \begin{Bmatrix} M_{ij} & \text{if} & i+i & \text{is} & \text{even} \\ -M_{ij} & \text{if} & i+j & \text{is} & \text{odd} \end{Bmatrix}$

Examples: **Find the Minor and cofactor of** $\begin{bmatrix} 2 & -3 & 4 \\ 4 & -5 & 6 \\ 3 & 7 & 8 \end{bmatrix}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 4 \\ 4 & -5 & 6 \\ 3 & 7 & 8 \end{bmatrix}$ then

M_{11} = minor of $a_{11} = \begin{vmatrix} -5 & 6 \\ 7 & 8 \end{vmatrix}$

M_{12} = minor of $a_{12} = \begin{vmatrix} 4 & 6 \\ 3 & 8 \end{vmatrix}$

M_{13} = minor of $a_{13} = \begin{vmatrix} 4 & -5 \\ 3 & 7 \end{vmatrix}$

M_{21} = minor of $a_{21} = \begin{vmatrix} -3 & 4 \\ 7 & 8 \end{vmatrix}$

M_{22} = minor of $a_{22} = \begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix}$

M_{23} = minor of $a_{23} = \begin{vmatrix} 2 & -3 \\ 3 & 7 \end{vmatrix}$

M_{31} = minor of a_{31} = $\begin{vmatrix} -3 & 4 \\ -5 & 6 \end{vmatrix}$

M_{32} = minor of a_{32} = $\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix}$

M_{33}= minor of a_{33}= $\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix}$

And ,

A_{11} = co-factor of M_{11}=$(-1)^{1+1}\begin{vmatrix} -5 & 6 \\ 7 & 8 \end{vmatrix} = -82$

A_{12}= co-factor of M_{12}= $-\begin{vmatrix} 4 & 6 \\ 3 & 8 \end{vmatrix} = -14$

A_{13} = co-factor of M_{13}= $+\begin{vmatrix} 4 & -5 \\ 3 & 7 \end{vmatrix} = 43$

A_{21} = co-factor of M_{21} = $-\begin{vmatrix} -3 & 4 \\ 7 & 8 \end{vmatrix} = 52$

A_{22} =co-factor of M_{22}= $+\begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} = 4$

A_{23} =co-factor of M_{23}= $-\begin{vmatrix} 2 & -3 \\ 3 & 7 \end{vmatrix} = 23$

A_{31}= co-factor of M_{31} = $+\begin{vmatrix} -3 & 4 \\ -5 & 6 \end{vmatrix} = 2$

A_{32} =co-factor of M_{32} = $-\begin{vmatrix} 2 & 4 \\ & 4 \\ 6 & \end{vmatrix} = 4$

A_{33}= co-factor of M_{33}= $+\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix} = 2$

Determinant of a square matrix of order 3 × 3.

Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3 × 3.

Then the determinant of the matrix A is defined to be the number $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$; where A_{11}, A_{12} and A_{13} are the cofactors of a_{11}, a_{12} and a_{13} respectively.

i.e. $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$

$\therefore\ |A| = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$

$= (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Which is called the value of determinant of |A| in the expansion along the elements of first row. The value of determinant will not be changed if by expanding form any rows or any columns with their signs given by $(-1)^{i+j}$; where i denotes the row and j denotes the column in which the element lies in the array.

For example, expanding |A| along the elements of second column,

we have

$$|A| = (-1)^{1+2} a_{12} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13})$$

Notes:

(i) While expanding a determinant of order 3 along the elements of any row or column we have to choose the appropriate signs as shown in the above problem.

(ii) Usually, it is convenient to expand a determinant about the row or column that contains most zeros.

For example,

If $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 7 & 3 \\ 5 & 0 & 4 \end{vmatrix}$, then expanding |A| about 2nd column,

we have, $|A| = -0 \begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} + 7 \begin{bmatrix} 1 & -1 \\ 5 & 4 \end{bmatrix} - 0 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = 7(4+5) = 7 \times 9 = 63$

(iii) A determinant of order higher than 3 can be evaluated by the expansion of the determinant along the elements of any row or column. For example,

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

$= a_{11}A_{11} + a_{12} + A_{12} + \dots. + a_{1j}A_{1j} + \dots. + a_{1n}A_{1n}$

$= a_{21}A_{21} + a_{22} + A_{22} + \dots. + a_{2j}A_{2j} + \dots. + a_{2n}A_{2n}$ etc.

$= a_{11}A_{11} + a_{21}A_{21} + \dots. + a_{i1}A_{i1} + \dots. + a_{n1}A_{n1}$ etc

Examples:

(i) Find the determinant of square matrix $\begin{bmatrix} 2 & -3 & 4 \\ 4 & -5 & 6 \\ 3 & 7 & 8 \end{bmatrix}$.

Let $|A| = \begin{vmatrix} 2 & -3 & 4 \\ 4 & -5 & 6 \\ 3 & 7 & 8 \end{vmatrix}$ Then expanding $|A|$ about 1^{st}row, we get.

$$|A| = 2\begin{vmatrix} -5 & 6 \\ 7 & 8 \end{vmatrix} - (-3)\begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix} + 4\begin{bmatrix} 4 & -5 \\ 3 & 7 \end{bmatrix}$$

$= 2(-40 - 42) + 3(32 - 18) + 4\,(28 + 15) = -168 + 42 + 172 = 46.$

(ii) Let $|A| = \begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & 5 \\ 4 & 2 & 1 \end{vmatrix}$. Then expanding $|A|$ about 1^{st} row,

we get $|A| = 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 5 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$

$= 2(3 - 10) + (1 - 20) = -14 - 19 = -33$

Rule of Sarrus

The value of determinant of order 3 × 3 be obtained by using sarrus rule. We need to understand that the sarrus rule does not exist higher than order 3.

For example: Find the determinant of following matrix by using sarrus rule.

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Then the Rule of Sarrus follows the following steps.

Step 1: Write down the three columns of the given determinant.

Step2: Rewrite the first two columns to make 4^{th} and 5^{th} columns as shown in the figure.

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Step 3: Find the products of the three elements lying on the diagonals from top to bottom containing three elements:

Step 4: Similarly find the product of the elements lying on the off-diagonal from bottom to top containing three elements:

Step 5: The difference of the sum obtained in step 3 and the sum obtained in step 4 gives the value of the determinant.

i.e. $|A| = (a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}a_{31} + a_{13}\,a_{21}\,a_{32}) - (a_{31}\,a_{22}\,a_{13} + a_{32}\,a_{23}\,a_{11} + a_{33}\,a_{21}\,a_{12})$.

Example

1. Let $A = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & 3 \end{vmatrix}$.

Then, for the value of |A| by the Rule of Sarrus, we write

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ -1 & 0 & 4 & -1 & 0 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

$\therefore\ |A| \quad = \{0 + 2 \times 4 + 3 \times (-1) \times 2\} - \{\ 0 + 2 \times 4 + 3 \times (-1) \times 2\}$

$= (8 - 6) - (8 - 6)$

$= 0$

Properties of Determinants

Here are some of the properties of the determinant which are discussed below:

Property 1

If all the elements of row (or column) of a determinant are zero, then the value of the determinant is zero. Let $|A| = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}$

Then, $|A| = 0 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - 0 \begin{vmatrix} a_1 & b_1 \\ a_3 & {}_{b}3 \end{vmatrix} + 0 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

Property 2

If rows and columns are interchanged then value of determinant remains same (value does not change). Therefore det(A)= det(A^T),

here A^T is transpose of matrix A. i.e. $|A| = |A^T|$

Let,Matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

$|A| = a_{11}.a_{22} - a_{21}.a_{21}$ .... (a)

Then, $A^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$|A^T| = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}.\ a_{22} - a_{12}.\ a_{21.}$ ....(b)

Hence, from (a) and (b), we see, $|A| = |A^T|$

Property 3

If all the elements of a row (or column) are zeros, then the value of the determinant is zero..Let $|A| = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}$.

Then, $|A| = 0\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - 0\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + 0\begin{vmatrix} a_1 & b_1 \\ a_2 & a_2 \end{vmatrix} = 0$

Property 4

If any two rows (or columns) of a determinate are interchanged, the resulting determinant is the negative of the original determinant.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$|C|$ be thedeterminant obtained by interchanging the 1st and 3rd rows of $|A|$.

i.e. $|C| = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$

Then, $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1\begin{vmatrix} a_2 & b_2 \\ a_3 & a_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & a_3 \end{vmatrix}$

(expanding about 1st row)

$= a_1(b_{2.}c_3 - b_3c_2) - b_1(c_3a_2 - c_2a_3) + c_1(a_2b_3 - a_3b_2)$

$= a_1b_2c_3 - a_1b_3c_2 - b_1c_3a_2 + b_1c_2a_3 + c_1a_2b_3 - c_1a_3b_2$........(i)

and $|C| = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$

$= a_1\begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix} - b_1\begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix} + c_1\begin{vmatrix} a_3 & c_3 \\ a_2 & b_2 \end{vmatrix}$

(expanding about 3rdrow)

$= a_1(b_3c_2 - b_2c_3) - b_1(c_2a_3 - c_3a_2) + c_1(a_3b_2 - a_2b_3)$

$= a_1b_3c_2 - a_1b_2c_3 - b_1c_2a_3 + b_1c_3a_2 + c_1a_3b_2 - c_1a_2b_3$

$= -(a_1b_2c_3 - a_1b_3c_2 - b_1c_3a_2 + b_1c_2a_3 + c_1a_2b_3 - c_1a_3b_2)$

$= -|A|$, from (i).

Property 5

If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

Let $|A| = \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$ be a determinant whose first and third column are identical.

Then

$|A| = a_1\begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & a_2 \\ a_3 & a_3 \end{vmatrix} + a_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

(expanding along 1^{st}row)

$= a_1(a_3b_2 - a_2b_3) - b_1(a_2a_3 - a_2a_3) + a_1(a_2b_3 - a_3b_2)$

$= a_1a_3b_2 - a_1a_2b_3) - b \times 0 + a_1a_2b_3 - a_1a_3b_2$

$= 0$

Property 6

If all the elements of any row (or column) of a determinant $|A|$ are multiplied by a constant k, then the value of the resulting determinant is k $|A|$.

Let $|A| = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$\therefore \quad |A| = a_1\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - a_2\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} + a_3\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}$

(expanding along 1^{st}column)...(i)

Then, $\begin{vmatrix} ka_1 & b_1 & a_1 \\ ka_2 & b_2 & a_2 \\ ka_3 & b_3 & a_3 \end{vmatrix} = ka_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - ka_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + ka_3\begin{bmatrix} a_1 & c_1 \\ b_2 & c_2 \end{bmatrix}$

(expanding along 1^{st} column)

$= k\left\{ a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right\} = k.|A|$ [from (i)]

Property 7

If each element of a row (or column) is equal to the same multiple of corresponding element of another row (or column), then the value of the determinant is zero.

Let $|A| = \begin{vmatrix} a_1 & ka_1 & c_1 \\ a_2 & ka_2 & c_2 \\ a_3 & ka_2 & c_2 \\ a_3 & ka_3 & c_3 \end{vmatrix}$.

Then, $|A| = k\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$ (by property 6)

$= k. 0$ ($\because$ $C_1 = C_2$, by property 5) $= 0$

Property 8

If each element of any row (or column) of a determinant is written as the sum of two (or more) terms, then the determinant can be written as the sum of two (or more) determinants.

i.e.

$$\begin{vmatrix} a_1+\alpha & b_1 & c_1 \\ a_2+\beta & b_2 & c_2 \\ a_3+\gamma & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b_3 & c_3 \end{vmatrix}$$

Here, $\begin{vmatrix} a_1+\alpha & b_1 & c_1 \\ a_2+\beta & b_2 & c_2 \\ a_3+\gamma & b_3 & c_3 \end{vmatrix}$

$=(a_1+\alpha)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (a_2+\beta)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3+\gamma)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

(expanding along 1st column)

$= \left\{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right\} + \left\{\alpha\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \beta\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + \gamma\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right\}$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b3 & c_3 \end{vmatrix}$

Property 9

If a scalar multiple of all the elements of a row (or column) of a determinant are added to the respective elements of any other row (or column), then the value of the determinant remains unchanged.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then,

$|B| = \begin{vmatrix} a_1+kc_1 & b_1 & c_1 \\ a_2+kc_2 & b_2 & c_2 \\ c_3+kc_3 & b_3 & c_3 \end{vmatrix}$ $(C_1 \to C_1 + kC_3)$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & b_1 & c_1 \\ kc_2 & b_2 & c_2 \\ kc_3 & b_3 & c_3 \end{vmatrix}$ (by property 8)

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$ (by property 6)

$= |A| + k.\,0$ $[\because C_1 = C_3$, by property 5]

$= |A|$

Worked out examples:

Example 1:

i) Without expanding show that

$$\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0$$

Solution: Here, L.H.S

$$\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix}$$

$$\begin{vmatrix} \sec^2\theta - \tan^2\theta & \tan^2\theta & 1 \\ \tan^2\theta - \sec^2\theta & \sec^2\theta & -1 \\ 38-36 & 36 & 2 \end{vmatrix} [\geq C_1 - C_2]$$

$$= \begin{vmatrix} 1 & \tan^2\theta & 1 \\ -1 & \sec^2\theta & -1 \\ 2 & 36 & 2 \end{vmatrix} = 0$$

$= \quad 0\ [\geq C_1 = C_3]$ Proved

ii) Without expending the determinant

Prove that: $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Solution: Here,

R.H.S $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$$= \frac{1}{xyz} \times xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \frac{1}{xyz} \begin{vmatrix} xyz & x & x^2 \\ xyz & y & y^2 \\ xyz & y & z^2 \end{vmatrix}$$

Taking common x, y, z from R_1, R_2, R_3 respectively.

$$= \frac{1}{xyz}.\ xyz \begin{vmatrix} yz & 1 & x \\ xz & 1 & y \\ xy & 1 & z \end{vmatrix}$$

$$= \begin{vmatrix} yz & 1 & x \\ xz & 1 & y \\ xy & 1 & z \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$$

$$= -x- \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \text{L.H.S Proved}$$

Example 2:

Prove that: $\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = 0$

Solution: Here,

$$\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & 1 & b+c \\ a+b+c & 1 & c+a \\ a+b+c & 1 & a+b \end{vmatrix} \quad [\geq C_1 \Delta C_1 + C_2]$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$$= (a+b+c) \times 0 \; [\geq C_1 = C_2]$$

$$= 0 \Rightarrow \text{R.H.S Proved}$$

Example-3

Evaluate: $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Solution: Here, $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} \quad [\geq c_1 \Delta c_1 - c_2 \text{ and } c_2 \Delta c_2 - c_3]$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ (a-b) & (b-c) & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ (a^2+ab+b^2) & (b^2+bc+c^2) & c^3 \end{vmatrix}$$

Expanding from R_1.

$$= (a-b)(b-c) \left\{ 0 - 0 + 1 \begin{vmatrix} 1 & 1 \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix} \right\}$$

$$= (a-b)(b-c)(b^2+bc+c^2 - a^2 - ab - b^2)$$

$$= (a-b)(b-c)\{bc - ab + c^2 - a^2\}$$

= $(a-b)(b-c)\{b(c-a)+(c-a)(c+a)\}$

= $(a-b)(b-c)(c-a)(b+c+a)$

= $(a-b)(b-c)(c-a)(a+b+c)$.

Example 4:

Prove that: $\begin{vmatrix} x & x^2 & y+z \\ y & y^2 & z+x \\ z & z^2 & x+y \end{vmatrix} = (y-z).(z-x)(x-y)(x+y+z)$

Solution: Here,

L.H.S $\begin{vmatrix} x & x^2 & y+z \\ y & y^2 & z+x \\ z & z^2 & x+y \end{vmatrix}$

= $\begin{vmatrix} x+y+z & x^2 & y+z \\ x+y+z & y^2 & z+x \\ x+y+z & z^2 & x+y \end{vmatrix}$ $[\geq C_1 \Delta C_1 + C_3]$

= $(x+y+z)\begin{vmatrix} 1 & x^2 & y+z \\ 1 & y^2 & z+x \\ 1 & z^2 & x+y \end{vmatrix}$

= $(x+y+z)\begin{vmatrix} 1 & x^2 & y+z \\ 0 & y^2-z^2 & z-y \\ 0 & z^2-x^2 & x-z \end{vmatrix}$ $[\geq R_2 \rightarrow R_2 - R_3 \text{ and } R_3 \rightarrow R_3 - R_1]$

= $(x+y+z)\begin{vmatrix} 1 & x^2 & (y+z) \\ 0 & (y+z)(y-z) & -(y-z) \\ 0 & (z+x)(z-x) & -(z-x) \end{vmatrix}$

= $(x+y+z).(y-z).(z-x)\begin{vmatrix} 1 & x^2 & (y+z) \\ 0 & y+z & -1 \\ 0 & z+x & -1 \end{vmatrix}$

Expanding from C_1

= $(x+y+z).(y-z).(z-x).\left\{1\begin{vmatrix} y+z & -1 \\ x+x & -1 \end{vmatrix} - 0 + 0\right\}$

= $(x+y+z).(y-z)(z-x)(-y-z+z+x)$

= $(x-y)(y-z)(z-x)(x+y+z)$

= RHS proved

Example 5

Prove that : $\begin{vmatrix} p & q & px+qy \\ p & r & qx+ry \\ px+qy & qx+ry & 0 \end{vmatrix} = (q^2-pr)(px^2+2qxy+ry^2)$

Solution: Here,

L.H.S

$$\begin{vmatrix} p & q & px+qy \\ p & r & qx+ry \\ px+qy & qx+ry & 0 \end{vmatrix}$$

Applying $C_3 \Delta C_3 - xC_1 - yC_2$

$$= \begin{vmatrix} p & q & o \\ q & r & o \\ px+qy & qx+ry & -px^2-qxy-qxy-ry^2 \end{vmatrix}$$

$$= \begin{vmatrix} p & q & o \\ q & r & o \\ px+qy & qx+ry & -(px^2+2qxy+ry^2) \end{vmatrix}$$

Expanding from C_3

$$\left\{0-0-(px^2+2qxy+ry^2)\begin{vmatrix} p & q \\ q & r \end{vmatrix}\right\}$$

$= \quad -(px^2+2qxy+ry^2)(pr-q^2)$

$= \quad (q^2-pr)(px^2+2qxy+ry^2)$ = R.H.S. Proved

Example 6

Prove that: $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution: Here,

L.H.S.

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $R_1 \Delta R_1 + R_2 + R_3$

$$\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \quad (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_1 \Delta C_1 - C_2$

$$= (a+b+c)\begin{vmatrix} 0 & 1 & 1 \\ 2b-b+c+a & b-c-a & 2b \\ 0 & 2c & c-a-b \end{vmatrix}$$

$$= \quad (a+b+c)^2 \begin{vmatrix} 0 & 1 & 1 \\ 1 & b-c-a & 2b \\ 0 & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \Delta C_2 - C_3$

$$= (a+b+c)^2 \begin{vmatrix} 0 & 0 & 1 \\ 1 & -(a+b+c) & 2b \\ 0 & (a+b+c) & (c-a-b) \end{vmatrix}$$

$$= \quad (a+b+c)^3 \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & (c-a-b) \end{vmatrix}$$

Expanding from C_3

$$= \quad (a+b+c)^3 \left\{ 0 - 0 + 1 \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \right\}$$

$$= \quad (a+b+c)^3 . (1\,(1+0))$$

$$= \quad (a+b+c)^3 \text{ R.H.S Proved}$$

Example 7

Prove that: $\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = (1 + x^2 + y^2 + z^2)$

Solution: Here,

$$\text{L.H.S} \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$$

Applying $R_1 \Delta x R_1$, $R_2 \Delta y R_2$ and $R_3 \Delta zR_3$.

$$= \begin{vmatrix} x(x^2+1) & x^2y & x^2z \\ xy^2 & y(y^2+1) & y^2z \\ xz^2 & yz^2 & z(z^2+1) \end{vmatrix}$$

Taking x, y and z from C_1, C_2 and C_3

$$xyz \begin{vmatrix} 1+x^2 & x^2 & x^2 \\ y^2 & 1+y^2 & y^2 \\ z^2 & z^2 & 1+z^2 \end{vmatrix}$$

Applying $R_1 \Delta R_1 + R_2 + R_3$

$$xyz \begin{vmatrix} 1+x^2+y^2+z^2 & 1+x^2+y^2+z^2 & 1+x^2+y^2+z^2 \\ y^2 & 1+y^2 & y^2 \\ z^2 & z^2 & 1+z^2 \end{vmatrix}$$

$$= \quad xyz\,(1 + x^2 + y^2 + z^2) \begin{vmatrix} 1 & 1 & 1 \\ y^2 & 1 + y^2 & y^2 \\ z^2 & z^2 & 1 + z^2 \end{vmatrix}$$

Applying $C_2 \Delta C_2 - C_3$ and $C_3 \Delta C_3 - C_1$

$$= \quad xyz\,(1 + x^2 + y^2 + z^2) \begin{vmatrix} 1 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & -1 & 1 \end{vmatrix}$$

Expanding from R_1

$$xyz\,(1 + x^2 + y^2 + z^2) \left\{ 1 \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} - 0 + 0 \right\}$$

$$= \quad xyz\,(1 + x^2 + y^2 + z^2)\,(1 - 0)$$

$= \quad xyz\,(1 + x^2 + y^2 + z^2)$ ***Proved***

8. Proved that: $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$$\text{L.H.S.} = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

Applying $C_1 \Delta\, C_1 + C_2 - C_3$

$$\begin{vmatrix} b+c+c+a-a-b & c+a & a+b \\ q+r+r+p-p-q & r+p & p+q \\ y+z+z+x-x-y & z+x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 2c & a+c & a+b \\ 2r & p+r & p+q \\ 2z & x+z & x+y \end{vmatrix} \qquad = \quad 2 \begin{vmatrix} c & a+c & a+b \\ r & p+r & p+q \\ z & x+z & x+y \end{vmatrix}$$

Applying $C_2 \Delta C_2 - C_1$ and $C_3 \Delta C_3 - C_2$

$$= \quad 2 \begin{vmatrix} c & a & b-c \\ r & p & q-r \\ z & x & y-z \end{vmatrix}$$

Applying $C_3 \Delta\, C_3 + C_1$

$$= \quad 2 \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix} \qquad = \quad -2 \begin{vmatrix} c & b & a \\ r & q & p \\ z & y & x \end{vmatrix} \; [\geq C_2 \leftrightarrow C_3]$$

$$= \quad -x-2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \; [\geq C_1 \leftrightarrow C_3] \qquad = \quad 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

= R.H.S. Proved

Exercise

1. Evaluate the following determinants:

a) $\begin{vmatrix} c & d \\ e & f \end{vmatrix}$ b) $\begin{vmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{vmatrix}$

c) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 3 & -2 \\ -1 & -4 & -5 \end{vmatrix}$ d) $\begin{vmatrix} 1 & 0 & 2 \\ 0 & 7 & 4 \\ 5 & 0 & 8 \end{vmatrix}$ e) $\begin{vmatrix} -2 & -3 & 4 \\ -7 & 2 & -5 \\ 8 & -1 & -4 \end{vmatrix}$

2. Find the following determinants by using sarrus rule.

a) $\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 4 \\ -2 & -3 & -1 \end{vmatrix}$ b) $\begin{vmatrix} 2 & -2 & 0 \\ -1 & 2 & 4 \\ -4 & 5 & 6 \end{vmatrix}$

3. Without expanding the determinants, prove that the following determinants.

a) $\begin{vmatrix} 6 & 3 & 12 \\ 4 & 8 & -4 \\ 10 & 5 & 20 \end{vmatrix} = 0$ b) $\begin{vmatrix} 20 & 11 & 5 \\ 24 & 13 & 6 \\ 28 & 17 & 7 \end{vmatrix} = 0$

c) $\begin{vmatrix} 2 & 4 & 7 \\ 6 & 1 & 9 \\ 12 & 13 & 30 \end{vmatrix} = 0$ d) $\begin{vmatrix} 51 & 61 & 71 \\ 5 & 6 & 7 \\ 1 & 10 & 1 \end{vmatrix} = 0$

e) $\begin{vmatrix} y+z & 1 & x \\ x+z & 1 & y \\ x+y & 1 & z \end{vmatrix} = 0$ f) $\begin{vmatrix} b & c & b+c \\ c & a & c+a \\ a & b & a+b \end{vmatrix} = 0$

g) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ac & ac(a+c) \\ 1 & ab & ab(a+b) \end{vmatrix} = 0$ h) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(c+b) \end{vmatrix} = 0$

i) $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$, where ω is an imaginary cube root of unity.

4. Prove the following:

i) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$ ii) $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

iii) $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ iv) $\begin{vmatrix} yz & x^2 & x^2 \\ y^2 & xz & y^2 \\ z^2 & z^2 & xy \end{vmatrix} = \begin{vmatrix} yz & xy & xz \\ xy & xz & yz \\ xz & yz & xy \end{vmatrix}$

v) $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a^2 & bc & a \\ b^2 & ac & b \\ c^2 & ab & c \end{vmatrix}$

5) Prove that the following determinants:

i) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

ii) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(bc+ac+ab)$

iii) $\begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix} = (x-a)(x-b)(x+a+b)$

iv) $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2$

v) $\begin{vmatrix} -x^2 & xy & xz \\ xy & -y^2 & yz \\ xz & yz & -z^2 \end{vmatrix} = 4x^2y^2z^2$

vi) $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$

vii) $\begin{vmatrix} a+x & b & c \\ a & b+y & c \\ a & b & c+z \end{vmatrix} = xyz\left(1+\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$

viii) $\begin{vmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{vmatrix} = (1+a^2+b^2+c^2)$

ix) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (abc+ab+bc+ac)$

x) $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

6. If $xyz + 1 = 0$ show that: $\begin{vmatrix} x & x^2 & x^3+1 \\ y & y^2 & y^3+1 \\ z & z^2 & z^3+1 \end{vmatrix} = 0$

7. Prove that : $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

8. Show that: $\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2 = \begin{vmatrix} a^2 & b^2 & b^2 \\ b^2 & a^2 & b^2 \\ b^2 & b^2 & a^2 \end{vmatrix}$

Where $a^2 = x^2 + y^2 + z^2$ and $b^2 = xy + yz + xz$.

9) Solve the equations:

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

10) Prove that the determinants

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$ is a perfect square and find its value.

Answers

1. (a) (af –de) (b) 1 (c) – 34 (d) 56 (e) 194
2. (a) – 23 (b) 4

Inverse of a matrix

We must have clear concept about the singular and non-singular matrix before defining inverse of a matrix.

Singular matrix: A square matrix 'A' is said to be singular matrix iff its determinant is zero.
for example; if

$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$. Then,

$|A| = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 3 \times 4 - 6 \times 2 = 12 - 12 = 0.$

$\therefore$ Matrix 'A' is a singular or $|A| = 0$.

Non-singular matrix: A square matrix that is invertible or that has non-zero determinant is called a non-singular matrix. For example; If $B = \begin{bmatrix} 3 & 4 \\ -2 & 6 \end{bmatrix}$ then

$|B| = \begin{vmatrix} 3 & 4 \\ -2 & 6 \end{vmatrix} = 3 \times 6 - (-2) \times 4 = 18 + 8 = 26$

$\therefore$ Matrix A is a non-singular as $|A| \neq 0$.

Note: All matrix do not have inverse only those square matrix that have non-zero determinants possess inverse.

Definition of inverse matrix: A square matrix A which is non-singular (i.e. $|A| \neq 0$) then there exists an $n \times n$ matrix B such that AB = BA = I, where I is unit matrix. Then A is said to be inverse matrix of B and vice-versa.

Example 1 : $P = \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$

Then, $P.Q = \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} . \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$

$= \begin{bmatrix} 8 \times 2 + 5 \times -3 & 8 \times -5 + 5 \times 8 \\ 3 \times 2 + 2 \times -3 & 3 \times -5 + 2 \times 8 \end{bmatrix} = \begin{bmatrix} 16 - 15 & -40 + 40 \\ 6 - 6 & -15 + 16 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$Q.P = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} . \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 8 - 5 \times 3 & 2 \times 5 - 5 \times 2 \\ -3 \times 8 + 8 \times 3 & -3 \times 5 + 8 \times 2 \end{bmatrix}$

$$= \begin{bmatrix} 16-15 & 10-10 \\ -24+24 & -15+16 \end{bmatrix} \quad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, PQ = QP = I

Hence P and Q are inverse matrices to each other.

Example:

Adjoint of a matrix: If A is any square matrix, then the adjoint of A is defined as the transpose of the matrix obtained by replacing the element of A by their corresponding co-factors.

$\therefore$ Adjoint of A (Adj. A) = Transpose of the cofactor matrix.

i.e. Adj. A = (co-factor of A)^T.

Example1 : Find the adjoint of the matrix $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Solution: Here,

Let A = $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$. Then,

$|A| = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 2 \times 2 - (-1) \times (-4) = 4 - 4 = 0$

A_{11} = Cofactor of 2 = 2

A_{12} = Cofactor of – 4 = – (– 1) = 1.

A_{21} = Cofactor of – 1 = – (– 4) = 4.

A_{22} = Cofactor of 2 = + (2) = 2.

Now,

$$\text{Adj } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

$\therefore$ Adj A = $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

Example 2: Compute the adjoint and the inverse of the matrix $\begin{pmatrix} 4 & -2 \\ -4 & 8 \end{pmatrix}$

Solution: Here,

Let P = $\begin{pmatrix} 4 & -2 \\ -4 & 8 \end{pmatrix}$

$|P| = \begin{vmatrix} 4 & -2 \\ -4 & 8 \end{vmatrix} = 4 \times 8 - (-4) \times (-2) = 32 - 8 = 24$

$|P| \neq 0$, P^{-1} exists.

P_{11} = Cofactor of 4 = 8

P_{12} = Cofactor of – 2 = – (– 4) = 4.

P_{21} = Cofactor of – 4 = – (– 2) = 2.

P_{22} = Cofactor of 8 = + (4) = 4.

Now,

$$\text{Adj } p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^T = \begin{pmatrix} 8 & 4 \\ 2 & 4 \end{pmatrix}^T = \begin{pmatrix} 8 & 2 \\ 4 & 4 \end{pmatrix}$$

$$p^{-1} = \frac{\text{Adj}(p)}{|\, p\, |} = \frac{\begin{pmatrix} 8 & 2 \\ 4 & 4 \end{pmatrix}}{24} = \frac{1}{12}\begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix}.$$

Example 3 : Find the inverse of the matrix: $\begin{bmatrix} -1 & 4 & 3 \\ 1 & 2 & -3 \\ 4 & 2 & 1 \end{bmatrix}$.

Solution: Here,

Let $M = \begin{bmatrix} -1 & 4 & 3 \\ 1 & 2 & -3 \\ 4 & 2 & 1 \end{bmatrix}$. Then,

$$|M| = \begin{vmatrix} -1 & 4 & 3 \\ 1 & 2 & -3 \\ 4 & 2 & 1 \end{vmatrix}$$

or, $|M| = -\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$

$= -1\,(2 + 6) - 4\,(1 + 12) + 3\,(2 - 8)$

$= -8 - 52 - 18$

$= -78$

$|M| \neq 0$ i.e. M is non-singular matrix and therefore inverse of matrix M exists.

If A_{11}, A_{12}, A_{13}............A_{33} are the cofactors of the element of M.

The cofactor of $A_{11} = +\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} = (2 + 6) = 8.$

The cofactor of $A_{12} = -\begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix} = -(1 + 12) = -13.$

The cofactor of $A_{13} = +\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = +(2 - 8) = -6.$

The cofactor of $A_{21} = -\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = -(4 - 6) = 2.$

The cofactor of $A_{22} = +\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} = (-2 - 12) = -14$

The cofactor of $A_{23} = -\begin{vmatrix} -1 & 4 \\ 4 & 2 \end{vmatrix} = -(-2 - 16) = 18$

The cofactor of $A_{31} = +\begin{vmatrix} 4 & 3 \\ 2 & -3 \end{vmatrix} = (-12 - 6) = -18.$

The cofactor of $A_{32} = -\begin{vmatrix} -1 & 3 \\ 1 & -3 \end{vmatrix} = -(3 - 3) = 0$

The cofactor of A_{33} = + $\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix}$ = – 2 – 4 = – 6

The matrix co-factor of M = $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 8 & -13 & -6 \\ 2 & -14 & 18 \\ -18 & 0 & -6 \end{pmatrix}$

Adj. M = $\begin{pmatrix} 8 & -13 & -6 \\ 2 & -14 & 18 \\ -18 & 0 & -6 \end{pmatrix}^T = \begin{pmatrix} 8 & 2 & -18 \\ -13 & -14 & 0 \\ -6 & 18 & -6 \end{pmatrix}$

$\therefore M^{-1} = \frac{1}{|M|}(\text{adj.}M) = \frac{1}{-78}\begin{pmatrix} 8 & 2 & -18 \\ -13 & -14 & 0 \\ -6 & 18 & -6 \end{pmatrix}$

Example 4: If A = $\begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ 4 & 2 & -1 \end{pmatrix}$, verify that A (AdjA) = (AdjA) A = |A|I.

Solution: Here, A = $\begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ 4 & 2 & -1 \end{pmatrix}$

We have, $a_{11} = 1$, $a_{12} = 0$, $a_{13} = 5$

$a_{21} = 2$, $a_{22} = -1$, $a_{23} = 3$

$a_{31} = 4$, $a_{32} = 2$, $a_{33} = -1$

A_{11} = Cofactor of a_{11} = + $\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$ = 1 – 6 = – 5

A_{12} = Cofactor of a_{12} = – $\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$ = – (– 2 – 12) = 14

A_{13} = Cofactor of a_{13}= + $\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}$ = (4 + 4) = 8

A_{21} = Cofactor of a_{21} = – $\begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix}$ = – (0 – 10_ = 10.

A_{22} = Cofactor of a_{22} = + $\begin{vmatrix} 1 & 5 \\ 4 & -1 \end{vmatrix}$ = – 1 – 20 = – 21.

A_{23} = Cofactor of a_{23} = – $\begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix}$ = – (2 – 0) = – 2

A_{31} = Cofactor of a_{31} = + $\begin{vmatrix} 0 & 5 \\ -1 & 3 \end{vmatrix}$ = (0 + 5) = 5

A_{32} = Cofactor of a_{32} = – $\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$ = – (3 – 10) = 7

A_{33} = Cofactor of a_{33} = + $\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix}$ = – 2 – 4 = – 6

Matrix co-factor of A = $\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} -5 & 14 & 8 \\ 10 & -21 & -2 \\ 5 & 7 & -1 \end{pmatrix}$

Adjoint of A = $\begin{pmatrix} -5 & 14 & 8 \\ 10 & -21 & -2 \\ 5 & 7 & -1 \end{pmatrix}^T = \begin{pmatrix} -5 & 10 & 5 \\ 14 & -21 & 7 \\ 8 & -2 & -1 \end{pmatrix}$

Now,

A. Adj A = $\begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ 4 & 2 & -1 \end{pmatrix} . \begin{pmatrix} -5 & 10 & 5 \\ 14 & -21 & 7 \\ 8 & -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1(-5) + 0 \times 14 + 5 \times 8 & 1 \times 10 + 0 \times -21 + 5x - 2 & 1 \times 5 + 0 \times 7 + 5 \times -1 \\ 2 \times (-5) - 1 \times 14 + 3 \times 8 & 2 \times 10 + (-1) \times (-21) + 3 \times (-2) & 2 \times 5 + (-1) \times 7 + 3 \times (-1) \\ 4 \times (-5) + 2 \times 14 - 1 \times 8 & 4 \times 10 + 2 \times (-21) + (-1) \times (-2) & 4 \times 5 + 2 \times 7 + (-1) \times (-1) \end{pmatrix}$

$= \begin{pmatrix} -5 + 0 + 40 & 10 + 0 - 10 & 5 + 0 - 5 \\ -10 - 14 + 24 & 20 + 21 - 6 & 10 - 7 - 3 \\ -20 + 28 - 8 & 40 - 42 + 2 & 20 + 14 + 1 \end{pmatrix} = \begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix} = 35 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Again,

(Adj. A) A = $\begin{pmatrix} -5 & 10 & 5 \\ 14 & -21 & 7 \\ 8 & -2 & -1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ 4 & 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} -5 + 20 + 20 & 0 - 10 + 10 & -25 + 30 - 5 \\ 14 - 42 + 28 & 0 + 21 + 14 & 70 - 63 - 7 \\ 8 - 4 - 4 & 0 + 2 - 2 & 40 - 6 + 1 \end{pmatrix} = \begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix} = 35 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Also,

$|A| = \begin{vmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ 4 & 2 & -1 \end{vmatrix}$

$= 1 \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} + 5 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}$

$= 1(1 - 6) - 0 + 5\ (4 + 4)$

$= -5 + 40$

$= 35$

$\therefore\ |A|.\ I = 35 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore$ A. (adj.A) = (adj. A). A |A| I. verified,

Exercise 3.3

1) Find the cofactors of the following matrices.

a) $\begin{bmatrix} 2 & 5 \\ 3 & -7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 5 & -1 \\ 4 & -2 & 3 \\ -1 & 3 & -2 \end{bmatrix}$

2) Find the adjoint of the following matrices;

a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 & 4 \\ -2 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix}$

3) Find the inverse of the following matrices;

a) $\begin{bmatrix} 3 & -4 \\ -3 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

4) If $p = \begin{bmatrix} 7 & -3 \\ 6 & 2 \end{bmatrix}$, prove that: $p^{-1} = \frac{1}{32}\begin{bmatrix} 2 & 3 \\ -6 & 7 \end{bmatrix}$

5) Prove that the matrices $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ are inverse to each other.

6) If a matrix $\begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are inverse of each other. Find the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

7) If $p = \begin{pmatrix} 1 & 5 \\ 3 & -7 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & 4 \\ -3 & 6 \end{pmatrix}$ verify that: $(PQ)^{-1} = Q^{-1}P^{-1}$.

8) If $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 5 \\ 0 & 1 & -3 \end{pmatrix}$, verify that

$A.(Adj.\ A) = (Adj.A).\ A = |A|.\ I$.

9) If $M = \begin{pmatrix} 1 & 3 & 2 \\ -4 & -2 & 0 \\ 3 & -2 & 5 \end{pmatrix}$, find M^{-1} and verify that: $M.M^{-1} = I$.

Answers

1(a)

$\begin{bmatrix} -7 & -5 \\ -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & 0 & 0 \\ 4 & 8 & 3 \\ 4 & -23 & -22 \end{bmatrix}$

2(a) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (b) $\begin{bmatrix} -5 & -4 & 4 \\ 0 & -8 & 23 \\ 0 & -3 & 0 \end{bmatrix}$

3(a) $\begin{bmatrix} \frac{-1}{3} & \frac{-2}{3} \\ -1 & \frac{-1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} 0.75 & -0.25 & 0.25 \\ -3.75 & 2.25 & -1.25 \\ 1.25 & -0.75 & 0.75 \end{bmatrix}$

Activities and Project work

1. Make a project report to show use of matrix in daily life taking an appropriate example.

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