Sequence and Series Continued (3^{rd} Part)

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Convergence and Divergence of a Sequence

Let $\{a_n\}$ be a sequence of numbers. The sequence is said to converge a finite limit $\boldsymbol{a}$(say) if to every positive number ε, there corresponds an integer N such that for all $n \in \mathbb{N}\ , n > N \quad \Rightarrow\ |a_n - \boldsymbol{a}| < \varepsilon$

If no such number a exists, $\{a_n\}$ diverges.

If $\{a_n\}$ converges to **a**, we write $\lim\limits_{n\to\infty} a_n = \boldsymbol{a}$

or, simply $a_n \to \boldsymbol{a}$ and $\boldsymbol{a}$ is called limit of the sequence.

Infinite Series:

Let, $\{a_n\}$ be a sequence of numbers, then an expression of the form

$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots\cdots\cdots$ is called an infinite series.

The number a_n is the n^{th} term of the series.

Let us define the sequences of partial sums as

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

.

.

.

$S_n = a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n = \sum_{i=1}^{n} a_i$

.

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is the sequence of partial sums of the infinite series. Thus the partial sums form a sequence $\{S_n\}$whose n^{th} term is S_n. If the sequence of the partial sum converges to a limit S(say), the infinite series **converges** and its sum is S.

So, $a_1 + a_2 + a_3 + \cdots\cdots\cdots + a_n = \sum_{n=1}^{\infty} a_i = S$

If the sequence of partial sums of the series does not converge then the series also **diverges**.

Convergence of Geometric Series

An infinite geometric series $a + ar + ar^2 + ar^3 + \ldots\ldots + a\, r^n \ldots\ldots \infty = \sum_{n=1}^{\infty} ar^{n-1}$

i) converges to $\frac{a}{1-r}$ if $|r| < 1$ and

ii) diverges to ∞ if $|r| > 1$ or $r = 1$.

Example:

i. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\cdots\cdots$ is convergent since $|r| = \frac{1}{3} < 1$

ii. $1 + 3 + 9 + 27 + \cdots\cdots\cdots$ is divergent since, $|r| = 3 > 1$

Absolute Convergence:

A series $\sum x_n$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values $\sum|x_n|$, converges.

Example: The geometric series, $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\cdots\cdots$ converges absolutely because the corresponding series of absolute values $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots\cdots$ converges.

1.2 Factorial Notation:

We are all familiar with multiplication. The factorial notation is a symbol that we use to represent a multiplication operation. But it is more than just a symbol. The factorial notation is the exclamation mark, and you will see it directly following a number. For example, you will see it as 5! or 3!. You read there as 'five factorial' and ' three factorial'. Factorial means that we multiply all the integers less than or equal to our chosen number. So, 5! means that

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$
$$= 5\,(5-1) \times (5-2) \times (5-3) \times (5-4)$$

$n! = n \times (n-1) \times (n-2) \times (n-3) \ldots 3.2.1$

Definition:

The continued product of the first n- positive integers is denoted by n! or $\lfloor n$

Remark: $0! = 1$

We know that;

$n! = n.(n - 1)!$

Putting $n = 1$. Then,

$1! = 1. (1 - 1)!$

Or, $1 \times 1 = 0!$

$\therefore 0! = 1.$

Derivatives

Definition: Let $y = f(x)$ be a continuous function. If Δx be the small increment in x and Δy be the corresponding small increment in y, then the ratio $\frac{\Delta y}{\Delta x}$ is called the average rate of change of y with respect to x on some neighborhood of x. Thus, when x changes from a to b then

$\Delta x = b - a$ and $\Delta y = f(b) - f(a)$.

Hence, the average rate of change of y with respect to x is $\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$.

When $\Delta x \to 0$ then the average rate of change of y with respect to x becomes **instantaneous** rate of change of y with respect to x, provided that this limit exists.

Thus, instantaneous rate of change = $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

Hence, the rate of change of y with respect to the infinitely many small change in x is called derivative.

Note: $\Delta x = 0.0000000000000\ldots\ldots\ldots\ldots..1 \to 0$

Derivative of a function

Definition: Let $y = f(x)$ be a continuous function defined in an open interval I. Let Δx be the small increment in x and Δy be the corresponding small increment in y. Then the derivative or the differential coefficient of f with respect to x is defined as the limiting value of $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, provided that the limit exists. The derivative of $y = f(x)$ with respect to x is denoted by $\frac{dy}{dx}$ or $\mathbf{f'(x)}$ or $\frac{d}{dx}[f(x)]$ or y' or y_1 or Dy

Note: Algorithm of finding derivative of any function f(x) form first principle or by definition

Let $\quad y = f(x) \quad \ldots\ldots(i)$

Let Δx and Δy be the **small** increments in x and y respectively. Then

$$y + \Delta y = f(x + \Delta x)$$

or, $\Delta y = f(x + \Delta x) - y$

or, $\Delta y = f(x + \Delta x) - f(x)$ [from (i)]

Dividing both side by Δx

$$\text{or, } \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Taking $\lim\limits_{\Delta x \to 0}$ on both sides,

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\text{Hence, } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\text{Or, } \frac{d}{dx}[f(x)] = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\text{Or, } f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If h is the small increment in x, then the derivative of y = f(x) with respect to x can be defined as $f'(x) = \lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided that this limit exists. This method of finding the derivative of function by definition is also called first principle method.

The derivative of y = f(x) at particular point x = a is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \text{ provided that the limit exists.}$$

If x = a+ h, then h = x – a. So, $h \to 0 \Rightarrow x \to a$. Then,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ provided that the limit exists.}$$

Definition: A function y = f(x) is said to be differentiable or derivable with respect to x if its derivative $f'(x)$ exists. The process of finding the derivative of a given function is called differentiation.

Find the derivation from first principle or by definition:

i. x

Let, $y = x$ ……(i)

Let Δx and Δy be the small increments in x and y respectively. Then

$$y + \Delta y = x + \Delta x$$

or, $\Delta y = x + \Delta x - y$

or, $\Delta y = x + \Delta x - x$ [from (i)

or $\Delta y = \Delta x$

Dividing both side by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x}$$

Taking $\lim\limits_{\Delta x \to 0}$ on both side,

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

Or, $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} 1$

Hence, $\frac{d(x)}{dx} = 1$

(ii) x^2

Let, $y = x^2$ ……(i)

Let Δx and Δx be the small increments in x and y respectively. Then

$$y + \Delta y = (x + \Delta x)^2$$

or, $\Delta y = (x + \Delta x)^2 - y$

or, $\Delta y = (x + \Delta x)^2 - x^2$ [from (i)

or, $\Delta y = 2.x.\ \Delta x + (\Delta x)^2$

Dividing both side by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x(2x+\Delta x)}{\Delta x}$$

Taking $\lim_{\Delta x \to 0}$ on both side,

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x}$$

$$\text{Or,} \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x + 0 = 2x$$

$$\text{Or,} \quad \frac{d(x^2)}{dx} = 2x + 0$$

$$\text{Hence,} \quad \frac{d(x^2)}{dx} = 2x$$

(iii) Let, $y = x^3$ ……(i)

Let Δx and Δx be the small increments in x and y respectively. Then

$$y + \Delta y = (x + \Delta x)^3$$

$$\text{or,} \quad \Delta y = (x + \Delta x)^3 - y$$

$$\text{or,} \quad \Delta y = (x + \Delta x)^3 - x^3 \quad \text{[from (i)}$$

$$\text{or,} \quad \Delta y = 3.x^2.\ \Delta x + 3.x.\ (\Delta x)^2 + (\Delta x)^3$$

Dividing both side by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x(3x^2+3.x.\Delta x+(\Delta x)^2)}{\Delta x}$$

Taking $\lim_{\Delta x \to 0}$ on both side,

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(3x^2+3.x.\Delta x+(\Delta x)^2)}{\Delta x}$$

$$\text{Or,} \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} 3x^2 + 3.x.\Delta x + (\Delta x)^2$$

$$\text{Or,} \quad \frac{d(x^2)}{dx} = 3\,x^2 + 3.\ x.\ 0 + 0^2$$

$$\text{Hence,} \quad \frac{d(x^3)}{dx} = 3x^2.$$

(iv) $y = f(x) = x^n$; $n \in \mathbf{Q}$.

Let Δx be a small increment in x and Δy be corresponding small increment in y. Then,

$$y + \Delta y = (x + \Delta x)^n$$

or, $$\Delta y = (x + \Delta x)^n - x^n$$

or, $$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

or, $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

or, $$\frac{dy}{dx} = \lim_{(x + \Delta x) \to x} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x} = n\, x^{n-1}$$

$\therefore$ $\boldsymbol{\frac{d(x^n)}{dx} = n\, x^{n-1}}$. This formula is called the power rule.

Note: For any rational number n, $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

Examples

$\frac{d(x)}{dx} = 1x^{1-1} = 1.\ X^0 = 1.1 = 1$

$\frac{d(x^2)}{dx} = 2\,x^{2-1} = 2x$

$\frac{d(x^3)}{dx} = 3\,x^{3-1} = 3\,x^2$

$\frac{d(x^4)}{dx} = 4\,x^{4-1} = 4x^3$

If $f(x) = 3x^2$

$\frac{d(3x^2)}{dx} = \frac{3\,d(x^2)}{dx} = 3.\ 2\,x^{2-1} = 6x$

Note: $\frac{d(1)}{dx} = \frac{d(2)}{dx} = \frac{d(3)}{dx} = \ldots \quad \ldots \quad \ldots = 0$

i.e., $\frac{d(Constant)}{dx}$

v.

Let, $y = 1$ ……(i)

Let Δx and Δy be the small increments in x and y respectively. Then

$$y + \Delta y = 1$$

or, $$\Delta y = 1 - y$$

or, $$\Delta y = 1 - 1 \quad \text{[from (i)}$$

or $\Delta y = 0$

Dividing both side by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x}$$

Taking $\lim_{\Delta x \to 0}$ on both side,

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

Or, $$\frac{dy}{dx} = \lim_{\Delta x \to 0} 0$$

Hence, $\frac{d(1)}{dx} = 0$

Examples:

i. Find the derivative of $\sqrt{x}$ from the first principle or by definition.

Let, $y = \sqrt{x}$ ……..(i)

Suppose, Δx be the small increment in x and Δy be the corresponding increment in y.

$$y + \Delta y = \sqrt{x + \Delta x}$$

or, $$\Delta y = \sqrt{x + \Delta x} - y$$

or, $$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} \text{ [from (i)]}$$

Dividing both sides by Δx,

or, $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$$

Taking $\lim_{\Delta x \to 0}$ on both sides,

Or, $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$$

Or, $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \times \frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

Or, $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{x+\Delta x-x}{\Delta x\,(\sqrt{x+\Delta x}+\sqrt{x})}$$

Or, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\,(\sqrt{x+\Delta x}+\sqrt{x})}$

Or, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{(\sqrt{x+\Delta x}+\sqrt{x})}$

Or, $\frac{dy}{dx} = \frac{1}{(\sqrt{x+0}+\sqrt{x})}$

Or, $\frac{dy}{dx} = \frac{1}{(\sqrt{x}+\sqrt{x})}$

Or, $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$

ii. Find the derivative of sinx from the first principle or by definition.

Let, $y = \sin x$ …….(i)

Supose, Δx be the small increment in x and Δy be the corresponding increment in y.

$$y + \Delta y = \sin(x+\Delta x)$$

or, $\Delta y = \sin(x+\Delta x) - y$

or $\Delta y = \sin(x+\Delta x) - \sin x$ [from (i)]

Dividing both sides by Δx,

or, $\frac{\Delta y}{\Delta x} = \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$

Taking $\lim_{\Delta x \to 0}$ on both sides,

Or, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$

Or, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos\frac{(x+\Delta x+x)}{2}.\sin\frac{(x+\Delta x-x)}{2}}{\Delta x}$

Formula: [$\sin C - \sin D = 2\cos\frac{(C+D)}{2} . \sin\frac{(C-D)}{2}$]

Or, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2\cos\frac{(2x+\Delta x)}{2}.\ \sin\frac{\Delta x}{2}}{\Delta x}$

Or, $\frac{dy}{dx} = \lim_{\Delta x \to 0} 2\cos\frac{(2x+\Delta x)}{2} \times \left[\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right] \times \frac{1}{2}$

Formula: $\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$

Or, $\frac{dy}{dx} = \cos\frac{(2x+0)}{2} \times [1]$

Or, $\frac{dy}{dx} = \cos\left(\frac{2x}{2}\right)$

Or, $\frac{d(sinx)}{dx} = \text{cosx}$

Some Important Formulae-

1. $\frac{dx^n}{dx} = nx^{n-1}$
2. $\frac{d(\text{Constant})}{dx} = 0$
3. $\frac{d}{dx}$ (sin x) = cos x
4. $\frac{d}{dx}$ (cos x) = – sin x
5. $\frac{d}{dx}$ (tan x) = $\sec^2 x$
6. $\frac{d}{dx}$ (see x) = sec x. tan x.
7. $\frac{d}{dx}$ (cot x) = – $\text{cosec}^2 x$
8. $\frac{d}{dx}$ (cosec x) = – cosec x. cot x
9. $\frac{de^x}{dx} = e^x$, $\frac{de^{ax}}{dx} = \frac{e^{ax}}{a}$
10. $\frac{d(\text{logx})}{dx} = \frac{1}{x}$

Some Important Rules

1. Sum Rule

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

2. General Power Rule:

If $y = u^n$ and $u = f(x)$ then $\frac{dy}{dx} = n.u^{n-1}.\frac{du}{dx}$.

3. The Product Rule

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

4. The Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}, \; v \neq 0$$

5. The Chain Rule (Derivative of a composite function)

If $y = f(u)$ and $u = g(x)$ are two differentiable functions

then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

6. Derivative of Parametric Functions

Let $x = f(t)$ and $y = g(t)$ be any two functions of t, t being a parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided that } \frac{dx}{dt} \neq 0.$$

Important Formula: $\frac{d(x^n)}{dx} = n\,x^{n-1}$

Above rules are described below in detail with example:

Techniques of Differentiation

To make writing of derivative of a function, a mechanical matter, we shall establish a few standard forms of derivatives which can then be generalized to evaluate the derivative of functions whose forms are known.

(i) The coefficient Rule

If f(x) is differentiable function and k is any constant then,

$$\frac{d}{dx}[k \cdot f(x)] = k.\frac{d}{dx}[f(x)]$$

Example:

Let $f(x) = 7x^3$. Then,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(7x^3) = 7 \cdot \frac{d}{dx}(x^3) = 7 \times 3\, x^{3-1} = 21\, x^2$$

(ii) The Sum Rule

If f(x) and g(x) are any two functions which are differentiable with respect to x, then so is their sum or difference. Also,

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Note: If u and v are two differentiable functions of x then

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

Example:

If $y = 10x^3 - 17x^2$ then find $\frac{dy}{dx}$.

Solution:

Here, $$\frac{dy}{dx} = \frac{d}{dx}(10x^3 - 17x^2)$$

$$= \frac{d}{dx}(10\, x^3) - \frac{d}{dx}(17x^2) = 10.\frac{d}{dx}(x^3) - 17\frac{d}{dx}(x^2)$$

$$= 10 \times 3x^{3-1} - 17 \times 2x^{2-1} = 30\, x^2 - 34x.$$

iii) The Product Rule

If f(x) and g(x) be two differentiable functions of x,

then $$\frac{d}{dx}[f(x). g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Hence, we conclude that if u and v are two differentiable functions of x then.

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Thus, $\frac{d}{dx}$ (1^{st} function × 2^{nd} function)

= 1^{st}Function × Derivative of 2^{nd} Function + 2^{nd}Function × Derivative of 1^{st} Function.

Example: y = $(x^3 - 3x).(x^2 + 4)$

Solution:

Here, $$\frac{dy}{dx} = \frac{d}{dx}[(x^3 - 3x)(x^2 + 4)]$$

$$= (x^3 - 3x)\frac{d}{dx}(x^2 + 4) + (x^2 + 4)\frac{d}{dx}(x^3 - 3x).$$

$$= (x^3 - 3x)(2x + 0) + (x^2 + 4).(3x^2 - 3)$$

$$= 2x(x^3 - 3x) + (x^2 + 4)(3x^2 - 3)$$

$$= 2x^4 - 6x^2 + 3x^4 - 3x^2 + 12x^2 - 12$$

$$= 5x^4 + 3x^2 - 12$$

iv) The Quotient Rule

If f(x) and g(x) be two differentiable functions of x and $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Note: From above rule we conclude that if u and v are any two differentiable functions of x and $v \neq 0$ then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$.

Thus, $\frac{d}{dx}\left(\frac{1^{st}\text{ Funciton}}{2^{nd}\text{ Function}}\right)$

$$= \frac{2^{nd}\text{Funciton} \times \text{Derivative of } 1^{st}\text{ Funciton} - 1^{st}\text{ function} \times \text{Derivative of } 2^{nd}\text{ Function}}{(2^{nd}\text{Function})^2}$$

Example: $\mathbf{y = \dfrac{3x + 4}{5x - 4}}$

Solution:

Here, $y = \dfrac{3x + 4}{5x - 4}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x+4}{5x-4}\right)$$

$$= \frac{(5x-4)\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(5x-4)}{(5x-4)^2} = \frac{(5x-4).3 - (3x+4).\,5}{(5x-4)^2}$$

$$= \frac{15x - 12 - 15x - 20}{(5x-4)^2} = \frac{-32}{(5x-4)^2}$$

(iii) General Power Rule:

If $y = u^n$ and $u = f(x)$ then $\dfrac{dy}{dx} = n.\, u^{n-1}.\dfrac{du}{dx}$.

Example: $(3x^2 - 5)^3$

Solution: Let, $y = (3x^2 - 5)^3$ and let $u = 3x^2 - 5$

By using general power rule,

$$\frac{dy}{dx} = \frac{d}{dx}(u^3) = \frac{d(u^3)}{du} \times \frac{du}{dx}$$

$$= 3u^2 \times \frac{du}{dx}$$

$$= 3u^2.\frac{d}{dx}(3x^2 - 5)$$

$$= 3u^2 \times 6x$$

$$= 18u^2x$$

$$= 18x(3x^2 - 5)^2$$

(vi) The Chain Rule (Derivative of a composite function)

If $y = f(u)$ and $u = g(x)$ are two differentiable functions

then, $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx}$

i.e. $\frac{d}{dx}[f(g(x)] = f'(g(x)).\ g'(x)$.

Example:

If $y = 2u^2 - 3u + 1$ and $u = 2x^2$ then find $\frac{dy}{dx}$.

Solution:

We have $y = 2u^2 - 3u + 1$ .... (i) and $u = 2x^2$ .... (ii)

Differentiating both sides of (i) w. r. t. u we get,

$$\frac{dy}{du} = 2\frac{d(u^2)}{du} - 3.\frac{du}{du} + \frac{d(1)}{du}$$

$$= 4u - 3$$

Differentiating both sides of (ii) w. r. t. x we get,

$$\frac{du}{dx} = 4x.$$

Hence by chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (4u - 3) \times 4x$$

$$= 4x\,(8x^2 - 3)$$

(vii) Derivative of Parametric Functions

Parametric functions are those functions in which both the variables x and y are expressed in terms of a third variable, called the parameter.

Let $x = f(t)$ and $y = g(t)$ be any two functions of t, t being a parameter.

In order to find $\frac{dy}{dx}$, we differentiate these two functions separately with respect to t and then apply the following relation:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided that } \frac{dx}{dt} \neq 0.$$

Example:

If $x = 2a(t^3 + 5)$ and $y = 5a^2\,(t^2 + 7)$

Solution:

We have, $x = 2a(t^3 + 5)$ .... (i) and

$y = 5a^2(t^2 + 7)$ .... (ii)

Differentiating both sides of (i) w. r. t. t we get,

$$\frac{dx}{dt} = 2a(3t^2 + 0) = 6at^2$$

Differentiating both sides of (ii) w.r.t.t we get,

$$\frac{dy}{dt} = 5a^2(2t + 0) = 10a^2t.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{10a^2t}{6at^2} = \frac{5a}{3t}$$

(viii) Derivative of Implicit Functions

A function given in the form $y = f(x)$, say $y = 10x^2 + 11x - 5$, is called an explicit function because the variable y is explicitly expressed as a function of x.

If the equation $f(x, y) = 0$ can be solved for y, we can explicitly write out the function $y = f(x)$ and find its derivative by the methods learned before. But if the equation $f(x, y) = 0$ cannot be solved for y explicitly, then it is an implicit function. However, we can differentiate term wise and solve for $\frac{dy}{dx}$. This process of finding $\frac{dy}{dx}$ without solving the equation for y is called implicit differentiation.

The following example illustrates the technique.

Example:

Find $\frac{dy}{dx}$ when $x^2 + 5xy + y^3 = 0$.

Differentiating both sides with respect to x we get,

$$\frac{d}{dx}(x^2) + 5\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = 0$$

$$\text{or, } 2x + 5\left(x\frac{dy}{dx} + y.\frac{dx}{dx}\right) + \frac{d}{dy}(y^3) \times \frac{dy}{dx} = 0$$

$$\text{or, } 2x + 5\left(x.\frac{dy}{dx} + y\right) + 3y^2\frac{dy}{dx} = 0$$

or, $2x + 5x\frac{dy}{dx} + 5y + 3y^2.\frac{dy}{dx} = 0$

or, $(5x + 3y^2)\frac{dy}{dx} = -(2x + 5y)$

$\therefore \frac{dy}{dx} = -\left(\frac{2x+5y}{5x+3y^2}\right)$

Power Series

Consider a function f that has a power series representation at x = a. Then the power series about (centered at) x = a is

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$
$$= c_0 + c_1 (x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots.$$

Where, the center a and the coefficients c_0, c_1, c_2, ..., c_n, ... are constants. Then the series is called power series and the function f(x) is called power series function.

A power series about x = a = 0 is a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2x^2 + \ldots + c_nx^n + \ldots.$$

We call f(x) a power series centered at x = 0 .

Taylor Series

Definition:

If f has derivative of all orders at x = a, then the **Taylor series** for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \cdots$$

Or,

The Taylor series for f(x) centered at x=c is

f(x) = $\sum_{n=0}^{\infty} \frac{f(n)(a)}{n!} (x-a)^n$

The Taylor series for f at x = a = 0 is known as the **Maclaurin series** for f.

i.e. if a = 0, then the series is called the **Maclaurin series** for f.

We use the notation $f^{(n)}$ to denote the nth order derivative of f.

Finding a Taylor series.

Find the Taylor series for f (x) = $\frac{1}{x}$ $at\ x = 1.$

Solution

For f (x) = $\frac{1}{x}$, the value of the function and its first four derivatives at x=1 are

f (x) = $\frac{1}{x}$ f (1) = 1

$f'(x)$ = - $\frac{1}{x^2}$ $f'(1)$ = -1 = - 1! [$d(x^{-1})/dx = (-1)\, x^{-1-1} = -\, x^{-2} = -1/x^2$]

$f''(x) = \frac{2}{x^3}$ $f''(x)$ = 2 = 2! [$d(x^{-2})/dx = (-2)\, x^{-2-1} = -\, 2x^{-3} = -2/x^3$]

$f'''(x)$ = - $\frac{3.2}{x^4}$ $f'''(1)$ = -3!

$f^{(4)}$ (x) = $\frac{4.3.2}{x^5}$ $f^{(4)}$ (1) = 4!

… … … … … …

That is, we have $f^{(n)}$ (1) = $(-1)^n$n! for all n≥ 0.

Therefore, the Taylor series for f at x =1 is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} (x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

One of the most important uses of infinite series is the potential for using an initial portion of the series for f to approximate f. We have seen, for example, that when we add up the first n terms of an alternating series with decreasing terms that the difference between this and the true value is at most the size of the next term. A similar result is true of many Taylor series.

Theorem 11.11.1 *Suppose that f is defined on some open interval I around a and suppose $f^{(N+1)}(x)$ exists on this interval. Then for each $x \neq a$ in I there is a value z between x and a so that*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!} (x-a)^{N+1}.$$

In another and easy way Taylor Series can be defined as follows:

Taylor's Theorem. *Let f be an $(n+1)$ times differentiable function on an open interval containing the points a and x. Then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some number c between a and x.

The function T_n defined by

$$T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n \quad \text{where } a_r = \frac{f^{(r)}(a)}{r!},$$

is called the *Taylor polynomial* of degree n of f at a. This can be thought of as a polynomial which approximates the function f in some interval containing a. The error in the approximation is given by the remainder term $R_n(x)$. If we can show $R_n(x) \to 0$ as $n \to \infty$ then we get a sequence of better and better approximations to f leading to a power series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which is known as the *Taylor series* for f. In general this series will converge only for certain values of x determined by the radius of convergence of the power series (see Note 17). When the Taylor polynomials converge rapidly enough, they can be used to compute approximate values of the function.

Theorem

Taylor's theorem with remainder

Let f be a function that can be differentiated n+1 times on an interval/containing the real number a. Let P_n be the nth Taylor polynomial of f at a and let

$$R_n(x) = f(x) - P_n(x)$$

be the nth remainder. Then for each x in the interval **I**, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

If their exists a real number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!} \; |x-a|^{n+1}$$

for all x in **I.**

Convergence of Taylor series

Suppose that f has derivatives of all orders on an interval containing a. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f(n)(a)}{n!}(x-a)^n$$

Converges to f (x) for all x in **I** if and only if

$$\lim_{n\to\infty} R_n(\text{x}) = 0 \text{ for all x in.}$$

Where, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^n$

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder R_n (x)→ 0. To prove that $R_n(x) \to 0$, we typically use the bond.

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

Maclaurin series

We have, Taylor Series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taking a = 0 in Taylor's theorem give us the expansion

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0))}{n!}x^n + R_n(x)$$

Where, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^n$

For some number c between 0 and x. For those values of x for which

$\lim_{n\to\infty} R_n$(x) = 0, we then obtain the following power series expansion

In short form Maclaurin series of f(x) can be written as

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ (Here $f^{(0)}(0)$ is defined to be f(0).)

In the next example, we find the Maclaurin series for e^x and sin x and show that these series converge to the corresponding functions for all real numbers by proving that the remainders R_n (x) → 0 for all real numbers x.

Finding Maclaurin series

For each of the following functions, find the Maclaurin series and its interval of convergence with remainder to prove that the Maclaurin series for f converges to f on that interval.

a. e^x

b. sin x

Solution:

a. Using the nth Maclaurin polynomial for e^x found as

$$e^x \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Since, if f(x) = e^x then $f^n(x) = e^x$ and $f^n(0) = 1$ for every natural n.

For any real number x. By combining this fact with the squeeze theorem, the result in $\lim_{n\to\infty} R_n(x)=0$

Explanation: - To find the values of x for which this is valid, we need to consider the remainder term (or use the Ratio Test alone)

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^n = \frac{x^{n+1}}{n+1!}\ e^c$

For some c between 0 and x. It follows from the Ratio Test that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Converges for any x and hence the sequence $\frac{x^n}{n!}$ Converges to 0. Therefore $R_n(x) \to 0$ for every $x \epsilon \mathbb{R}$, so the Maclaurin Series expansion is valid for every x.

Example. To find the Maclaurin series of the sine function we need to find its derivative of order n.

F(x) = sin x f(0) = 0

f′ (x) = cos x $f'(0) = 1$

f″ (x) = -sin x $f''(0) = 0$

f''' (x)= -cos x $f'''(0) = -1$

$f^{(4)}$ (x)= sin x $f^{(4)}(0) = 0$

It follows that $f^{(n)}(0) = 0$ if n is even, and alternates as 1, -1,1,-1…

for n= 1,3,5,7….

Hence the Maclaurin series expansion is

Sin x= x $-\frac{x^3}{3!}+\frac{x^5}{5!}-$ ...

Sin x = $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

a. Using the nth Maclaurin polynomial for sin x found in example below, we find that the Maclaurin series for sin x is given by

sinx =$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.

In order to apply the ratio test, consider

$\frac{|a_n+1|}{a_n} = \frac{|x|^{2n+3}}{(2n+1)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)}$.

Then $\lim_{n\to\infty}\frac{|x|^2}{(2n+3)(2n+2)}$ for all x, we obtain the interval of convergence as (-∞, ∞).

To show that the Maclaurin series converges to sin x, look at R_n (x).

For each x here exists a real number c between 0 and x such that

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

$\left|f^{(n+1)}(c)\right| \leq 1$ for all integers n and all numbers c, we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

Basic (and important) Maclaurin series,

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ $(x \in \mathbb{R})$

sin x = $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $(x \in \mathbb{R})$

cos x = $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ $(x \in \mathbb{R})$

$(1+x)^a = 1+ax+\frac{a(a-1)x^2}{2!}+\ldots = \sum_{n=0}^{\infty}\binom{a}{n}x^n$) $(x < 1)$

Where, $\binom{a}{n}$ = a(a-1) … (a- (n-1))/ n! $(a \in \mathbb{R})$

log(1+x) = $x-\frac{x^2}{2}+\frac{x^3}{3!}-\ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$ $(x < 1)$

-log(1-x) = $x+\frac{x^2}{2}+\frac{x^3}{3}-\ldots = \sum_{n=1}^{\infty}\frac{x^n}{n}$ $(x < 1)$

Example:

Find the Maclaurin series for

f(x) = cos(x)

Solution

We compute:

Given, f(x) – cosx, f(0) = cos0 =1

$f'(x)$ = -sinx, $f'(0)$ = -sin0 = 0

$f''(x)$ = - (cosx), $f''(0)$ = -(cos0) = -1

$f'''(x)$ = - (-sinx) = sinx, $f'''(0)$ = sin0 =0

Similarly,

$f^{(iv)}(0)$ = 1, $f^{(v)}(0)$ = 0 , $f^{(vi)}(0)$ = -1, $f^{(vii)}(0)$ = 0

$f^{(viii)}(0) = 1$, $f^{(ix)}(0) = 0$, $f^{(x)}(0) = -1$, $f^{(11)} = 0$

Hence, we have the series is

f(x)= f(0) + $f'(0)$ x+ $\frac{f''(0)}{2!}x^2$+…+$\frac{f^{(n)}(0))}{n!}x^n$ +…..

cosx =1-$x^2/2 + x^4$/4! - x^6/ 6! + x^8/8! -…

We see that series is

$$cos\ (x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

Note:

For, 2,4,6, … … nth term = a + (n-1) d = 2 + (n-1).2 = 2 + 2n -2 =2n

For, 1,3.5,7, … … … nth term = a + (n-1) d = 1 + (n-1).2 = 1 + 2n -2 = 2n -1

For, 3,5,7, …. ….. nth term = a + (n-1) d = 3 + (n-1).2 = 3 + 2n -2 = 2n+1

Exercises :

Find the Taylor series expansion for

A. sinx centered at x = 0

B. lnx centered at x = 0

Taylor polynomials

We have, Taylor series for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\ (x-a)^n = f(a) + \ f'(a)(x-a) + \frac{f''(a)}{2!} \qquad (x-a)^2 + \qquad \cdots$$
$$+ \frac{f^{(n)}(a)}{n!}\ (x-a)^n + \cdots$$

The nth partial sum of the Taylor series for a function **f** at **a** is known as the nth Taylor polynomial. For example, the 0^{th},1^{st}, 2^{nd} and 3^{rd} partial sums of the Taylor serried are given by

$P_o(x)$ =f (a)

P_1 (x) = f (a)+ f'(a) (x-a)

P_2 (x) = f(a) +f'(a) (x-a) + $\frac{f''(a)}{2!}$(x-a)

P_3(x) = f(a) +f'(a) (x-a) +$\frac{f''(a)}{2!}(x-a)^2$+$\frac{f'''(a)}{3!}(x-a)^3$ respectively.

These partials sums are known as the 0^{th},1^{st},2^{nd} and 3^{rd} Taylor polynomial.

For example, the 0^{th}, 1^{st}, 2^{nd} these polynomials are known as Maclaurin polynomial for f at 0. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition:

If f has n derivatives at x = a, then the nth Taylor polynomial for f at a is

$P_n(x) = f(a) + f'(a)\,(x\text{-}a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n$

The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f.

Example

Finding Taylor polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for f(x)= lnx at x =1.

Solution

f(x) = lnx f(1) = 0

$f'(x) = \frac{1}{x}$ $f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $f''(1) = -1$

$f'''(x)=\frac{2}{x^3}$ $\qquad$ $f'''(1) = 2$

$P_0(x) = f(1) = 0$

$P_1(x) = f(1) + f'(2)(x-1) = x-1$

$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2,$

$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$

$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

The graphs of y= f(x) and three first three Taylor polynomials are shown in figure 6.5.

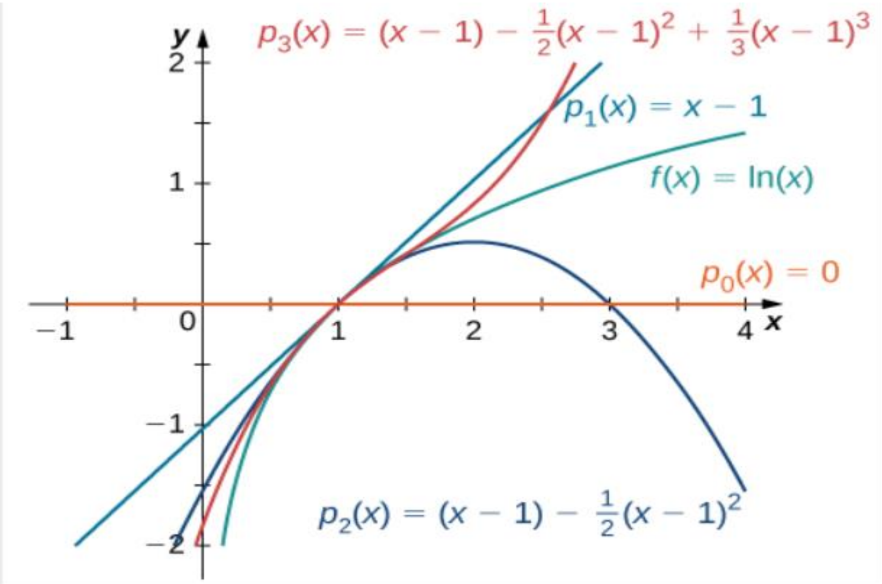

Figure 6.5 The function $y = \ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$ are plotted on this graph.

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, and p_2 and p_3. Find a formula for the nth Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. f(x)= e^x
b. f(x)= sin x
c. f(x) =cos x

Solution:

a. Since f (x) = e^x, we know that f(x) = f''(x) = f'''(x) =…= $f^{(n)}$(x) = e^x for all positive integers n.

Therefore, f(0) =f''(0) =f'''(0) =...= f^n(0) = 1

For all positive integers n. Therefore, we have

$P_0(X) = f(0) = 1$,

$P_1(x) = f(0) + f'(0)\,x = 1+x$,

$P_2(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2$

$P_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$

$P_n(x) = f(0) + f'(0)\,x + \frac{f'''(0)}{2}x^2 + \frac{f'''(0)'}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

$= \sum_{k=0}^{n} \frac{x^k}{k!}$

The function and the first three Maclaurin polynomials are shown in figure 6.6.

$$= \sum_{k=0}^{n} \frac{x^k}{k!}.$$

The function and the first three Maclaurin polynomials are shown in Figure 6.6.

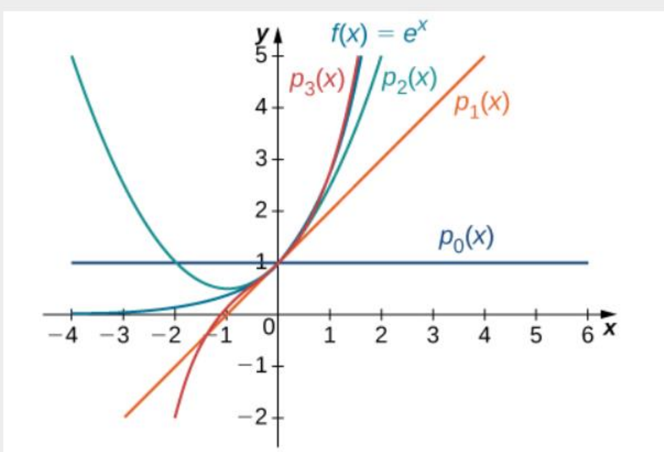

Figure 6.6 The graph shows the function $y = e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.

a. For f(x) =sin x, the value of the function and its four derivatives at x= 0 are given as follows:

f(x) = sinx, f(0) = 0

f'(x) = cosx, f'(0)=1

f'' (x)= -sinx, f''(0) = 0

f''' (x)= -cosx, f''' (0) = -1

$f^{(4)}$ (x) = sin x, $f^{(4)}$(0) =0.

Since the fourth derivative is sin x, the pattern repeats. That is, $f^{(2m)}$ (0)= 0 and $f^{(2m+1)}$ (0) = $(-1)^m$ for m ≥0. Thus, we have

P_0 (x) = 0

P_1(x) = 0+x = x,

P_2(x) = 0+x+ 0 = x,

P_3 (x) = 0+x+ 0-$\frac{1}{3!}x^3$= x- $\frac{x^3}{3!}$

P_4 (x) = o+x+0 -$\frac{1}{3!}x^3$+0=x-$\frac{x^3}{3!}$,

P_5(x)= 0+x+-$\frac{1}{3!}x^3$+0+$\frac{1}{5!}x^5$=x-$\frac{x^3}{3!}$+$\frac{x^5}{5!}$,

And for m≥0,

P_{2m+1} (x) = P_{2m+2}(x)

= x-$\frac{x^3}{3!}$+$\frac{x^5}{5!}$ - … +$(-1)^m$ $\frac{x^{2m+1}}{(2m+1)!}$

= $\sum_{k=0}^{m}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Graphs of the function and its Maclaurin polynomials are shown in Figure 6.7.

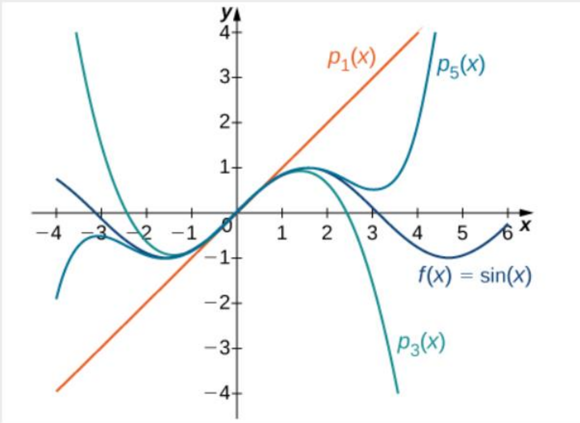

Figure 6.7 The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.

a. f(x) = cos x, the values of the function and its four derivatives at x= 0 are given as follows:

$f(x) = \cos x \qquad f(0) = 1$

$F'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{4}(0) = 1.$

Since the four derivatives cosx, the pattern repeats.

In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for m ≥ 0. Therefore,

$P_0(x) = 1$

$P_1(x) = 1 + 0 = 1,$

$P_2(x) = 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!}$

$P_3(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!},$

$P_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$

$P_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$

And for $n \geq 0$, $P_{2m}(x) = P_{2m+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!}$

Graphs of the function and the Maclaurin polynomials appear in Figure 6.8.

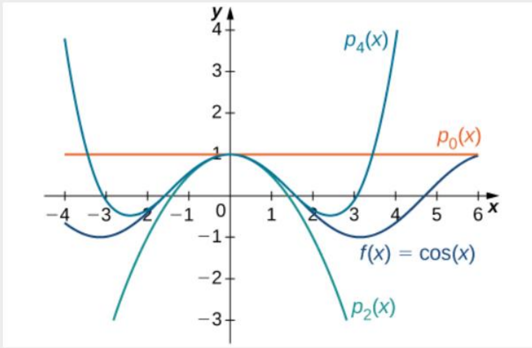

Figure 6.8 The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.

And so on.