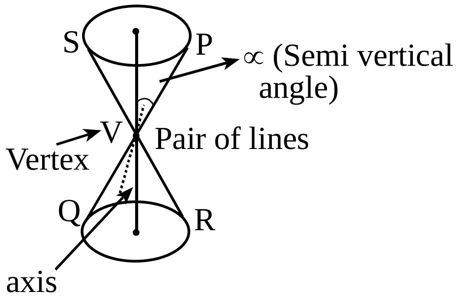
5

Conic Section

A section cut off from a circular (not necessary a right circular) cone by a plane in various way is a conic section. Its shape depends upon the position of the plane.

Consider a double right circular (nappe) cone of semi vertical angle α and let it be cut by a plane inclined at an angle θ to the axis of the cone. We will get different sections (curves) as follows.



Case I: If the plane passes through the vertex O(0,0)

The curve of intersection is pair of straight lines passing through the vertex which are

i) Real and distinct for $\theta < \alpha$

ii) Coincident for $\theta = \alpha$ i.e. plane touches the cone

iii) Imaginary for $\theta > \alpha$

Case – II :- If the plane does not pass through the vertex O(0,0)

The curve of intersection is called

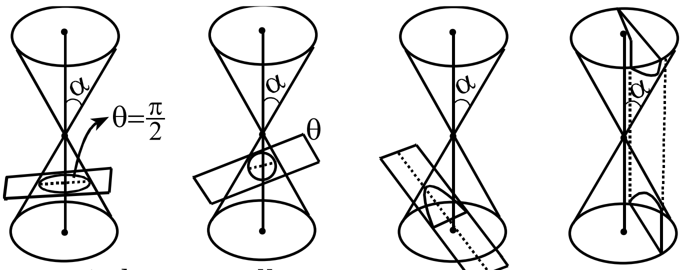
i) a circle if $\theta = \frac{\pi}{2}$ i.e. plane intersects cone perpendicular to axis

ii) a parabola for $\theta = \alpha$ i.e. if the plane not passing through vertex is parallel to the generator PQ

iii) an ellipse for $\theta > \alpha \left(\theta \neq \frac{\pi}{2}\right)$ i.e. if the plane cuts both the generating lines PQ and RS (with the axis greater then semi-vertical angles)

iv) a hyperbola for $\theta < \alpha$ i.e. if the plane cuts both the cones (angles between the axis and the plane is less than the semi vertical-angle)

Thus, we may get the section either as a pair of straight line, a circle, a parabola, an ellipse or hyperbola depending upon the different positions of the cutting plane. The curves of intersection are called the conic sections which have been clearly shown in the following figures.



For circle ($\theta = \frac{\pi}{2}$) For ellipse ($\frac{\pi}{2} > \theta > \alpha$) For parabola ($\theta = \alpha$) For parabola ($\theta > \alpha$)

Definitions

i) **Conic section**:- A conic section or conic is the locus of a point (say P) which moves in such a way that its distance from a fixed point (say S) always bears a constant ratio to the perpendicular distance from a fixed line all being in the same plane.

ii) **Focus:-** The fixed point is called focus of the conic section.

iii) **Directrix:-** The fixed straight line is called the directrix of the conic section.

I n general every conic has four foci, two of them are real and the other two are imaginary. Due to two real foci, every conic has two directrix corresponding to each real focus

iv) **Eccentricity:-** The constant ratio is called eccentricity of the conic section and is denoted by e.

On the basis of e, conic sections are classified as follows.

a) for $e \to \infty$, the conic is obtained as straight line

b) for $e > 1$, the conic is obtained as hyperbola

c) for $e = 1$, the conic section is obtained as parabola

d) for $e < 1$, the conic section is obtained as ellipse

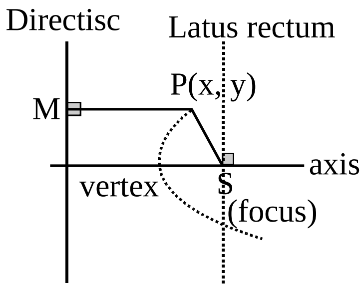
e) If $e = 0$, the conic section is obtained as circle

v) Axis:- The straight line passing through the focus and perpendicular to the directrix is called axis of the conic.

vi) Vertex:- The point of intersection of the conic section and the axis are called vertices of the conic.

vii) Centre:- The point which bisects every chord of conic section through it is called centre of conic section.

viii) Latus-Rectum:- The chord passing through the focus and perpendicular to the axis is called latus rectum



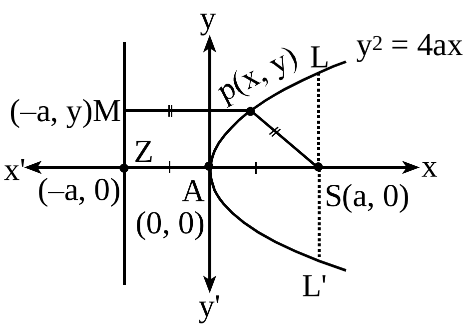
Here,

$e = \frac{PS}{PM}$. For parabola e = 1. So, PS = PM

Equation of the parabola in its standard form

Let, xx^1 be axis, S(a, 0) be focus, A(0, 0) be vertex and MZ be the directrix of the parabola. Let P(x, y) be any point on the parabola,

PM ⊥ MZ.



By the definition of parabola AS = AZ.

So, co-ordinates of Z is (– a, 0) and M is (– a, y)

Now, PS = PM (By definition)

$\Rightarrow \quad PS^2 = PM^2$

$\Rightarrow \quad (x - a)^2 + (y - 0)^2 + = \{x - (-a)\}^2 + (y - y)^2$

$\Rightarrow \quad x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2 + 0$

$\Rightarrow \quad y^2 = 4ax$, is the equation of parabola in standard form.

Tracing of the parabola $y^2 - 4ax$, a > 0

Given equation can be written as $y = \pm 2\sqrt{ax}$

a. **Origin**: The curve passes through origin and the tangent at origin is x = 0.

b. **Intersection with the axes**: The curve meets the co–ordinate axes only at the origin.

c. **Region**: For every negative value of x, the value of y is imaginary, therefore no part of the curve lies left of y –axis.

d. **Symmetry**: For every positive value of x, there are two equal and opposite value of y.

e. **Position occupied**: As $x \to \infty$, $y \to \infty$. Therefore, the curve extends to infinity to the right of axis of y.

***Note**: Equation of directrix as per above figure is $x = -a \Rightarrow x + a = 0$*

Some more terms

a. **Double ordinate**: A chord passing through P and perpendicular to the axis of parabola is called double ordinate.

b. **Latus Rectum**: A double ordinate through the focus is called latus rectum. Hence, latus rectum of a parabola is a chord passing through the focus and perpendicular to the axis.

Form above figure, when we put x = a in $y^2 = 4ax$, we get, $y = \pm 2a$

Hence two ends of latus rectum are L (a, 2a) and $L^1(a, -2a)$.

Also, **length of latus rectum** = 2a + 2a = 4a

Note: [Latus rectum = 2 [length of perpendicular from the focus on directrix]

c. **Focal distance of any point** : The distance of P(x, y) from the focus S is called the focal distance of the point P.

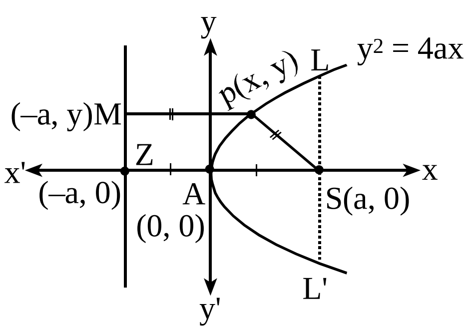
$\therefore$ Focal distance (SP) = $\sqrt{(x - a^2) + (y - 0)^2}$

$= \sqrt{(x - a)^2 + y^2} = \sqrt{(x - a^2) + 4ax} = \sqrt{(x + a)^2} = x + a.$

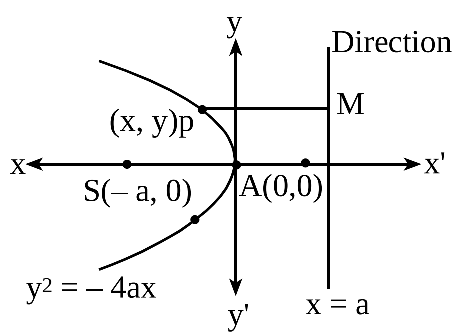
d. **Focal chord** : A chord of a parabola passing through the focus is called focal chord.

All the standard forms of parabola

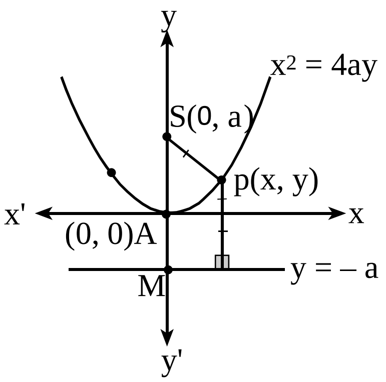
1. $y^2 = 4ax$



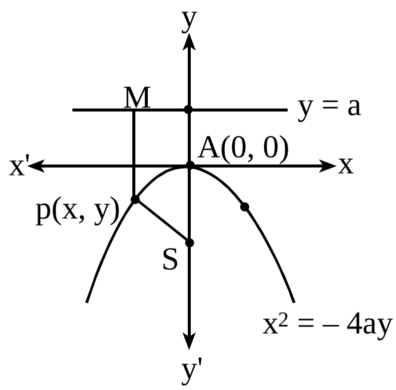
2. $y^2 = -4ax$



3. $x^2 = 4ay$



4. $x^2 = -4ay$



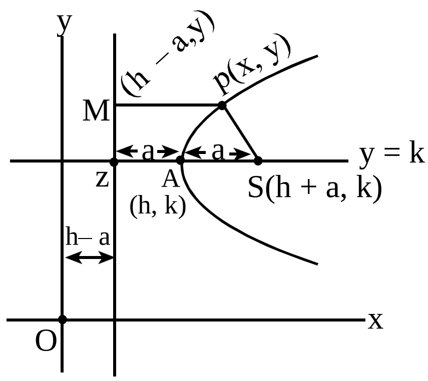
Regarding the shapes of the curves in the four-standard form, their corresponding results are given below in the table:

Equation	Co–ordinate of Vertex	Co–ordinate of focus	Equation of directrix	Equation of the axis	Length of latus rectum	Focal distance of point P(x, y)	Opens	Symmetry (about)
$y^2 = 4ax$	(0, 0)	(a, 0)	$x = -a$	$y = 0$	4a	$x + a$	Right	x – axis
$x^2 = -4ax$	(0, 0)	(–a, 0)	$x = a$	$y = 0$	4a	$a - x$	left	x– axis
$x^2 = -4ay$	(0, 0)	(0, a)	$y = -a$	$x = 0$	4a	$y + a$	Up	y – axis
$x^2 = -4ay$	(0, 0)	(0, –a)	$y = a$	$x = 0$	4a	$a - y$	down	y – axis

Equation of a parabola with its axis parallel to x–axis and vertex at any point (h, k)

Let a be the distance between vertex and focus, axis $y = k$ is parallel to x – axis. By definition

PS = PM



$\Rightarrow \quad PS^2 = PM^2$

$\Rightarrow \quad \{x - (h + a)\}^2 + (y - k)^2 = \{x - (h - a)\}^2$

$\Rightarrow \quad (x - h)^2 - 2(x - h)\, a + a^2 + (y - k)^2$

$\quad = \{(x - h) + a\}^2$

$\Rightarrow \quad (x - h)^2 - 2(x - h)\, a + a^2 + (y - k)^2$

$\quad = (x - h)^2 + 2\,(x - h)\, a + a^2$

$\Rightarrow \quad (y - k)^2 = 4a(x - h)$ is the required equation.

Note: Equation of a parabola with its axis parallel to the y–axis and the vertex at (h, k) is $(x - h)^2 = 4a(y - k)$

Corresponding results of above two equations are as follows

Equation	Results	Focus	Directrix	Axis	Latus Rectum	Symmetry (about)	Opens
$(y - k)^2 = 4a(x - h)$	(h, k)	(h + a, k)	$x = h - a$	$y = k$	4a	$y = k$	Left
$(x - h)^2 = 4a(y - k)$	(h, k)	(h, k + a)	$y = k - a$	$x = h$	4a	$x = h$	Above

Worked out Examples

1. Find the vertex, focus, equation of directrix, axis and length of latus rectum for the following parabolas.

a. $y^2 = 12x$

b. $x^2 = 12y$

c. $y^2 = 6y - 12x + 45$

d. $x^2 + 2y - 3x + 5 = 0$

Solution:-

a. Given $y^2 = 12x$ …………… (i)   Comparing with $y^2 = 4ax$, we get, $a = 3$

(i) Vertex = (0, 0)

(ii) Focus = (a, 0) = (3, 0)

(iii) Equation of directrix is, $x = -a \Rightarrow x = -3 \Rightarrow x+3 = 0$

(iv) Axis is x-axis, i.e. $y = 0$

(v) Latus rectum = $|4a| = |4\times 3| = 12$

b. Given $x^2 = 12y$ ………………… (i)

Comparing (i) with $x^2 = 4ay$, we get, $a = 3$

(i) Vertex = (0, 0)

(ii) Focus = (0, a) = (0, 3)

(iii) Equation of directrix is, $y = -a \Rightarrow y = -3 \Rightarrow y + 3 = 0$

(iv) Latus rectum = $|4a| = |4 \times 3| = 12$

c. Given, $y^2 = 6y - 12x + 5$

$\Rightarrow y^2 - 6y + a = -12x + 45 + 9 \rightarrow (y-3)^2 = -12x + 54$

$\Rightarrow (y-3)^2 = (-12)\left(x - \frac{9}{2}\right)$ …… (i)

Comparing with $(y-k)^2 = 4a(x-h)$, we get, $h = \frac{9}{2}$, $k = 3$,

$4a = -12 \Rightarrow a = -3$

(i) Vertex $= (h, k) = \left(\frac{9}{2}, 3\right)$

(ii) Focus $= (h + a, k) = \left(\frac{9}{2} - 3, 3\right) = \left(\frac{3}{2}, 3\right)$

(iii) Equation of directrix is, $x = h - a \Rightarrow x = \frac{9}{2} - (-3) \Rightarrow 2x - 15 = 0$

(iv) Length of Latus rectum $= |4a| = |4 \times (-3)| = |-12| = 12$

d. Given parabola is

$x^2 + 2y - 3x + 5 = 0$

$\Rightarrow \quad x^2 - 3a = -2y - 5$

$\Rightarrow \quad x^2 - 2. x . \frac{3}{2} + \left(\frac{3}{2}\right)^2 = -2y - 5 + \left(\frac{3}{2}\right)^2$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 = -2\left(y + \frac{11}{8}\right)$ …… (i)

Comparing with $(x-h)^2 = 4a(y-k)$ we get,

$h = \frac{3}{2}$, $k = \frac{-11}{8}$, $4a = -2 \rightarrow a = \frac{-1}{2}$

(i) Vertex $= (h, k) = \left(\frac{3}{2}, \frac{-11}{8}\right)$

(ii) Focus $= (h, k + a) = \left(\frac{3}{2}, \frac{-11}{8} \frac{-1}{2}\right) = \left(\frac{3}{2}, \frac{-15}{8}\right)$

(iii) Equation of directrix is, $y = k - a \rightarrow y = \frac{-11}{8} \; \frac{-1}{2} \Rightarrow 8y + 15 = 0$

(iv) Length of Latus rectum $= |4a| = \left|4 \times \left(\frac{-1}{2}\right)\right| = |-2| = 2$

2. Find the equation of the parabola satisfying the following conditions

a. Focus at (4, 0), vertex at (0 ,0)

b. Focus at (0 ,4), vertex (0 ,0)

Solution :

a. Focus = (a, 0) = (4, 0) $\Rightarrow$ a = 4

We have equation of parabola is

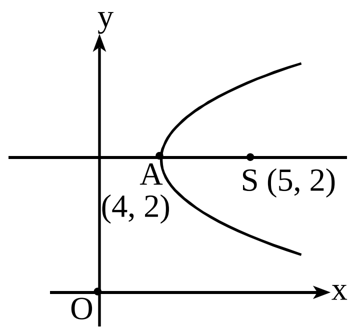
$y^2 = 4ax \Rightarrow y^2 = 4 \times 4 \times x \Rightarrow y^2 = 8x$

b. Focus = (0, a) = (0, – 4) $\Rightarrow$ a = – 4

We have, equation of parabola is,

$x^2 = 4ay \Rightarrow x^2 = 4 \times (-4) \times y \Rightarrow x^2 = -16y$

3. Find the equation of parabola whose vertex is (4, 2) and focus is (5, 2)



Solution: Since y – co–ordinate of the vertex and the focus are equal, the axis is parallel to x – axis. Here, (h, k) = (4, 2) = vertex and focus = (h + a, k) = (5, 2)

Then, h + a = 5 $\Rightarrow$ 4 + a = 5 $\Rightarrow$ a = 1

Now, equation of the parabola is $(y - k)^2 = 4a(x - 4)$

$\Rightarrow (y - 2)^2 = 4 \times 1. (x - 4)$

$\Rightarrow y^2 - 4y + 4 = 4x - 16$

$\Rightarrow y^2 - 4y - 4x + 20 = 0$ is required parabola.

3. Find the equation of the parabola whose focus is (1, – 1) and vertex is (2, 1). Also find its axis and latus rectum.

Solution:

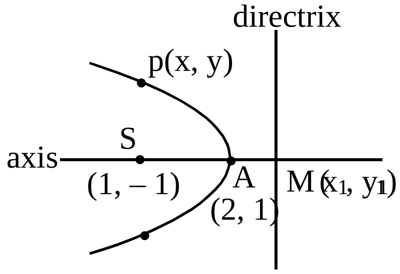
Let, M (x_1, y_1) be any point on the directrix.

$\therefore \frac{x_1 + 1}{2} = 2$ and $\frac{y_1 + (-1)}{2} = 1$

$\Rightarrow x_1 = 3, y_1 = 3$

$\therefore$ axis meets directrix at (3, 3)

Slope of axis $(m_1) = \frac{1 - (-1)}{2 - 1} = 2$



Slope of directrix (m_2) = $\frac{1}{m_1}$ = $\frac{-1}{2}$

Equation of directrix is $y - 3 = \frac{-1}{2}(x - 3) \Rightarrow x + 2y - 9 = 0$

Let, P(x, y) be any point on the parabola, then

Distance of P from focus = Perpendicular distance of P from directrix

$$\Rightarrow \sqrt{(x-1)^2 + (y+1)^2} = \pm\left[\frac{x + 2y - 9}{\sqrt{1^2 + 2^2}}\right]$$

Squaring,

$$(x - 1)^2 + (y + 1)^2 = \frac{(x + 2y - 9)^2}{5}$$

$\Rightarrow 4x^2 + y^2 - 4xy + 8x + 4by - 71 = 0$

Now,

Slope of axis (m_1) = 2

And axis passes through (1, -1), then equation of axis is

$y - (-1) = 2(x - 1) \Rightarrow 2x - y - 3 = 0$

Also,

Latus rectum = 2 [perpendicular length from focus on the directrix]

= 2 [length of the perpendicular from (1, -1) on $x + 2y - 9 = 0$]

$$= 2\left|\frac{1 - 2 - 9}{\sqrt{1 + 4}}\right| = 4\sqrt{5}$$

4. Find the equation of the parabola in which ends of the latus rectum have the co-ordinates (-1,5) and (-1,-11) and the vertex is (-5,-3)

Solution:- Since the end points of latus rectum are (-1, 5) and (-1, -11). So, the equation of latus rectum is x = -1. Hence, latus rectum is parallel to y – axis. So, axis is parallel to x-axis.

Also, vertex = (h, k) = (-5, -3)

Now, equation of parabola is

$(y - k)^2 = 4a(x + h)$

$\Rightarrow (y + 3)^2 = 4a(x + 5)$…….(i)

Equation (i) also passes through (-1, 5) then

$(5 + 3)^2 = 4a\,(-1 + 5) \Rightarrow a = 4$

Putting the value of 'a' is (i)

$(y + 3\,)^2 = 4 \times 4\,(x + 5)$

$\Rightarrow y^2 + 6y - 16x - 71 = 0$ is the required parabola

Exercise

1. Find vertex, focus, axis, equation of directrix and length of latus rectum of the following parabolas.

   a. $y^2 = 8x$

   b. $y^2 - 4y - 3x + 1 = 0$-

   c. $4x^2 + y = 0$

   d. $x^2 + y = 6x - 14$

2. Find the equation of the parabola whose focus is the point (2, 3) and directrix is the line $x - 4y + 3 = 0$. Also find the length of its latus rectum.

3. Find the equation of the parabola if

   a. Focus is at (0, – 3) and vertex is at (0, 0).

   b. Focus is at (0, – 3) and vertex is at (– 1, – 3).

   c. Focus is at (– 6, – 6) and vertex is at (– 2, 2).

Answer

1. a. (0, 0), (2, 0), y = 0, x = – 2, 8

   b. $(-1, 2), \left(\frac{-1}{4}, 2\right), y = -2, x = -\frac{7}{4}, 3$

   c. $(0, 0), \left(0, -\frac{1}{16}\right), x = 0, y = \frac{1}{16}, \frac{1}{4}$

   d. $(3, -5), \left(3, \frac{-21}{4}\right), x = 3, 4y + 19 = 0$

2. $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0, \frac{14}{\sqrt{17}}$

3. a. $x^2 - 12y$

   b. $y^2 + 6y - 4x + 5 = 0$

   c. $(2x - y)^2 + 4(26x + 37y - 31) = 0$

Multiple Choice Questions

Choose the best answer

1. The parabola $y^2 = -4ax$ with positive 'a' opens

(a) up (b) down (c) left (d) right

2. The directrix of $y^2 = -4ay$ is

a) $x = a$ (b) $x = -a$ (c) $y = -a$ (d) $y = a$

3. $x^2 = 4ay$ is symmetric about

(a) y-axis (b) x-axis (c) both y-axis and x-axis (d) none

4. Focus of the parabola $x^2 + 10y = 0$ is

(a) (0, 0) (b) $\left(\frac{3}{2}, 0\right)$ (c) $\left(0, \frac{3}{2}\right)$ (d) none

Answers

1. c   2. d   3. a   4. b

Activities and project work

1) A prepare a concrete material to show parabola by using thread and nail in wooden panel.
2) Prepare a project work to show the application of parabolic curve in everyday life.

Application of Parabola

Parabola is in a projectile thrown shape. So, it has many application. Parabolic reflectors have the property that the light rays or sound waves coming parallel to its axis converge at the focus and then it reflects them parallel to the axis. Due to this property, parabolic reflectors are used in cars, automobiles, loudspeakers, solar cookers, telescopes etc.

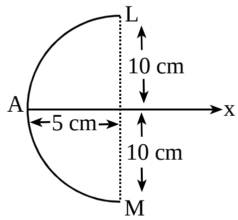
If the road way of a suspension bridge is loaded uniformly per horizontal meters, the suspension cable hangs in the form of arcs which closely approximate to the parabolic arcs. Therefore, parabolic arcs are used in suspension cable bridge construction.

Example-1

If a parabolic reflector is 20 cm in diameter and 5 cm deep, find its focus.

Solution:-

Let LAM be the parabolic reflector such that LM is its diameter and AN is its depth. It is given that AN = 5 cm and LM = 20 cm



$\therefore$ LN = 10 cm

Taking A as origin, AX along x-axis and a line through A perpendicular to AX as y=axis, let the equation of the reflector be $y^2 = 4ax$ ………… (i)

The point L has co-ordinates (5, 10) and lies on (i)

$\therefore$ $10 = 4a \times 5 \Rightarrow a = 5$.

So, the equation of the reflector is $y^2 = 20x$.

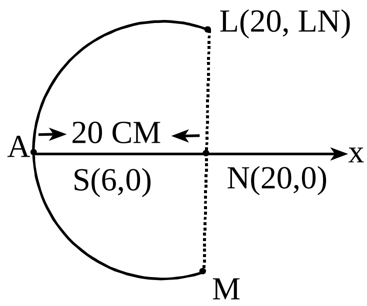
Its focus is at (5, 0) i.e. at the point N.

Hence, the focus is at the mid-point of the given diameter.

Example-2

The focus of a parabolic mirror is at distance 6 cm from its vertex. If the mirror is 20 cm deep, find the distance LM as given in figure below,

Solution: Let the axis of the mirror be along the positive direction of x-axis and the vertex A be the origin.



Since, the focus is at a distance of 6 cm from the vertex. Then, the co-ordinates of the focus are (6, 0). Therefore, the equation of the parabolic section is, $y^2 = 24x$ [Putting a = 6 in $y^2 = 4ax$]

Since, L(20, LN) lies on this parabola,

$(LN)^2 = 24 \times 20 \Rightarrow LN = 4\sqrt{30}$

$\therefore$ $LM = 2LN = 8\sqrt{30}$ cm.

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