Solution of partial Differential Equation (P.D.E)

→ The differential equation having two or more independent variable is known as partial differential equation.

→ The physical phenomena in applied and engineering fall into a category of partial differential equation.

eg: Heat flow problem, fluid flow analysis, electrical potential distribution etc.

→ The general form of P.D.E. involving two independent variable is,

$$a \frac{\delta^2 f}{\delta x^2} + b \frac{\delta^2 f}{\delta x . \delta y} + c \frac{\delta^2 f}{\delta y^2} = F\left(x, y, f, \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}\right) \quad \text{--- eq}^n \text{ (1)}$$

where co-effecients a, b & c may be constant or function of x and y.

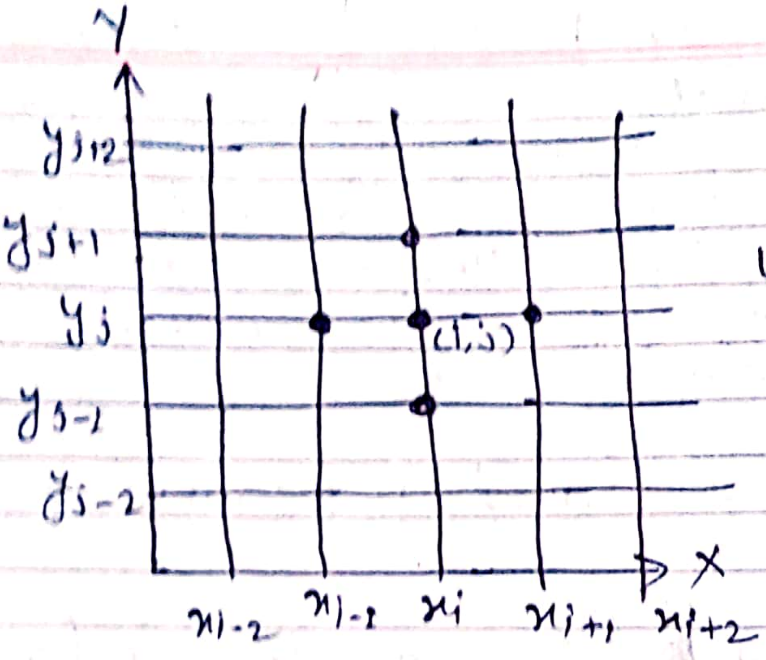
→ Equation (1) can be classified into three types.

1. Elliptic if $b^2 - 4ac < 0$.
2. parabolic if $b^2 - 4ac = 0$.
3. Hyperbolic if $b^2 - 4ac > 0$.

→ The finite difference & Finite element can be used to solve P.D.E.

Derivation of Difference Equation.

→ Consider two dimensional solution domain in which the domain is split into regular rectangular grids of width 'h' and height 'K'.



where
$x_{i+1} = x_i + h$
$y_{j+1} = y_j + k$

fig: 2D finite difference grid

→ In finite difference method, we replace the derivative that occur in P.D.E by their finite difference equivalent.

→ The first and second derivative are given as follows according to central difference approximation.

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h}$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2h} \quad \text{--- eq}^n \; 2$$

$$f''(x_i) = \frac{f(x_i+h) - 2f(x_i) + f(x_i-h)}{h^2}$$

$$f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \quad \text{--- eq}^n \; 3$$

→ When f is function of two variables x & y then partial derivative with respect to x & y is

$$\frac{\delta f(x_i, y_j)}{\delta x} = f_x(x_i, y_j) = \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{2h}$$

$$\text{or, } f_{x,ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h} \quad \text{--- eq}^n \text{ (4)}$$

$$\frac{\delta f(x_i, y_j)}{\delta y} = f_y(x_i, y_j) = \frac{f(x_i, y_{j+1}) - f(x_i, y_{j-1})}{2k}$$

$$\text{or, } f_{y,ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k} \quad \text{--- eq}^n \text{ (5)}$$

$$\frac{\delta^2 f(x_i, y_j)}{\delta x^2} = f_{xx}(x_i, y_j) = \frac{f(x_{i+1}, y_j) - 2f(x_i, y_j) + f(x_{i-1}, y_j)}{h^2}$$

$$\text{or, } f_{xx,ij} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} \quad \text{--- eq}^n \text{ (6)}$$

$$\frac{\delta^2 f(x_i, y_j)}{\delta y^2} = f_{yy}(x_i, y_j) = \frac{f(x_i, y_{j+1}) - 2f(x_i, y_j) + f(x_i, y_{j-1})}{k^2}$$

$$\text{or, } f_{yy,ij} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{k^2} \quad \text{--- eq}^n \text{ (7)}$$

$$\frac{\delta^2 f(x_i, y_j)}{\delta x . \delta y} = \frac{f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_{j-1}) - f(x_{i-1}, y_{j+1}) + f(x_{i-1}, y_{j-1})}{4hk}$$

or, $f_{xy,ij} = \dfrac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4hk}$ — eqⁿ (8)

Elliptic Equation.

→ There are two types of equation ($b^2 - 4ac < 0$)
(1) Laplace's equation &
(2) poisson's equation.

Laplace's equation

→ When $a = 1$, $b = 0$ & $c = 1$ and $F(x, y, f, \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}) = 0$ then eqⁿ (1) becomes,

$$\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = 0 \quad \text{— eq}^n \ (9)$$

or, $\Delta^2 f = 0$

where, the operator

$\Delta^2 = \left(\dfrac{\delta^2}{\delta x^2} + \dfrac{\delta^2}{\delta y^2}\right)$ is caused Laplacian operator and the eqⁿ (5) is caused Laplace's equation.

→ Using eqⁿ (6) & eqⁿ (7),

$$\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{k^2} = 0$$

→ If consider the square grid then
$h = k$..

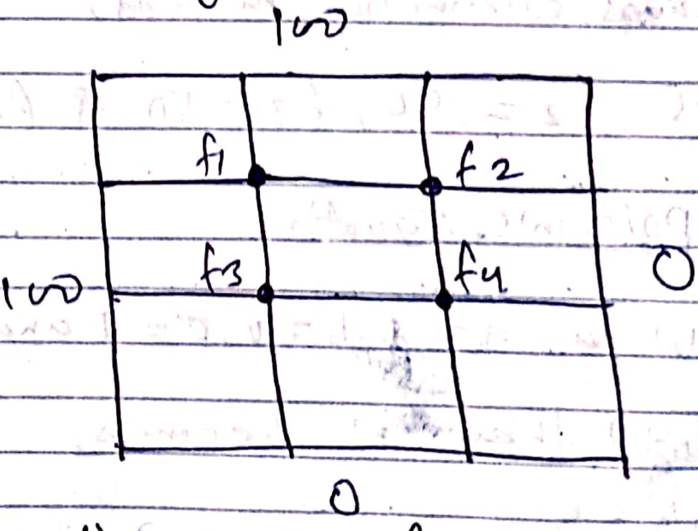
So, $\nabla^2 f_{i,j} = \dfrac{f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j}}{h^2} = 0$

or, $\nabla^2 f_{i,j} = f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j} = 0$ — eqn (10)

→ The equation (10) contains four neighbouring points around the central point (x_i, y_j). So, this eqn is known as five-point difference formula for Laplace's equation.

Example: Consider a steel plate of size 15 cm × 15 cm. If two of sides are held at 100°c and other two sides are held at 0°c, what are the steady-state temprature at interior points assuming a grid size of 5cm×5cm.

Soln



The system of equations are as follows,

At point 1: $f_2 + f_3 + 100 + 100 - 4f_1 = 0$

At point 2: $f_1 + f_4 + 100 + 0 - 4f_2 = 0$

At point 3: $f_1 + f_4 + 100 + 0 - 4f_3 = 0$

At point 4: $f_2 + f_3 + 0 + 0 - 4f_4 = 0$

i.e,

$$-4f_1 + f_2 + f_3 = -200$$
$$f_1 - 4f_2 + f_4 = -100$$
$$f_1 - 4f_3 + f_4 = -100$$
$$f_2 + f_3 - 4f_4 = 0$$

Solving above eqn as,

$$\left[\begin{array}{cccc:c} -4 & 1 & 1 & 0 & -200 \\ 1 & -4 & 0 & 1 & -100 \\ 1 & 0 & -4 & 1 & -100 \\ 0 & 1 & 1 & -4 & 0 \end{array}\right]$$

By using Gauss elimination method,

$f_1 = 75$, $f_2 = 50$, $f_3 = 50$ & $f_4 = 25$.

Poisson's Equation

- In eqn (1), when $a = 1$, $b = 0$, $c = 1$ and $F = \left(x, y, f, \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}\right)$ $= g(x,y)$ then it becomes,

$$\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = g(x,y) \quad \text{— eqn (11)}$$

or, $\nabla^2 f = g(x,y)$

Where,

$\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$ is called poisson's operator & eq^n (11) is called poisson's equation.

→ using eq^n (6) & eq^n (7)

$$\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{k^2} = g(x,y)$$

→ If we consider the square grid then $h = k$

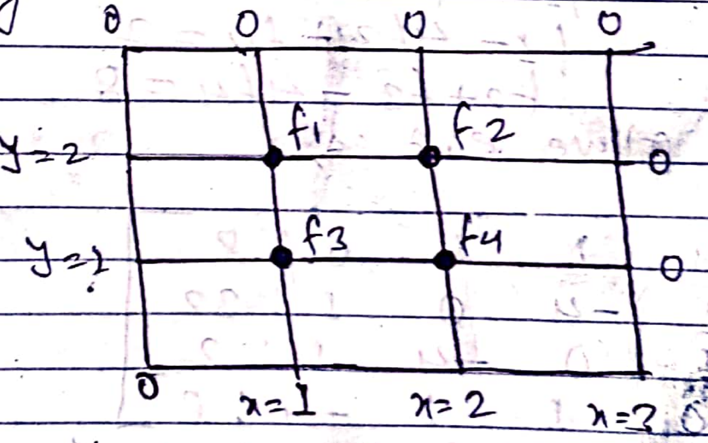
$$f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j} = h^2 g(x,y)$$

Example: Solve the poisson equation

$$\nabla^2 f = 2x^2y^2$$

over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$.

Soln



Since,

$$g(x,y) = 2x^2y^2$$

The system of equations are,

At point 1: $f_2 + f_3 + 0 + 0 - 4f_1 = (1)^2 (2x^2y^2)$

$= 2 * (1)^2 * (2)^2$

or, $-4f_1 + f_2 + f_3 = 8$

At point 2: $f_1 + f_4 + 0 + 0 - 4f_2 = (1)^2 * (2 * 2^2 * 2^2)$

or, $f_1 - 4f_2 + f_4 = 32$

At point 3: $f_1 + f_4 + 0 + 0 - 4f_3 = (1)^2 * (2 * (1)^2 * (1)^2)$

or, $f_1 - 4f_3 + f_4 = 2$

At point 4: $f_2 + f_3 + 0 + 0 - 4f_4 = (1)^2 * (2 * 2^2 * 1^2)$

or, $f_2 + f_3 - 4f_4 = 8$

Rearranging the equation, we get.

$$-4f_1 + f_2 + f_3 = 8$$
$$f_1 - 4f_2 + f_4 = 32$$
$$f_1 - 4f_3 + f_4 = 2$$
$$f_2 + f_3 - 4f_4 = 8$$

Solving above eqn as,

$$\left[\begin{array}{cccc:c} -4 & 1 & 1 & 0 & 8 \\ 1 & -4 & 0 & 1 & 32 \\ 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & 1 & -4 & 8 \end{array}\right]$$

By using Gauss elimination method,

$$\left.\begin{array}{l} f_1 = -22/4 \\ f_2 = -43/4 \\ f_3 = -13/4 \\ f_4 = -22/4 \end{array}\right\} \underline{\text{Ans}}$$