3

Sequence and Series

Learning outcomes/objectives:

After the completion of this chapter, the students will be enaable to

i) define sequence and series.

ii) classify sequence and series (arithmetic, geometric and harmonic)

iii) solve the problems related to arithmetic, geometric, harmonic sequence and series.

iv) establish relation among A.M., G.M. and H.M.

v) find the sum of infinite geometric series.

vi) find the sum of finite natural numbers, sum of squares of first n-natural numbers, sum of cubes of first n-natural numbers and related problems.

vii) define convergence and divergenge series with suitable examples,

viii) Taylor's Theorem, Taylor's Series, Taylor's Polynomial, Maclaurin Series, Exponential Series and their expansionf for given functions.

1.5.1 Introduction

In mathematics, the word 'sequence' is used widely in all sphere of mathematics. In fact, a collection of objects is listed in a 'sequence'. The collection of objects are arranged in an order. It means, first, member, second member, third member and so on. The amount of money deposited in the bank over a certain period of time form a sequence. The depreciation amount of certain machinery items form a sequence. Sequences have many important applications in the human civilization.

Hence, **sequence is an arrangement of objects in a definite order, rulr or in a visible pattern seperating with comma.**

Example-1: 2, 4, 6, 8, ... where two consecutive terms are in **equal difference** 2.

Example-2: 2, 4, 8, 16 .... where two consecutive terms are in **equal ratio** 2.

In this chapter, we study sequence, series, kinds of sequences i.e. arithmetic sequence, geometric sequence and harmonic sequence. Also, relationship between A.M., G.M. and H.M. and sum of the infinite to series.

1.5.2 Sequences

Before defining sequence, let us consider the following example:

Let us consider the successive quotients which are obtained in the division of 10 by 3 at different steps of division. In this process, we get 3, 3.3, 3.33, 3.333 ..... and so on. Which forms a sequence the number occurring in a sequence is called a term in a sequence. We denote the terms of the sequence by a_1, a_2, a_3 ..... a_n ..... etc. The first term is the number of first position, second term is the number of second position and n^{th} term is the number of n^{th} position, it is denoted by t_n. In the above example,

$a_1 = 3$, $a_2 = 3.3$, $a_3 = 3.33$ ....., $a_5 = 3.3333$ etc. A sequence containing finite number of terms is called a finite sequence and a sequence containing infinite number of terms is called an infinite sequence.

Often, sequence is possible to express in a rule which gives the various terms of the sequence. Let us consider the multiple of 4 i.e. 4, 8, 12, 16, 20 .....

Here,

$a_1 = 4 = 4 \times 1$

$a_2 = 8 = 4 \times 2$

$a_3 = 12 = 4 \times 3$

. . .

. . .

. . .

$a_n = 4 \times n$ and so on.

So, n^{th} term $(a_n) = 4n$ where n is the set of natural number which gives any number of terms.

In all the cases we cannot find a rule or formula of a sequence. For example;

1, 1, 2, 3, 5, 8 ..... has no **visible pattern.** But, sequence is generated by recurrence relation given by;

$a_1 = a_2 = 1$

$a_3 = a_1 + a_2 = 1 + 1 = 2$

$a_4 = a_2 + a_3 = 1 + 2 = 3$

$a_5 = a_3 + a_4 = 2 + 3 = 5$

$a_6 = a_4 + a_5 = 3 + 5 = 8$

. . .

. . .

. . .

$a_n = a_{n-2} + a_{n-1}$, $n > 2$

Thus, sequence is called **Fibonacci sequence**.

In every sequence, we should not except a formula. Such case, just by verbal, we generate a sequence i.e. by theoretical rule, from the above description we define sequence as A sequence is a set of number which are **connected by a fixed rule** and separated by comma (,). For example; 2, 4, 6, 8, 10 ..... where each of term is obtained by adding 2 to the preceding term. **Or,**

A sequence can be regarded as a function whose domain is the set of natural numbers and range is a set of real number, $\mathbb{R}$ i.e. $f : \mathbb{N} \to \mathbb{R}$. Sometimes, we use the functional notation $f(n)$ for t_n.

Example: A function f is defined as $f : \mathbb{N} \to \mathbb{R}$ by $f(x) = 4x - 3$, find sequence.

Here,

$f(x) = 4x - 3$ [$\because \mathbb{N} = \{1, 2, 3, 4 ....\}$]

$f(1) = 4 \times 1 - 3 = 1$

$f(2) = 4 \times 2 - 3 = 5$

$f(3) = 4 \times 3 - 3 = 9$

. . .

. . .

$f(n) = 4.n - 3$

Hence, the required sequence is

$\{f(1), f(2), f(3), ..., f(n)\} = \{1, 5, 9 ....4n\text{-}3\}$

$= \{(x, f(x) : x = 1, 2, 3 .....n)\}$

The sequence is denoted by $\{f_n\}$ rather than (fn). Since, the term $\{f\text{n}\}$ is called the n^{th} term or the general term of the sequence. It is sometimes denoted by t_n.

1.5.3 Series

Let a_1, a_2, a_3 ....., a_n be a given sequence. then, the expression $a_1 + a_2 + a_3 + ... + a_n + ...$ is called the series associated with the given sequence. If the number of terms is the sequence are limited then it is called finite sequence and if the number of terms in the sequence are unlimited or uncountable then it is called infinite sequence and the series are finite and infinite according as it is finite and infinite sequence.

Mainly, sequences are written in the compact form called sigma notation. The symbol used to denote sigma notation is Greek letter 'Σ' (Summation). The abbreviation form to denote series $a_1 + a_2 + a_3 + ... + a_n$ is $\sum\limits_{k=1}^{n} a_n$.

By studying the above notion of series. It can be redefined by the following way.

A series is a sequence of numbers that is connected by + or – signs. The word 'Series' is said to **represent the sum** of the numbers, and not the sum itself.

1.5.4 Progression

Progression are sets of numbers which are arranged in some definite rule and a sequence in a progresssion has a specific formula to calculate its n^{th} term. Since, a sequence can be based on logical rule, like as; set of prime numbers {2, 3, 5, 7, 11} some examples of progression:

i) 2, 6, 10, 14 .....

ii) $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$ .....

iii) – 3, 0, 3, 6 .....

Note: All the progressions are sequences but its converse is not always true.

1.5.5 Types of sequences or progressions

There are various types of sequences but in this level, we do concentrate only three types of sequences which are listed below:

a) Arithmetic progression (A.P.)

A sequence is said to be an arithmetic progression (A.P.) if the difference between a term to its preceding term is always same or constant throughout a whole sequence. The constant term is called a **common difference (d)** of an A.P.

In other word, A sequence $a_1, a_2, a_3 \,.....\, a_n \,.....$ is called arithmetic sequence if $a_{n+1} = a_n + d$ for all $x \in N$, where a_1 is the first term and 'd' is the common difference of an A.P.

General term of an Arithmetic progression.

Let $t_1 = a$ be the first term, d be the **common difference** of an A.P. $t_2, t_3, t_4, t_5 \,.....\, t_n$ be the second, third, fourth, ... and n^{th} term of an Arithmetic progression.

1^{st} term = $(t_1) = a = a + 0 = a + (1-1)$

Second term $(t_2) = t_1 + d = a + d = a + (2-1)\, d$

Third term $(t_3) = t_2 + d = a + d + d = a + 2d = a + (3-1)\, d$

Fourth term $(t_4) = t_3 + d = a + 2d + d = a + 3d = a + (4-1)\, d$

....

n^{th} term $(t_n) = a + (n-1) \,.\, d$

$\therefore$ **n^{th} term $(t_n) = a + (n-1) \,.\, d$**

Note:- In A.S. move forward adding d and move backward subtracting d. Then we get, ..., $a - 2d, a - d, a, a + d, a + 2d,$ ... ...

Sum of the first n-terms of an A.P.

Let 'a' be the first term and 'd' be the common difference of an A.P., then the first n terms of this sequence are; $a, a + d, a + 2d, a + 3d \,.....\, l$. Where the last term or the n^{th} term denoted by l or a_n

---

$\therefore \quad a_n = a + (n-1) \,.\, d$

$\boldsymbol{l = a + (n-1) \,.\, d}$

And, s_n be its sum.

$s_n = a + (a + d) + (a + 2d) + \,........\, + (l - d) + l \,.....\, (i)$

Again, the series can be written in the reverse order.

$s_n = l + (l - d) + (l - 2d) + \,......\, + (a + d) + a$

Adding eq^n (i) and eq^n(ii); we get

$2s_n = (a + l) + (a + l) + (a + l) + \,.....\, + (a + l) + (a + l)$

or, $2s_n = n(a + l)$

or, $s_n = \frac{n}{2}(a + l)$

or, $s_n = \frac{n}{2}[a + a + (n - 1) . d] = \frac{n}{2}[2a + (n - 1) . d]$

$\therefore$ $\mathbf{s_n = \frac{n}{2}[2a + (n - 1) . d]}$ is the sum of n terms in A. S.

Properties of Arithmetic sequence:

i) If a constant is added to each term of an A.P. the resulting sequence is also an A.P. For example:

If a, b, c, d, ... be an A.S. and k a constant number not equal to zero, then. a + k, b + k, c + k, d + k ..... are in A.P.

ii) If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P. If a, b, c, d ..... be an A.S. and $k \neq 0$. Then,

a – k, b – k, c – k, d – k, ... are in A.P.

iii) If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P. If a, b, c, d, ... be an A.S. and $k \neq 0$. Then,

ak, bk, ck, dk, ... are in A.P.

iv) If each term of an A.P. is dividend by a non-zero constant then the resulting sequence is also an A.P. If a, b, c, d, ... be an A.P. and $k \neq 0$. Then,

$\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}$, ... are in A.P.

Example:1

In an A.P. if p^{th} term is q and the q^{th} term is p, where $p \neq q$, find the m^{th}term.

Solution: Here,

$t_n = a + (n - 1) . d$

$t_p = a + (p - 1) . d$

$q = a + (p - 1) . d$ ..... (i)

And,

$t_q = a + (q - 1) . d$

or, $p = a + (q - 1) . d$ ..... (ii)

Solving eq^n (i) and eq^n (ii), we get

$p - q = a + qd - d - a - pd + d$

or, $p - q = qd - pd$

or, $(p - q) = (q - p) . d$

or, $d = \frac{(p - q)}{(q - p)} = - \frac{(p - q)}{(p - q)} = - 1$

Putting the value of d in eq^n (i)

$q = a + (p - 1) \times - 1$

or, $q = a - p + 1$

or, $a = p + q - 1$

$\therefore$ m^{th} term $(t_m) = a + (m - 1) . d$

$= p + d - 1 + (m - 1) \times - 1$

$= p + q - 1 - m + 1$

$= (p + q - m)$.

Example: 2

If the sum of n terms of an A.P. is $np + \frac{1}{2} n(n - 1) .Q$, where p and Q are constants, find the common difference

Solution: Here,

Let, $a_1, a_2, a_3 ..... a_n$ be the given A.P. Then,

$S_n = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$

or, $s_n = np + \frac{1}{2}n(n - 1) . Q$

Therefore,

$s_1 = a_1 = p$, $s_2 = a_1 + a_2 = p + p + Q = 2p + Q$

So that, $a_2 = 2p + Q - p = p + Q$

Hence, the common difference is given by

$d = a_2 - a_1 = p + Q - p = Q$.

1.5.6 Geometric sequence/progression and series

A sequence of terms in which the ratio between any two successive terms is the same throughout a whole sequence is called geometric sequence. The ratio is called **common ratio** and denoted by '**r**'.

In other word, A sequence $a_1, a_2, a_3 ....., a_n .....$ is called geometric progression, if each term is non-zero and $\frac{a_{k+1}}{a_k} = r$ (constant), for $k \geq 1$. By letting $a_1 = a$, we obtain a geometric progression; a, ar, ar^2, ar^3 ..... where a is first term and r is common ratio of a G.P.

General term of a Geometric sequence

Let us consider a G.P. with non-zero first term 'a' and **common ratio** 'r'. Then, $t_1, t_2, t_3, t_4, \ldots t_n$ be the first term, second term, third term, fourth term, ... n^{th} terms of an G.P. Then,

$t_1 = a = a \cdot 1 = a \cdot r^0 = a \cdot r^{1-1}$

$t_2 = a \cdot r = a \cdot r^{2-1}$

$t_3 = a \cdot r^2 = a \cdot r^{3-1}$

$t_4 = a \cdot r^3 = a \cdot r^{4-1}$

$t_5 = a \cdot r^4 = a \cdot r^{5-1}$

. . .

. . .

. . .

$t_n = ar^{n-1}$

$\therefore$ **n^{th} term of a G.P. $(t_n) = ar^{n-1}$**

If l is the last term then $\boldsymbol{l = ar^{n-1}}$

Note:- In G.S. move forward multiplying by r and move backward dividing by r. Then we get, ..., $\frac{a}{r^2}$, $\frac{a}{r}$, a, ar, ar^2, ...

Sum of n^{th} terms of a G.P.

Let us consider a be the first term, r be the common ratio and s_n be the sum of the first n-terms of a geometrical sequence. Then,

$s_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$ ..... (i)

Multiplying each term by 'r' we get.

$rs_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$ ..... (ii)

Subtracting eq^n (ii) from eq^n (i), we get,

$s_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$

$- \quad rs^n = \pm ar \pm ar^2 \pm ar^3 + \ldots \pm ar^{n-1} \pm ar^n$

$(1 - r)\, s_n = a + 0 + 0 + 0 + \ldots + 0 - ar^n$

or, $(1 - r)s_n = a - ar^n$

or, $\mathbf{s_n = \dfrac{a(1 - r^n)}{1 - r}}$, **$r \neq 1$. Then**

Case-I: If r < 1 then $s_n = \dfrac{a(1-r^n)}{1-r}$

Case-II: If r > 1, then $s_n = \dfrac{a(r^n-1)}{(r-1)}$

If *l* be the last term of a G.P. then *l* = ar^{n-1}

$$\therefore \quad s_n = \frac{ar^n - a}{(r-1)} = \frac{ar^{n-1}\,.\,r - a}{(r-1)} = \frac{lr - a}{(r-1)} : r > 1$$

Also, $s_n = \dfrac{-(a - lr)}{-(1-r)} = \dfrac{a - lr}{(1-r)} : r < 1$

Properties of Geometric sequence.

If a, b, c, d ..... are the terms of a G.S. and k is a constant i.e. $k \neq 0$. Then,

i) If each term of a G.P. is multiplied by a constant the resulting sequence is also a G.P. i.e.

ak, bk, ck, dk, ... is a G.S.

ii) If each term of a G.P. is divided by a non-zero constant the resulting sequence is also a G.P. i.e.

$\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}$, ... is a G.P.

iii) If each term of a G.P. is raised the power k then the resulting sequence is also a G.P. i.e.

a^k, b^k, c^k, d^k, ... is a G.P.

Example: 1

Find the 10^{th} term and n^{th} terms of the G.P. 5, 25, 125, ...

Solution: Here,

$a = 5,\ r = \dfrac{t_2}{t_1} = \dfrac{25}{5} = 5$

Thus, 10^{th} term $(t_{10}) = ar^{10-1} = 5.\,(5)^9 = 5^{10}$

n^{th} term $(t_n) = ar^{n-1} = 5.\,(5)^{n-1} = 5^n$

Example: 2

Find the sum of first n terms and the sum of first 5 terms of the geometric series: $1 + \frac{2}{3} + \frac{4}{9} + ...$

Solution: Here,

Let s_n be the sum of the series:

$$s_n = 1 + \frac{2}{3} + \frac{4}{9} + ...$$

$$a = 1 \text{ and } r = \frac{t_2}{t_1} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$$

Therefore,

$$s_n = \frac{a(1 - r^n)}{(1 - r)}$$

$$\text{So, } S_5 = \frac{1\left[1 - \left(\frac{2}{3}\right)^5\right]}{1 - \frac{2}{3}} = \frac{1 - \frac{32}{243}}{\frac{3-2}{3}} = \frac{\frac{211}{243}}{\frac{1}{3}} = \frac{211}{243_{81}} \times \frac{3}{1} = \frac{211}{81}.$$

$$\text{Also, } S_n = \frac{1\left[1 - \left(\frac{2}{3}\right)^n\right]}{1 - \frac{2}{3}} = 3\left[1 - \left(\frac{2}{3}\right)^n\right]$$

1.5.7 Harmonic Sequence (H. S.) / Harmonic progression (H.P.)

A sequence of numbers $a_1, a_2, a_3, ...$ such that their **reciprocals** $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ...$ form an arithmetic sequence (number separated by a common difference).

There is no separate formulas to find the n^{th} term and sum to n^{th} term of the harmonic sequence. To find out these terms, we need to connect with arithmetic sequence.

$$\therefore \quad n^{th} \text{ tem of H.S.} = \frac{1}{n^{th} \text{ term of A.S.}}$$

Note: H.S. changes into A.S. taking the reciprocal of each term.

Properties of Harmonic sequence

If a, b, c, d, ... are the terms of a H.S. and k be non-zero terms such that $k \neq 0$. Then,

i) If each term of a H.S. is multiplied by a constant $k \neq 0$, the resulting sequence is also a H.S. i.e. ak, bk, ck, dk, ... is a H.S.

ii) If each term of a H.S. is divided by a constant $k \neq 0$, the resulting sequence is also a H.S. i.e $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}$, ... is a H.S.

Collection of Formulae:-

1) For A.S, ($d \neq 0$)

n^{th} term $(t_n) = a + (n - 1) . d$

$s_n = \frac{n}{2} [2a + (n - 1) . d]$ is the sum of n terms in A. S.

2) For G.S. ($r \neq 1$)

n^{th} term of a G.P. $(t_n) = ar^{n-1}$

Sum of n terms in G.S. is as follows:

Case-I: If $r < 1$ then $s_n = \frac{a(1 - r^n)}{1 - r}$

Case-II: If $r > 1$, then $s_n = \frac{a(r^n - 1)}{(r - 1)}$

3) For H.S.

$\therefore$ n^{th} tem of H.S. $= \frac{1}{n^{th} \text{ term of A.S.}}$

1.5.8 Means

Any number of term or terms between the first term and last term of the sequences are called means. Indeed, A finite sequence has two or more than two means between first and last term. If the terms of the sequence are in A.P., G.P. and H.P. they are called A.M., G.M. and H.M. respectively.

i) Any number of terms between first term and last term of an arithmetic sequence is called arithmetic mean which is written as A.M. in abbreviation form.

ii) Any number of terms between first term and last term of a Geometric sequence is called geometric mean which is written as G.M. in abbreviation form.

iii) Any number of term or terms between first term and last term of a Harmonic sequence is called Harmonic mean written as H.M. in abbreviation form.

Formula for means:

Formula: 1

If a and b be any two given numbers then, the A.M., G.M. and H.M. between them are given below:

i) A.M. = $\frac{a+b}{2}$ ii) G.M. = $\sqrt{ab}$ or $(G.M.)^2$ = ab

iii) H.M. = $\frac{2ab}{a+b}$

Proof:

i) If A be the Arithmetic mean between a and b. Then, a, A, b are in A.P.

So, A – a = b – A [common difference]

or, A + A = a + b

or, 2A = (a + b)

or, A = $\frac{a+b}{2}$

$\therefore$ A.M. = $\frac{a+b}{2}$

ii) If G be the Geometric mean between a and b. Then, a, G, b are in G.P.

So, $\frac{G}{a} = \frac{b}{G}$ [common ratio]

or, G^2 = ab

$\therefore$ G.M. = $\sqrt{ab}$

iii) If H be the Harmonic mean between a and b.

Then, a, H, b are in H.P. So, $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

or, $\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$ [common difference]

or, $\frac{1}{H} + \frac{1}{H} = \frac{1}{a} + \frac{1}{b}$

or, $\frac{1+1}{H} = \frac{b+a}{ab}$

or, $\frac{2}{H} = \frac{a+b}{ab}$

or, $H = \frac{2ab}{a+b}$

$\therefore$ H.M. $= H = \frac{2ab}{a+b}$

Formula: 2

There are n Arithmetic means between a and b then common difference (d) $= \frac{(b-a)}{(n+1)}$

Solution: Here,

Let, $m_1, m_2, m_3, \ldots\ldots m_n$ be the n arithmetic means between a and b. Then,
a, m, $m_2, m_3, \ldots\ldots m_n$, b are in A.S. So,

a, $m_1, m_2, m_3, m_4, \ldots\ldots\ldots\ldots, m_n$, b are in A.S.

First term $(t_1) = a$

Last term $(t_n) = b$

Number of terms (n) = (n + 2)

We know that,

$t_n = t_1 + (n-1) \cdot d$

or, $b = a + (n + 2 - 1) \cdot d$

or, $b - a = (n + 1) \cdot d$

or, $d = \frac{(b-a)}{(n+1)}$

$\therefore$ 1^{st} mean $(m_1) = t_1 + d = a + \frac{(b-a)}{(n+1)}$

2^{nd} mean $(m_2) = m_1 + d = a + \frac{(b-a)}{(n+1)} + \frac{b-a}{(n+1)}$

$= a + 2 \cdot \frac{(b-a)}{(n+1)}$

3^{rd} mean $= m_2 + d = a + 2 \cdot \frac{(b-a)}{(n+1)} + \frac{(b-a)}{(n+1)}$

$= a + 3 \frac{(b-a)}{(n+1)}$

....

....

m^{th} mean = a + md

$$= a + m \cdot \frac{(b-a)}{(n+1)}$$

Formula: 3

There are n G.M.'S between two numbers a and b. Then, $r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

Solution: Here,

Let G_1, G_2, G_3 ....., G_n be n G.M.'S between a and b. Then a, G_1, G_2, G_3 ....., G_n, b are in G.S.

First term $(t_1) = a$

No. of terms (N) = (n + 2)

Last term $(t_n) = b$

We know that,

$t_n = ar^{N-1}$

or, $b = a. r^{n+2-1}$

or, $\frac{b}{a} = r^{(n+1)}$

or, $r^{n+1} = \frac{b}{a}$

or, $r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

$\therefore$ 1^{st} G.M. $(G_1) = a \cdot r = a \cdot \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

2^{nd} G.M. $(G_2) = a. r^2 = a \cdot \left(\frac{b}{a}\right)^{\frac{2}{(n+1)}}$

3^{rd} G.M. $(G_3) = a \cdot r^3 = a.\left(\frac{b}{a}\right)^{\frac{3}{(n+1)}}$

....

....

$$n^{th} \text{ G.M. } (G_n) = a \,.\, r^n = a \,.\, \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$$

Formula: 4

The A.M., G.M. and H.M. between any two **unequal positive** numbers satisfy the following relations:

a) $(G.M.)^2 = (A.M.) \times (H.M.)$

b) A.M. > G.M. > H.M.

Proof:

Let a and b be two unequal positive numbers. Then,

Arithmetic mean (A.M.) $= \frac{(a + b)}{2}$

Geometric mean (G.M.) $= \sqrt{ab}$

Harmonic mean (H.M.) $= \frac{2ab}{(a + b)}$

Now,

For (a)

$$\text{R.H.S.} = (A.M.) \times (H.M.) = \frac{(a+b)}{2} \times \frac{2ab}{(a+b)}$$

$$= \frac{2ab}{2}$$

$$= ab$$

$$= (\sqrt{ab})^2$$

$$= (G.M.)^2 = \text{L.H.S.}$$

$\therefore$ $(G.M.)^2 = (A.M.) \times (H.M.)$ Proved

For (b)

$$\text{A.M.} - \text{G.M.} = \frac{(a+b)}{2} - \sqrt{ab}$$

$$= \frac{(a+b) - 2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2}{2}$$

$$= \frac{(\sqrt{a} - \sqrt{b})^2}{2} > 0$$

[∵ The difference of the square of two unequal numbers is always greater than > 0]

A.M. – G.M. > 0

or, A.M. > G.M. ....(i)

Again, from first part i.e. (a)

$(\text{A.M.}) \times (\text{H.M.}) = (\text{G.M.})^2$

or, A.M. × H.M. = G.M. × G.M.

or, $\frac{\text{A.M.}}{\text{G.M.}} = \frac{\text{G.M.}}{\text{H.M.}}$

From first part; A.M. > G.M.

So, obviously G.M. > H.M. ....(ii) [see $\frac{8}{4} = \frac{4}{2}$]

Combining (i) and (ii), we get

A.M. > G.M. > H.M.

Worked out Examples:

Example: 1

Find the 15th term under $t_n = (n-1) . (2-n) . (n+3)$

Solution: Here,

$t_n = (n-1) . (2-n) . (n+3)$

When, n = 15, Then,

$t_{15} = (15-1) . (2-15) (15+3)$

$= 14 \times -13 \times 18$

$= -3276.$

Example: 2

The sum of n^{th} term of two arithmetic progressions are in the ratio (3n +8) : (7n + 15). Find the ratio of their 12^{th} term.

Solution: Here,

Let, a_1, a_2 and d_1, d_2 be first term and common difference of two arithmetic sequences. Then,

Sum of n^{th} term of first sequence $(S_n) = n/2[\, 2a_1 + (n - 1) \,.\, d_1]$

Sum of n^{th} term of Second sequence $(S_{n'}) = n/2[2\, a_2 + (n' - 1) \,.\, d_2]$

We know that,

$$\frac{S_n}{S_{n'}} = \frac{3n + 8}{7n + 15}$$

$$\text{or, } \frac{\frac{n}{2}[2a_1 + (n - 1) \,.\, d_1]}{\frac{n}{2}[2a_2 + (n - 1) \,.\, d_2]} = \frac{3n + 8}{7n + 15}$$

$$\text{or, } \frac{2a_1 + (n - 1) \,.\, d_1}{2a_2 + (n - 1) \,.\, d_2} = \frac{3n + 8}{7n + 15}$$

12^{th} terms of two sequences

$$\text{or, } \frac{2a_1 + (12 - 1) \,.\, d_1}{2a_2 + (12 - 1) \,.\, d_2} = \frac{3 \times 12 + 8}{7 \times 12 + 15}$$

$$\text{or, } \frac{2a_1 + 11d_1}{2a_2 + 11d_2} = \frac{44}{99} = \frac{4}{9}$$

$$\text{or, } \frac{2a_1 + 11d_1}{2a_2 + 11d_2} = \frac{4}{9}$$

$\therefore$ Their ratios are 4 : 9

Example: 3

If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then find the sum of the first (p + q) terms.

Solution: Here,

Let a and d be the first term and common difference of an A.P.

Sum of p^{th} term $(s_p) = \frac{n}{2}[2a + (p - 1) \,.\, d]$

Sum of q^{th} term $(s_q) = \frac{n}{2}[2a + (q - 1) . d]$

By the question,

$s_p = s_q$

$\frac{p}{2}$ $(p - 1) . d] = \frac{q}{2}[2a + (q - 1) . d]$

or, $p[2a + (p - 1) . d] = q[2a + (q - 1) . d]$

or, $2ap + p^2d - pd = 2aq + q^2d - qd$

or, $2a (p - q) + d (p^2 - q^2) - d (p - q) = 0$

or, $2a (p - q) + d (p + q) (p - q) - d (p - q) = 0$

or, $(p - q) (2a + pd + qd - d) = 0$

Since, $(p - q) \neq 0$. So,

$2a + pd + qd - d = 0$

or, $d(p + q - 1) + 2a = 0$

or, $2a = - d(p + q - 1)$

Then,

$$\text{Sum of } (p + q) \text{ terms} = \frac{(p+q)}{2}[2a + (p + q - 1)d]$$

$$= \frac{(p+q)}{2}[- d(p + q - 1) + (p + q - 1)d]$$

$$= \frac{(P+q)}{2} \times 0$$

$$= 0$$

$\therefore$ Sum of $(p + q)$ terms $= 0$

Example: 4

If p is A.M. between q and r, q is G.M. between r and p their prove that r will be H.M. between p and q.

Solution: Here,

As p is A.M. between q and r.

$p = \frac{q+r}{2}$ .....(i)

As q is G.M. between r and p

$q^2 = rp$ .....(ii)

Now, from (i)

$p = \frac{(q + r)}{2}$

or, $2p = (q + r)$

or, $2pq = q(q + r)$ [Multiplying both sides by q]

or, $2pq = q^2 + qr$

or, $2pq = rp + qr$ [from (ii)]

or, $2pq = r(p + q)$

or $r = \frac{2pq}{(p + q)}$

$\therefore$ r is the H.M. between p and q

Example: 5

If a, b, c are in G.P. prove that $\frac{1}{(a + b)}, \frac{1}{2b}, \frac{1}{(b + c)}$ are in A.P.

Solution: Given,

If a, b, c are in G.P. Then,

$b^2 = ac$ ..... (i)

Now,

$\frac{1}{(a + b)}, \frac{1}{2b}, \frac{1}{(b + c)}$ will be in A.P.

if $\frac{1}{2b} - \frac{1}{(a + b)} = \frac{1}{(b + c)} - \frac{1}{2b}$

or, $\frac{a + b - 2b}{2b(a + b)} = \frac{2b - b - c}{2b(b + c)}$

or, $\frac{a - b}{a + b} = \frac{b - c}{b + c}$

or, $(a - b)(b + c) = (a + b)(b - c)$

or, $ab + ac - b^2 - bc = ab - ac + b^2 - bc$

or, $ac + ac = b^2 + b^2$

or, $2ac = 2b^2$

or, $b^2 = ac$ which is true by ..... (i)

Hence, $\frac{1}{(a + b)}, \frac{1}{2b}, \frac{1}{(b + c)}$ are in A.P.

Example: 6

If a, b, c are in H.P. prove that 2a – b, b, 2c – b are in G.P.

Solution: Here,

If a, b, c are in H.P.

$$b = \frac{2ac}{(a + c)}$$

Now,

$$(2a - b)\,(2c - b) = \left(2a - \frac{2ac}{a + c}\right).\left(2c - \frac{2ac}{a + c}\right)$$

$$= \frac{(2a^2 + 2ac - 2ac)}{(a + c)} \cdot \frac{(2ac + 2c^2 - 2ac)}{(a + c)}$$

$$= \frac{2a^2}{(a + c)} \times \frac{2c^2}{(a + c)}$$

$$= \frac{4a^2c^2}{(a + c)^2}$$

$$= \left(\frac{2ac}{(a + c)}\right)^2 = b^2$$

$\therefore$ $(2a - b) . (2c - b) = b^2$. This shows that (2a – b), b, (2c – b) are is G.P.

Example: 7

The sum of three numbers in A.P. is 36. When the numbers are increased by 1, 4, 43 respectively, the resulting numbers are in G.P. Find the numbers.

Solution: Here,

Let three numbers in A.P. be **a – d, a, a + d**. Then,

$a - d + a + a + d = 36$

or, $3a = 36$

or, $a = 12$

Since, a – d + 1, a + 4, a + d + 43 are in G.P.

i.e. 12 – d + 1, 12 + 4, 12 + d + 43 are in G.P.

i.e. 13 – d, 16, 55 + d are in G.P.

or, $\dfrac{16}{13 - d} = \dfrac{55 + d}{16}$

or, $256 = (13 - d) . (55 + d)$

or, $256 = 13(55 + d) - d(55 + d)$

or, $256 = 715 + 13d - 55d - d^2$

or, $256 = 715 - 40d - d^2$

or, $d^2 + 40d + 256 - 715 = 0$

or, $d^2 + 40d - 459 = 0$

or, $d^2 + 51d - 9d - 459 = 0$

or, $d (d + 51) - 9 (d + 51) = 0$

or, $(d - 9) . (d + 51) = 0$

Either,

$d - 9 = 0$ or, $d + 51 = 0$

or, $d = 9$ or, $d = -51$

When $d = -51$, $a = 12$

$a - d = 12 + 51 = 63$

$a = 12$

$a + d = 12 - 51 = -39$

When $d = 9$, $a = 12$

$a - d = 12 - 9 = 3$

$a = 12$

$a + d = 12 + 9 = 21$

Required three numbers are 63, 12, – 39 and 3, 12, 21

Example: 8

If a^2, b^2, c^2 are in A.P. then prove that $b + c$, $c + a$, $a + b$ are in H.P.

Solution: Here,

a^2, b^2, c^2 are in A.P.

$2b^2 = a^2 + c^2$

Adding $(2ab + 2bc + 2ca)$ both sides, we get.

$2b^2 + 2ab + 2bc + 2ca = a^2 + c^2 + 2ab + 2bc + 2ca$

or, $2b (b + a) + 2c (b + a) = (a + c)^2 + 2b (a + c)$

or, $(a + b) (2b + 2c) = (a + c) (a + c + 2b)$

or, $2(a + b) (b + c) = (a + c) (a + c + 2b)$

or, $\dfrac{2}{a+c} = \dfrac{(a+c+2b)}{(a+b)\,(b+c)}$

or, $\dfrac{2}{(a+c)} = \dfrac{(a+c+b+b)}{(a+b)\,(b+c)}$

or, $\dfrac{2}{(a+c)} = \dfrac{(a+b+b+c)}{(a+b)\,(b+c)}$

or, $2\,(a+b)\,(b+c) = (a+c)\,.\,(a+b+b+c)$

or, $\dfrac{2(a+b)\,(b+c)}{\{(a+b)+(b+c)\}} = (a+c)$

$\therefore$ $(a+c) = \dfrac{2(a+b)\,(b+c)}{\{(a+b)+(b+c)\}}$

Hence, b + c, c + a, a + b are is H.P. Proved.

Example: 9

If A be the arithmetic mean and H, the H.M. between two quantities a and b show that: $\dfrac{a-A}{a-H} \times \dfrac{b-A}{b-H} = \dfrac{A}{H}$

Solution: Here,

Let a and b be two numbers. Then,

A.M. (A) = $\dfrac{(a+b)}{2}$

G.M. (G) = $\sqrt{ab}$

H.M. (H) = $\dfrac{2ab}{a+b}$

L.H.S. $\dfrac{a-A}{a-H} \times \dfrac{b-A}{b-H}$

$$= \frac{a - \dfrac{(a+b)}{2}}{a - \dfrac{2ab}{a+b}} \times \frac{b - \dfrac{(a+b)}{2}}{b - \dfrac{2ab}{a+b}}$$

$$= \frac{\dfrac{2a-a-b}{2}}{\dfrac{a^2+ab-2ab}{(a+b)}} \times \frac{\dfrac{2b-a-b}{2}}{\dfrac{ab+b^2-2ab}{(a+b)}}$$

$$= \frac{\frac{(a-b)}{2}}{\frac{a^2-ab}{(a+b)}} \times \frac{\frac{-a+b}{2}}{\frac{b^2-ab}{(a+b)}}$$

$$= \frac{(a-b)}{2} \times \frac{(a+b)}{a(a-b)} \times \frac{(b-a)}{2} \times \frac{(a+b)}{b(b-a)}$$

$$= \frac{(a+b)\,(a+b)}{2 \times 2ab}$$

$$= \frac{(a+b)}{2} \times \frac{(a+b)}{2ab}$$

$$= A \times \frac{1}{\frac{2ab}{(a+b)}}$$

$$= \frac{A}{H}$$

= R.H.S proved.

Example: 10

If x, y, z be in A.P. y, z, p be in G.P. and z, p, q be in H.P. prove that x, z, q are in G.P.

Solution: Here,

As, x, y, z are in A.P.

$y = \frac{x+z}{2}$ ..... (i)

As y, z, p be in G.P.

$z^2 = y\,p$ ..... (ii)

As z, p, q be in H.P.

$p = \frac{2zq}{z+q}$ ..... (iii)

From (ii)

$z^2 = y\ \ p$

or, $z^2 = \frac{(x+z)}{2} \times \frac{2zq}{(z+q)}$

or, $z = (x+z) \cdot \frac{q}{(z+q)}$

or, $z^2 + zq = qx + zq$

or, $z^2 = qx$

$\therefore$ x, z, q are in G.P. proved.

Example: 11

If a, b, c are the x^{th} term, y^{th} term and z^{th} term respectively of a series in G.P., then show that: $a^{y-z}.b^{z-x}.c^{x-y} = 1$.

Solution: Given,

x^{th} term $(t_x) = a$, y^{th} term $(t_y) = b$, z^{th} term $(t_z) = c$

We know that,

$t_x = Ar^{x-1}$

or, $a = A.r^{x-1}$ ..... (i)

Also, $t_y = A.r^{y-1}$

or, $b = A.r^{y-1}$ ..... (ii)

And,

$t_z = A.r^{z-1}$

or, $z = A.r^{z-1}$ ..... (iii)

L.H.S $a^{y-z}.b^{z-x}.c^{x-y}$

$= (A.r^{x-1})^{(y-z)}.(A.r^{y-1})^{(z-x)}.(A.r^{z-1})^{(x-y)}$

$= A^{y-z}.r^{(x-1).(y-z)}.A^{z-x}.r^{(y-1)(z-x)}.A^{x-y}.r^{(z-1)(x-y)}$

$= A^{y-z+z-x+x-y}.r^{(x-1)(y-z)+(y-1)(z-x)+(z-1)(x-y)}$

$= A^o.r^{xy-xz-y+z+yz-xy-z+x+xz-yz-x+y}$

$= 1 \times r^o$

$= 1 \times 1$

$= 1$

$=$ R.H.S proved.

Example: 12

If a, b, c are three positive numbers in harmonic sequence show that $a^5 + c^5 > 2b^5$

Solution: Here,

A.M. between a^5 and c^5 is $= \frac{a^5 + c^5}{2}$

G.M. between a^5 and $c^5 = \sqrt{a^5c^5} = (a^5c^5)^{\frac{1}{2}}$

Then,

A.M. > G.M.

or, $\frac{a^5 + c^5}{2} > (a^5c^5)^{\frac{1}{2}}$ ..... (i)

G.M. between a and c is $\sqrt{ac}$

H.M. between a and c is b.

Then,

$\sqrt{ac} > b$

or, $(ac)^{\frac{1}{2}} > b$

or, $(ac)^{\frac{5}{2}} > b^5$ ..... (ii)

From (i) and (ii)

$\frac{a^5 + c^5}{2} > (ac)^{\frac{5}{2}} > b^5$

or, $\frac{a^5 + c^5}{2} > b^5$

or, $a^5 + c^5 > 2b^5$ proved.

Exercise - 5.1

1. Find the value of p so that $8p - 4$, $3p - 2$ and $2p + 7$ will be in A.P.

2. In an A.P., if p^{th} term is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of first pq terms is $\frac{1}{2}(pq + 1)$, where $p \neq q$

3. The ratio of the sums of p and q terms of an A.P. is $p^2 : q^2$. Show that the ratio of p^{th} and q^{th} term is $(2p - 1) : (2q - 1)$.

4. If the sum of x terms of an A.P. is $3x^2 + 5x$ and its m^{th} term is 164, find the value of m.

5. The sum of first three terms of a G.P. is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

6. If A.M. and G.M. of two positive numbers x and y are 10 and 8 respectively, find the numbers.

7. If G be the geometric mean between two unequal positive numbers a and b, prove that:

a) $\frac{1}{G-a}+\frac{1}{G-b}=\frac{1}{G}$

b) $\frac{1}{G^2-a^2}+\frac{1}{G^2-b^2}=\frac{1}{G^2}$

8. If G and H be the geometric mean and Harmonic mean between two unequal positive numbers a and b prove that.

a) $\frac{1}{H-a}+\frac{1}{H-b}=\frac{1}{a}+\frac{1}{b}$

b) $\frac{a-A}{a-H}\times\frac{b-A}{b-H}=\frac{A}{H}$

9. a) If a be the Arithmetic mean between b and c, b the G.M. between a and c, Prove that c will be the H.M. between a and b.

b) If x, y, z be in A.P., y, z, x be in H.P., Prove that z, x, y are in G.P.

c) If v, w, x are in A.P. w, x, y are in G.P. and x, y, z are in H.P., Prove that v, x, z are in G.P.

10. If x, y, z are in A.P., G.P. or H.P. Show that y^2 is greater than, equal to or less than xz.

11. The A.M. between two unequal positive numbers a and b exceeds their G.M. by 2 and G.M. exceeds H.M. by 1.6. Find the numbers.

12. The sum of first three numbers in A.P. is 36. When the numbers are increased by 1, 4, 43 respectively, the resulting numbers are in G.P. Find the numbers.

13. a) $p^x = q^y = r^z$ and p, q, r are in G.P. then prove that x, y, z are in H.P.

b) If one G.M. 'G' and two A.M. 'S a and b are inserted between two given positive numbers, prove that $G^2 = (2a - b) . (2b - a)$.

14. a) If b + c, c + a, a + b are in H.P., then prove that $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$ are in A.P.

b) If x^2, y^2, z^2 are in A.P. then prove that

$\frac{1}{y+z}$, $\frac{1}{x+z}$, $\frac{1}{x+y}$ are also in A.P.

15. If x, y, z are in H.P. Prove that

a) $\frac{yz}{y+z}, \frac{xz}{x+z}, \frac{xy}{x+y}$ are in H.P.

b) $2x - y, y, 2z - y$ are in G.P.

c) $x(y + z), y(x + z), z(x + y)$ are in A.P.

16. If a, b, c are in H.P. Prove that:

$\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

17. If three unequal positive numbers x, y, z are in H.P., Prove that

i) $x^3 + z^3 > 2y^3$

ii) $x^n + z^n > 2y^n$

18. If the sum of the first n terms of a G.P. is 'S', their product is P and the sum of their reciprocals is R, then prove that $p^2 = \left(\frac{S}{R}\right)^n$.

Answers

1. $P = -\frac{7}{4}$ 4. M = 27, 5. $r = -\frac{3}{4}, \frac{4}{3}, -1, \frac{3}{4}$ and $r = -\frac{4}{3}, \frac{3}{4}, -1, \frac{4}{3}$ 6. x = 16, y = 4 or x = 4, y = 16 11. a = 16, b = 4 or, a = 4, b = 16 12. 3,12,21.

1.5.9 Sum of infinite series:

Infinite geometric series is the sum of all the terms of an infinite geometric sequence. For examples:

$$100 + 50 + 25 + \frac{25}{2}, ...$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + ... + \frac{1}{2}.\left(\frac{1}{3}\right)^{n-1} + ...$$

Here, $a = \frac{1}{2}$, $r = \frac{t_2}{t_1} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$

Then,

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1}{2}\left(\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}\right)$$

$$= \frac{1}{2} \times \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

$$= \frac{3}{4} - \frac{3}{4} \cdot \frac{1}{3^n}$$

$$= \frac{3}{4} - \frac{1}{4 \cdot 3^{n-1}}$$

To get the sum of infinite geometric series, we increase the value of n. When n increases, $\frac{1}{3^{n-1}}$ decreases. When n increased sufficiently, $\frac{1}{3^{n-1}}$ decreases, sufficiently. Thus when $n \to \infty$ $S_n \to \frac{3}{4}$.

In general, the sum of an infinite geometric progression whose first term 'a' and common ratio 'r' ($-1 < r < 1$) i.e. $|r| < 1$ is $S_\infty = \frac{a}{1 - r}$

Proof:

Let, $a + ar + ar^2 + ar^3 + \ldots\ldots + ar^n \ldots\ldots \infty$ is called an infinite geometric series. Now, the sum of n-terms of a G.S. is

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

or, $$S_n = \frac{a - ar^n}{(1 - r)}$$

or, $$S_n = \frac{a}{1 - r} - \frac{ar^n}{(1 - r)} \ldots\ldots \text{(i)}$$

Since, $-1 < r < 1$, therefore, r^n decrease as n increases and $r^n \to 0$ as $n \to \infty$

$$\therefore \quad \frac{ar^n}{1 - r} \to 0 \text{ as } n \to \infty$$

$$\therefore \quad S_\infty = \frac{a}{(1 - r)}$$

Hence, the sum of infinite geometric series is $\mathbf{S_\infty = \frac{a}{1 - r}}$ **where $|r| < 1$**.

Worked out examples

Example: 1

Prove that: $6^{\frac{1}{2}} . 6^{\frac{1}{4}} . 6^{\frac{1}{8}} .... = 6.$

Solution: Here,

At first sum to $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ is

$$S_\infty = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$S_\infty = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$\text{L.H.S} = \;= 6^{\frac{1}{2}} . 6^{\frac{1}{4}} . 6^{\frac{1}{8}} ...$$

$$= 6^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...}$$

$$= 6^1$$

$$= 6$$

= R.H.S proved.

Example: 2

Express $0.\overline{3}$ as a rational number using geometrical series.

Solution: Here,

$0.\overline{3} = 0.3333$

$= 0.3 + 0.03 + 0.003 + 0.0003 + ...$

$$= \frac{0.3}{1 - 0.10} = \frac{0.3}{0.90} = \frac{3}{9} = \frac{1}{3}$$

Example: 3

The sum to infinity of a geometric series is 15, and the first term is 3. Find the common ratio.

Solution: Here,

Sum of infinity G.S. (S_∞) = 15

First term (a) = 3

Common ratio (r) = ?

We know that,

$$S_\infty = \frac{a}{1-r}$$

or, $15 = \dfrac{3}{1-r}$

or, $5 = \dfrac{1}{1-r}$

or, $5 - 5r = 1$

or, $4 = 5r$

$\therefore$ $r = \dfrac{4}{5}$

Example: 4

If $| x | < 1$, $y = x + x^2 + x^3 \ldots\ldots \infty$. Prove that $x = \dfrac{y}{1+y}$

Solution: Here,

$| x | < 1$

$\therefore$ $y = x + x^2 + x^3 + \ldots\ldots + \infty$

or, $y = \dfrac{x}{1-x}$ [Since, $S_\infty = \dfrac{a}{1-r}$]

or, $y - xy = x$

or, $x + yx = y$

or, $y = x(1 + y)$

or, $x = \dfrac{y}{1+y}$

$\therefore$ $x = \dfrac{y}{1+y}$ proved.

Example: 5

A square is drawn by joining the mid-points of the sides of a given square. A third square is drawn inside the second square in the same way and the process continues indefinitely. If a side of the first square is 16 cm, determine the sum of the areas of all the squares.

Solution:

Let ABCD be a square of side 16cm as shown as in figure. Let EFGH be the second square drawn by joining the mid-points of the sides of given square ABCD.

BF = EB = 8 cm

$EF^2 = EB^2 + BF^2 = 8^2 + 8^2 = 128$

So, EF = $8\sqrt{2}$

Area of the first square ABCD = 16^2 = 256 sq.cm

Area of the second square EFGH = $\left(8\sqrt{2}\right)^2$ = 128 sq.cm

Hence, infinite G.S. of area is 256 + 128 + ...

where, first term (a) = 256 and common ration(r) = 128/256 = 0.5

We have,

$S_\infty = \frac{a}{1-r} = \frac{256}{1-0.5}$ = 512 sq.cm

Example: 6

A side o f an equilateral triangle is 16cm. Second equilateral triangle is formed by joining the mid-points of first equilateral triangle and third equilateral triangle is also formed by joining the mid-points of second equilateral triangle. If this process continues indefinitely, find the sum of the perimeters of all the triangles.

Solution:-

The mid-points of its sides are joined to from another triangle whose mid-points are joined to from another triangle. This process continues indefinitely.

Side of 2nd equilateral triangle =8 cm (line segment joining mid-point of two sides of a triangle is parallel to third side and half of it).

Similarly, side of third equilateral triangle =4cm and so on.

Perimeter of 1st triangle =3. 16 = 48cm

Perimeter of 2nd triangle = 3.8 = 24 cm

Perimeter of 3rd triangle = 3. 4 = 12cm.

Sum of perimeter of all triangles = 48 + 24 + 12 + ...

This is an infinite G. S. where, firs term(a)= 48 and common ratio = 24/48 = 1/2 = 0.5

Sum of an infinity G.S = $S_\infty = \frac{a}{1-r} = \frac{48}{1-0.5}$ = 96 cm.

Exercise

1. Determine whether the given geometric infinite series have sums.
   a) $1 + 4 + 4^2 + 4^3 + \ldots\ldots\ldots\ldots$
   b) $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots\ldots\ldots$
   c) $4 + 2 + 1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$
   d) $5 + \sqrt{5} + 1 + \ldots\ldots\ldots\ldots\ldots\ldots$
   e) $1 + (-1) + 1 + (-1) + \ldots.$
2. Find the sum of each of the following geometric series.
   a) $81 + 27 + 9 + 3 + \ldots\ldots\ldots\ldots$
   b) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots\ldots\ldots\ldots$
   c) $5^{-1} + 5^{-2} + 5^{-3} + \ldots\ldots\ldots\ldots$
   d) $16 - 8 + 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots$
   e) $0.6 + 0.06 + 0.006 + 0.006 + \ldots\ldots\ldots$
3. a) The sum to infinity of a geometric series is 14, and the first term is 2. Find the common ratio.
   b) The sum of an infinite number of terms in G.S. is 15, and the sum of their squares is 45; find the series.
   c) The sum of first two terms of an infinite G.P. is 5 and each term is three times the sum of succeeding terms. Find the series.
4. Prove that: $4^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 4^{\frac{1}{16}} \ldots\ldots\ldots\ldots = 4$
5. If $|y| < 1$, $x = y + y^2 + y^3 + \ldots\ldots\ldots\ldots$ to ∞, prove that : $y = \frac{x}{1 + x}$
6. If $x = \frac{1}{1 + y}$, $y > 0$, prove that: $x + x^2 + x^3 + \ldots\ldots\ldots\ldots \infty = \frac{1}{y}$
7. A rubber ball is dropped from a height of 16 cm. At each rebounds three-fourth of the distance falls. If it continues to fall of the previous fall, how far will it travel before coming to rest?

   Hint: While falling down, Infinite G.S. is $16 + 16 \times \frac{3}{4} + \ldots.. = 16 + 12 + \ldots$

   where, $a = 16$, $r = \frac{3}{4}$, $S_\infty = \frac{a}{1 - r} = \frac{16}{1 - \frac{3}{4}} = 64$

While rebouncing up, Infinite G.S. is $12 + 1 \times \frac{3}{4} + \ldots = 12 + 9 + \ldots$

$a = 12,\ r = \frac{3}{4},\ S_\infty = \dfrac{a}{1-r} = \dfrac{12}{1-\frac{3}{4}} = 48$

Total distance covered = 64 + 48 = 112 cm

8. The sides of square is 32 cm. A second square is formed by joining the mid-points of each sides and Second Square is also formed by repeating the same process. If this process is continued indefinitely, what is the sum of the perimeters of all the squares?

Hint:- Infinite G.S. is $4 \times 32 + 4 \times 8\sqrt{2}, \ldots = 128 + 32\sqrt{2}$

$a = 4 \times 32 = 128,\ r = \dfrac{32\sqrt{2}}{128} = \dfrac{\sqrt{2}}{4}$

$S_\infty = \dfrac{a}{1-r} = \dfrac{128}{1-\frac{\sqrt{2}}{4}} = \dfrac{512}{4-\sqrt{2}} = \dfrac{512(4+\sqrt{2})}{15}$ cm

Answer :

1.b) 2 1.c) 8, 1.d) $\dfrac{25 + 5\sqrt{5}}{4}$ e) $\dfrac{1}{2}$ 2.a) $\dfrac{243}{2}$ 2b. $\dfrac{3}{4}$ 2.c) $\dfrac{1}{4}$ 2.d) $\dfrac{32}{3}$ 2.e) $\dfrac{2}{3}$

Activities and Project work

1. Take a square of arbitrary measure assuming its one square unit area make another square joining the mid points of all sides. Continue this process as per as possible. Prepare a project model to verify the sum of area of all the possible squares.

♦●♦●♦