Solution of Non linear Equation

Error in Numerical Calculus.

(i) Exact:

The numbers having exact digits are called exact numbers. For example 2, 99, 2040, 7/2 are exact numbers, i.e they have termenating digits.

(ii) Approximate:

There are some numbers such as 4/3 = 1.3333........ $\sqrt{2}$ = 1.414228 .... and π = 3.141592 ...... which cannot be expressed by a finite number of digits are called approximate numbers. These may be approximated by numbers 1.3333, 1.4142 and 3.1416 respectively. Such numbers which represents the given number to a certain degree of accuracy are called approximate numbers.

Significant Digits

⇒ If we write 2/7 as 0.285714 and π as 3.14159 then we say the number contain six significant digits.

⇒ All non zero digits are significant.

⇒ All zeros occuring between non zero digits are significant digits.

⇒ Trailing zeros following a decimal point are significant. Example 3.50, 65.0 and 0.230 all have 3 significant digit each.

⇒ Zeros between the decimal point and preceding a non zero digit are not significant. Example 0.0001234, 0.001234 all have 4 sd.

⇒ when the decimal point is not written, trailing zeros are not taken to be significant. Example 7.56×10^4, 7.560×10^4, 7.5600×10^4 has 3, 4 and 5 significant digits respectively.

Hence the concept of accuracy and precision are closely related to significant digits.

Accuracy:

Accuracy reffered to the number of significant digits in a value. Example, the number 68.214 is accurate upto 5 significant digit.

Procesion:

Precision refers to the number of decimal positions i.e the order of magnitude of the last digit in a value. Example, the number 68.214 has a precision of 0.001 or 10^{-3}.

* Which of the following number has the greatest precision?
(a) 4.3201 (b) 4.32 (c) 4.320106

→ 4.320106

Inherent Error:-

Error that are present in the data supplied to the model. It is also known as input error which consist of data error and conversion error.

(a) Data error.

It is also known as emporical error which arises when data for a problem are obtained by some experimental means and, are, therefore of limited accuracy and precision.

(b) Conversion error.

It is also known as representation which arises due to limitation of computers to store the data exactly. The decimal number 0.1 has a non terminating binary like $(0.1)_{10}$ has not terminating binary form 0.0001100 -110011001 ..... but the computers retains only a specified number of bits. Thus if we add 10 such numbers in a computer the result will not be exactly 0.1 because of round off error during the conversion of 0.1 to binary form.

* Represent the decimal number 0.1 and 0.4 in binary form with an accur of 8 binary digits add them & then convert the result to decimal form.

Soln,

$$(0.1)_{10} = (0.00011001)_2$$
$$(0.4)_{10} = \underline{(0.01100110)_2}$$
$$(0.01111111)_2$$
$$= 0.25 + 0.125 + 0.0625 + 0.03125 + 0.015625 + 0.0078125 + 0.0$$
$$= (0.49609375)_{10}$$

Note that the answer should be 0.5, but it is not due to error conversion from decimal to binary form.

Numerical error:

It is also known as procedural error that are introd during the process of implementation of numerical method. It i of two types.

(a) Round off error

This error arises from the process of rounding off n during the computation. for example 2.5397482, due to rou off the number 2.54 & 2.53 both creates error.

(b) chopping.

In chopping the extra bits are dropped. This called truncating the numbers. Example, if we are using a computer with a fix word length of 4 digits then t numbers like 59.7893 will be stored as 59.78.

Truncation error:

It is the error resulting from the truncation of the numerical process. This error arises from using an appro-tion in place of an exact mathematical procedure. Example sum of series $S = \sum_{i=0}^{\infty} a_i x^i$ replace by the finite sum $\sum_{i=0}^{n} a_i x^i$.

Absolute and relative error.

Let us suppose that the true value of a data item is denoted by x_t and its approximate value is denoted by x_a then they are related as,

True value x_t = approximate value x_a + Error

$\therefore$ Error $= x_t - x_a$

Error may be positive or negative depending on the value of x_t and x_a.

In error analysis what is important is the magnitude of error and not the sign which we called absolute error. It is denoted as

$$e_a = |x_t - x_a|$$

Hence the relative error is,

$$e_r = \frac{e_a}{|x_t|} = \frac{|x_t - x_a|}{|x_t|}$$

$$= \left| 1 - \frac{x_a}{x_t} \right|$$

e_r can also be expressed as the percentage relative error as

$$\text{Percent } e_r = e_r \times 100$$

Given that True value $x_t = 0.005$, and approximate value $x_a = 0.0$ calculate e_a, e_r and Percent e_r.

Soln,

Given $x_t = 0.005$

$x_a = 0.015$

Then, absolute error $e_a = |x_t - x_a|$

$= |0.005 - 0.015|$

$= 0.01$.

relative error $= \dfrac{e_a}{|x_t|}$

$$= \frac{0.01}{10.0051}$$
$$= 2.$$

Then,

$$\text{percent } e_r = e_r \times 100$$
$$= 2 \times 100$$
$$= 200\%$$

Meaning of solution of Non-linear equation.

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots a_{n-1}x + a_n$ where a's are constants ($a_0 \neq 0$) and n is a positive integer is called a polynomial in x of degree n. The polynomial $f(x) = 0$ is called an algebric equation of degree n.

The value α of x which satisfies $f(x) = 0$ is called a root of $f(x) = 0$. Geometrically a root of $f(x) = 0$ is the value of x where the graph of $y = f(x)$ crosses the x-axis.

The process of finding the roots of an equation is known as solution of that equation.

Any function of one variable which doesnot graph as a straight line in two dimension or any function of two variables which doesnot graph as a plane in 3 dimension can be said to be non linear.

$y = 8x + 5$ is a linear function. OR

a function $f(x)$ is said to be non linear if the response of the dependent variable y is not in direct or exact proportion to the changes in the independent variable x. for example $y = x$ is a non linear function.

Types of Equations

① Algebric equation.

An equation of type $y = f(x)$ is said to be algebraic if it can be expressed in the form $f_n y_n + f_{n-1} y_{n-1} + \cdots + f_1 y_1 + f_0 = 0$.

Example,

$3x + 5y - 21 = 0$ – linear.

$2x + 3xy + 25 = 0$ – non linear.

$x^3 - xy - 3y^2 = 0$ – non linear.

② Polynomial equation.

These are simple type of algebraic eqn. It can be expressed as $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 = 0$ is called nth degree polynomial and has n roots. The roots may be real and different, real and repeated and complex numbers.

③ Transcendental Equation.

These are non algebraic equation which include trigonometric, exponential and logrithmic functions. Examples are.

$2\sin x - x = 0$

$e^{-x}\sin x - \frac{1}{2}x = 0$

$\log x - 1 = 0$

Methods of solution of Non-linear equation.

There are different methods to find the roots of non linear equation.

1. (i) Direct analytical method.
2. (ii) Graphical method
3. (iii) Trial and error method
4. (iv) Iteration method.

(i) Direct analytical method.

It is only used for certain simple cases. It cannot be used for $2\sin x - x = 0$. In this method the roots can be found without calculation, so it is used for simple equations only like ~~$2x^2+5x-7=0$~~ $x^2-3x+2=0$

(ii) Graphical methods

This methods involves plotting the given function in a graph and to find the point where the function cuts the x axis. This method gives only approximate values.

(iii) Trial and Error method.

This method involves a series of guesses for x, each time evaluating the function to see weather it is closed to 0. The value of x that causes the function value closer to 0 is one of the approximate root of the eqn.

This method is time consuming & has low accuracy.

Solve the equation $2x^3+x^2-13x+6=0$ by trial & error method.

Soln,

By inspection we find $x=2$ satisfy the given equation Therefore 2 its root, i.e. $(x-2)$ is a factor of above equation. So dividing this polynomial by $(x-2)$, we get the quotient $2x^2+5x-3$ and remainder as zero.

Equating the quotient to 0 we get

$$2x^2+5x-3=0$$

solving this quadratic eqn we get,

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4\times 2(-3)}}{2\times 2}$$

$$= \frac{-5 \pm 7}{4}$$

$$= \frac{-5+7}{4}, \quad \frac{-5-7}{4}$$

$$= \frac{1}{2}, \quad -3$$

Hence the root of given equations are 2, -3, 1/2.

(iv) Iterative method

It is also known as algorithmetic approaches. It usually begins with and approximate value of the root known as the initial guess which is then successively corrected iteration by iteration. The process of iteration stopes when the desired level of accuracy is obtained. It can be grouped into two catagories

(1) Bracketing method

It is also known as interpolating method. It has two catagories.

(a) Bisection method

(b) False position method.

(2) Open end method

It is also known as extrapolating method.

(a) Half Interval method (Bisection method)

Bisection method is one of the simplest and most reliable type of iterative method for the solution of non linear equation. This method is also known as binary chopping or Half interval mtd.

This method is based on the repeated applicat of intermediate value property. If $f(x)$ is real & continuous in the interval $a < x < b$, and $f(a)$ & f are of opposite sign, i.e.

$$f(a) * f(b) < 0$$

Then there is atleast one real root in the interval between a and b. (There may be more than one root in the interval.)

let $x_1 = a$ and $x_2 = b$. let us define another point x_0 to be the mid point between a and b. i.e,

$$x_0 = \frac{x_1 + x_2}{2} \quad \text{-------} (1)$$

Now there exist the following 3 conditions.

(1) If $f(x_0) = 0$ we have root at x_0.
(2) If $f(x_0) \cdot f(x_1) < 0$ there is a root between x_0 and x_1
(3) If $f(x_0) \cdot f(x_2) < 0$ there is a root between x_0 and x

by testing the sign of the function at mi point, we can deduce which part of the interval contains the root. It is illustrated by figure as fol

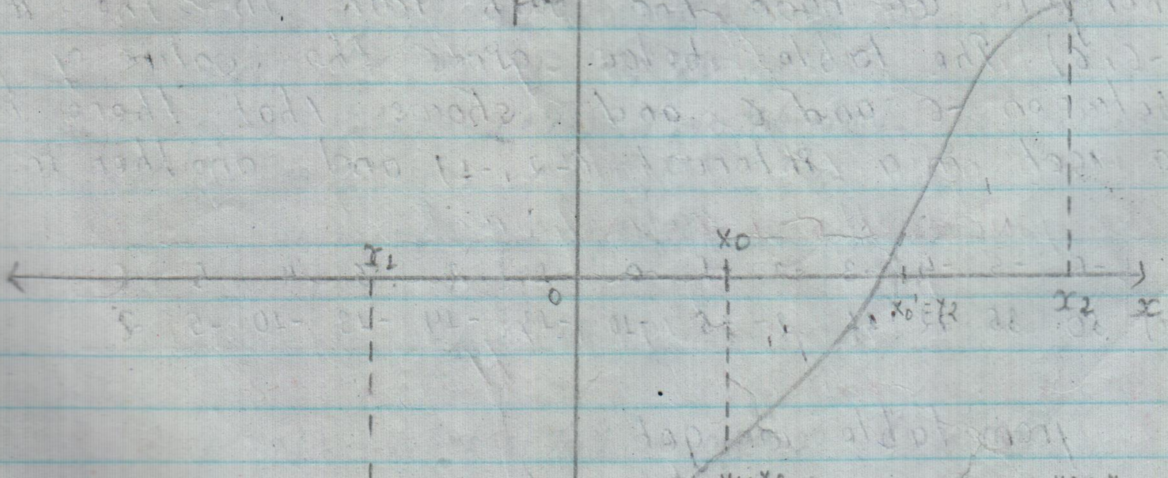


figure shows that $f(x_0)$ and $f(x_2)$ are of opposite sign. So a root lies between x_0 and x_2. we can further devide this sub interval into 2 halfs to locate a new sub interval containing the root this process can be continued until the interval containing the root is ~~als~~ as small as we desire.

*** Find the root of the equation by Half interval method.**

$$x^2 - 4x - 10 = 0.$$

Solution,

Given equation is $x^2 - 4x - 10 = 0$

The first step to guess 2 initial values that would bracket a root. Using the formula $x_{max} = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$ we can decide the maximum absolute solⁿ.

So,

$$x_{max} = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

$$= \sqrt{\left(\frac{-4}{1}\right)^2 - 2\left(\frac{-10}{1}\right)}$$

$$= \sqrt{36}$$

$$= \pm 6$$

Therefore we have the both root in the interval (-6, 6). The table below gives the value of $f(x)$ between -6 and 6 and shows that there is a root in a interval (-2, -1) and another in (5 6).

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	50	35	22	11	2	-5	-10	-13	-14	-13	-10	-5	2

from table we get

$x_1 = -2$ and $x_2 = -1$.

So $x_0 = \frac{-2 + -1}{2}$

$xe^{x} - 1 = 0$ in the interval [0, 1]

10^{th} ($x_0 = 0.5673$

$= -3/2$

$= -1.5$

Now, $f(x_0) = f(-1.5) = -1.75$

$f(x_1) = f(-2) = 2$

Now the root lies in between -1.5 and -2.

So $x_2 = -1.5$ $x_1 = -2$, then.

$$x_0 = \frac{-1.5 + (-2)}{2}$$

$$= -1.75$$

Now,

$f(x_0) = f(-1.75) = 0.0625$

$f(x_1) = f(-2) = 2$

$f(x_2) = f(-1.5) = -1.75$

Since $f(-1.5) \cdot f(x_0) < 0$ the roots lies between interval $(-1.75, -1.5)$.

$$\text{So } x_0 = \frac{-1.75 + (-1.5)}{2}$$

$$= -1.625$$

Now,

$f(x_0) = f(-1.625) = -0.859375$

$f(x_1) = f(-1.75) = 0.0625$

$f(x_2) = f(-1.5) = -1.75$

Since $f(-1.625) \cdot f(-1.75) < 0$, so the roots lies between interval $(-1.625, -1.75)$.

$$x_0 = \frac{-1.75 + (-1.625)}{2}$$

$$= -1.6875$$

Now,

$f(x_0) = f(-1.6875) = -0.4023$

$f(x_1) = f(-1.75) = 0.0625$

Since $f(-1.6875) \cdot f(-1.75) < 0$ the roots lies between the interval $(-1.75, -1.6875)$.

$$x_0 = \frac{-1.75 + (-1.6875)}{2}$$

$$= -1.71875$$

Now,

$f(x_0) = f(-1.71875) = -0.0170$

$f(x_1) = f(-1.75) = 0.0625$

Since $f(x_0) \cdot f(x_1) < 0$, so the roots lies in between $(-1.75, -1.71875)$.

$$x_0 = \frac{-1.75 + (-1.71875)}{2}$$

$$= -1.734375$$

Now,

$f(x_0) = f(-1.734375) = -0.054443359$

$f(x_1) = f(-1.75) = 0.0625$

Since $f(x_0) \cdot f(x_1) < 0$ so the roots lies in between the interval $(-1.75 - 1.734375)$.

So

$$x_0 = \frac{-1.75 + (-1.734375)}{2}$$

$$= -1.7421875$$

Now,

$f(x_0) = f(-1.7421875) = 0.00397$

Hence after 7th iteration,
We found the required root.
i.e

$$x_0 = -1.7421875$$

* Find the root of the equation $xe^{x} - 1 = 0$ in the interval [0,1]

Soln,

Given $x_1 = 0$, $x_2 = 1$

first we know $f(x_1) \cdot f(x_2) < 0$ So, root lies in between this two values, then

$$x_0 = \frac{0+1}{2}$$

$$= 0.5$$

then,

$f(x_0) = f(0.5) = 0.5e^{0.5} - 1 = -0.17564$

$f(x_2) = f(1) = 1 \cdot e^{1} - 1 = 1.71828$

Here $f(x_0) \cdot f(x_2) < 0$ So root lies in between (0.5, 1), then

So

$$x_0 = \frac{0.5+1}{2}$$

$$= 0.75$$

then,

$f(x_0) = f(0.75) = 0.75e^{0.75} - 1 = 0.58775$

$f(x_1) = f(0.5) = 0.5e^{0.5} - 1 = -0.17564$

Here $f(x_0) \cdot f(x_1) < 0$ So root lies in between (0.5, 0.75)

So,

$$x_0 = \frac{0.5+0.75}{2}$$

$$= 0.625$$

then,

$f(x_0) = f(0.625) = 0.625e^{0.625} - 1 = 0.16765$

$f(x_1) = f(0.5) = 0.5e^{0.5} - 1 = -0.17564$

Here $f(x_0) \cdot f(x_1) < 0$ So root lies in between (0.5, 0.6

So,

$$x_0 = \frac{0.5+0.625}{2}$$

$$= 0.5625$$

Then,

$f(x_0) = f(0.5625) = 0.5625e^{0.5625} - 1 = -0.01278$

$f(x_2) = f(0.625) = 0.625e^{0.625} - 1 = 0.16765$

Here $f(x_0) \cdot f(x_2) < 0$ so root lies in between $(0.5625, 0.625)$

So,

$$x_0 = \frac{0.5625 + 0.625}{2}$$

$$= 0.59375$$

Then,

$f(x_0) = f(0.59375) = 0.59375e^{0.59375} - 1 = 0.07514$

$f(x_1) = f(0.5625) = 0.5625e^{0.5625} - 1 = -0.01278$

Here $f(x_0) \cdot f(x_1) < 0$ so roots lies in between $(0.59375, 0.5625)$

So,

$$x_0 = \frac{0.5625 + 0.59375}{2}$$

$$= 0.57813$$

Then,

$f(x_0) = f(0.57813) = 0.03391$

$f(x_1) = f(0.5625) = -0.01278$

Here $f(x_0) \cdot f(x_1) < 0$ so roots lies in between $(0.57813, 0.5625)$

So,

$$x_0 = \frac{0.5625 + 0.57813}{2}$$

$$= 0.57032$$

Then,

$f(x_0) = f(0.57032) = 0.00879$

Hence after 8^{th} iteration, the required root is

$x_0 = 0.57032$

* Find the root of the equation $x^3-4x-9=0$ in between (2,3)

Sol^n,

Here given that $x_1=2$ and $x_2=3$. ~~then~~, So,

$$x_0 = \frac{2+3}{2}$$

$$= 2.5$$

then,

$f(x_0) = f(2.5) = -3.375$

$f(x_2) = f(3) = 6$

Here $f(x_0) f(x_2) < 0$ so roots lies in between (2.5, 3)

So,

$$x_0 = \frac{2.5+3}{2}$$

$$= 2.75$$

then,

$f(x_0) = f(2.75) = 0.79688$

$f(x_1) = f(2.5) = -3.375$

Here, $f(x_0) \cdot f(x_1) < 0$ so roots lies in between (2.5, 2.75)

So,

$$x_0 = \frac{2.5+2.75}{2}$$

$$= 2.625$$

then

$f(x_0) = f(2.625) = -1.41211$

$f(x_2) = f(2.75) = 0.79688$

Here $f(x_0) \cdot f(x_2) < 0$ So the roots lies in between (2.625, 2.75

So,

$$x_0 = \frac{2.625+2.75}{2}$$

$$= 2.6875$$

then,

$f(x_0) = f(2.6875) = -0.33911$

$f(x_2) = f(2.75) = 0.79688$

Here $f(x_0) \cdot f(x_2) < 0$ so roots lies in between $(2.6875, 2.75)$

So,

$$x_0 = \frac{2.6875 + 2.75}{2}$$

$$= 2.71875$$

then,

$f(x_0) = f(2.71875) = 0.22092$

$f(x_1) = f(2.6875) = -0.33911$

Here $f(x_0) \cdot f(x_1) < 0$ so roots lies in between $(2.6875, 2.71875)$

So,

$$x_0 = \frac{2.6875 + 2.71875}{2}$$

$$= 2.70313$$

then,

$f(x_0) = f(2.70313) = -0.0611$

$f(x_2) = f(2.71875) = 0.22092$

Here $f(x_0) \cdot f(x_2) < 0$ so the roots lies in between $(2.70313, 2.71875)$

So,

$$x_0 = \frac{2.70313 + 2.71875}{2}$$

$$= 2.71094$$

then $f(x_0) = f(2.71094) = 0.07947$

$f(x_1) = f(2.70313) = -0.0611$

Here $f(x_0) \cdot f(x_1) < 0$ so root lies in between $(2.70313, 2.71094)$

So,

$$x_0 = \frac{2.70313 + 2.71094}{2}$$

$$= 2.70704$$

then

$f(x_0) = f(2.70704) = 0.0091$

Hence after 8^{th} iteration the required root is

$$x_0 = 2.70704$$

* Solve by using bisection method $\sin x = 1/x$, root lies in between $x = 1$ and $x = 1.5$ (measure in radian) carry out computation upto 7^{th} iteration.

$(1^r = 57.3^\circ)$

Soln.

Given, $\sin x = \frac{1}{x}$

or $x \sin x - 1 = 0$

$\therefore f(x) = x \sin x - 1$, then, we know the root lies in between 1 and 1.5, so,

$$x_0 = \frac{1 + 1.5}{2}$$

$$= 1.25$$

then,

$f(x_0) = f(1.25) = 0.18623$

$f(x_1) = f(1) = -0.15853$

Here $f(x_0) \cdot f(x_1) < 0$ so root lies in between $(1, 1.25)$

So,

$$x_0 = \frac{1 + 1.25}{2}$$

$$= 1.125$$

then,

$f(x_0) = f(1.125) = 0.01505$

$f(x_1) = f(1) = -0.15853$

Here $f(x_0) \cdot f(x_1) < 0$ so root lies in between $(1, 1.125)$

So,

$$x_0 = \frac{1 + 1.125}{2}$$

$$= 1.0625$$

then,

$f(x_0) = f(1.0625) = -0.07183$

$f(x_2) = f(1.125) = 0.01505$

Here $f(x_0) \cdot f(x_2) < 0$ so root lies in between $(1.0625, 1.125)$

So,

$$x_0 = \frac{1.0625 + 1.125}{2}$$

$= 1.09375$

Then,

$f(x_0) = f(1.09375) = -0.02836$

$f(x_2) = f(1.125) = 0.01505$

Here $f(x_0) \cdot f(x_2) < 0$ so root lies in between $(1.09375, 1.125)$

So,

$$x_0 = \frac{(1.09375 + 1.125)}{2}$$

$= 1.10938$

Then,

$f(x_0) = f(1.10938) = -0.00664$

$f(x_2) = f(1.125) = 0.01505$

Here $f(x_0) \cdot f(x_2) < 0$ so the root lies in between $(1.10938, 1.125)$

So,

$$x_0 = \frac{1.10938 + 1.125}{2}$$

$= 1.11719$

Then,

$f(x_0) = f(1.11719) = 0.00421$

$f(x_1) = f(1.10938) = -0.00664$

Here $f(x_0) \cdot f(x_1) < 0$ so root lies in between $(1.10938, 1.11719)$

So,

$$x_0 = \frac{1.10938 + 1.11719}{2}$$

$= 1.11329$

then

$f(x_0) = f(1.11329) = -0.0012$

Hence after 7^{th} iteration the required root is

$x_0 = 1.11329$

Algorithm for bisection method

1. → Decide initial values for x_1 and x_2 and stopping criteria ϵ.
2. → Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$.
3. → If $f_1 * f_2 > 0$ x_1 and x_2 donot bracket any root, and go to step 7, Otherwise continue.
4. → Compute $x_0 = (x_1 + x_2)/2$ and compute $f_0 = f(x_0)$.
5. → If $f_1 \cdot f_0 < 0$ root lies in between f_1 and f_2. Set $x_2 = x_0$ else set $x_1 = x_0$ and set $f_1 = f_0$.
6. → If absolute value of $(x_2 - x_1)/x_2$ is less than ϵ then root $= (x_1 + x_2)/2$. write the value of root and goto step 7.
   → else goto step 4.
7. → stop.

Convergence of bisection method.

In bisection method, the interval containing the root is reduced by a factor of 2. If the procedure is repeated n times then the interval containing the root is reduced to the size $x_2 - x_1 / 2^n = \Delta x / 2^n$.

After n iteration, the root must lies within $\pm \Delta x / 2^n$ of our estimate.

This means that the error bound at nth iteration is,

$$E_n = \left| \frac{\Delta x}{2^n} \right|$$

Simillarly

$$E_{n+1} = \left| \frac{\Delta x}{2^{n+1}} \right|$$

$$= \frac{E_n}{2}$$

This means that the error decreases linearly with each step by a factor of 0.5. Hence bisection method is linearly convergent.

Advantages of bisection method.

⇒ It is always convergent.

⇒ Error can be controlled. (Decreases step by step)

Disadvantages of bisection method.

⇒ Convergence is generally slow.

⇒ choosing a guess closed to the root may result in needing many iteration to converge.

⇒ Cannot find roots of some equation. Eg $f(x) = x^2$

⇒ may seek a singularity point as root. $f(x) = 1/x$

Secant method:

Secant method uses two initial estimates but doesnot required that they must bracket the root & also it doesnot required the condition $f(x_0) \cdot f(x_1) < 0$

Here the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration two must recent approximations to the root are used to find the next approximation. Also it is not necessary that the interval must contain the root.

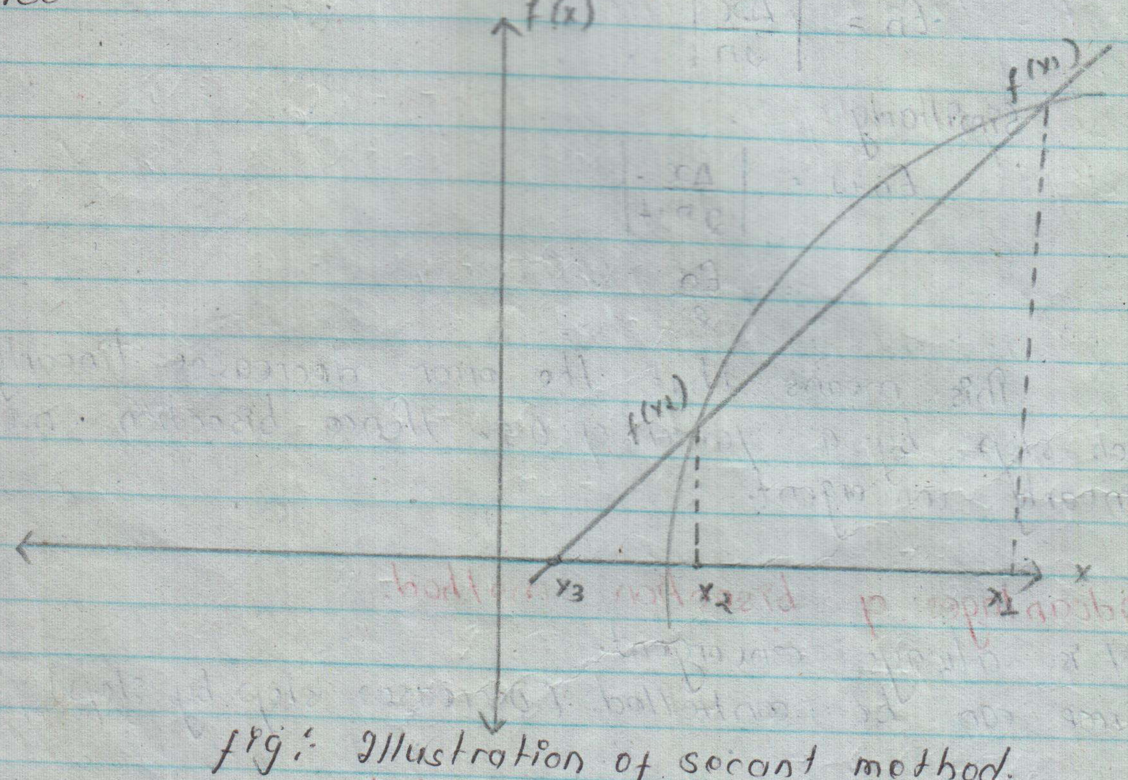


fig: Illustration of secant method.

If x_1 and x_2 are the initial limits of the interval, we can write equation of the curve joining these points as,

$$y - f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2)$$

Then the abscissa (x-axis) of the point where it crosses the x axis, $(y = 0)$ is given by

$$0 - f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_3 - x_2)$$

$$-f(x_2)(x_2 - x_1) = f(x_2) - f(x_1) \cdot (x_3 - x_2)$$

or $$(x_3 - x_2) = -\frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

or, $$x_3 = -\frac{(x_2 - x_1)}{f(x_2) - f(x_1)} \cdot f(x_2) + x_2$$

$$= x_2 - \frac{f(x_2) \cancel{+ f(x_1)}\,(x_2 - x_1)}{f(x_2) - f(x_1)}$$

Algorithm for secant method

1. Decide two initial points x_1 and x_2, accuracy level E.
2. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$
3. Compute $x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1}$
4. Test for accuracy of x_3. If absolute value, i.e $\left|\frac{x_3 - x_2}{x_3}\right| > E$ then set $x_1 = x_2$ and $f_1 = f_2$
   $x_2 = x_3$ and $f_2 = f(x_3)$
   go to step 3.
   otherwise,
   set root $= x_3$, print result.
5. Stop.

* Use secant method to find the root of the equation $x^2 - 4x - 10 = 0$.

Soln,

Given, equation is,

$$f(x) = x^2 - 4x - 10$$

then let us consider $x_1 = 4$ and $x_2 = 2$

then $f(x_1) = f(4) = -10$ -ve

$f(x_2) = f(2) = -14$ -ve,

$$= 6.66667 - \frac{7.077781 \times (6.66667 - 5)}{7.077781 - (-5)}$$

$$= 6.66667 - 1.0014496$$

$$= 5.65$$

Then new interval is $(x_1, x_2) = (6.66667, 5.65)$

f_1

$$f(x_1) = f(6.66667) = 7.07781$$

$$f(x_2) = f(5.65) = -0.6775$$

fourth iteration,

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 5.65 - \frac{(-0.6775)(5.65 - 6.66667)}{-0.6775 - 7.07781}$$

$$= 5.65 - (-0.08146)$$

$$= 5.0731$$

Hence after 4th iteration

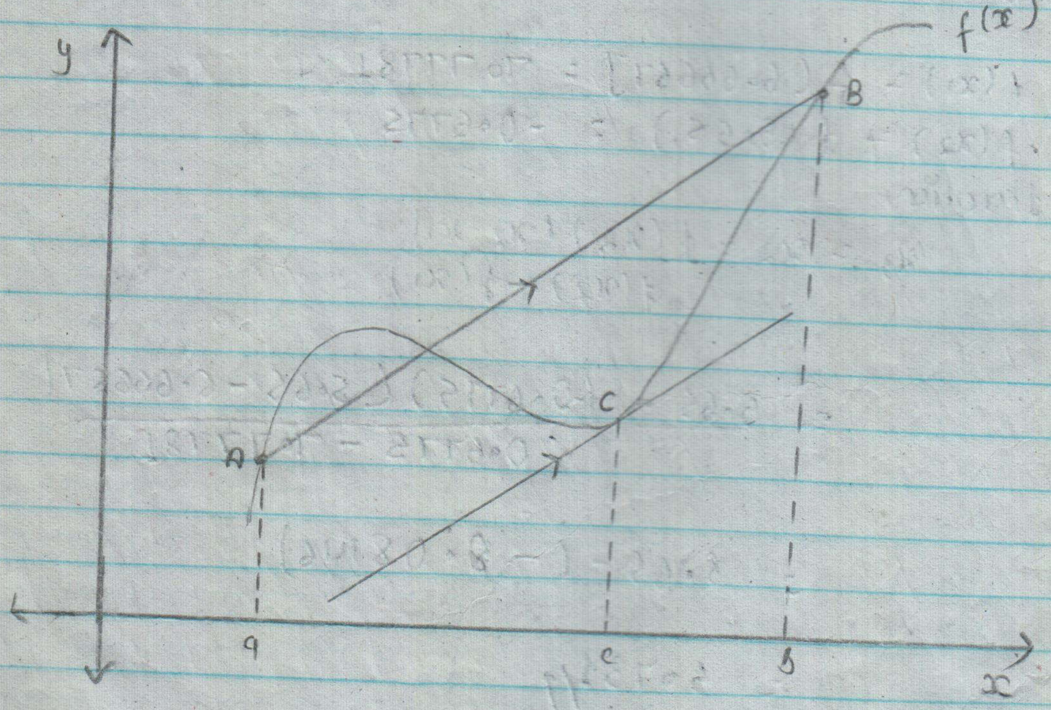
the required root is

$$x = 5.0731$$

Mean value theorem.

If a function f is continuous on the closed interval $[a, b]$ where $a < b$ and differentiable on open interval (a, b) then there exist a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Convergance of secant method

The formula for secant method for iteration is

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{---(1)}$$

let x_r be the actual root of $f(x)$ and e_i be the error in the estimate of x_i then,

$$x_{i+1} = e_{i+1} + x_r$$
$$x_i = e_i + x_r$$
$$x_{i-1} = e_{i-1} + x_r$$

substituting these values in eq^n (1) we get,

$$e_{i+1} + x_r = e_i + x_r - \frac{f(x_i)\,(e_i + x_r - e_{i-1} - x_r)}{f(x_i) - f(x_{i-1})}$$

$$e_{i+1} = \frac{e_{i-1} f(x_i) - e_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{---} \; (2)$$

according to mean value theorom, there exist atleast one point $x = R_i$ in the interval x_i & x_r such that $f'(R_i) = \frac{f(x_i) - f(x_r)}{x_i - x_r}$

we know that,

$f(x_r) = 0$

then we know,

$x_i - x_r = e_i$, and therefore,

$$f'(R_i) = \frac{f(x_i)}{e_i}$$

$\therefore f(x_i) = f'(R_i) \cdot e_i$

simillarly,

$$f(x_{i-1}) = f'(R_{i-1}) \cdot e_{i-1}$$

substituting this in numerator of eq^n (2), we get.

$$e_{i+1} = e_i e_{i-1} \frac{f'(R_i) - f'(R_{i-1})}{f(x_i) - f(x_{i-1})}$$

That is, we can say,

$$e_{i+1} \propto e_i e_{i-1} \quad \text{----} \; (3)$$

We know that, the order of convergence of an iteration process is p if

$$e_i \propto e_{i-1}^{\,p} \quad \text{----} \; (4)$$

and $e_{i+1} \propto e_i^{\,p}$ ---- (5)

substituting for e_{i+1} and e_i in eq^n (3) we get,

$$e_i \propto e_{i-1}^{\frac{p+1}{p}} \quad \text{---} \; (6)$$

Comparing the relation (4) and (6), we observe that

$$p = \frac{p+1}{p}$$

that is,

$$p^2 - p - 1 = 0$$

which has the solution,

$$p = \frac{1 \pm \sqrt{5}}{2}$$

$$= \pm 1.618$$

Since p is always +ve, we have,

$$p = 1.618$$

It follows that the order of convergance of secant method is 1.618 and the convergen is reffered to as super linear convergance. (<2)

Advantages of Secant method.

(i) It converges faster than linear rate, so it is more rapidly convergant than bisection method.

(ii) it doesnot require the use of derivative of the function.

(iii) It require only one function evaluation per iteratio as compare to Netwons method which required 2.

Disadvantages of secant method.

(i) It maynot converge.

(ii) There is no guranted error bound for the computed iteration.

Newton - Raphson Method

Consider a graph of function $f(x)$ as shown in the figure.

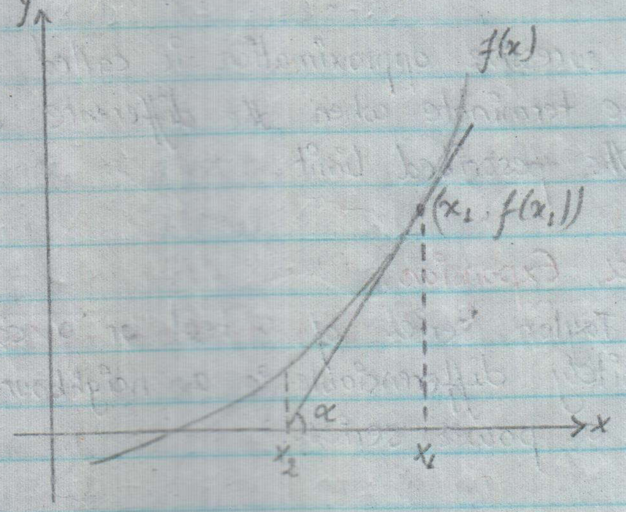


fig: Illustration of Newton - Raphson Method

let us assume that x_1 is the first approximation of the root $f(x) = 0$. Draw a tangent at the curve $f(x)$ at $x = x_1$ as shown in the figure above. The point of intersection of this tangent with x-axis gives the second approximation to the ~~gor~~ root. let this point of intersection be x_2.

The slope of the tangent is given by,

$$\tan\alpha = \frac{f(x_1)}{x_1 - x_2} = f'(x_1) \quad \text{------ (1)}$$

$$x_1 - x_2 = \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This is called Newton - Raphson formula.

The next approximation will be,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This method of sucessive approximation is called Newton-Raphson method. The process will be terminate when the difference between two sucession values is within the pescribed limit.

Taylor series Expansion

The Taylor series of a real or a complex valued function $f(x)$, that is infinitely differenciable in a neighbour of a real or a complex number a is a power series.

$$f(x+a) = \cancel{f(x)} = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots\cdots$$

which can be written as,

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Q. Derive the Newton-Raphson formula using the Taylor series Expansion.

Soln

Assume that x_n is an estimate of a root of a function $f(x)$. Consider the small interval h such that,

$$h = x_{n+1} - x_n$$

we can express $f(x_{n+1})$ using Taylor series expansion as follows:

$$f(x_{n+1}) = f(x_n) + f'(x_n)h + f''(x_n)\frac{h^2}{2!} + \cdots\cdots$$

If we neglect the terms containing the second order and higher order derivatives, we get,

$$f(x_{n+1}) = f(x_n) + f'(x_n)h$$

If x_{n+1} is a root of $f(x)$, then

$$f(x_{n+1}) = 0 = f(x_n) + f'(x_n)h$$

Then,

$$h = -\frac{f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

Therefore,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Algorithm for Newton-Raphson Method

1) Assign an initial value to x say x_1
2) Evaluate $f(x_1)$ and $f'(x_1)$
3) find the input estimate of x_1

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

4) Check for the accuracy of the latest estimate
compare relative error to the predefine value E

if $\left|\frac{x_2 - x_1}{x_2}\right| \leq E$

stop;

otherwise

continue;

5) Replace x_1 by x_2 and repeate the step 3 and 4
6) stop

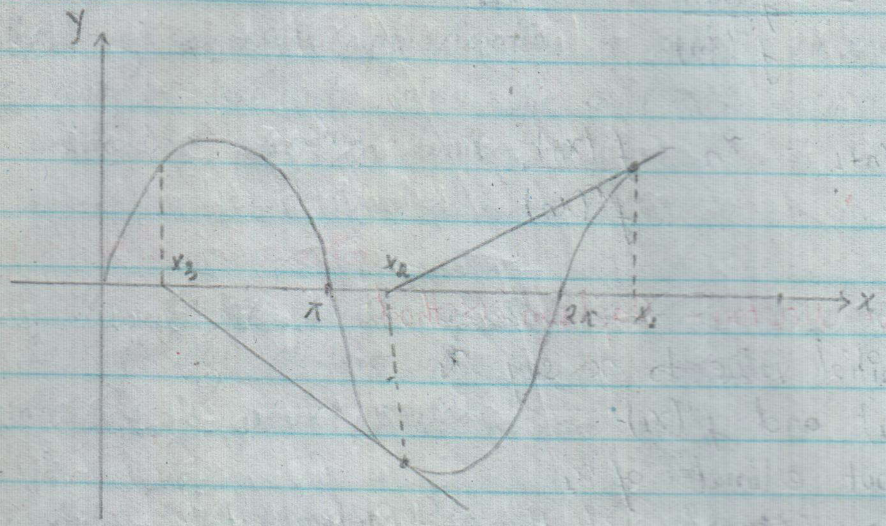
Advantages of Newton-Raphson method

1) It converges fast (Quadratic convergence), if it converges.
2) Requires only one guess.

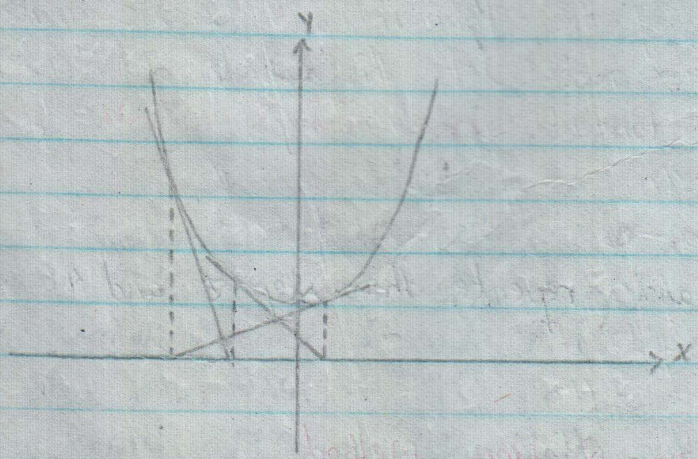
Disadvantages of Newton-Raphson method

1) Complication will arises if the derivative $f'(x_n) = 0$. In such cases a new initial value for x must be choosen to continue the procedure.

2) Root juming might take place.
for example:
$$f(x) = \sin x = 0$$



3) Oscillates local maxima or minima.
for example: $x^2 + 2 = 0$



4) The method is very expensive i.e. it needs the function evaluation and derivative evaluation.

Q. Find the root of the given function using Newton-Raphson method.
$f(x) = x^2 - 3x + 2$ for $x = 0$

Soln.

Here, Given that

$$f(x) = x^2 - 3x + 2$$

and initial point, $x = 0$

Now,

$$f'(x) = 2x - 3$$

$$\therefore f'(0) = 2\times0 - 3 = -3$$

and $f(0) = 0^2 - 3\times0 + 2 = 2$

Then,

$$x_2 = 0 + 2/3 = 0.66667$$

Again,

$$f'(0.66667) = 2\times0.66667 - 3 = -1.66667$$

$$f(0.66667) = 0.44444$$

Then,

$$x_3 = 0.66667 + \frac{0.44444}{1.66667} = 0.93333$$

Again,

$$f'(0.93333) = 2\times0.93333 - 3 = -1.13333$$

$$f(0.93333) = 0.07111$$

Then,

$$x_4 = 0.93333 + \frac{0.07111}{1.13333} = 0.99608$$

Again,

$$f(0.99608) = 0.00394$$

$$f'(0.99608) = -1.00184$$

Then,

$$x_5 = 0.99608 + \frac{0.00394}{1.00184} = 1.00001$$

Hence the required root is

$$x \simeq 1.00001$$

Convergence of Newton - Raphson method

let x_n be an estimate of a root of the function $f(x)$. If x_n and x_{n+1} are close to each other then using Taylor's series expansion, we can state,

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(R)}{2}(x_{n+1} - x_n)^2 \quad \text{-----(1)}$$

where R lies somewhere between x_n to x_{n+1} and the third and higher order derivatives have been dropped.

let us assume that the exact root of $f(x)$ is x_r. So, that $x_{n+1} = x_r$.

$\therefore f(x_{n+1}) = 0$

substituting this value in (1)

$$0 = f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(R)}{2}(x_r - x_n)^2 \quad \text{-------(2)}$$

we know that the Newton's iterative formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

solving for $f(x_n)$, we get,

$$f(x_n) = f'(x_n)(x_n - x_{n+1}) \quad \text{-------(3)}$$

Substituting equation (3) in (2), we get,

$$0 = f'(x_n)(x_n - x_{n+1}) + f'(x_n)(x_r - x_n) + \frac{f''(R)}{2}(x_r - x_n)^2$$

$$0 = f'(x_n)(x_r - x_{n+1}) + \frac{f''(R)}{2}(x_r - x_n)^2 \quad \text{------(4)}$$

we know that the error in the estimate of x_{n+1} is given by,

$$e_{n+1} = x_r + x_{n+1}$$

Similarly,

$$e_n = x_r - x_n$$

Now,

substituting these in equation (4)
we get,

$$0 = f'(x_n)\, e_{n+1} + \frac{f''(R)}{2} e_n^2$$

Rearranging the term,
we get,

$$e_{n+1} = -\frac{f''(R)}{2f'(x_n)} e_n^2 \quad \text{------- (5)}$$

Equation 5 shows that the error is roughly proportional to the square of error in the previous iteration. Therefore, Newton-Raphson method is said to ~~be~~ have quadratic convergence.

Complex Root

Complex root of an equation occurs in pairs in conjugate form i.e. if $\alpha + i\beta$ is a root of $f(x) = 0$ then, $\alpha - i\beta$ is also its root.

In otherword, if $[x-(\alpha+i\beta)]$ and $[x-(\alpha-i\beta)]$ are factors of $f(x)$, then,

$$[x-(\alpha+i\beta)][x-(\alpha-i\beta)]$$
$$= (x-\alpha-i\beta)(x-\alpha+i\beta)$$
$$= x^2 - x\alpha + ix\beta - x\alpha + \alpha^2 - i\alpha\beta - ix\beta + i\alpha\beta - i^2\beta^2$$
$$= x^2 - 2x\alpha + \alpha^2 + \beta^2$$

is also a factor of $f(x)$. $x^2 - 3x + 2 = 0$

fixed point Iterative method

Any function in the form of $f(x) = 0$ ------ (1)
can be manipulated such that x is on the left hand side of the equation as shown below,

$$x = g(x) \quad \text{------ (2)}.$$

Here, eq^n (1) and (2) are equivalent so a root of equation (1) is also of equation (2). The root of equation (2) is given by the point of intersection of the curve $y = x$ and $y = g(x)$. This point of intersection is known as the fixed point of $g(x)$.

The above transformation can be obtained either by algebraic manipulation of the given equation or by simply adding x to both side of the equation.

for example:

1) $x^2 + x - 2 = 0$

can be written as,

$$x = 2 - x^2$$

2) $\tan x = 0$

can be written as

$$x = \tan x + x$$

This equation $x = g(x)$ is known as the fixed point equation. If x_0 is the initial guess to a root, then the next approximation is given by,

$$x_1 = g(x_0)$$

further approximation is given by

$$x_2 = g(x_1)$$

In general form,

$$x_{n+1} = g(x_n)$$

where $n = 0, 1, 2$ ------

which is called the fixed point iteration formula and the method of solution is known as the method successive approximation or method direct substitution.

Algorithm for fixed point iteration method.

Rearrange the given function $f(x) = 0$ into equivalent form $x = g(x)$.

1) Read the initial value x_1 and required error E.
2) set $x_2 = g(x_1)$
3) If $\left| \frac{x_2 - x_1}{x_2} \right| < E$

print x_2 as root

else

set $x_1 = x_2$

4) goto step 2.

5) stop.

}

Q. Find the ~~equation~~ root of equation $x^2+x-2=0$ using the fixed point iteration method.

Soln

The given equation can be represented as,

$$x = 2-x^2$$

let us start with an initial value of $x_0 = 0$.

$$x_1 = 2-0 = 2$$

$$x_2 = 2-4 = -2$$

$$x_3 = 2-(-2)^2 = 2-4 = 2$$

Since, $x_3 - x_2 = 0$, -2 is one of the root of the equation.

Again,

let us assume that, $x_0 = -1$, then

$$x_1 = 2-1 = 1$$

$$x_2 = 2-1 = 1$$

∴ Another root of the given equation is 1.

Q. Evaluate the square root of 5 using the equation $x^2-5=0$ by applying the fixed point iteration method.

Soln

let us reorganized the function as follows:

$$x = \frac{5}{x}$$

Let, $x_0 = 1$, then,

$$x_1 = \frac{5}{x_0} = \frac{5}{1} = 5$$

$x_2 = \frac{5}{x_1} = \frac{5}{5} = 1$

$x_3 = \frac{5}{x_2} = \frac{5}{1} = 5$

$x_4 = \frac{5}{x_3} = \frac{5}{5} = 1$

The process doesnot converges to the solution. This type of divergence is known as oscillatory divergence.

Again,

let us consider another form of g(x) as follows:

$x = x^2 + x - 5$

let, $x_0 = 0$, Then

$x_1 = x_0^2 + x_0 - 5 = 0^2 + 0 - 5 = -5$

$x_2 = x_1^2 + x_1 - 5 = (-5)^2 + (-5) - 5 = 25 - 5 - 5 = 15$

$x_3 = x_2^2 + x_2 - 5 = (15)^2 + (15) - 5 = 235$

$x_4 = (235)^2 + (235) - 5 = 55455$

Again, it doesnot ~~convergence~~ rather it diverges rapidly. This type of divergence is known as monoton divergence.

Again,

let us try the third form of g(x).

$2x = \frac{5}{x} + x$

$\therefore x_1 = \frac{x_0 + 5/x_0}{2}$

let us assume $x_0 = 1$.

So, $x_1 = \frac{1 + 5/1}{2} = 3$

$x_2 = \frac{3 + 5/3}{2} = 2.33$

$x_3 = \frac{2.33 + 5/2.33}{2} = 2.23$

$x_4 = \frac{2.23 + 5/2.23}{2} = 2.23$

$x_5 = \frac{2.23 + 5/2.23}{2} = 2.23$

$x_6 = \frac{2.23 + 5/2.23}{2} = 2.23$ //

This time the process converges rapidly to the solution. the square root of 5 is 2.2361.

Q. Evaluate the root using the equation $2x - \log_{10} x = 7$ by applying the fixed point iteration method.

Soln/

let us reorganized the function as follow

$$x = \frac{7 + \log_{10} x}{2}$$

let us assume, $x_0 = 1$,

$x_1 = \frac{7 + \log_{10} x_0}{2} = \frac{7 + \log_{10} 1}{2} = 3.5$

$x_2 = \frac{7 + \log_{10} x_1}{2} = \frac{7 + \log_{10} 3.5}{2} = 3.772$

$x_3 = \frac{7 + \log_{10} x_2}{2} = \frac{7 + \log_{10} 3.772}{2} = 3.788$

$$x_4 = \frac{7 + \log_{10} x_3}{2} = \frac{7 + \log_{10} 3.788}{2} = 3.789$$

$$x_5 = \frac{7 + \log_{10} x_4}{2} = \frac{7 + \log_{10} 3.789}{2} = 3.789.$$

Therefore, the root of the given equation is 3.789.

Q. Evaluate the square root using the equation $\cos x = 3x - 1$ by applying the fixed point iteration method.

Soln/

Let us reorganised the function as follows:-

$$x = \frac{1}{3}(\cos x + 1)$$

let, us assume. $x_0 = 0$

$$x_1 = \frac{1}{3}(\cos 0 + 1) = 0.6667$$

$$x_2 = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$$

$$x_3 = \frac{1}{3}(\cos 0.5953 + 1) = 0.6093$$

$$x_4 = \frac{1}{3}(\cos 0.6093 + 1) = 0.6067$$

$$x_5 = \frac{1}{3}(\cos 0.6067 + 1) = 0.6072$$

$$x_6 = \frac{1}{3}(\cos 0.6072 + 1) = 0.6071$$

$$x_7 = \frac{1}{3}(\cos 0.6071 + 1) = 0.6071$$

Therefore the root of the given equation is found to be 0.6071//

Convergence of fixed point iteration.

Convergence of fixed point iteration depends on the nature of $g(x)$. Following figure represents the various pattern of behaviour.

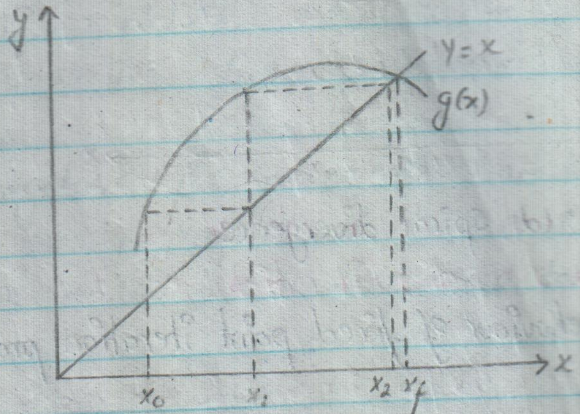


fig : (a) Monotone convergence

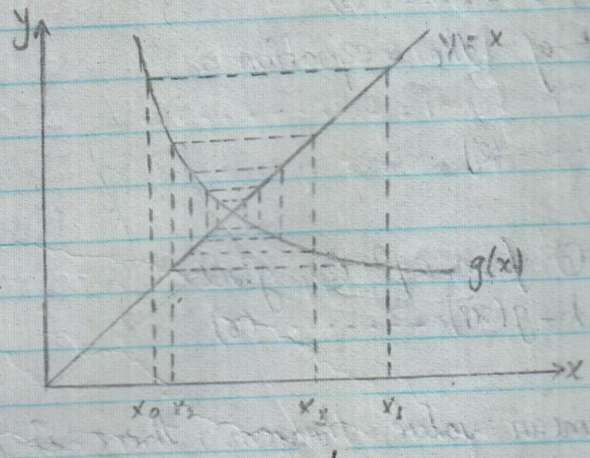


fig : (b) Spiral convergence

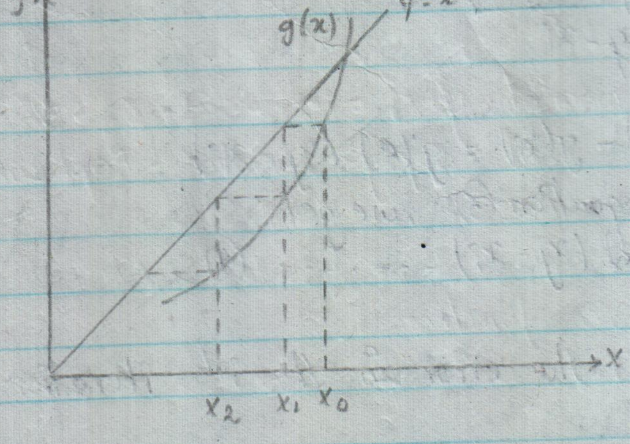


fig : (c) Monotone divergence

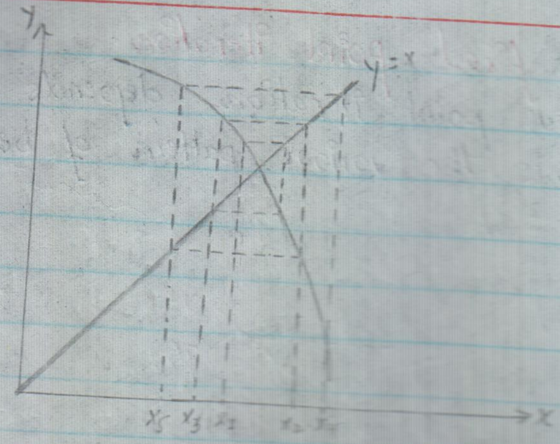


fig : (d) Spiral divergence

fig : Pattern of behaviour of fixed point iteration process

The iteration formula is

$$x_{i+1} = g(x_i) \text{ --------- } (1)$$

Let x_f be a root of the equation, then,

$$x_f = g(x_f) \text{ --------- } (2)$$

Substracting eq^n ① from eq^n ② gives,

$$x_f - x_{i+1} = g(x_f) - g(x_i) \text{ ------- } (3)$$

According to the mean value theorem, there is at least one point, say $x = R$, in the interval x_f and x_i. Such that

$$g'(R) = \frac{g(x_f) - g(x_i)}{x_f - x_i}$$

This gives, $g(x_f) - g(x_i) = g'(R)\,(x_f - x_i)$

Substituting this in equation ③, we get,

$$x_f - x_{i+1} = g'(R)\,(x_f - x_i) \text{ ------- } (4)$$

If e_i represent the error in the i^{th} iteration then the equation ④ becomes

$$e_{i+1} = g'(R)\, e_i \text{ ------ } (5)$$

This shows that the error will decreases with each iteration only if $g'(R) < 1$.

from equation ⑤ we can conclude that

① Error decreases if $g'(R) < 1$

② Error grows if $g'(R) > 1$

③ If $g'(R)$ is positive, the convergence is monotonic as in the figure a.

④ If $g'(R)$ is negative, the convergence will be oscillatory as in figure b

⑤ The error is roughly proportional to the error in the previous step, the fixed point method is therefore said to be linearly convergent.

Solving Polynomials.

Newton's method / Horner's Role :

Synthetic Division :

A polynomial of degree n can be expressed as

$P(x) = (x - x_r)q(x)$

where, x_r is a root of polynomial $p(x)$.

$q(x)$ is the quotient polynomial of degree $n-1$.

Once a root is found we can use this root to find a lower degree polynomial $q(x)$ by dividing $P(x)$ by $(x - x_r)$ using a process known as synthetic division. Hence, the activity of reducing the degree of polynomial is known as deflection.

Again, the quotient polynomial $q(x)$ can be used to determines the other roots of $P(x)$. A further deflation can be perform and the process can be continue until the degree is reduced to 1.

Synthetic Division of a polynomial

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ by $(x - \alpha)$ is done as follows :

α	a_0	a_1	a_2	-------	a_{n-1}	a_n
		αb_0	αb_1	-------	αb_{n-2}	αb_{n-1}
	a_0 $(= b_0)$	$a_1 + \alpha b_0$ $(= b_1)$	$a_2 + \alpha b_1$ $(= b_2)$	-------	$a_{n-1} + \alpha b_{n-2}$ $(= b_{n-1})$	$a_n + \alpha b_{n-1}$ $(= R)$

Hence, quotient $= b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1}$

Algorithm for fixed point iteration

1. Read the degree of polynomial n.
2. Read the coefficient a_i for $i = 0, 1, 2, \ldots, n$.
3. Read the initial guess say x_0.
4. Set $b_0 = a_0$ and $c_0 = b_0$.
5. Set $b_i = a_i + b_{i-1} x_1$ for $i = 1, 2, \ldots, n$.
6. Set $c_i = b_i + c_{i-1} x_1$ for $i = 1, 2, \ldots, n$
7. Set, $x_1 = x_1 - \dfrac{b_n}{c_{n-1}}$
8. If $\left|\dfrac{b_n}{c_{n-1}}\right| < E$,

   print x_1 as root

   else

   go to step 5.

Q. Divide the polynomial $f(x) = x^3 + x^2 - 3x - 3$ by $(x-2)$ using synthetic division and apply Newton - Raphson method.

Soln,

Here,

the given function is

$$f(x) = x^3 + x^2 - 3x - 3$$

Now,

first iteration - 1.

2	1	1	-3	-3
		2	6	6
	1	3	3	3 = R = $f(x_1)$
		2	10	
	1	5	13 = $f'(x_1)$	

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2 - \frac{3}{13}$$

$$= 1.769231$$

Second iteration - 2

1.769231	1	1	-3	-3
		1.769231	4.89941	3.36049
	1	2.769231	1.89941	0.360493 = R = $f(x_2)$
		1.769231	8.02959	
	1	4.53846	9.928997 = $f'(x_2)$	

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.769231 - \frac{0.360493}{9.928997}$$

$$= 1.73292.$$

Third-iteration - x_3.

1.73292	1	1	-3	-3	
	↓	1.73292	4.73593	3.00823	
	1	2.73292	1.73593	0.00823	$-f(x_3)$
	↓	1.73292	7.73834		
	1	4.46584	9.47487		$f'(x_3)$

$$\therefore x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.73292 - \frac{0.00823}{9.47487}$$

$$= 1.73292 + 0.00087$$

$$= 1.73379$$

fourth iteration,

1.73379	1	1	-3	-3	
	↓	1.73379	4.73982	3.01648	
	1	2.73379	1.73982	0.01648	$=f(x_4)$
	↓	1.73379	7.74185		
	1	4.46758	9.48567		$=f'(x_4)$

$$\therefore x_5 = 1.073379 - \frac{0.001648}{9.048567}$$

$$= 1.078379 - 0.00174$$

$$= 1.073205$$

Fifth iteration

1.073205	1	1	-3	-3	
	↓	1.073205	4.073205	2.99999	
	1	2.073205	1.073205	-0.00001	$-f(x_5)$
	↓	1.073205	7.073204		
	1	4.04641	9.046409		$-f'(x_5)$

Hence

$$x_6 = 1.073205 - \frac{(-0.00001)}{9.046409}$$

$$= 1.073205 + 0.00000106$$

$$= 1.0732051$$

Hence the required root after 5th iteration is

$$1.073205$$

Horner's Rule

Let us consider the evalution of a polynomials by using Horner's rule as follows :-

$$f(x) = ((\text{-----}((a_n x + a_{n-1})x + a_{n-2})x + \text{-----} + a_1)x + a_0) \text{ ----- } ①$$

Here, the innermost expression $a_n x + a_{n-1}$ is evaluated first Horner's method is also known as nested multiplication and is implemented using the following algorithm.

$P_n = a_n$

$P_{n-1} = P_n x + a_{n-1}$

$P_j = P_{j+1} x + a_j$

$P_1 = P_2 x + a_1$

$f(x) = P_0 = P_1 x + a_0$.

Q. Evaluate the polynomial $f(x) = x^3 - 4x^2 + x + 6$ using Horner's rule at $x = 2$.

Soln/

Here, $n = 3$; $a_3 = 1$, $a_2 = -4$, $a_1 = 1$ and $a_0 = 6$

Then,

$P_3 = a_3 = 1$

$P_2 = P_3 x + a_{n-1} = 1 \times 2 + (-4) = -2$

$P_1 = (-2) \times 2 + 1 = -3$

$P_0 = (-3) \times 2 + 6 = 0$

So, $f(2) = 0$.

Chapter - 2

Interpolation and Approximation :-

Interpolation

Suppose we are given the following values of $y = f(x)$ for a set of values of x:

x :	x_0	x_1	x_2	....	x_n
y :	y_0	y_1	y_2	....	y_n

Then the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation. Thus, interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called Intrapolation.

In interpolation, curve / function passes through all the data points, while in approximation it is not necessary for the curve to pass through the individual points.

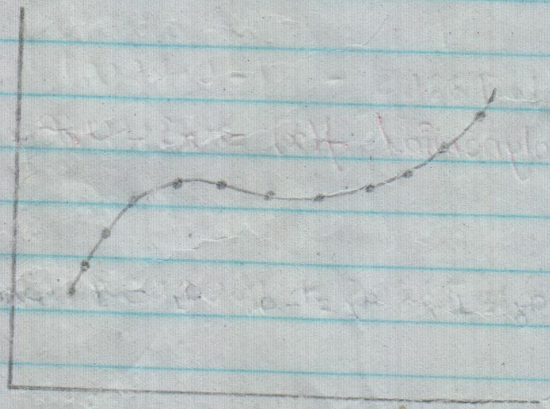


fig: Interpolation

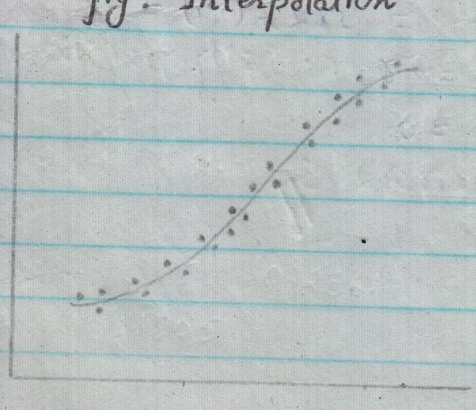


fig: Approximation

Linear Interpolation

The straight form of interpolation that approximate two data points by a straight line is a linear interpolation.

If $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are the two given points then these two points can be connected linearly shown in figure below

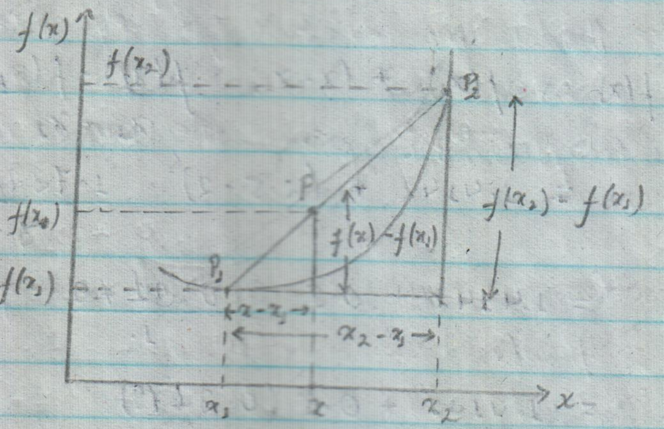


fig: Graphical Representation of linear interpolation

Using the concept of similar triangle we can show that

$$\frac{f(x)-f(x_1)}{f(x_2)-f(x_1)} = \frac{x-x_1}{x_2-x_1}$$

i.e. $$\frac{f(x)-f(x_1)}{x-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

solving for $f(x)$, we get.

$$f(x) = f(x_1) + (x-x_1)\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)$$

Q. Table below gives the square roots of integers:

x	1	2	3	4	5
$f(x)$	1	1.4142	1.7321	2	2.2361

Determine the root of 2.5.

Soln

Here, the given value lies in between 2 and 3.
Therefore,

$x_1 = 2$ and $x_2 = 3$ and $x = 2.5$

$\therefore f(x_1) = 1.4142$

$f(x_2) = 1.7321.$

Then,

$$f(x) = f(x_1) + (x - x_1)\,\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= 1.4142 + (2.5 - 2)\,\frac{1.7321 - 1.4142}{3 - 2}$$

$$= 1.4142 + 0.5\,\frac{0.3179}{1}$$

$$= 1.4142 + 0.5\,(0.3179)$$

$$= 1.4142 + 0.15896$$

$$\therefore f(x) = 1.57315$$

So, the root of the 2.5 is 1.57315.

Q. What is the value at 4.5

Soln

Here, the given value lies in 4 and 5.
Therefore,

$x_1 = 4$

$x_2 = 5$

$f(x_1) = 2$

$f(x_2) = 2.2361.$

then,

$$f(x) = f(x_1) + (x - x_1)\,\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= 2 + (4.5 - 4)\,\frac{2.2361 - 2}{5 - 4}$$

$$= 2 + 0.5\,(0.2361)$$

$$= 2.11805$$

Lagrang's Interpolation

If $y = f(x)$ takes the values $f_0, f_1 ------ f_n$ corresponding to $x = x_0, x_1, ------ x_n$ and if the points $(x_0, f_0), (x_1, f_1), --------- (x_n, f_n)$ can be imagined to be data values connected by a curve of function $p(x)$ satisfying the condition

$$P(x_k) = f_k \quad \text{for } k = 0, 1, ------ n$$

Then the function $P(x_k)$ is called interpolation function.

Hence, an interpolation function is a curve that passes through the data points.

If $y = f(x)$ takes the values $f_0, f_1 ------- f_n$ corresponding to $x = x_0, x_1, ------- x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)-------(x-x_n)}{(x_0-x_1)(x_0-x_2)-------(x_0-x_n)} f_0 + \frac{(x-x_0)(x-x_2)-----(x-x_n)}{(x_1-x_0)(x_1-x_2)----(x_1-x_n)}$$

$$f_1 + ------------ + \frac{(x-x_0)(x-x_1)------(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)-----(x_n-x_{n-1})} f_n \quad \text{——— (1)}$$

This is known as Lagranges interpolation.

Proof:

Let $y = f(x)$ be a function which takes the values $(x_0, f_0) (x_1, f_1) ---- (x_n, f_n)$. Since there are $n+1$ pair of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n.

Let this polynomial be of the form

$$y = f(x) = a_0 (x-x_1)(x-x_2) -------- (x-x_n) + a_1 (x-x_0)(x-x_2) ---- (x-x_n) + a_2 (x-x_0)(x-x_1)(x-x_3) ------ (x-x_n) + ---- ------ + a_n (x-x_0)(x-x_1) ------ (x-x_{n-1}) \quad \text{——— (2)}$$

putting $x = x_0$, $y = f_0$ in equation (2) we get,

$$f_0 = a_0 (x_0-x_1)(x_0-x_2) ------ (x_0-x_n)$$

$$\therefore a_0 = \frac{f_0}{(x_0-x_1)(x_0-x_2) ------ (x_0-x_n)}$$