UNIT 5

Vectors

Learning Outcomes:

After the completion of this unit, the students will be enable to

(i) introduce vectors in space
(ii) operate the algebra of vectors in space
(iii) to find length(magnitude/modulus valu), Dinstance between two points, unit vector direction cosines and define null vector,
(iv) find scalars and vector product of two vectors.
(v) find the angle between two vectors,
(vi) apply properties of scalar product of vectors in trigonometry and
(vii) define and find the scalar and vector tripple product of three vectors

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Vectors

Physical Quantities:– Those quantities which can be quantified with measurement or measurable quantities are called physical quantities. A physical quantity can be expressed as the combination of a numerical value and a unit. For example, mass can be quantified as n Kg where n is the numerical value (or magnitude) and Kg is the unit.

Regarding **magnitude** and **direction**, physical quantities can be classified into two categories:

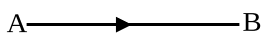
i) Scalars   ii) Vectors.

Scalars :– Physical quantities having magnitude but no direction are called scalars. For example: mass, length, distance, time, temperature, volume, density, work etc.

Vectors:– Physical quantities having both the magnitude and direction are called vectors. For example: force, velocity, acceleration, displacement, momentum etc.

<u>Representation of a vector</u>

A vector is represented by a directed line segment such that the length of the line segment represent the magnitude of the vector in a certain scale and the direction of arrow marked at one end denotes the direction of the vector.



In above figure, the directed line segment from A to B represents a vector and it is written as $\overrightarrow{AB}$. A is the initial point and B is the terminal point of the vector $\overrightarrow{AB}$.

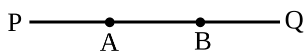
The **modulus or module or magnitude** of a vector $\overrightarrow{AB}$ is the positive number which is the measure of its length and is denoted by $\left|\overrightarrow{AB}\right|$ or AB.

Every vector has **three characteristics**:

i) **Magnitude**:- Magnitude $\overrightarrow{AB}$ is the measure of the length of vector $\overrightarrow{AB}$ and it is denoted by $\left|\overrightarrow{AB}\right|$ or AB. So, $|\overrightarrow{AB}| = |\overrightarrow{BA}|$ or AB = BA.

But, $\overrightarrow{AB} = -\overrightarrow{BA}$

ii) **Direction**:- The direction of a vector $\overrightarrow{AB}$ is parallel to a given line. The line parallel to $\overrightarrow{AB}$ is called parallel support. In the following figure,OQ is support of AB,



iii) **Sense**:- The sense of $\overrightarrow{AB}$ is from P to Q and that of $\overrightarrow{BA}$ is from Q to P. Thus, the sense of a directed line segment is from its initial point to the terminal point.

<u>**Types of vectors**</u>

1) Zero vector or Null vector:- A vector whose initial and terminal points are coincident is called the null vector. The **magnitude of the zero vector is zero** and it can have any arbitrary direction and any line as its line of support. In other words, the direction of a zero vector is in determinate. It is denoted by $\overrightarrow{0}$. In a plane $\overrightarrow{0} = (0, 0)$ and in a space $\overrightarrow{0} = (0, 0, 0)$.

2) Proper vector:- A vector which is not a zero vector is called a proper vector.

3) Unit vector:- A vector whose modulus (magnitude) is unity is called a unit vector. The unit vector in the $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ direction of a non-zeor vector $\overrightarrow{a}$ is denoted by $\hat{a}$ and read as a cap. eg:- The vectors (1, 0), (0,1) are examples of a unit vector in a plane, Similarly, (1,0,0), (0,– 1,0), $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are unit vectors in a space.

4) **Equal vectors:-** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have (i) same (equal) length (magnitude), (ii) same direction and sense and written as $\vec{a} = \vec{b}$.

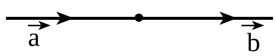
5) **Negative (opposite) of a vector:-** A vector having same magnitude as that of a given vector but opposite direction is called the negative vector of the given vector. e.g.:- If $\overrightarrow{AB}$ is a given vector then $-\overrightarrow{BA}$ is negative of $\overrightarrow{AB}$.

6) Parallel vector **(Important):-** Two vectors $\vec{a}$ and $\vec{b}$ are said to be parallel to each other if $\vec{a} = \lambda \vec{b}$ or $\vec{b} = \lambda \vec{a}$, where, λ is any real number.

There are two types of parallel vectors:

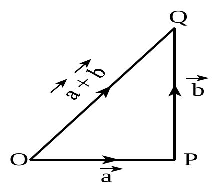
7) **Collinear vector:-** Two or more vectors are said to be collinear if they are parallel to the same straight line (support). (**Parallel Vector जस्तै ल ।)**

Collinear vectors are parallel to the same vector and the initial point of one vector coincides with the terminal point of another vector in a common point.



Triangle Law of Vector Addition

$\overrightarrow{OQ} = \overrightarrow{OP} + PQ = \vec{a} + \vec{b}$



Also, $\overrightarrow{OQ}$ is resultant vector of $\vec{a}$ and $\vec{b}$.

PROPERTIES OF ADDITION OF VECTORS

Some properties of addition of vectors are as follows:

i) Commutativity: For any two vectors $\vec{a}$ and $\vec{b}$ we have $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

ii) Associativity: For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ we have,

$(\vec{a}+\vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

iii) Existence of additive identity: For every vector, we gave the zero vector $\vec{0}$ such that $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$.

iv) Existence of additive inverse: For every vector $\vec{a}$, there corresponds its negative $-\vec{a}$ such that $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + (\vec{a})$

SUBTRACTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ are two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ is denoted by $\vec{a} - \vec{b}$, i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Example

For any two vectors $\vec{a}$ and $\vec{b}$, prove that

i) $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$

ii) $|\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$

iii) $|\vec{a}-\vec{b}| \ge |\vec{a}| - |\vec{b}|$

PROPETIES OF MULTIPLICATION OF VECTORS BY A SCALAR

The following are properties of multiplication of vectors by scalars.

For vectors $\vec{a}, \vec{b}$ and scalar m, n we have

i) $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$

ii) $(-m)(-\vec{a}) = m\vec{a}$

iii) $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$

iv) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

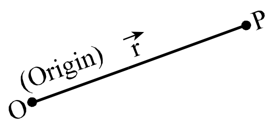
v) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

POSITION VECTOR

Definition: If a point 'O' is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector of P with respect to O.

If we say that the position vector of P is $\vec{r}$ with respect to some origin O.

Then $\overrightarrow{OP} = \vec{r}$.



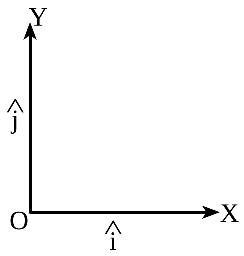
$\overrightarrow{AB}$ in Terms of Position Vector of A and B

Let, , $\overrightarrow{OA} = (x_1, y_1, z_1)$ and $\overrightarrow{OB} = (x_2, y_2, z_2)$, then

$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Unit vectors along mutually perpendicular co-ordinate axis:

i) Unit vectors along mutually perpendicular co-ordinate axes **in plane(in two dimension):**

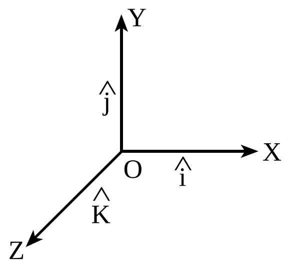


Let $\hat{i}$ and $\hat{j}$ be two unit vectors along positive x-axis (i.e. OX) and along y-axis (i.e. OY) respectively.

Then, $\hat{i} = (1,0)$ and $\hat{j} = (0,1)$

Where, $\hat{i}$ and $\hat{j}$ are read as i cap and j cap respectively.

ii) Unit vectors along mutually perpendicular co-ordinate axes **in space(in three dimensions)**



Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be three unit vectors along positive x-axis (i.e. OX) along y-axis (i.e. OY) and along z-axis (i.e. OZ) respectively.

Then, $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$ and $\hat{k} = (0,0,1)$

Components of a Vector in Plane(Two Dimensions) (Position Vector)

Let P(x, y) be a point in a plane with reference to OX and OY as the coordinate axes as shown in the given figure. Then OM = x and PM = y. Let $\hat{i}$, $\hat{j}$ be unit vectors along OX and OY respectively. Then $\overrightarrow{OM} = x\hat{i}$ and $\overrightarrow{MP} = y\hat{j}$ vectors, $\overrightarrow{OM}$ and $\overrightarrow{MP}$ are known as the components of $\overrightarrow{OP}$ along x-axis and y-axis respectively.

i. Position Vector

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

$$\Rightarrow \overrightarrow{OP} = \boldsymbol{x\hat{i} + y\hat{j} = (x, y)}$$

ii. Magnitude/modulus value

$$\therefore \quad |\overrightarrow{OP}| = \mathbf{OP} = \sqrt{\mathbf{x^2+y^2}}$$

Thus, if a point P in a plane has coordinates (x, y) , then

i) $\overrightarrow{OP} = x\hat{i} + y\hat{j} = (x, y)$, it is position vector.

ii) $|\overrightarrow{OP}| = \sqrt{x^2+y^2} = OP$, it is magnitude.

Vector and Distance(Length) Joining Two Points in Plane (Two Dimension)

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points, then

$$\overrightarrow{AB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} = (x_2 - x_1, y_2 - y_1)$$

Also, $|\overrightarrow{AB}| = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ = Distance or Length or Magnitude

Example: 1

Let 'O' be the origin and let P (–4, 3) be a point in the xy-plane. Express $\overrightarrow{OP}$ in terms of vectors $\hat{i}$ and $\hat{j}$. Also, find $|\overrightarrow{OP}|$

Solution:

The position vector of point P is $-4\hat{i} + 3\hat{j}$

$$\therefore \overrightarrow{OP} = -4\hat{i} + 3\hat{j}$$

$$\Rightarrow |\overrightarrow{OP}| = \sqrt{(-4)^2 + 3^2} = 5$$

Example: 2

If the position vector $\vec{a}$ of a point (12, n) is such that $|\vec{a}| = 13$, find the value of n.

Solution:

The position vector of the point (12, n) is $12\hat{i} + n\hat{j}$

$$\therefore \vec{a} = 12\hat{i} + n\hat{j}$$

$$|\vec{a}| = \sqrt{12^2 + n^2}$$

Given,

$$|\vec{a}| = 13$$

$$\Rightarrow \sqrt{12^2 + n^2} = 13$$

Squaring,

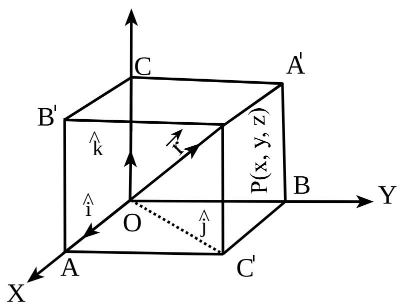
$$\Rightarrow 144 + n^2 = 169$$

$$\Rightarrow n^2 = 25$$

$$\Rightarrow n = \pm 5$$

Position Vector of a Point in Space(Three Dimension)

i. The position vector of a point P(x, y, z) in space is given by



$$\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

ii. Magnitude/Modulus Value

$$\therefore OP = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

Distance/Length between Two Points in Space

Let P (x, y, z) and Q (x_2, y_2, z_2) be two points. Then,

$\overrightarrow{PQ}$ = Position vector of Q - position vector of P

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\therefore$ PQ $= |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ = Magnitude/Modulus value

Thus, the distance between the points P (x_1, y_1, z_1) and Q = (x_2, y_2, z_2)

is given by: $|\overrightarrow{PQ}| = PQ = \sqrt{(x_2 - x_1) + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Unit vector along a vector (Important)

a) In Plane:-

Let, P(x, y) be any point in a plane then

$\overrightarrow{OP} = (x, y) = x\vec{i} + y\vec{j}$

and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$ = OP = magnitude

The unit vector along $\overrightarrow{OP}$ is denoted by $\widehat{OP}$ and defined by

$$\widehat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} \Rightarrow \widehat{OP} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = \left(\frac{1}{\sqrt{x^2+y^2}}, \frac{1}{\sqrt{x^2+y^2}}\right)$$

Where, $\widehat{OP}$ is read as OP cap.

b) In space:–

Let, P(x, y, z) be any point in a space then

$$\overrightarrow{OP} = (x, y, z) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2} = OP =$ mgnitude

Then unit vector along $\overrightarrow{OP}$ is denoted by $\widehat{OP}$ is denoted by $\widehat{OP}$ and defined by

$$\widehat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$$

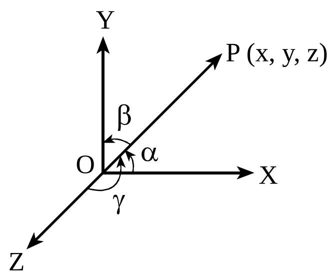
$\Rightarrow \widehat{OP} = \dfrac{x\,\vec{i} + y\,\vec{j} + \vec{k}z}{\sqrt{x^2 + y^2 + z^2}}$, where, $\widehat{OP}$ in read as OP cap.

$$\Rightarrow \widehat{OP} = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

Direction cosines of a vector.

Let $\overrightarrow{OP} = (x, y, z) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.

Let α, β and γ be respectively be the angles made by the line OP with three mutually perpendicular straight lines OX, OY, OZ respectively then cos α, cos β and cos γ are said to be direction cosines of $\overrightarrow{OP}$ and these cosines are also respectively denoted by ℓ, m, n.



Then,

$$\ell = \cos\alpha = \frac{x}{|\overrightarrow{OP}|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$m = \cos\beta = \frac{y}{|\overrightarrow{OP}|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{and} \quad n = \cos\gamma = \frac{z}{|\overrightarrow{OP}|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

are the direction cosines (d.c's) of $\overrightarrow{OP}$. In fact, ℓ, m and n are the components of $\widehat{OP}$.

Also, $\ell^2 + m^2 + n^2 = \frac{x^2}{x^2 + y^2 + z^2} + \frac{y^2}{x^2 + y^2 + z^2} + \frac{z^2}{x^2 + y^2 + z^2}$

$= \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1.$

This means that $|\hat{OP}| = 1$.

Illustrative Examples:

Example 1: Find the magnitude of the vector $\vec{a} = 3\vec{i} - 2\vec{j} + 2\hat{k}$.

Solution:

We have, $|\vec{a}| = \sqrt{(3)^2 + (-2)^2 + (6)^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7.$

Example 2: Find the unit vector in the direction of $3\vec{i} - 6\vec{j} + 2\hat{k}$.

Solution:

Let $\vec{a} = 3\vec{i} - 6\vec{j} + 2\hat{k}$

$|\vec{a}| = \sqrt{3^2 + (-6)^2 + 2^2} = 7.$

So, the unit vector in the direction of $\vec{a}$ is given by

$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}$

$= \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$

Worked Out Example:

If $\overrightarrow{OP} = \hat{i} - 3\hat{j} + 7\hat{k} = (1, -3, 7)$ and $\overrightarrow{OQ} = 5\hat{i} - 2\hat{j} + 4\hat{k} = (5, -2, 4)$, find

i) $\overrightarrow{PQ}$ (Position vector) (ii) magnitude of $\overrightarrow{PQ}$

iii) unit vector along $\overrightarrow{PQ}$ and (iv) direction cosines of $\overrightarrow{PQ}$.

Solution:

i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5, -2, 4) - (1, 3, -7) = (4, -5, 11) = 4\hat{i} - 5\hat{j} + 11\hat{k}$

ii) Magnitude of $\overrightarrow{PQ}$ (or, modulus of $\overrightarrow{OP}$)

$= |\overrightarrow{PQ}| = \sqrt{(4)^2 + (-5)^2 + (11)^2} = 9\sqrt{2}$.

iii) Unit vector along $\overrightarrow{PQ} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

$$= \frac{(4, -5, 11)}{9\sqrt{2}} = \left(\frac{4}{9\sqrt{2}}, \frac{-5}{9\sqrt{2}}, \frac{11}{9\sqrt{2}}\right)$$

iv) Direction cosines of $\overrightarrow{PQ}$ are $\dfrac{4}{9\sqrt{2}}, \dfrac{-5}{9\sqrt{2}}$ and $\dfrac{11}{9\sqrt{2}}$.