Comau - Exponential Mapping Method

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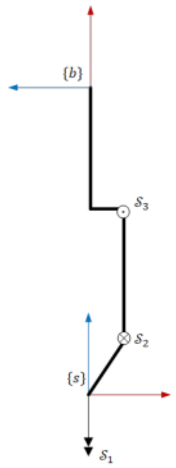
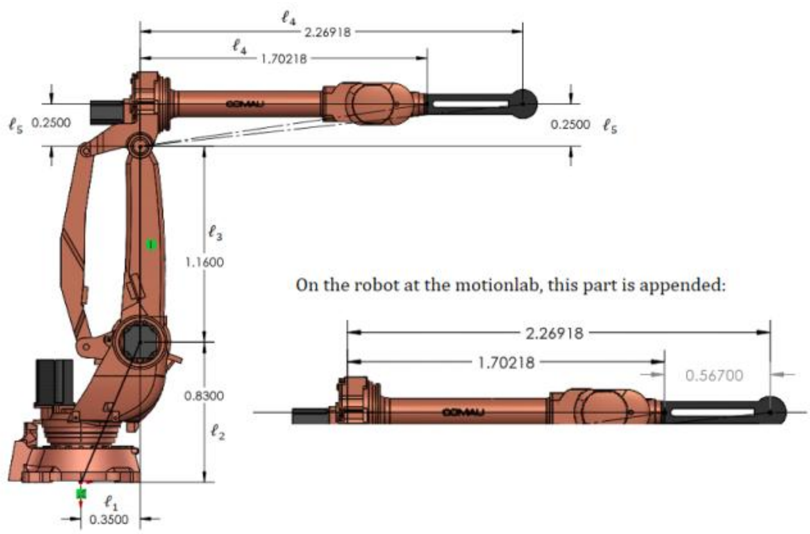


Forward kinematics using screws $\mathcal{S} = \begin{bmatrix}\mathbf{w}\\ \mathbf{v}\end{bmatrix}$, $\|\mathbf{w}\| = 1$, $\mathbf{v} = -\mathbf{w} \times \mathbf{p}$
The vector $\mathbf{w}$ is a unit vector along the screw axis.
The vector $\mathbf{p}$ is a position vector from $\{s\}$ to any any point on the screw axis.

Note that $\{s\} = \{\mathbf{0}, \hat{\mathbf{x}}_s, \hat{\mathbf{y}}_s, \hat{\mathbf{z}}_s\}$ is the fixed base frame and $\{b\} = \{\mathbf{o}_b, \hat{\mathbf{x}}_b, \hat{\mathbf{y}}_b, \hat{\mathbf{z}}_b\}$ is the moving end-effector frame.
Note the directions of the screws, where $\otimes$ denotes a vector into the page and $\odot$ denotes a vector out of the page.

$$\mathcal{S}_1 = \begin{bmatrix}-\hat{\mathbf{z}}_s\\ \mathbf{v}_1\end{bmatrix}, \qquad \mathbf{v}_1 = \mathbf{0}$$
$$\mathcal{S}_2 = \begin{bmatrix}\hat{\mathbf{y}}_s\\ \mathbf{v}_2\end{bmatrix}, \qquad \mathbf{v}_2 = -\hat{\mathbf{y}}_s \times (\ell_1\hat{\mathbf{x}}_s + \ell_2\hat{\mathbf{z}}_s)$$
$$\mathcal{S}_3 = \begin{bmatrix}-\hat{\mathbf{y}}_s\\ \mathbf{v}_3\end{bmatrix}, \qquad \mathbf{v}_3 = \hat{\mathbf{y}}_s \times (\ell_1\hat{\mathbf{x}}_s + (\ell_2 + \ell_3)\hat{\mathbf{z}}_s)$$

The zero pose of the robot
$$\mathbf{H}_b^s(\mathbf{0}) = \begin{bmatrix}\mathbf{R}_b^s(\mathbf{0}) & \mathbf{r}_b^s(\mathbf{0})\\ \mathbf{0} & 1\end{bmatrix} = \begin{bmatrix}\mathbf{R}_y(-\pi/2) & \begin{bmatrix}(\ell_1 - \ell_5)\\ 0\\ \ell_2 + \ell_3 + \ell_4\end{bmatrix}\\ \mathbf{0} & 1\end{bmatrix}$$



On the robot at the motionlab, this part is appended:

Forward kinematics can be written
$$\mathbf{H}_b^s(\mathbf{q}) = \exp(\tilde{\mathcal{S}}_1 q_1)\exp(\tilde{\mathcal{S}}_2 q_2)\exp\left(\tilde{\mathcal{S}}_3(q_2 + q_3)\right)\mathbf{H}_b^s(\mathbf{0})$$
$$= e^{\tilde{\mathcal{S}}_1 q_1}\, e^{\tilde{\mathcal{S}}_2 q_2}\, e^{\tilde{\mathcal{S}}_3(q_2+q_3)}\,\mathbf{H}_b^s(\mathbf{0})$$
where the $\tilde{\square}$ (tilde) operator denotes a Lie algebra
$$\tilde{\mathcal{S}} = \begin{bmatrix}\tilde{\mathbf{w}} & \mathbf{v}\\ \mathbf{0} & 0\end{bmatrix} \in se(3), \qquad \tilde{\mathbf{w}} = \begin{bmatrix}0 & -w_3 & w_2\\ w_3 & 0 & -w_1\\ -w_2 & w_1 & 0\end{bmatrix} \in so(3)$$

Notice that the third scew is rotated by angle $(q_2 + q_3)$!

Table when using $L_4 = 1492.18$

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      x          z       q2        q3
   -------    ------    -----   -------

    700.63    1483.7      -75       -86
   -757.56    2643.2      -75       -10
   -33.831      3433      -20       -10
    2871.3    77.808       95      -125
    649.95    -518.9       95   -223.95
    649.97    151.55    52.94   -213.94
    1842.2      2240        0       -90
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Table from doc

Pos	X	Z	Ax.2	Ax.3
	[mm]	[mm]	[deg]	[deg]
1	700,63	1483,71	-75°	-86°
2	-757,56	2643,15	-75°	-10°
3	-33,82	3432,97	-20°	-10°
4	2871,3	77,81	95°	-125°
5	650	-518,93	95°	-223,95°
6	650	151,56	52,94°	-213,94°
7	1842,18	2240	0°	-90°

Velocity and acceleration using twist $\mathcal{V} = \begin{bmatrix}\boldsymbol{\omega}_b^s\\ \mathbf{v}_{b/s}^s\end{bmatrix}$, where $\boldsymbol{\omega}_b^s$ is the angular velocity of $\{b\}$ in $\{s\}$.
$$\mathcal{V} = \begin{bmatrix}\boldsymbol{\omega}_b^s\\ \mathbf{v}_{b/s}^s\end{bmatrix} = \mathbf{J}(\mathbf{q})\,(\dot{\mathbf{q}} + \dot{q}_2\hat{\mathbf{z}}_s)$$
$$\dot{\mathcal{V}} = \frac{d\mathcal{V}}{dt} = \begin{bmatrix}\dot{\boldsymbol{\omega}}_b^s\\ \dot{\mathbf{v}}_{b/s}^s\end{bmatrix} = \frac{\partial \mathbf{J}}{\partial \mathbf{q}}\,\dot{\mathbf{q}}\,(\dot{\mathbf{q}} + \dot{q}_2\hat{\mathbf{z}}_s) + \mathbf{J}(\mathbf{q})(\ddot{\mathbf{q}} + \ddot{q}_2\hat{\mathbf{z}}_s)$$

The Jacobian is built using screws mapped to adjacent joints (the previous screw axis)
$$\mathbf{J}(\mathbf{q}) = [\mathcal{S}_1 \quad \mathrm{Ad}_{\mathbf{H}_1(\mathbf{q})}(\mathcal{S}_2) \quad \mathrm{Ad}_{\mathbf{H}_2(\mathbf{q})}(\mathcal{S}_3)] \in \mathbb{R}^{6\times 3}$$
where
$$\mathrm{Ad}_{\mathbf{H}(\mathbf{q})} = \begin{bmatrix}\mathbf{R}(\mathbf{q}) & \mathbf{0}\\ \tilde{\mathbf{r}}(\mathbf{q})\mathbf{R}(\mathbf{q}) & \mathbf{R}(\mathbf{q})\end{bmatrix}, \qquad \mathbf{H}_i(\mathbf{q}) = \begin{bmatrix}\mathbf{R}_i(\mathbf{q}) & \mathbf{r}_i(\mathbf{q})\\ \mathbf{0} & 1\end{bmatrix} \in SE(3)$$
The elements in $SE(3)$, i.e. $\mathbf{H}_1$ and $\mathbf{H}_2$ are elements mapping joints to the base frame $\{s\}$.

Note that the vector $\mathbf{v}_{b/s}^s$ is <u>not</u> the translational velocity of $\{b\}$ in $\{s\}$. That would just be $\mathbf{v}_b^s = \dot{\mathbf{r}}_b^s$.
Rather, it is the velocity of $\{b\}$ when mapped back to the origin of $\{s\}$. Hence, to calculate the velocity and acceleration:
$$\dot{\mathbf{r}}_b^s = \mathbf{v}_{b/s}^s + \boldsymbol{\omega}_b^s \times \mathbf{r}_b^s$$
$$\ddot{\mathbf{r}}_b^s = \dot{\mathbf{v}}_{b/s}^s + \dot{\boldsymbol{\omega}}_b^s \times \mathbf{r}_b^s + \boldsymbol{\omega}_s \times \dot{\mathbf{r}}_b^s$$

To calculate the orientation of $\{b\}$, we assume an order of rotation: XYZ, (roll-pitch-yaw)
$$\boldsymbol{\varphi} = \begin{bmatrix}\phi_x\\ \phi_y\\ \phi_z\end{bmatrix}, \qquad \mathbf{R} = \mathbf{R}_x(\phi_x)\mathbf{R}_y(\phi_y)\mathbf{R}_z(\phi_z)$$

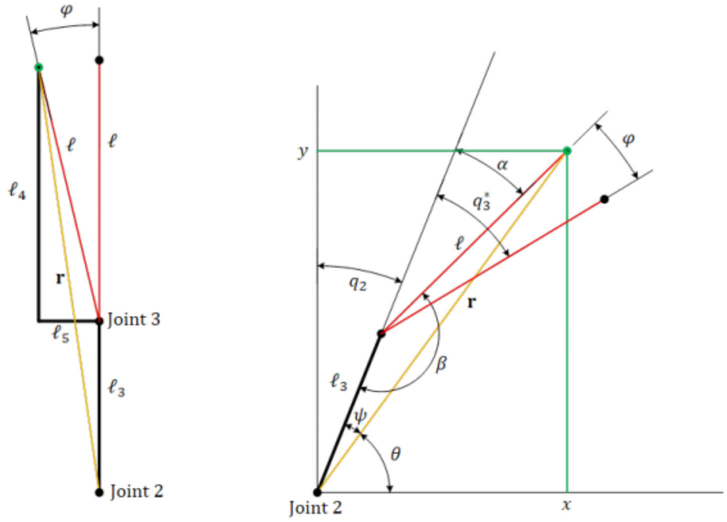
We can use the elements of $\mathbf{R}$ to determine consistent values of $\boldsymbol{\varphi}$, but with a catch whenever $\phi_y = \pm 90°$.

At that point, $\mathbf{R}$ gets the form $\mathbf{R}=\begin{bmatrix} 0 & 0 & \pm 1 \\ c_x s_z \pm s_x c_z & c_x c_z \mp s_x s_z & 0 \\ s_x s_z \mp c_x c_z & c_x s_z \pm s_x c_z & 0 \end{bmatrix}$ and we cannot determine a unique ϕ_x, ϕ_z.

$$2(c_x s_z + s_x c_z) = r_{21} + r_{32}$$

If we need the rates of change of $\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}$, then we can find a relation $\boldsymbol{\omega}_b^s = \mathbf{T}(\mathbf{q})\dot{\boldsymbol{\varphi}}$, using XYZ order of rotation.

Inverse kinematics can be determined by carefully investigating the figures below.
The figure represents the geometry of the distal two links of the Comau robot when the robot is at it's zero pose.



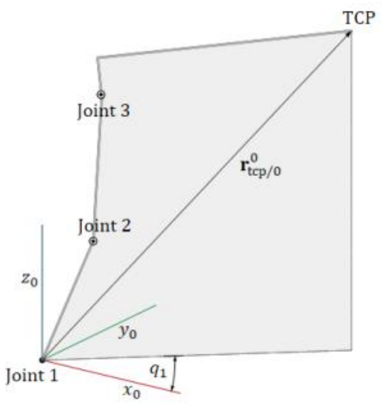
From the figure we can obtain the following geometric relationships

$$\theta + \psi + q_2 = \pi/2$$
$$\alpha + \beta = \pi$$
$$\alpha = q_3^* - \varphi$$
$$\cos(\psi) = \frac{\ell_3^2 + \|\mathbf{r}\|^2 - \ell^2}{2\ell_3\|\mathbf{r}\|}$$
$$\cos(\beta) = \frac{\ell_3^2 + \ell^2 - \|\mathbf{r}\|^2}{2\ell_3\ell}$$
$$\varphi = \operatorname{atan}(\ell_5/\ell_4)$$

Having found the expression for q_3^* we need to negate and subract q_2 due to the closed-loop behaviour of the Comau robot:

$$q_3 = -q_3^* - q_2$$

The joint variable q_1 is found directly from the vector $\mathbf{r}_{\text{tcp}/0}^0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as $q_1 = \operatorname{atan2}(y, x)$, where $\mathbf{r}_{\text{tcp}/0}^0$ denotes the position vector from the base frame $\{0\}$ to the TCP (tool center point) of the robot, represented in the base frame.



The inverse velocity and acceleration is more straightforward as these are linear in their parameters:

$$\mathcal{V} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \quad \Rightarrow \quad \dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})\mathcal{V}$$
$$\dot{\mathcal{V}} = \frac{\partial \mathbf{J}}{\partial \mathbf{q}}\dot{\mathbf{q}}^2 + \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} \quad \Rightarrow \quad \ddot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q})\left(\dot{\mathcal{V}} - \frac{\partial \mathbf{J}}{\partial \mathbf{q}}\dot{\mathbf{q}}^2\right)$$