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e the basic definitions which inclu
and methods to solve heat equation nam
d, Separation of Variables Method, Laplace's Trans
hod and Finite Element Method. The detail meshing of some
mains are also discussed in this chapter.

transfer

al energy exchange among physical systems is termed as heat transfer.
ansfer have three main modes [1-13]

- Convection
- Radiation
- Conduction

1.3 Heat transfer by convection

Convection relate heat transfer between a surface and a liquid or gas in motion. As the fluid or gas travels faster, the convective heat transfer increases. Convection is of two types.

- Natural convection.
- Forced convection.

Example

- Boiling water- if we take a pot full of water and boil it on the burner then the bottom of the pot is heated so the water at the bottom is also heated due to the hotness of the bottom of the pot. Afterwards, the density of the hot water is low as compared to the cooler water so the cooler water at the top of the goes down and the hot water rises up into the top of the pot. This process is again repeated.
- Ice melting- suppose we have an ice cube then because of difference in temperature, it will melt. By this process the ice will start melting into liquid form.

1.4 Heat transfer by radiation

As we know that conduction and convection need matter to transfer heat. In radiation mode, the heat is transfer by electromagnetic waves. Radiation is a procedure of heat transfer that does not depends upon any touching between the heat origin and the heated body.

Example

- Our body get warm from the heat of the sun.
- Our body get warm from the light bulb.
- Our body get warm from fire.

1.5 Heat transfer by conduction

Conduction occurs when two objects at different temperatures are in contact with each other. Heat flows from the warmer to the cooler object until they both attain the same temperature.

Example

- By touching a stove and get burned
- Ice is getting melt due to hand heat

1.6 Methods to solve heat equation

- Adomian Decomposition Method
- Separation of Variables Method
- Laplace's Transform
- Lie Symmetric Method
- Finite Element Method

1.6.1 Adomian decomposition method

George Adomian was developed (ADM). The ADM has been e recent years. Generally in this field of applied mathematics, Partic the area of the series, by using this method this is proved to be stu can easily handle a vast class of linear or nonlinear, ordinary or p d linear and nonlinear integral equations. The deca so that, it gives us differ

$), $

$\sin x,$

$) = \frac{1}{2!}t^2 \sin x,$

the series form $m(x,t)$ is

$$m_0(x,t)+m_1(x,t)+m_2(x,t)+\dots$$

$$=\sin(x)\left(1-t+\frac{1}{2!}t^2-\cdots\right),$$

and in a close form we can write

$$m(x,t)=e^{-t}\sin x.$$

1.6.2 Separation of variables method

PDE to ODE

The linear partial differential equation is homogeneous and it can be solved by the method of separation of variables. We use this method to solve the heat conduction problems and other type of many different problems.

Important thing of the separation of variables method is that it is efficiently put back the partial differential equation by a system of ordinary differential equation that can be solved easily. Dissimilar the decay method, the separation of variables method engage certain expectations and change by formulas in dealing partial

Boundary condition must be homogeneous

differential equations. In notable, the separation of variables method requires that the BC must be homogeneous. For inhomogeneous BC, we use some tools to transform in homogeneous boundary condition.

Example

Solve the initial boundary value problem by using the separation of variable method.

$$
\begin{array}{lll}
\text{PDE} & v_t = v_{xx}, & 0 \le x \le \pi, \\
\text{BC} & v_x(0,t) = 0, & t \ge 0, \\
 & v_x(\pi,t) = 0, & t \ge 0, \\
IC & v(x,0) = 2 + 3\cos x. &
\end{array}
\tag{1.7}
$$

Solution:

Given that the BC are Neumann boundary conditions. Now we suppose

$$v(x,t) = X(x)T(t), \tag{1.8}$$

using equation (1.8) into equation (1.7) we obtain ordinarily differential equations in terms $T(t)$ and $X(x)$ that is

$$T'(t) + \lambda^2 T(t) = 0, \tag{1.9}$$

and

$$X''(x) + \lambda^2 X(x) = 0, \tag{1.10}$$

the solution of equation (1.9) and equation (1.10) are

$$T(t) = Ce^{-\lambda^2 t}, \tag{1.11}$$

and

$$X(x) = A\cos\lambda x + B\sin\lambda x, \tag{1.12}$$

now we find the values of A, B and λ, then we using the boundary conditions

$$\begin{aligned} v_x(0,t) &= X'(0)T(t) = 0, \Rightarrow X'(0) = 0, \\ v_x(\pi,t) &= X'(\pi)T(t) = 0, \Rightarrow X'(\pi) = 0, \end{aligned} \tag{1.13}$$

using equation (1.13) into equation (1.12) we have

B=0,

and

$$\lambda \sin \pi\lambda = 0,$$

which gives λ_n by

$$\lambda = 0, or\ \lambda_n = n, \qquad n = 1,2,3, \ldots$$

and therefore $\lambda_n = n, n = 0,1,2, \ldots$,Where $\lambda = 0$ is involve since it will not give the trivial solution $v(x,t)$ =0 .

So

$$X_n(x) = \cos nx,$$

$$T_n(t) = e^{-n^2}t,\ \text{n=0,1,2,....,}$$

using the principle of superposition, the linear combination of these solution is also a solution

$$v(x,t) = \sum_{n=0}^{\infty} C_n e^{-n^2 t} \cos nx,$$

or

$$v(x,t) = C_0 + C_1 e^{-t} \cos x + C_2 e^{-4t} \cos 2x + \cdots,$$

comparing coefficients, so we find

$$C_0 = 2,\ \ C_1 = 3,\ \ C_m = 0,\ m \geq 2,$$

The solution takes the from

$$v(x,t) = 2 + 3e^{-t} \cos x.$$

1.6.3 Laplace transform

Laplace was French mathematician, astronomers and physicists who applied the Newtonian theory of gravitation of the solar system. He played a vital role in the development of metric system as well. The Laplace transform was proposed by Laplace. Further, the Laplace transform is deeply used in engineering applications, especially where the driving force is discontinues. It is usually used in process control. The Laplace change give a beneficial means of dealing different types of differential equations when evident IC are given, when the origin values.

The Laplace change (L) ,of some function f(t) for t>0 is described by the following integral from the limit zero to infinity;

$$L\{f(t)\}=\int_0^{\infty} e^{-st} f(t)dt,$$

At the end the function of S, which we write as F(S). In another words we say that the Laplace change f (t) in terms of F(S) can be written as

$$L\{f(t)\} = F(S)$$

Example

$$
\begin{array}{lll}
\text{PDE} & v_t = v_{xx}, & 0 \le x \le 2, \\
\text{BC} & v(0,t) = 0, & t \ge 0, \\
 & v(2,t) = 0, & t \ge 0, \\
IC & v(x,0) = 3\sin(2\pi x). &
\end{array}
\tag{1.14}
$$

Solution

Using Laplace transform and apply initial condition we have

$$\frac{d^2V}{dx^2}(x,S) = SV(x,S) - V(x,0) = SV(x,S) - 3\sin(2\pi x), \tag{1.15}$$

the equation (1.15), we can be written as

$$\frac{d^2V}{dx^2}(x,S) - SV(x,S) = -3\sin(2\pi x),$$

now consider homogeneous equation

$$\frac{d^2V}{dx^2}(x,S) - SV(x,S) = 0, \tag{1.16}$$

we can write the characteristic equation

D^2-S=0

D^2=S

D=$\pm\sqrt{S}$

the solution can be written as;

$V_h(x,S)=c_1e^{\sqrt{S}x}+c_2e^{-\sqrt{S}x}$.

Suppose that the solution of non-homogeneous problem can be written as

$$V_p(x,S)=A\cos(2\pi x)+B\sin(2\pi x), \tag{1.17}$$

so,

$$\frac{d}{dx}\left(V_p(x,S)\right)=-2\pi A\sin(2\pi x)+2\pi B\cos(2\pi x),$$
$$\frac{d^2V}{dx^2}\left(V_p(x,S)\right)=-(2\pi)^2A\cos(2\pi x)-(2\pi)^2B\sin(2\pi x),$$

Therefore, the equation becomes

$$-(2\pi)^2A\cos(2\pi x)-(2\pi)^2B\sin(2\pi x)-S\left(A\cos(2\pi x)+B\sin(2\pi x)\right)=-3\sin(2\pi x),$$
$$\left(-(2\pi)^2-S\right)\left(A\cos(2\pi x)+B\sin(2\pi x)\right)=-3\sin(2\pi x),$$

from this we conclude that

$$-\left(S+(2\pi)^2\right)A=0 \quad , \qquad \text{and} \qquad -\left(S+(2\pi)^2\right)=-3$$
$$A=0, \qquad\qquad B=\frac{-3}{\left(S+(2\pi)^2\right)}.$$

Now we have the general solution

2 Chapter # 2

Two Dimensional Heat Equation: Finite Element Simulations

2.1 Introduction

In this chapter we study the heat transfer in a two dimensional partially heated trapezium solid surface. The trapezium is made up of Aluminum 6063-T83 material. The transfer of heat is due to conduction mode. We used Finite Element Method for solution of established heat equation. The results are presented with the help of graphs.

2.2 Problem statement

We have considered a two dimensional solid surface of trapezium shape, see Fig. 2.1(a). The length measures are taken in Xcm, Ycm and Zcm. The trapezium geometry is centrally holed with circle having radius $0.3cm$ and the circle is centered at (X,Y)=(2,1.5). The centrally holed trapezium is made-up of Aluminum 6063-T83. The Aluminum 6063-T83 is one of the type of Aluminum-6063 and furnished via T83 temper. Such material claim lesser ductility as compared to rest of variants of Aluminum-6063. The alloy composition of Aluminum 6063-T83 is 97.5 to 99.4 Aluminum (Al), 0.45 to 0.9 Magnesium (Mg), 0.2 to 0.6 Silicon (Si), 0 to 0.35 Iron (Fe),) 0 to 0.1 Chromium (Cr), Titanium (Ti), Zinc (Zn), Copper (Cu) and 0.15 are residuals. Moreover, some fundamental properties of Aluminum 6063-T83 are given in Table-1. The upper wall of the trapezium is of length $3cm$ with vertices (1,3) and (3,3). The lower wall of the trapezium is of length $4cm$ having vertices (0,0) and (4,0). The left and right walls of trapezium are of positive and negative slops that is $m_1 = 3$ and $m_2 = -3$ respectively. Since

$$V(x,S)=c_1e^{\sqrt{S}x}+c_2e^{-\sqrt{S}x}+\frac{3}{S+4\pi^2}\sin(2\pi x), \quad (1.18)$$

we know that the Laplace transform of the boundary condition give

$$v(0,t)=0 \quad => \quad V(0,S)=0, \quad (1.19)$$

$$v(2,t)=0 \quad => \quad V(2,S)=0,$$

Using equation (1.19) into equation (1.18), we have

$$0=V(0,S)=c_1+c_2, \qquad 0=V(2,S)=c_1e^{\sqrt{S}2}+c_2e^{-\sqrt{S}2}, \quad (1.20)$$

the equation (1.20) impels $c_1=c_2=0$, we have

$$V(x,S)=\frac{3}{S+4\pi^2}sin(2\pi x),$$

now we apply inverse Laplace transform, then we have

$$V(x,t)=L^{-1}(\frac{3}{S+4\pi^2}sin(2\pi x)),$$

$$=3e^{-4\pi^2t}sin(2\pi x).$$

1.6.4 Lie symmetric method

Lie Symmetry is proposed by Sophus Lie. This method has many applications and order reduction is one of them. One can find detail in this direction by following

Our intention is to examine the heat transfer in such configuration. For this purpose we have to utilize the heat equation. In component form the three dimensional heat equation takes the form

$$\rho(U,V,W,t)c_1(U,V,W,t)\frac{\partial T(U,V,W,t)}{\partial t} = k(U,V,W,t)\begin{pmatrix} \dfrac{\partial^2 T(U,V,W,t)}{\partial U^2} \\ +\dfrac{\partial^2 T(U,V,W,t)}{\partial V^2} \\ +\dfrac{\partial^2 T(U,V,Wt)}{\partial W^2} \end{pmatrix} + \tag{2.1}$$

$$\rho(U,V,W,t)Q(U,V,W,t),$$

since the material properties namely, $\rho(U,V,W,t)$, $c_1(U,V,W,t)$, $k(U,V,W,t)$ are assumed to be independent of space variables and time, therefore we may consider them constant that is;

$$\begin{aligned} &\rho = \rho(U,V,W,t) = \text{constant},\ c_1 = c_1(U,V,W,t) = \text{constant}, \\ &k = k(U,V,W,t) = \text{constant}, \end{aligned} \tag{2.2}$$

further, in the absence of heat source ($Q(U,V,W,t)$) and using Eq. (2.2) into Eq. (2.1), one can obtain

$$\rho c_1 \frac{\partial T(U,V,W,t)}{\partial t} = k\left(\frac{\partial^2 T(U,V,W,t)}{\partial U^2} + \frac{\partial^2 T(U,V,W,t)}{\partial V^2} + \frac{\partial^2 T(U,V,W,t)}{\partial W^2}\right), \tag{2.3}$$

the heat flow along Z-direction is zero, therefore $T(U,V,W,t) = T(U,V,t)$. In the light of this fact the Eq. (2.3) further reduces to;

$$\rho c_1 \frac{\partial T(U,V,t)}{\partial t} = k\left(\frac{\partial^2 T(U,V,t)}{\partial U^2} + \frac{\partial^2 T(U,V,t)}{\partial V^2}\right), \tag{2.4}$$

the Eq. (2.4) is generally acceptable to examine the time dependent heat transfer in two dimensional solid surfaces. In this case the boundary conditions are as follows;

$$\left.\begin{aligned}
&T(U,V,t)=T_0=293.15K\} \text{ when } t=0, \\
&T(U,V,t)=\begin{cases} T_0=293.15K, & 0\le U<1 \\ T_{H_1}=1000K, & 1\le U<3 \\ T_0=293.15K, & 3\le U\le 4 \end{cases} \\
&\frac{\partial T(U,V,t)}{\partial n}=0, \begin{cases} V=3U, & 0\le U\le 1 \\ V=-3U+12, & 3\le U\le 4 \end{cases} \\
&T(U,V,t)=T_{H_2}=1000K, \quad (U-2)^2+(V-1.5)^2=(0.3)^2
\end{aligned}\right\} \text{when } t>0. \tag{2.5}$$

2.3 Analysis

The two dimensional trapezium shape geometry made up of Aluminum 6063-T83 is considered for the heat transfer analysis. The surface has circular hole at centroid of trapezium. The heat flow is time dependent. For inspecting heat transfer patterns, the whole physical system is translated in terms of mathematical model. The Eqs. (2.4)-(2.5) are constructed in this regard. For better description we are interested to solve the problem with the help of Finite Element Method (FEM). We have consider a trapezium shape geometry with circular hole at center. Therefore the final geometry is treated as single (01) domain with ten (10) boundaries and ten (10) vertices. The Fig. 2.1(b) is constructed in this direction. We have consider two cases firstly the heater H_1 has length of $2cm$ with temperature $T_{H_1}=1000K$ and later on the computations were performed for the

the circle is of radius 0.3$cm$and centered at (2, 1.5) with area 0.282 cm^2 and area of trapezium is 9 cm^2 . Therefore the centrally holed trapezium area will be $(9-0.282)=8.718cm^2$. Initially the whole system is at room temperature. The trapezium upper wall is assumed to be cold. It is at room temperature that is $T_0 = 293.15K$ while the two lengths of lower wall $L_1[(0,0)\text{ to }(1,0)]$ and $L_2[(3,0)\text{ to }(4,0)]$ are also considered to be cold that is $T_0 = 293.15K$. In between these two lengths L_1 and L_2 we have incorporated a heater H_1 having temperature $T_{H_1} = 1000K$. The both left and right walls are assumed to adiabatic walls. Further, the circumference (1.88 cm) of centrally circle is treated as heater H_2. The temperature of heater H_2 is same as the temperature of a heater H_1 that is $T_{H_1} = 1000K = T_{H_2}$.

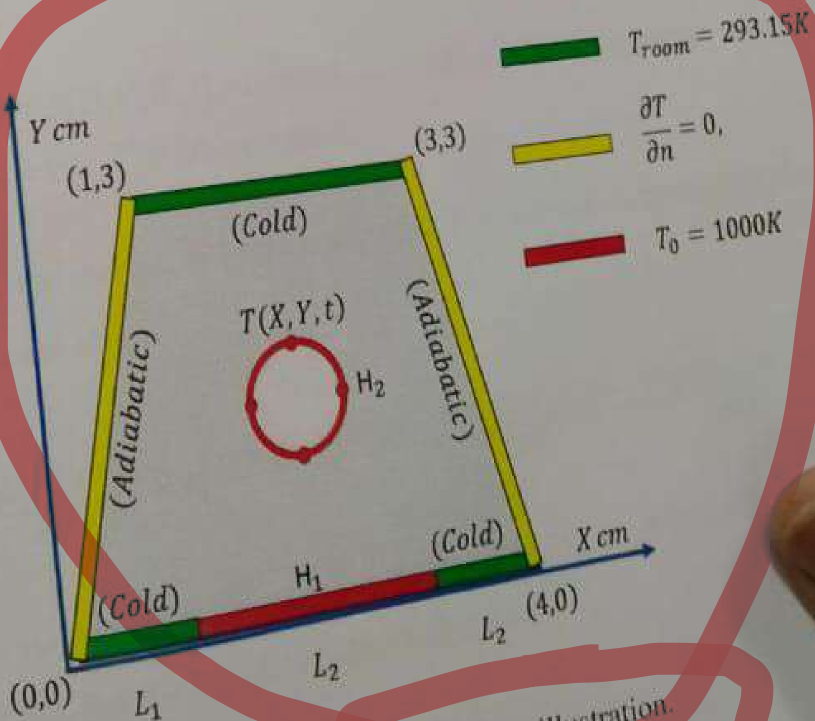

Fig. 2.1(a). Problem illustration.

heater H_2 of length $3cm$ with same temperature that is $T_{H_2} = 1000K$. The entire domain for first case consists of 804 triangular domain elements and 84 triangular boundary elements, see Fig. 2.1(c). For the second case when heater length is increase by one unit that is 3cm, the still geometry has ten (10) boundaries and ten (10) vertices, see Fig. 2.1(e) but the meshing is little enhanced. The geometry with heater of length $3cm$ is meshed with 836 triangular domain elements and 84 boundary triangular elements. The Fig. 2.1(f) is plotted in this direction. The problem given by Eqs. (2.4)-(2.5) is solved by Finite Element Method and the outcomes are offered with the aid of graphs. In detail, Figs. 2.2-2.6 are plotted to examine the surface temperature of trapezium shape two dimensional solid geometry for various values of time. The temperature is measure in Kelvin (K) and time in second (s). To be more specific, Fig. 2.2 gives the surface temperature of solid shape trapezium surface at $t = 0.2s$. One can see that the circumference of inner cylinder and heater with length transfer heat with same magnitude. From lower wall it was expected that the heat transfer towards outer surface of inner cylinder. From Fig. 2.3 it can be seen that after few seconds the transfer of heat via both heated lower wall and outer surface of cylinder form a union. At this moment heat covers more than 80% portion of cavity. Fig. 2.4 and Fig. 2.5 shows the heat transfer for the value of time $t = 50s$.Fig. 2.4 reports the surface temperature visualization while the Fig. 2.5 identifies the strength of temperature at different portion of cavity with the aid of isotherms plots. Fig. 2.6 and Fig. 2.7 shows the results when we have taken a heater of length 3cm being incorporated at lower wall of cavity. In detail, the Fig. 2.6 shows the surface temperature for the time t=50sand the corresponding isotherms are provided in Fig. 2.7.From these figures it is concluded that the significant amount of heat transfer is observed for the large length of heater with same temperature as compared to heater having lesser length for each step of time (t).

2.4 Graphical outcomes

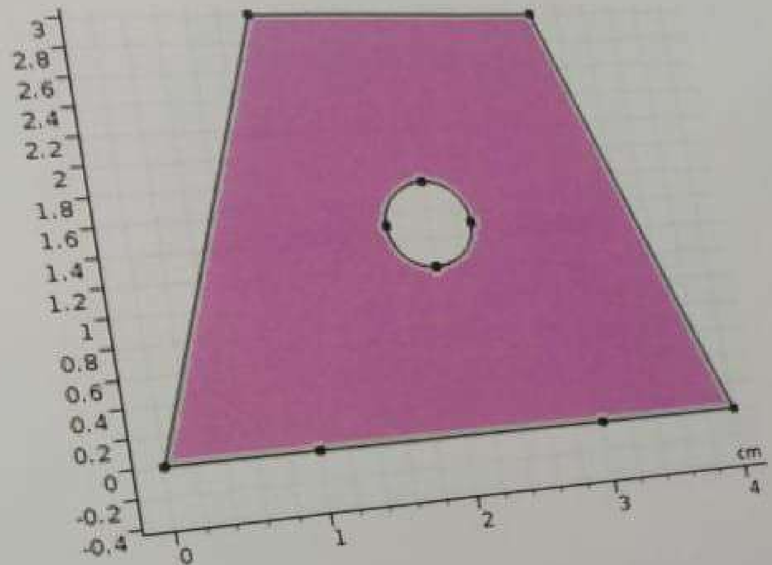

Fig. 2.1(b). Trapezium domain for H_1.

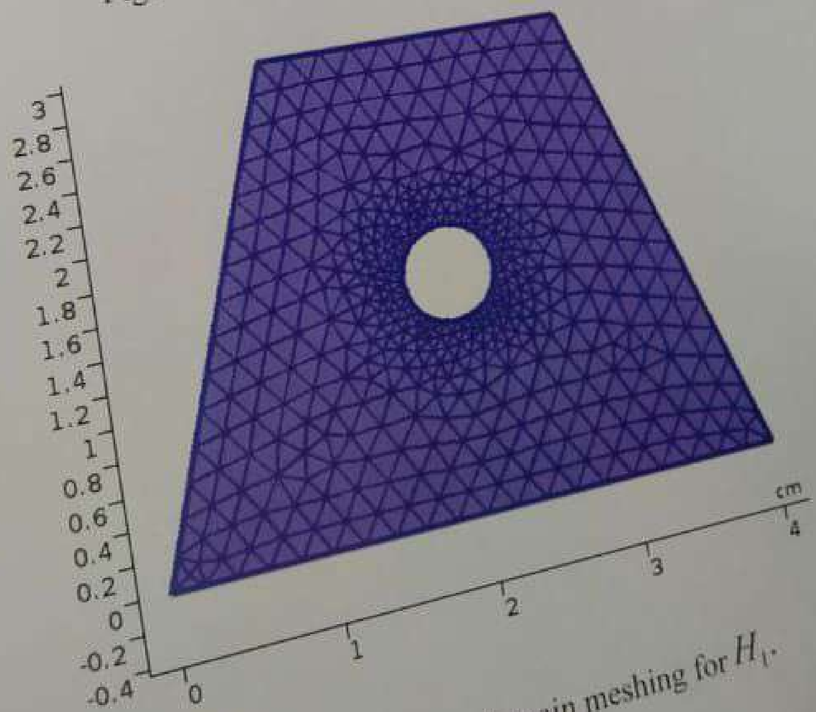

Fig. 2.1(c). Trapezium domain meshing for H_1.

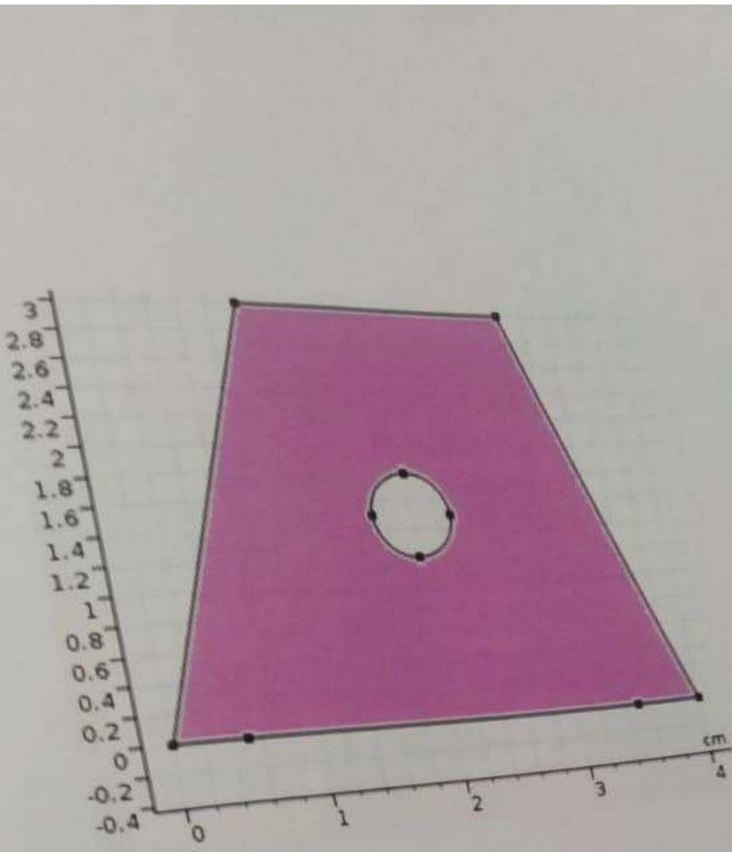

Fig. 2.1(e). Trapezium domain for H_2.

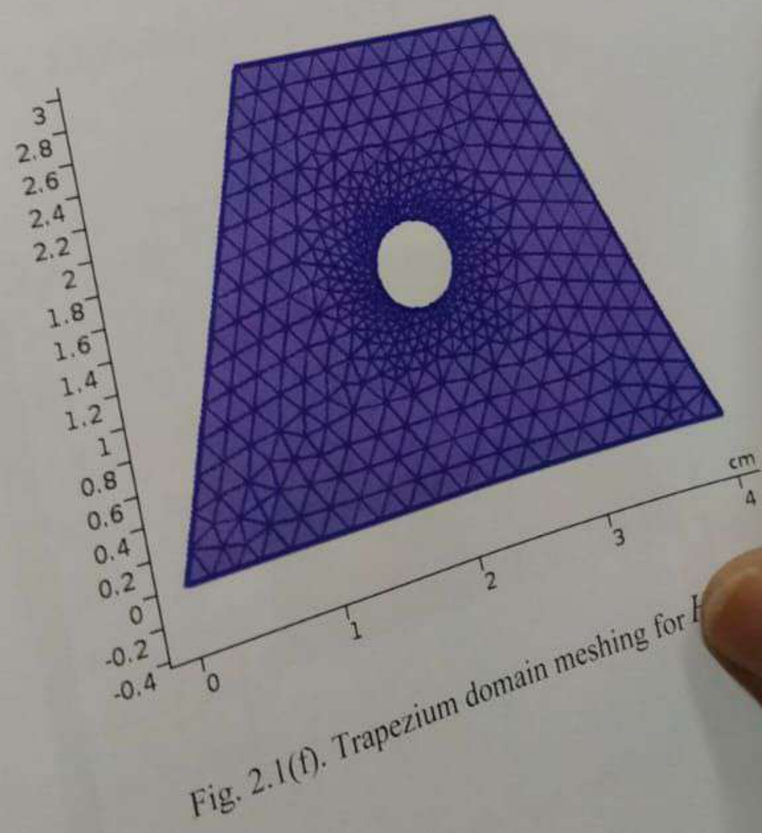

Fig. 2.1(f). Trapezium domain meshing for H

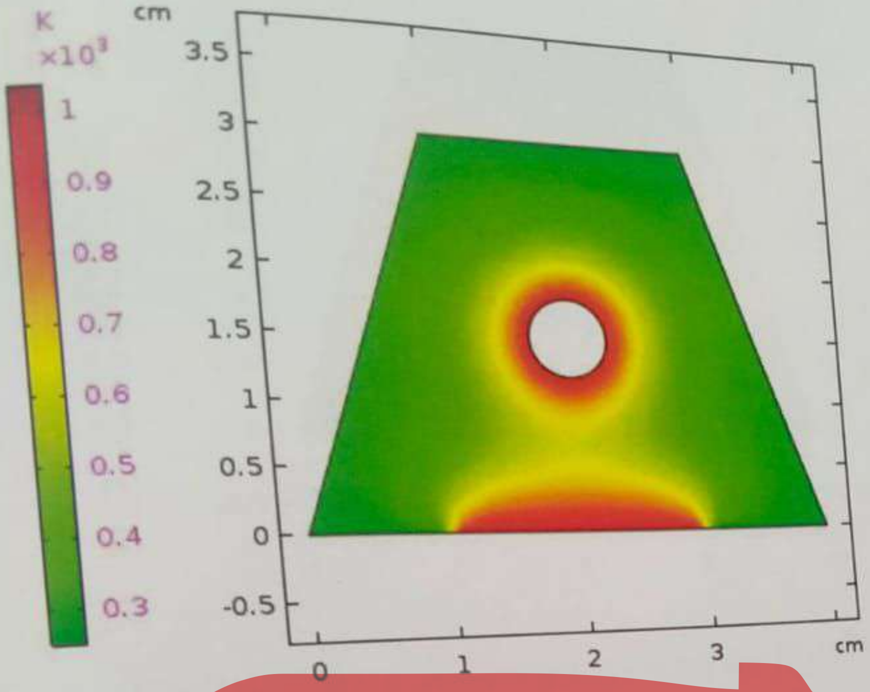

Fig. 2.2. Surface temperature at $t = 0.2s$ for H_1.

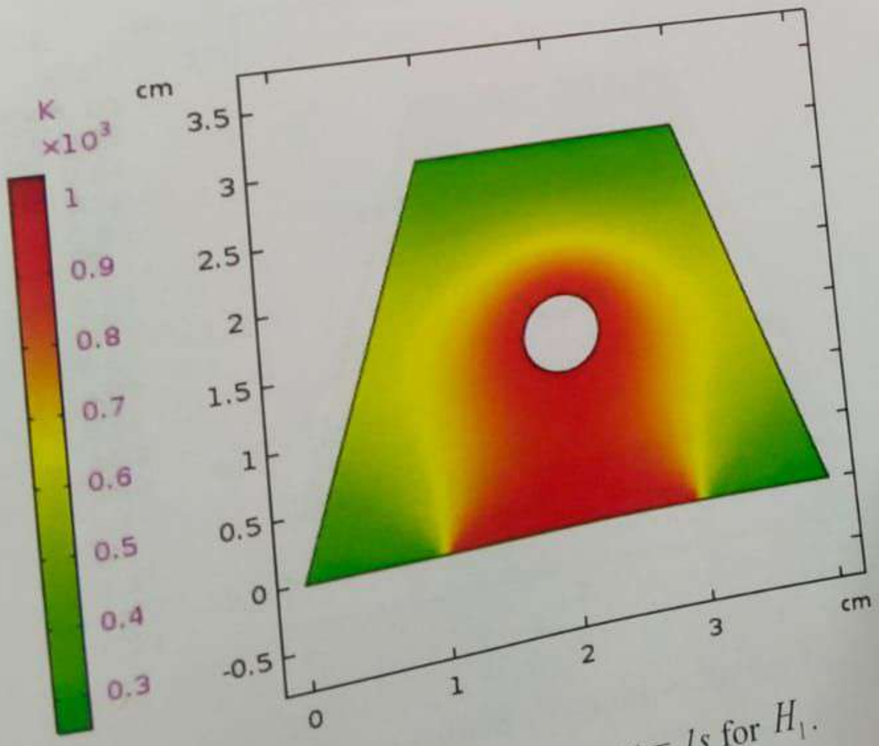

Fig. 2.3. Surface temperature at $t = 1s$ for H_1.

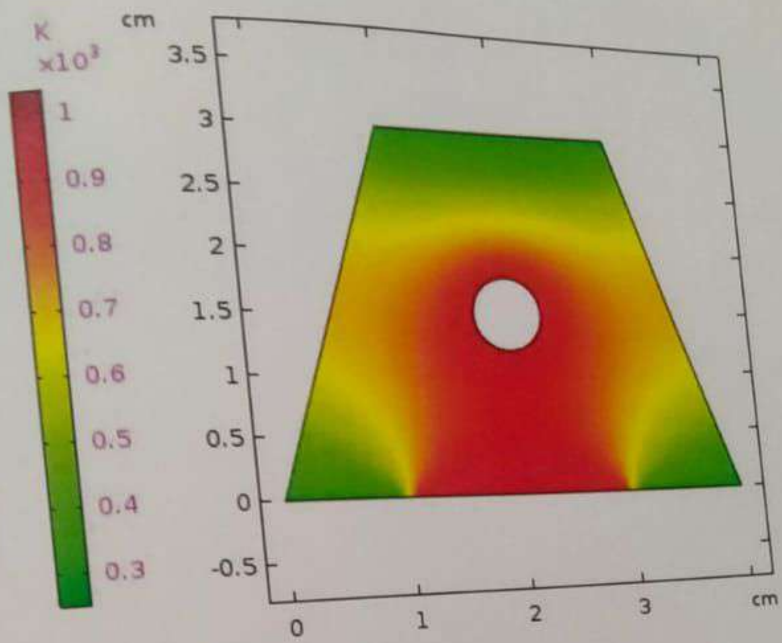

Fig. 2.4. Surface temperature at $t = 50s$ for H_1.

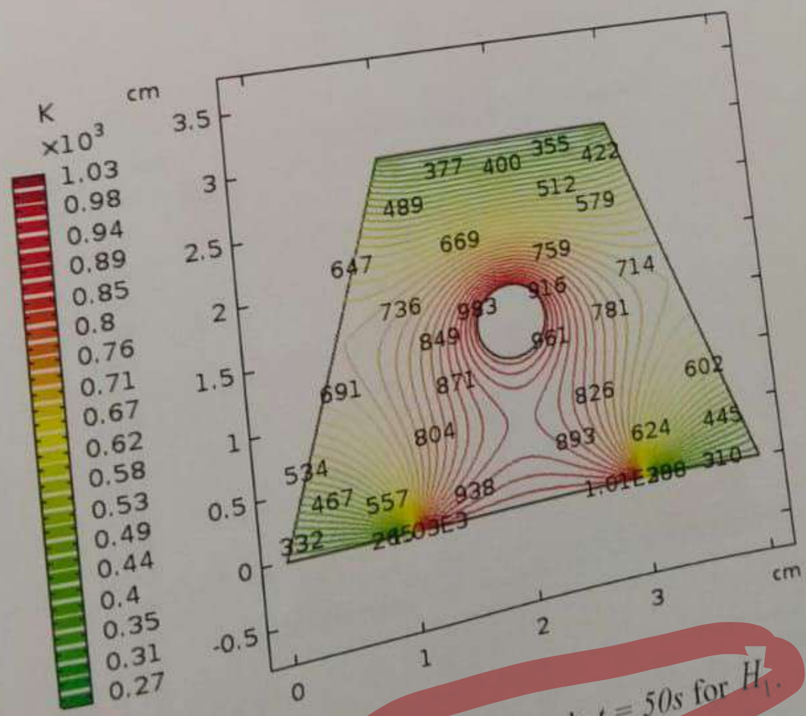

Fig. 2.5. Isotherms pattern towards $t = 50s$ for H_1.

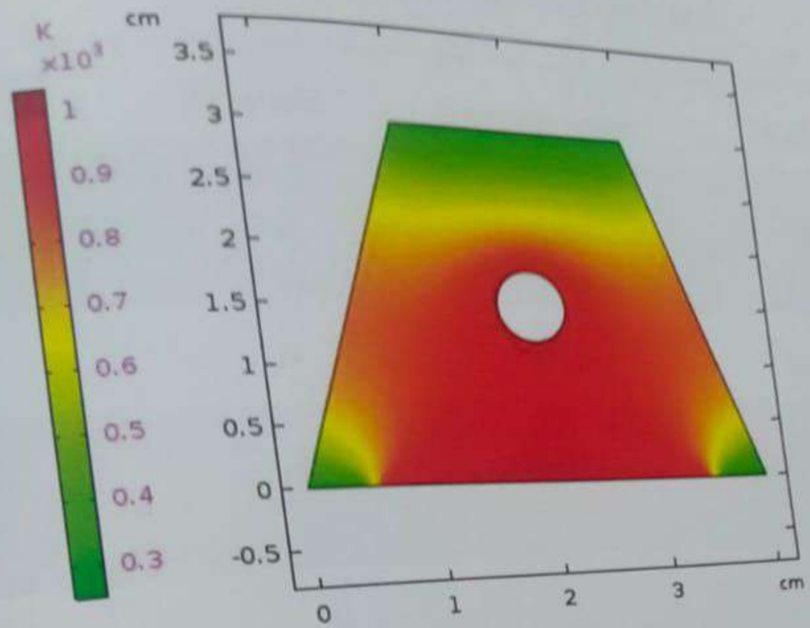

Fig. 2.6. Surface temperature at $t = 50s$ for H_2.

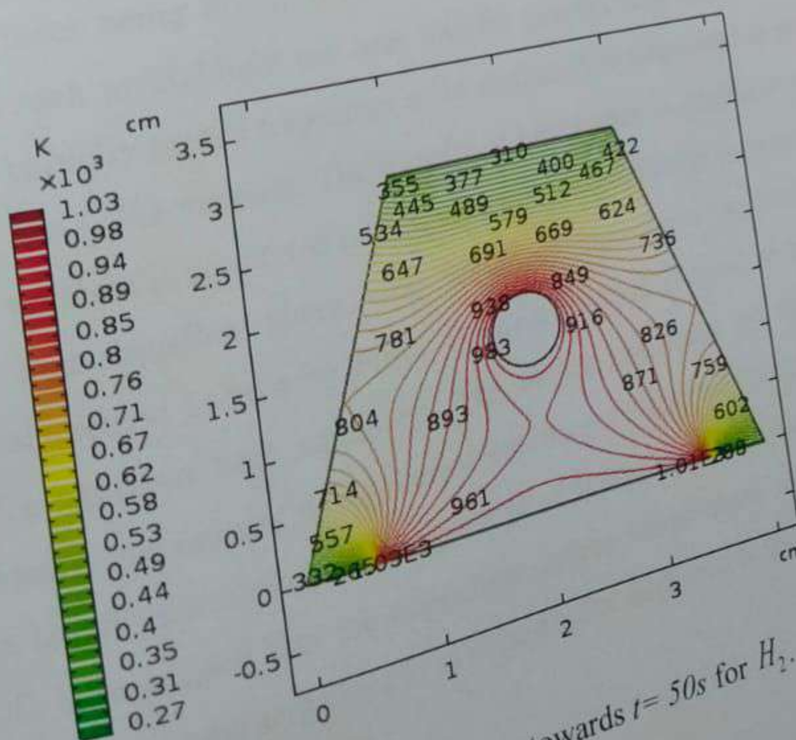

Fig. 2.7. Isotherms patterns towards $t= 50s$ for H_2.

Table 1. Electrical properties of Aluminum 6063-T83.

Property	Symbols	Numeric value	Unit
Heat capacity at constant pressure	c_1	900	J/(kg.K)
Density	ρ	2700	kg/m^3
Thermal conductivity	k	201	W/(m.K)

2.5 Conclusion

The current pagination contains a first step towards heat transfer visualization in a convoluted surfaces being involved in daily life. In this attempt we execute a numerical approach to highlight the heat transfer process in a two dimensional centrally hole partially heated trapezium solid surface. The trapezium is made up of Aluminum 6063-T83 material. The transfer of heat is due to conduction mode. The physical problem is converted into mathematical formulation in terms of two dimensional heat equation along with boundary constrains. A well-reputed computational method known by Finite Element Method is used to report the solution of established heat equations. The obtain outcomes are offered via graphical trends. For novelty enhancement comparative analysis is also reported for distinct length heaters being incorporated at lower wall of trapezium solid geometry. It is concluded that the trapezium surface temperature shows inciting values for the heater with large length as compared to heater with lesser length.

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4	"Heat Conduction", Springer Nature, 2008 Publication	%1
5	english.eagetutor.com Internet Source	%1
6	Khalil Ur Rehman, M.Y. Malik, Mostafa Zahri, Qasem M. Al-Mdallal, Mohammed Jameel, M. Imran Khan. "Finite element technique for the analysis of buoyantly convective multiply connected domain as a trapezium enclosure with heated circular obstacle", Journal of Molecular Liquids, 2019 Publication	%1