(1)

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Assignment 4
PDEs Fourier Series,

Find the fourier Series of following ftn?

(a) $f(x) = \begin{cases} x & -\pi < x < 0 \\ h & 0 < x < \pi \end{cases}$ h is constant

$$a_0 = \frac{1}{L}\int_{L}^{u} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} h\,dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{0}{2} - \frac{(\pi)^2}{2}\right] + \frac{h}{\pi}\left[\pi - 0\right]$$

$$a_0 = -\frac{\pi}{2} + h$$

$$a_k = \frac{1}{L}\int_{L}^{u} f(x)\cos\frac{k\pi x}{L}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cos kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} h\cos kx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos kx\,dx + h\int_{0}^{\pi}\cos kx\,dx\right.$$

$$= \frac{1}{\pi}\left[\frac{1}{k^2}\left(1 - \cos k\pi\right) + \frac{h}{k}\left(\sin k\pi - 0\right)\right]$$

$$a_k = \frac{1}{\pi}\left[\frac{1}{k^2} + (-1)(-1)^k\frac{1}{k^2}\right] = \frac{1}{\pi}\left[\frac{1}{k^2} + (-1)^{k+1}\frac{1}{k^2}\right]$$

$$b_k = \frac{1}{L}\left[\int_{L}^{u} f(x)\sin\frac{k\pi x}{L}\,dx\right]$$

$$b_k = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin x\,dx + \int_{0}^{\pi} h\sin kx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin kx\,dx + h\int_{0}^{\pi}\sin kx\,dx\right]$$

(2)

$$= \frac{1}{\pi}\left[\frac{\pi}{k}(-1)^k + \frac{1}{k^2}\sin kx\Big|_{-\pi}^{0} + \frac{h}{k}\left[(-1)^{k+1}+1\right]\right]$$

$$b_k = \frac{1}{\pi}\left[\frac{\pi(-1)^k}{k} + \frac{h}{k}\left[(-1)^{k+1}+1\right]\right]$$

$$f(x) \approx -\frac{\pi}{4} + \frac{h}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi}\left((-1)^{k+1}+1\right)\cos kx + \frac{(-1)^k}{k}$$

$$+ \frac{h}{k\pi}\left((-1)^{k+1}+1\right)\sin kx$$

(b) $f(x) = x + \sin x$

Since function is odd, so we can only find fourier series of sin

$$f(x) \approx \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{L}$$

$$b_k = \frac{1}{L}\int_{L}^{L} f(x)\sin\frac{k\pi x}{L}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\sin x)\sin kx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin kx + \int_{-\pi}^{\pi}\sin x\sin kx\,dx\right]$$

$$= \int_{-\pi}^{\pi} x\sin kx\,dx \quad \text{— (A)}$$

$$I = \int_{-\pi}^{\pi}\sin x\sin kx\,dx \quad \text{— (B)}$$

Solving (A) $\int_{-\pi}^{\pi} x\sin kx\,dx$

$f(x) = x \qquad g(x) = \frac{1}{k}(-\cos kx)$

$f'(x) = 1 \qquad g'(x) = \sin kx.$

③

$$= -x\frac{1}{k}\cos kx \Big|_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos kx\,dx$$

$$= \frac{-\pi}{k}(-1)^k - \frac{\pi}{k}(-1)^k + \frac{1}{k^2}(0)$$

$$= \frac{-2\pi}{k}(-1)^k$$

Solving (B)

$I = \int_{-\pi}^{\pi} \sin x \sin kx\,dx$ is an orthogonal so its equal to zero.

$$= \frac{1}{\pi}\left[-\frac{2\pi}{k}(-1)^k + 0\right]$$

$$= \frac{-2}{k}(-1)^k$$

$$f(x) \approx \sin x + \sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx$$

(c) $f(x) = e^x \qquad -\pi < x < \pi$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{\pi}(e^x)\Big|_{-\pi}^{\pi} = \frac{2\sinh\pi}{\pi}$$

$$b_k = \frac{1}{L}\int_{-\pi}^{\pi} e^x \sin kx\,dx$$

$$I = \int_{-\pi}^{\pi} \sin kx\,dx$$

$f(x) = \sin kx \qquad g(x) = e^x$

$f'(x) = \cos kx(k) \qquad g'(x) = e^x$

$$I = \sin kx\, e^x \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos kx\,(k)\,e^x\,dx$$

$$= \sin kx\, e^x \Big|_{-\pi}^{\pi} - (ke^x\cos kx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^x(-\sin kx)k^2\,dx)$$

$$= 0 - ke^x\cos kx\Big|_{-\pi}^{\pi} - k^2 I$$

(4)

$$I = \frac{1}{\pi}\left[-k e^{\pi} \cos k\pi + k e^{-\pi} \cos k\pi - k^2 I\right]$$

$$= -k\left[(-1)^k e^{\pi} - e^{-\pi}(-1)^k\right] - k^2 I$$

$$I + k^2 I = -k(-1)^k (2\sinh\pi)$$

$$I = \left(\frac{1}{1+k^2}\right)\left(-k(-1)^k 2\sinh\pi\right)$$

$$b_k = \frac{1}{\pi}\left[\frac{1}{1+k^2}\left(-k(-1)^k 2\sinh\pi\right)\right]$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x) \cos kx \, dx$$

$$I = \int_{-\pi}^{\pi} e^x \cos kx \, dx$$

$f(x) = \cos kx$ $\quad g(x) = e^x$

$f'(x) = -\sin kx (k)$ $\quad g'(x) = e^x$

$$= \left[\cos kx \, e^x \Big|_{-\pi}^{\pi} + k\int_{-\pi}^{\pi} \sin kx \, e^x \, dx\right]$$

$$= \left[e^{\pi}(-1)^k - e^{-\pi}(-1)^k + k(0) - k^2 I\right]$$

$$I(1+k^2) = (-1)^k 2\sinh\pi$$

$$a_k = \left[(-1)^k 2\sinh\pi \frac{1}{1+k^2}\right]\frac{1}{\pi}$$

$$f(x) \approx \frac{\sinh\pi}{\pi}\left[1 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{1+k^2}(\cos kx - k\sin kx)\right]$$

(d) $f(x) = 1 + x + x^2$ $\quad -\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\pi + \frac{\pi^2}{2} + \frac{\pi^3}{3} - \left(-\pi + \frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$$

5

$$= 1 + \frac{\pi}{2} + \frac{\pi^2}{3} \cancel{*} 1 \cancel{+} - \frac{\pi}{2} + \frac{\pi^2}{3}$$

$$a_0 = 2 + \frac{2\pi^2}{3}$$

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos kx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + x + x^2) \cos kx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (\cos kx + x \cos kx + x^2 \cos kx) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos kx \, dx$$

let $\frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, dx$ — (1)

$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx \, dx$ — (2) ; $\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos kx \, dx$ — (3)

By eq (1)

$$= \frac{1}{\pi} \sin kx (k) \Big|_{-\pi}^{\pi} = 0$$

By eq (2)

$$= \frac{1}{\pi} \left[x \sin kx \left(\frac{1}{k}\right) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin kx \left(\frac{1}{k}\right) dx \right.$$

$$= \frac{1}{\pi} \left[\cos kx \left(\frac{1}{k^2}\right) \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{k^2} (\cos \pi k - \cos \pi k) \right] = 0$$

By eq (3)

$$= \frac{1}{\pi}\left\{ x^2 \sin kx \, \frac{1}{k} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2 \sin kx \left(\frac{1}{k}\right) dx \right]$$

$$= \frac{1}{\pi}\left(-2x \cos kx \left(\frac{1}{k^2}\right) - \int_{-\pi}^{\pi} -2 \cos kx \frac{1}{k^2} \, dx \right) \Big]$$

$$= \frac{1}{\pi}\left\{ \frac{2}{k^2} \left(2\pi (-1)^k\right) \right] = \frac{4\pi (-1)^k}{\pi k^2}$$

$$b_k = \frac{1}{L} \int_{L}^{u} f(x) \sin kx \, dx$$

$$= \frac{1}{k} \int_{-\pi}^{\pi} (1+x+x^2) \sin kx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \sin kx \, dx + \int_{-\pi}^{\pi} x \sin kx \, dx + \int_{-\pi}^{\pi} x^2 \sin kx \, dx \right.$$

$$= \frac{1}{\pi} (-\cos kx) \frac{1}{k} \Big|_{-\pi}^{\pi} + \frac{1}{\pi}\left[x(-\cos kx) \frac{1}{k} \Big|_{-\pi}^{\pi} \right.$$

$$\left. + \int_{-\pi}^{\pi} \cos kx \frac{1}{k} \, dx \right| + \frac{1}{\pi}\left(x^2 (-\cos kx \frac{1}{k}) \right) \Big|_{-\pi}^{\pi}$$

$$\left. + \int_{-\pi}^{\pi} 2x \cos kx \frac{1}{k} \, dx \right]$$

$$= \frac{1}{\pi k}\left\{ -2\pi (-1)^k \right] + \frac{1}{\pi}\left[2 \cos kx \frac{1}{k^3} \right\} \Big|_{-\pi}^{\pi}$$

$$= -\frac{2\pi (-1)^k}{\pi k} + \frac{1}{\pi}\left(2 \cos k\pi \frac{1}{k^3} - 2 \cos k\pi \frac{1}{k^3} \right]$$

$$b_k = \frac{-2\pi (-1)^k}{\pi k}$$

$$f(x) \approx 2 + \frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \left\{ \frac{4\pi (-1)^k \cos kx}{\pi k^2} - \frac{(-2\pi (-1)^k) \sin kx}{\pi k} \right\}$$

⑦

$$f(x) \approx \frac{6k + 2k\pi^2 + 6}{3k} \quad \sum_{k=1}^{\infty} (-1)^k \left(\frac{2\cos kx}{k} - \sin kx \right)$$