1D HEAT CONDUCTION IN A UNIFORM ROD WITH DIRICHLET'S BOUNDARY CONDITIONS (Separation of Variable)

Separation of Variable Solution

To solve the heat equation $\partial u/\partial t = \alpha * \partial^2 u/\partial x^2$ with the given initial and boundary conditions using separation of variables, we assume that the solution can be expressed as a product of two separate functions, one dependent on time (t) and the other dependent on the spatial variable (x). Let's denote the solution as

$$u(x, t) = X(x) * T(t).$$

We can then substitute this into the heat equation and rearrange to obtain:

$$(1/\alpha) * T'(t)/T(t) = X''(x)/X(x)$$

Since the left side of the equation depends only on time (t) and the right side depends only on the spatial variable (x), they must be equal to a constant, which we'll denote as $-\lambda^2$:

$$(1/\alpha) * T'(t)/T(t) = X''(x)/X(x) = -\lambda^2$$

Now, we have two separate ordinary differential equations (ODEs) to solve:

1. $T'(t)/T(t) = -\alpha\lambda^2$
2. $X''(x)/X(x) = -\lambda^2$

Solving the first ODE, we find:

$$T'(t)/T(t) = -\alpha\lambda^2$$
$$\int T'(t)/T(t)\, dt = -\alpha\lambda^2 \int dt$$
$$\ln|T(t)| = -\alpha\lambda^2 t + C_1$$
$$T(t) = C_2 * \exp(-\alpha\lambda^2 t)$$

Here, C_1 and C_2 are constants determined by the initial condition.

Solving the second ODE, we find:

$$X''(x)/X(x) = -\lambda^2$$
$$X''(x) + \lambda^2 X(x) = 0$$

The general solution to this ODE is given by:

$$X(x) = A * \cos(\lambda x) + B * \sin(\lambda x)$$

Here, A and B are constants determined by the boundary conditions.

Now, we apply the boundary conditions:

$$u(0,t) = X(0) * T(t) = 0$$

This implies $X(0) = 0$, which gives us $A = 0$.

$$u(L,t) = X(L) * T(t) = 1$$

This implies $X(L) = 1$, which gives us $B * \sin(\lambda L) = 1$.

From this, we can solve for λL:

$$\lambda L = \sin^{(-1)}(1/B)$$

Now, we have the expressions for $X(x)$ and $T(t)$:

$$X(x) = B * \sin(\lambda x)$$
$$T(t) = C_2 * \exp(-\alpha\lambda^2 t)$$

Combining them, the solution to the heat equation with the given initial and boundary conditions is:

$$u(x, t) = X(x) * T(t) = B * \sin(\lambda x) * C_2 * \exp(-\alpha\lambda^2 t)$$

To determine the constants B and C_2, we need to use the initial condition:

$$u(x, 0) = f(x) = B * \sin(\lambda x) * C_2$$

Comparing this with the given initial condition $u(x, 0) = f(x)$, we can determine B and C_2 by matching the functional form of $f(x)$.

Note that λ can take different values determined by the boundary condition $\lambda L = \sin^{(-1)}(1/B)$. The sum of these solutions, for different λ values, will give the complete solution to the heat equation.

Since this should hold for all x, we can equate the terms without x:

$$f(x) = B * C_2$$

Therefore, we have $B * C_2 = f(x)$.

Now, substituting $B = f(x) / C_2$ into the spatial part of the solution, we get:

$$X(x) = (f(x) / C_2) * \sin(\lambda x)$$

Hence, the unique solution to the heat equation with the given initial and boundary conditions is:

$$u(x, t) = X(x) * T(t)$$

$$u(x, t) = (f(x) / C_2) * \sin(\lambda x) * C_2 * \exp(-\alpha\lambda^2 t)$$

$$u(x, t) = f(x) * \sin(\lambda x) * \exp(-\alpha\lambda^2 t)$$

where $\lambda L = \sin^{-1}(1 / (f(x) / C_2))$.

Please note that the exact value of λ will depend on the specific function f(x) and the boundary conditions.