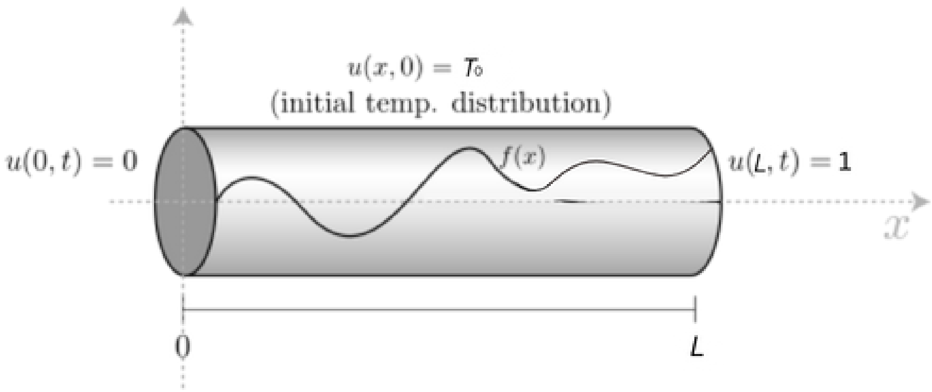
1D HEAT CONDUCTION IN A UNIFORM ROD WITH DIRICHLET'S BOUNDARY CONDITIONS (Separation of Variable)

Heat Equation (One Dimension)

Heat equation is a *parabolic* partial differenital equaiton. This heat equation in one dimension is used to model the *temprature distribution* in a one dimensional rod.



The equation is given by:

$$s\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

- $u(x,t)$ is the temprature at position x (spatial variable) and time t
- $\frac{\partial u}{\partial t}$ is the partial derivative with respect to time t
- k is the rate of heat transfer in a material
- $\frac{\partial^2 u}{\partial x^2}$ is the second partial derivaitve of u with respect ot spatial variable x

Initial & Boundary Conditions (IBCs)

This equation describes the dissipation of heat for $0 \le x \le L$ and $t \ge 0$. Our goal is to solve it for the temperature $u(x,t)$.

Initially the temperature is a nonzero constant, so the initial condition is

$$u(x,0) = T_0$$

Also, the temperature is zero at the left boundary, and nonzero at the right boundary, so the boundary conditions are

$$u(0,t) = 0 \quad ; \quad u(L,t) = 1$$

Steps to Solve at Matlab

1. Code the equation
2. Initial conditions & boundary conditions

3. Select a suitable solution mesh before calling the solver `pdepe`

Code Equation

Before you can code the equation, you need to make sure that it is in the form that the `pdepe` solver expects:

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right).$$

In this form, the heat equation is

$$1 \cdot \frac{\partial u}{\partial t} = x^0 \frac{\partial}{\partial x}\left(x^0 \frac{\partial u}{\partial x}\right) + 0.$$

So the values of the coefficients are as follows:

- $m = 0$
- $c = 1$
- $f = \frac{\partial u}{\partial x}$
- $s = 0$

The value of m is passed as an argument to `pdepe`, while the other coefficients are encoded in a function for the equation, which is

```
function [c,f,s] = heatpde(x,t,u,dudx)
c = 1;
f = dudx;
s = 0;
end
```

(Note: All functions are included as local functions at the end of the example.)

Code Initial Condition

The initial condition function for the heat equation assigns a constant value for u_0. This function must accept an input for x, even if it is unused.

```
function u0 = heatic(x)
u0 = sin;
end
```

Code Boundary Conditions

The standard form for the boundary conditions expected by the `pdepe` solver is

$$p(x,t,u) + q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0.$$

Written in this form, the boundary conditions for this problem are

$u(0,t) + (0 \cdot f) = 0,$

$(u(L,t) - 1) + (0 \cdot f) = 0.$

So the values for p and q are

- $p_L = u_L, \qquad q_L = 0.$
- $p_R = u_R - 1, \quad q_R = 0.$

The corresponding function is then

```
function [pl,ql,pr,qr] = heatbc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
pr = ur - 1;
qr = 0;
end
```

Select Solution Mesh

Use a spatial mesh of 20 points and a time mesh of 30 points. Since the solution rapidly reaches a steady state, the time points near $t = 0$ are more closely spaced together to capture this behavior in the output.

```
L = 1;
x = linspace(0,L,20);
t = [linspace(0,0.05,20), linspace(0.5,5,10)];
```

Solve Equation

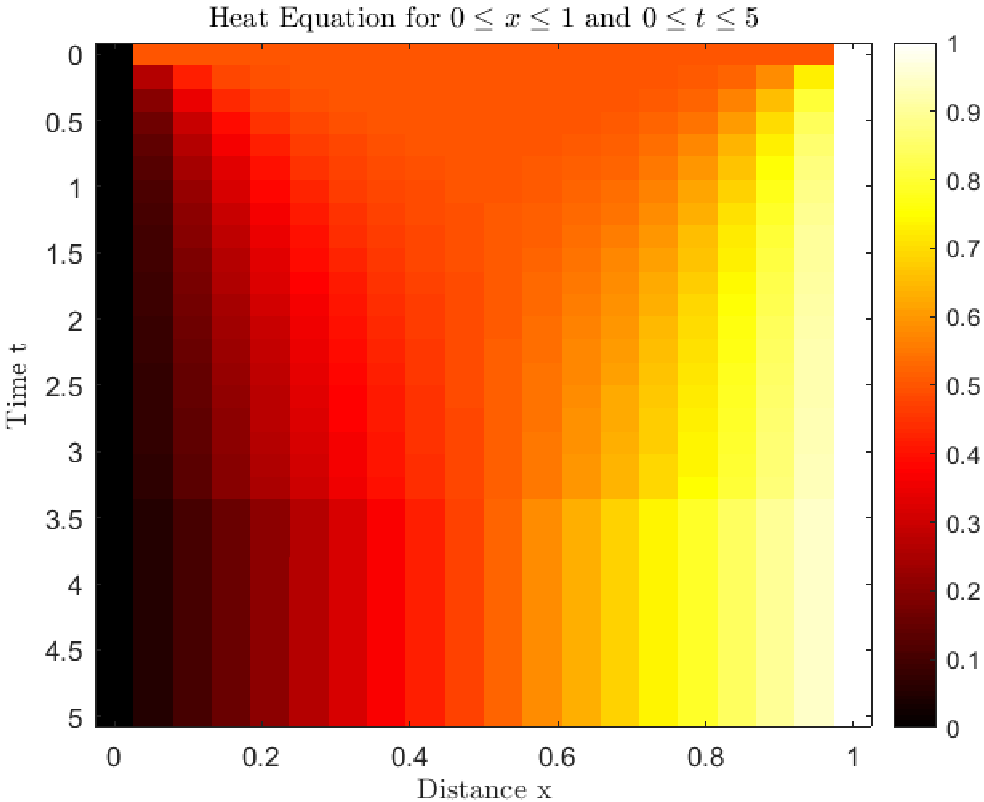
Finally, solve the equation using the symmetry m, the PDE equation, the initial condition, the boundary conditions, and the meshes for x and t.

```
m = 0;
sol = pdepe(m,@heatpde,@heatic,@heatbc,x,t);
```

Plot Solution

Use `imagesc` to visualize the solution matrix.

```
colormap hot
imagesc(x,t,sol)
colorbar
xlabel('Distance x','interpreter','latex')
ylabel('Time t','interpreter','latex')
title('Heat Equation for $0 \le x \le 1$ and $0 \le t \le 5$','interpreter','latex')
```


Heat Equation for $0 \leq x \leq 1$ and $0 \leq t \leq 5$
Time t
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1