ラグランジュ乗数を λ_i, s.t. $i = 1, \cdots, n$ とする．ラグランジュ関数を L とおくと,

$$L = \sum_{c=1}^{C} \sum_{i=1}^{n} (u_{ci})^{\theta} d_{ci} + \sum_{c=1}^{C} \lambda_i \left(\sum_{c=1}^{c} u_{ci} - 1 \right)$$

最適性の必要条件は,

$$\frac{\partial L}{\partial u_{ci}} = \theta (u_{ci})^{\theta - 1} d_{ci} + \lambda_i = 0$$

$x_i \neq b_c$ のとき

$$u_{ci} = \left(\frac{-\lambda_i}{\theta d_{ci}} \right)^{\frac{1}{\theta - 1}} \tag{1}$$

$$\left(\frac{-\lambda_i}{\theta} \right)^{\frac{1}{\theta - 1}} = u_{ci} (d_{ci})^{\frac{1}{\theta - 1}} \tag{2}$$

$\sum_{c=1}^{C} u_{ci} = 1$ が成り立つので

$$\sum_{c=1}^{C} \left(\frac{-\lambda_i}{\theta d_{ci}} \right)^{\frac{1}{\theta-1}} = 1 \tag{3}$$

(3) において c を l とすると，

$$\sum_{l=1}^{C} \left(\frac{-\lambda_i}{\theta d_{li}} \right)^{\frac{1}{\theta-1}} = 1 \tag{4}$$

(2)(4) より，

$$\sum_{\ell=1}^{C} u_{ci} \left(\frac{d_{ci}}{d_{li}} \right)^{\frac{1}{\theta-1}} = 1$$

$$u_{ci} = \left(\sum_{l=1}^{C} \left(\frac{d_{ci}}{d_{li}} \right)^{\frac{1}{\theta-1}} \right)^{-1}$$

$$u_{ci} = \left(\sum_{l=1}^{C} \left(\frac{\|x_i - b_c\|^2}{\|x_i - b_l\|^2} \right)^{\frac{1}{\theta-1}} \right)^{-1} \tag{5}$$

証明終

$$
\begin{aligned}
J_{fcm} &= \sum_{c=1}^{c} \sum_{i=1}^{n} u_{ci}^{\theta} d_{ci} \\
&= \sum_{c=1}^{C} \sum_{i=1}^{n} u_{ci}^{\theta} \left\| x_i - b_c \right\|^2
\end{aligned}
\tag{6}
$$

最適性の必要条件は

$$
\frac{\partial}{\partial b_c} \left(\sum_{i=1}^{n} u_{ci}^{\theta} \left\| x_i - b_c \right\|^2 \right) = \sum_{i=1}^{n} u_{ci}^{\theta} (-2) \left(x_i - b_c \right) = 0 \tag{7}
$$

$$
\sum_{i=1}^{n} u_{ci}^{\theta} x_i = \sum_{i=1}^{n} u_{ci}^{\theta} b_c \tag{8}
$$

から

$$
b_c = \frac{\sum_{i=1}^{n} u_{ci}^{\theta} x_i}{\sum_{i=1}^{n} u_{ci}^{\theta}} \tag{9}
$$

を得る.