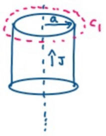
5.22 A long cylindrical conductor whose axis is coincident with the z-axis has a radius a and carries a current characterized by a current density $\mathbf{J} = \hat{\mathbf{z}}J_0/r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic field $\mathbf{H}$ for

(a) $0 \le r \le a$

(b) $r > a$



1) $I = \int_S J \cdot ds$

$I = \int_0^{2\pi}\int_0^{r} \hat{z}\frac{J_0}{r}\cdot\hat{z}\, r\,dr\,d\phi$

$I = \int_0^{2\pi} J_0 r\, d\phi = J_0 2\pi r$

$\oint_c H\cdot dl = I = \int_0^{2\pi} H_1 \cdot r\,d\phi$

$J_0 2\pi r = H_1 2\pi r$

$H_1 = J_0 \quad \vec{H}_1 = J_0\hat{\phi}$

b) $I = \int_S J\cdot ds$

$I = \int_0^{2\pi}\int_0^{a} \hat{z}\frac{J_0}{r}\cdot\hat{z}\, r\,dr\,d\phi$

$I = 2\pi J_0 a$

$I = \oint_c H\cdot dl$

$J_0 a 2\pi = \int_0^{2\pi} H \cdot r\,dr = 2\pi r H$

$H = \frac{J_0 a}{r} \quad \vec{H} = \hat{\phi}\frac{J_0 a}{r}$

7.36. The electric motor shown in Figure 7.39 has field

$H = \frac{10^6}{r}\sin 2\phi\,\hat{r}$ A/m

Calculate the flux per pole passing through the air gap if the axial length of the pole is 20 cm.

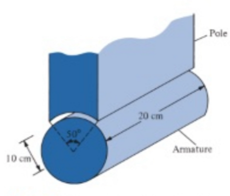


FIGURE 7.39 Electric motor pole of Problem 7.36.

$B = \mu_0 H$

$B = \mu_0 \frac{10^6}{r}\sin 2\phi\,\hat{r}$

$flux = \oint_S B\cdot ds$

$= \int_0^{0.2}\int_0^{50^\circ} \mu_0 \frac{10^6}{r}\sin 2\phi\, r\,d\phi\,dz$

$flux = \mu_0(10^6)\int_0^{0.2} dz \int_0^{50} \sin 2\phi\, d\phi$

$= \mu_0(10^6)(0.2)\left[\frac{-\cos 2\phi}{2}\right]_0^{50}$

$= 0.25\left[-\frac{\cos(100)}{2} + \frac{\cos 0}{2}\right]$

$= 0.146\ wb$

7.24 Let $\mathbf{H} = y^2\mathbf{a}_x + x^2\mathbf{a}_y$ A/m. (a) Find $\mathbf{J}$. (b) Determine the current through the strip $z = 1, 0 < x < 2, 1 < y < 5$.

7.24) $H = y^2 a_x + x^2 a_y$ A/m

a) Find J.

$J = \nabla\times H = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 0\end{vmatrix} = a_x\left(0 - \frac{\partial}{\partial z}x^2\right) + a_y\left(0 - \frac{\partial}{\partial z}y^2\right) + a_z\left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right)$

$= a_z(2x - 2y) = 2(x-y)a_z$

(b) $I = \int_S J\,ds = \iint(2x-2y)\,dx\,dy = 2\int_1^5 dy\int_0^2 x\,dx - 2\int_0^2 dx\int_1^5 y\,dy$

$= 2(4)\left(\frac{x^2}{2}\Big|_0^2\right) - 2(2)\left(\frac{y^2}{2}\Big|_1^5\right)$

7.19 (a) State Ampère's circuit law.

(b) A hollow conducting cylinder has inner radius a and outer radius b and carries current I along the positive z-direction. Find $\mathbf{H}$ everywhere.

b) For $\rho < a$ $\int H dl = I_{enc} \longrightarrow H = 0$.

For $a < \rho < b$

$H_\phi \cdot 2\pi\rho = \frac{I\pi(\rho^2 - a^2)}{\pi(b^2 - a^2)}$

$H_\phi = \frac{I}{2\pi\rho}\left(\frac{\rho^2 - a^2}{b^2 - a^2}\right)$

For $\rho > b$, $H_\phi \cdot 2\pi\rho = I \rightarrow H_\phi = \frac{I}{2\pi\rho}$

$H_\phi = \begin{cases} 0 & \rho < a \\ \frac{I}{2\pi\rho}\left(\frac{\rho^2 - a^2}{b^2 - a^2}\right), & a < \rho < b \\ \frac{I}{2\pi\rho}, & \rho > b.\end{cases}$

7.32 Determine the magnetic flux through a rectangular loop ($a \times b$) due to an infinitely long conductor carrying current I as shown in Figure 7.37. The loop and the straight conductors are separated by distance d.

7.25 Let $\mathbf{H} = k_o\left(\frac{\rho}{a}\right)\mathbf{a}_\phi$, $\rho < a$, where k_o is a constant. (a) Find $\mathbf{J}$ for $\rho < a$. (b) Find $\mathbf{H}$ for $\rho > a$.

7.34 In free space, the magnetic flux density is

$\mathbf{B} = y^2\mathbf{a}_x + z^2\mathbf{a}_y + x^2\mathbf{a}_z$ Wb/m^2

(a) Show that $\mathbf{B}$ is a magnetic field

(b) Find the magnetic flux through $x = 1, 0 < y < 1, 1 < z < 4$.

(c) Calculate $\mathbf{J}$.

(d) Determine the total magnetic flux through the surface of a cube defined by $0 < x < 2, 0 < y < 2, 0 < z < 2$.

5.24 In a certain conducting region, the magnetic field is given in cylindrical coordinates by

$$\mathbf{H} = \hat{\boldsymbol{\phi}}\frac{4}{r}[1 - (1+3r)e^{-3r}]$$

Find the current density $\mathbf{J}$.

Ampere's Law $\nabla\times H = J$

$J = \hat{r}\left(\frac{1}{r}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial H_r}{\partial\phi}\right)$ (no H_z; H_ϕ has no z; no H_r; no H_z; no H_r)

$J = \hat{z}\left(\frac{1}{r}\right)\frac{\partial}{\partial r} rH_\phi$

$= \hat{z}\left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r} 4[1 - (1+3r)e^{-3r}]\right]$

$= \hat{z}\left(\frac{1}{r}\right)\left[4[(3)e^{-3r} + (1+3r)\cdot -3e^{-3r}]\right]$

$= \hat{z}\left(\frac{1}{r}\right) 4\left[-[3e^{-3r} - 3e^{-3r} - 9re^{-3r}]\right]$

$= \hat{z}\left(\frac{1}{r}\right) 4[+9re^{-3r}]$

$= \hat{z}\ 36e^{-3r}$ A/m^2

7.32.

$B = \frac{\mu_0 I}{2\pi\rho}a_\phi$

$\psi = \int B\,ds = \int_{\rho=d}^{d+a}\int_{z=0}^{b}\frac{\mu_0 I}{2\pi\rho}d\rho\,dz$

$= \frac{\mu_0 I b}{2\pi}\ln\frac{d+a}{d}$.

7.25

(a) $J = \nabla\times H = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)a_z$

(b) For $\rho > a$,

$\int H\cdot dl = I_{enc} = \int J\cdot ds = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{2k_o}{a}\rho\,d\rho\,d\phi$

$= \frac{2K_o}{a}(2\pi)\frac{\rho^2}{2}\Big|_0^a$

$= H_\phi 2\pi\rho = 2\pi K_o a \longrightarrow H_\phi = \frac{k_o a}{\rho}$

$H = K_o\left(\frac{a}{\rho}\right)a_\phi,\ \rho > a$

7.27. Assume in a conductor, $\boldsymbol{H} = 10^3 r^2\hat{\boldsymbol{\phi}}$ A/m. (a) Find $\boldsymbol{J}$. (b) Calculate the current through the surface $0 < r < 2, 0 < \phi < 2\pi, z = 0$.

(radius of $c = 2$) $z = 0$

$\nabla\times H = J$

$J = \hat{r}\left(\frac{1}{r}\right)\left(\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right) + \phi\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) + z\left(\frac{1}{r}\right)\left(\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial H_r}{\partial\phi}\right)$

$J = \hat{z}\left(\frac{1}{r}\right)\left(\frac{\partial}{\partial r}\right)(10^3 r^3)$

$= \hat{z}\left(\frac{1}{r}\right)(3(10^3)r^2)$

$J = 3\cdot 10^3 r\,\hat{z}$ A/m^2

b) $I = \oint H\cdot dl$

$= \int_0^{2\pi} 10^3 r^2\phi\cdot r\,d\phi$

$= 10^3 r^3 2\pi$

$= 10^3(2)^3 2\pi$

$= 50.265$ KA

or

$I = \int_S J\cdot ds$

$I = \int_0^{2\pi}\int_0^2 3\cdot 10^3(r)\hat{z}\cdot r\,dr\,d\phi$

$= \int_0^{2\pi} 3\cdot 10^3\frac{2^3}{3}\hat{z}\,d\phi$

$= 10^3 2^3 2\pi = 50.265$ KA

7.34

(a) $\nabla\cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$

showing that B satisfy Maxwell equation.

(b) $dS = dy\,dz\,a_x$

$\psi = \int B ds = \int_{z=1}^{4}\int_{y=0}^{1} y^2 dy\,dz = \frac{y^3}{3}\Big|_0^1 (z)\Big|_1^4 = 1$ Wb

(c) $\nabla\times H = J \rightarrow J = \nabla\times\frac{B}{\mu_0}$.

$\nabla\times B = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2\end{vmatrix}$, so $J = \frac{-2}{\mu_0}(z a_x + x a_y + y a_z)$

(d) Since $\nabla\cdot B = 0$,

$\psi = \int_S B\cdot ds = \int\nabla\cdot B\,dv = 0$

7.28. A cylindrical conductor of radius $a = 1$ cm carries current I which produces $H = 4r\hat{\phi}$ A/m. Find I.

$I = \oint_c H\cdot dl$

$= \int_0^{2\pi} 4r\hat{\phi}\cdot r\,d\phi$

$= 4r^2 2\pi = 4(0.01)^2(2\pi)$

$I = 2.513$ mA

$J = \nabla\cdot H$

$J = \hat{r}\left(\frac{1}{r}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial H_r}{\partial\phi}\right)$

$J = \hat{z}\left(\frac{1}{r}\right)\left(\frac{\partial}{\partial r}\right)(4r^2)$

$J = 8\hat{z}$

$I = \int_S J\cdot ds = \int_0^{2\pi}\int_0^{a} 8\hat{z}\cdot r\,dr\,d\phi$

or

Prob. 7.30

Consider the two-wire transmission line whose cross section is illustrated in Figure 7.36. Each wire is of radius 2 cm, and the wires are separated 10 cm. The wire centered at (0, 0) carries a current of 5 A, while the other centred at (10 cm, 0) carries the return current. Find H at

(a) (5 cm, 0)

(b) (10 cm, 5 cm)

Let $\vec{H} = \vec{H}_1 + \vec{H}_2$ where H_1 and H_2 are due to the wires centered at $x = 0$ and $x = 10$ cm.

(a) Magnetic field around a long, straight wire:

$\vec{H} = \frac{I}{2\pi r}\hat{\phi},\ \hat{\phi} = \hat{l}\times\hat{r}$

For $\vec{H}_1$, $r = 5$ cm, $\hat{\phi} = \hat{l}\times\hat{r} = \hat{z}\times\hat{x} = \hat{y}$

$\vec{H}_1 = \frac{5}{2\pi(5\cdot 10^{-2})}\hat{y} = \frac{50}{\pi}\hat{y}$

For $\vec{H}_2$, $r = 5$ cm, $\hat{\phi} = \hat{l}\times\hat{r} = -\hat{z}\times -\hat{x} = \hat{y}$

$\vec{H}_2 = \vec{H}_1$ $\therefore \vec{H} = 2H_1 = \frac{100}{\pi}\hat{y} = 31.83\hat{y}$

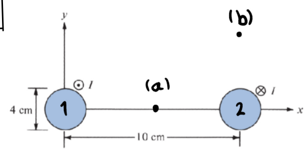


FIGURE 7.36 Two-wire line of Problem 7.30.

(b) For H_1, $r = \sqrt{10^2 + 5^2} = 5\sqrt{5}$

$\vec{r} = (10-0)\hat{x} + (5-0)\hat{y} = 10\hat{x} + 5\hat{y}$

$\hat{r} = \frac{10\hat{x} + 5\hat{y}}{\sqrt{10^2 + 5^2}} = \frac{2\hat{x} + \hat{y}}{\sqrt{5}}$

$\hat{\phi} = \hat{z}\times\frac{2\hat{x} + \hat{y}}{\sqrt{5}} = \frac{2\hat{y} - \hat{x}}{\sqrt{5}}$

$\vec{H}_1 = \frac{5}{2\pi(5\sqrt{5}\cdot 10^{-2})}\left(\frac{2\hat{y} - \hat{x}}{\sqrt{5}}\right) = -3.183\hat{x} + 6.366\hat{y}$

For $\vec{H}_2$, $r = 5$ cm, $\hat{r} = -\hat{z}\times\hat{y} = \hat{x}$

$\vec{H}_2 = \frac{5}{2\pi(5\cdot 10^{-2})}\hat{x} = 15.92\hat{x}$

$\therefore \vec{H} = \vec{H}_1 + \vec{H}_2 = 12.3\hat{x} + 6.37\hat{y}$

7.38. If $\boldsymbol{B} = \frac{2}{R^3}\cos\theta\hat{\boldsymbol{R}} + \frac{1}{R^3}\sin\theta\hat{\boldsymbol{\theta}}$ Wb/m^2, find the magnetic flux through the spherical cap $R = 1, \theta < \pi/3$.

Solution

In spherical coordinates, we can write:

$$\psi = \int \boldsymbol{B}\cdot d\boldsymbol{s}_{\boldsymbol{R}}, \quad d\boldsymbol{s}_{\boldsymbol{R}} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\boldsymbol{R}}$$

$$\psi = \iint\frac{2}{R^3}\cos\theta\,R^2\sin\theta\,d\theta\,d\phi\Bigg|_{R=1} = 2\int_0^{2\pi}d\phi\int_0^{\frac{\pi}{3}}\cos\theta\sin\theta\,d\theta$$

$$\psi = 2(2\pi)\int_0^{\frac{\pi}{3}}\sin\theta\,d(\sin\theta) = 4\pi\frac{\sin^2\theta}{2}\Big|_0^{\pi/3} = 2\pi\sin^2\left(\frac{\pi}{3}\right) = 4.7123\ \text{Wb}$$