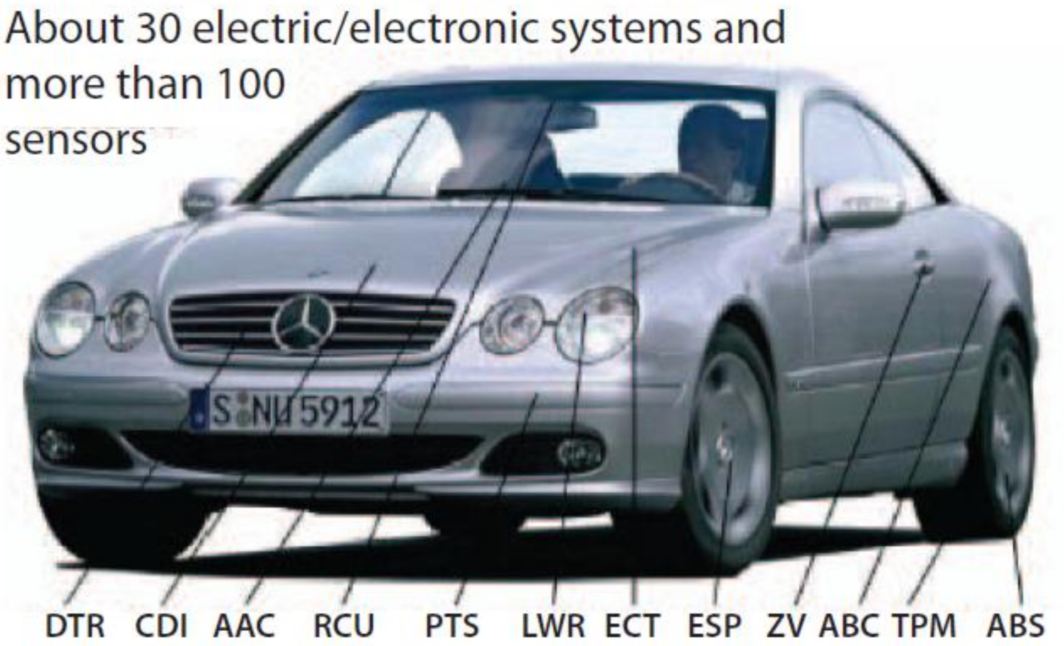

System	Abbrev.	Sensors	System	Abbrev.	Sensors
Distronic	DTR	3	Common-rail diesel injection	CDI	11
Electronic controlled transmission	ECT	9	Automatic air condition	AAC	13
Roof control unit	RCU	7	Active body control	ABC	12
Antilock braking system	ABS	4	Tire pressure monitoring	TPM	11
Central locking system	ZV	3	Elektron. stability program	ESP	14
Dyn. beam levelling	LWR	6	Parktronic system	PTS	12

Figure TF7-1: Most cars use on the order of 100 sensors. (Courtesy Mercedes-Benz.)

3. ELECTROSTATICS

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Maxwell's Equations

Gauss's law
$$\nabla \cdot \mathbf{D} = \rho_V,$$

Faraday's law
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

Gauss's law for magnetism
$$\nabla \cdot \mathbf{B} = 0,$$

Ampère's law
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

For most materials:

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

E : Electric field intensity (V/m)

D : Electric flux density (C/m^2)

ε : permittivity of a material (F/m)

H : Magnetic field intensity (A/m)

B : Magnetic flux density (Wb/m^2) or (T)

μ : Permeability of a material (H/m)

J : Conduction current density (A/m^2)

σ : Conductivity of a material (S/m)

ρ_V : Volume charge density (C/m^3)

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Maxwell's Equations

$$\nabla \cdot \mathbf{D} = \rho_v,$$

$$\nabla \times \mathbf{E} = -\cancel{\frac{\partial \mathbf{B}}{\partial t}}\ 0,$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \cancel{\frac{\partial \mathbf{D}}{\partial t}}\ 0.$$

For simple materials:

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

D = electric flux density

B = magnetic flux density

Under *static* conditions, none of the quantities appearing in Maxwell's equations are functions of time (i.e., $\partial/\partial t = 0$). ***This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that ρ_v and J are constant in time.*** Under these circumstances, the time derivatives of **B** and **D** in Eqs. (4.1b) and (4.1d) vanish, and Maxwell's equations reduce to

Electrostatics

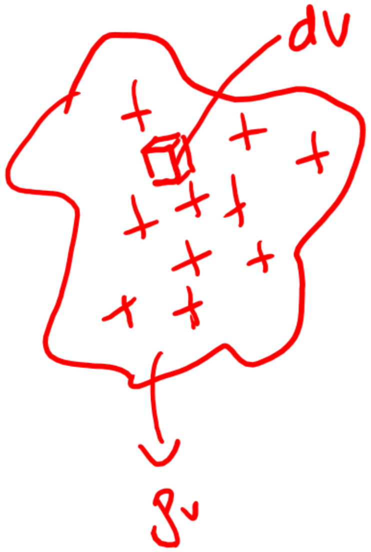
$$\nabla \cdot \mathbf{D} = \rho_v, \quad (4.2a)$$
$$\nabla \times \mathbf{E} = 0. \quad (4.2b)$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0, \quad (4.3a)$$
$$\nabla \times \mathbf{H} = \mathbf{J}. \quad (4.3b)$$

Electric and magnetic fields become decoupled under static conditions.

Charge Distributions

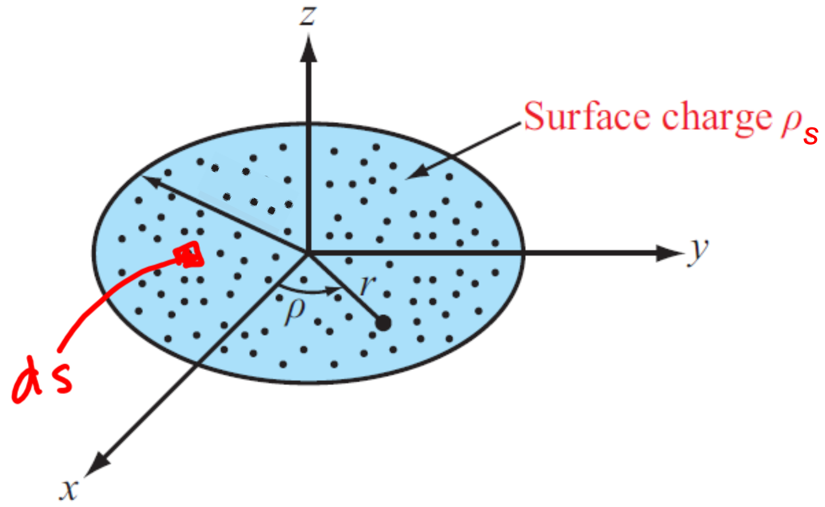


Volume charge density:

$$\rho_{\mathrm{v}} = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta q}{\Delta \mathcal{V}} = \frac{dq}{d\mathcal{V}} \qquad (\mathrm{C/m^3})$$

Total Charge in a Volume

$$Q = \int_{\mathcal{V}} \rho_{\mathrm{v}} \, d\mathcal{V} \qquad (\mathrm{C})$$



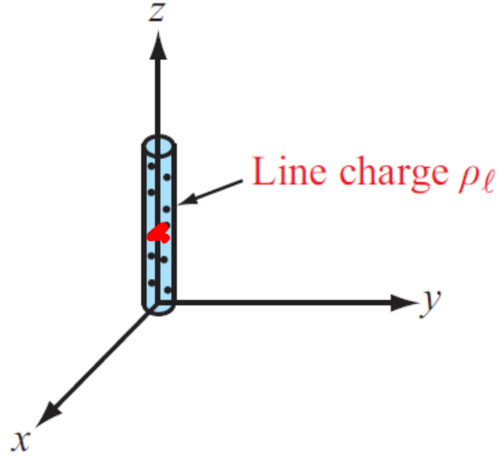
Surface and Line Charge Densities

$$\rho_{\mathrm{s}} = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\mathrm{C/m^2})$$

$$Q = \int_s \rho_s \, ds$$

$$\rho_{\ell} = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\mathrm{C/m})$$

$$Q = \int_\ell \rho_\ell \, d\ell$$



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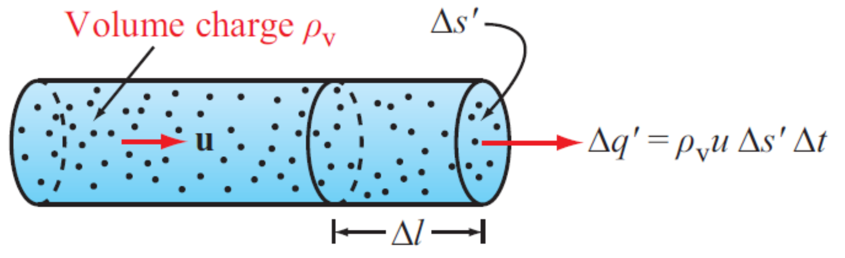
Current Density

$\Delta l = u\,\Delta t$

The amount of charge that crosses the tube's cross-sectional surface $\Delta s'$ in time Δt is therefore

$$\Delta q' = \rho_v \, \Delta V = \rho_v \, \underline{\Delta l} \, \Delta s' = \rho_v \underbrace{u \, \Delta s' \, \Delta t}. \qquad (4.8)$$



(a)

For a surface with any orientation:

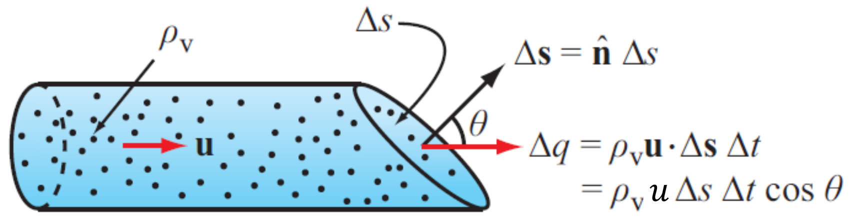
$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} \, \Delta t, \qquad (4.9)$$

where $\Delta \mathbf{s} = \hat{\mathbf{n}} \, \Delta s$ and the corresponding total current flowing in the tube is

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}, \qquad (4.10)$$

where

$$\boxed{\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2) \qquad (4.11)}$$



(b)

Figure 4-2: Charges with velocity $\mathbf{u}$ moving through a cross section $\Delta s'$ in (a) and Δs in (b).

J is the current density

$C/m^3 \times m/s = C/s \times 1/m^2 = A/m^2$

In general,

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad (\text{A}). \qquad (4.12)}$$

Total Current

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Convection vs. Conduction

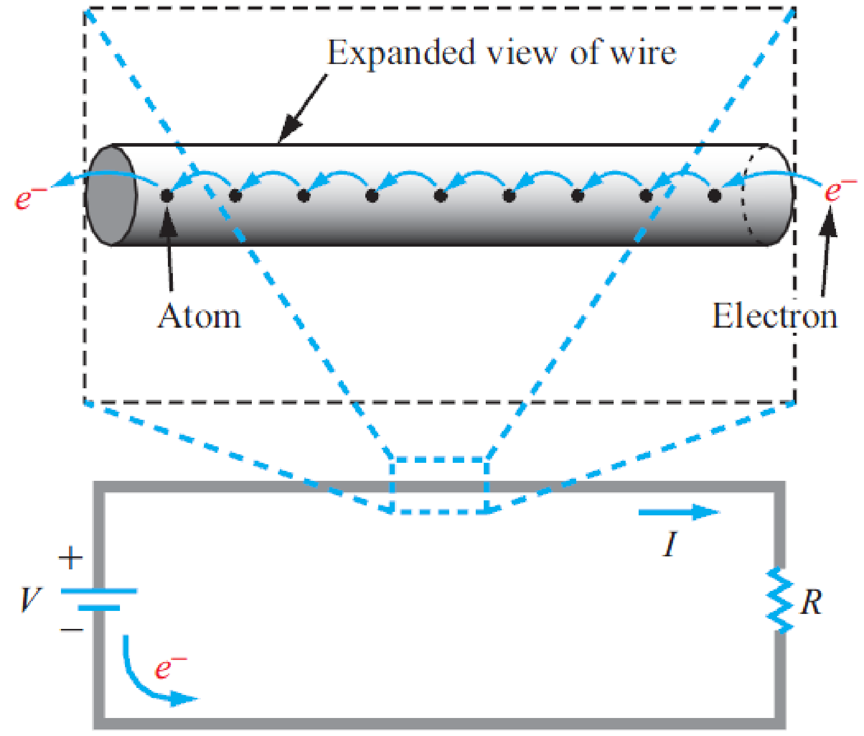
When the current is due to the movement of electrically charged matter, this is called a convection current, and $\mathbf{J}$ is the convection current density. e.g. particles in a charged cloud in the atmosphere.

When the current is due to the movement of charged particles relative to their host material, $\mathbf{J}$ is the conduction current density. e.g. a metal wire.

This movement of electrons from atom to atom constitutes a conduction current. The electrons that emerge from the wire are not necessarily the same electrons that entered the wire at the other end.

$V = IR$

Conduction current obeys Ohm's Law, whereas convection current does not.



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Coulomb's Law

Electric field at point P due to single charge, q

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2} \qquad \text{(V/m)}$$

Electric force on a test charge, q', placed at *P*

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}$$

Electric flux density **D**

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad (\text{C/m}^2)$$

where $\varepsilon = \varepsilon_r \varepsilon_0$

$$\varepsilon_0 = 8.85 \times 10^{-12} \simeq (1/36\pi) \times 10^{-9} \qquad \text{(F/m)}$$

ε_r is the relative permittivity (dielectric constant) of a material

A point charge creates an electric field in the whole space around it

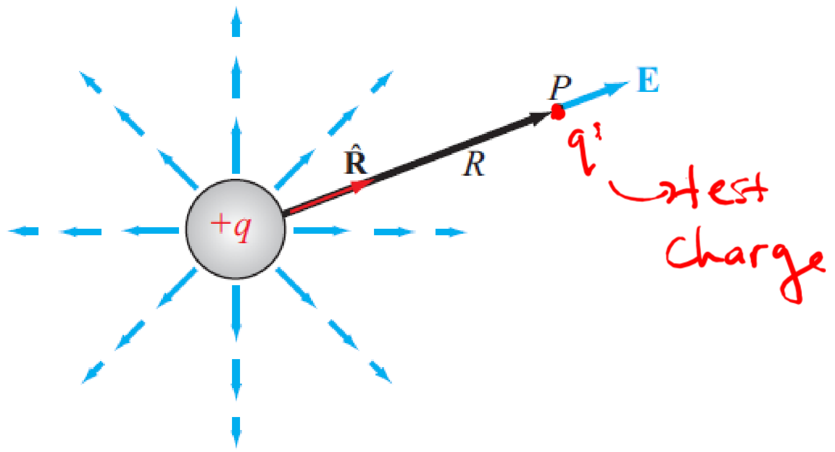

Figure 4-3: Electric-field lines due to a charge q.

*If ε is independent of the magnitude of **E**, then the material is said to be **linear** because **D** and **E** are related linearly, and if it is independent of the direction of **E**, the material is said to be **isotropic**.*

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Electric Field Due to Multiple Charges

$$\vec{E}_1 = \widehat{\vec{R}-\vec{R}_1}\,\frac{q_1}{4\pi\varepsilon|\vec{R}-\vec{R}_1|^2} = \frac{(\vec{R}-\vec{R}_1)}{|\vec{R}-\vec{R}_1|}\,\frac{q_1}{4\pi\varepsilon|\vec{R}-\vec{R}_1|^2}$$

The electric field at point P due to charge q_1 is:

$$\mathbf{E}_1 = \frac{q_1(\mathbf{R}-\mathbf{R}_1)}{4\pi\varepsilon|\mathbf{R}-\mathbf{R}_1|^3} \quad \text{(V/m)}. \qquad (4.17a)$$

Similarly, the electric field at P due to q_2 alone is

$$\mathbf{E}_2 = \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{4\pi\varepsilon|\mathbf{R}-\mathbf{R}_2|^3} \quad \text{(V/m)}. \qquad (4.17b)$$

The electric field obeys the principle of linear superposition.

Hence, the total electric field $\mathbf{E}$ at P due to q_1 and q_2 is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$
$$= \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right]. \qquad (4.18)$$

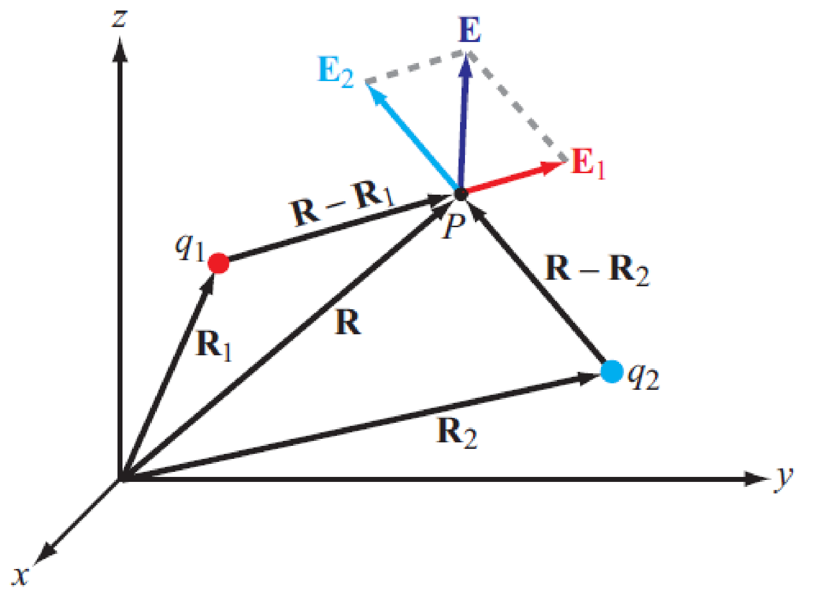

Figure 4-4: The electric field $\mathbf{E}$ at P due to two charges is equal to the vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$.

Coulomb's law for the electric field due to *N* point charges

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i(\mathbf{R}-\mathbf{R}_i)}{|\mathbf{R}-\mathbf{R}_i|^3} \quad \text{(V/m)}.$$

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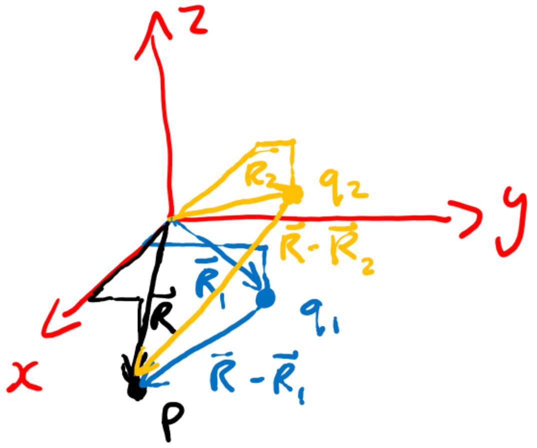
Example: Electric Field due to Two Point Charges

Example 4-3: Electric Field Due to Two Point Charges

Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates (1, 3, −1) and (−3, 1, −2), respectively. Find (a) the electric field **E** at (3, 1, −2) and (b) the force on a 8×10^{-5} C charge located at that point. All distances are in meters.



a) Total E-field is:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[q_1 \frac{(\vec{R}-\vec{R}_1)}{|\vec{R}-\vec{R}_1|^3} + q_2 \frac{(\vec{R}-\vec{R}_2)}{|\vec{R}-\vec{R}_2|^3}\right]$$

$$|\vec{R}-\vec{R}_1|^3 = \left(\sqrt{2^2+(-2)^2+1^2}\right)^3 = 27$$

$$|\vec{R}-\vec{R}_2|^3 = \left(\sqrt{6^2}\right)^3 = 216$$

$\vec{R}_1 = \hat{x} + 3\hat{y} - \hat{z}$

$\vec{R}_2 = -3\hat{x} + \hat{y} - 2\hat{z}$

$R = 3\hat{x} + \hat{y} - 2\hat{z}$

$\vec{R}-\vec{R}_1 = 2\hat{x} - 2\hat{y} - \hat{z}$

$\vec{R}-\vec{R}_2 = 6\hat{x}$

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Example: Electric Field due to Two Point Charges

Example 4-3: Electric Field Due to Two Point Charges

Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates $(1, 3, -1)$ and $(-3, 1, -2)$, respectively. Find (a) the electric field $\mathbf{E}$ at $(3, 1, -2)$ and (b) the force on a 8×10^{-5} C charge located at that point. All distances are in meters.

b) $\vec{F} = q'\vec{E}$

$$\boxed{= \frac{2\hat{x} - 8\hat{y} - 4\hat{z}}{27\pi\varepsilon_0} \times 10^{-10} \text{ N}}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[(2\times10^{-5})\frac{(2\hat{x}-2\hat{y}-\hat{z})}{27} + (-4\times10^{-5})\frac{(6\hat{x})}{216}\right]$$

$$\boxed{= \frac{\hat{x} - 4\hat{y} - 2\hat{z}}{108\pi\varepsilon_0} \times 10^{-5} \ \frac{\text{V}}{\text{m}}}$$

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Electric Field Due to Charge Distributions

Electric field due to:

a differential amount of charge $dq = \rho_v \, d\mathcal{V}'$ contained in a differential volume $d\mathcal{V}'$ is

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{4\pi\varepsilon R'^2}, \qquad (4.20)$$

$$\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2}$$

(volume distribution). (4.21a)

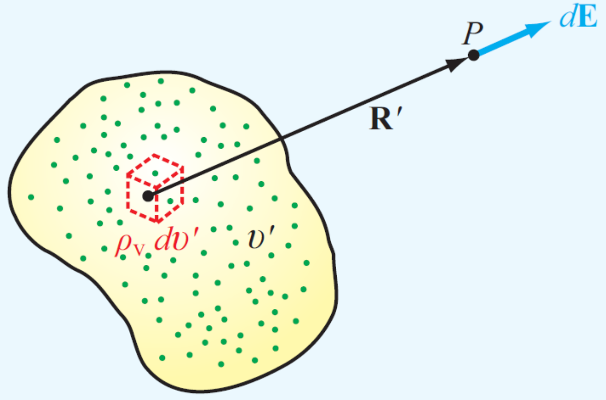

Figure 4-5 Electric field due to a volume charge distribution.

For surface distribution: $dq = \rho_s \, ds'$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(surface distribution)}, \qquad (4.21b)$$

For line distribution: $dq = \rho_\ell \, d\ell'$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2} \quad \text{(line distribution)}. \qquad (4.21c)$$

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Example: Electric Field due to a Ring of Charge

Example 4-4: Electric Field of a Ring of Charge

A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_ℓ. The ring resides in free space and is positioned in the x–y plane as shown in Fig. 4-6. Determine the electric field intensity $\mathbf{E}$ at a point $P = (0, 0, h)$ along the axis of the ring at a distance h from its center.

$$\vec{E} = \int_{\ell'} d\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{\ell'} \hat{R}' \frac{\rho_\ell \, d\ell'}{|\vec{R}'|^2}$$

Consider E-field due to differential ring segment 1 with cylindrical coordinates $(b, \phi, 0)$.

$$d\ell = b\, d\phi \,, \quad dq = \rho_\ell \, d\ell = \rho_\ell b \, d\phi$$

$$\vec{R}'_1 = -b\hat{r} + h\hat{z} \,, \quad |\vec{R}'_1| = \sqrt{b^2+h^2} \qquad \hat{R}'_1 = \frac{\vec{R}'_1}{|\vec{R}'_1|} = \frac{-b\hat{r} + h\hat{z}}{\sqrt{b^2+h^2}}$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \hat{R}'_1 \frac{\rho_\ell b \, d\phi}{|\vec{R}'_1|^2} = \frac{\rho_\ell b}{2\cancel{4}\pi\varepsilon_0} \frac{(\cancelto{0}{-b\hat{r}} + \cancelto{2h}{h}\hat{z})}{(b^2+h^2)^{3/2}} \, d\phi$$

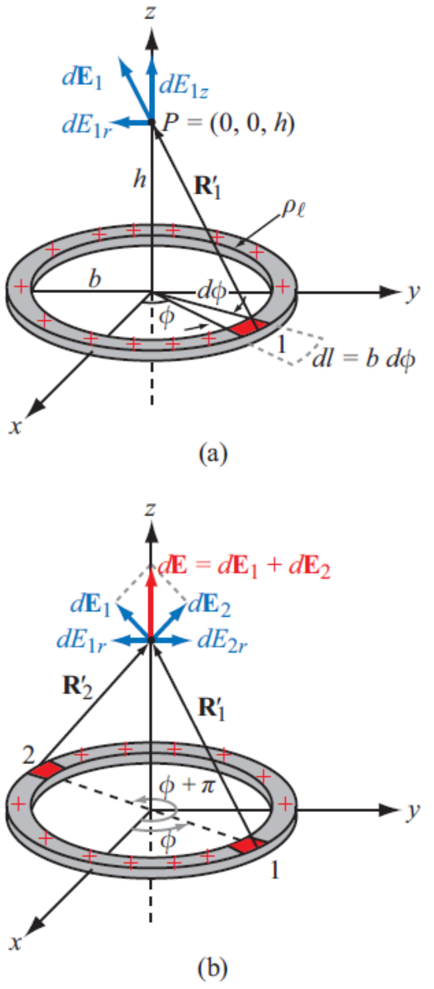

Figure 4-6: Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

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Example: Electric Field due to a Ring of Charge

Consider 2 ring segments at opposite location

$\longrightarrow$ r components cancel, z component doubles

$$d\vec{E} = d\vec{E}_1 + d\vec{E}_2 = \hat{z}\,\frac{\rho_\ell b h}{2\pi\varepsilon_0}\,\frac{1}{(b^2+h^2)^{3/2}}\,d\phi$$

$$\vec{E} = \int_{\ell'} d\vec{E} = \hat{z}\,\frac{\rho_\ell b h}{2\pi\varepsilon_0 (b^2+h^2)^{3/2}} \int_0^{\pi} d\phi \quad \pi$$

$$= \hat{z}\,\frac{\rho_\ell b h}{2\varepsilon_0 (b^2+h^2)^{3/2}} = \boxed{\hat{z}\,\frac{h}{4\pi\varepsilon_0 (b^2+h^2)^{3/2}}\,Q}$$

$$Q = 2\pi b \rho_\ell$$

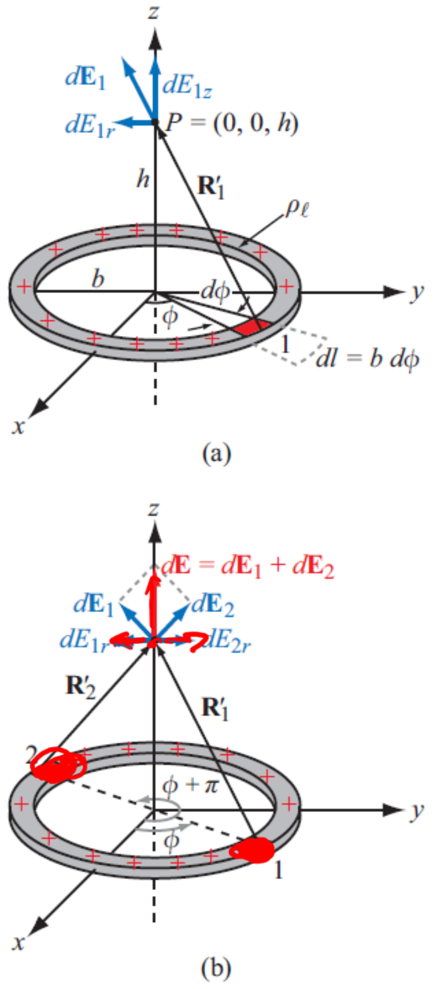

Figure 4-6: Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

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Example: Electric Field due to a Circular Disk of Charge

Example 4-5: Electric Field of a Circular Disk of Charge

Find the electric field at point P with Cartesian coordinates $(0, 0, h)$ due to a circular disk of radius a and uniform charge density ρ_s residing in the x–y plane (Fig. 4-7). Also, evaluate $\mathbf{E}$ due to an infinite sheet of charge density ρ_s by letting $a \to \infty$.

Consider disk as a set of concentric rings.

We know from previous example:

$$\vec{E}_{ring} = \hat{z}\,\frac{h}{4\pi\varepsilon_0 (b^2+h^2)^{3/2}}\,Q$$

Now, $dq = \rho_s\, ds = \rho_s\,(2\pi a\, dr)$

$$d\vec{E} = \hat{z}\,\frac{h}{2\,\cancel{4\pi}\varepsilon_0 (r^2+h^2)^{3/2}}\,(\cancel{2\pi}\,\rho_s\, r\, dr)$$

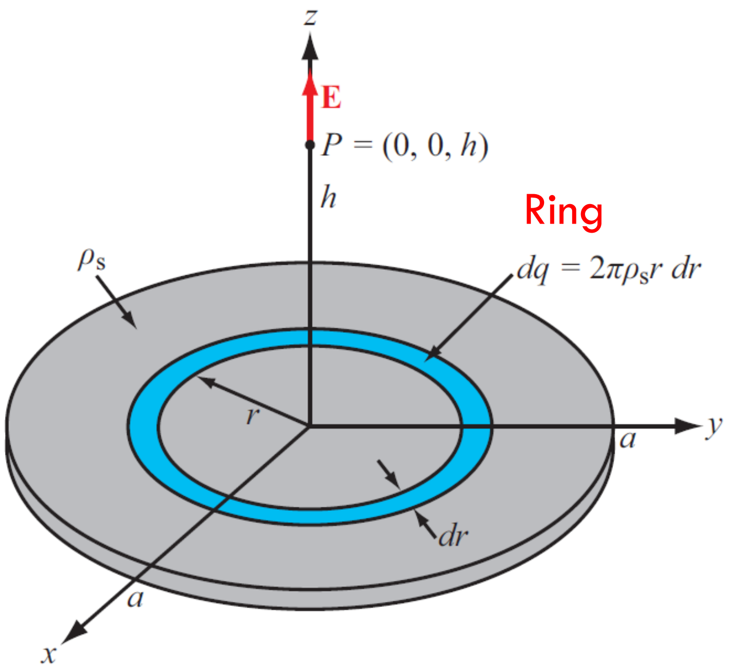

Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

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Example: Electric Field due to a Circular Disk of Charge

$$\vec{E}_{disk} = \int_S d\vec{E} = \hat{z}\,\frac{\rho_s h}{2\varepsilon_0}\int_0^a \frac{r}{(r^2+h^2)^{3/2}}\,dr$$

$$= \pm\hat{z}\,\frac{\rho_s}{2\varepsilon_0}\left[1 - \underbrace{\cancel{\frac{|h|}{\sqrt{a^2+h^2}}}}_{0}\right] \qquad \begin{array}{l} + : h>0 \\ - : h<0 \end{array}$$

For an infinite sheet of charge,

$$\vec{E} = \pm\hat{z}\,\frac{\rho_s}{2\varepsilon_0}$$

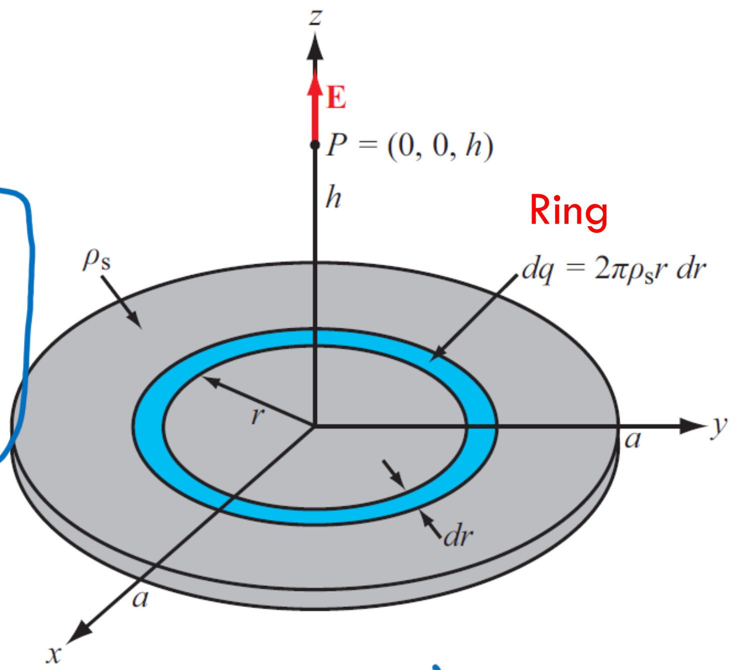
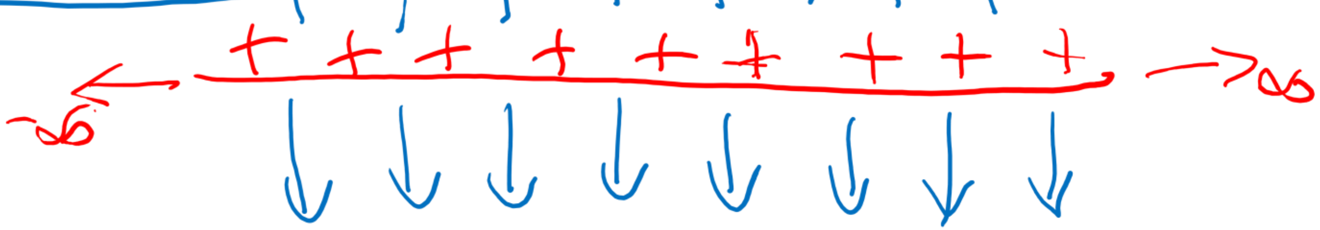


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).



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Gauss's Law

$$\nabla \cdot \mathbf{D} = \rho_v$$

(Differential form of Gauss's law),

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D} \, d\mathcal{V} = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = Q \quad \text{RHS} \qquad (4.27)$$

Application of the divergence theorem to the LHS gives:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D} \, d\mathcal{V} = \oint_S \mathbf{D} \cdot d\mathbf{s}. \quad \text{LHS} \qquad (4.28)$$

Comparison of Eq. (4.27) with Eq. (4.28) leads to

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

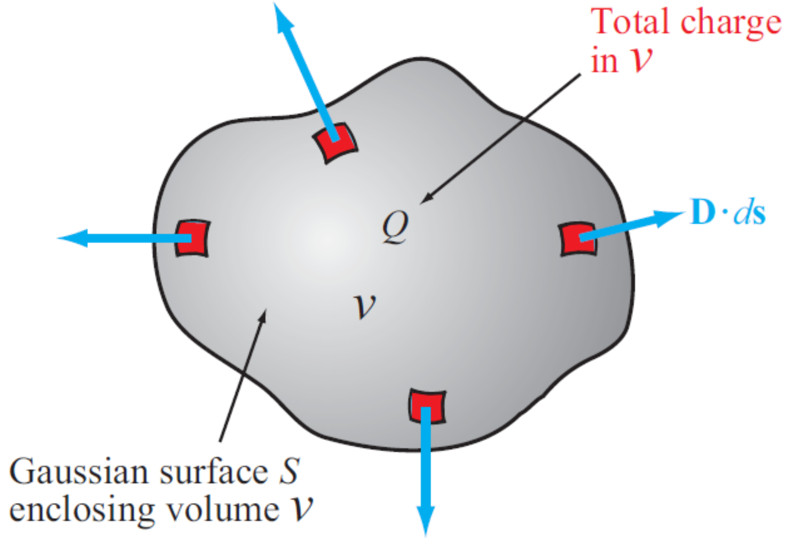

Figure 4-8: The integral form of Gauss's law states that the outward flux of **D** through a surface is proportional to the enclosed charge Q.

The integral form of Gauss's law is illustrated diagrammatically in Fig. 4-8; for each differential surface element $d\mathbf{s}$, $\mathbf{D} \cdot d\mathbf{s}$ *is the electric field flux flowing outward of* $\mathcal{V}$ *through* $d\mathbf{s}$, *and the total flux through surface* S *equals the enclosed charge* Q. *The surface* S *is called a* ***Gaussian surface***.

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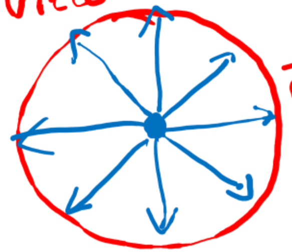
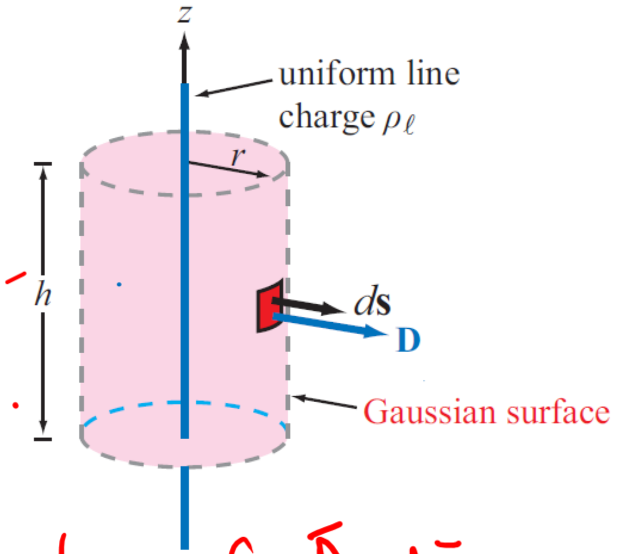
Applying Gauss's Law

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

Top view





Gauss' law gives us a convenient method to determine $\vec{D}$, but only if there's symmetry in the charge distribution.

Choose a Gaussian surface such that integration of $\vec{D} \cdot d\vec{s}$ over surface (i.e. $\oint_S \vec{D} \cdot d\vec{s}$, flux through surface) is easy.

→ Over each subsurface, $\vec{D}$ should be constant in mag. and its direction should be either parallel or normal to $d\vec{s}$

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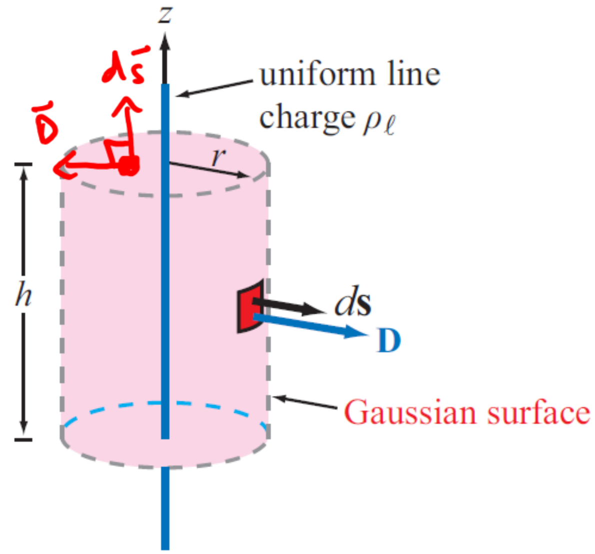
Example: Electric Field due to an Infinite Line Charge

Example 4-6: Electric Field of an Infinite Line Charge

Use Gauss's law to obtain an expression for **E** due to an infinitely long line with uniform charge density ρ_ℓ that resides along the z-axis in free space.



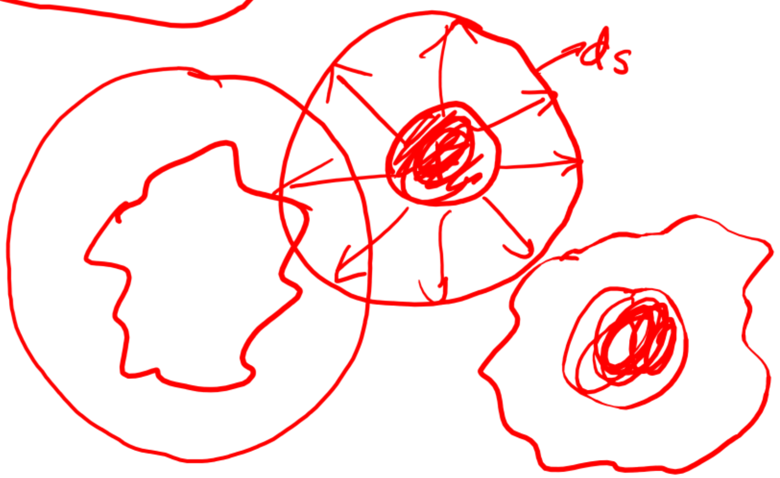
$$\boxed{\oint_S \vec{D}\cdot d\vec{s} = Q}$$

For curved surface : $\int \vec{D}\cdot d\vec{s} = \int D\,ds\,\cancel{\cos 0^\circ}^{\,1} = D\int ds = D(2\pi r h)$

For top/bottom ends: $\int \vec{D}\cdot d\vec{s} = \int D\,ds\,\cancel{\cos 90^\circ}^{\,0} = 0$

$$\oint \vec{D}\cdot d\vec{s} = 2\pi r h D = \rho_\ell h$$

$$\Rightarrow D = \frac{\rho_\ell}{2\pi r} \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon_0} = \hat{r}\,\frac{\rho_\ell}{2\pi\varepsilon_0 r}$$



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Electric Scalar Potential

When q is moved from 1 pt. to another in E-field (eg. in elec. cct.) work done is voltage diff. V between 2 pts.

*The term "voltage" is short for "voltage potential" and synonymous with **electric potential**.*

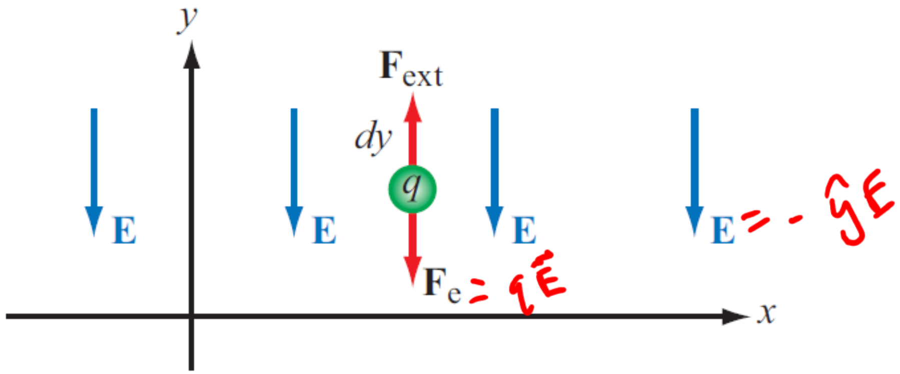

Figure 4-11: Work done in moving a charge q a distance dy against the electric field $\mathbf{E}$ is $dW = qE\,dy$.

Minimum force needed to move charge against **E** field:

$$\mathbf{F}_{ext} = -\mathbf{F}_e = -q\mathbf{E}. \qquad (4.34)$$

The work done, or energy expended, in moving any object a vector differential distance $d\mathbf{l}$ while exerting a force $\mathbf{F}_{ext}$ is

or N·m

$$dW = \mathbf{F}_{ext} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J)}. \qquad (4.35)$$

Work, or energy, is measured in joules (J). If the charge is moved a distance dy along $\hat{\mathbf{y}}$, then

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\,dy = qE\,dy. \qquad (4.36)$$

The differential electric potential energy dW per unit charge is called the ***differential electric potential*** (or differential voltage) dV. That is,

E (V/m)

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V)}. \qquad (4.37)$$

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Electric Scalar Potential

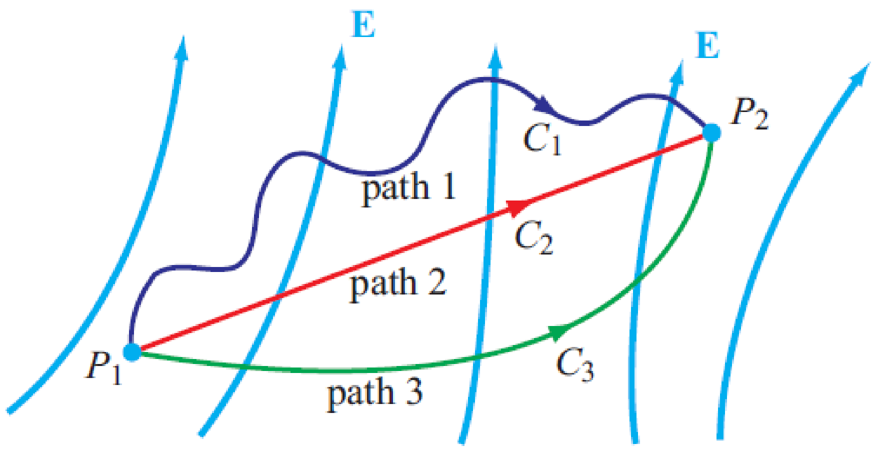

Figure 4-12: In electrostatics, the potential difference between P_2 and P_1 is the same irrespective of the path used for calculating the line integral of the electric field between them.

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{(Electrostatics).} \qquad (4.40)$$

Electric potential (OR potential diff.)
in moving q from P_1 to P_2
→ integrate $dV = -\vec{E} \cdot d\vec{l}$ along
ANY path between P_1 and P_2.

A vector field whose line integral along any closed path is zero is called a ***conservative*** *or an* ***irrotational*** *field. Hence, the electrostatic field* **E** *is conservative.*

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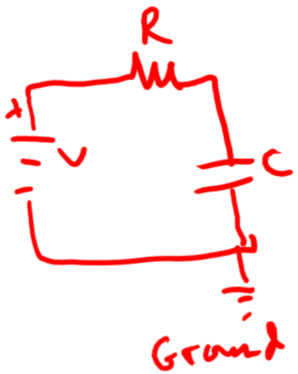
Electric Potential Due to Charges

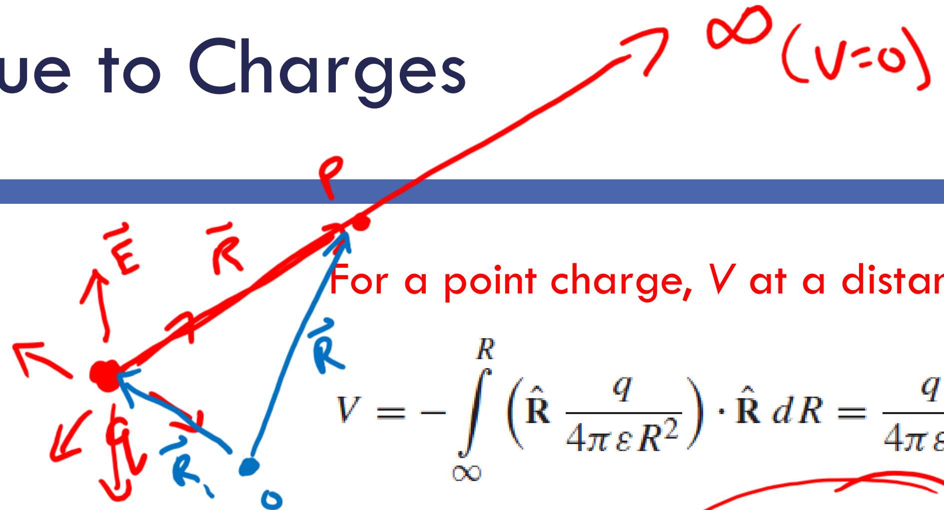


$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

In electric circuits, we usually select a convenient node that we call ground and assign it zero reference voltage. In free space and material media, we choose infinity as reference with $V = 0$. Hence, at a point P:

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \quad \text{(V)}. \qquad (4.43)$$



For a point charge, V at a distance R is:

$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}}\, dR = \frac{q}{4\pi\varepsilon R} \quad \text{(V)}. \qquad (4.45)$$

If q at arbitrary $\vec{R}_1$,

$\Rightarrow V = \dfrac{q}{4\pi\varepsilon|\vec{R}-\vec{R}_1|}$

$$V = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \quad \text{(V)}. \qquad (4.47)$$

For N point charges.

For continuous charge distributions:

$dq = \rho_v dV'$

$dq = \rho_s ds'$

$dq = \rho_\ell dl'$

$$V = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho_v}{R'}\, d\mathcal{V}' \quad \text{(volume distribution)}, \qquad (4.48a)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_s}{R'}\, ds' \quad \text{(surface distribution)}, \qquad (4.48b)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_\ell}{R'}\, dl' \quad \text{(line distribution)}. \qquad (4.48c)$$

$R' = |\vec{R} - \vec{R}_i|$

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Relating **E** to V

From (4.37):

$$dV = -\mathbf{E} \cdot d\mathbf{l}. \qquad (4.49)$$

For a scalar function V, Eq. (3.73) gives

$$dT = \nabla T \cdot d\mathbf{l}. \qquad (3.73)$$

$$dV = \nabla V \cdot d\mathbf{l}, \qquad (4.50)$$

where ∇V is the gradient of V. Comparison of Eq. (4.49) with Eq. (4.50) leads to

$$\boxed{\mathbf{E} = -\nabla V. \qquad (4.51)}$$

*This differential relationship between V and **E** allows us to determine **E** for any charge distribution by first calculating V and then taking the negative gradient of V to find **E**.*

$$V = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|}$$

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Example: Electric Field of an Electric Dipole

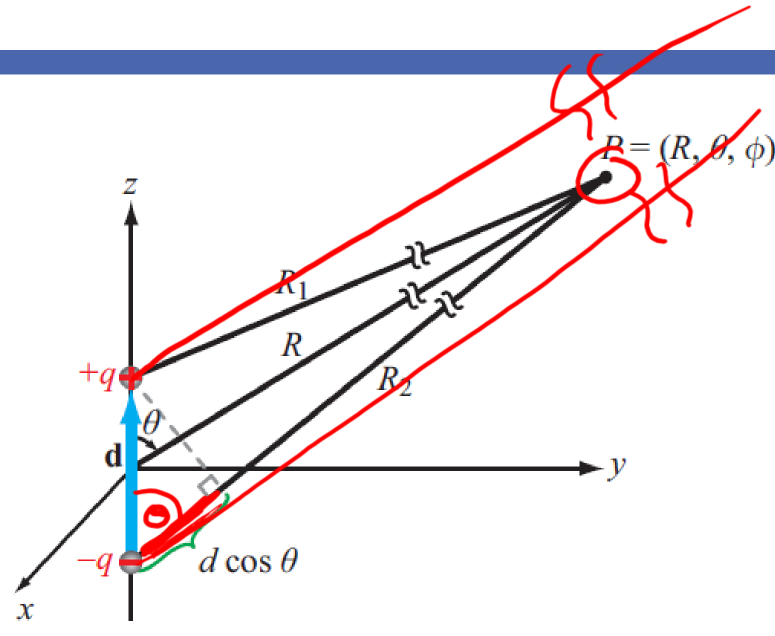
$$V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|\vec{R}-\vec{R}_i|}$$

For 2 pt. charges (free space): $V = \frac{1}{4\pi\varepsilon_0}\left(\frac{+q}{R_1} + \frac{-q}{R_2}\right)$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right)$$

Since for electric dipole typically $d \ll R$, then R_1 and R_2 approx. parallel.

$R_2 - R_1 \approx d\cos\theta$, $\underset{\approx R}{R_1}\underset{\approx R}{R_2} \approx R^2 \Rightarrow V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}$



(a) Electric dipole

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Example: Electric Field of an Electric Dipole

Generalizing to arbitrarily oriented dipole :

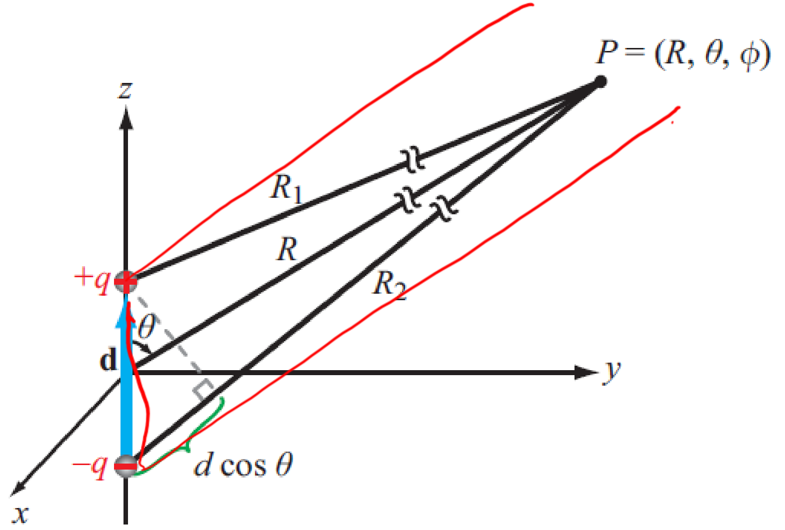
$qd\cos\theta = q\vec{d}\cdot\hat{R} = \vec{p}\cdot\hat{R}$

where $\vec{p} = q\vec{d}$ is dipole moment.

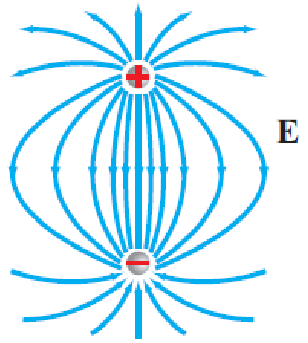
$$\boxed{V = \frac{\vec{p}\cdot\hat{R}}{4\pi\varepsilon_0 R^2}}$$

$$\vec{E} = -\nabla V \xrightarrow[\text{coor.}]{\text{spherical}} \vec{E} = -\left(\hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\cancel{\frac{\partial V}{\partial \phi}}^{0}\right)$$

$$\boxed{= \frac{qd}{4\pi\varepsilon_0 R^3}\left(\hat{R}\,2\cos\theta + \hat{\theta}\sin\theta\right)}$$



(a) Electric dipole



(b) Electric-field pattern

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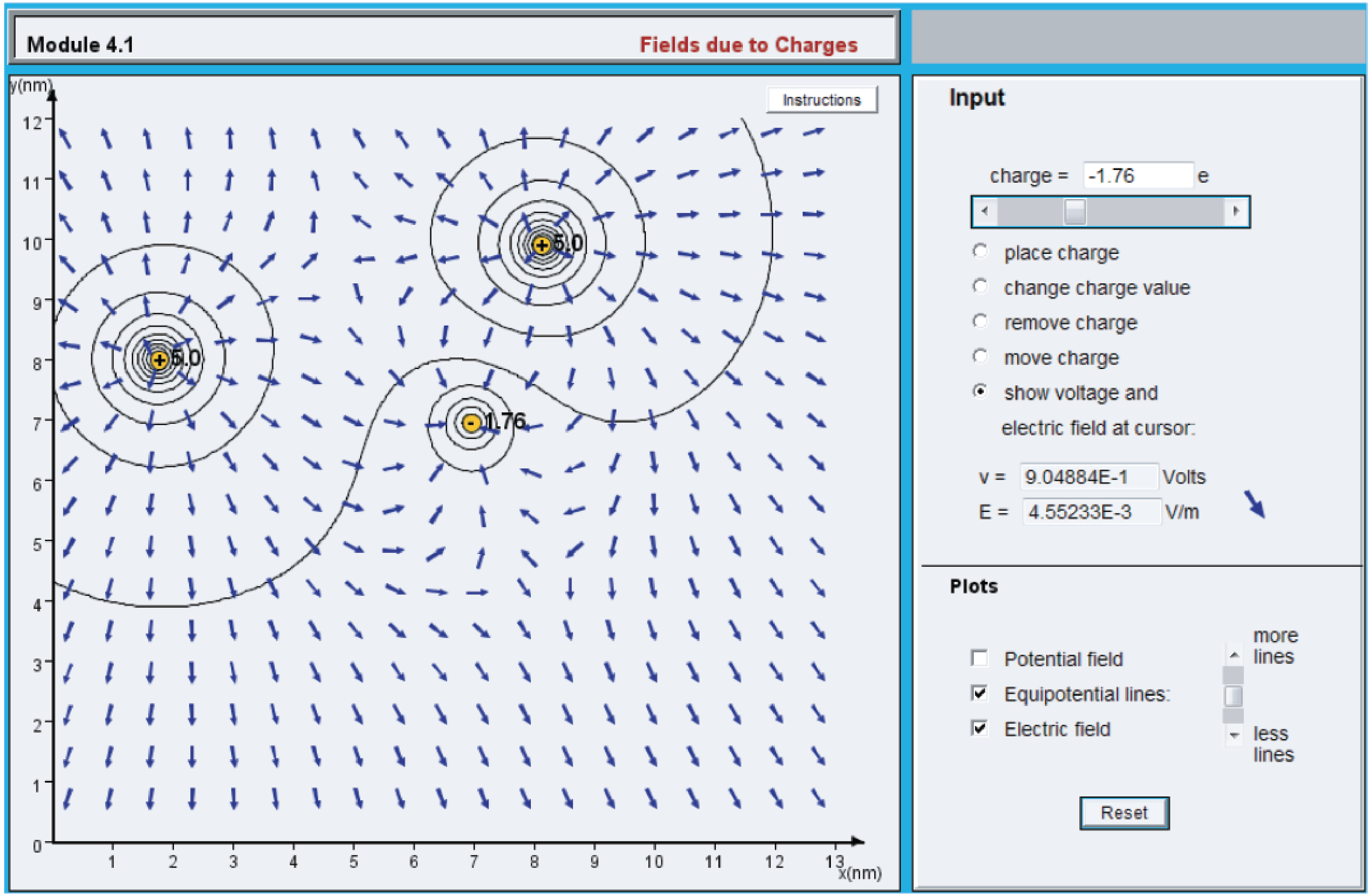
CD Module 4.1 Fields due to Charges For any group of point charges, this module calculates and displays the electric field **E** and potential *V* across a 2-D grid. The user can specify the locations, magnitudes and polarities of the charges.

https://em8e.eecs.umich.edu



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Poisson's & Laplace's Equations

With $\mathbf{D} = \varepsilon\mathbf{E}$, the differential form of Gauss's law given by Eq. (4.26) may be cast as

$\nabla \cdot \vec{D} = \rho_v$

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon}. \qquad (4.57)$$

Inserting Eq. (4.51) in Eq. (4.57) gives

$\vec{E} = -\nabla V$

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon}. \qquad (4.58)$$

Given Eq. (3.110) for the Laplacian of a scalar function V,

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}, \qquad (4.59)$$

Eq. (4.58) can be cast in the abbreviated form

$$\boxed{\nabla^2 V = -\frac{\cancel{\rho_v}^{\,0}}{\varepsilon} \qquad \text{(Poisson's equation).} \qquad (4.60)}$$

This is known as ***Poisson's equation***. For a volume $\mathcal{V}'$ containing a volume charge density distribution ρ_v, the solution for V derived previously and expressed by Eq. (4.48a) as

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v}{R'} \, d\mathcal{V}' \qquad (4.61)$$

In the absence of charges:

$$\boxed{\nabla^2 V = 0 \qquad \text{(Laplace's equation),}}$$

Poisson's and Laplace's Eqns are useful for finding V in region with boundaries where V is known (eg. between plates of capacitor).

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Conduction Current

σ

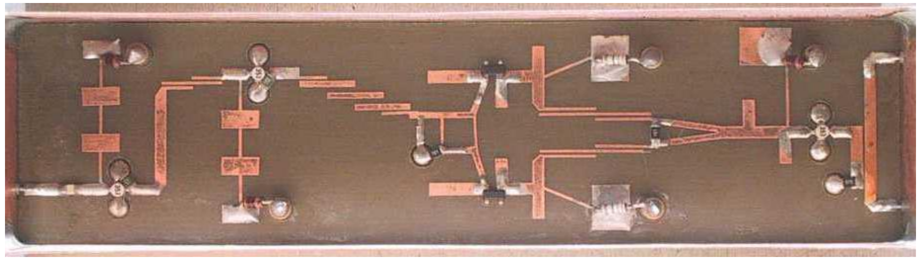
The conductivity of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field.

Conduction current density:

$$\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2) \qquad (\text{Ohm's law}),$$

$\propto I \quad \propto \frac{1}{R} \quad \propto V$ $\qquad I = \frac{1}{R} V \Rightarrow V = IR$

A ***perfect dielectric*** *is a material with $\sigma = 0$. In contrast, a* ***perfect conductor*** *is a material with $\sigma = \infty$. Some materials, called superconductors, exhibit such a behavior.*



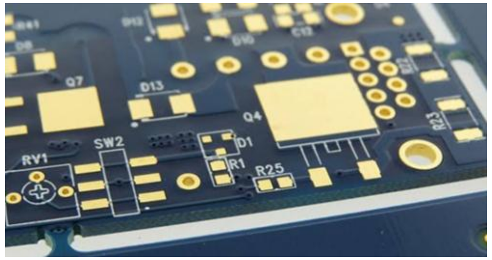


Table 4-1: Conductivity of some common materials at 20°C.

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^{7}
Copper	5.8×10^{7}
Gold	4.1×10^{7}
Aluminum	3.5×10^{7}
Iron	10^{7}
Mercury	10^{6}
Carbon	3×10^{4}
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

Note how wide the range is, over 24 orders of magnitude

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Conductivity

Drift velocity : $\vec{u}_e = -\mu_e \vec{E}$, $\vec{u}_h = \mu_h \vec{E}$

$\vec{J} = \vec{J}_e + \vec{J}_h = \rho_{ve}\vec{u}_e + \rho_{vh}\vec{u}_h = (-\rho_{ve}\mu_e + \rho_{vh}\mu_h)\vec{E} = \sigma \vec{E}$

$$\sigma = -\rho_{ve}\mu_e + \rho_{vh}\mu_h = (N_e\mu_e + N_h\mu_h)e \quad \text{(S/m)} \quad \text{(semiconductor)}, \tag{4.67a}$$

and its unit is siemens per meter (S/m). For a good conductor, $N_h\mu_h \ll N_e\mu_e$, and Eq. (4.67a) reduces to

$$\sigma = -\rho_{ve}\mu_e = N_e\mu_e e \quad \text{(S/m)} \quad \text{(conductor)}. \tag{4.67b}$$

$\vec{J} = \sigma \vec{E}$

In view of Eq. (4.66), in a perfect dielectric with $\sigma = 0$, $\mathbf{J} = 0$ regardless of $\mathbf{E}$, and in a perfect conductor with $\sigma = \infty$, $\mathbf{E} = \mathbf{J}/\sigma = 0$ regardless of $\mathbf{J}$.

That is,

Perfect dielectric:	$\mathbf{J} = 0$,
Perfect conductor:	$\mathbf{E} = 0$.

ρ_{ve} = volume charge density of electrons

ρ_{vh} = volume charge density of holes

μ_e = electron mobility

μ_h = hole mobility

N_e = number of electrons per unit volume

N_h = number of holes per unit volume

Absolute charge of an electron or hole:

$e = 1.602 \times 10^{-19}$ C

$$\mathbf{J} = \sigma\mathbf{E} \quad \text{(A/m}^2\text{)} \quad \text{(Ohm's law)}, \tag{4.66}$$

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Example: Conduction Current in a Copper Wire

Example 4-8: Conduction Current in a Copper Wire

A 2-mm-diameter copper wire with conductivity of 5.8×10^7 S/m and electron mobility of 0.0032 (m^2/V·s) is subjected to an electric field of 20 (mV/m). Find (a) the volume charge density of the free electrons, (b) the current density, (c) the current flowing in the wire, (d) the electron drift velocity, and (e) the volume density of the free electrons.

a) For conductor: $\sigma = -\rho_{ve}\mu_e \Rightarrow \rho_{ve} = -\frac{\sigma}{\mu_e} = -\frac{5.8\times10^{7}}{0.0032} = -1.81\times10^{10}\ \frac{C}{m^3}$

b) $\vec{J} = \sigma\vec{E} = (5.8\times10^{7})(20\times10^{-3}\hat{x}) = 1.16\times10^{6}\,\hat{x}\ A/m^2$

c) $I = JA = J(\pi r^2) = J\left(\pi\frac{d^2}{4}\right) = 3.64\ A$

d) $\vec{u}_e = -\mu_e\vec{E} = (-0.0032)(20\times10^{-3}\hat{x}) = -6.4\times10^{-5}\,\hat{x}\ \frac{m}{s}$

e) $\sigma = -\rho_{ve}\mu_e = N_e\mu_e e \Rightarrow N_e = -\frac{\rho_{ve}}{e} = -\frac{(-1.81\times10^{10})}{(1.602\times10^{-19})} = 1.13\times10^{29}\ \frac{\text{electrons}}{m^3}$

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Resistance

Longitudinal Resistor

Applying a potential difference V across the conductor creates an electric field:

$$\vec{E} = \hat{x}E_x$$

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E}\cdot d\mathbf{l}$$

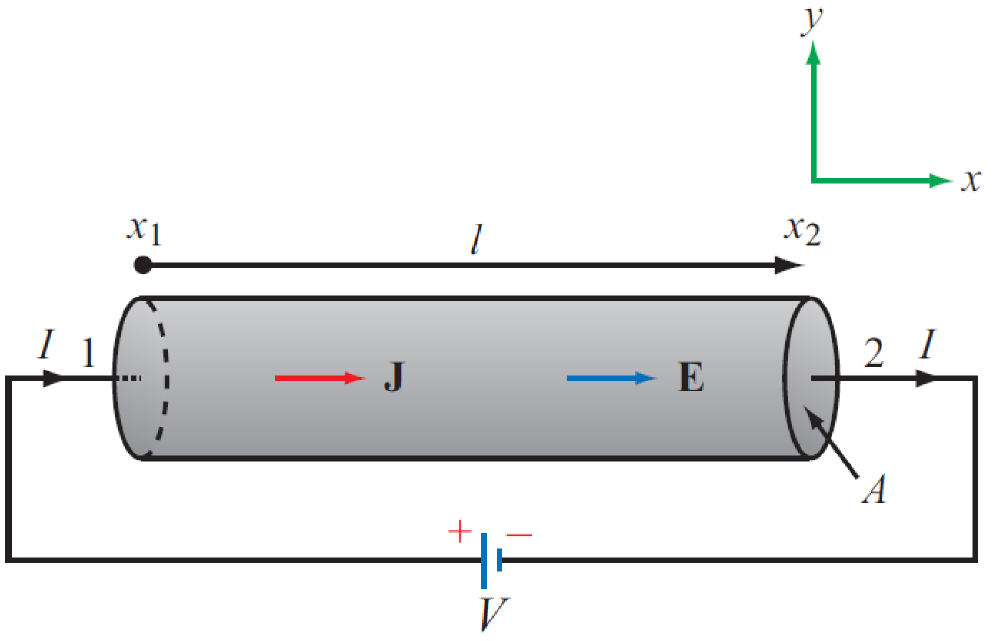
$$= -\int_{x_2}^{x_1} \hat{\mathbf{x}}E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \qquad \text{(V)}. \qquad (4.68)$$

Using Eq. (4.63), the current flowing through the cross section A at x_2 is

$$I = \int_A \mathbf{J}\cdot d\mathbf{s} = \int_A \sigma\mathbf{E}\cdot d\mathbf{s} = \sigma E_x A \qquad \text{(A)}. \qquad (4.69)$$

From $R = V/I$, the ratio of Eq. (4.68) to Eq. (4.69) gives

$$R = \frac{l}{\sigma A} \qquad (\Omega). \qquad (4.70)$$



For any conductor:

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E}\cdot d\mathbf{l}}{\int_S \mathbf{J}\cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E}\cdot d\mathbf{l}}{\int_S \sigma\mathbf{E}\cdot d\mathbf{s}}.$$

Conductance $G = \frac{1}{R} = \frac{\sigma A}{l}$ (Ω^{-1} or S)

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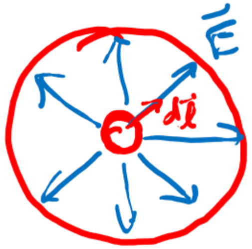
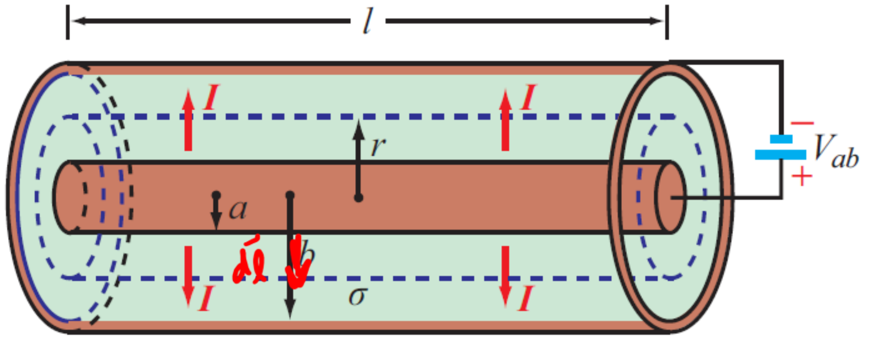
Example: Conductance of a Coaxial Cable

Example 4-9: Conductance of Coaxial Cable

The radii of the inner and outer conductors of a coaxial cable of length l are a and b, respectively (Fig. 4-15). The insulation material has conductivity σ. Obtain an expression for G', the conductance per unit length of the insulation layer.

Cross section





Current density through insulator at r:

$$\vec{J} = \hat{r}\frac{I}{A} = \hat{r}\frac{I}{2\pi r l}$$

$$\vec{J} = \sigma\vec{E} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = \hat{r}\frac{I}{2\pi\sigma r l}$$

$$V_{ab} = -\int_b^a \vec{E}\cdot d\vec{l} = -\int_b^a \frac{I}{2\pi\sigma l}\frac{\hat{r}\cdot\hat{r}\,dr}{r} = \frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right)$$

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab}l} = \frac{2\pi\sigma}{\ln(b/a)} \quad (\mathrm{S/m})$$

$G' = 0$ if insulting layer is air (free space)

Joule's Law

$\Delta W = \vec{F}_e \cdot \Delta \vec{l}_e + \vec{F}_h \cdot \Delta \vec{l}_h$

$\Delta P = \frac{\Delta W}{\Delta t} = \vec{F}_e \cdot \frac{\Delta \vec{l}_e}{\Delta t} + \vec{F}_h \cdot \frac{\Delta \vec{l}_h}{\Delta t} = \overbrace{(\rho_{ve} \vec{E} \cdot \vec{u}_e + \rho_{vh} \vec{E} \cdot \vec{u}_h)}^{\vec{E} \cdot \vec{J}} \Delta V$

$\vec{u}_e$ $\vec{u}_h$

The power dissipated in a volume containing electric field **E** and current density **J** is:

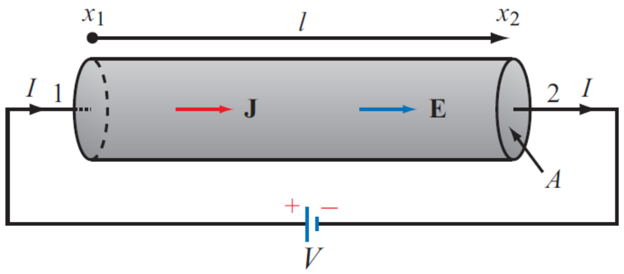
$$P = \int_V \mathbf{E} \cdot \mathbf{J} \, dV \qquad \text{(W)} \qquad \text{(Joule's law)}$$

This can be expressed in terms of the electric field, from Ohm's law:

$\to \vec{J} = \sigma \vec{E}$

$$P = \int_V \sigma |\mathbf{E}|^2 \, dV$$

For a wire:



For the resistance of a wire that we considered previously, Joule's Law simplifies to:

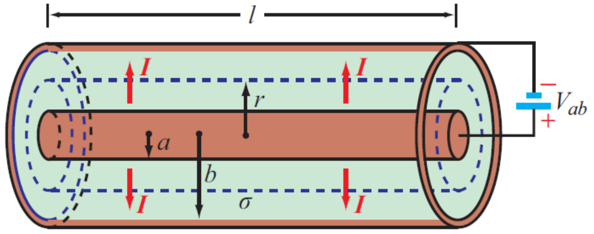
$$P = \int_V \sigma |\mathbf{E}|^2 \, dV = \int_A \sigma E_x \, ds \int_l E_x \, dl$$
$$= \underbrace{(\sigma E_x A)}_{I} \underbrace{(E_x l)}_{V} = IV = I^2 R$$

see slide 3-30

For a coaxial cable:

$$P = I^2 \underbrace{R}_{} = I^2 \frac{\ln(b/a)}{2\pi\sigma l}$$

see slide 3-31



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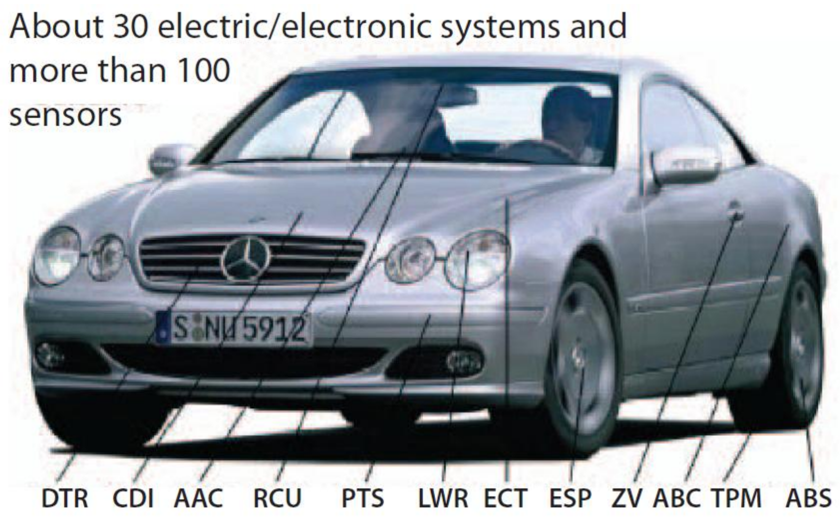
Tech Brief: Resistive Sensors

An ***electrical sensor*** is a device capable of responding to an applied stimulus by generating an electrical signal whose voltage, current, or some other attribute is related to the intensity of the stimulus.

Typical stimuli : temperature, pressure, position, distance, motion, velocity, acceleration, concentration (of a gas or liquid), blood flow, etc.

Sensing process relies on measuring resistance, capacitance, inductance, induced electromotive force (emf), oscillation frequency or time delay, etc.



System	Abbrev.	Sensors	System	Abbrev.	Sensors
Distronic	DTR	3	Common-rail diesel injection	CDI	11
Electronic controlled transmission	ECT	9	Automatic air condition	AAC	13
Roof control unit	RCU	7	Active body control	ABC	12
Antilock braking system	ABS	4	Tire pressure monitoring	TPM	11
Central locking system	ZV	3	Elektron. stability program	ESP	14
Dyn. beam levelling	LWR	6	Parktronic system	PTS	12

Figure TF7-1: Most cars use on the order of 100 sensors. (Courtesy Mercedes-Benz.)

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Piezoresistivity

The Greek word ***piezein*** means to press

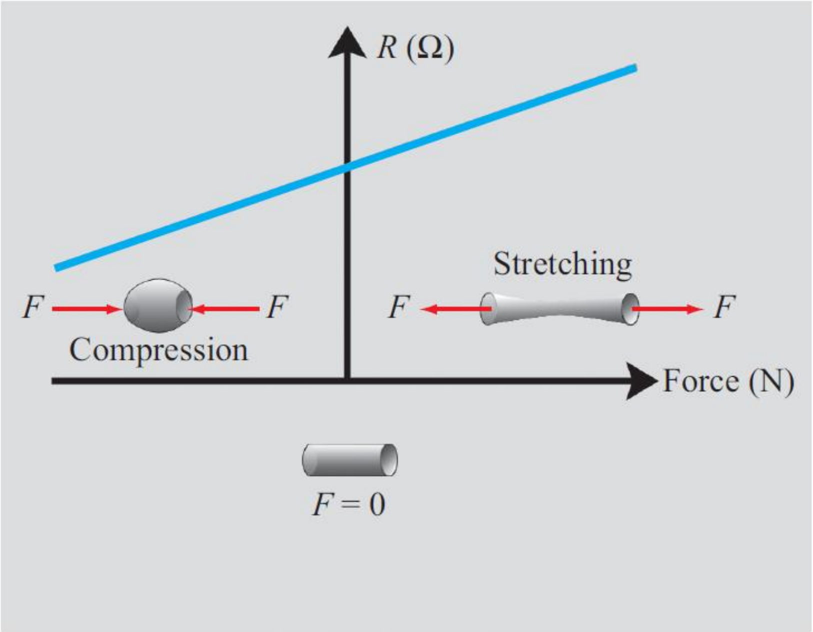

Figure TF7-2: Piezoresistance varies with applied force.

For wire conductor: $R = \frac{\ell}{\sigma A}$

Stretch : $\uparrow \ell$, $\downarrow A \Rightarrow \uparrow R$

Compress : $\downarrow \ell$, $\uparrow A \Rightarrow \downarrow R$

$$R = R_0 \left(1 + \frac{\alpha F}{A_0}\right)$$

R_0 = resistance when $F = 0$
F = applied force
A_0 = cross-section when $F = 0$
α = piezoresistive coefficient of material

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Piezoresistors

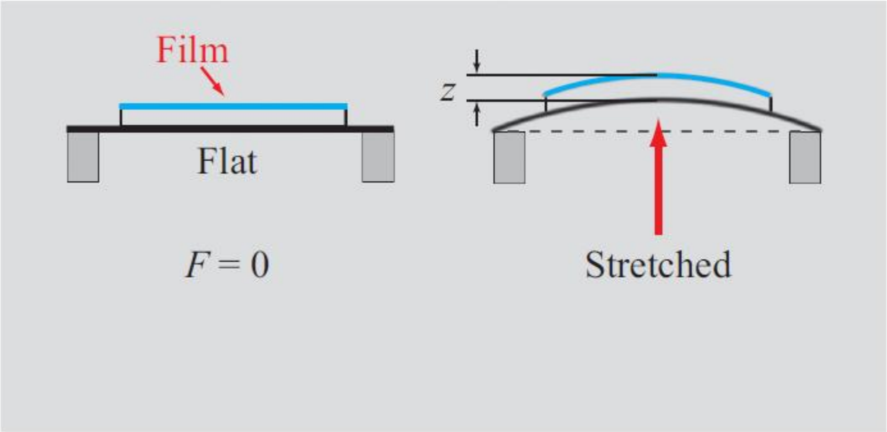

Figure TF7-3: Piezoresistor films.

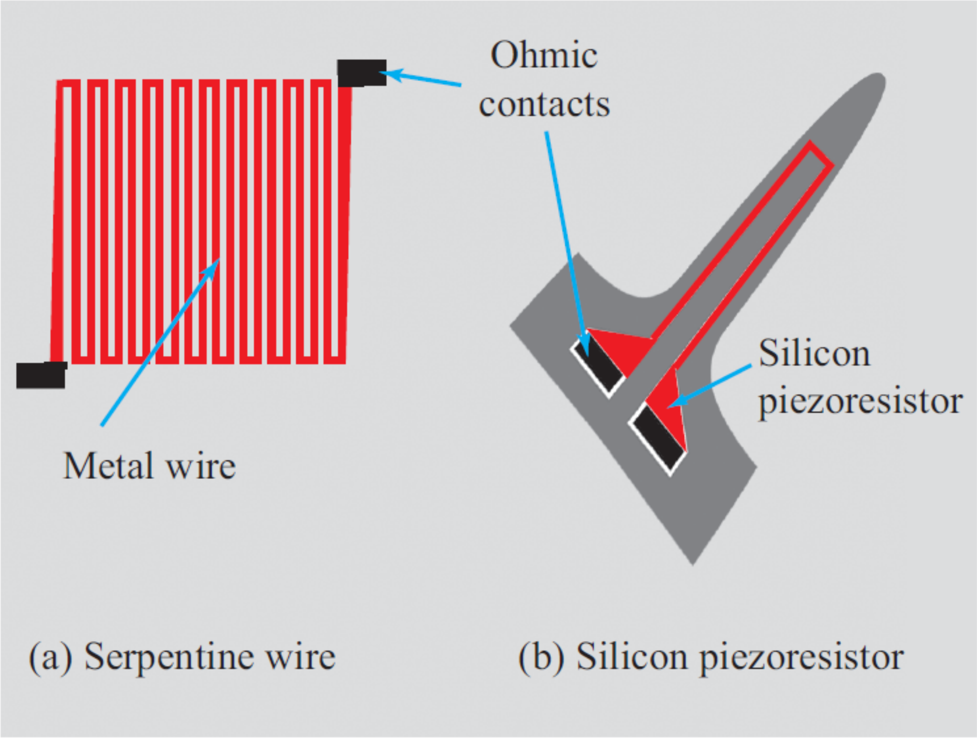

Figure TF7-4: Metal and silicon piezoresistors.

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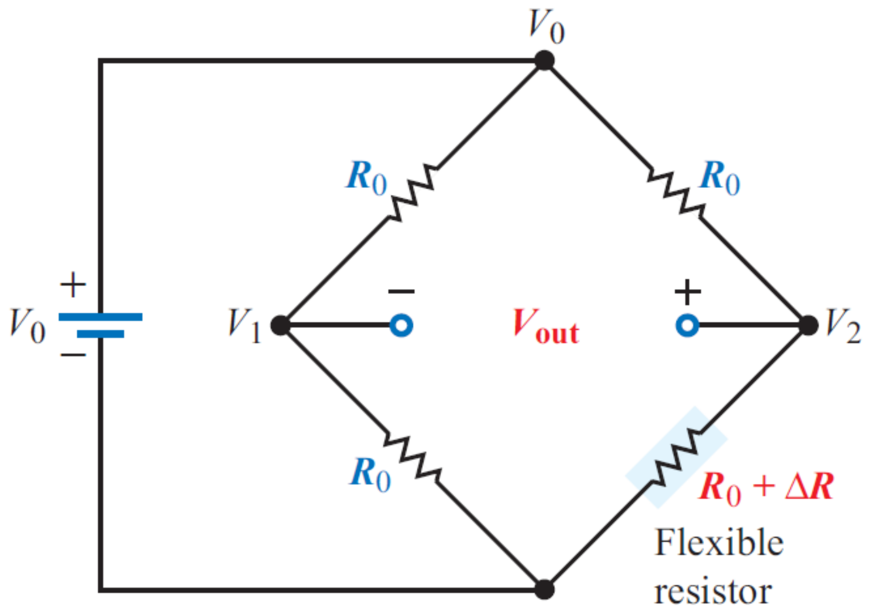
Wheatstone Bridge



$$V_{out} = \frac{V_0}{4}\left(\frac{\Delta R}{R_0}\right)$$

Figure TF7-5: Wheatstone bridge circuit with piezoresistor.

A Wheatstone bridge is a high-sensitivity circuit for measuring small changes in resistance

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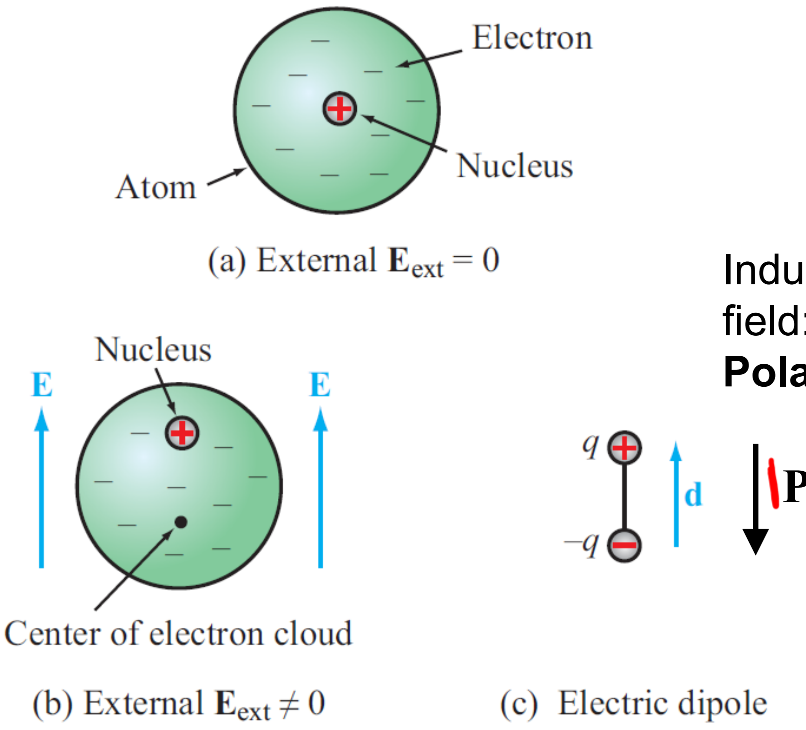
Dielectric Materials



Induced electric field:
Polarization field

$|\mathbf{P}| < E_{ext}$

Figure 4-16: In the absence of an external electric field **E**, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance d.

Resulting E-field is less than $\vec{E}_{ext}$

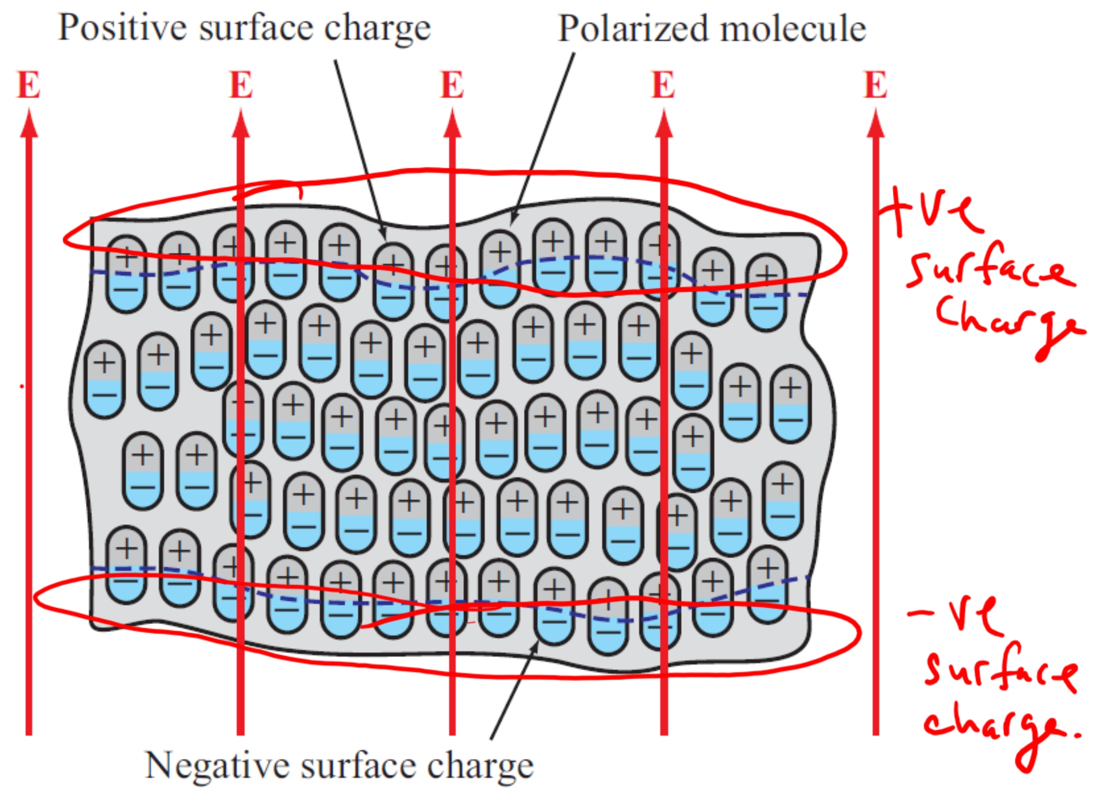


Figure 4-17: A dielectric medium polarized by an external electric field **E**.

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Polarization Field

In free-space, the electric flux density is:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

In the presence of microscopic dipoles inside a dielectric medium, this relationship becomes:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad (4.83)$$

$\mathbf{P}$ = electric flux density induced by $\mathbf{E}$ or otherwise the electric polarization field → degree of polarizability of material.

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \qquad (4.84)$$

where χ_e is the electric susceptibility of the material.
Inserting Eq. (4.84) into Eq. (4.83), we have:

$$\begin{aligned}\mathbf{D} &= \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} \\ &= \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}, \\ &= \varepsilon_0 \varepsilon_r \mathbf{E}\end{aligned}$$

$\varepsilon_r = 1 + \chi_e$

; ε_r = relative permittivity or dielectric constant

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DiElectric Breakdown

The dielectric strength E_{ds} is the largest magnitude of $\mathbf{E}$ *that the material can sustain without breakdown.*

In dielectric breakdown, electrons detach from molecules and create conduction current.

Material sustains permanent damage due to e collisions with molecular structure.

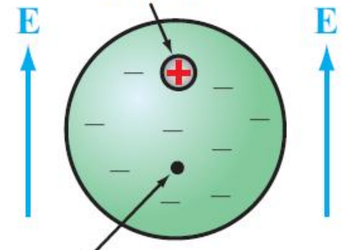

Table 4-2: Relative permittivity (dielectric constant) and dielectric strength of common materials.

Material	Relative Permittivity, ε_r	Dielectric Strength, E_{ds} (MV/m)
Air (at sea level)	1.0006	3
Petroleum oil	2.1	12
Polystyrene	2.6	20
Glass	4.5–10	25–40
Quartz	3.8–5	30
Bakelite	5	20
Mica	5.4–6	200

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

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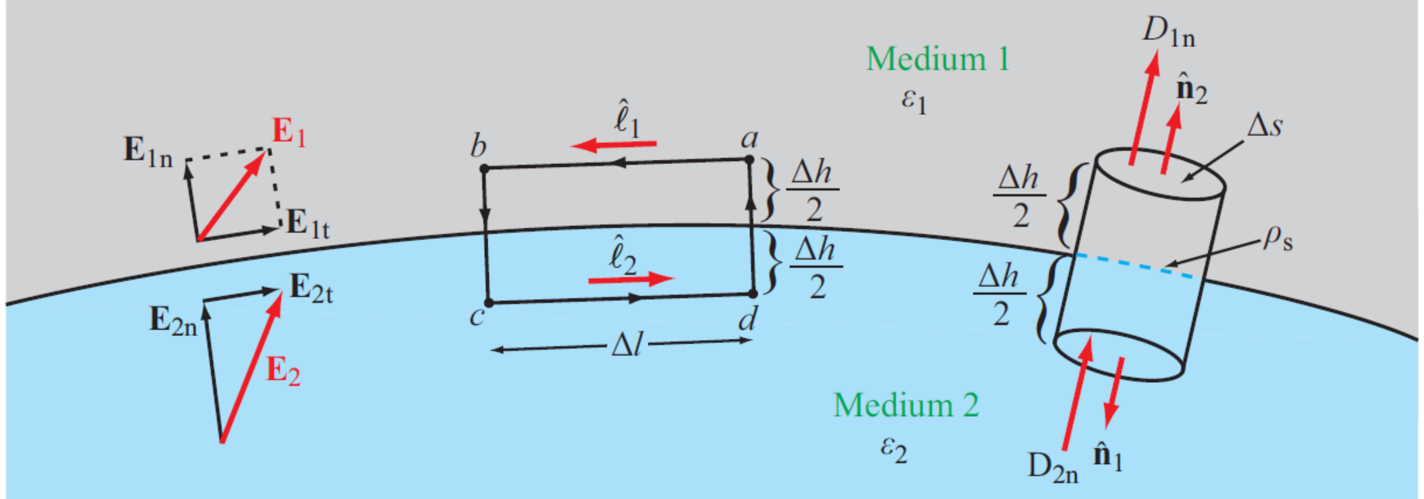
Electric Boundary Conditions



BC determine how tangential / normal fields components are related across boundary.

- Although an electric field can be continuous between neighbouring media, it can also be discontinuous at the boundary between them.
- The boundary conditions that we will define here for **E, D** and **J** hold for boundaries between any media, i.e. between two dielectrics or between a conductor and a dielectric.
- Note that any of the two media can be air.

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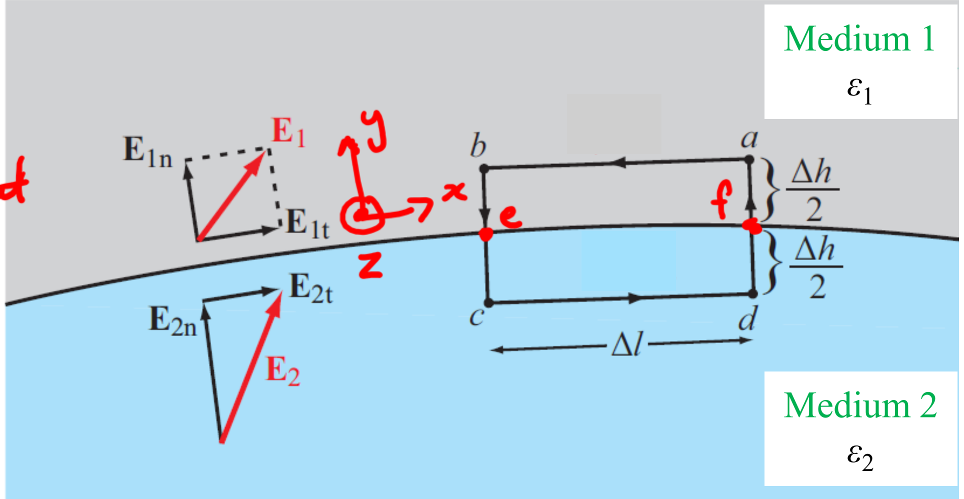
Derivation of Electric Boundary Conditions (Tangential)

Assume: $\hat{x} \parallel \vec{E}_{1t} \parallel \vec{E}_{2t}$, $\hat{y} \parallel \vec{E}_{1n} \parallel \vec{E}_{2n}$

Δh, Δl are small $\Rightarrow$ boundary line straight



Apply conservative property of E-field to $\square abcda$

$$\oint_C \vec{E} \cdot d\vec{l} = \int_a^b (\vec{E}_{1t}+\vec{E}_{1n}) \cdot d\vec{l} + \int_b^e (\vec{E}_{1t}+\vec{E}_{1n}) \cdot d\vec{l} + \int_e^c (\vec{E}_{2t}+\vec{E}_{2n}) \cdot d\vec{l}$$

$$+ \int_c^d (\vec{E}_{2t}+\vec{E}_{2n}) \cdot d\vec{l} + \int_d^f (\vec{E}_{2t}+\vec{E}_{2n}) \cdot d\vec{l} + \int_f^a (\vec{E}_{1t}+\vec{E}_{1n}) \cdot d\vec{l} = 0$$

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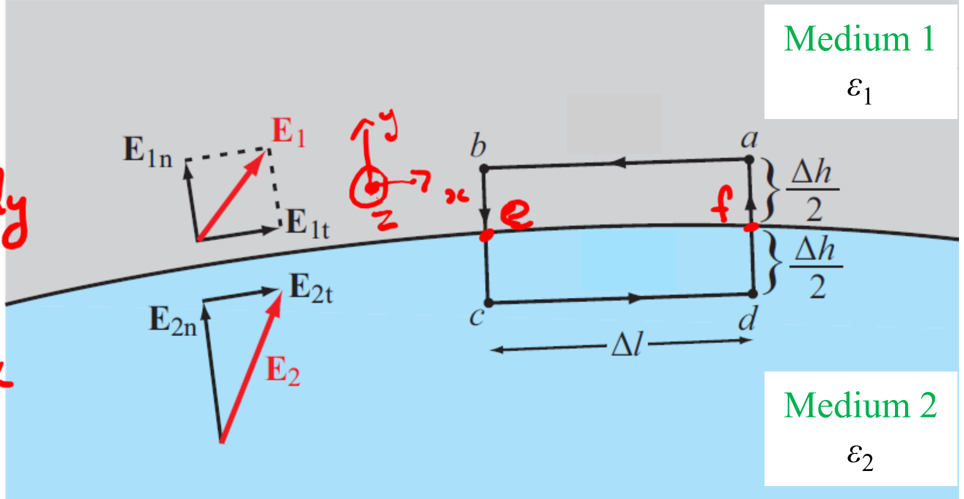
Derivation of Electric Boundary Conditions (Tangential)



In general : $d\vec{l} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

$$\oint_C \vec{E}\cdot d\vec{l} = \int_a^b (\vec{E}_{1t}\cdot\hat{x} + \cancelto{0}{\vec{E}_{1n}\cdot\hat{x}})\,dx + \int_b^e (\cancelto{0}{\vec{E}_{1t}\cdot\hat{y}} + \vec{E}_{1n}\cdot\hat{y})\,dy$$

$$+ \int_e^c (\cancelto{0}{\vec{E}_{2t}\cdot\hat{y}} + \vec{E}_{2n}\cdot\hat{y})\,dy + \int_c^d (\vec{E}_{2t}\cdot\hat{x} + \cancelto{0}{\vec{E}_{2n}\cdot\hat{x}})\,dx$$

$$+ \int_d^f (\cancelto{0}{\vec{E}_{2t}\cdot\hat{y}} + \vec{E}_{2n}\cdot\hat{y})\,dy + \int_f^a (\cancelto{0}{\vec{E}_{1t}\cdot\hat{y}} + \vec{E}_{1n}\cdot\hat{y})\,dy = 0$$

$$= E_{1t}\cancelto{-\Delta l}{\int_a^b dx} + E_{1n}\cancelto{-\Delta h/2}{\int_b^e dy} + E_{2n}\cancelto{-\Delta h/2}{\int_e^c dy} + E_{2t}\cancelto{\Delta l}{\int_c^d dx} + E_{2n}\cancelto{\Delta h/2}{\int_d^f dy} + E_{1n}\cancelto{\Delta h/2}{\int_f^a dy} = 0$$

$$\Rightarrow E_{2t}\cancel{\Delta l} = E_{1t}\cancel{\Delta l} \Rightarrow \boxed{\vec{E}_{1t} = \vec{E}_{2t}}$$ Continuous across boundary.

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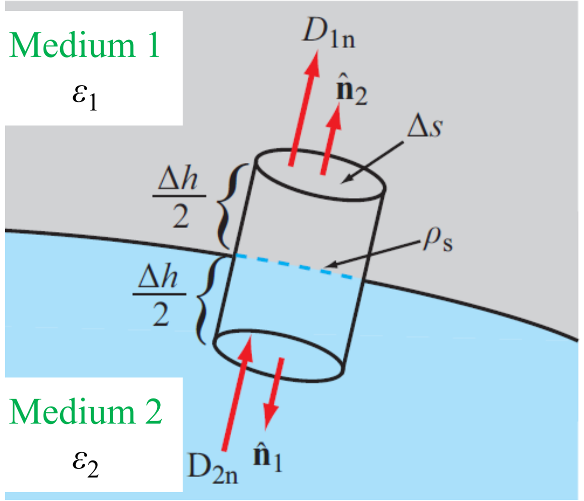
Derivation of Electric Boundary Conditions (Normal)



Apply Gauss' law : $\oint_S \vec{D} \cdot d\vec{s} = Q$

Letting $\Delta h \to 0$, flux through curved surface is 0, if there are free charges. Only charge is ρ_s.

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{top} \vec{D}_1 \cdot \hat{n}_2 ds + \int_{bot} \vec{D}_2 \cdot \hat{n}_1 ds = \rho_s \Delta s$$

$$= \vec{D}_1 \cdot \hat{n}_2 \cancel{\Delta s} + \vec{D}_2 \cdot \hat{n}_1 \cancel{\Delta s} = \rho_s \cancel{\Delta s}$$

$$= \vec{D}_1 \cdot \hat{n}_2 + \vec{D}_2 \cdot (-\hat{n}_2) = \rho_s$$

$$\hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

Since D_{1n} and D_{2n} are the normal components of $\vec{D}_1$ and $\vec{D}_2$ along $\hat{n}_2$:

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

Discontinuous across boundary

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Electric Boundary Conditions

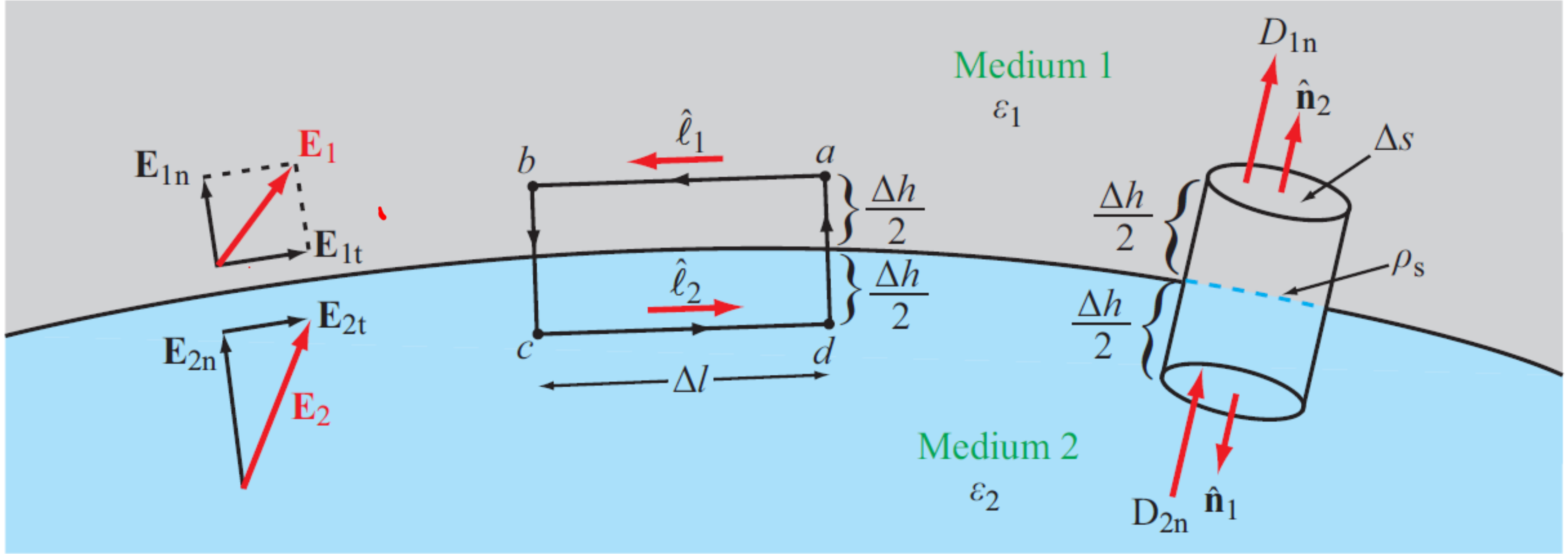

Figure 4-18: Interface between two dielectric media.

$$\boxed{\mathbf{E}_{1t} = \mathbf{E}_{2t} \qquad \text{(V/m)}. \qquad (4.90)}$$

$$\boxed{\frac{\mathbf{D}_{1t}}{\varepsilon_1} = \frac{\mathbf{D}_{2t}}{\varepsilon_2}\,. \qquad (4.91)}$$

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad (\text{C/m}^2).$$

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2). \qquad (4.94)}$$

*The normal component of **D** changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.*

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Summary of Electric Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Perfect conductor
$(\sigma = \infty)$
$\bar{E} = \frac{\bar{J}}{\sigma} = 0$

$E_t = 0$
$E_n = 0$

- Remember that in a perfect conductor with $\sigma = \infty$, $\mathbf{E} = \mathbf{J}/\sigma = 0$ regardless of $\mathbf{J}$.

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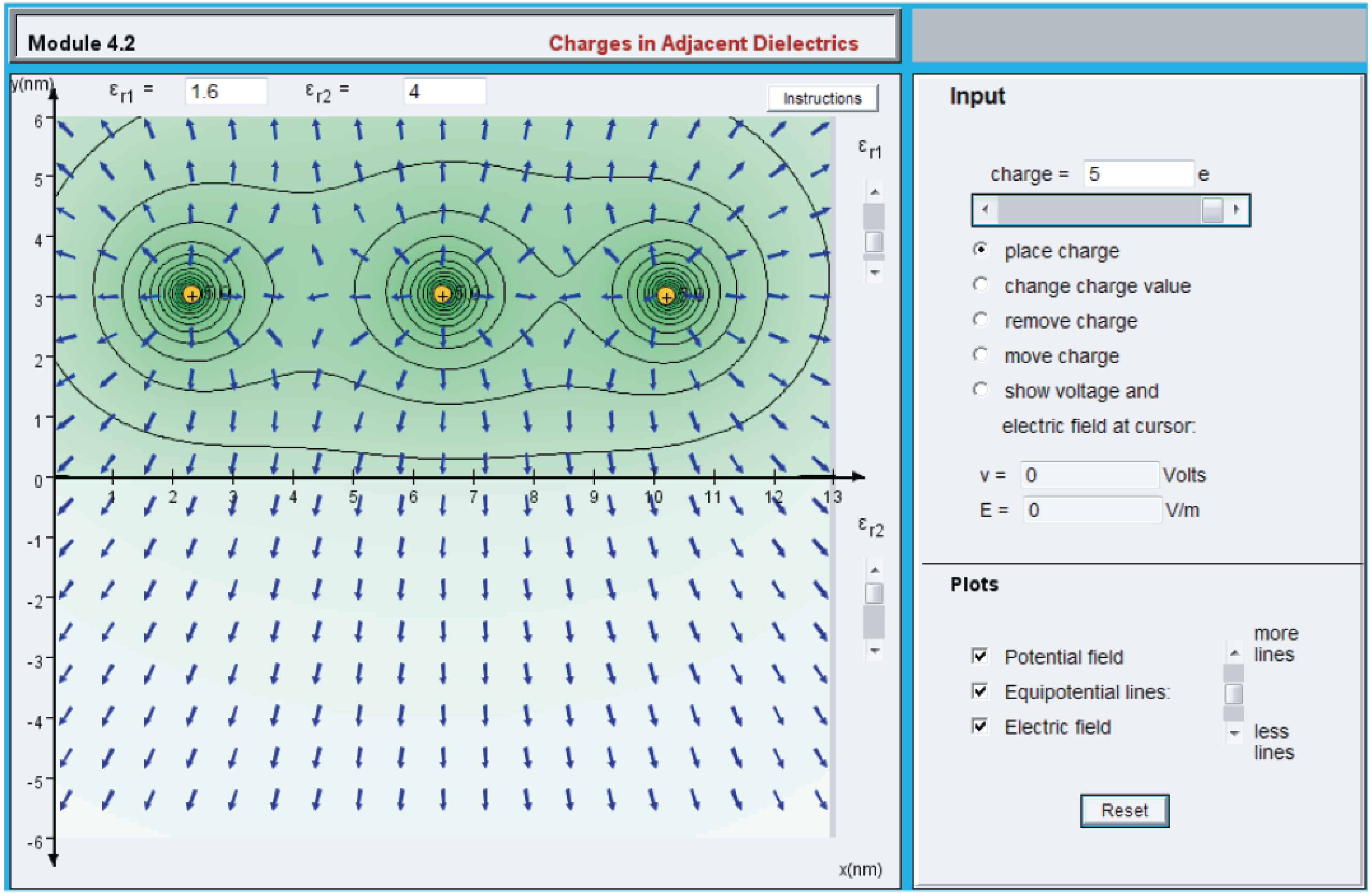
CD Module 4.2 Charges in Adjacent Dielectrics In two adjoining half-planes with selectable permittivities, the user can place point charges anywhere in space and select their magnitudes and polarities. The module then displays **E**, V, and the equipotential contours of V.

https://em8e.eecs.umich.edu



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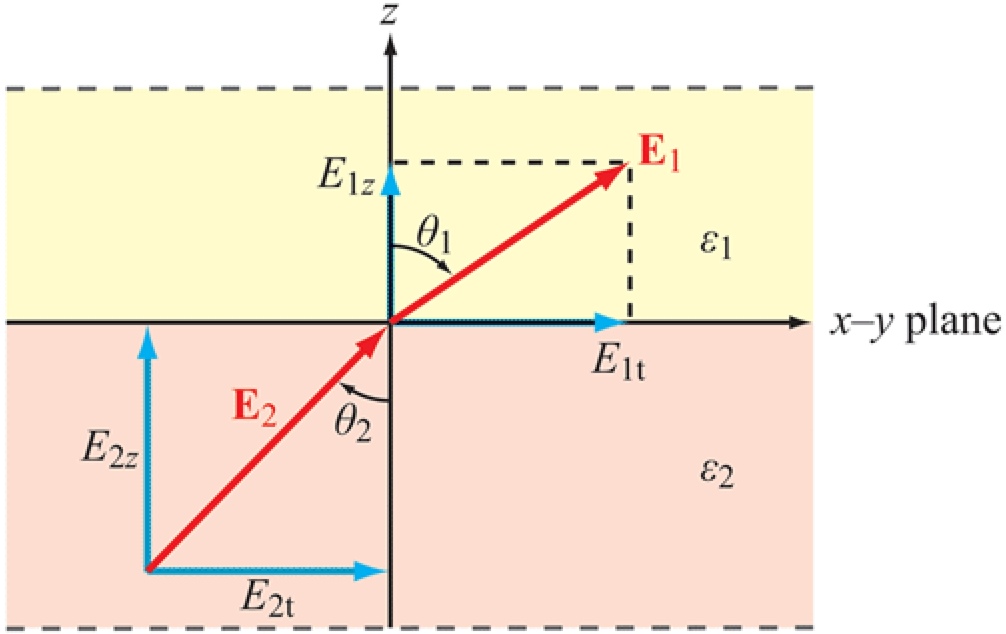
Example: Application of Boundary Conditions

The x–y plane is a charge-free boundary separating two dielectric media with permittivities ε_1 and ε_2, as shown in Fig. 4-21. If the electric field in medium 1 is

$$\mathbf{E}_1 = \hat{\mathbf{x}}E_{1x} + \hat{\mathbf{y}}E_{1y} + \hat{\mathbf{z}}E_{1z},$$

find (a) the electric field $\mathbf{E}_2$ in medium 2 and (b) the angles θ_1 and θ_2.



a) $\vec{E}_2 = \hat{x}E_{2x} + \hat{y}E_{2y} + \hat{z}E_{2z}$

1^{st} BC : $E_{2t} = E_{1t} \Rightarrow E_{2x} = E_{1x}$

$E_{2y} = E_{1y}$

2^{nd} B : $D_{1n} - D_{2n} = \cancel{\rho_s}^{0} \Rightarrow D_{1n} = D_{2n} = D_{1z} = D_{2z} \Rightarrow \varepsilon_1 E_{1z} = \varepsilon_2 E_{2z}$

$\Rightarrow E_{2z} = \frac{\varepsilon_1}{\varepsilon_2} E_{1z}$

$\vec{E}_2 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}\frac{\varepsilon_1}{\varepsilon_2}E_{1z}$ //

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Example: Application of Boundary Conditions

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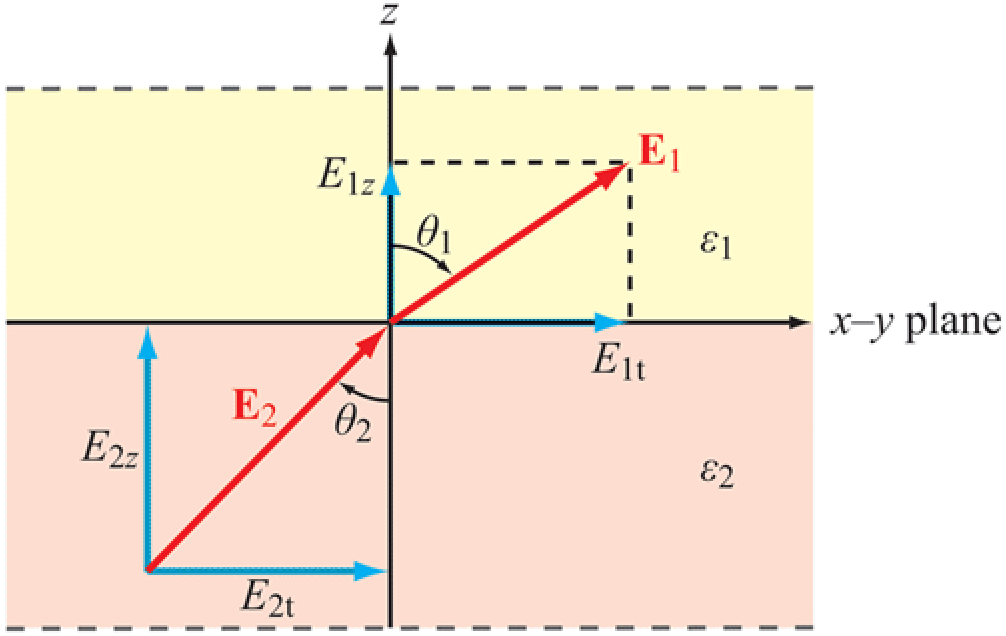

b) $E_{1t} = \sqrt{E_{1x}^2 + E_{1y}^2}$, $E_{2t} = \sqrt{E_{2x}^2 + E_{2y}^2}$

$\tan\theta_1 = \dfrac{E_{1t}}{E_{1z}} = \dfrac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}} \longrightarrow (1)$

$\tan\theta_2 = \dfrac{E_{2t}}{E_{2z}} = \dfrac{\sqrt{E_{2x}^2 + E_{2y}^2}}{E_{2z}} \rightarrow (2)$

$E_{1x} = E_{2x}$, $E_{1y} = E_{2y}$

$E_{2z} = \dfrac{\varepsilon_1}{\varepsilon_2} E_{1z}$

$(2) \div (1)$ $\boxed{\dfrac{\tan\theta_2}{\tan\theta_1} = \dfrac{\varepsilon_2}{\varepsilon_1}}$ Snell's Law.

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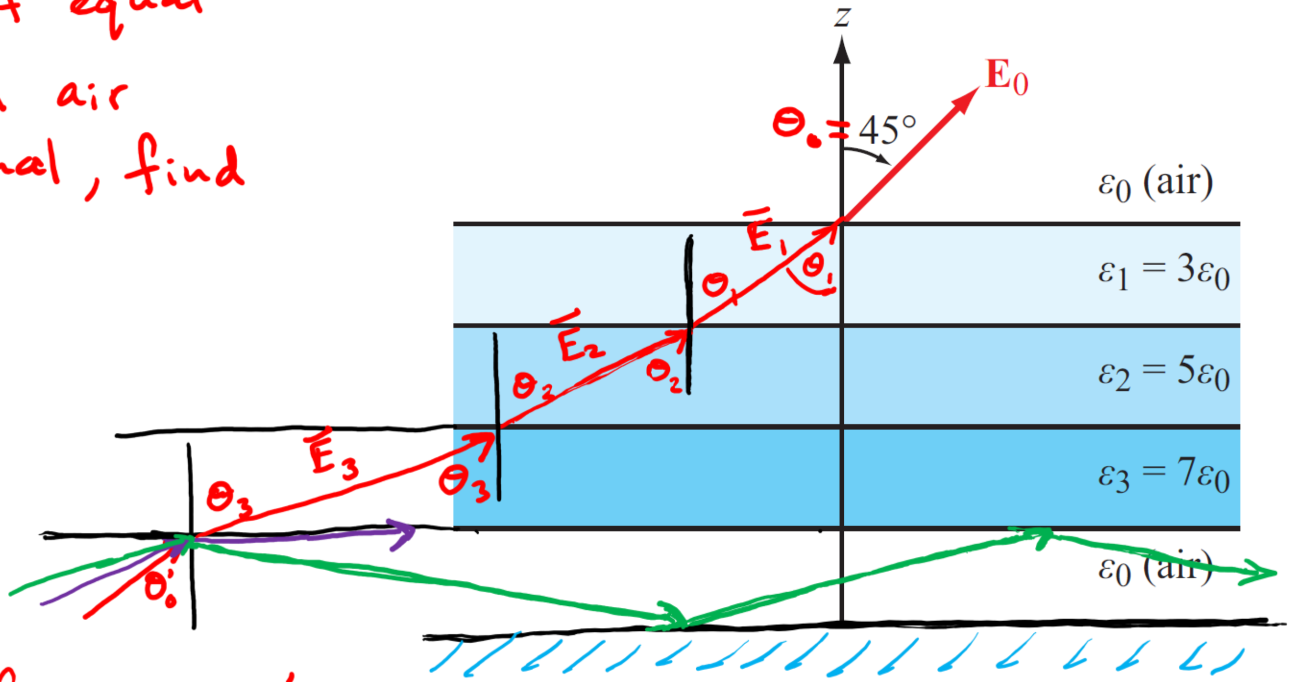
Example: Application of Snell's Law

Consider 3 planar dielectric slabs of equal thickness, but different ε. If $\bar{E}_o$ in air makes an angle of 45° with normal, find the angle in each of the layers.



$$\theta_1 = \tan^{-1}\left(\frac{\varepsilon_1}{\varepsilon_0}\tan\theta_o\right) = 71.6°$$

$$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right) = 78.7°$$

$$\theta_3 = \tan^{-1}\left(\frac{\varepsilon_3}{\varepsilon_2}\tan\theta_2\right) = 81.9°$$

$$\theta_o' = \tan^{-1}\left(\frac{\varepsilon_o}{\varepsilon_3}\tan\theta_3\right) = 45° \Rightarrow \text{Therefore, } \theta_o = \theta_o'$$

Snell's law : $\dfrac{\tan\theta_1}{\tan\theta_o} = \dfrac{\varepsilon_1}{\varepsilon_o}$

By increasing incident angle and/or ε, total internal reflection (TIR) can be achieved $\Rightarrow$ fiber optic communication.

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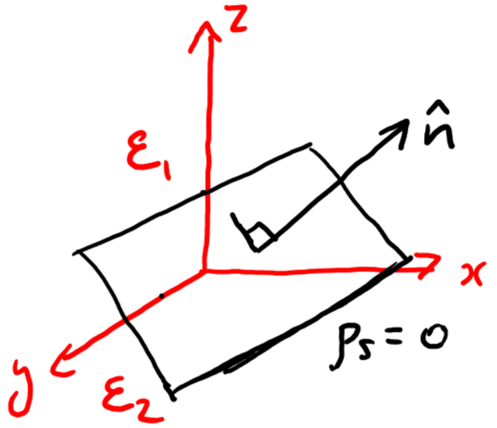
Example: Boundary Conditions on an inclined plane

Regions 1 & 2 have $\varepsilon_1 = 2\varepsilon_0$ & $\varepsilon_2 = 5\varepsilon_0$

The regions are separated by plane $x + 2y + z = 1$

If $\vec{E}_1 = 20\hat{x} - 10\hat{y} + 40\hat{z}$, find: a) $\vec{E}_{1t}$ & $\vec{E}_{1n}$

b) $\vec{E}_2$



$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$

a) Let $f(x,y,z) = x + 2y + z - 1 = 0$

direction of steepest ascent on surface at given point

$\nabla f = \hat{x} + 2\hat{y} + \hat{z} \Rightarrow \hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}(\hat{x} + 2\hat{y} + \hat{z})$

$E_{1n} = (\vec{E}_1 \cdot \hat{n})\hat{n} = \frac{1}{\sqrt{6}}(20 - 20 + 40)\frac{1}{\sqrt{6}}(\hat{x} + 2\hat{y} + \hat{z})$

$= 6.67\hat{x} + 13.3\hat{y} + 6.67\hat{z}$ V/m

$\vec{E}_{1t} = \vec{E}_1 - \vec{E}_{1n} = 13.3\hat{x} - 23.3\hat{y} + 33.3\hat{z}$ V/m

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Example: Boundary Conditions on an inclined plane

b) 1st BC : $\vec{E}_{2t} = \vec{E}_{1t} = 13.3\hat{x} - 23.3\hat{y} + 33.3\hat{z}$ V/m

2nd BC : $\vec{D}_{2n} = \vec{D}_{1n}$ (since $\rho_s = 0$)

$$\varepsilon_2 \vec{E}_{2n} = \varepsilon_1 \vec{E}_{1n}$$

$$\Rightarrow \quad E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} \vec{E}_{1n}$$

$$= \frac{2\cancel{\varepsilon_0}}{5\cancel{\varepsilon_0}} (6.67\hat{x} + 13.3\hat{y} + 6.67\hat{z})$$

$$= 2.7\hat{x} + 5.3\hat{y} + 2.7\hat{z}$$

$$\Rightarrow \vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} = 16\hat{x} - 18\hat{y} + 36\hat{z} \text{ V/m}$$

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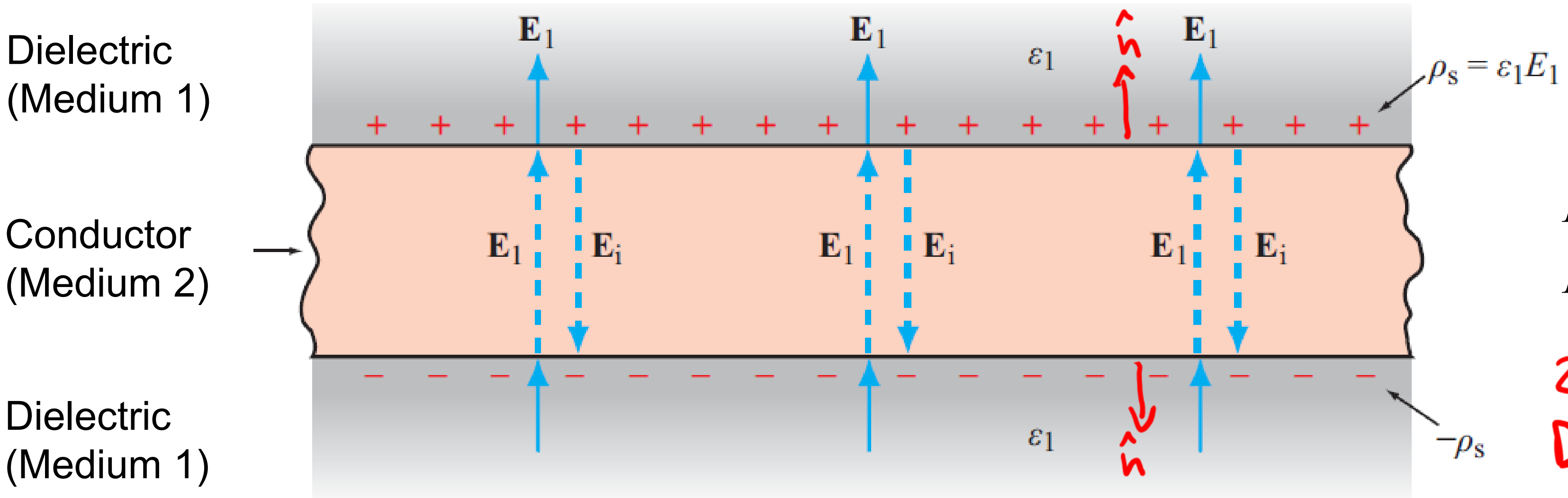
Conductors

Net electric field inside a perfect conductor is zero



1st BC:

$E_{1t} = \cancel{E_{2t}}^{0} = 0$

$D_{1t} = 0$

$$E_{2t} = 0$$

$$E_{2n} = 0$$

2nd BC:

$D_{1n} - \cancel{D_{2n}}^{0} = \rho_s$

$D_{1n} = \varepsilon_1 E_{1n} = \rho_s$

$\bar{D}_1 = \varepsilon_1 \bar{E}_1 = \hat{n}\rho_s$

- When we place a conductor in an external electric field E_1, a positive charge density ρ_s accumulates on the top surface, and a negative charge density $-\rho_s$ on the bottom surface.
- These charges induce an internal electric field $E_i = -E_1$.
- Consequently, the total electric field inside the conductor is zero.

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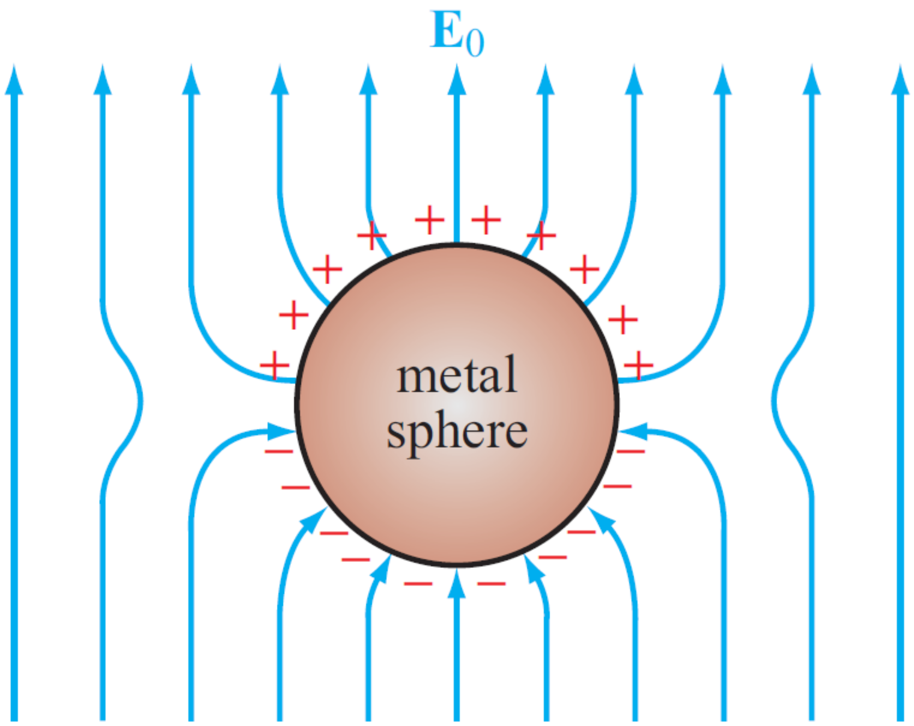
Field Lines at a Conductor Boundary



Inside the sphere:

$$E_{2t} = 0$$
$$E_{2n} = 0$$

In the air:

$$E_{1t} = 0 \quad \text{(1st BC)}$$
$$E_{1n} = \rho_s / \varepsilon_1 \quad \text{(2nd BC)}$$

Figure 4-21: Metal sphere placed in an external electric field $\mathbf{E}_0$.

Because only the normal component of the electric field remains in the air around the sphere, at the conductor boundary the E-field direction is always perpendicular to the conductor surface.

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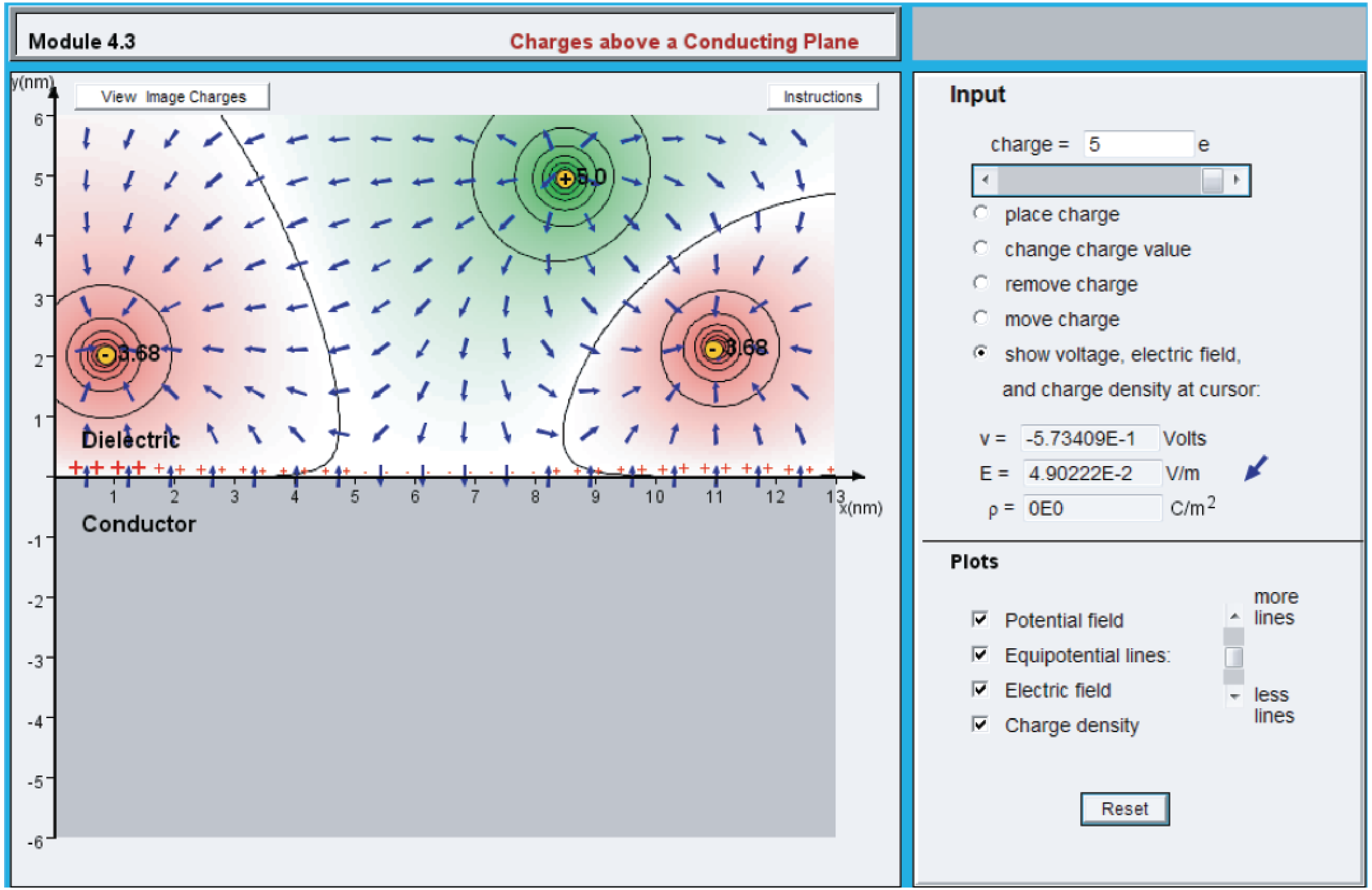
CD Module 4.3 Charges above a Conducting Plane

When electric charges are placed in a dielectric medium adjoining a conducting plane, some of the conductor's electric charges move to its surface boundary, thereby satisfying the boundary conditions outlined in Table 4-3. This module displays **E** and V everywhere and ρ_s along the dielectric-conductor boundary.



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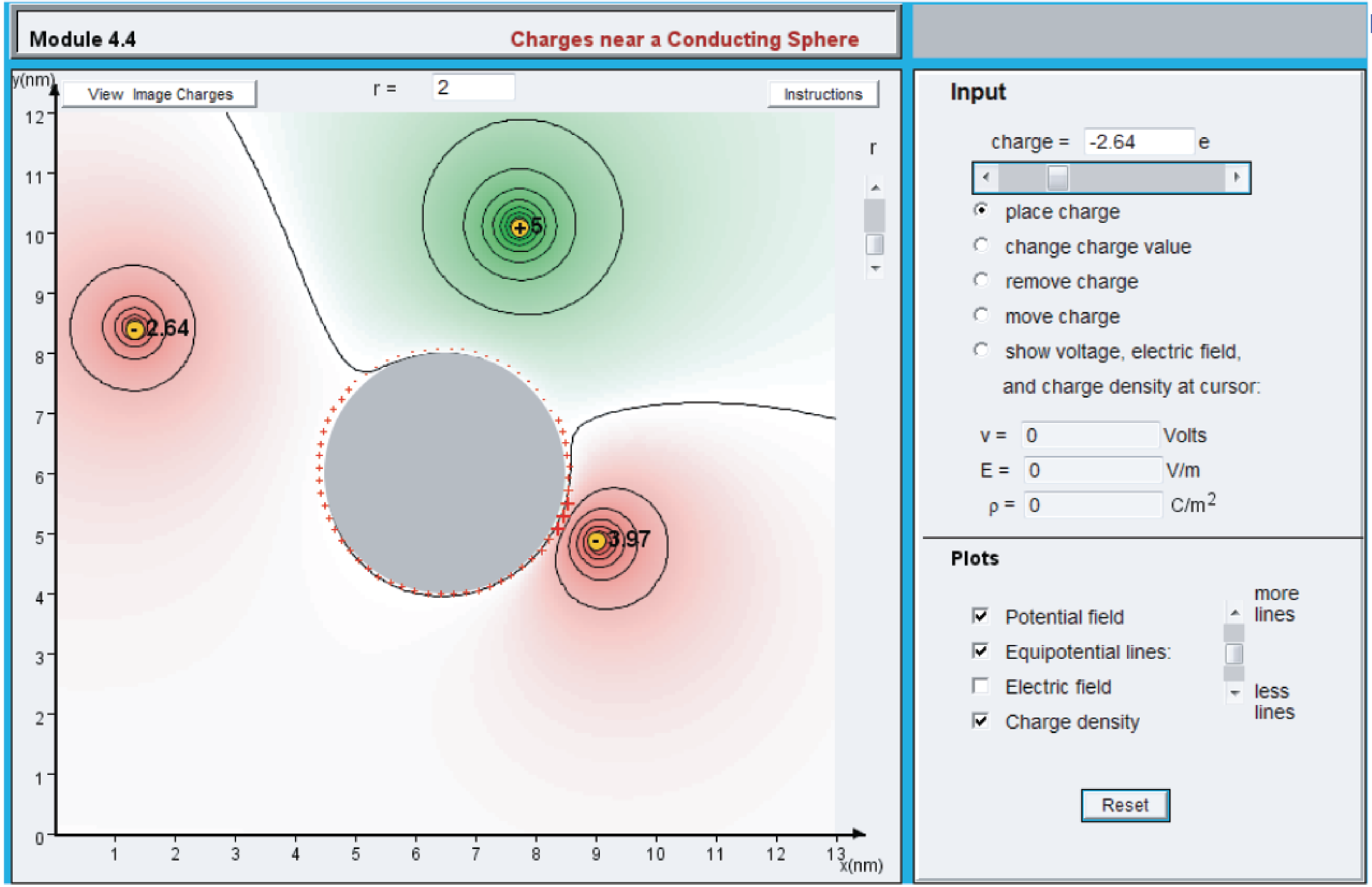
CD Module 4.4 Charges near a Conducting Sphere

This module is similar to Module 4.3, except that now the conducting body is a sphere of selectable size.



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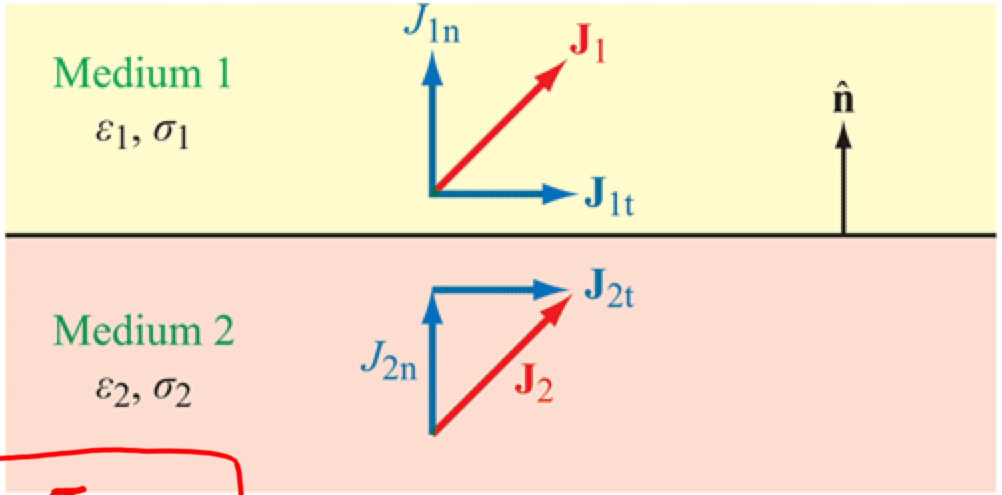
Conductor-Conductor Boundary

- The general case of the boundary between two media—neither of which is a perfect dielectric or a perfect conductor.



E-fields give rise to J:

$\vec{J}_1 = \sigma_1 \vec{E}_1 \quad , \quad \vec{J}_2 = \sigma_2 \vec{E}_2$

1st BC: $\vec{E}_{1t} = \vec{E}_{2t} \Rightarrow \dfrac{\vec{J}_{1t}}{\sigma_1} = \dfrac{\vec{J}_{2t}}{\sigma_2} \Rightarrow \boxed{\vec{J}_{1t} = \dfrac{\sigma_1}{\sigma_2}\vec{J}_{2t}}$

2nd BC: $D_{1n} - D_{2n} = \rho_s \Rightarrow \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s \Rightarrow \varepsilon_1 \dfrac{J_{1n}}{\sigma_1} - \varepsilon_2 \dfrac{J_{2n}}{\sigma_2} = \rho_s$

If $J_{1n} \neq J_{2n}$, then ρ_s not constant in time → violates electrostatics

$\therefore$ J has to be continuous ($J_{1n} = J_{2n}$) $\Rightarrow \boxed{J_{1n}\left(\dfrac{\varepsilon_1}{\sigma_1} - \dfrac{\varepsilon_2}{\sigma_2}\right) = \rho_s}$

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Example: Boundary Conditions on a Cylindrical Conductor

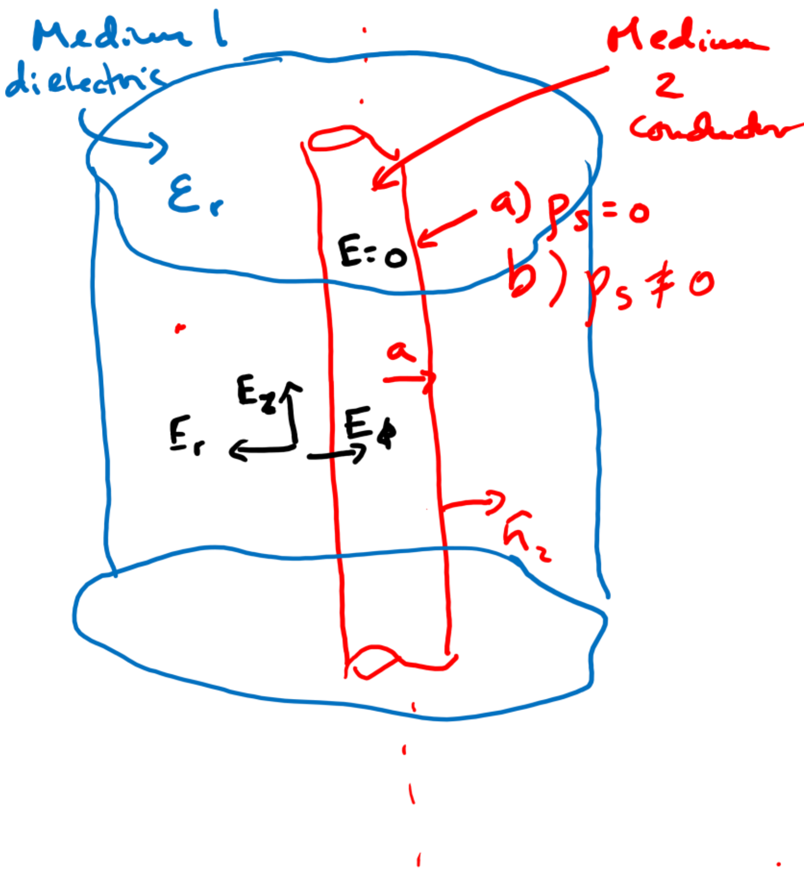
a) For $r > a$, $\bar{E}_{1t} = -\hat{\phi}\,\frac{\cos\phi}{r^2}$. Find $\bar{E}_1$

$$\bar{E}_1 = \bar{E}_{1n} + \bar{E}_{1t} = \hat{r}E_r - \hat{\phi}\,\frac{\cos\phi}{r^2}$$

Apply Gauss' law: $\nabla \cdot \bar{D}_1 = \cancel{\rho_s}^{\,0}$

$$\frac{1}{\cancel{r}}\frac{\partial}{\partial r}(rE_r) + \frac{1}{\cancel{r}}\frac{\partial}{\partial \phi}\left(-\frac{1}{r^2}\cos\phi\right) = 0$$

$$\frac{\partial}{\partial r}(rE_r) = -\frac{1}{r^2}\sin\phi$$



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Example: Boundary Conditions on a Cylindrical Conductor

$$\int \frac{\partial}{\partial r}(rE_r)\,dr = -\sin\phi \int \frac{1}{r^2}\,dr$$

$$rE_r = \frac{1}{r}\sin\phi$$

$$E_r = \frac{1}{r^2}\sin\phi$$

$$\boxed{\bar{E}_1 = \hat{r}\frac{1}{r^2}\sin\phi - \hat{\phi}\frac{1}{r^2}\cos\phi}$$

b) Now, conductor has $\rho_s \neq 0$.
Find ρ_s.

1^{st} BC : $E_{1t} = \cancel{E_{2t}}^{0} = 0$

On the surface of the conductor:

$$\bar{E}_1 = \hat{r}\frac{1}{r^2}\sin\phi$$

2^{nd} B : $\hat{n}_2 \cdot (\bar{D}_1 - \cancel{\bar{D}_2}^{0}) = \underline{\rho_s}$

$$\hat{n}_2 = \hat{r}$$

$$\rho_s = \hat{r}\cdot\bar{D}_1\Big|_{r=a} = \varepsilon_1 E_1\Big|_{r=a}$$

$$\boxed{\rho_s = \frac{\varepsilon_r\varepsilon_0}{a^2}\sin\phi \quad C/m^2}$$

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Capacitance

When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor.

The ***capacitance*** of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad \text{(C/V or F)}, \qquad (4.105)$$

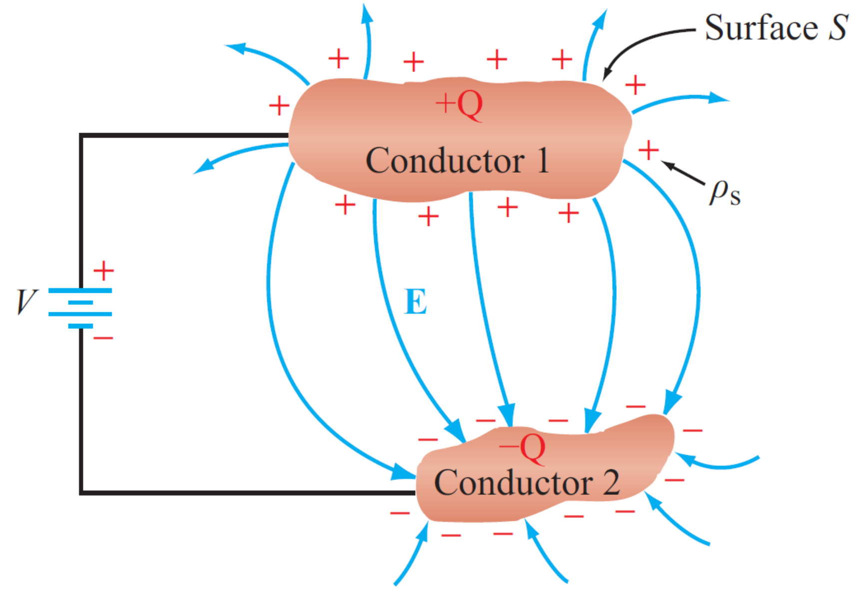

Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

The charges on the two conductors' surfaces give rise to an electric field between the them. Since the tangential component of **E** always goes to zero on a conductor's surface, **E** is always perpendicular to the conducting surfaces.

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Capacitance

For any two-conductor configuration:

$C = \frac{Q}{V}$

$$C = \frac{\displaystyle\int_S \overbrace{\varepsilon \mathbf{E}}^{\vec{D}} \cdot d\mathbf{s}}{-\displaystyle\int_l \mathbf{E} \cdot d\mathbf{l}} \quad \text{(F)}, \quad (4.109)$$

} Gauss' law

} Potential

$V = V_{12} = -\int_{P_2}^{P_1} \vec{E} \cdot d\vec{l}$

For any resistor:

If medium between the 2 conductors is imperfect diele with σ, I can flow and there R.

$$R = \frac{\overbrace{-\displaystyle\int_l \mathbf{E} \cdot d\mathbf{l}}^{V}}{\underbrace{\displaystyle\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}_{I}} \quad (\Omega). \qquad (4.110)$$

For a medium with uniform σ and ε, the product of Eqs. (4.109) and (4.110) gives

$$RC = \frac{\varepsilon}{\sigma}. \qquad (4.111)$$

This simple relation allows us to find R if C is known, or vice versa.

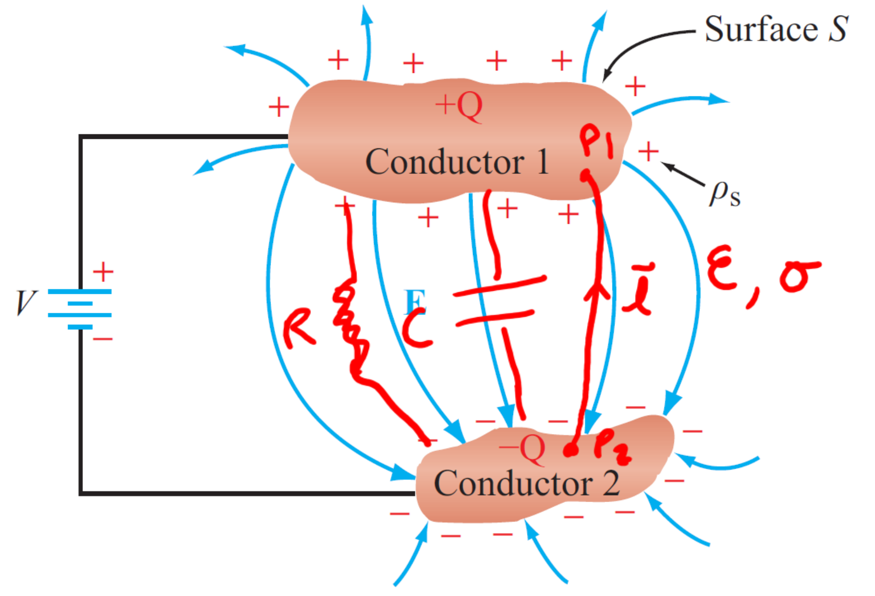

Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

Since $\vec{E}$ is in numerator and denominator of Eq. 4.109, the C is independent of $|\vec{E}|$, but rather it is dep. on geometry and ε.

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Example: Capacitance of a Parallel-Plate Capacitor

$$C = \frac{Q}{V}$$

$$V = -\int_{z=0}^{z=d} \vec{E}\cdot d\vec{l} = -\int_{0}^{d} (-E\hat{z})\cdot(dz\,\hat{z})$$

$$V = Ed$$

Applying Gauss' law:

$$\oint_S \varepsilon\vec{E}\cdot d\vec{s} = Q$$

$$\varepsilon E A = Q$$

$$E = \frac{Q}{\varepsilon A}$$

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon A}{d} = \varepsilon_0\varepsilon_r\frac{A}{d}}$$

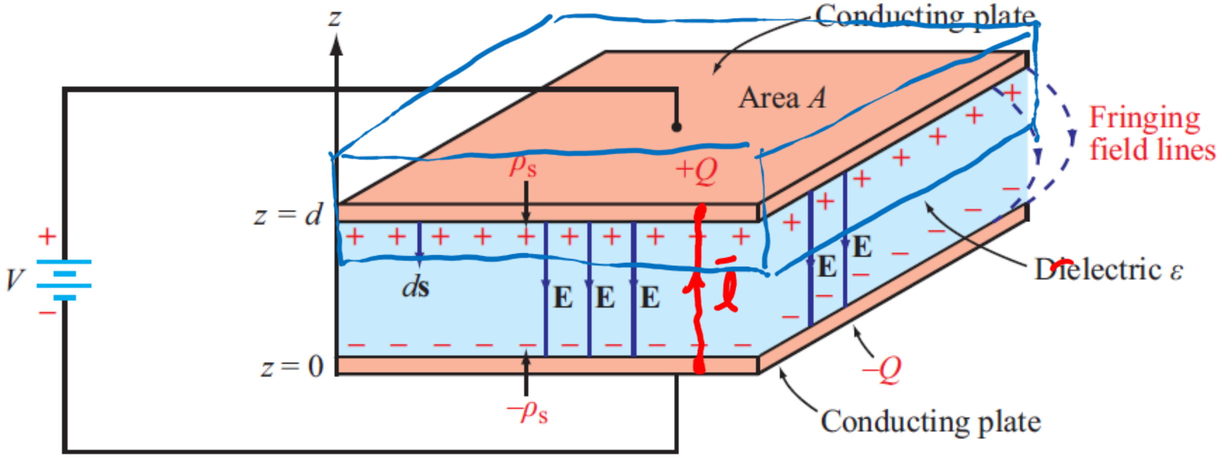

Figure 4-24: A dc voltage source connected to a parallel-plate capacitor (Example 4-11).

We can neglect fringing effects because $A \gg d$, so most of E-field is between the 2 plates.

Example: Capacitance pu Length of a Coaxial Line

↳ per unit

Application of Gauss's law: $\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$

$$\oint_S \varepsilon E_r(-\hat{r}) \cdot d\vec{s}_r = Q$$

$$\oint_S \varepsilon E_r(-\hat{r}) \cdot (\hat{r}\, r\, d\phi\, dz) = Q$$

$$-\varepsilon r E_r \int_0^{2\pi} d\phi \int_0^{\ell} dz = Q$$

$$-\varepsilon r E_r (2\pi)(\ell) = Q \Rightarrow E_r = \frac{-Q}{2\pi\varepsilon r\ell}$$

$$\vec{E} = -\hat{r}\frac{Q}{2\pi\varepsilon r\ell}$$

$$V = -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \left(-\hat{r}\frac{Q}{2\pi\varepsilon r\ell}\right) \cdot \hat{r}\, dr = \frac{Q}{2\pi\varepsilon\ell}\ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon\ell}{\ln(b/a)} \quad F$$

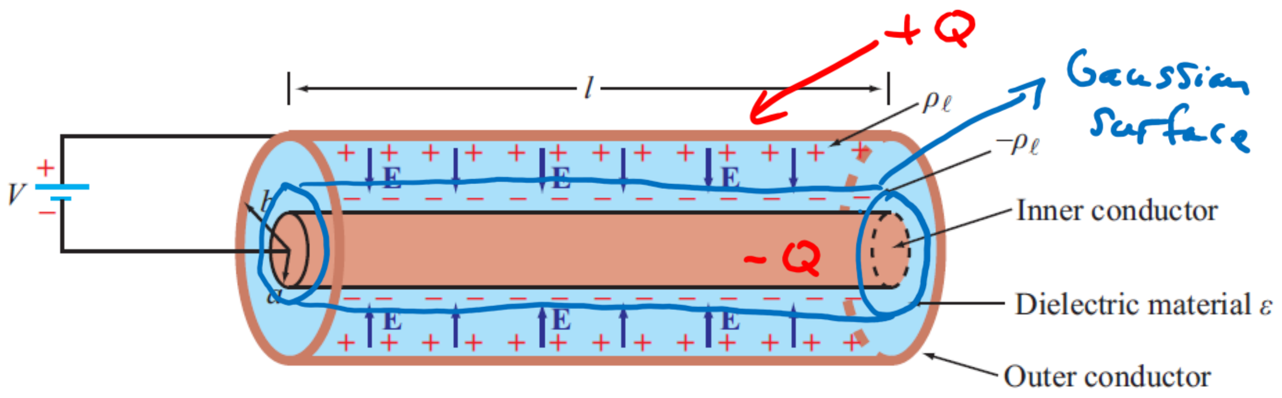

Figure 4-25: Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Q is the total charge on the inside of the outer cylinder, and –Q is the total charge on the outside surface of the inner cylinder

$$\Rightarrow \boxed{\text{Capacitance pu length is: } C' = \frac{C}{\ell} = \frac{2\pi\varepsilon}{\ln(b/a)} \text{ F/m}}$$

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Electrostatic Potential Energy

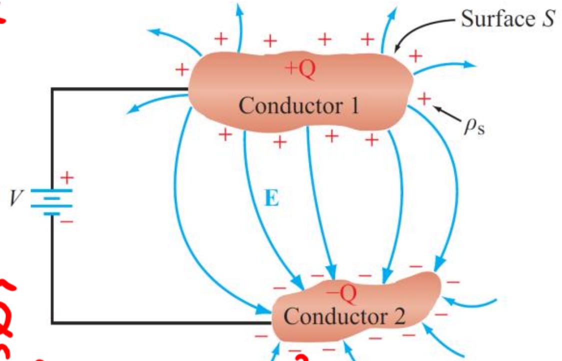
When we connect a source across a capacitor, it charges up. Where does the energy from the source go though?

- It's stored in the dielectric medium between the 2 conductors in form of electrostatic potential energy.
- Work to transfer dq between conductors:

$dW_e = v\,dq = \frac{q}{C}dq \Rightarrow W_e = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$

intermediate voltage

final voltage



Electrostatic potential energy density (Joules/volume)

$$w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2}\varepsilon E^2 \qquad (\text{J/m}^3).$$

volume

Energy stored in a capacitor

$$W_e = \tfrac{1}{2}CV^2 \qquad (\text{J}).$$

Total electrostatic energy stored in a volume

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon E^2 \, d\mathcal{V} \qquad (\text{J})$$

For a parallel-plate capacitor: $W_e = \frac{1}{2}\left(\frac{\varepsilon A}{d}\right)(Ed)^2 = \frac{1}{2}\varepsilon E^2 (Ad)$

$= \frac{1}{2}\varepsilon E^2 \mathcal{V}$ volume

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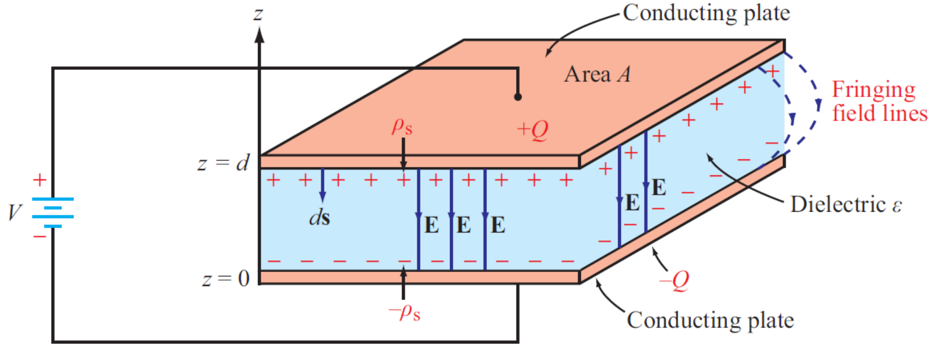
Tech Brief 8: Supercapacitors

For a traditional parallel-plate capacitor, what is the maximum attainable energy density?



Energy density is given by: $W' = \dfrac{\left(\frac{1}{2}CV^2\right)}{\rho A d}$, where $C = \dfrac{\varepsilon A}{d}$

$$W' = \frac{\varepsilon V^2}{2\rho d^2} \quad \text{(J/kg)}$$

$= \dfrac{\left(\frac{1}{2}\frac{\varepsilon A}{d}V^2\right)}{\rho A d}$

$= \dfrac{\varepsilon V^2}{2\rho d^2}$

ε = permittivity of insulation material
V = applied voltage
ρ = density of insulation material
d = separation between plates

Mica has one of the highest dielectric strengths ~2 x 10^8 V/m.
If we select a voltage rating of 1 V and a breakdown voltage of 2 V (50% safety), this will require that d be no smaller than 10 nm.
For mica, $\varepsilon = 6\varepsilon_0$ and $\rho = 3 \times 10^3$ kg/m^3 .

$V_{br} = E_{ds} d$

$d = \dfrac{V_{br}}{E_{ds}} = \dfrac{2V}{2\times10^8 \,\frac{V}{m}} = 10^{-8}\,m = 10nm$

Hence:

W' = 90 J/kg = 2.5 x10^{-2} Wh/kg.

By comparison, a lithium-ion battery has W' = 1.5 x 10^2 Wh/kg, almost 4 orders of magnitude greater. But very slow to charge/discharge rates.

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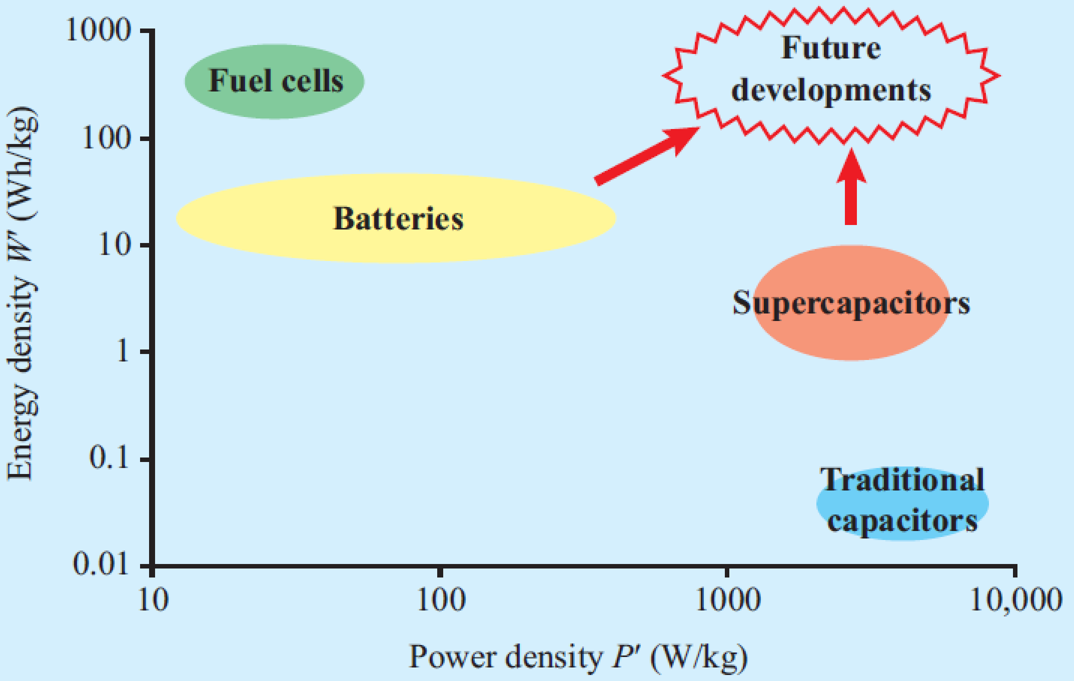
Energy Comparison

Energy Storage Devices

Feature	Traditional Capacitor	Supercapacitor	Battery
Energy density W' (Wh/kg)	$\sim 10^{-2}$	1 to 10	5 to 150
Power density P' (W/kg)	1,000 to 10,000	1,000 to 5,000	10 to 500
Charge and discharge rate T	10^{-3} sec	$\sim$ 1 sec to 1 min	$\sim$ 1 to 5 hrs
Cycle life N_c	∞	$\sim 10^6$	$\sim 10^3$



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A supercapacitor is a “hybrid” battery/capacitor

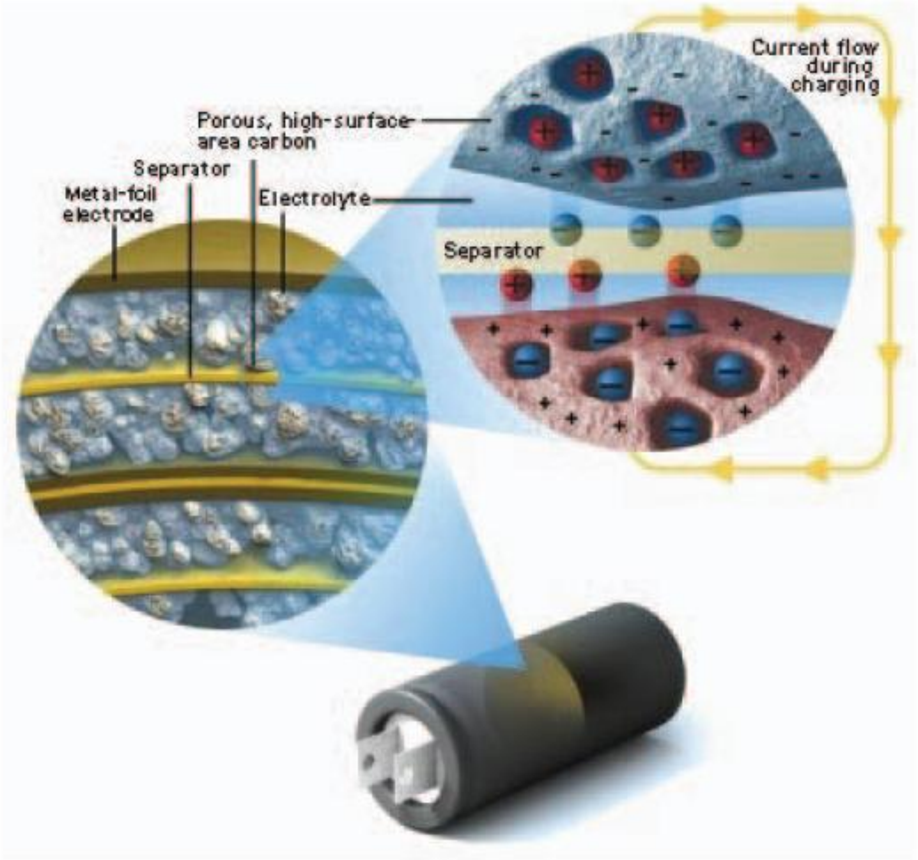

Figure TF8-1: Cross-sectional view of an electrochemical double-layer capacitor (EDLC), otherwise known as a supercapacitor. (Courtesy of Ultracapacitor.org.)

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Users of Supercapacitors

Figure TF8-2: Examples of systems that use supercapacitors. (Courtesy of Railway Gazette International; BMW; NASA; Applied Innovative Technologies.)

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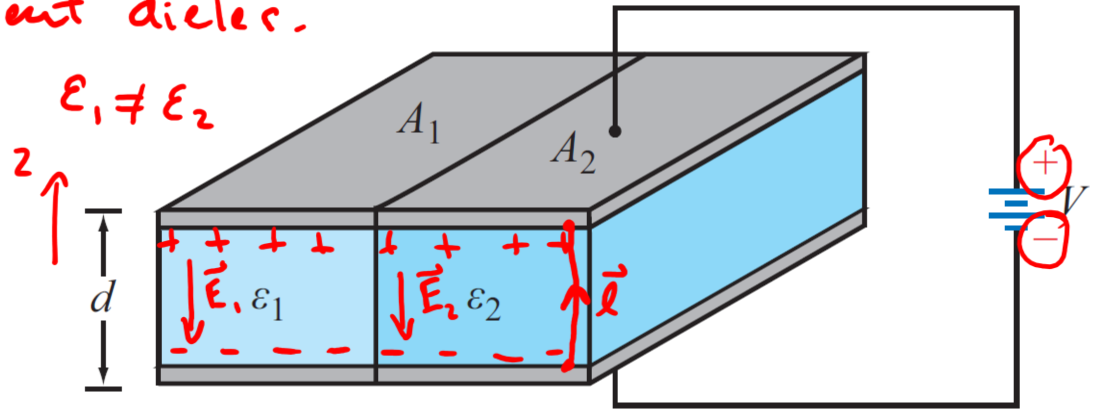
Example: Parallel-Plate Capacitor (C in Parallel)

Consider a parallel-plate cap. with 2 adjacent dielec.

$\varepsilon_1 \neq \varepsilon_2$



a) Find $\vec{E}_1$ and $\vec{E}_2$.

$$V = -\int_\ell \vec{E}_1 \cdot d\vec{\ell} = -\int_\ell (-E_1 \hat{z}) \cdot d\vec{\ell} = E_1 d$$

$\Rightarrow E_1 = \frac{V}{d}$ Similarly, $E_2 = \frac{V}{d}$

$\vec{E}_1 = -\frac{V}{d}\hat{z}$ $\vec{E}_2 = -\frac{V}{d}\hat{z}$

b) Calculate energy stored in each section.

Electrostatic energy stored in volume V: $W_e = \frac{1}{2}\int \varepsilon E^2 dV$

$$W_{e1} = \frac{1}{2}\varepsilon_1 E_1^2 V_1 = \frac{1}{2}\varepsilon_1\left(\frac{V}{d}\right)^2 (A_1 d) = \frac{1}{2}\varepsilon_1 V^2 \frac{A_1}{d} = \frac{1}{2} C_1 V^2 \quad \therefore C_1 = \frac{\varepsilon_1 A_1}{d}$$

$$W_{e2} = \frac{1}{2}\varepsilon_2 E_2^2 V_2 = \frac{1}{2}\varepsilon_2\left(\frac{V}{d}\right)^2 (A_2 d) = \frac{1}{2}\varepsilon_2 V^2 \frac{A_2}{d} = \frac{1}{2} C_2 V^2 \quad \therefore C_2 = \frac{\varepsilon_2 A_2}{d}$$

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Example: Parallel-Plate Capacitor (C in Parallel)

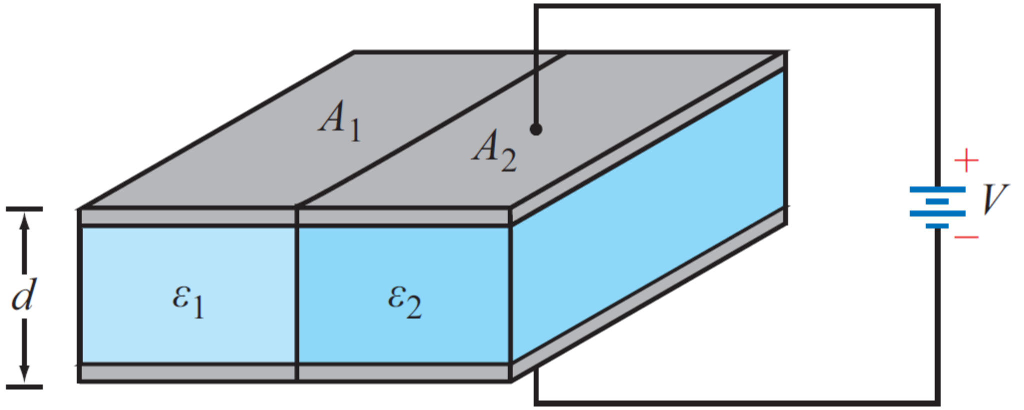
c) Use total energy stored in cap. to find an expression for total C.

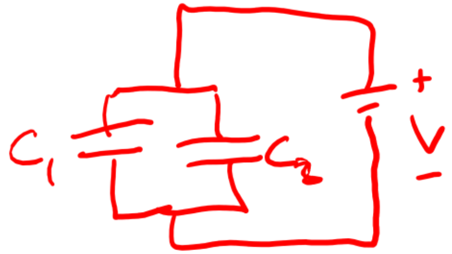
$$W_e = W_{e1} + W_{e2}$$

$$= \frac{1}{2}\varepsilon_1 V^2 \frac{A_1}{d} + \frac{1}{2}\varepsilon_2 V^2 \frac{A_2}{d}$$

$$= \frac{1}{2}V^2\left(\frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d}\right) = \frac{1}{2}C_{tot}V^2$$

$$\Rightarrow C_{tot} = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d} = C_1 + C_2$$ //





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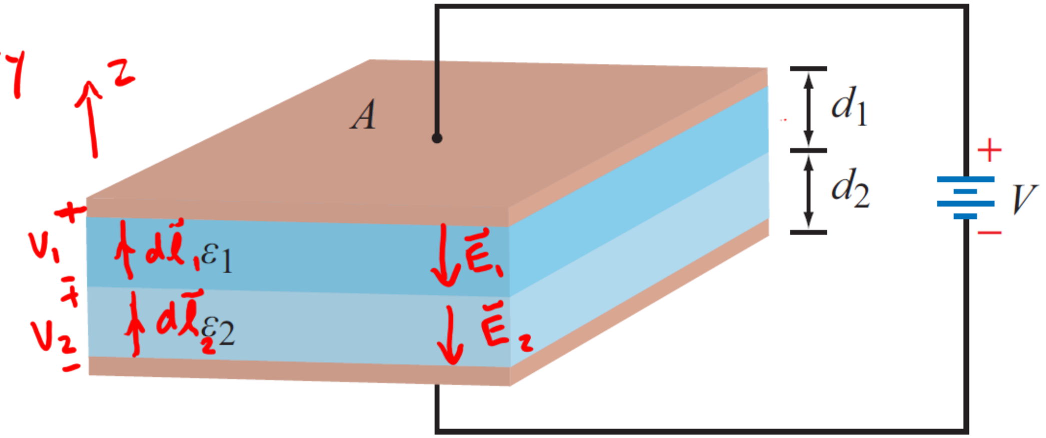
Example: Parallel-Plate Capacitor (C in Series)

Consider a parallel-plate cap. with 2 stacked dielec. Assume zero surface charge at boundary between dielecs.



a) Find $\vec{E}_1$ and $\vec{E}_2$.

$$V_1 = -\int_{l_1} \vec{E}_1 \cdot d\vec{l}_1 = -\int_0^{d_1} (-\vec{E}_1 \hat{z}) \cdot dz\hat{z} = E_1 d_1$$

$$V_2 = -\int_{l_2} \vec{E}_2 \cdot d\vec{l}_2 = -\int_0^{d_2} (-\vec{E}_2 \hat{z}) \cdot dz\hat{z} = E_2 d_2$$

At the boundary between dielecs.

$$2^{nd}\ BC: D_{1n} - D_{2n} = \cancel{\rho_s}\, 0 \Rightarrow D_{1n} = D_{2n} \Rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \Rightarrow E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n}$$

$$V = V_1 + V_2 = E_1 d_1 + E_2 d_2 = E_1 d_1 + \left(\frac{\varepsilon_1}{\varepsilon_2} E_1\right) d_2$$

$$\vec{E}_1 = \frac{-V}{\left(d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2\right)} \hat{z} \qquad \text{Similarly, } \vec{E}_2 = \frac{-V}{\left(d_2 + \frac{\varepsilon_2}{\varepsilon_1} d_1\right)} \hat{z}$$

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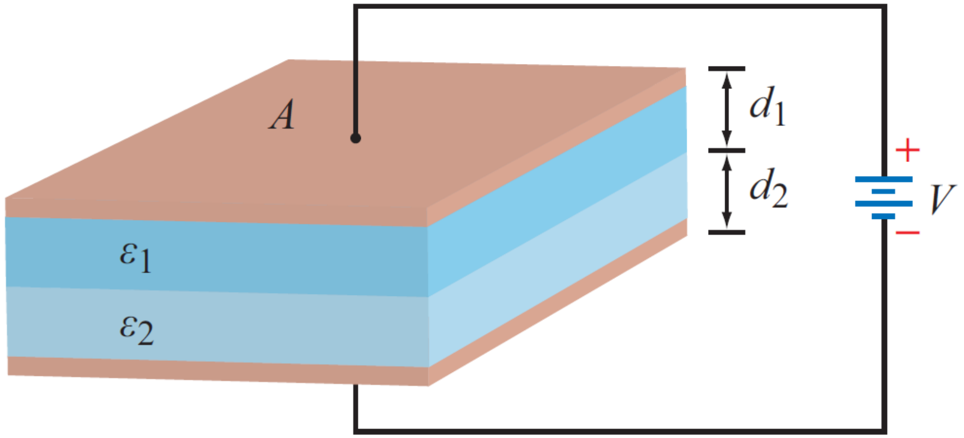
Example: Parallel-Plate Capacitor (C in Series)

b) Calculate energy stored in each dielec layer.
Find total C.



$$W_e = \frac{1}{2}\int_V \varepsilon E^2 \, dV$$

$$W_{e1} = \frac{1}{2}\varepsilon_1 E_1^2 V_1 = \frac{1}{2}\varepsilon_1\left(\frac{V}{d_1 + \frac{\varepsilon_1}{\varepsilon_2}d_2}\right)^2 A d_1$$

$$= \frac{1}{2}V^2\left(\frac{\varepsilon_1\varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}\right)$$

$$W_{e2} = \frac{1}{2}\varepsilon_2 E_2^2 V_2 = \frac{1}{2}\varepsilon_2\left(\frac{V}{d_2 + \frac{\varepsilon_2}{\varepsilon_1}d_1}\right)^2 A d_2 = \frac{1}{2}V^2\left(\frac{\varepsilon_1^2\varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right)$$

$$\Rightarrow W_e = W_{e1} + W_{e2} = \frac{1}{2}V^2 \underbrace{\left[\frac{\varepsilon_1\varepsilon_2^2 A d_1 + \varepsilon_1^2\varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right]}_{C_{tot}} = \frac{1}{2} C_{tot} V^2$$

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Example: Parallel-Plate Capacitor (C in Series)

$$C_{tot} = \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}$$

$$= \varepsilon_1 \varepsilon_2 A \frac{\cancel{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^{\cancel{2}}}$$

$$= \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2} \frac{\times \frac{A}{d_1 d_2}}{\times \frac{A}{d_1 d_2}} = \frac{\overbrace{\left(\frac{\varepsilon_1 A}{d_1}\right)}^{C_1} \overbrace{\left(\frac{\varepsilon_2 A}{d_2}\right)}^{C_2}}{\underbrace{\left(\frac{\varepsilon_2 A}{d_2}\right)}_{C_2} + \underbrace{\left(\frac{\varepsilon_1 A}{d_1}\right)}_{C_1}}$$

$$\Rightarrow C_{tot} = \frac{C_1 C_2}{C_1 + C_2}$$

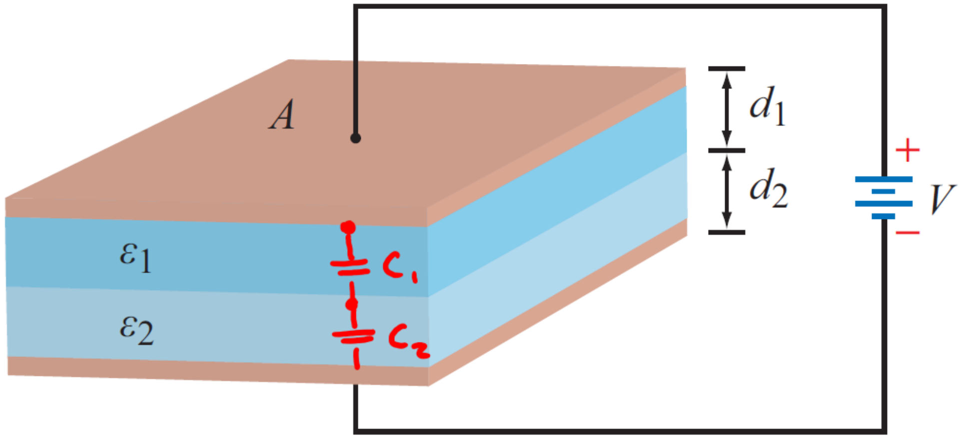

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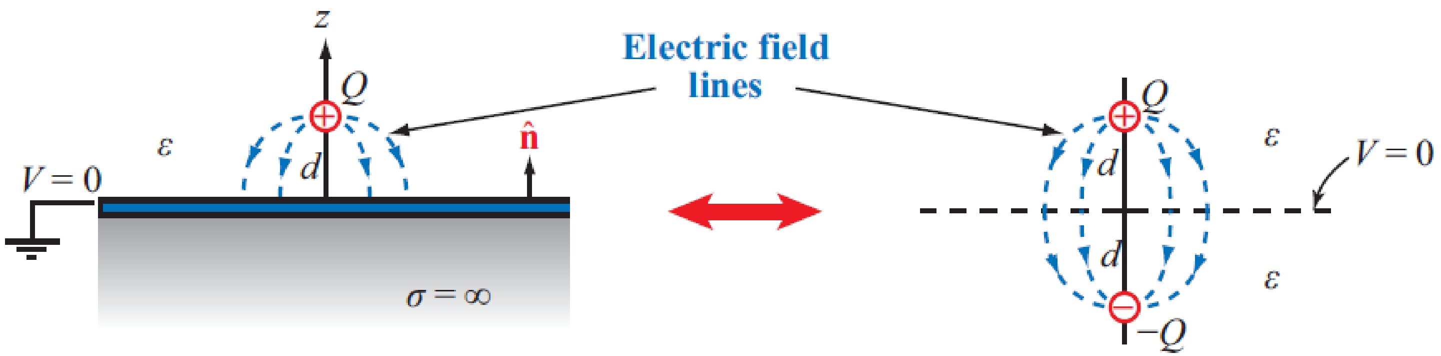
Image Method



(a) Charge Q above grounded plane

(b) Equivalent configuration

Figure 4-26: By image theory, a charge Q above a grounded perfectly conducting plane is equivalent to Q and its image $-Q$ with the ground plane removed.

Image method simplifies calculation for **E** and *V* due to charges near conducting planes.

Applies also to continuous charge distribution.

1. For each charge Q, add an image charge –Q
2. Remove conducting plane
3. Calculate field due to all charges

Example: Image Method

Example 4-13: Image Method for Charge Above Conducting Plane

Use image theory to determine $\mathbf{E}$ at an arbitrary point $P = (x, y, z)$ in the region $z > 0$ due to a charge Q in free space at a distance d above a grounded conducting plate residing in the $z = 0$ plane.

Solution: In Fig. 4-28, charge Q is at $(0, 0, d)$ and its image $-Q$ is at $(0, 0, -d)$. From Eq. (4.19), the electric field at point $P = (x, y, z)$ due to the two charges is given by

Coulomb's law:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\mathbf{R}_1}{R_1^3} + \frac{-Q\mathbf{R}_2}{R_2^3}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z-d)}{[x^2 + y^2 + (z-d)^2]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z+d)}{[x^2 + y^2 + (z+d)^2]^{3/2}}\right]$$

for $z \geq 0$.

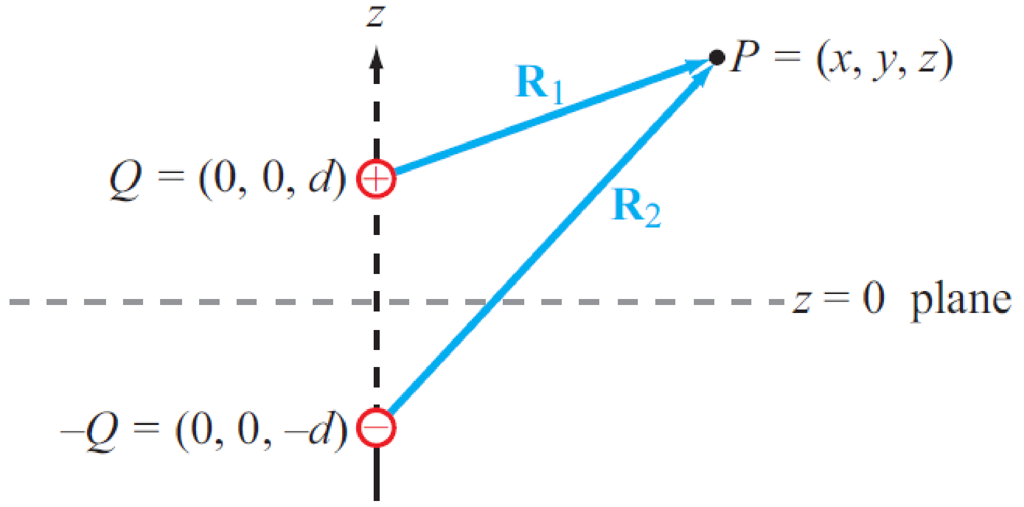


Figure 4-28: Application of the image method for finding $\mathbf{E}$ at point P (Example 4-13).

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Tech Brief 9: Capacitive Sensors

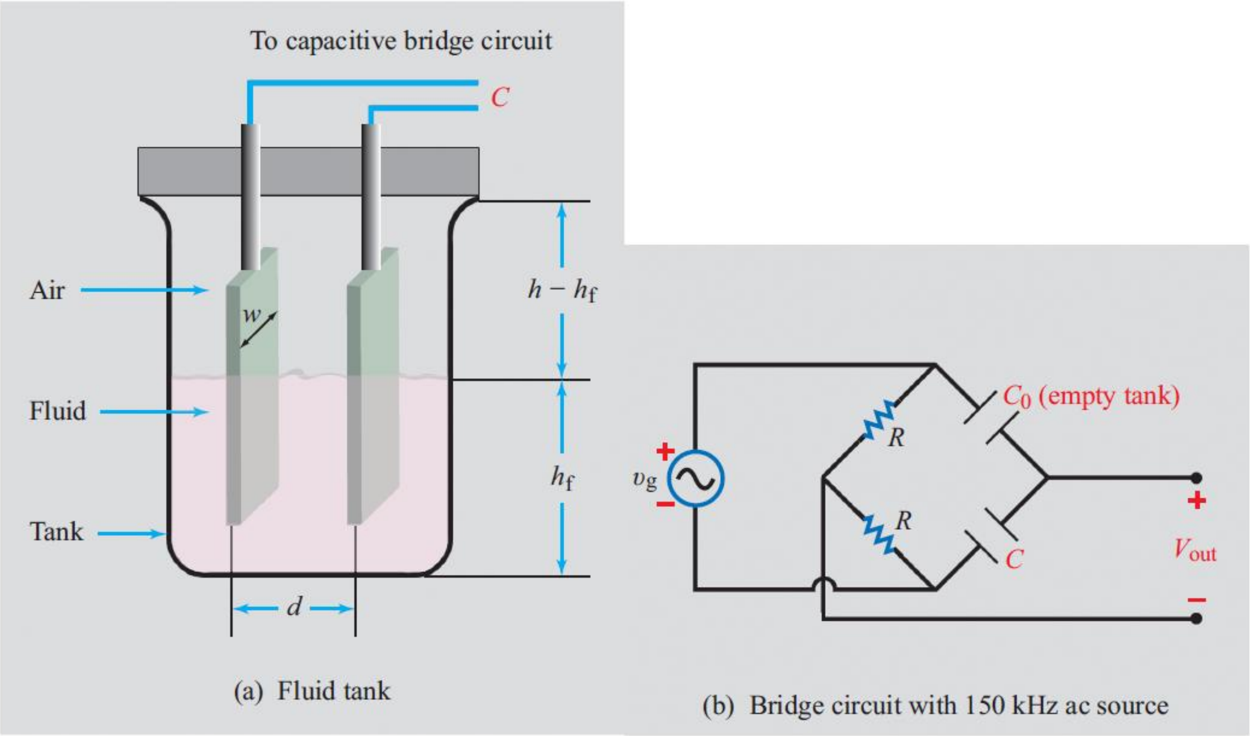

Figure TF9-1: Fluid gauge and associated bridge circuit, with C_0 being the capacitance that an empty tank would have and C the capacitance of the tank under test.

Fluid Gauge

The two metal electrodes in Fig. TF9-1(a), usually rods or plates, form a capacitor whose capacitance is directly proportional to the ***permittivity*** of the material between them. If the fluid section is of height h_f and the height of the empty space above it is $(h - h_f)$, then the overall capacitance is equivalent to two capacitors in parallel, or

$$C = C_f + C_a = \varepsilon_f w \frac{h_f}{d} + \varepsilon_a w \frac{(h - h_f)}{d},$$

where w is the electrode plate width, d is the spacing between electrodes, and ε_f and ε_a are the permittivities of the fluid and air, respectively. Rearranging the expression as a linear equation yields

$$C = kh_f + C_0,$$

$K = (\varepsilon_f - \varepsilon_a) w / d$, $C_0 = \varepsilon_a w h / d$

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Humidity Sensor

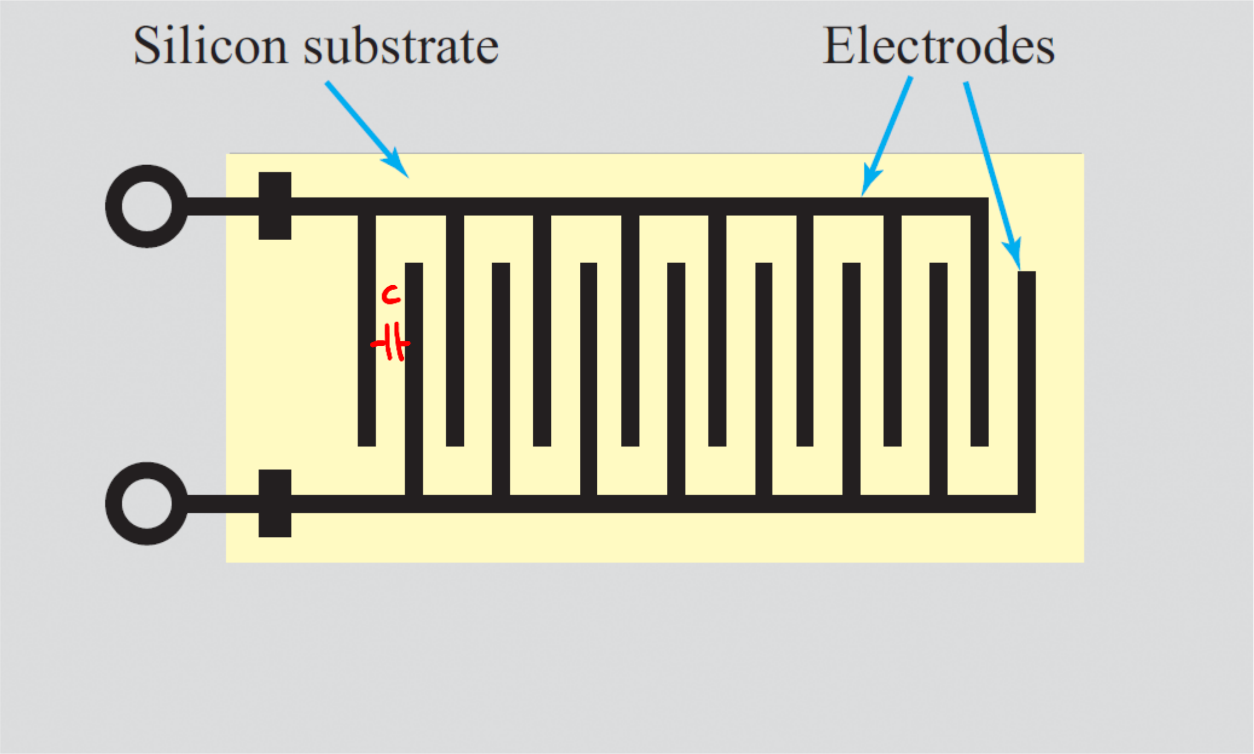

Figure TF9-2: Interdigital capacitor used as a humidity sensor.

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Pressure Sensor

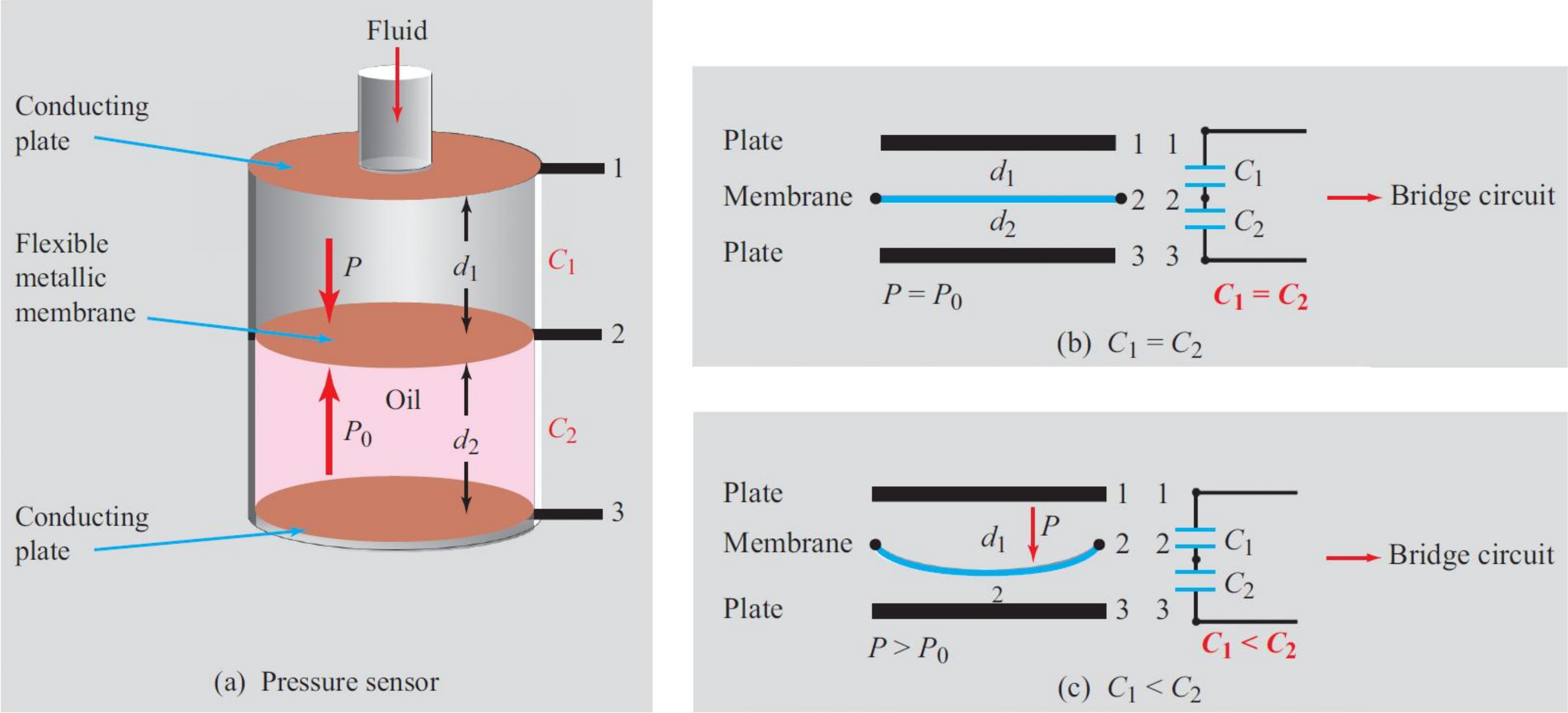

Figure TF9-3: Pressure sensor responds to deflection of metallic membrane.

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Planar capacitors

Non contact capacitance sensors

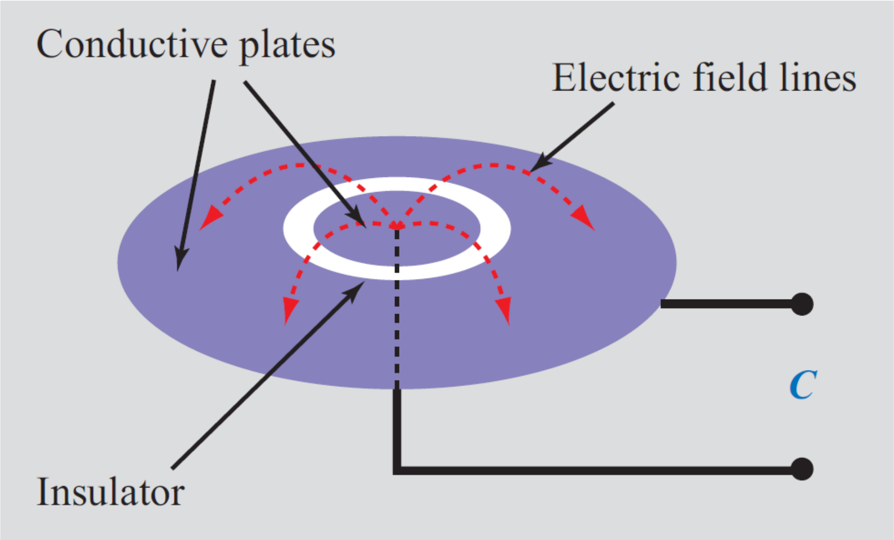

Figure TF9-4: Concentric-plate capacitor.

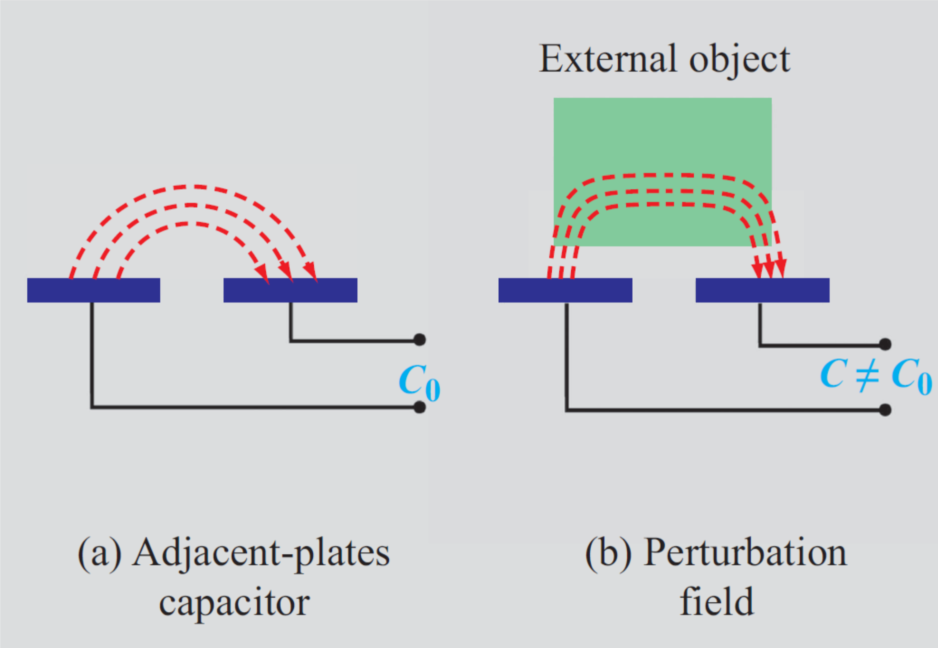

Figure TF9-5: (a) Adjacent-plates capacitor; (b) perturbation field.

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Fingerprint Imager

2D array of adjacent-plates capacitors.
(~ 65 um)

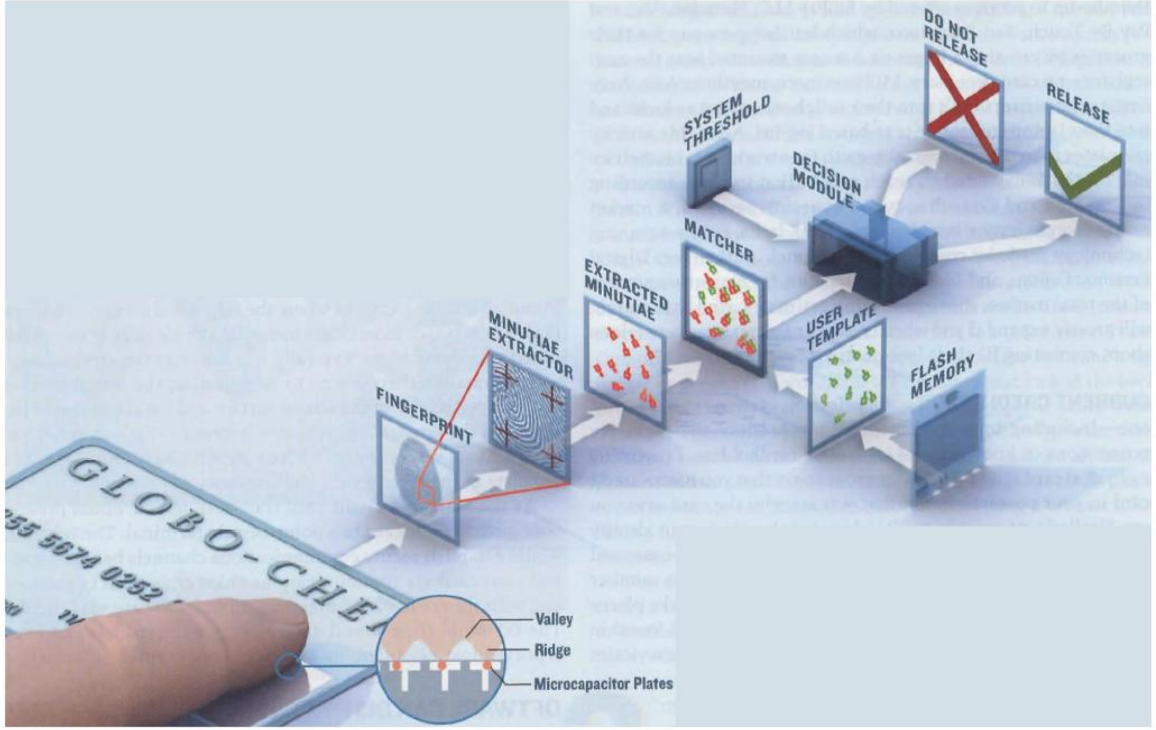

Figure TF9-6: Elements of a fingerprint matching system. (Courtesy of IEEE Spectrum.)

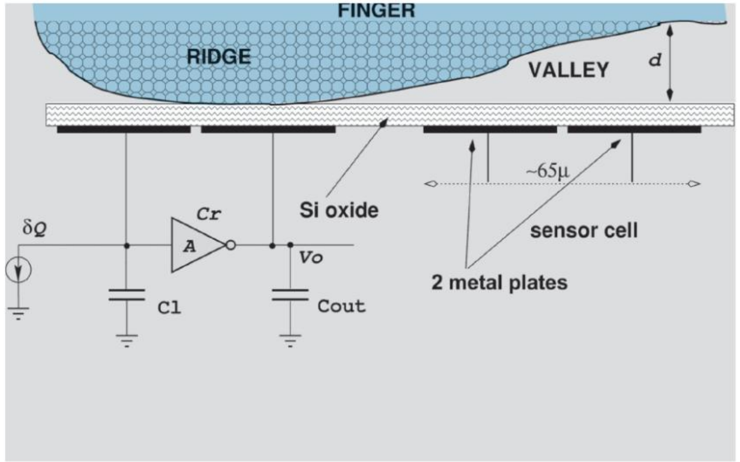

Figure TF9-7: Fingerprint representation. (Courtesy of Dr. M. Tartagni, University of Bologna, Italy.)

References

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- Sections: 4-1 to 4-11