4.0.0 Continuous Random Variables

4.0.0 Introduction

Remember that discrete random variables can take only a countable number of possible values. On the other hand, a continuous random variable X has a range in the form of an interval or a union of non-overlapping intervals on the real line (possibly the whole real line). Also, for any $x \in \mathbb{R}$, $P(X = x) = 0$. Thus, we need to develop new tools to deal with continuous random variables. The good news is that the theory of continuous random variables is completely analogous to the theory of discrete random variables. Indeed, if we want to oversimplify things, we might say the following: take any formula about discrete random variables, and then replace *sums* with *integrals*, and replace *PMFs* with probability density functions (*PDFs*), and you will get the corresponding formula for continuous random variables. Of course, there is a little bit more to the story and that's why we need a chapter to discuss it. In this chapter, we will also introduce mixed random variables that are mixtures of discrete and continuous random variables.

$X = [0, 1]$

4.1.0 Cumulative Distribution Function (CDF)

I choose a real number uniformly at random in the interval $[a, b]$, and call it X. By uniformly at random, we mean all intervals in $[a, b]$ that have the same length must have the same probability. Find the CDF of X.

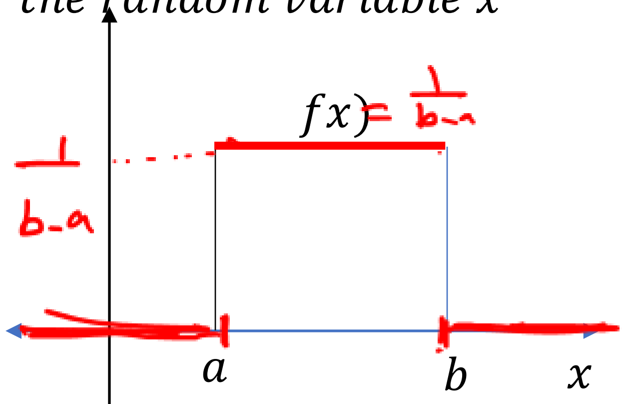
$$Let\ f(x) = \frac{1}{b-a} = \frac{1}{Length\ of\ the\ interval\ [a,b]}$$ *represents the uniform probability of the random variable x*

$P(a \le X \le b) = \frac{1}{b-a}$

$$\boxed{f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & x < a \text{ or } x > b \end{cases}}$$

This is the probability density function of f(x)

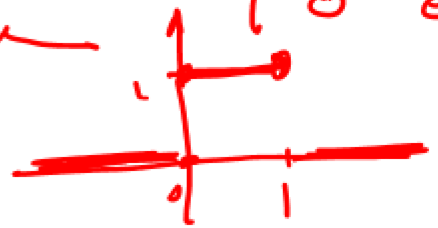
PDF



Discrete $f_X(x) = \begin{cases} \frac{1}{n} & n = 1, \text{---} \\ 0 & \text{else} \end{cases}$ PMF

Let X = Cont. R.V. on $[0, 1]$

PDF = $f_X(x) = \begin{cases} \frac{1}{1-0} & 0 \le x \le 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

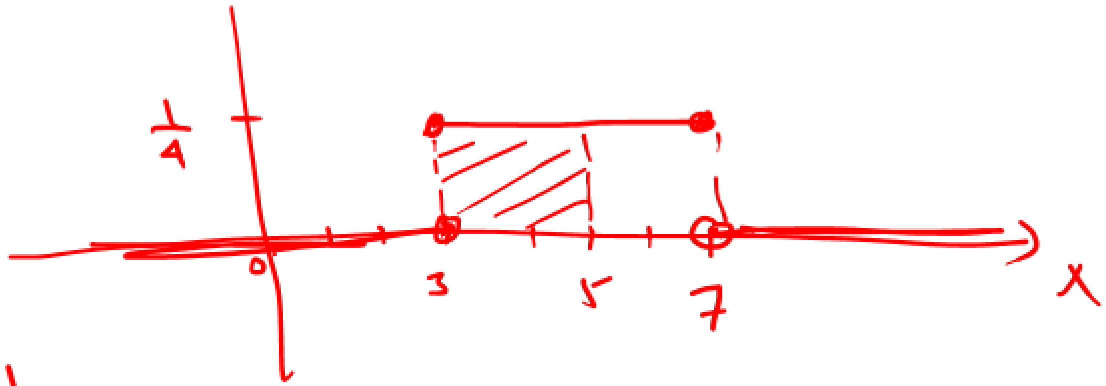


Ex : If X = a Uniform Cont. R.V on $[3, 7]$

$$PDF = \begin{cases} \frac{1}{7-3} & 3 \le x \le 7 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{4} & 3 \le x \le 7 \\ 0 & \text{else} \end{cases}$$

$$CDF = \begin{cases} 0 & x < 3 \\ \frac{x-3}{7-3} & 3 \le x \le 7 \\ 1 & x > 7 \end{cases} = \begin{cases} 0 & x < 3 \\ \frac{x-3}{4} & 3 \le x \le 7 \\ 1 & x > 7 \end{cases}$$

$$P(X \le 5) = F_X(x=5) = \frac{5-3}{4} = \frac{2}{4} = \frac{1}{2}$$



$$P(\underline{3 \le X \le 7}) = \frac{1}{4}$$

$P(X \le 5) = ?$ $P(X \le 10)$

4.1.0 Cumulative Distribution Function (CDF)

I choose a real number uniformly at random in the interval $[a, b]$, and call it X. By uniformly at random, we mean all intervals in $[a, b]$ that have the same length must have the same probability. Find the CDF of X.

Let $f(x) = \dfrac{1}{b-a}$ *represents the uniform probability of the random variable x*

Then, the CDF $\boxed{F_X(x) = P(X \leq x)}$ $=$ *Area to the left of* $X = x$ *as shown in the graph*

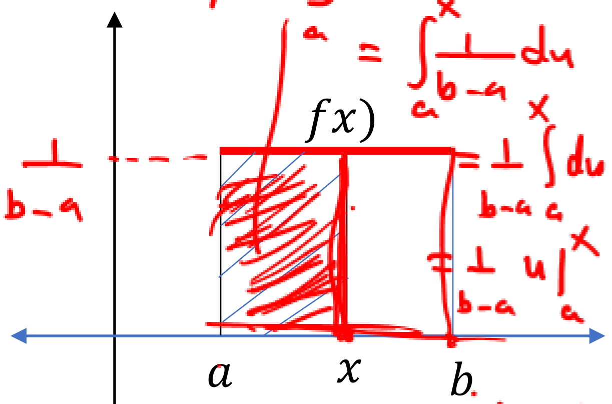
$$= \frac{x-a}{b-a}$$

$$A = \int_a^b f(x)\,dx$$

$$\text{Hence the CDF}, F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \boxed{\dfrac{x-a}{b-a}} & a < x < b \\ 1 & x \geq b \end{cases}$$

F must be Cont.

$$A = \int_a^x f(u)\,du = \int_a^x \frac{1}{b-a}\,du = \frac{1}{b-a}\int_a^x du = \frac{1}{b-a}\,u\Big|_a^x = \frac{1}{b-a}(x-a) = \frac{x-a}{b-a}$$



$$f_X(x) = \text{PDF} = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

4.1.1 Probability Density Function (PDF)

Definition 4.2

Consider a continuous random variable X with an absolutely continuous CDF $F_X(x)$. The function $f_X(x)$ defined by

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x), \qquad \text{if } F_X(x) \text{ is differentiable at } x$$

is called the probability density function (PDF) of X.

$$\text{CDF} = F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \text{PDF}\,dx$$

$$\text{CDF} = \int_{-\infty}^{x} \text{PDF} = \int_{-\infty}^{x} f_X(x)\,dx \Rightarrow f_X(x) = F'_X(x)$$

4.1.1 Probability Density Function (PDF)

Since the PDF is the derivative of the CDF, the CDF can be obtained from PDF by integration (assuming absolute continuity):

$$\text{CDF} = F_X(x) = \int_{-\infty}^{x} f_X(u)du.$$

$P(X<x)$

Also, we have

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_{a}^{b} f_X(u)du.$$

In particular, if we integrate over the entire real line, we must get 1, i.e.,

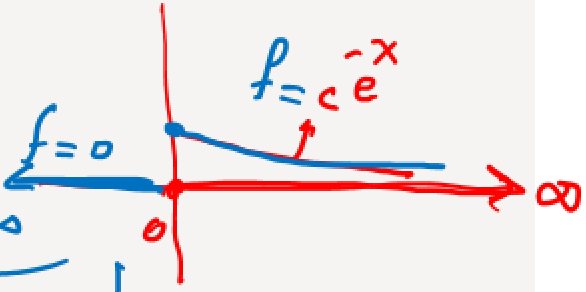
$$\int_{-\infty}^{\infty} f_X(u)du = 1.$$

Example 4.2

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} ce^{-x} & \boxed{x \geq 0} \\ 0 & \text{otherwise} \end{cases}$$

$x<0$ $-\infty$



where c is a positive constant.

a. Find c.
b. Find the CDF of X, $F_X(x)$.
c. Find $P(1 < X < 3)$.

$$\boxed{f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{else} \end{cases}}$$ PDF

a. To find c, we can use Property 2 above, in particular

$$1 = \int_{-\infty}^{\infty} f_X(u)du$$
$$= \int_0^{\infty} ce^{-u}du$$
$$= c\left[-e^{-x}\right]_0^{\infty}$$
$$= c.$$

$$\int e^{\sin x} \cos x\, dx = e^{\sin x} + C$$

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

$$\Rightarrow \int_{-\infty}^{0} f_X(x)\,dx + \int_0^{\infty} f_X(x)\,dx = 1$$

$$\Rightarrow \cancel{\int_{-\infty}^{0} 0\,dx} + \int_0^{\infty} ce^{-x}\,dx = 1$$

$$\Rightarrow \lim_{t\to\infty} \int_0^t ce^{-x}\,dx = 1$$

$$\Rightarrow \lim_{t\to\infty} (-1)c \int_0^t (-1)e^{-x}\,dx = 1$$

$$\Rightarrow \lim_{t\to\infty} -ce^{-x}\Big|_0^t = 1$$

$$\Rightarrow \lim_{t\to\infty} \cancel{-ce^{-t}} + ce^{0} = 1$$

$$\boxed{c = 1}$$

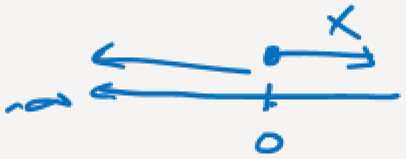
Example 4.2

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

a. Find c.
b. Find the CDF of X, $F_X(x)$.
c. Find $P(1 < X < 3)$.



$$\text{CDF} = F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)du$$

$$= \int_{-\infty}^{x} e^{-u} du = \cancel{\int_{-\infty}^{0} 0\, du} + \int_{0}^{x} e^{-u} du$$

$$= -e^{-u}\Big|_0^x = -e^{-x} + e^{0}$$

$$= -e^{-x} + 1$$

$$= 1 - e^{-x}$$

b. To find the CDF of X, we use $F_X(x) = \int_{-\infty}^{x} f_X(u)du$, so for $x < 0$, we obtain $F_X(x) = 0$. For $x \geq 0$, we have

$$F_X(x) = \int_0^x e^{-u} du = 1 - e^{-x}.$$

Thus,

$$F_X(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{CDF} = F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

Example 4.2

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

a. Find c.
b. Find the CDF of X, $F_X(x)$.
c. Find $P(1 < X < 3)$.

c. We can find $P(1 < X < 3)$ using either the CDF or the PDF. If we use the CDF, we have

$$P(1 < X < 3) = F_X(3) - F_X(1) = \left[1 - e^{-3}\right] - \left[1 - e^{-1}\right] = e^{-1} - e^{-3}.$$

Equivalently, we can use the PDF. We have

$$\begin{aligned} P(1 < X < 3) &= \int_1^3 f_X(t)dt \\ &= \int_1^3 e^{-t}dt \\ &= e^{-1} - e^{-3}. \end{aligned}$$

① $P(1 < X < 3) = \int_1^3 f_X(x)\,dx$

$= \int_1^3 e^{-x}\,dx = -e^{-x}\Big|_1^3$

$= -e^{-3} + e^{-1} = e^{-1} - e^{-3}$

$= \frac{1}{e} - \frac{1}{e^3}$

② $P(1 < X < 3) = P(1 \leq X \leq 3) = F_X(3) - F_X(1)$

$= (1 - e^{-3}) - (1 - e^{-1})$

$= \cancel{1} - e^{-3} - \cancel{1} + e^{-1} = -e^{-3} + e^{-1}$

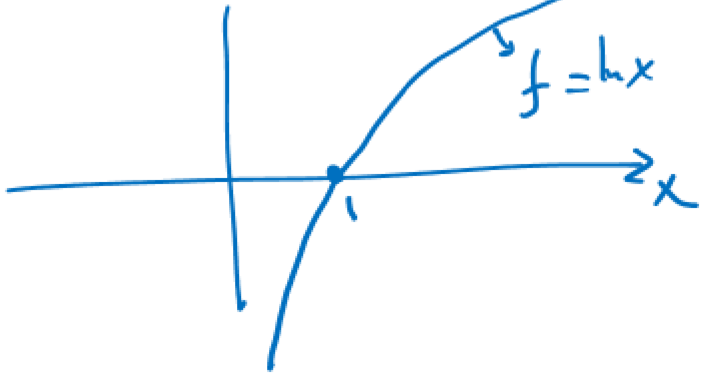
$= e^{-1} - e^{-3} = \frac{1}{e} - \frac{1}{e^3}$

Range

The range of a random variable X is the set of possible values of the random variable. If X is a continuous random variable, we can define the range of X as the set of real numbers x for which the PDF is larger than zero, i.e,

$$R_X = \{x | f_X(x) > 0\}.$$

$f(x) = \ln x$



$R_x = \{x > 1\}$

4.1.2 Expected Value and Variance

As we mentioned earlier, the theory of continuous random variables is very similar to the theory of discrete random variables. In particular, usually summations are replaced by integrals and PMFs are replaced by PDFs. The proofs and ideas are very analogous to the discrete case, so sometimes we state the results without mathematical derivations for the purpose of brevity.

Remember that the expected value of a discrete random variable can be obtained as

$$E(X) = \sum_{x_k \in R_X} x_k P_X(x_k). = \sum x \underbrace{f_X(x)}_{\text{PMF}}$$

Now, by replacing the sum by an integral and PMF by PDF, we can write the definition of expected value of a continuous random variable as

$$\boxed{EX = \int_{-\infty}^{\infty} x f_X(x) dx}$$

Example

Let $X \sim Uniform(a, b)$. Find EX.

Solution

As we saw, the PDF of X is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & x < a \text{ or } x > b \end{cases}$$

other.

$f = \frac{1}{b-a}$

$-\infty$ a b

so to find its expected value, we can write

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_a^b x\left(\frac{1}{b-a}\right) dx \\ &= \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b \\ &= \frac{a+b}{2}. \end{aligned}$$

$$E(x) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \cancel{\int_{-\infty}^{a} x(0)\, dx} + \int_a^b x \frac{1}{b-a}\, dx + \cancel{\int_b^{\infty} x(0)\, dx}$$

0 0

$$= \int_a^b \frac{x}{b-a}\, dx = \frac{1}{b-a}\int_a^b x\, dx$$

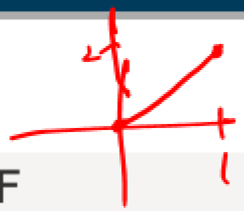
$$= \frac{1}{b-a}\left[\frac{x^2}{2}\Big|_a^b\right] = \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right] = \frac{1}{2\cancel{(b-a)}}\cancel{(b-a)}(b+a)$$

$$= \frac{b+a}{2} = \frac{a+b}{2}$$

This result is intuitively reasonable: since X is uniformly distributed over the interval $[a, b]$, we expect its mean to be the middle point, i.e., $\boxed{EX = \frac{a+b}{2}}$.

Example



Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$f = 0$ $2x$ 0

0 1

Find the expected value of X.

Solution

We have

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx$$
$$= \int_0^1 x(2x) dx$$
$$= \int_0^1 2x^2 dx$$
$$= \frac{2}{3}.$$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \cancel{\int_{-\infty}^{0} x(0)\, dx} + \int_0^1 x(2x)\, dx + \cancel{\int_1^{\infty} x(0)\, dx}$$

$$= \int_0^1 2x^2\, dx = 2\frac{x^3}{3}\Big|_0^1$$

$$= \frac{2}{3} - 0 = \frac{2}{3}$$

Expected Value of a Function of a Continuous Random Variable

Remember the law of the unconscious statistician (LOTUS) for discrete random variables:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k) \tag{4.2}$$

Now, by changing the sum to integral and changing the PMF to PDF we will obtain the similar formula for continuous random variables.

Law of the unconscious statistician (LOTUS) for continuous random variables:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{4.3}$$

PDF

As we have seen before, expectation is a linear operation, thus we always have

Example

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$f = x + \frac{1}{2}$ 0 0

0 1

Find $E(X^n)$, where $n \in \mathbb{N}$.

Solution

Using LOTUS we have

$$\begin{aligned} E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x) dx \\ &= \int_0^1 x^n (x + \tfrac{1}{2}) dx \\ &= \left[\frac{1}{n+2} x^{n+2} + \frac{1}{2(n+1)} x^{n+1} \right]_0^1 \\ &= \frac{3n+4}{2(n+1)(n+2)}. \end{aligned}$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

$$E(X^n) = \int_{-\infty}^{0} \cancel{x^n (0)}\, dx + \int_0^1 x^n (x+\tfrac{1}{2})\, dx + \int_1^{\infty} \cancel{x^n (0)}\, dx$$

$$= \int_0^1 x^n (x + \tfrac{1}{2})\, dx = \int_0^1 x^{n+1} + \frac{x^n}{2}\, dx$$

$$\left. \frac{x^{n+2}}{n+2} + \frac{x^{n+1}}{2(n+1)} \right|_0^1 = \frac{1^{n+2}}{n+2} + \frac{1^{n+1}}{2(n+1)} - 0$$

$$= \frac{1}{n+2} + \frac{1}{2(n+1)}$$

Variance of Continuous RV

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Variance

Remember that the variance of any random variable is defined as

$$\mathrm{Var}(X) = E\left[(X-\mu_X)^2\right] = E(X^2) - (EX)^2. = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx - \left[\int_{-\infty}^{\infty} x\, f_x(x)\,dx\right]^2$$

So for a continuous random variable, we can write

$$\mathrm{Var}(X) = E\left[(X-\mu_X)^2\right] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$$

$$= EX^2 - (EX)^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \mu_X^2$$

Example

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{3}{x^4} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$f=0$ $\quad -\infty \quad 1 \quad \frac{3}{x^4}$

Find the mean and variance of X.

Solution

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_{1}^{\infty} \frac{3}{x^3} dx \\ &= \left[-\frac{3}{2} x^{-2} \right]_1^{\infty} \\ &= \frac{3}{2}. \end{aligned}$$

Next, we find EX^2 using LOTUS,

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_{1}^{\infty} \frac{3}{x^2} dx \\ &= \left[-3x^{-1} \right]_1^{\infty} \\ &= 3. \end{aligned}$$

Thus, we have

$$\text{Var}(X) = EX^2 - (EX)^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

$$E(X) = \text{mean} = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$= \cancel{\int_{-\infty}^{1} x(0) dx} + \int_{1}^{\infty} x \frac{3}{x^4} dx = 3\int_1^{\infty} x^{-3} dx$$

$$= 3 \lim_{t\to\infty} \int_1^t x^{-3} dx = 3 \lim_{t\to\infty} \frac{x^{-2}}{-2} \Big|_1^t$$

$$= 3\left[\lim_{t\to\infty} \frac{t^{-2}}{-2} + \frac{(1)^{-2}}{2}\right]$$

$$= 3\left[\cancel{\lim_{t\to\infty} -\frac{1}{2t^2}} + \frac{1}{2}\right] = \frac{3}{2}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = E(X^2) - \left(\frac{3}{2}\right)^2 = \boxed{E(X^2) - \frac{9}{4}}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \cancel{\int_{-\infty}^{1} x^2(0) dx} + \int_1^{\infty} x^2 \frac{3}{x^4} dx = \int_1^{\infty} \frac{3}{x^2} dx$$

$$= 3 \lim_{t\to\infty} \left[\int_1^t x^{-2} dx\right] = 3 \lim_{t\to\infty} \left[\frac{x^{-1}}{-1}\Big|_1^t\right] = 3 \lim_{t\to\infty} \left[\cancel{-\frac{1}{t}} + 1\right] = \boxed{3}$$

4.1.3 Functions of Continuous Random Variables

If X is a continuous random variable and $Y = g(X)$ is a function of X, then Y itself is a random variable. Thus, we should be able to find the CDF and PDF of Y. It is usually more straightforward to start from the CDF and then to find the PDF by taking the derivative of the CDF. Note that before differentiating the CDF, we should check that the CDF is continuous. As we will see later, the function of a continuous random variable might be a non-continuous random variable. Let's look at an example.

Example

Let X be a $Uniform(0, 1)$ random variable, and let $Y = e^X$.

a. Find the CDF of Y.
b. Find the PDF of Y.
c. Find EY.

First, note that we already know the CDF and PDF of X. In particular,

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

It is a good idea to think about the range of Y before finding the distribution. Since e^x is an increasing function of x and $R_X = [0, 1]$, we conclude that $R_Y = [1, e]$. So we immediately know that

$$F_Y(y) = P(Y \le y) = 0, \qquad \text{for } y < 1,$$

$$F_Y(y) = P(Y \le y) = 1, \qquad \text{for } y \ge e.$$

a. To find $F_Y(y)$ for $y \in [1, e]$, we can write

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(e^X \le y) \\ &= P(X \le \ln y) && \text{since } e^x \text{ is an increasing function} \\ &= F_X(\ln y) = \ln y && \text{since } 0 \le \ln y \le 1. \end{aligned}$$

To summarize

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 1 \\ \ln y & \text{for } 1 \le y < e \\ 1 & \text{for } y \ge e \end{cases}$$

$$CDF, F_X(x) = \begin{cases} 0 & if\ x < a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{elw},\ x<a, x>b \end{cases}$$

$$PDF = f_{X(0,1)}(x) = \begin{cases} \frac{1}{1-0} & 0 \le x \le 1 \\ 0 & \text{elw} \end{cases} = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elw} \end{cases} \quad f_X(x)$$

$$CDF = F_X(x) = \begin{cases} 0 & x<0 \\ \frac{x-0}{1-0} & 0 \le x \le 1 \\ 1 & x>1 \end{cases} = \begin{cases} 0 & x<0 \\ x & 0 \le x \le 1 \\ 1 & x>1 \end{cases}$$

$\boxed{R_X = [0,1]}$

But $Y = e^X$

when $x = 0 \Rightarrow y = e^0 = 1$

$x = 1 \Rightarrow y = e^1 = e$

$\boxed{R_Y\ [1, e]}$

y: 1, e

$$\boxed{F_Y(y) = \begin{cases} 0 & y<1 \\ \ln y & 1 \le y \le e \\ 1 & y>e \end{cases}}$$

CDF (Y)

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$? = F_Y(y) \qquad 1 \le y \le e$

$$F_Y(y) = P(Y \le y) = P(e^X < y) = P(\cancel{\ln} \cancel{e}^X < \ln y) = P(X < \ln y)$$

$$= F_X(\ln y)$$

$$= \ln y$$

Example

Let X be a $Uniform(0, 1)$ random variable, and let $Y = e^X$.

a. Find the CDF of Y.
b. Find the PDF of Y.
c. Find EY.

CDF

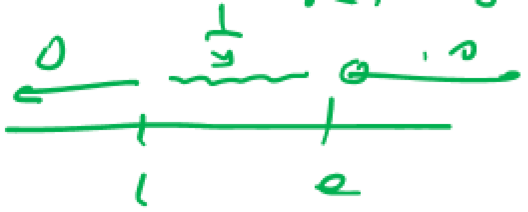
$$F_Y(y) = \begin{cases} 0 & \text{for } y < 1 \\ \ln y & \text{for } 1 \leq y < e \\ 1 & \text{for } y \geq e \end{cases}$$

b. The above CDF is a continuous function, so we can obtain the PDF of Y by taking its derivative. We have

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{y} & \text{for } 1 \leq y \leq e \\ 0 & \text{otherwise} \end{cases}$$

Note that the CDF is not technically differentiable at points 1 and e, but as we mentioned earlier we do not worry about this since this is a continuous random variable and changing the PDF at a finite number of points does not change probabilities.

$$\text{PDF} = F'_y(y) = \begin{cases} \frac{1}{y} & 1 \leq y \leq e \\ 0 & \text{else} \quad y<1 \sim y>e \end{cases}$$



Example

Let X be a $Uniform(0,1)$ random variable, and let $Y = e^X$.

a. Find the CDF of Y.
b. Find the PDF of Y.
c. Find EY.

c. To find the EY, we can directly apply LOTUS,

$$E[Y] = E[e^X] = \int_{-\infty}^{\infty} e^x f_X(x)dx$$
$$= \int_0^1 e^x dx$$
$$= e - 1.$$

For this problem, we could also find EY using the PDF of Y,

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)dy$$
$$= \int_1^e y\frac{1}{y}dy$$
$$= e - 1.$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$
$$= \cancel{\int_{-\infty}^{1} y(0)dy} + \int_1^e y\frac{1}{y}dy + \cancel{\int_e^{\infty} y(0)dy}$$
$$= y\Big|_1^e = \boxed{e-1}$$

$$E(Y) = E(e^X) = \int_{-\infty}^{\infty} e^x f_X(x)\,dx$$
$$= \cancel{\int_{-\infty}^{0} e^x(0)dx} + \int_0^1 e^x(1)dx + \cancel{\int_1^{\infty} e^x(0)dx}$$
$$= e^x\Big|_0^1 = e^1 - e^0 = \boxed{e-1}$$

4.2.1 Uniform Distribution

We have already seen the uniform distribution. In particular, we have the following definition:

> A continuous random variable X is said to have a *Uniform* distribution over the interval $[a, b]$, shown as $X \sim Uniform(a, b)$, if its PDF is given by
>
> $$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$x < a$ $\quad a \le x \le b \quad$ $x > b$

$-\infty \quad a \quad b \quad \infty$

We have already found the CDF and the expected value of the uniform distribution. In particular, we know that if $X \sim Uniform(a, b)$, then its CDF is given by equation 4.1 in example 4.1, and its mean is given by

$$\boxed{EX = \frac{a+b}{2}} = \mu$$

To find the variance, we can find EX^2 using LOTUS:

$$\begin{aligned} EX^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\ &= \int_a^b x^2 \left(\frac{1}{b-a}\right) dx \\ &= \frac{a^2+ab+b^2}{3}. \end{aligned}$$

Therefore,

$$\boxed{\begin{aligned} \mathrm{Var}(X) &= EX^2 - (EX)^2 \\ &= \frac{(b-a)^2}{12}. \end{aligned}} \quad \sigma_X^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{a} x^2 f_X(x)\,dx + \int_a^b x^2 f_X(x)\,dx + \int_b^{\infty} x^2 f_X(x)\,dx$$

$$= \cancel{\int_{-\infty}^{a} x^2 (0)\,dx} + \int_a^b x^2 \frac{1}{b-a}\,dx + \cancel{\int_b^{\infty} x^2 (0)\,dx}$$

$0 \qquad 0$

$$= \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left.\frac{x^3}{3}\right|_a^b = \frac{1}{3(b-a)}(b^3-a^3)$$

$$= \frac{1}{3(\cancel{b-a})}(\cancel{b-a})(b^2+ab+a^2) = \frac{b^2+ab+a^2}{3}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

$$= E(X^2) - \left(\frac{a+b}{2}\right)^2$$

$$= E(X^2) - \frac{a^2+2ab+b^2}{4}$$

$$\therefore \mathrm{Var}(X) = \frac{b^2}{3} + \frac{ab}{3} + \frac{a^2}{3} - \frac{a^2}{4} - \frac{2ab}{4} - \frac{b^2}{4} = \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

Problem 5

Suppose X has a continuous uniform distribution over the interval $[\underset{a}{1.5}, \underset{b}{5.5}]$. *Determine the following:*

$$f_X(x) = \text{PMF} = \begin{cases} \dfrac{1}{5.5-1.5} & 1.5 \le x \le 5.5 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} \dfrac{1}{4} & 1.5 \le x \le 5.5 \\ 0 & \text{else} \end{cases}$$

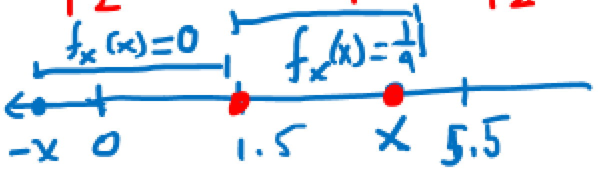
(a) $E(X), V(X)$,

(b) Value for x such that $P(-x < X < x) = 0.90$.

a) $E(X) = \dfrac{a+b}{2} = \dfrac{1.5+5.5}{2} = \dfrac{7}{2}$

$V(X) = \dfrac{(b-a)^2}{12} = \dfrac{(5.5-1.5)^2}{12} = \dfrac{4^2}{12} = \dfrac{16}{12} = \dfrac{4}{3} = \sigma^2 \Rightarrow \text{standard dev} = \sqrt{\dfrac{4}{3}} = \dfrac{2}{\sqrt{3}}$

b) $P(-x < X < x)$



$$= \int_{-x}^{x} \text{PMF}\,dx = \int_{-x}^{x} f_X(x)\,dx = \int_{1.5}^{x} \frac{1}{4}\,du = \frac{u}{4}\Big|_{1.5}^{x} = \frac{x}{4} - \frac{1.5}{4} = \frac{x-1.5}{4}$$

Since $P(-x < X < x) = 0.9$

$\Rightarrow \dfrac{x-1.5}{4} = 0.9 \Rightarrow x - 1.5 = 3.6 \rightarrow x = 3.6 + 1.5 = \boxed{5.1}$

Problem 5

Suppose X has a continuous uniform distribution over the interval $[1.5, 5.5]$. *Determine the following:*

(a) $E(X), V(X)$,

(b) *Value for x such that* $P(-x < X < x) = 0.90$.

$$PDF = f(x) = \begin{cases} \dfrac{1}{5.5 - 1.5} = \dfrac{1}{4} & 1.5 \leq x \leq 5.5 \\ 0 & otherwise \end{cases}$$

$$(a)\ E(x) = \frac{a+b}{2} = \frac{1.5 + 5.5}{2} = 3.5 \ \ and \ \ V(x) = \frac{(b-a)^2}{12} = \frac{(5.5 - 1.5)^2}{12} = \frac{4}{3}$$

$$(b)\ P(-x < X < x) = P(1.5 \leq X < x) \int_{1.5}^{x} \frac{1}{4} dx = \frac{x - 1.5}{4} = 0.9, then\ x = (4)(0.9) + 1.5 = 5.1$$

Problem 3

Suppose that X follows a continuous uniform distribution on $[-3, 3]$. Find x such that $P(X < x) = \frac{1}{2}$.

a b

PMF $= f_X(x) = \begin{cases} \frac{1}{3-(-3)} & -3 \le x \le 3 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{6} & -3 \le x \le 3 \\ 0 & \text{else} \end{cases}$

$f=0 \quad f=\frac{1}{6} \quad f=0$ PMF. ✓

$-3 \quad x \quad 3$

CDF $= F_X(x) = \begin{cases} 0 & x < -3 \\ \frac{x-(-3)}{3-(-3)} & -3 \le x \le 3 \\ 1 & x > 3 \end{cases} = \begin{cases} 0 & x < -3 \\ \frac{x+3}{6} & -3 \le x \le 3 \\ 1 & x > 3 \end{cases}$

$F = 0$
$x < -3$

$-3 \quad 3$

use PMF to find x

$P(X<x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{-3} \cancel{0}\,dx + \int_{-3}^{x} \frac{1}{6}\,dx = \frac{x}{6}\Big|_{-3}^{x} = \frac{x}{6} - \frac{-3}{6} = \frac{x+3}{6}$

but $P(X<x) = \frac{1}{2} \Rightarrow \frac{x+3}{6} = \frac{1}{2} \Rightarrow x+3 = 3 \Rightarrow \boxed{x=0}$

use CDF

$P(X<x) = F_X(x)$; $-3 \le x \le 3$

$= \frac{x+3}{6}$

but $P(X<x) = \frac{1}{2} \Rightarrow \frac{x+3}{6} = \frac{1}{2} \Rightarrow \boxed{x=0}$

4.2.2 Exponential Distribution

The exponential distribution is one of the widely used continuous distributions. It is often used to model the time elapsed between events. We will now mathematically define the exponential distribution, and derive its mean and expected value. Then we will develop the intuition for the distribution and discuss several interesting properties that it has.

A continuous random variable X is said to have an *exponential* distribution with parameter $\lambda > 0$, shown as $X \sim Exponential(\lambda)$, if its PDF is given by

$f = \lambda e^{-\lambda x}$

$$\text{PMF} = f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$x \leq 0$

0 $\quad$ $x > 0$

Figure 4.5 shows the PDF of exponential distribution for several values of λ.

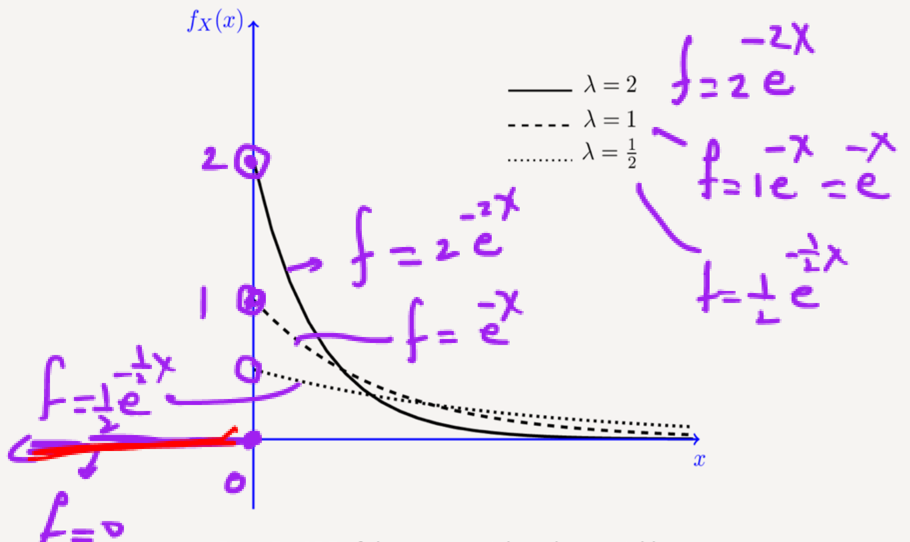
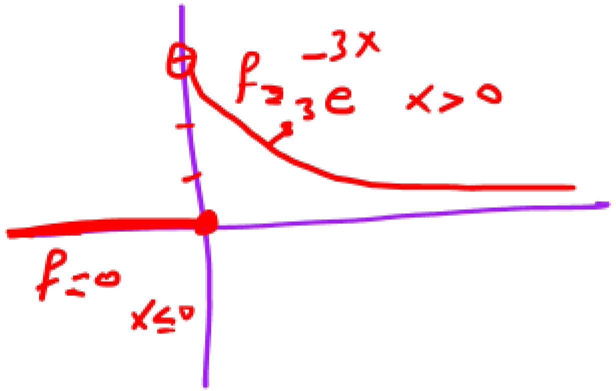


Fig.4.5 - PDF of the exponential random variable.

$Exp(\lambda = 3)$

$$\text{PMF} = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{else} \end{cases}$$



Expected Value of the Exponential Distribution Function E(x)

It is convenient to use the unit step function defined as

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\int e^{\sin x} \cos x \, dx = e^{\sin x} + c$

so we can write the PDF of an $Exponential(\lambda)$ random variable as

$$f_X(x) = \lambda e^{-\lambda x} u(x).$$

Let us find its CDF, mean and variance. For $x > 0$, we have

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = \boxed{1 - e^{-\lambda x}}.$$

So we can express the CDF as

$$F_X(x) = \left(1 - e^{-\lambda x}\right) u(x).$$

Let $X \sim Exponential(\lambda)$. We can find its expected value as follows, using integration by parts:

$$\begin{aligned} EX &= \int_0^\infty x \lambda e^{-\lambda x} dx \\ &= \frac{1}{\lambda} \int_0^\infty y e^{-y} dy \qquad \text{choosing } y = \lambda x \\ &= \frac{1}{\lambda} \Big[-e^{-y} - y e^{-y} \Big]_0^\infty \\ &= \frac{1}{\lambda}. \end{aligned}$$

$E(x) = \int_{-\infty}^{\infty} x f_x(x) \, dx$

$f = 0$ $\quad$ $f = \lambda e^{-\lambda x}$

$-\infty \quad 0 \quad \infty$

$= \int_{-\infty}^{0} x f_x(x) \, dx + \int_0^{\infty} x f_x(x) \, dx$

$= \cancel{\int_{-\infty}^{0} x (0) \, dx} + \int_0^{\infty} x \lambda e^{-\lambda x} \, dx = \lim_{t \to \infty} \int_0^t x \lambda e^{-\lambda x} \, dx$

By Parts let $\boxed{u = x}$ $\quad dv = \lambda e^{-\lambda x} dx$

$du = dx \quad v = -\int -\lambda e^{-\lambda x} dx$

$= -e^{-\lambda x}$

$\int x \lambda e^{-\lambda x} dx = -x e^{-\lambda x} - \frac{1}{\lambda} \int -\lambda e^{-\lambda x} dx$

$= -x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x}$

$= \lim_{t \to \infty} -x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^t = \lim_{t \to \infty} e^{-\lambda x} \left(-x - \frac{1}{\lambda}\right) \Big|_0^t$

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$$= \lim_{t \to \infty} e^{-\lambda t}\left(-t-\frac{1}{\lambda}\right) - e^{-\lambda(0)}\left(0-\frac{1}{\lambda}\right)$$

$$= \lim_{t\to\infty} \cancel{\frac{-t-\frac{1}{\lambda}}{e^{\lambda t}}} + \frac{1}{\lambda} = \frac{1}{\lambda}$$

0

If X is exp. R.V $\Rightarrow E(X) = \mu_x = \frac{1}{\lambda}$

If X is an exponential R.V

$\checkmark\ f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{else} \end{cases}$

PMF

$\checkmark$ CDF $= F_X(x) = \begin{cases} 0 & x < 0 \\ \boxed{1 - e^{-\lambda x}} & x > 0 \end{cases}$

$F_X(x) = P(X < x)$, $x > 0$

$f = 0 \quad \lambda e^{-\lambda x}$

$0 \quad x$

$$= \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \cancel{\int_{-\infty}^{0} 0\,dx} + -\int_0^x -\lambda e^{-\lambda x}\,dx = -\int_0^x (-\lambda) e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_0^x = -e^{-\lambda x} + e^{-\lambda(0)}$$

$$= -e^{-\lambda x} + 1 = \boxed{1 - e^{-\lambda x}}$$

$E(X) = \frac{1}{\lambda} = \mu = \sigma$

$Var(X) = \sigma^2 = \frac{1}{\lambda^2}$

Stand dev $= \sigma = \frac{1}{\lambda}$

Variance of the Exponential Distribution Function Var(x)

Now let's find Var(X). We have

$$\begin{aligned} EX^2 &= \int_0^\infty x^2 \lambda e^{-\lambda x} dx \\ &= \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-y} dy \\ &= \frac{1}{\lambda^2} \Big[-2e^{-y} - 2ye^{-y} - y^2 e^{-y} \Big]_0^\infty \\ &= \frac{2}{\lambda^2}. \end{aligned}$$

Thus, we obtain

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

> If $X \sim Exponential(\lambda)$, then $EX = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

$$Var(X) = E(X^2) - E(X)^2$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x^2} dx - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

If X is exp. R.V then

$$E(X) = \frac{1}{\lambda} = \mu$$

$$\boxed{\mu = \sigma = \frac{1}{\lambda}}$$

$$Var(X) = \frac{1}{\lambda^2} = \sigma^2$$

$$\text{Stand dev} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$$

Example

Let $U \sim Uniform(0, 1)$ and $X = -\ln(1-U)$. Show that $X \sim Exponential(1)$.

$a \quad b \qquad f_X(x) \qquad \lambda=1$

$$F_X(x) = \boxed{CDF_{exp} = 1 - e^{-x}}$$

$$f_U(u) = \begin{cases} \frac{1}{1-0} & 0 \le u \le 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$

PDF

$$CDF = F_U(u) = \begin{cases} 0 & u < 0 \\ \frac{u-0}{1-0} & 0 \le u \le 1 \\ 1 & u > 1 \end{cases} = \begin{cases} 0 & u < 0 \\ u & 0 \le u \le 1 \\ 1 & u > 1 \end{cases}$$

show that $CDF(x) = 1 - e^{-x}$

$$CDF(x) = F_X(x) = P(X < x) = \cancel{\int_{-\infty}^{x} f_X(x)\,dx}$$

bec $f_X(x)$ unknown

$$= P(-\ln(1-u) < x)$$

$$= P(\ln(1-u)^{-1} < x)$$

$$= P\left(\ln\frac{1}{1-u} < x\right) = P\left(e^{\ln\frac{1}{1-u}} < e^{x}\right)$$

$$= P\left(\frac{1}{1-u} < e^{x}\right) = P\left(1-u > \frac{1}{e^{x}}\right) = P\left(-u > \frac{1}{e^{x}} - 1\right)$$

$$= P\left(u < 1 - \frac{1}{e^{x}}\right) = P(u < 1 - e^{-x})$$

$$= F_U(1 - e^{-x})$$

$$= 1 - e^{-x}$$

$$\therefore F_X(x) = \boxed{1 - e^{-x}}$$

$CDF(X) = CDF(exp)$. $\therefore X \sim EXP(\lambda = 1)$

Let $U \sim Uniform(0,1)$ and $X = -\ln(1-U)$. Show that $X \sim Exponential(1)$.

Solution

First note that since $R_U = (0,1)$, $R_X = (0,\infty)$. We will find the CDF of X. For $x \in (0,\infty)$, we have

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(-\ln(1-U) \leq x) \\ &= P\left(\frac{1}{1-U} \leq e^x\right) \\ &= P(U \leq 1-e^{-x}) = 1-e^{-x}, \qquad bec.\ \ 0 < 1-e^{-x} < 1 \end{aligned}$$

which is the CDF of an $Exponential(1)$ random variable.

Problem 1

Suppose the number of customers arriving at a store obeys a Poisson distribution with an average of λ customers per unit time. That is, if Y is the number of customers arriving in an interval of length t, then $Y \sim Poisson(\lambda t)$. Suppose that the store opens at time $t = 0$. Let X be the arrival time of the first customer. Show that $X \sim Exponential(\lambda)$.

Solution

We first find $P(X > t)$:

$$\begin{aligned} P(X > t) &= P(\text{No arrival in } [0, t]) \\ &= e^{-\lambda t} \frac{(\lambda t)^0}{0!} \\ &= e^{-\lambda t}. \end{aligned}$$

Thus, the CDF of X for $x > 0$ is given by

$$F_X(x) = 1 - P(X > x) = 1 - e^{-\lambda x},$$

so we can write the PDF of an $Exponential(\lambda)$ random variable as

$$f_X(x) = \lambda e^{-\lambda x} u(x).$$

Let us find its CDF, mean and variance. For $x > 0$, we have

$$F_X(x) = \int^{x} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}.$$

4.2.3 Normal (Gaussian) Distribution

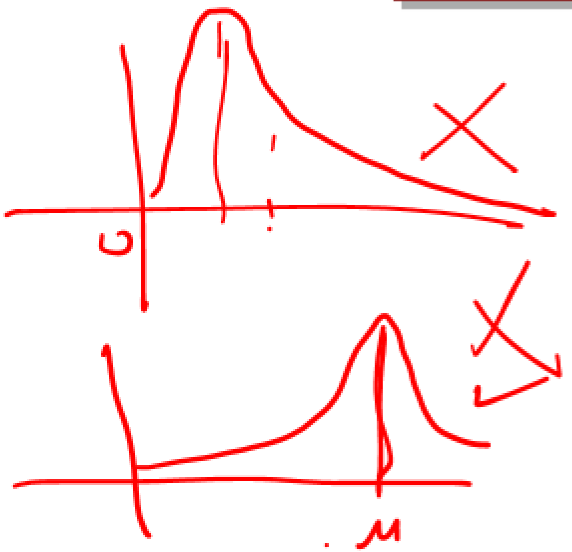
The normal distribution is by far the most important probability distribution. One of the main reasons for that is the *Central Limit Theorem* (CLT) that we will discuss later in the book. To give you an idea, the CLT states that if you add a large number of random variables, the distribution of the sum will be approximately normal under certain conditions. The importance of this result comes from the fact that many random variables in real life can be expressed as the sum of a large number of random variables and, by the CLT, we can argue that distribution of the sum should be normal. The CLT is one of the most important results in probability and we will discuss it later on. Here, we will introduce normal random variables.

We first define the **standard normal random variable**. We will then see that we can obtain other normal random variables by *scaling* and *shifting* a standard normal random variable.

A continuous random variable Z is said to be a *standard normal (standard Gaussian)* random variable, shown as $Z \sim N(0,1)$, if its PDF is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}, \qquad \text{for all } z \in \mathbb{R}.$$

The $\frac{1}{\sqrt{2\pi}}$ is there to make sure that the area under the PDF is equal to one. We will verify that this holds in the solved problems section. Figure 4.6 shows the PDF of the standard normal random variable.



The $\frac{1}{\sqrt{2\pi}}$ is there to make sure that the area under the PDF is equal to one. We will verify that this holds in the solved problems section. Figure 4.6 shows the PDF of the standard normal random variable.

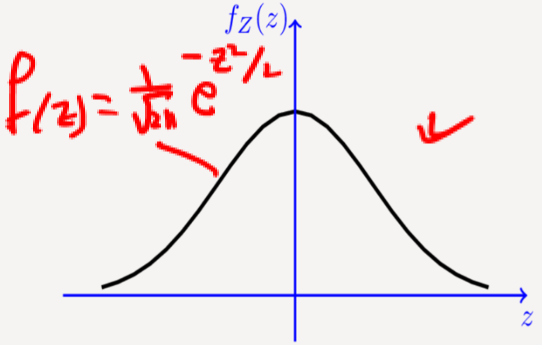


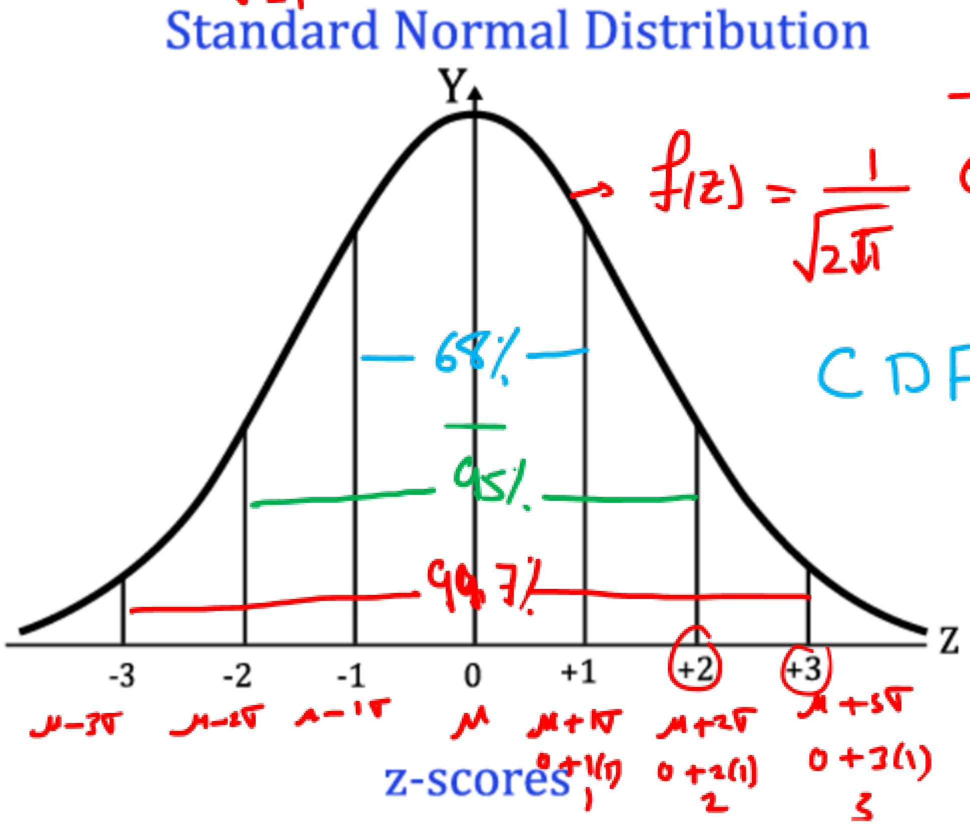
Fig.4.6 - PDF of the standard normal random variable.

Mean and Variance of the Standard Normal Distribution Function

If $Z \sim N(\overset{\mu}{0}, \overset{\sigma}{1})$, then $EZ = 0$ and $\mathrm{Var}(Z) = 1$.

$\sigma^2 = 1$

$\sigma = 1$



$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = f_x(x) = PDF.$$

$$CDF = \int_{-\infty}^{x} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}_{\text{non integrable}} dz$$

CDF of the standard normal

To find the CDF of the standard normal distribution, we need to integrate the PDF function. In particular, we have

$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left\{-\frac{u^2}{2}\right\} du.$$

This integral does not have a closed form solution. Nevertheless, because of the importance of the normal distribution, the values of $F_Z(z)$ have been tabulated and many calculators and software packages have this function. We usually denote the standard normal CDF by Φ.

The CDF of the standard normal distribution is denoted by the Φ function:

$$\mathrm{CDF}(z) = \Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \underbrace{\cancel{\int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\}}}_{\text{impossible to integrate}} du.$$

$= \underline{\phi(z)} = P(Z < z)$ use table

$\therefore \phi(z) = P(Z < z)$

= Area to the left of z in the table

As we will see in a moment, the CDF of any normal random variable can be written in terms of the Φ function, so the Φ function is widely used in probability. Figure 4.7 shows the Φ function.

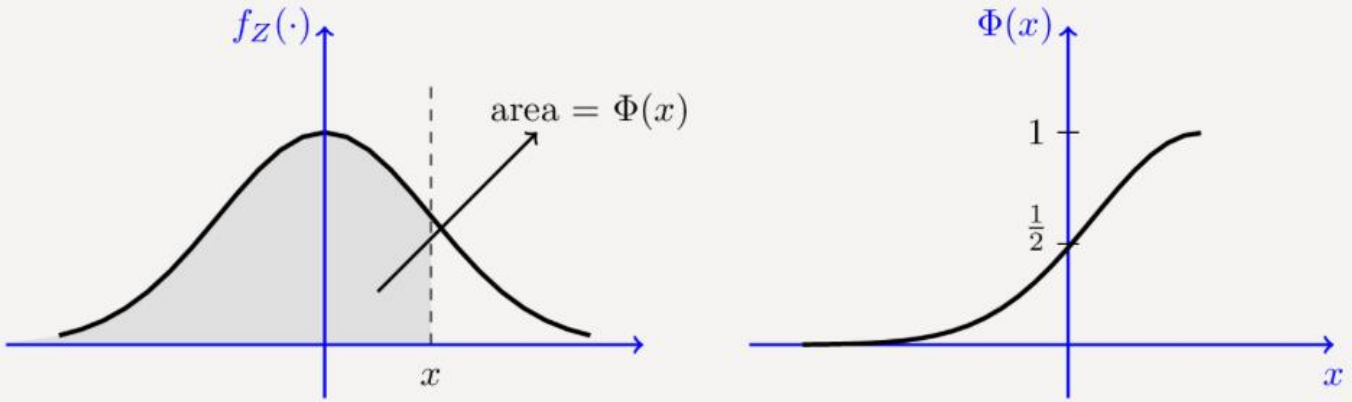
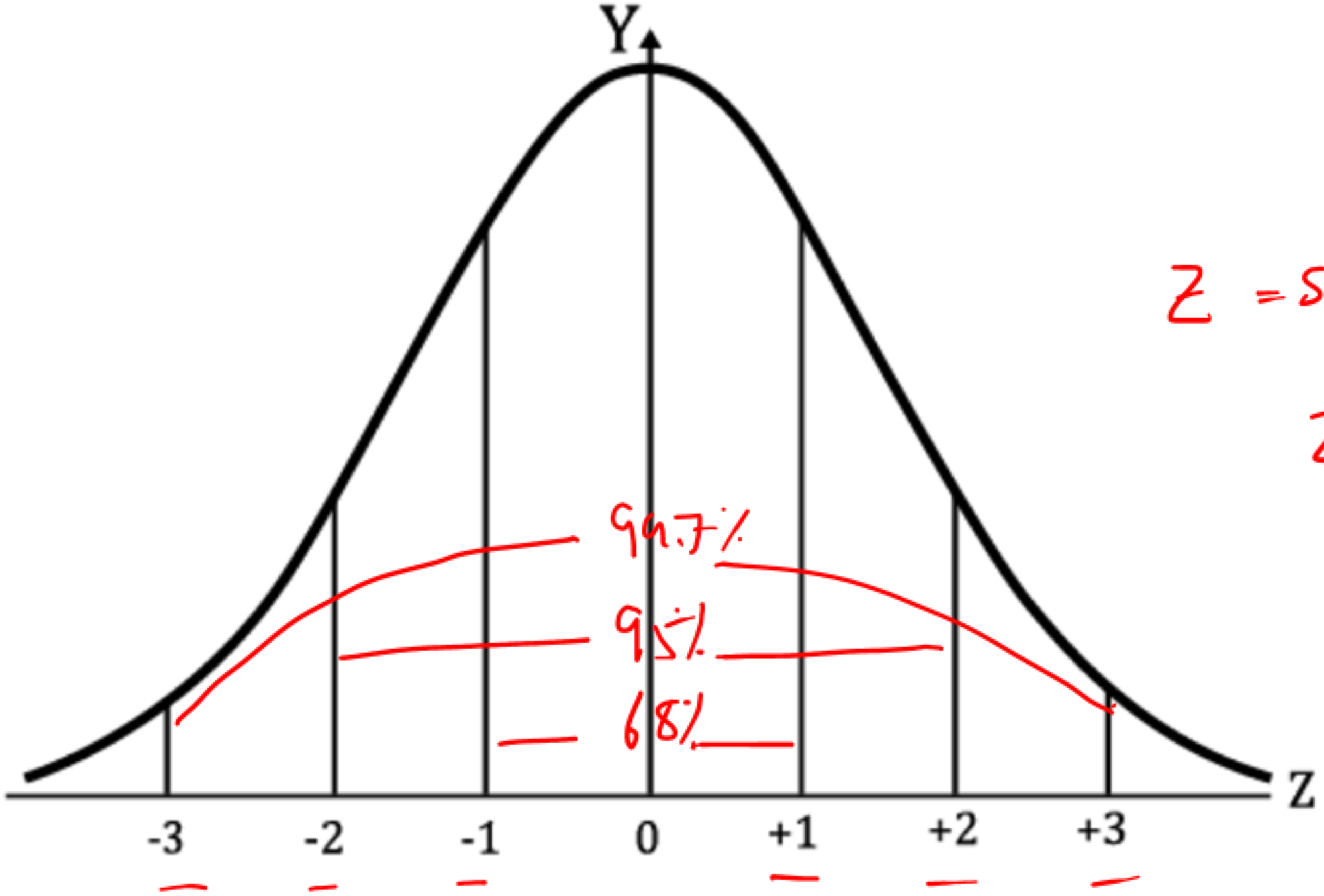


Fig.4.7 - The Φ function (CDF of standard normal).

Here are some properties of the Φ function that can be shown from its definition.

1. $\lim_{x \to \infty} \Phi(x) = 1, \quad \lim_{x \to -\infty} \Phi(x) = 0;$
2. $\Phi(0) = \frac{1}{2};$
3. $\Phi(-x) = 1 - \Phi(x)$, for all $x \in \mathbb{R}$.

Standard Normal Distribution



z-scores

Z = SNG

Z (0, 1)

μ var(Z) = 1

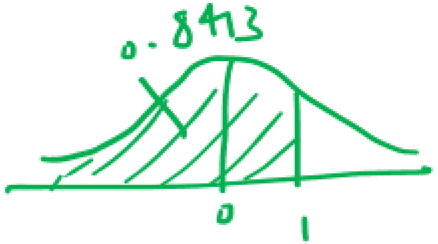
σ = 1

Examples

Compute the following probabilities:

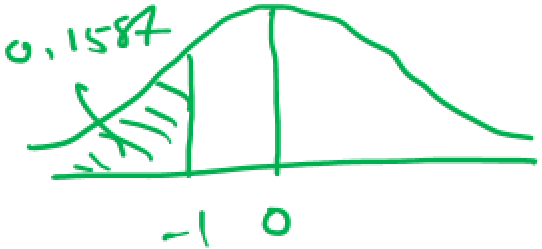
$\Phi(0) =$ Area to the left of $Z=0$

$= 0.5 = P(Z<0)$



$\Phi(1) =$ Area to the left of $Z=1$

$= 0.8413 = P(Z<1)$



$\Phi(-1) = 1-\Phi(1) = 1-0.8413 = 0.1587$

$= P(Z<-1)$

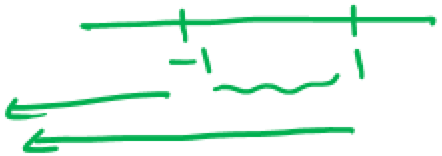
Examples

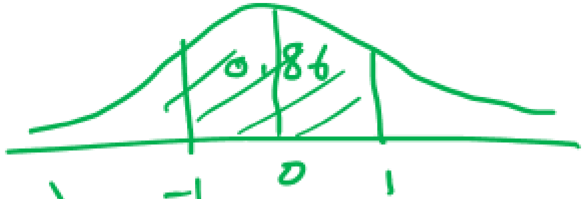
Compute the following probabilities:

$P(-1 \le z \le 1) =$ $P(Z \le 1) - P(Z \le -1) = P(Z \le 1) - [1 - P(Z \le 1)]$

$= \underline{P(Z \le 1)} - 1 + \underline{P(Z \le 1)} = 2P(Z \le 1) - 1 = 2\phi(1) - 1$

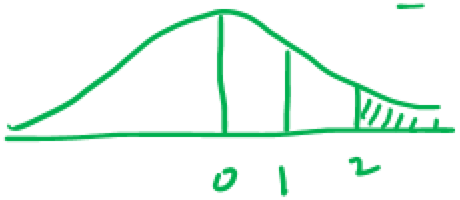
$= 2(0.8413) - 1 = 0.6826$





$P(Z \ge 2) =$ $1 - P(Z < 2) = 1 - 0.9772 = 0.0228$

$= 1 - \phi(2)$



$P(-2 \le Z \le 2)$

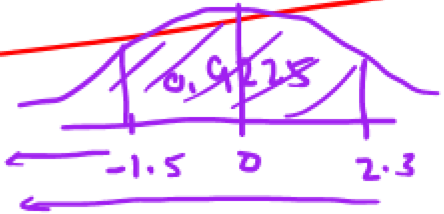
$= 2P(Z \le 2) - 1$

$= 2\phi(2) - 1$

$= 2(0.9772) - 1 = 0.9544$

≈ 0.95

$P(-1.5 \le Z \le 2.3)$



$= P(Z \le 2.3) - P(Z \le -1.5)$

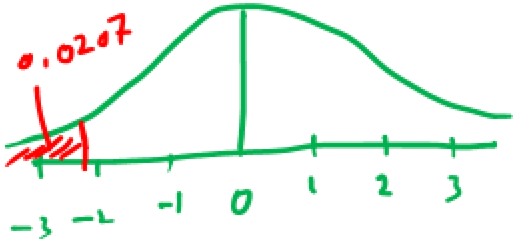
$= P(Z \le 2.3) - (1 - P(Z \le 1.5)) = 0.9893 - (1 - 0.9332) = \underline{0.9225}$

UMASS AMHERST

$P(Z \le -2.04)$

$= 1 - P(Z \le 2.04)$

$= 1 - 0.9793 = 0.0207$

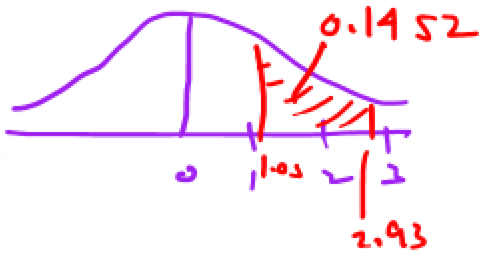


$1.5 = 1.5 + 0.00$

$2.3 = 2.3 + 0.00$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

$P(1.05 \le Z \le 2.93)$



$= P(Z \le 2.93) - P(Z \le 1.05)$

$2.93 = 2.9 + 0.03$

$= 0.9983 - 0.8531$

$1.05 = 1.0 + 0.05$

$= 0.1452 \times 100\%$

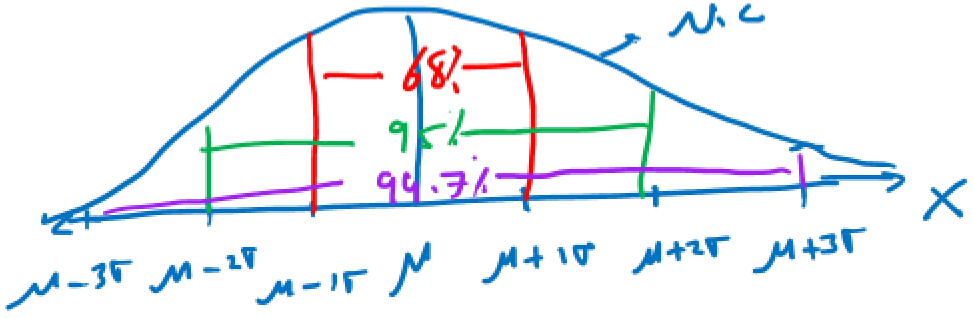
$= 14.5\%$

Normal Radom Variable

Let $\boxed{X = \sigma Z + \mu}$ be a random variable.

Since X depends on the Standard Normal variable Z $\Rightarrow$ X is normal R.V. with mean μ and $var(X) = \sigma^2$ and stand. dev. $= \sigma$

$\therefore \quad X \sim N(\mu, \sigma^2)$ Read: X is a Normal R.V. with mean $= \mu$ and var $= \sigma^2$



$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

but $X = \sigma Z + \mu$

$X - \mu = \sigma Z \Rightarrow \boxed{Z = \frac{X-\mu}{\sigma}}$

$P(Z \le z)$ use table.

Normal random variables

Now that we have seen the standard normal random variable, we can obtain any normal random variable by shifting and scaling a standard normal random variable. In particular, define

$$X = \sigma Z + \mu, \qquad \text{where } \sigma > 0.$$

Then

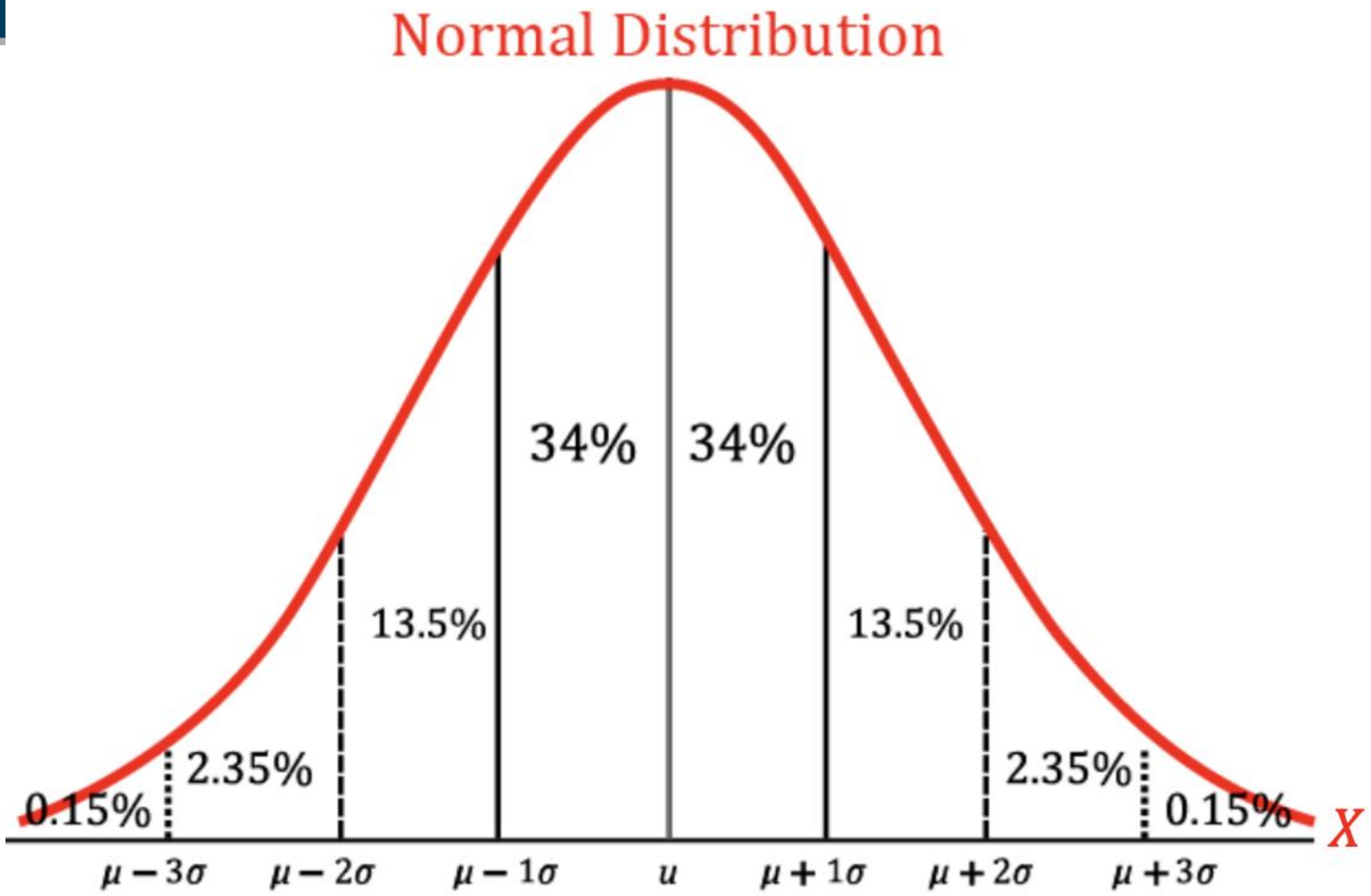
$$EX = \sigma EZ + \mu = \mu,$$

$$\mathrm{Var}(X) = \sigma^2 \mathrm{Var}(Z) = \sigma^2.$$

We say that X is a normal random variable with mean μ and variance σ^2. We write $X \sim N(\mu, \sigma^2)$.

> If Z is a standard normal random variable and $X = \sigma Z + \mu$, then X is a normal random variable with mean μ and variance σ^2, i.e,
>
> $$X \sim N(\mu, \sigma^2).$$

Conversely, if $X \sim N(\mu, \sigma^2)$, the random variable defined by $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable, i.e., $Z \sim N(0, 1)$. To find the CDF of $X \sim N(\mu, \sigma^2)$, we can write


Normal Distribution
34%
34%
13.5%
13.5%
2.35%
2.35%
0.15%
0.15%
X
μ − 3σ
μ − 2σ
μ − 1σ
u
μ + 1σ
μ + 2σ
μ + 3σ

To find the CDF of $X \sim N(\mu, \sigma^2)$, we can write

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(\sigma Z + \mu \leq x) \qquad \left(\text{where } Z \sim N(0,1)\right) \\ &= P\left(Z \leq \frac{x-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right). \end{aligned}$$

To find the PDF, we can take the derivative of F_X,

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma} \Phi'\left(\frac{x-\mu}{\sigma}\right) \qquad \text{(chain rule for derivative)} \\ &= \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}. \end{aligned}$$

If X is a normal random variable with mean μ and variance σ^2, i.e, $X \sim N(\mu, \sigma^2)$, then

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

$$F_X(x) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$P(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

The CDF of the standard normal distribution is denoted by the Φ function:

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du.$$

Example

Let $X \sim N(-5, 4)$.

$\qquad = \quad \mu \quad \sigma^2$

a. Find $P(X < 0)$.
b. Find $P(-7 < X < -3)$.
c. Find $P(X > -3 | X > -5)$.

X is a normal random variable with $\mu = -5$ and $\sigma = \sqrt{4} = 2$, thus we have

a. Find $P(X < 0)$:

$$P(X < 0) = F_X(0)$$
$$= \Phi\left(\frac{0-(-5)}{2}\right)$$
$$= \Phi(2.5) \approx 0.99$$

If X is a normal random variable with mean μ and variance σ^2, i.e, $X \sim N(\mu, \sigma^2)$, then

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

$$F_X(x) = P(X \le x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

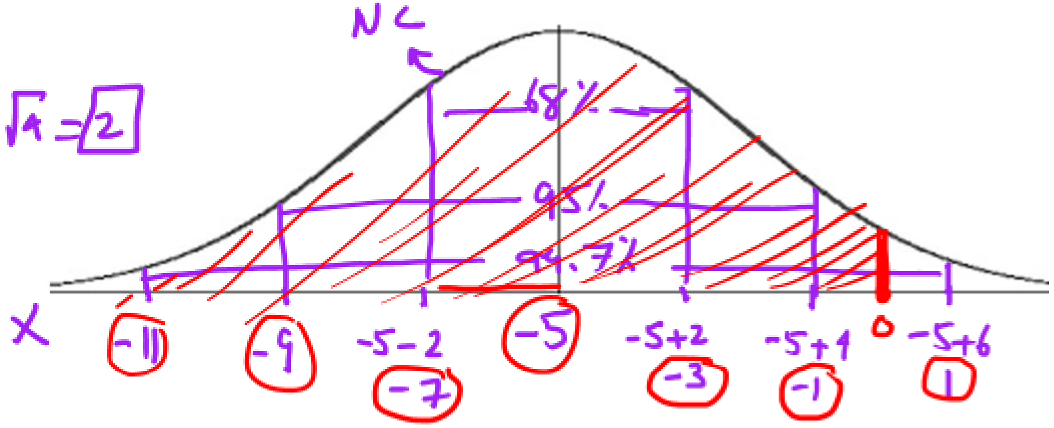
$$P(a < X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

X has mean $\mu = -5$ $\quad var(X) = 4 = \sigma^2 \Rightarrow$ stand dev $= \sigma = \sqrt{4} = \boxed{2}$

$P(-7 < X < -3) = 0.68$

$P(-9 < X < -1) = .95$

$P(-11 < X \le 1) = 0.997.$

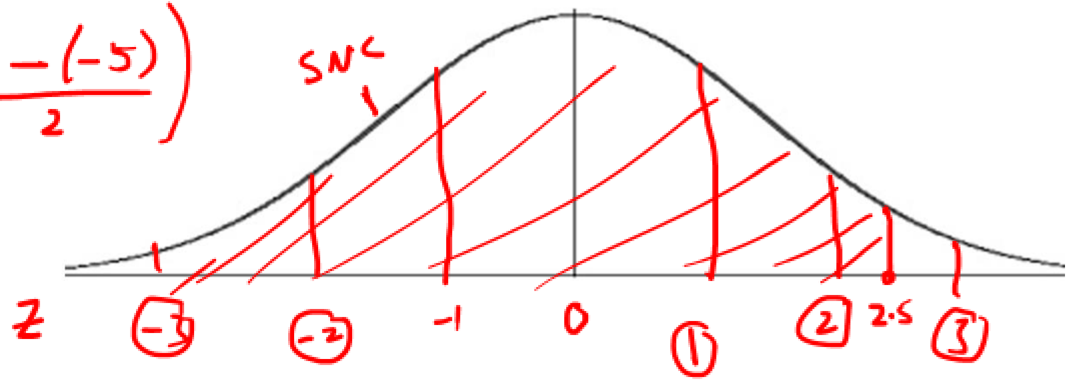


a) $P(X < 0) = P\left(\frac{X-\mu}{\sigma} < \frac{0-\mu}{\sigma}\right) = P\left(Z < \frac{0-(-5)}{2}\right)$

$= P\left(Z < \frac{5}{2}\right) = P(Z < 2.5)$

$= 0.9938 \times 100\%$

$\approx 99.38\%$



Example

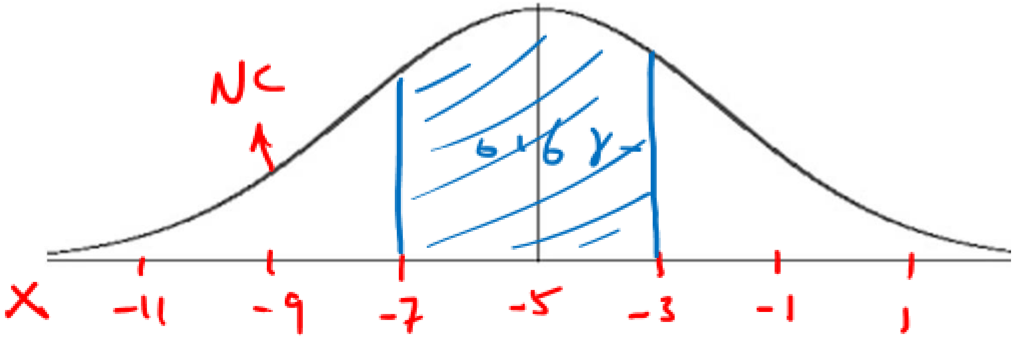
Let $X \sim N(-5, 4)$.

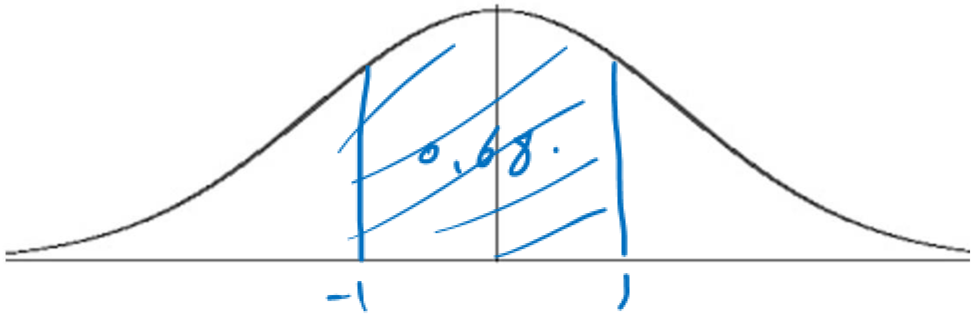
a. Find $P(X < 0)$.
b. Find $P(-7 < X < -3)$.
c. Find $P(X > -3 | X > -5)$.

b) $P(-7 < X < -3) = 0.68$

b. Find $P(-7 < X < -3)$:

$$
\begin{aligned}
P(-7 < X < -3) &= F_X(-3) - F_X(-7) \\
&= \Phi\left(\frac{(-3)-(-5)}{2}\right) - \Phi\left(\frac{(-7)-(-5)}{2}\right) \\
&= \Phi(1) - \Phi(-1) \\
&= 2\Phi(1) - 1 \qquad \left(\text{since } \Phi(-x) = 1 - \Phi(x)\right) \\
&\approx 0.68
\end{aligned}
$$





Example

Let $X \sim N(-5, 4)$.

a. Find $P(X < 0)$.
b. Find $P(-7 < X < -3)$.
c. Find $P(X > -3|X > -5)$.

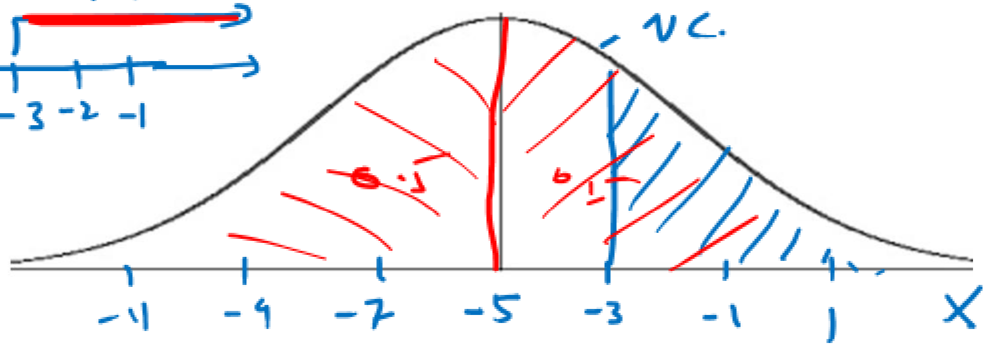
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$X > -5 \cap X > -3 = X > -3$

c. Find $P(X > -3|X > -5)$:

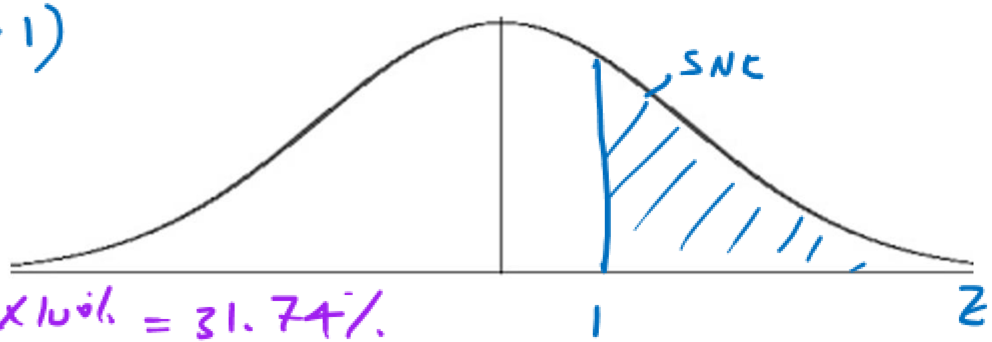
$$P(X > -3|X > -5) = \frac{P(X>-3, X>-5)}{P(X>-5)} = \frac{P(X>-3)}{P(X>-5)} = \frac{1-\Phi\left(\frac{(-3)-(-5)}{2}\right)}{1-\Phi\left(\frac{(-5)-(-5)}{2}\right)} = \frac{1-\Phi(1)}{1-\Phi(0)} \approx \frac{0.1587}{0.5} \approx 0.32$$

$$c)\ P(X > -3/X \geq -5) = \frac{P(X>-3 \cap X>-5)}{P(X>-5)} = \frac{P(X>-3)}{P(X>-5)}$$



$$P(X>-3) = P\left(\frac{X-\mu}{\sigma} > \frac{-3-\mu}{\sigma}\right) = P\left(Z > \frac{-3-(-5)}{2}\right) = P(Z>1)$$

$$= 1 - P(Z<1) = 1 - 0.8413 = \boxed{0.1587}$$



$P(X>-5) = 0.5$

$$\therefore\ P(X>-3/X>-5) = \frac{0.1587}{0.5} = 0.3174 \times 100\% = 31.74\%$$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

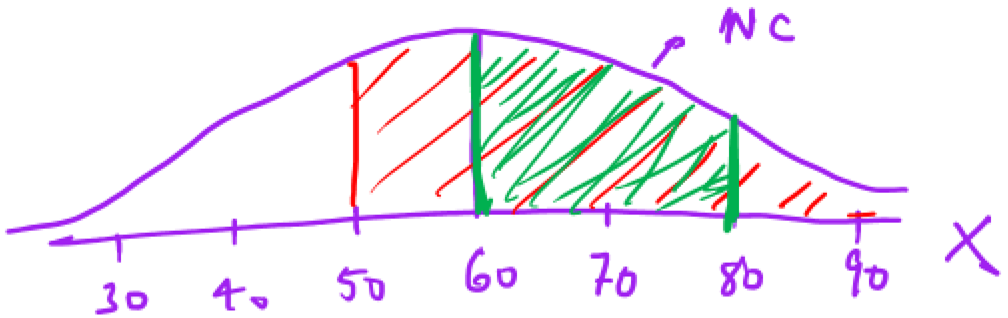
$2.5 = 2.5 + 0.00$ →

Problem 4. A continuous random variable X follows a normal distribution with mean $\mu = 60$ and standard deviation $\sigma = 10$. Determine the following probabilities

(a) **(8 points)** $P(X > 50)$.

(b) **(4 points)** $P(X = 60)$.

(c) **(8 points)** $P(60 \leq X < 80 \mid X > 50)$.

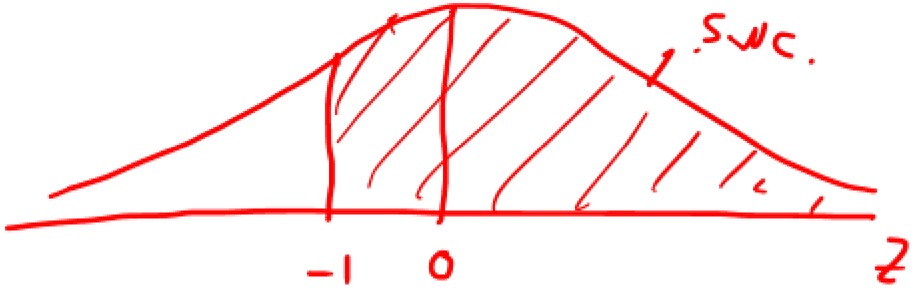


$X \sim N(60, 10^2) = N(60, 100)$

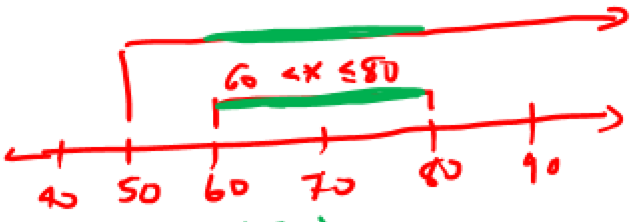
a) $P(X>50) = P\left(\frac{X-\mu}{\sigma} > \frac{50-60}{10}\right) = P(Z > -1)$

$= 1 - P(Z<-1) = 1 - [1 - P(Z<1)] = P(Z<1)$

$= 0.8413$



b) $P(X=60) = 0$



c) $P(60 \leq X \leq 80 / X > 50)$

$= \frac{P(60 \leq X \leq 80 \cap X>50)}{P(X>50)} = \frac{P(60\leq X \leq 80)}{P(X>50)} = \frac{\quad}{0.8413}$

$P(Z>a) = 1 - P(Z<a)$

$P(Z<-a) = 1 - P(Z<a)$.

$$P(60 \le X \le 80) = \frac{0.95}{2} = 0.4\underline{75}$$

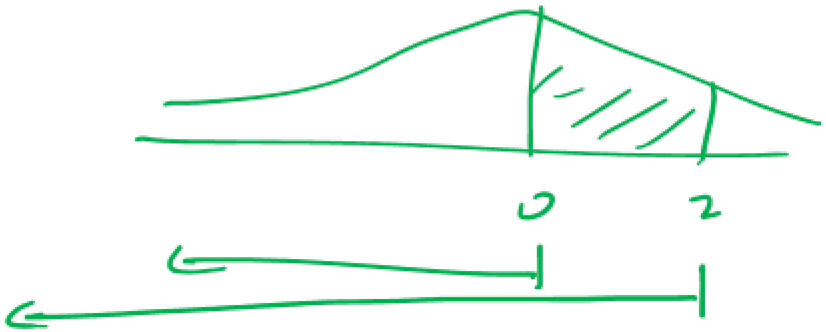
$$P(a \le Z \le b) = P(Z < b) - P(Z < a)$$

$$P(60 \le X \le 80) = P\left(\frac{60-60}{10} \le \frac{X-\mu}{\sigma} \le \frac{80-60}{10}\right)$$

$$= P(0 \le Z \le 2) = P(Z<2) - P(Z<0)$$

$$= 0.9772 - 0.5$$

$$= 0.4772$$



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Problem 4. A continuous random variable X follows a normal distribution with mean $\mu = 60$ and standard deviation $\sigma = 10$. Determine the following probabilities

(a) (**8 points**) $P(X > 50)$.

Do the standardizing transformation

$$Z = \frac{X - 60}{10}. \qquad \textit{2 points.}$$

Then

$$\begin{aligned} P(X > 50) =& P\left(Z > \frac{50-60}{10}\right) \\ =& P(Z > -1) \\ =& 1 - \Phi(-1) \\ =& 1 - 0.158655 = 0.841345. \qquad \textit{6 points.} \end{aligned}$$

(b) (**4 points**) $P(X = 60)$.

Since X is a continuous random variable, 2 points.

$$P(X = 60) = 0. \qquad \textit{2 points.}$$

(c) (**8 points**) $P(60 \leq X < 80 \,|\, X > 50)$.

We have

$$\begin{aligned} P(60 \leq X < 80|X > 50) =& \frac{P\Big(60 \leq X < 80 \cap X > 50\Big)}{P(X > 50)} \\ =& \frac{P(60 \leq X < 80)}{P(X > 50)}. \qquad \textit{2 points.} \end{aligned}$$

As in Question (a), we have

$$\begin{aligned} P(60 \leq X < 80) =& P\left(\frac{60-60}{10} \leq Z < \frac{80-60}{10}\right) \\ =& P(0 \leq Z < 2) \\ =& \Phi(2) - \Phi(0) \\ =& 0.977250 - 0.5 = 0.477250. \qquad \textit{5 points.} \end{aligned}$$

Therefore,

$$P(60 \leq X < 80|X > 50) = \frac{0.477250}{0.841345} = 0.5672. \qquad \textit{1 point.}$$