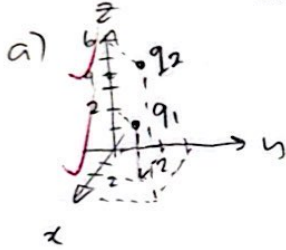


$$R_1 = (2, 2, 2)$$

Q1. Charge $q_1 = -6\pi\epsilon_0$ C is located at point (2 m, 2 m, 2 m) in Cartesian coordinates, and charge q_2 is located at point (3 m, 3 m, 6 m). $R_2 = (3, 3, 6)$

- (a) Draw the locations of the charges in the Cartesian coordinate system. [2]
 (b) Using Coulomb's law, what should charge q_2 be so that the total electric field \vec{E} at point (0, 0, 4 m) has no x-component? [5]
 (c) Find the force \vec{F} acting on a third charge $q_3 = 10$ C located the same point (0, 0, 4 m). [3]



b) $E_{1x} = E_{2x}$ @ (0, 0, 4)

$$\frac{1}{4\pi\epsilon} \left(\frac{q_1(R-R_1)_x}{|R-R_1|^3} \right) = \frac{1}{4\pi\epsilon} \left(\frac{q_2(R-R_2)_x}{|R-R_2|^3} \right)$$

$$\frac{-6\pi\epsilon_0(-2)}{2^3} = \frac{q_2(-2)}{3^3}$$

$$q_2 = \frac{-6\pi\epsilon_0(-2)(3^3)}{2^3(-2)}$$

$$= -\frac{6\pi\epsilon_0(54)}{24} = -\frac{27}{2}\pi \left(\frac{1}{36\pi} \times 10^{-9} \right) = -\frac{27}{72} \times 10^{-9}$$

$$= -0.375 \times 10^{-9} \text{ C}$$

c) $E = \frac{1}{4\pi\epsilon} \left[\frac{q_1(R-R_1)}{|R-R_1|^3} + \frac{q_2(R-R_2)}{|R-R_2|^3} \right]$ OK

$$= \frac{1}{4\pi\epsilon} \left[\frac{-6\pi\epsilon_0(-2\hat{x} - 2\hat{y} + 2\hat{z})}{(2^2 + 2^2 + 2^2)^{3/2}} + \left(-\frac{27}{2}\epsilon_0\right) \frac{(-2\hat{x} - 3\hat{y} - 2\hat{z})}{(3^2 + 3^2 + 2^2)^{3/2}} \right]$$

$$= \frac{1}{4\pi\epsilon} \left[\frac{-6\pi\epsilon_0(-2\hat{x} - 2\hat{y} + 2\hat{z})}{(12)^{3/2}} - \frac{27\epsilon_0(-2\hat{x} - 3\hat{y} - 2\hat{z})}{2(22)^{3/2}} \right]$$

$$F = q_3 E = -\frac{3}{2(12)^{3/2}} (-2\hat{x} - 2\hat{y} + 2\hat{z}) - \frac{27}{8(22)^{3/2}} (-2\hat{x} - 3\hat{y} - 2\hat{z})$$

= ? -2

Q2. In a rectangular box defined by the region $0 \leq x \leq 1$ m, $0 \leq y \leq 2$ m, $0 \leq z \leq 3$ m, the electric flux density is given by:

$$\vec{D} = 8x\hat{x} - 3y^2\hat{y} + z^3\hat{z} \text{ C/m}^2$$

- (a) Using the differential form of Gauss' Law, find the volume charge density, ρ_v . [4]
 (b) Use ρ_v from part (a) to find the total charge Q contained in the box. [6]

a) $Q = \int_V \nabla \cdot \vec{D} dV = \int_V \rho_v dV$

$\nabla \cdot \vec{D} = \rho_v$

$(8x, -3y^2, z^3)$

$\int_V \rho_v dV = \rho_v \int_0^1 dx \int_0^2 dy \int_0^3 dz = 6\rho_v = Q$

$\oint_S \vec{D} \cdot d\vec{s} = Q$

$\nabla \cdot \vec{D} = \frac{\partial}{\partial x} 8x + \frac{\partial}{\partial y} (-3y^2) + \frac{\partial}{\partial z} z^3$
 $= 8 - 6y + 3z^2 = \rho_v$

$\rho_v = 8 - 6y + 3z^2$

$\frac{\partial}{\partial x} (8x) = 8$

$\rho_v = \frac{22}{6R} (8x\hat{x} - 3y^2\hat{y} + z^3\hat{z})$

$\rho_v = \frac{1}{R} \left(\frac{88}{3}x\hat{x} - 11y^2\hat{y} + \frac{22}{6}z^3\hat{z} \right)$

b) $Q = 6\rho_v = \frac{1}{R} (176x\hat{x} - 66y^2\hat{y} + 22z^3\hat{z})$

$Q = 6\rho_v = 6(8 - 6y + 3z^2) = 48 - 36y + 18z^2$

-0.5

Q1. The potential distribution in a dielectric medium with permittivity $\epsilon = 9\epsilon_0$ is given in cylindrical coordinates by

$$V = \frac{1}{r^2} e^{-6z} \cos(\phi) \quad \text{V}$$

$$\frac{\partial V}{\partial \phi} = -\frac{1}{r^2} e^{-6z} \sin \phi$$

(a) Find the electric field \vec{E} due to this potential distribution. [4]

(b) Evaluate \vec{E} at point $P_1 (r_1, \phi_1, z_1) = (0.1, 2\pi, 0.5)$. [2]

(c) Evaluate the electric flux density \vec{D} at point P_1 . [2]

(d) Evaluate the electric polarization field \vec{P} at point P_1 . [2]

$$\frac{\partial V}{\partial z} = -\frac{6}{r^2} e^{-6z} \cos \phi$$

$$\begin{aligned} \text{a) } \vec{E} &= -\nabla V = -\left(\hat{r} \frac{\partial V}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial V}{\partial \phi} + \hat{z} \frac{\partial V}{\partial z}\right) \\ &= -\left(\hat{r} \left(-\frac{3}{r^3} e^{-6z} \cos \phi\right) + \hat{\phi} \left(-\frac{1}{r^3} e^{-6z} \sin \phi\right) + \hat{z} \left(-\frac{6}{r^2} e^{-6z} \cos \phi\right)\right) \\ &= \hat{r} \frac{3}{r^3} e^{-6z} \cos \phi + \hat{\phi} \frac{1}{r^3} e^{-6z} \sin \phi + \hat{z} \frac{6}{r^2} e^{-6z} \cos \phi \quad \text{V/m} \end{aligned}$$

$$\left(-\frac{1}{2}\right)$$

$$\begin{aligned} \text{b) } \vec{E}_1 &= \hat{r} \frac{3}{(0.1)^3} e^{-6(0.5)} \cos(2\pi) + \hat{\phi} \frac{1}{(0.1)^3} e^{-6(0.5)} \sin(2\pi) + \hat{z} \frac{6}{(0.1)^2} e^{-6(0.5)} \cos(2\pi) \\ &= \hat{r} 149.36 + \hat{z} 29.87 \quad \text{V/m} \end{aligned}$$

$$99.8 \hat{r} + \dots$$

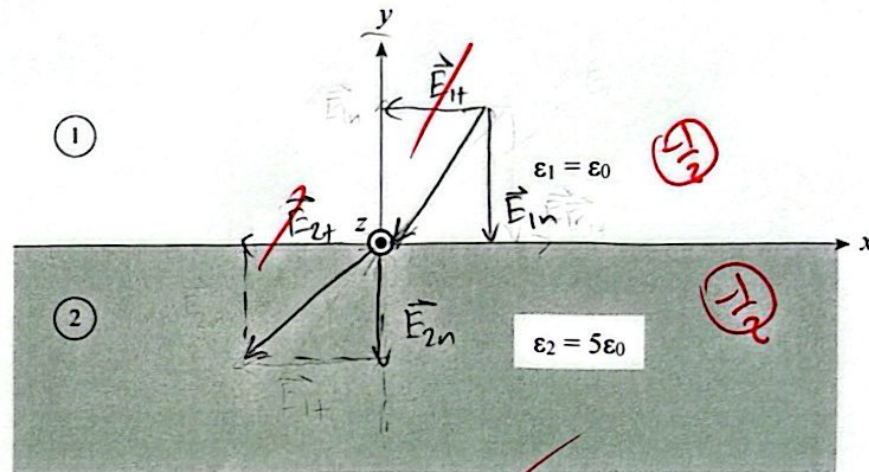
$$\begin{aligned} \text{c) } \vec{D}_1 &= \epsilon \vec{E}_1 = (9\epsilon_0) (149.36, 29.87) \\ &= \hat{r} 11.89 + \hat{z} 2.38 \quad \text{nC/m}^2 \end{aligned}$$

$$\text{d) } \epsilon_r = 9 = 1 + \chi_e \quad \chi_e = 9 - 1 = 8$$

$$\begin{aligned} \vec{P}_1 &= \epsilon_0 \chi_e \vec{E}_1 = \left(\frac{10^{-9}}{36\pi}\right) (8) (149.36, 29.87) \\ &= \hat{r} 10.57 + \hat{z} 2.11 \quad \text{nC/m}^2 \end{aligned}$$

Q2. The figure below shows the boundary between two dielectric media that extends along the xz -plane. Medium 1 above the boundary has a permittivity ϵ_1 , and Medium 2 below the boundary has a permittivity ϵ_2 . The electric field in Medium 1 is given by $\vec{E}_1 = 10\hat{x} - 6\hat{y} + 8\hat{z}$ V/m.

- Find the polarization field in Medium 1, \vec{P}_1 . [3]
- On the diagram below, draw the normal and tangential components of the electric field in Medium 1, \vec{E}_{1n} and \vec{E}_{1t} . [1]
- Use the electric boundary conditions to find the electric field in Medium 2, \vec{E}_2 . [5]
- On the diagram below, draw \vec{E}_2 and the normal and tangential components of the electric field in Medium 2, \vec{E}_{2n} and \vec{E}_{2t} . [1]



a) $\epsilon_r = 1 = 1 + \chi_c$, $\chi_c = 0$, $\vec{P}_1 = 0$ C/m² ✓

b)

c) $\vec{E}_t = \vec{E}_{2t} = 10\hat{x} + 8\hat{z}$ V/m

$\epsilon_1 \vec{E}_{1n} = \epsilon_2 \vec{E}_{2n}$ (no ρ_v)

$\vec{E}_{2n} = \frac{\epsilon_1}{\epsilon_2} \vec{E}_{1n} = \frac{\epsilon_0}{5\epsilon_0} (-6\hat{y}) = -1.2\hat{y}$ V/m

$\vec{E}_2 = 10\hat{x} - 1.2\hat{y} + 8\hat{z}$ V/m ✓

Q1. A charge with a value of $q = 10 \text{ mC}$ is travelling with a velocity of $\vec{u} = 3\hat{x} - 6\hat{y} + 7\hat{z}$ in both an external electric field and an external magnetic field, given by $\vec{E} = 4xy^2\hat{x} + xz^3\hat{y} - 10yz\hat{z} \text{ V/m}$ and $\vec{H} = \frac{1}{\mu_0}(2x^2\hat{x} - y^3\hat{y} + 2z^2\hat{z}) \text{ A/m}$, respectively. At point P (2, 0, 6):

- Find the electric force \vec{F}_e acting on the charge. [3]
- Find the magnetic force \vec{F}_m acting on the charge. [3]
- Find the total electromagnetic force \vec{F} acting on the charge. [2]
- Sketch the force \vec{F} in the Cartesian coordinate system. [2]

$$a) \vec{F}_e = q\vec{E} = (10 \times 10^{-3} \text{ C}) (4xz^2\hat{x} + xz^3\hat{y} - 10yz\hat{z} \text{ V/m})$$

$$= 40xz^2\hat{x} + 10xz^3\hat{y} - 100yz\hat{z} \text{ mN} = 4.32\hat{y} \text{ N}$$

$$b) \vec{F}_m = q\vec{u} \times \vec{B}$$

$$\vec{B} = \mu_0 \vec{H} = 2x^2\hat{x} - y^3\hat{y} + 2z^2\hat{z}$$

$$= (10 \times 10^{-3} \text{ C}) (3\hat{x} - 6\hat{y} + 7\hat{z}) \times (2x^2\hat{x} - y^3\hat{y} + 2z^2\hat{z})$$

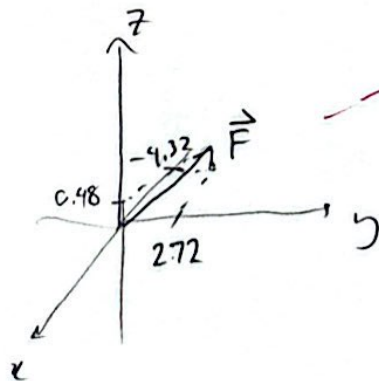
$$= (30\hat{x} - 60\hat{y} + 70\hat{z} \times 10^{-3}) \times (2x^2\hat{x} - y^3\hat{y} + 2z^2\hat{z})$$

$$= (-120z^2 + 70y^3)\hat{x} - (60z^2 - 140x^2)\hat{y} + (-30y^3 + 120x^2)\hat{z} \text{ mN} = -4.32\hat{x} - 1.6\hat{y} + 0.48\hat{z} \text{ N}$$

$$c) \vec{F} = \vec{F}_e + \vec{F}_m = (40xz^2 - 120z^2 + 70y^3)\hat{x} + (10xz^3 - 60z^2 + 140x^2)\hat{y} + (-100yz - 30y^3 + 120x^2)\hat{z} \text{ mN}$$

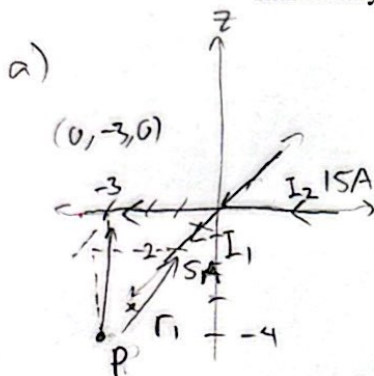
$$= -4.32\hat{x} + 2.72\hat{y} + 0.48\hat{z} \text{ N}$$

d)



Q2. The x- and y-axes, respectively, carry filamentary currents of 5 A along \hat{x} and 15 A along $-\hat{y}$.

- (a) Sketch the two filamentary currents in the Cartesian coordinate system. [2] 2
 (b) Use the Biot-Savart law to find the total magnetic field \vec{H} at point (2, -3, -4) m due to both filamentary currents. [8] 7.5



$$r_{p1} = \sqrt{(2-0)^2 + (-3-0)^2 + (-4-0)^2} = 5 \text{ m}$$

b)

$$\vec{B}_1 = \hat{\phi} \frac{\mu_0 I_1}{2\pi r_{p1}} \quad \hat{\phi} = \hat{z} \times \hat{r} = \hat{z} \times \left(\frac{4\hat{x} + 3\hat{y}}{5} \right) = -\frac{4}{5}\hat{y} + \frac{3}{5}\hat{z}$$

$$= \frac{(4\pi \times 10^{-7})(5\text{A})}{2\pi (5\text{m})} \left(-\frac{4}{5}\hat{y} + \frac{3}{5}\hat{z} \right)$$

$$= \left(-\frac{8}{5}\hat{y} + \frac{6}{5}\hat{z} \right) \times 10^{-7}$$

$$r_{p2} = \sqrt{(2-0)^2 + (-3-(-3))^2 + (-4-0)^2} = \sqrt{20}$$

$$\vec{B}_2 = \hat{\phi} \frac{\mu_0 I_2}{2\pi r_{p2}} \quad \hat{\phi} = \hat{z} \times \hat{r} = (-\hat{y}) \times \left(\frac{4\hat{x} - 2\hat{z}}{\sqrt{20}} \right) = -\frac{4}{\sqrt{20}}\hat{x} - \frac{2}{\sqrt{20}}\hat{z}$$

$$= \frac{(4\pi \times 10^{-7})(15\text{A})}{2\pi \sqrt{20}} \left(-\frac{4}{\sqrt{20}}\hat{x} - \frac{2}{\sqrt{20}}\hat{z} \right)$$

$$= (-6\hat{x} - 3\hat{z}) \times 10^{-7}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \left(-6\hat{x} - \frac{8}{5}\hat{y} - \frac{9}{5}\hat{z} \right) \times 10^{-7} \quad / 4\pi \times 10^{-7}$$

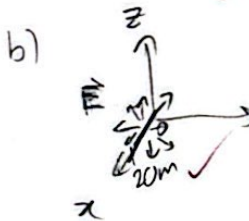
$$\vec{H} = \frac{\vec{B}}{\mu_0} = -\frac{3}{2\pi}\hat{x} - \frac{2}{5\pi}\hat{y} - \frac{9}{20\pi}\hat{z} \text{ A/m}$$

$$= -0.477\hat{x} - 0.127\hat{y} - 0.143\hat{z} \text{ A/m}$$

Q1. A wire extending along $0 < x < 20$ m has a linear charge density of $\rho_l = 50x^3 \mu\text{C/m}$.

- Find the total charge, Q , on the line. [2]
- Draw the wire in the Cartesian coordinate system, and draw vectorially the electric field distribution around the wire. [1]
- Using Coulomb's law, find the electric field \vec{E} at point $(0, 0, 100 \text{ km})$ in Cartesian coordinates. [4]
- Find the force \vec{F} acting on a point charge $q_1 = 5 \text{ C}$ located the same point $(0, 0, 100 \text{ km})$. [2]
- Draw the vector force \vec{F} on the diagram from part (b). [1]

a) $Q = \int \rho_l dx = \int_0^{20} 50x^3 \times 10^{-9} dx = 50 \frac{x^4}{4} \times 10^{-9} \text{ C} \Big|_0^{20} = 2 \text{ mC} \times -1/2$

b)  $\vec{R} = (x, 0, 0) - (0, 0, 100) = (x, 0, -100)$

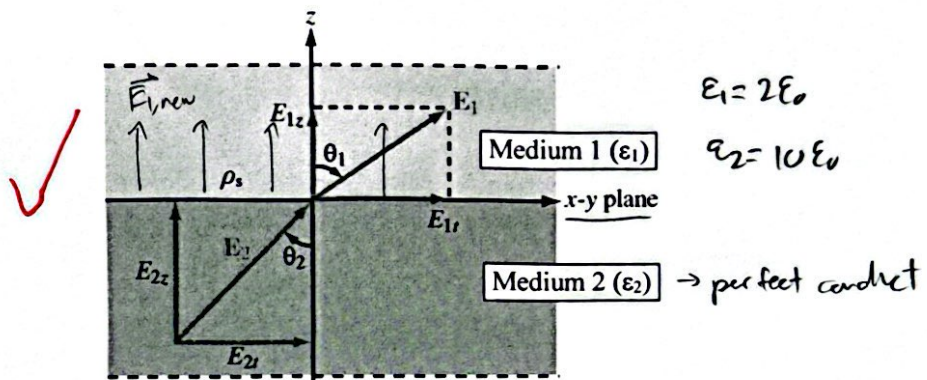
c) $\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_l d\vec{l}}{R^2} = \frac{1}{4\pi\epsilon_0} \int_0^{20} \frac{50x^3}{(100)^2} dx = \frac{50 \times 10^{-9}}{4\pi\epsilon_0 (100)^2} \left[\frac{x^4}{4} \Big|_0^{20} \right] \hat{R}$

$= \frac{(50 \times 10^{-9})(40000)}{4\pi(\frac{10^{-9}}{36\pi})(100)^2} \hat{R} = \frac{(50)(9)(40000)}{(100)^2} \hat{R}$

d) $\vec{F} = q\vec{E} = (5\text{C})(18 \times 10^{-6} \text{ V/m}) = 90 \mu\text{N}$

Q2. The figure below shows the boundary between two dielectric media that extends along the x-y plane. Medium 1 above the boundary has a permittivity ϵ_1 , and Medium 2 below the boundary has a permittivity ϵ_2 . The electric field in Medium 2 is given by $\vec{E}_2 = -10\hat{x} + 5\hat{y} + 20\hat{z}$ V/m, the surface charge density on the boundary is $\rho_s = 8\epsilon_0$ C/m², and the relative permittivities of the two media are $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 10$.

- Use the electric boundary conditions to find the electric field in Medium 1, \vec{E}_1 . [4]
- Medium 2 is now replaced with a perfect conductor. Thus, now the boundary is between Medium 1, a dielectric with permittivity ϵ_1 , and Medium 2, a perfect conductor. The surface charge density on the boundary remains at $\rho_s = 8\epsilon_0$ C/m². Find the new electric field in Medium 1, $\vec{E}_{1,new}$. Sketch $\vec{E}_{1,new}$ in the figure below. [4]
- For the scenario of part (b) above, calculate the energy densities for both Medium 1 and Medium 2. You can leave your answers in terms of ϵ_0 . [2]



a) $\vec{E}_1 = \vec{E}_2 = -10\hat{x} + 5\hat{y}$

$\epsilon_1 \vec{E}_{1n} - \epsilon_2 \vec{E}_{2n} = \rho_s$

$$\vec{E}_{1n} = \frac{\rho_s + \epsilon_2 \vec{E}_{2n}}{\epsilon_1} = \frac{8\epsilon_0}{2\epsilon_0} + \frac{10\epsilon_0}{2\epsilon_0} (20\hat{z}) = 100\hat{z} + 4\hat{z}$$

$$\vec{E}_1 = -10\hat{x} + 5\hat{y} + 100\hat{z} + 4\hat{z} \text{ V/m}$$

$$\left(-\frac{1}{2}\right)$$

b) $\vec{E}_{1t,new} = \vec{E}_{2t} = 0$

$\epsilon_1 \vec{E}_{1n,new} = \rho_s$

$$\vec{E}_{1n,new} = \frac{\rho_s}{\epsilon_1} = \frac{8\epsilon_0}{2\epsilon_0} = 4$$

$$\vec{E}_{1,new} = 4 \text{ V/m } \hat{z}$$

c) $w_e = \frac{1}{2} \epsilon |\vec{E}|^2$

$$w_{e1} = \frac{1}{2} (2\epsilon_0) |4|^2 = 16\epsilon_0 \text{ J/m}^3$$

$$w_{e2} = \frac{1}{2} \epsilon (0)^2 = 0 \text{ J/m}^3$$

Q3. The potential distribution in free space is given in spherical coordinates by:

$$V = \frac{1}{R^3} \cos \phi \sin \theta \quad V$$

$$\frac{\partial V}{\partial R} = -\frac{3}{R^4} \cos \phi \sin \theta$$

$$\frac{\partial V}{\partial \theta} = \frac{1}{R^3} \cos \phi \cos \theta$$

$$\frac{\partial V}{\partial \phi} = -\frac{1}{R^3} \sin \phi \sin \theta$$

(a) Find the electric field \vec{E} due to this potential distribution. [6]

(b) Evaluate \vec{E} at point $P_1 (R_1, \theta_1, \phi_1) = (1, \pi/4, \pi/4)$. [1]

(c) Find the electric flux density \vec{D} . [2]

(d) Evaluate \vec{D} at point $P_1 (R_1, \theta_1, \phi_1) = (1, \pi/4, \pi/4)$. [1]

$$\begin{aligned} a) \vec{E} &= -\nabla V = -\left(\hat{R} \frac{\partial V}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi}\right) \\ &= -\left(\hat{R} -\frac{3}{R^4} \cos \phi \sin \theta + \hat{\theta} \frac{1}{R^4} \cos \phi \cos \theta + \hat{\phi} -\frac{1}{R^4} \sin \phi\right) \\ &= \hat{R} \frac{3}{R^4} \cos \phi \sin \theta - \hat{\theta} \frac{1}{R^4} \cos \phi \cos \theta + \hat{\phi} \frac{1}{R^4} \sin \phi \end{aligned}$$

$$\begin{aligned} b) \vec{E} &= \hat{R} \frac{3}{1^4} \cos\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) - \hat{\theta} \frac{1}{1^4} \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{4}\right) + \hat{\phi} \frac{1}{1^4} \sin\left(\frac{\pi}{4}\right) \\ &= \hat{R} 3\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \hat{\theta} \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \hat{\phi} \frac{1}{\sqrt{2}} \\ &= \hat{R} \frac{3}{2} - \hat{\theta} \frac{1}{2} + \hat{\phi} \frac{1}{\sqrt{2}} \quad \text{V/m} = \hat{R} 1.5 - \hat{\theta} 0.5 + \hat{\phi} 0.707 \text{ V/m} \end{aligned}$$

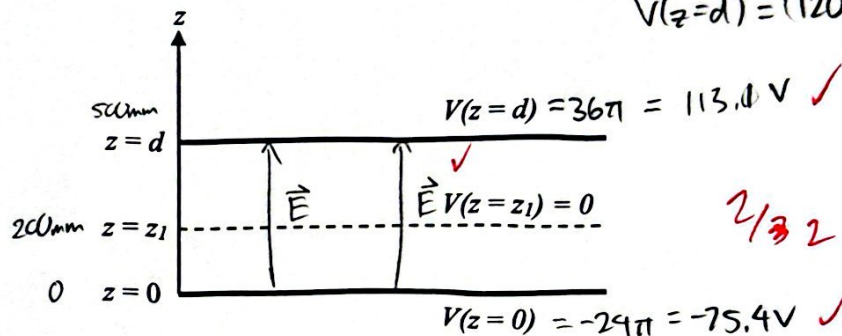
$$\begin{aligned} c) \vec{D} &= \epsilon_0 \vec{E} = \hat{R} \frac{1}{12\pi R^4} \cos \phi \sin \theta - \hat{\theta} \frac{1}{36\pi R^4} \cos \phi \cos \theta \\ &\quad + \hat{\phi} \frac{1}{36\pi R^4} \sin \phi \quad \text{nC/m} \end{aligned}$$

$$\begin{aligned} d) \vec{D} &= \epsilon_0 \vec{E} = \left(\frac{10^{-9}}{36\pi}\right) \vec{E} = \hat{R} 0.1326 - \hat{\theta} 0.0442 + \hat{\phi} 0.0625 \text{ pC/m}^2 \\ &\quad - 0.5 \end{aligned}$$

Q4. Two conducting plates are located at $z = 0$ and $z = d = 500$ mm, as shown in the figure below. The zero-voltage reference is at $z = z_1 = 200$ mm, and the electric field is given by

$$\vec{E} = 120\pi \hat{z} \quad \text{V/m}$$

- (a) Using Laplace's equation, calculate the voltage on the top conductor, $V(z = d)$. [5] 1.5
 (b) Calculate the voltage on the bottom conductor, $V(z = 0)$. [3] 3
 (c) On the diagram below, write down the values for $V(z = d)$ and $V(z = 0)$ and sketch the electric fields between the two conducting plates [2]. 2



a) $\nabla^2 V = 0$

$$\frac{dV}{dz} = \int \frac{dV^2}{dz^2} dz = \int 0 dz = A$$

$$V = \int \frac{dV}{dz} dz = \int A dz = Az + B$$

$$V(z=z_1) = 0 \text{ V}$$

$$\left. \begin{aligned} V(z=d) &= 36\pi = A(0.5) + B \\ V(z=z_1) &= 0 = A(0.2) + B \end{aligned} \right\}$$

$$\begin{aligned} 36\pi &= 0.5A \\ A &= 120\pi \end{aligned} \quad \begin{aligned} B &= 36\pi - (120\pi)(0.5) \\ &= -24\pi \end{aligned}$$

$$V = 120\pi z - 24\pi$$

b) $V(z=0) = 120\pi(0) - 24\pi$
 $= -24\pi = -75.4 \text{ V}$

c)