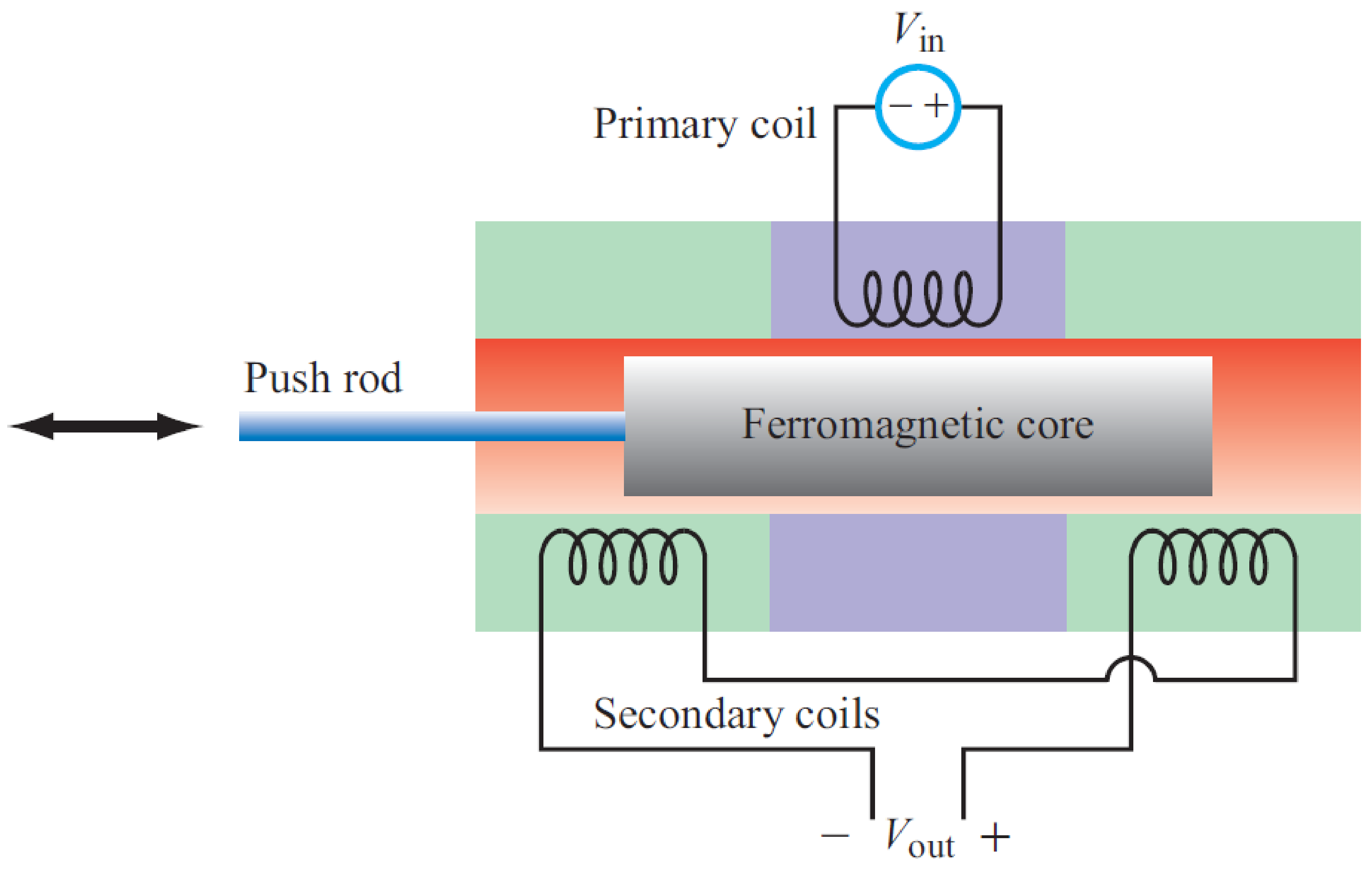

Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

4. MAGNETOSTATICS

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Comparison of Electric and Magnetic Attributes

For simple materials:

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

$\mathbf{D}$ = electric flux density
$\mathbf{B}$ = magnetic flux density

μ = permeability

In this course, for the dielectrics and conductors that we will consider, we assume that

$\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m

That is, we assume that the relation between $\mathbf{B}$ and $\mathbf{H}$ is linear.

Table 5-1: Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

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Electric and Magnetic Forces $\vec{F}_e = q\vec{E}$

From experimental observations, we know that a charge that is moving with a velocity **u** in a magnetic field **H** has a magnetic force $\mathbf{F_m}$ (with respect to the magnetic flux density, **B** = μ**H**):

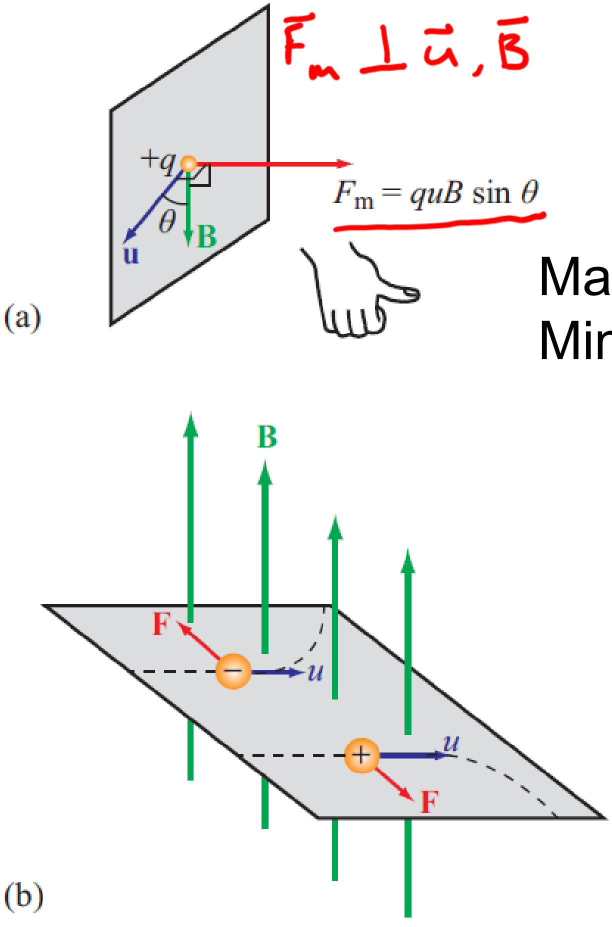
Magnetic Force $\left[\frac{N}{C\cdot m/s}\right] = [T]$

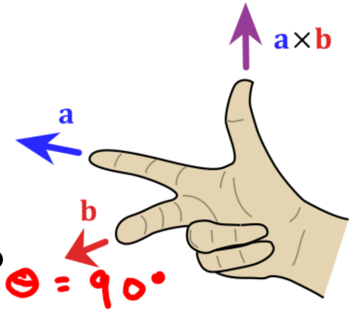
$$\mathbf{F}_{\rm m} = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)} \qquad (5.3)$$

If we have both electric and magnetic fields:

Electromagnetic force
(Lorentz force)

$$\mathbf{F} = \mathbf{F}_{\rm e} + \mathbf{F}_{\rm m} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$





Max? $\theta = 90°$
Min? $\theta = 0°$

±*q* determines the direction of **F**

Figure 5-1: The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both **B** and **u** and (b) depends on the charge polarity (positive or negative).

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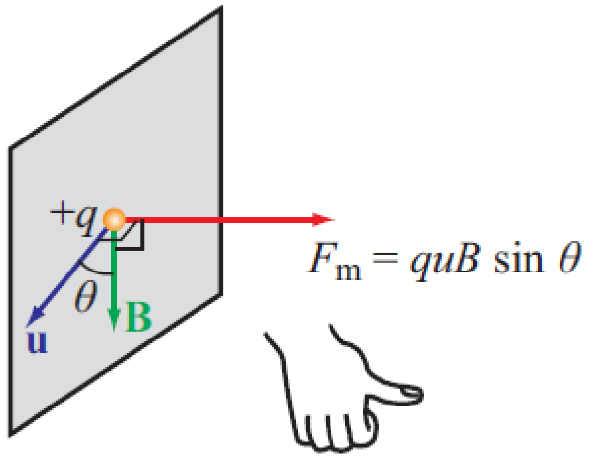
Electric and Magnetic Forces

$$\mathbf{F} = \mathbf{F}_\mathrm{e} + \mathbf{F}_\mathrm{m} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

- The electric force is always in the direction of **E**, while the magnetic force is always in the direction *perpendicular* to **B**.
- The electric force acts on a static or moving charge, while the magnetic force acts only on a moving charge.
- Energy is required to move a charge by an electric force, while a magnetic force does not produce any work when the charge is moved a distance $d\mathbf{l} = \mathbf{u}dt$.

0 since $\vec{F}_m \perp \vec{u}$

$$dW = \mathbf{F}_\mathrm{m} \cdot d\mathbf{l} = (\mathbf{F}_\mathrm{m} \cdot \mathbf{u})\, dt = 0.$$



- Because no work is produced, a magnetic field cannot change the kinetic energy of a charge. It can only change its direction, but not its speed.

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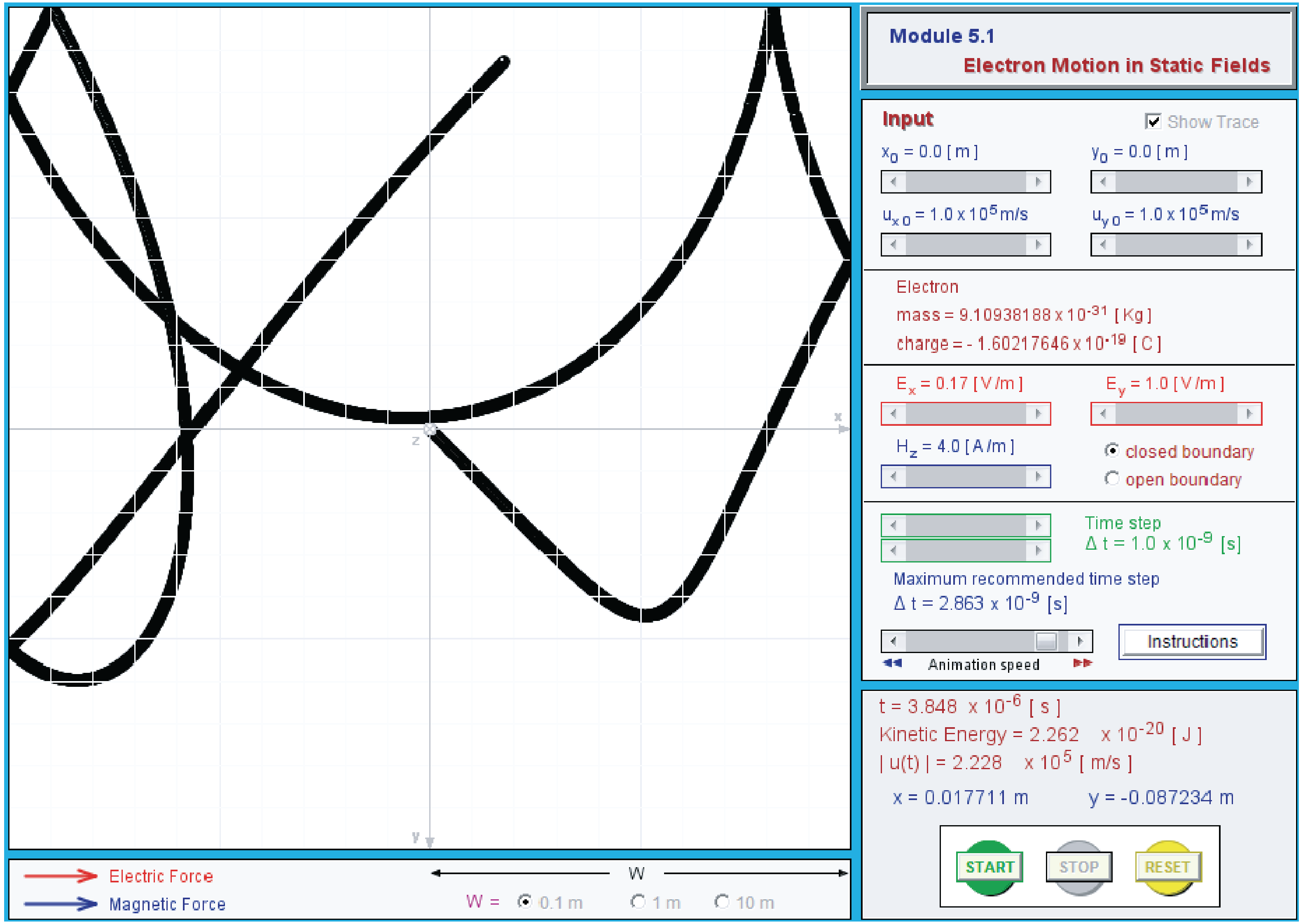
Module 5.1: Electron Motion in Static Fields
https://em8e.eecs.umich.edu/jsmodules/ch5/mod5_1.html

Coordinate system:
x-axis to the right
y-axis downwards
z-axis into the page

The electron has a negative charge

Can apply:
- Only an E-field
- Only an H-field
- Both E and H fields simultaneously

Can show the electric and magnetic forces or the trace of the electron.



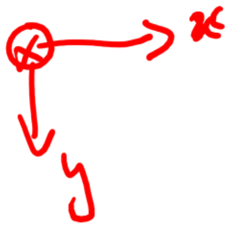
Module 5.1: Electron Motion in Static Fields

Case 1:

Ux= -1e5 m/s , uy = 0

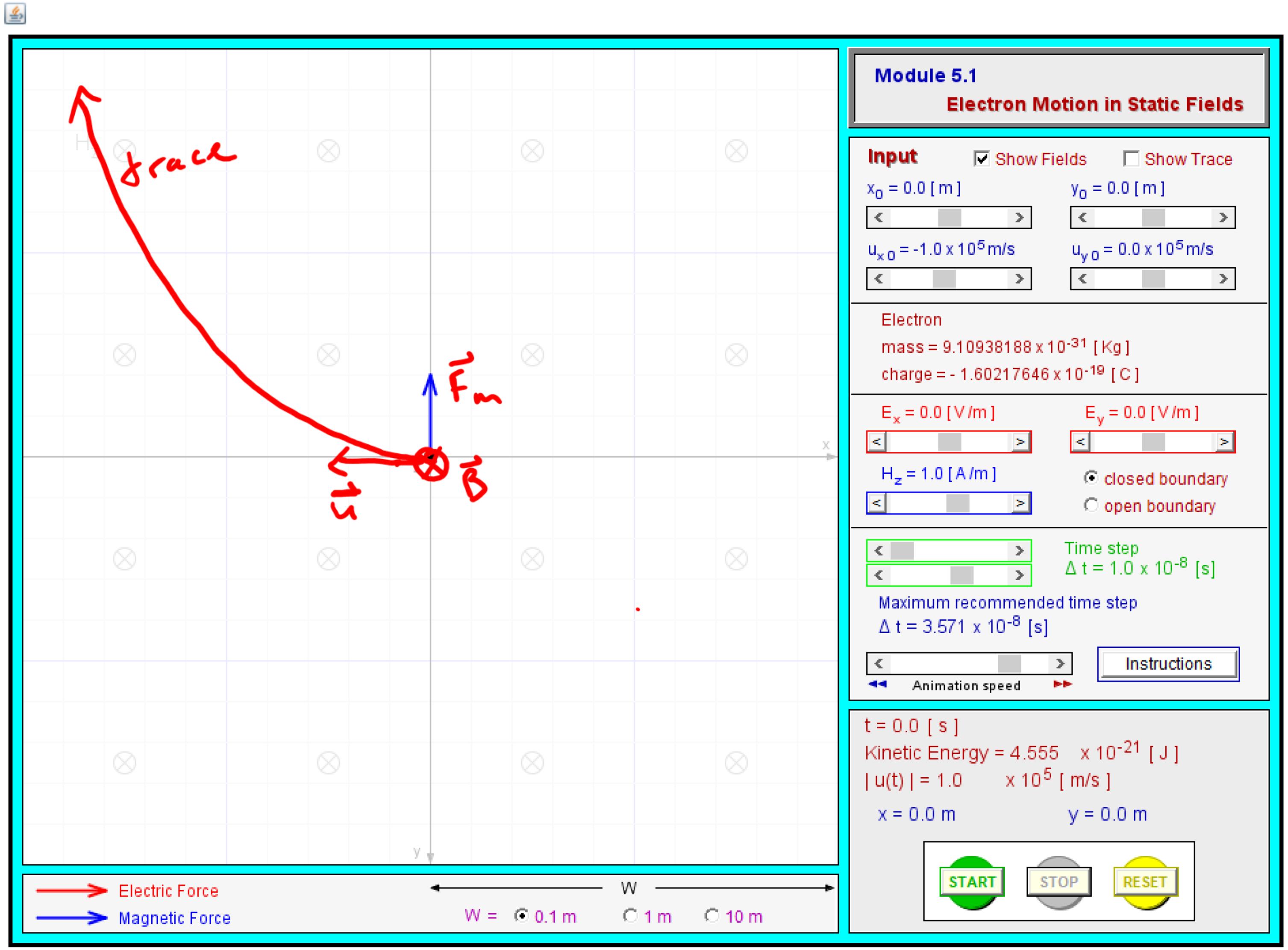
Hz = 1 A/m, (into page)

Ex = Ey = 0



Can see magnetic force direction or the trace of the electron.

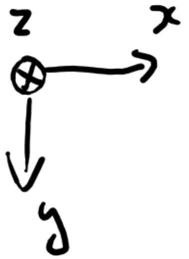
$$\vec{F}_m = q\vec{u} \times \vec{B}$$



Module 5.1: Electron Motion in Static Fields

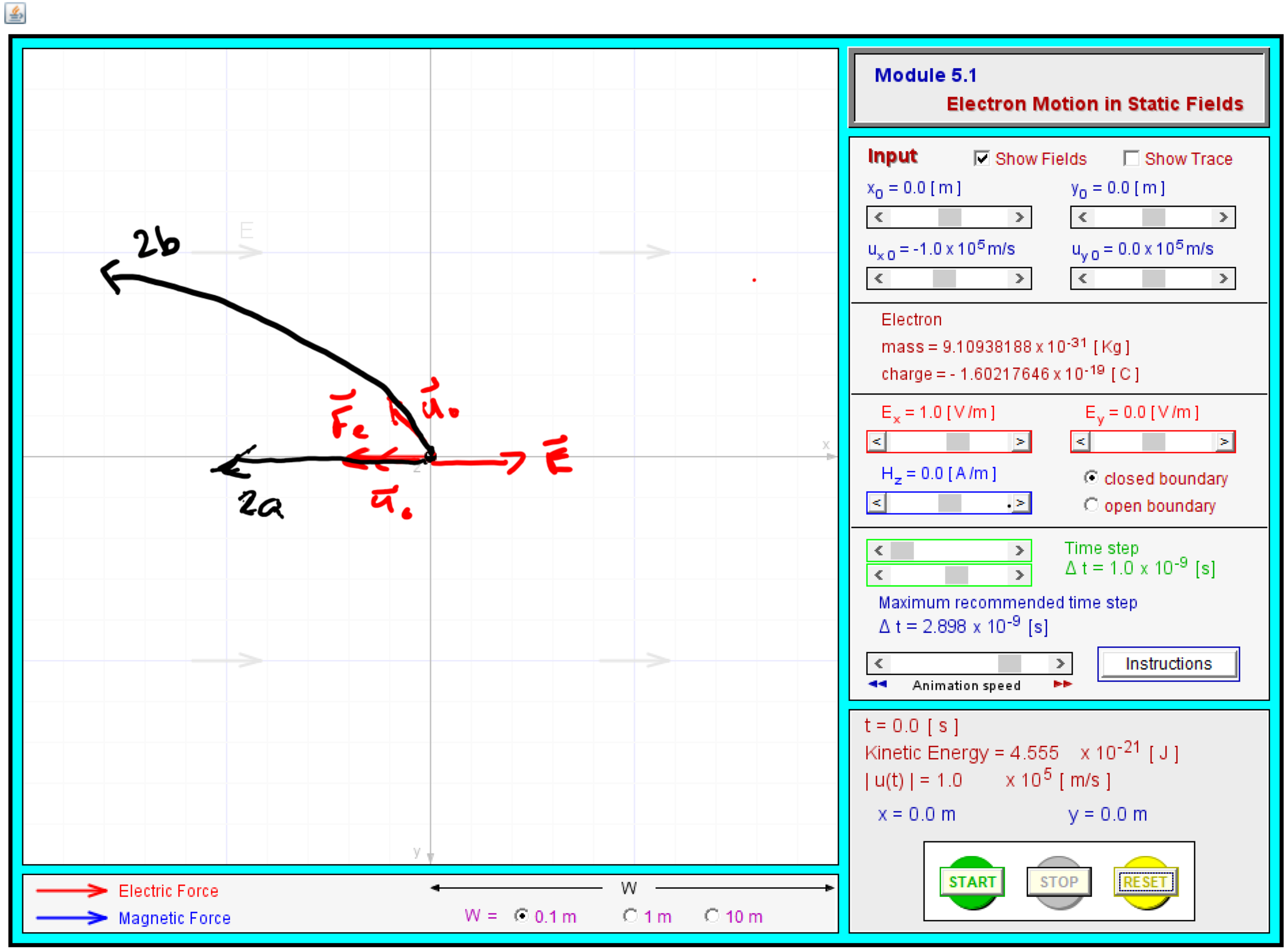
Case 2a:
Ux= -1e5 m/s , uy = 0
Ex = 1 V/m
Ey = 0
Hz = 0

$\vec{F}_e = q\vec{E}$



Case 2b:
Ux= -1e5 m/s , uy = -1e5 m/s
Ex = 1 V/m
Ey = 0
Hz = 0

Can see electric force direction or the trace of the electron.



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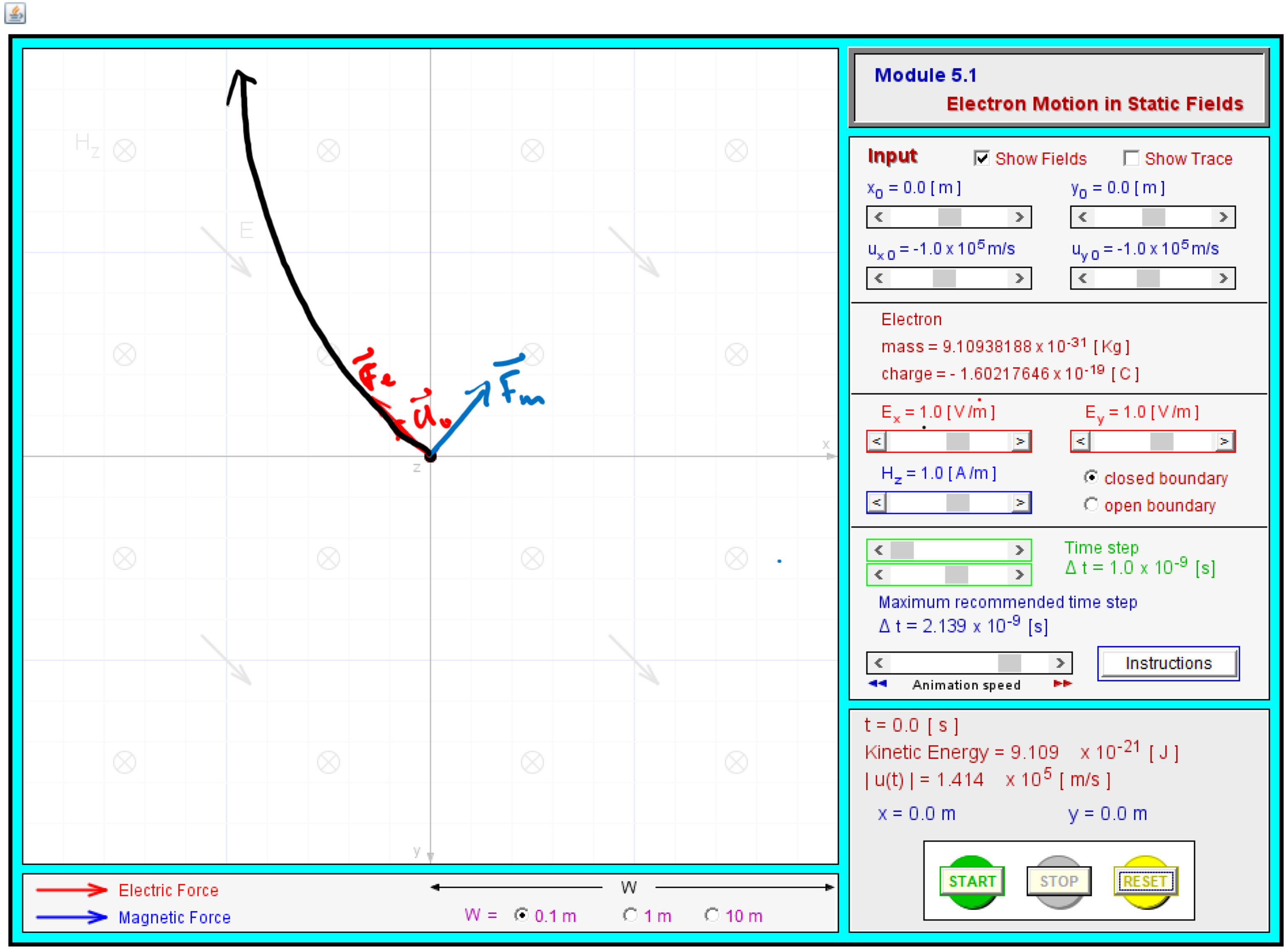
Module 5.1: Electron Motion in Static Fields

Case 3:
ux= -1e5 m/s , uy = -1e5 m/s
Ex = Ey = 1 V/m
Hz = 1 A/m

Can see both the electric and the magnetic force directions or the trace of the electron.



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Magnetic Force on a Current Element

Consider the differential force on electrons moving in a wire in a static B-field:

$$d\mathbf{F}_{\mathbf{m}} = dQ\, \mathbf{u}_{\mathbf{e}} \times \mathbf{B} = \underbrace{-N_e e A dl\, \mathbf{u}_{\mathbf{e}}}_{I} \times \mathbf{B}$$

N_e : moving e's per unit vol.

A : cross-sectional area

$\vec{u}_e$: drift vel. of e's

Differential force dF_m of a differential current element $I\, d\, \mathbf{l}$:

$$d\mathbf{F}_{\mathbf{m}} = I\, d\,\mathbf{l} \times \mathbf{B} \qquad \text{(N)} \qquad (5.9)$$

For a closed circuit of contour C that carries a current I, the total magnetic force is

$$\boxed{\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N).} \qquad (5.10)}$$

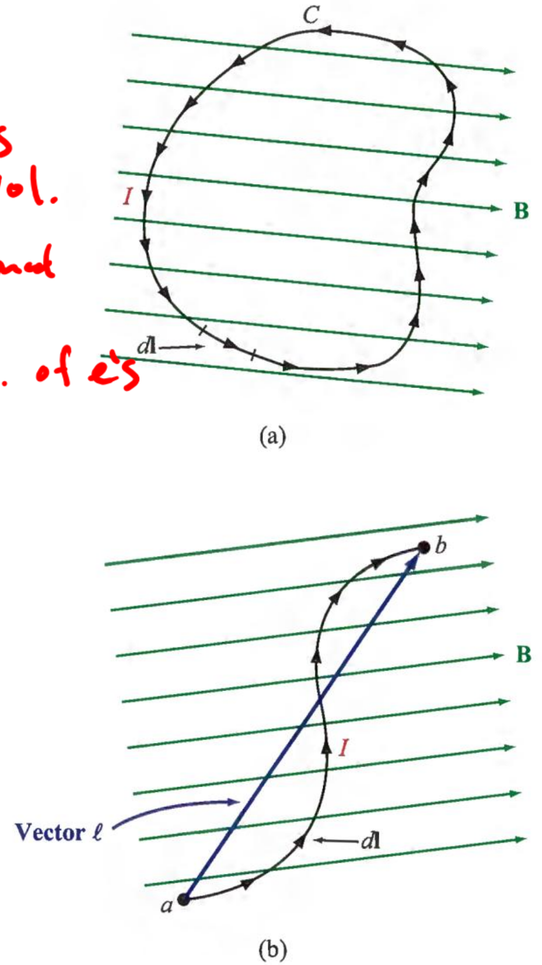

Figure 5-3: In a uniform magnetic field, (a) the net force on a closed current loop is zero because the integral of the displacement vector $d\mathbf{l}$ over a closed contour is zero, and (b) the force on a line segment is proportional to the vector between the end point ($\mathbf{F}_{\mathrm{m}} = I\boldsymbol{\ell} \times \mathbf{B}$).

Magnetic Force on a Current-Carrying Line

$$\boxed{\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N).} \qquad (5.10)}$$

If the closed wire shown in Fig. 5-3(a) resides in a uniform external magnetic field **B**, then **B** can be taken outside the integral in Eq. (5.10), in which case

$$\mathbf{F}_{\mathrm{m}} = I \left(\cancelto{0}{\oint_C d\mathbf{l}} \right) \times \mathbf{B} = 0. \qquad (5.11)$$

*This result, which is a consequence of the fact that the vector sum of the infinitesimal vectors d***l** *over a closed path equals zero, states that the total magnetic force on any closed current loop in a uniform magnetic field is zero.*

Along a line: $\mathbf{F}_{\mathrm{m}} = I \left(\int_\ell d\mathbf{l} \right) \times \mathbf{B} = I\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}},$ (5.12)

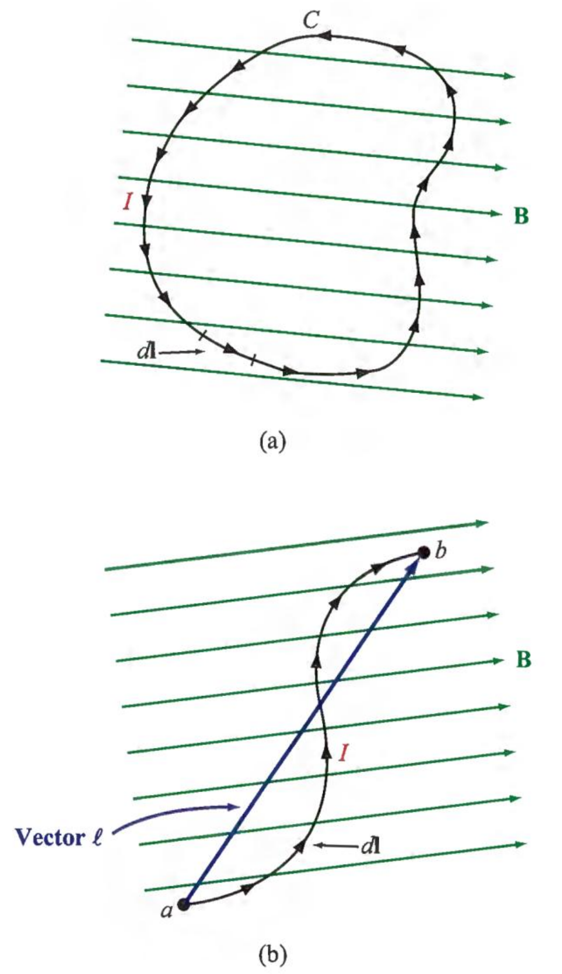

Figure 5-3: In a uniform magnetic field, (a) the net force on a closed current loop is zero because the integral of the displacement vector $d\mathbf{l}$ over a closed contour is zero, and (b) the force on a line segment is proportional to the vector between the end point ($\mathbf{F}_{\mathrm{m}} = I\boldsymbol{\ell} \times \mathbf{B}$).

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Magnetic Force on a Wire

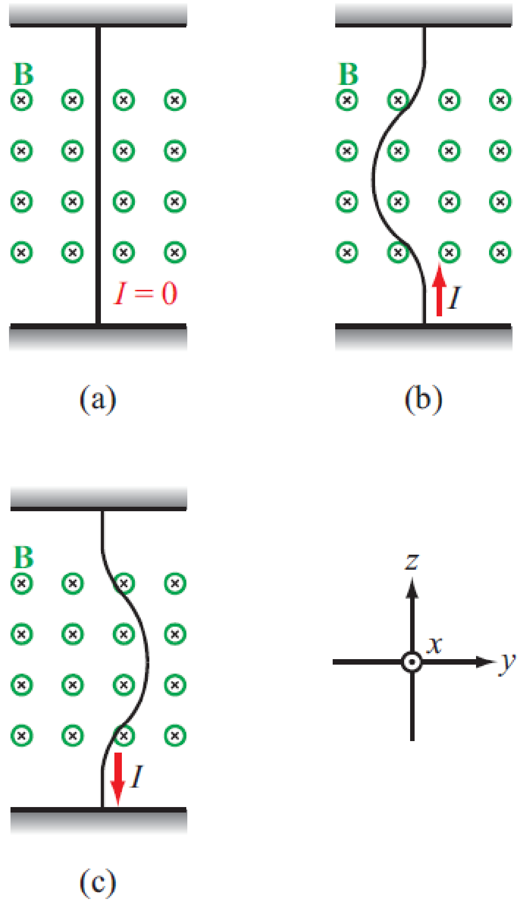
Magnetic Force

$$\mathbf{F}_\mathrm{m} = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)} \qquad (5.3)$$

(a) When $I = 0$, u = 0, then there is no magnetic force on the wire.
(b) When current flows in the $+z$ direction, then the force is then in the $-y$ direction.
(c) When the current flows in the $-z$ direction, then the force is then in the $+y$ direction.

The above arguments can also be given using the current along the wire.

$$\mathbf{F}_\mathrm{m} = I\left(\int_\ell d\mathbf{l}\right) \times \mathbf{B} = I\boldsymbol{\ell} \times \mathbf{B}, \qquad (5.12)$$



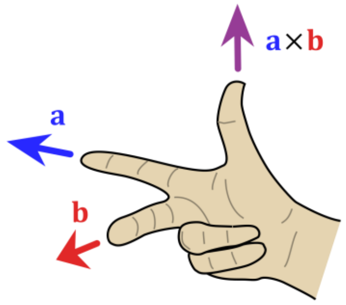


Figure 5-2: When a slightly flexible vertical wire is placed in a magnetic field directed into the page (as denoted by the crosses), it is (a) not deflected when the current through it is zero, (b) deflected to the left when I is upward, and (c) deflected to the right when I is downward.

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Torque

When we apply a force on a body that can pivot about a fixed axis, this will rotate around the axis. The torque is given by:

$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \qquad (\mathrm{N{\cdot}m})$$

$= |\vec{d}||\vec{F}|\sin\theta$ Max when $\theta = 90°$

d = Moment arm

F = Force

T = Torque

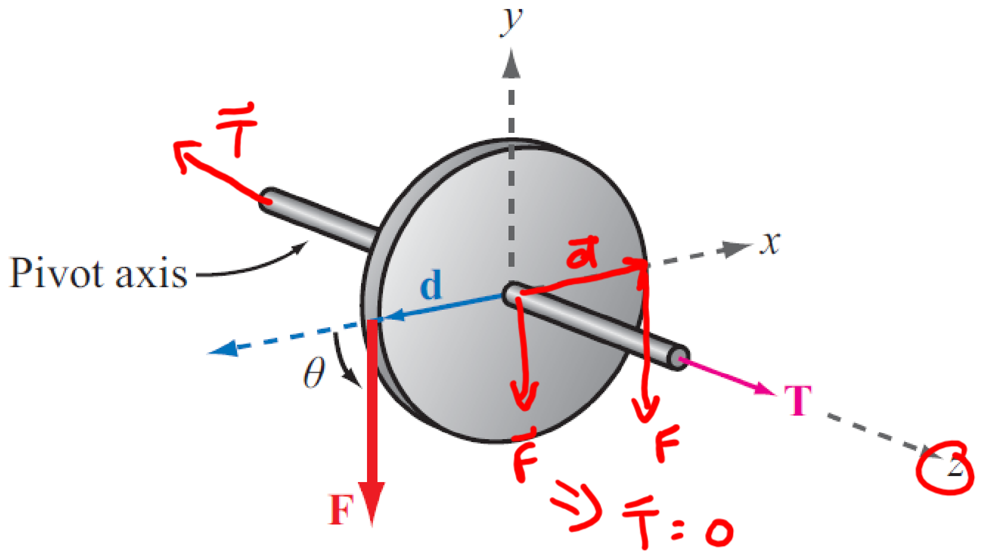

Figure 5-5: The force **F** acting on a circular disk that can pivot along the z-axis generates a torque $\mathbf{T} = \mathbf{d} \times \mathbf{F}$ that causes the disk to rotate.

$\vec{T}$ along $+\hat{z}$: CCW rotation

$\vec{T}$ along $-\hat{z}$: CW rotation.

These directions are governed by the following ***right-hand rule:*** *when the thumb of the right hand points along the direction of the torque, the four fingers indicate the direction that the torque tries to rotate the body.*

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Magnetic Torque on Current Loop

Consider a loop in the *xy*-plane that carries a current I. It can pivot around the *y*-axis. An external uniform magnetic field is applied $\vec{B} = \hat{x} B_0$

Recall $\vec{F}_m = I\vec{l} \times \vec{B}$

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}} I b B_0,$$

$$\vec{F}_2 = I(-\hat{x}a) \times (\hat{x}B_0) = 0$$

$$\mathbf{F}_3 = I(\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}} I b B_0.$$

$$\vec{F}_4 = I(\hat{x}a) \times (\hat{x}B_0) = 0$$

Magnetic torque:

$$\mathbf{T} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3$$

$$= \left(-\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (\hat{\mathbf{z}} I b B_0) + \left(\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (-\hat{\mathbf{z}} I b B_0)$$

$$= \hat{\mathbf{y}} I a b B_0 = \hat{\mathbf{y}} I A B_0,$$

Area of Loop

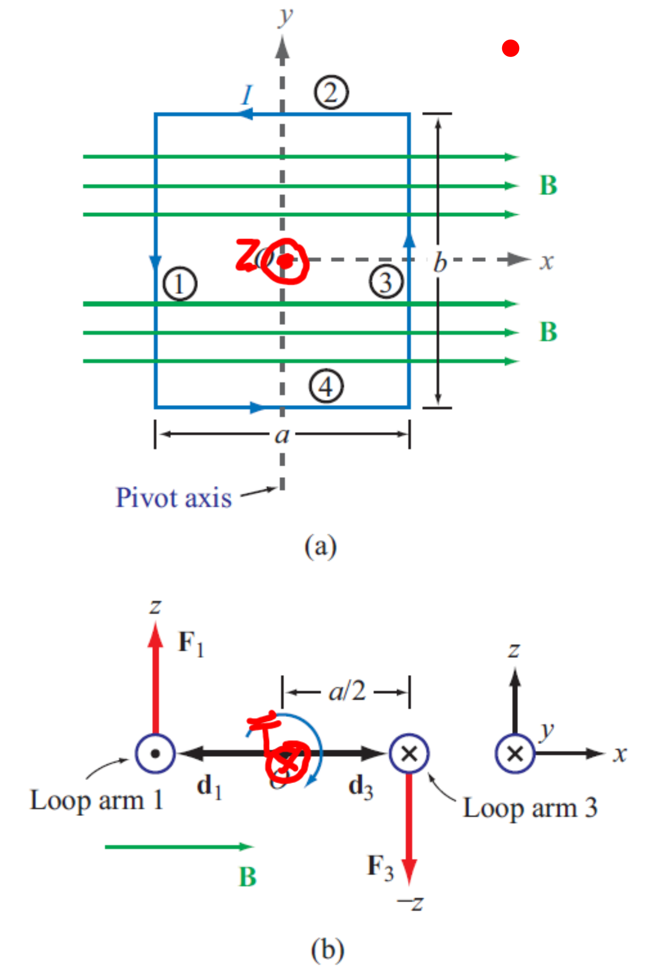

Figure 5-6: Rectangular loop pivoted along the *y*-axis: (a) front view and (b) bottom view. The combination of forces $\mathbf{F}_1$ and $\mathbf{F}_3$ on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).

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Inclined Loop

Now, let's have a look at a more general example of an inclined loop where the direction of **B** is at an angle θ with respect to the loop's surface normal $\hat{\boldsymbol{n}}$.

- Now, there will be forces acting on all 4 arms of the rectangular loop.
- Forces $\mathbf{F}_2$ and $\mathbf{F}_4$ are equal in magnitude and opposite in direction, and along the y-axis. The net torque is equal to zero.
- The currents in arms 1 and 3 are always perpendicular to **B** regardless of the angle θ. Therefore, we still have as before:

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}}IbB_0, \quad \mathbf{F}_3 = I(\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}}IbB_0.$$

- Only the moment arm magnitude $d=(a/2)\sin\theta$ changes as θ changes.

$T = IAB_0 \sin\theta \Rightarrow$ Max for $\theta = 90°$
Zero for $\theta = 0°$

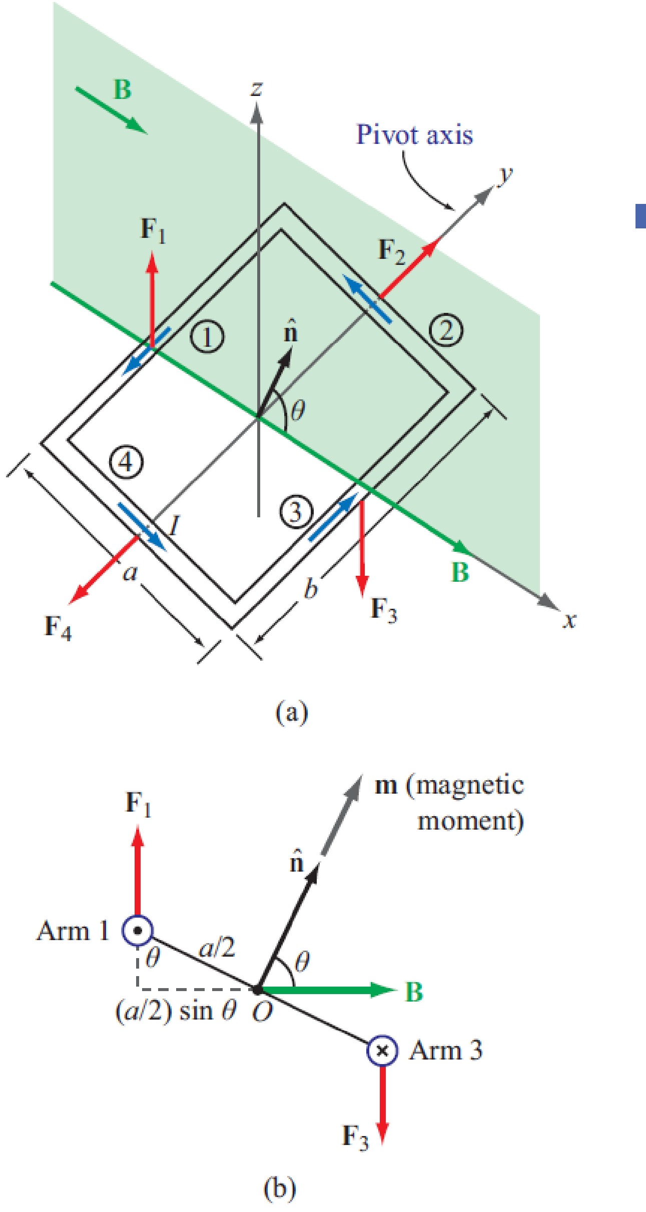

Figure 5-7: Rectangular loop in a uniform magnetic field with flux density **B** whose direction is perpendicular to the rotation axis of the loop, but makes an angle θ with the loop's surface normal $\hat{\mathbf{n}}$.

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Inclined Loop

For a loop with N turns and whose surface normal is at angle θ relative to the **B** direction:

$$T = NIAB_0 \sin\theta. \qquad (5.18)$$

The quantity NIA is called the ***magnetic moment*** m of the loop. Now, consider the vector

↳ strength of interaction with $\bar{B}$.

$$\boxed{\mathbf{m} = \hat{\mathbf{n}}\, NIA = \hat{\mathbf{n}}\, m \qquad (\text{A}\cdot\text{m}^2), \qquad (5.19)}$$

where $\hat{\mathbf{n}}$ is the surface normal of the loop and governed by the following ***right-hand rule: when the four fingers of the right hand advance in the direction of the current I, the direction of the thumb specifies the direction of $\hat{\mathbf{n}}$.*** In terms of **m**, the torque vector **T** can be written as

$$\boxed{\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}). \qquad (5.20)}$$

Valid for any orientation of $\bar{B}$, and loop of any shape.

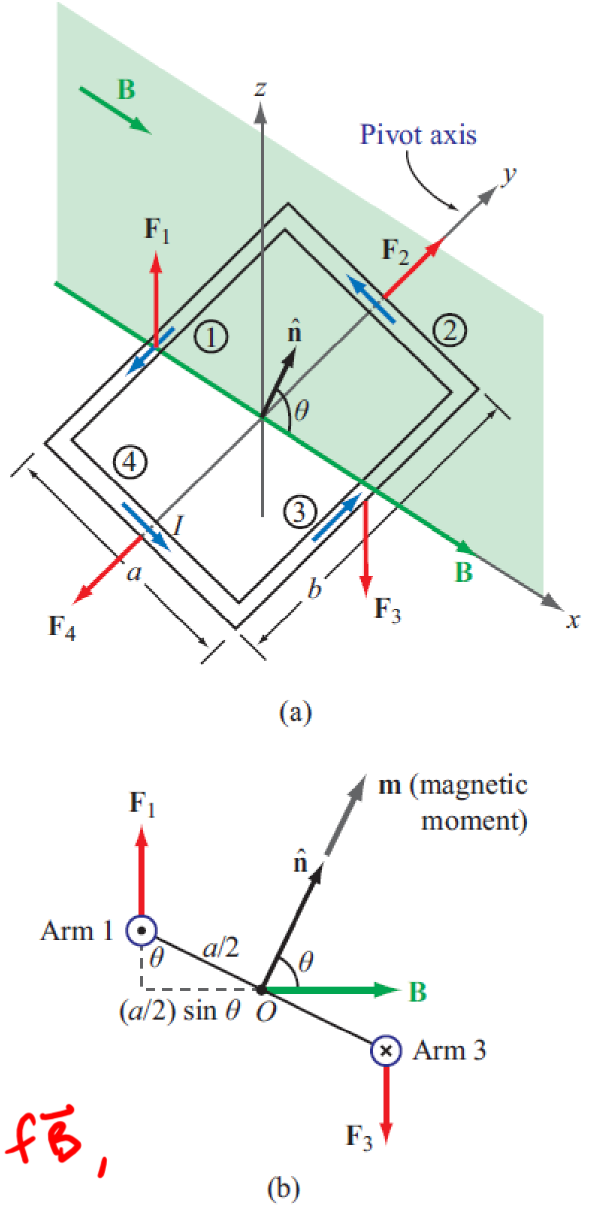

Figure 5-7: Rectangular loop in a uniform magnetic field with flux density **B** whose direction is perpendicular to the rotation axis of the loop, but makes an angle θ with the loop's surface normal $\hat{\mathbf{n}}$.

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The Biot-Savart Law

- Oested's experiments showed that currents induce magnetic fields that form closed loops around wires.
- The Biot-Savart law allows us to relate the magnetic field **H** at any point in space to the current I that generates **H**.

The differential magnetic field that is generated by a steady current I flowing through a differential length vector $d\mathbf{l}$:

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

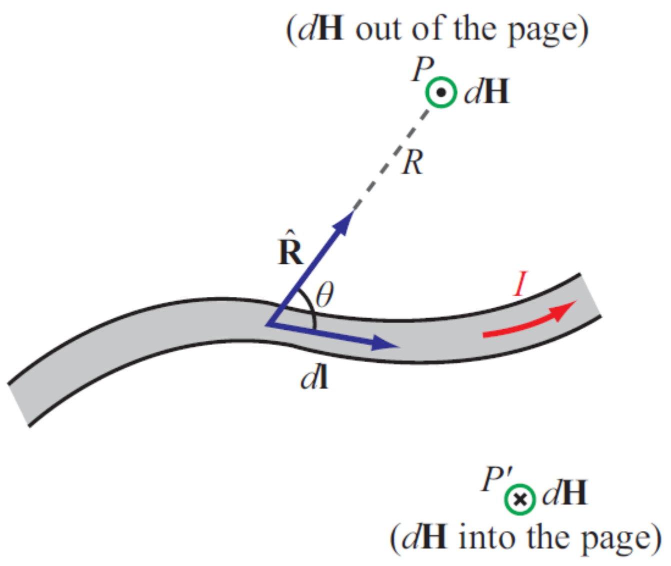

Figure 5-8: Magnetic field $d\mathbf{H}$ generated by a current element $I\ d\mathbf{l}$. The direction of the field induced at point P is opposite to that induced at point P'.

For entire length of wire:

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}, \qquad (5.22)$$

l is the path along which I exists.

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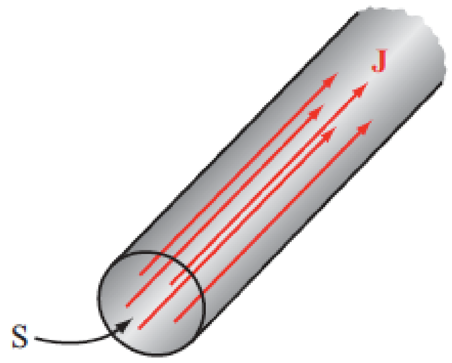
Magnetic Field due to Current Densities

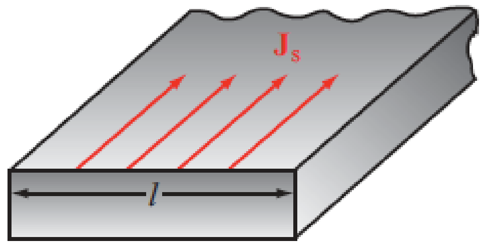
$$I\,d\mathbf{l} \leftrightarrow \mathbf{J}_s\,ds \leftrightarrow \mathbf{J}\,dV$$

$$\mathbf{H} = \frac{1}{4\pi}\int_S \frac{\mathbf{J}_s \times \hat{\mathbf{R}}}{R^2}\,ds \quad \text{(surface current)}, \quad \text{(A/m)}$$

$$\mathbf{H} = \frac{1}{4\pi}\int_V \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2}\,dV \quad \text{(volume current)}. \quad \text{(A/m)}$$



(a) Volume current density $\mathbf{J}$ in A/m^2



(b) Surface current density $\mathbf{J}_s$ in A/m

Figure 5-9: (a) The total current crossing the cross section S of the cylinder is $I = \int_S \mathbf{J}\cdot d\mathbf{s}$. (b) The total current flowing across the surface of the conductor is $I = \int_l J_s\,dl$.

Example: Magnetic Field of Linear Conductor

A linear conductor of length l carries a current I along the z axis. Determine the magnetic flux density **B** at point P located at a distance r in the xy-plane.

$$\mathbf{H} = \frac{I}{4\pi}\int_l \frac{d\mathbf{l}\times\hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}, \qquad (5.22)$$

Cylindrical coordinates.

$d\vec{l} = \hat{z}\,dz$

$\vec{R} = \hat{r}R\sin\theta + \hat{z}R\cos\theta$

$\hat{R} = \hat{r}\sin\theta + \hat{z}\cos\theta$

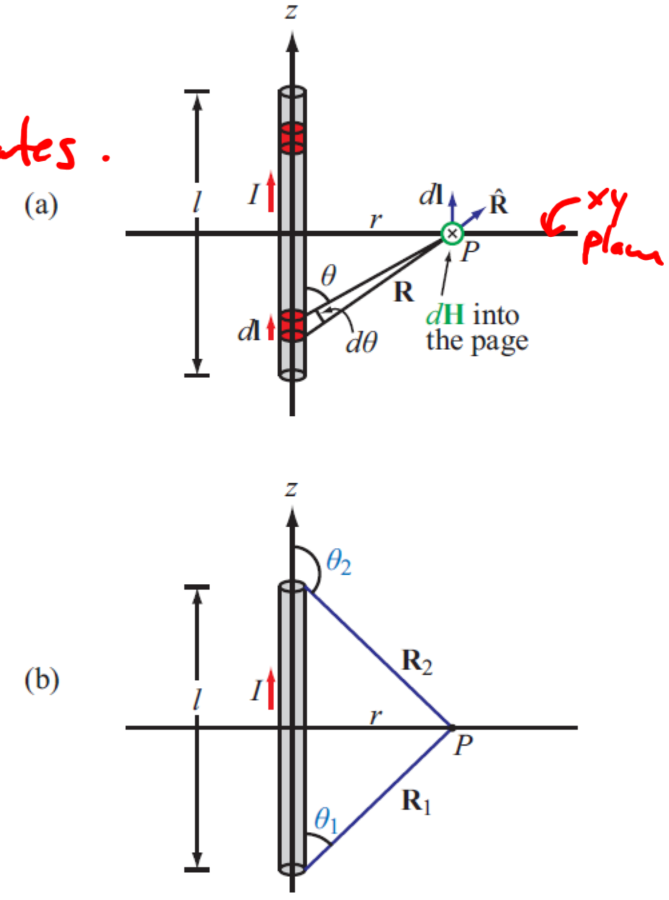


Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

$$d\vec{l}\times\hat{R} = \begin{vmatrix}\hat{r} & \hat{\phi} & \hat{z}\\ 0 & 0 & dz\\ \sin\theta & 0 & \cos\theta\end{vmatrix} = \hat{r}(0) - \hat{\phi}(0-\sin\theta\,dz) + \hat{z}(0) = \hat{\phi}\sin\theta\,dz$$

$$\vec{H} = \frac{I}{4\pi}\int_{-l/2}^{+l/2}\frac{d\vec{l}\times\hat{R}}{R^2} = \hat{\phi}\frac{I}{4\pi}\int_{-l/2}^{+l/2}\frac{\sin\theta}{R^2}\,dz \quad (1)$$

Convert integration variable from z to θ

$\sin\theta = r/R \Rightarrow R = \frac{r}{\sin\theta} = r\csc\theta$ (2)

$\cos\theta = -z/R \Rightarrow z = -R\cos\theta = -\left(\frac{r}{\sin\theta}\right)\cos\theta = \frac{-r}{\tan\theta} = -r\cot\theta$

$\Rightarrow \frac{dz}{d\theta} = -r(-\csc^2\theta) = r\csc^2\theta \Rightarrow dz = r\csc^2\theta\,d\theta$ (3)

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Example: Magnetic Field of Linear Conductor

Substituting ② + ③ into ① :

$$\vec{H} = \hat{\phi}\frac{I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{\sin\theta\,\cancel{r}\,\cancel{\csc^3\theta}\,d\theta}{r^{\cancel{2}}\,\cancel{\csc^2\theta}} = \hat{\phi}\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

$$= \hat{\phi}\frac{I}{4\pi r}\left(\cos\theta_1 - \cos\theta_2\right)$$

$$\cos(x) = -\cos(\pi - x)$$

$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2+(\frac{l}{2})^2}}, \quad \cos\theta_2 = -\cos\theta_1 = \frac{-l/2}{\sqrt{r^2+(l/2)^2}}$$

$$\vec{B} = \mu\vec{H} = \hat{\phi}\frac{\mu I}{4\pi r}\frac{l}{\sqrt{r^2+(l/2)^2}} = \hat{\phi}\frac{\mu I l}{2\pi r\sqrt{4r^2+l^2}}$$

For an infinitely long wire in air ($l \gg r$)

$$\boxed{\vec{B} = \hat{\phi}\frac{\mu_0 I}{2\pi r}}$$

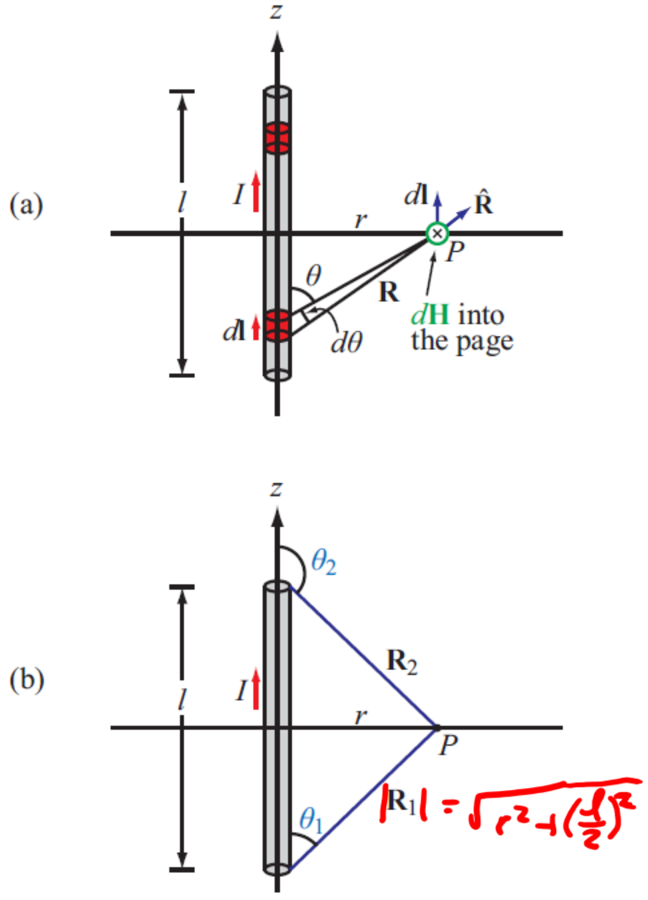

Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $I\,d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

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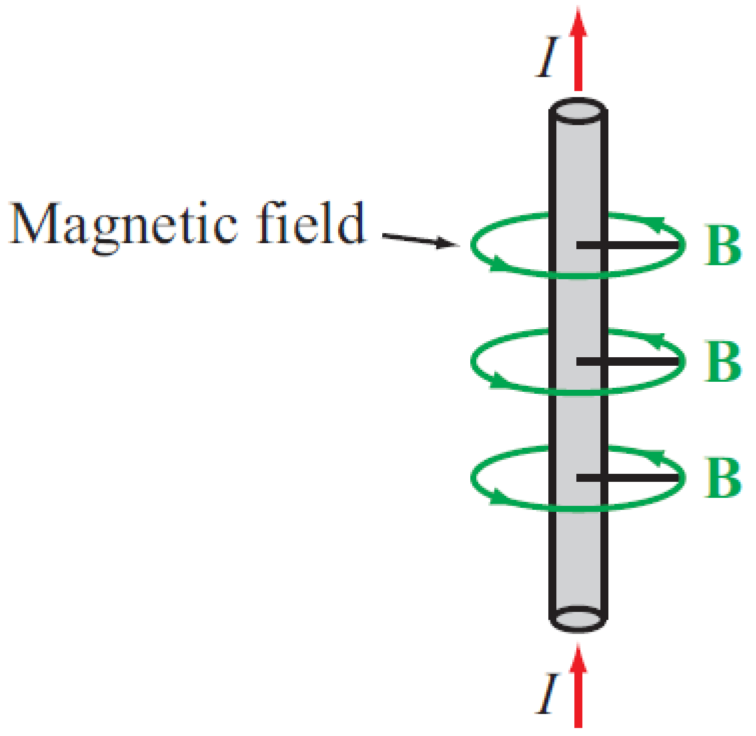
Magnetic Field of a Long Conductor

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r} \qquad \text{(infinitely long wire).}$$



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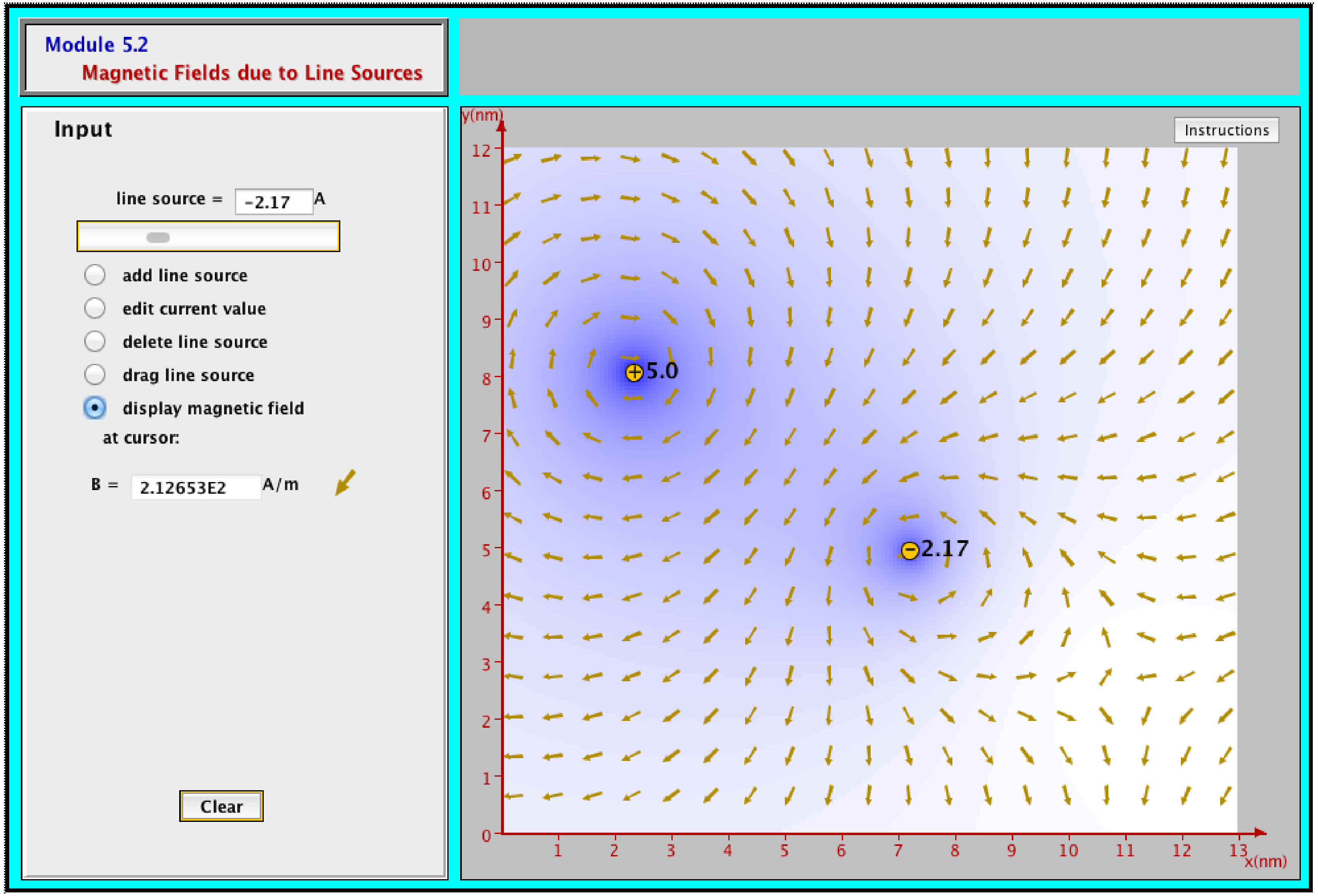
Module 5.2 Magnetic Fields Due to Line Sources You can place *z*-directed linear currents anywhere in the display plane (*x*–*y* plane), select their magnitudes and directions, and then observe the spatial pattern of the induced magnetic flux $\mathbf{B}(x, y)$. https://em8e.eecs.umich.edu/jsmodules/ch5/mod5_2.html



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Example: Magnetic Field of a Loop

A circular loop of radius a carries a steady current I. Determine the magnetic field **H** on the axis of the loop.

From Biot-Savart, the magnitude of field due to dl is:

$$dH = \frac{I}{4\pi R^2}\left|d\vec{l} \times \hat{R}\right| = \frac{I\,dl}{4\pi(a^2+z^2)}$$

d**H** is in the *rz*-plane, and therefore it has components dH_r and dH_z.

z-components of the magnetic fields due to d**l** and dl' add because they are in the same direction, but their *r*-components cancel.

∴ Only have to consider dH_z component.

Hence for element dl:

$$d\vec{H} = \hat{z}\,dH_z = \hat{z}\,dH\cos\theta = \hat{z}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}dl$$

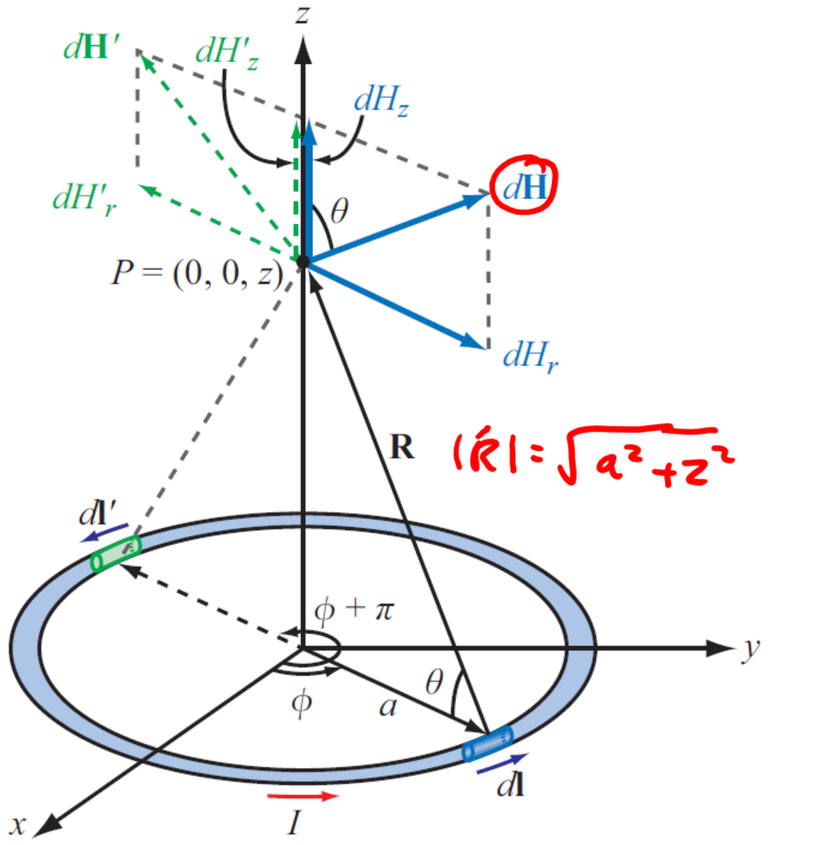

Figure 5-12: Circular loop carrying a current I (Example 5-3).

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Example: Magnetic Field of a Loop

For the entire loop:

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi(a^2+z^2)} \oint dl = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi(a^2+z^2)} (2\pi a). \quad (5.33)$$

Upon using the relation $\cos\theta = a/(a^2+z^2)^{1/2}$, we obtain

$$\boxed{\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2(a^2+z^2)^{3/2}} \quad \text{(A/m)}.} \quad (5.34)$$

At the center of the loop ($z = 0$), Eq. (5.34) reduces to

$$\boxed{\mathbf{H} = \hat{\mathbf{z}} \frac{I}{2a} \quad (\text{at } z = 0),} \quad (5.35)$$

and at points very far away from the loop such that $z^2 \gg a^2$, Eq. (5.34) simplifies to

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2|z|^3} \quad (\text{at } |z| \gg a). \quad (5.36)$$

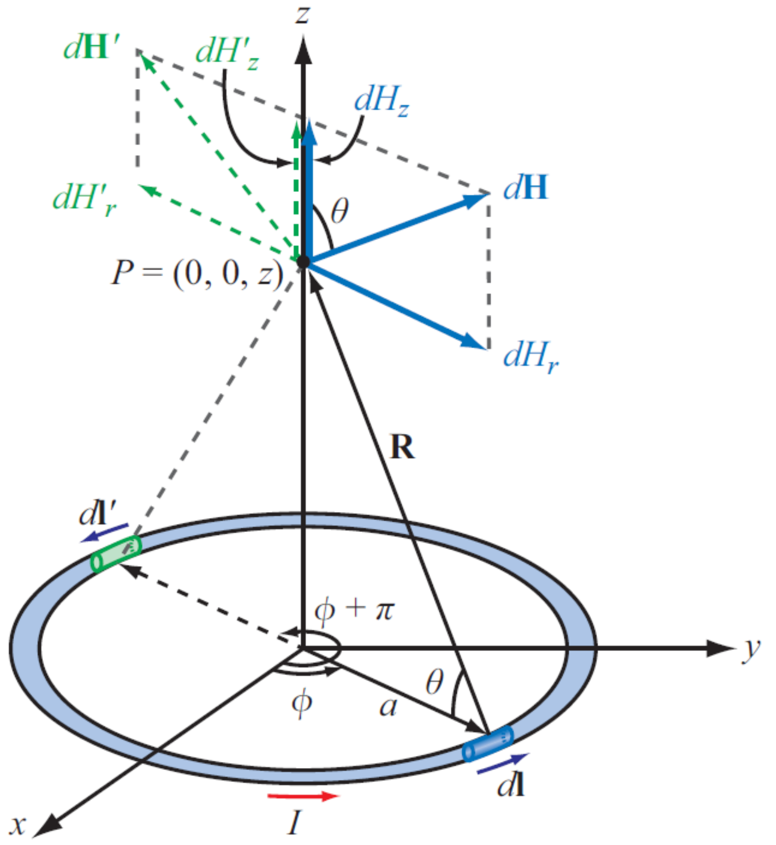

Figure 5-12: Circular loop carrying a current I (Example 5-3).

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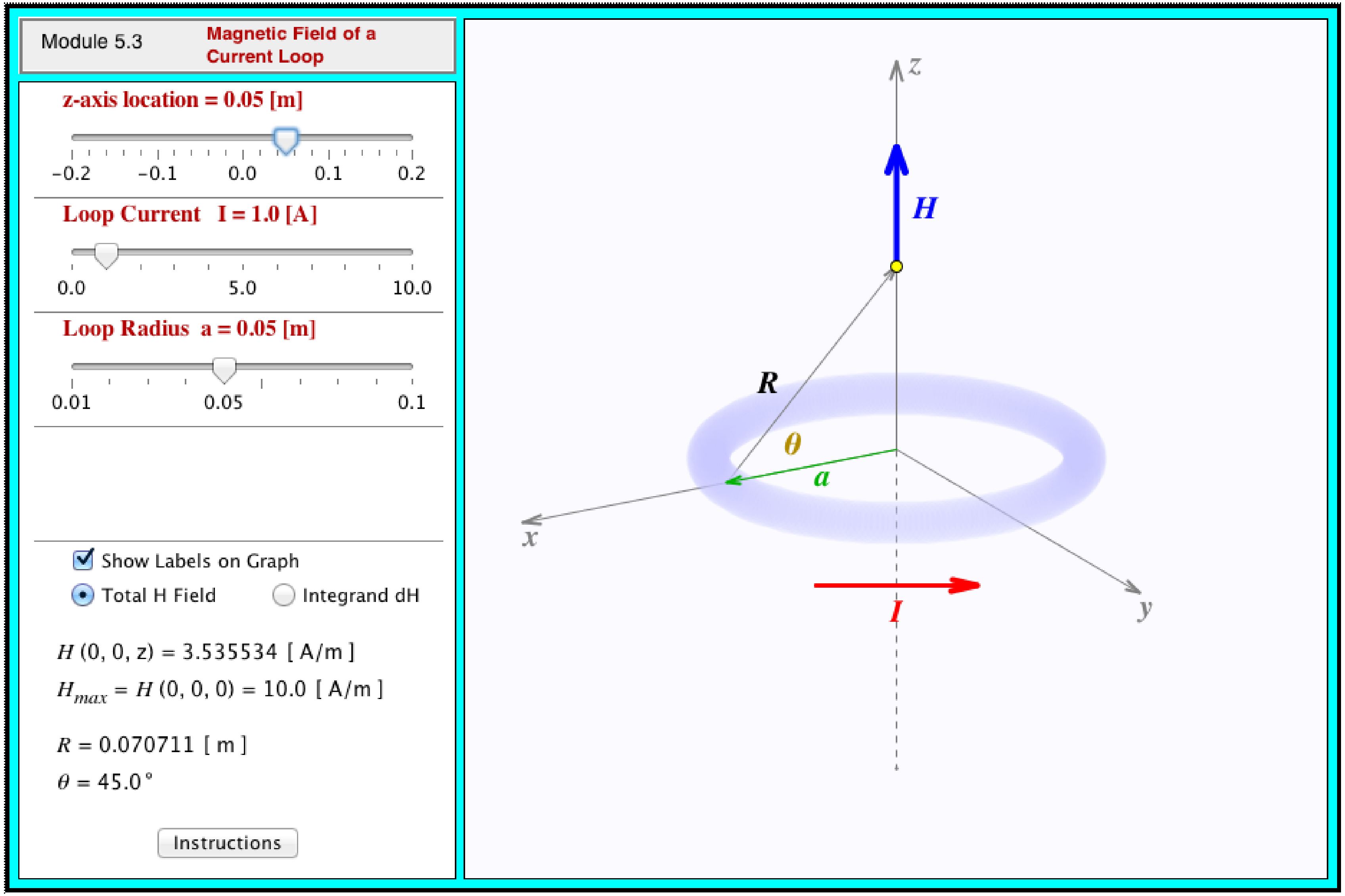
Module 5.3 Magnetic Field of a Current Loop Examine how the field along the loop axis changes with loop parameters.

https://em8e.eecs.umich.edu/jsmodules/ch5/mod5_3.html



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Magnetic Dipole

$\vec{H} = \hat{z}\dfrac{Ia^2}{2|z|^3}$ at $|z| >> a$

$= \hat{z}\dfrac{m}{2\pi|z|^3}$ (cyl. coor).

$m = NIA$

$m = I\pi a^2$

$\vec{E} = \dfrac{p}{4\pi\varepsilon_0 R^3}(\hat{R}\,2\cos\theta + \hat{\theta}\sin\theta)$

$\vec{H} = \dfrac{m}{4\pi R^3}(\hat{R}\,2\cos\theta + \hat{\theta}\sin\theta)$

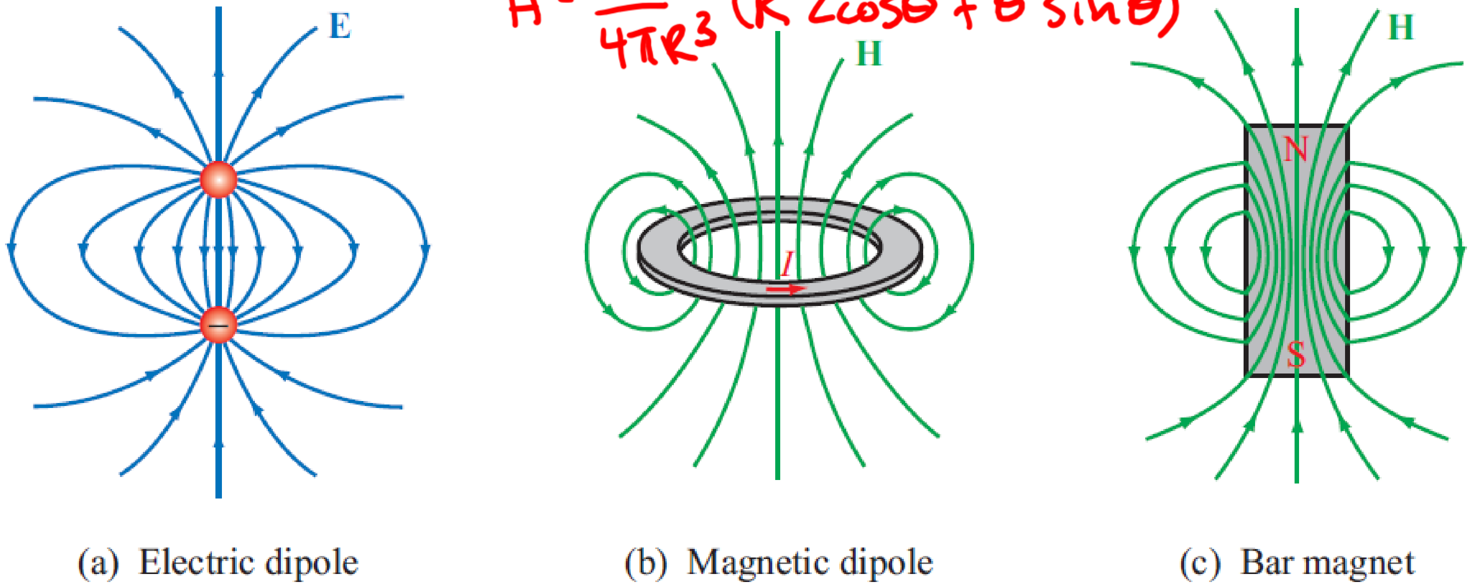

Figure 5-13: Patterns of (a) the electric field of an electric dipole, (b) the magnetic field of a magnetic dipole, and (c) the magnetic field of a bar magnet. Far away from the sources, the field patterns are similar in all three cases.

For an observation point very far away from a circular current loop, the magnetic field pattern is similar to the electric field pattern of an electric dipole. Therefore, the loop is called a *magnetic dipole*.

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Forces on Parallel Conductors

We have already seen that the magnetic force on a current element that is in an external magnetic field **B** = μ**H** is:

$$\mathbf{F}_m = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}. \qquad (5.10)$$

We've also seen that the current on the wire produces its own magnetic field. So, if we place two current-carrying conductors in each other's vicinity, each will exert a magnetic force on the other.

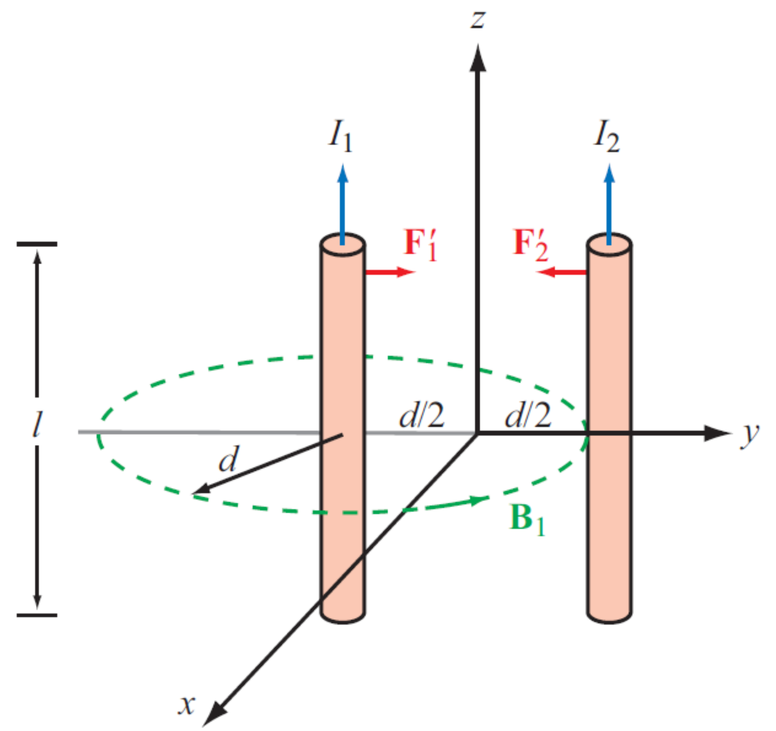

Figure 5-14: Magnetic forces on parallel current-carrying conductors.

Our two wires are parallel, they are separated by a distance *d*, and have currents I_1 and I_2 in the z-direction.
B_1 = magnetic field due to I_1 at the location of conductor 2.
B_2 = magnetic field due to I_2 at the location of conductor 1.

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Forces on Parallel Conductors

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$

$$\mathbf{B}_1 = -\hat{\mathbf{x}} \frac{\mu_0 I_1}{2\pi d} . \quad (5.39)$$

The force $\mathbf{F}_{21}$ exerted on a length l of wire I_2 due to its presence in field $\mathbf{B}_1$ may be obtained by applying Eq. (5.12):

$\vec{F} = I\vec{l} \times \vec{B}$

$$\mathbf{F}_{21} = I_2 l \hat{\mathbf{z}} \times \mathbf{B}_1 = I_2 l \hat{\mathbf{z}} \times (-\hat{\mathbf{x}}) \frac{\mu_0 I_1}{2\pi d}$$
$$= -\hat{\mathbf{y}} \frac{\mu_0 I_1 I_2 l}{2\pi d} , \quad (\mathrm{N}) \quad (5.40)$$

and the corresponding force per unit length is

$$\mathbf{F}'_{21} = \frac{\mathbf{F}_{21}}{l} = -\hat{\mathbf{y}} \frac{\mu_0 I_1 I_2}{2\pi d} . \quad (\mathrm{N/m}) \quad (5.41)$$

$\vec{F}_{12} = I_1 l \hat{z} \times \vec{B}_2$

$\vec{B}_2 = \hat{x} \frac{\mu_0 I_2}{2\pi d}$

A similar analysis performed for the force per unit length exerted on the wire carrying I_1 leads to

$$\mathbf{F}'_{12} = \hat{\mathbf{y}} \frac{\mu_0 I_1 I_2}{2\pi d} . \quad (5.42)$$

Parallel wires attract each other with equal force if their currents are in the same direction, and repel each other with equal force if their currents are in opposite directions.

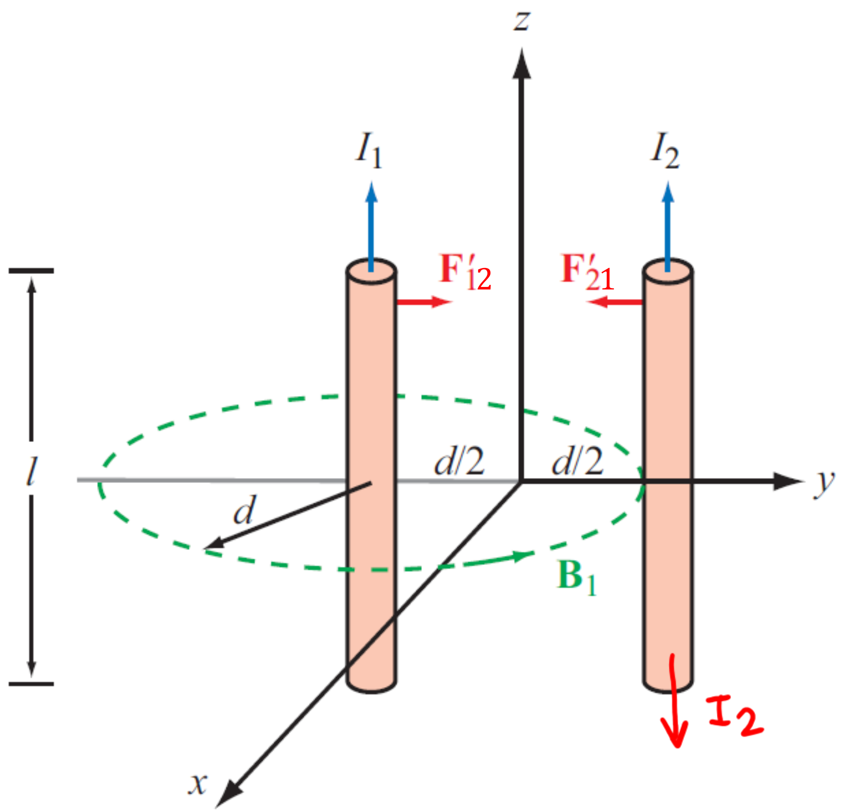
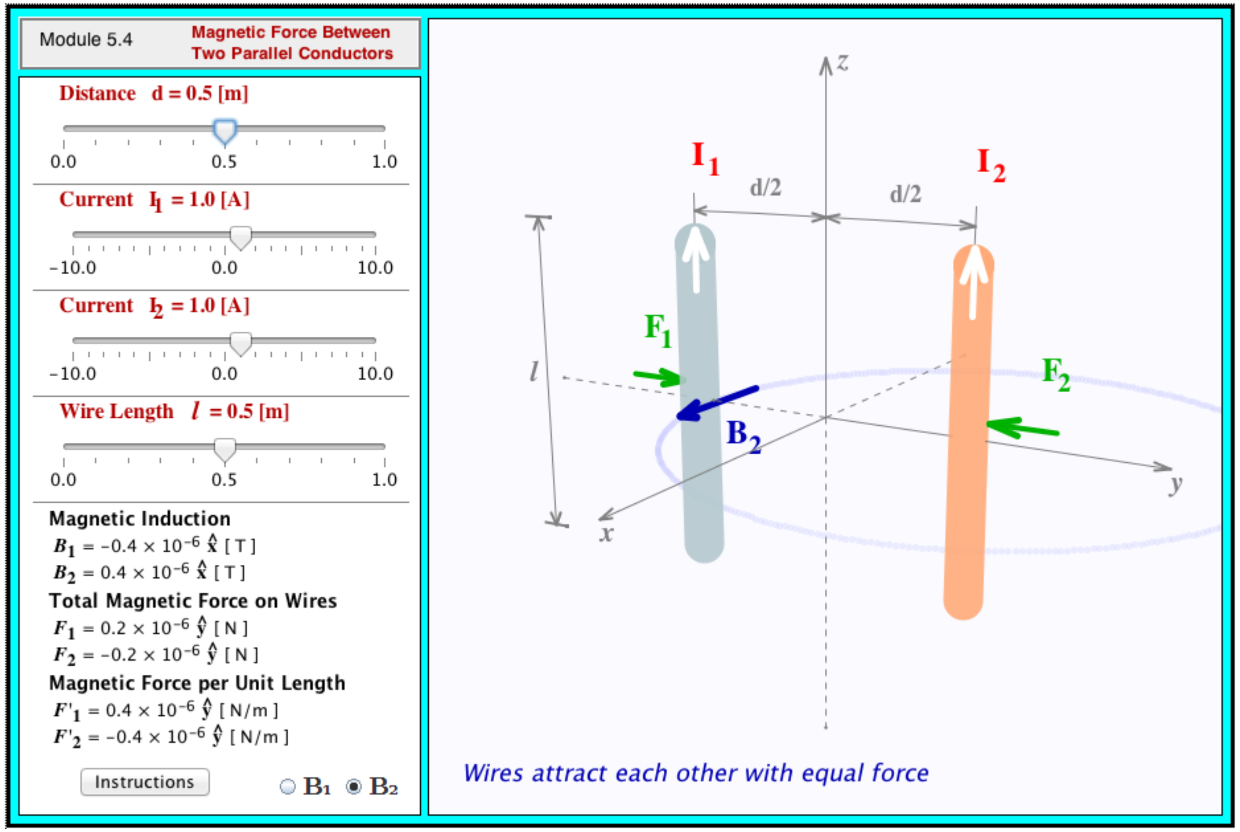


Figure 5-14: Magnetic forces on parallel current-carrying conductors.

Let's Play!

Module 5.4 Magnetic Force between Two Parallel Conductors Observe the direction and magnitude of the force exerted on parallel current-carrying wires.

https://em8e.eecs.umich.edu/jsmodules/ch5/mod5_4.html



Pop Quiz!

Q1: The parallel conductors _____ if their currents are in the same direction, and _____ if their currents are in opposite directions.

a) attract, attract
b) attract, repel
c) repel, attract
d) repel, repel

Q2: The force on the conductor carrying I_2 is $\overrightarrow{F_2}$. Now, suppose the conductor is rotated so that it is parallel to the x axis. What would be the force on it now?

a) $\overrightarrow{F_1}$
b) $\frac{1}{2}\overrightarrow{F_2}$
c) $\frac{1}{2}\overrightarrow{F_1}$
d) 0

Q3: The parallel conductors have equal currents in the same direction. Which equation is true if I_2 is then doubled?

a) $\overrightarrow{F_1} = -\overrightarrow{F_2}$
b) $\overrightarrow{F_1} = 2\overrightarrow{F_2}$
c) $\overrightarrow{F_2} = 2\overrightarrow{F_1}$
d) $\overrightarrow{F_1} = \overrightarrow{F_2}$

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Tech Brief: Electromagnets

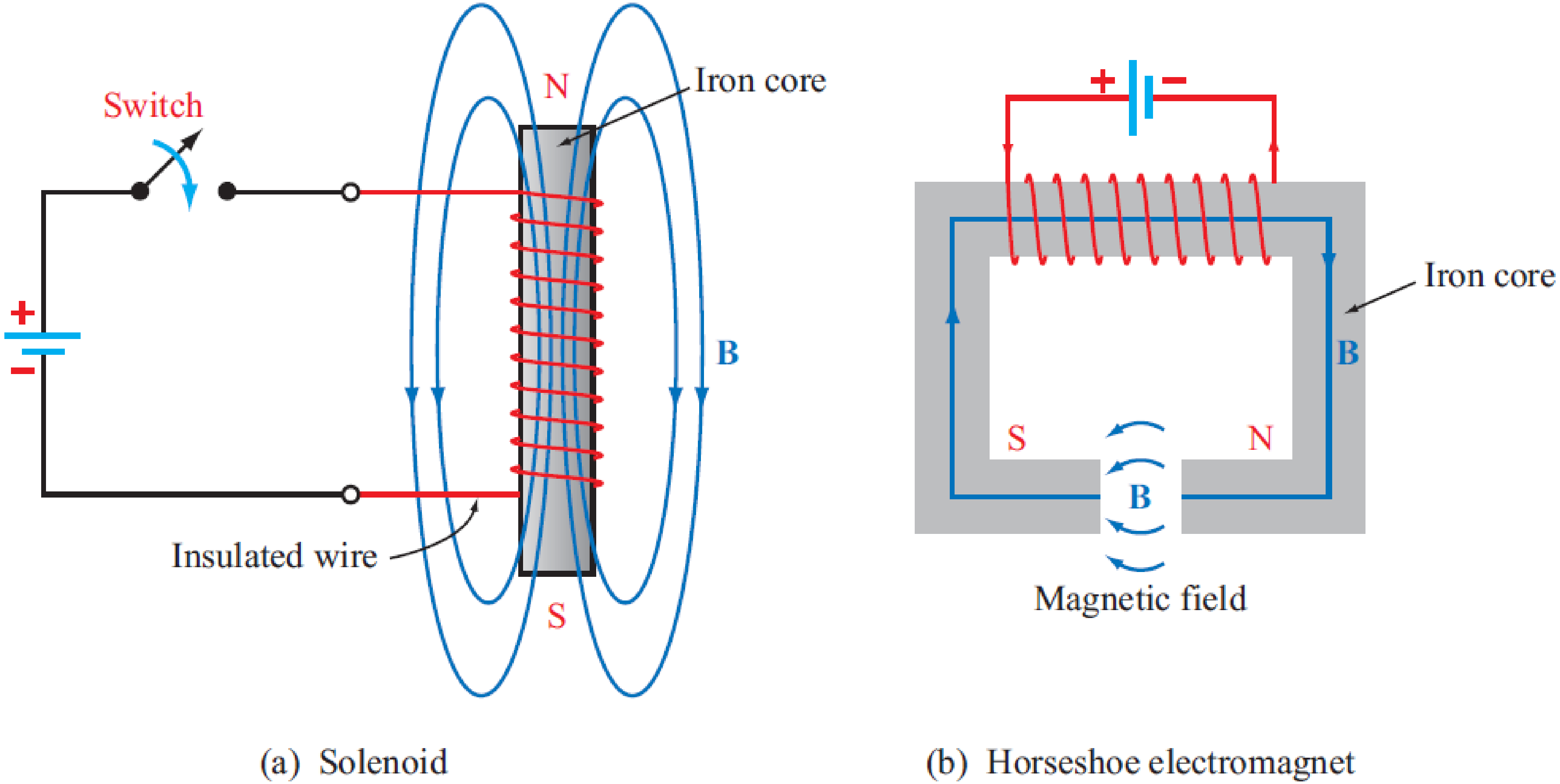

Figure TF10-1: Solenoid and horseshoe magnets.

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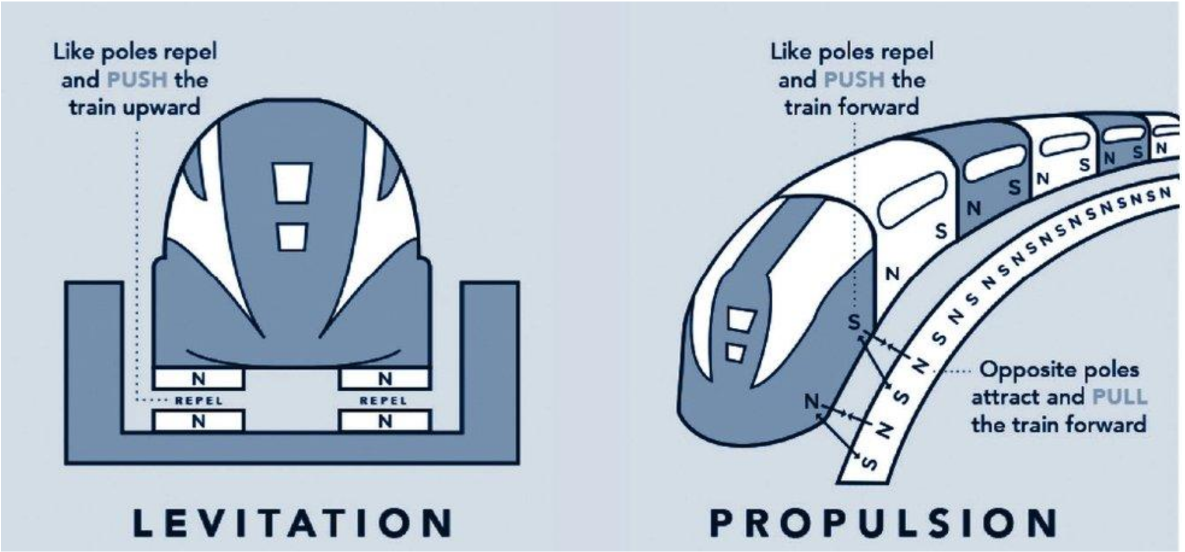
Magnetic Levitation

Maglev trains



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Gauss's Law for Magnetism

In Week 2's lecture, through Gauss's law we saw that we can find the electric fields associated with electric charges.

$$\nabla \cdot \mathbf{D} = \rho_v$$

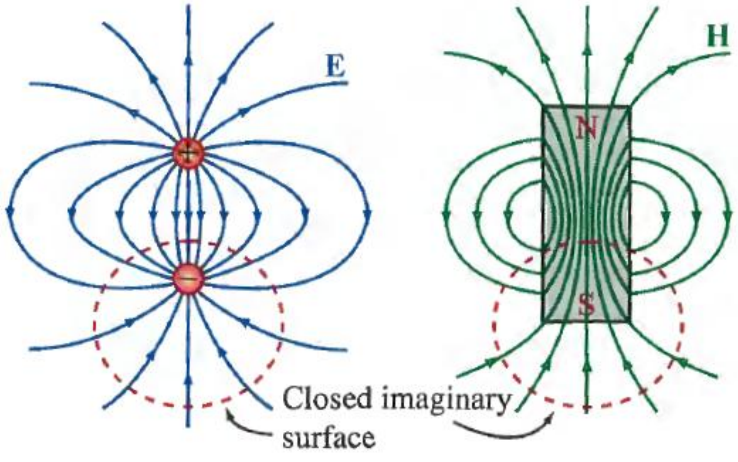
(Differential form of Gauss's law),

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

The magnetostatic counterpart is known as Gauss's law for magnetism:

$$\nabla \cdot \mathbf{B} = 0 \quad \leftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0. \qquad (5.44)$$



(a) Electric dipole (b) Bar magnet

- The magnetic equivalent of an electric point charge *q* or charge density ρ_v (i.e. a "magnetic charge") does not exist.
- There is also no magnetic current.
- The hypothetical magnetic analogue of an electric point charge is a magnetic monopole. However, magnetic monopoles always occur in pairs, i.e. as magnetic dipoles.
- However many times we split a permanent magnet, each piece will have a N & S pole (down to the atomic level).

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Ampère's Law

In Week 3's lecture, we saw through Kirchoff's voltage law that the electrostatic field is conservative.

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{(Electrostatics).} \qquad (4.40)$$

The magnetostatic counterpart is known as Ampere's law:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

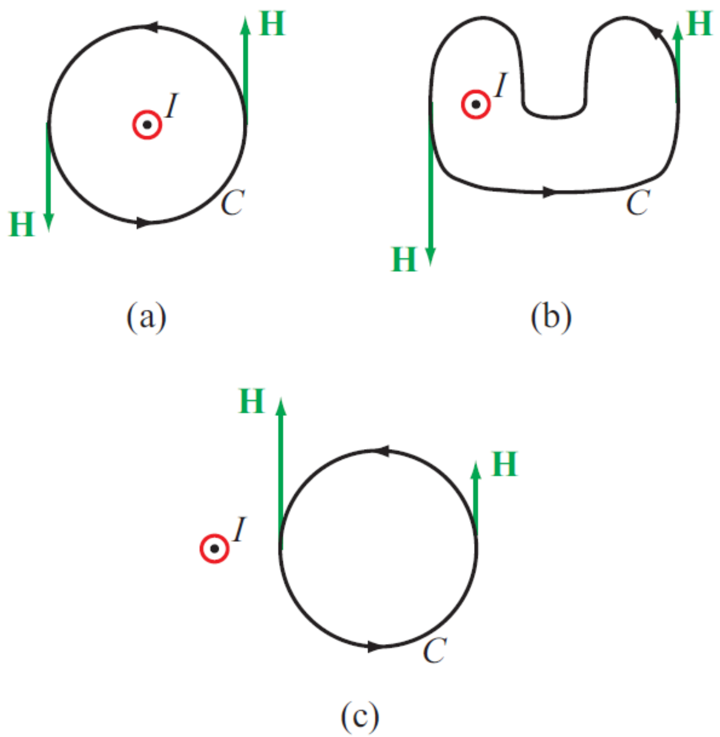

Figure 5-16: Ampère's law states that the line integral of **H** around a closed contour C is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of **H** is zero for the contour in (c) because the current I (denoted by the symbol $\odot$) is not enclosed by the contour C.

Sign convention for contour path C

I and **H** satisfy the right-hand rule for the Biot-Savart law. The direction of I is aligned with the direction of the thumb, and the direction of C is chosen along that of the other four fingers.

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Internal Magnetic Field of a Long Conductor

A long wire of radius a carries a steady current I.
Find the magnetic field **H** at a distance r.

For $r < a$ (Contour C_1 inside the wire)

$$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1,$$

LHS: $$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = \int_0^{2\pi} H_1(\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}) r_1 \, d\phi = 2\pi r_1 H_1.$$

Here, $\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1$, $d\mathbf{l}_1 = \hat{\boldsymbol{\phi}} r_1 \, d\phi$. I_1 is the fraction of the total current I flowing through the area enclosed by C_1. I_1 is equal to the total current I multiplied by the ratio of the area enclosed by C_1 to the total cross-sectional area of the wire:

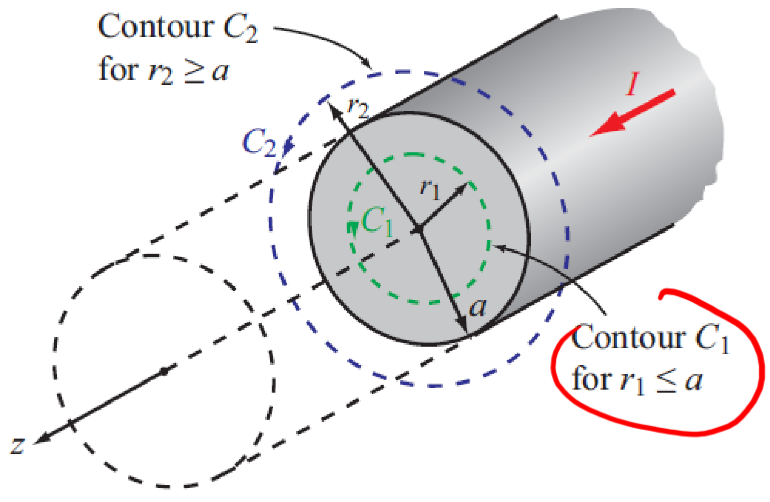
S.A bounded by C_1

RHS: $$I_1 = \left(\frac{\pi r_1^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I.$$

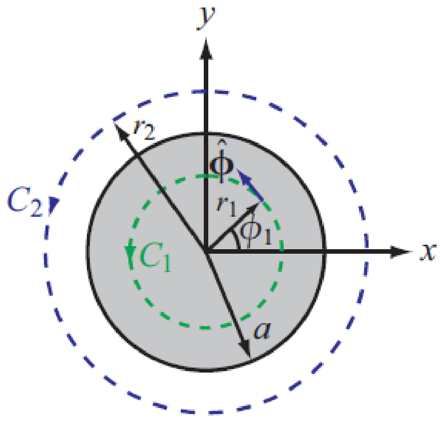
Total S.A.

Solving for $\mathbf{H}_1$ we get:

$$\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1 = \hat{\boldsymbol{\phi}} \frac{r_1}{2\pi a^2} I \qquad (\text{for } r_1 \leq a). \qquad (5.49a)$$



(a) Cylindrical wire



(b) Wire cross section

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External Magnetic Field of a Long Conductor

$$\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1 = \hat{\boldsymbol{\phi}} \frac{r_1}{2\pi a^2} I \qquad (\text{for } r_1 \leq a). \qquad (5.49\text{a})$$

For $r > a$ (Contour C_2 outside the wire)

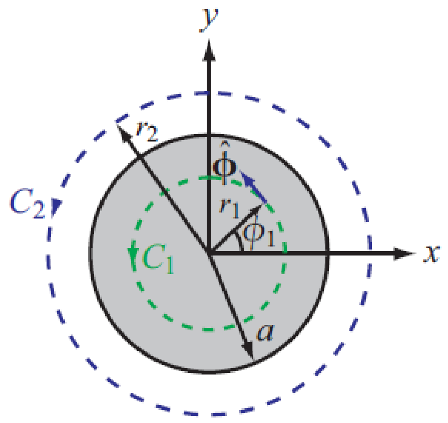
For $r = r_2 \geq a$, we choose contour C_2, which encloses all of the current I. Here, $\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2$, $d\mathbf{l}_2 = \hat{\boldsymbol{\phi}} r_2 \, d\phi$ and we have:

$$\oint_{C_2} \mathbf{H}_2 \cdot d\mathbf{l}_2 = 2\pi r_2 H_2 = I,$$

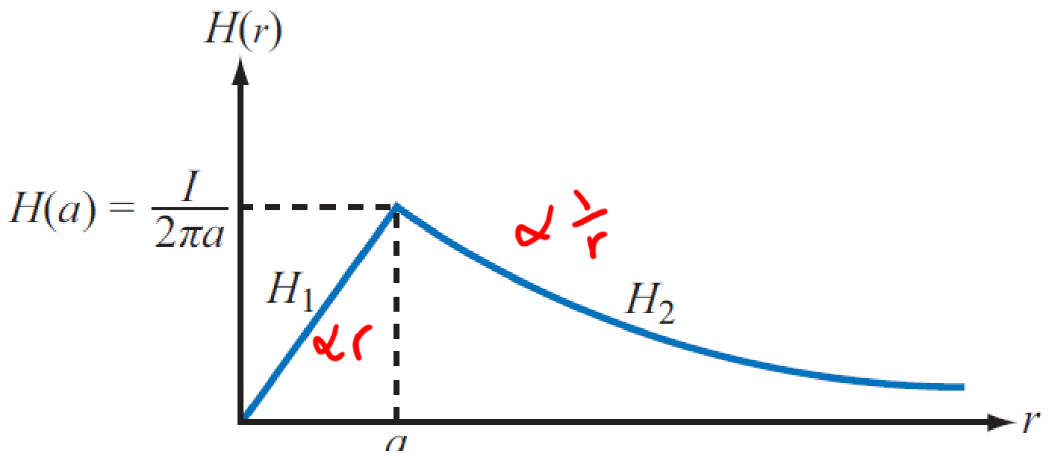
which gives us:

$$\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2 = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r_2} \qquad (\text{for } r_2 \geq a). \qquad (5.49\text{b})$$

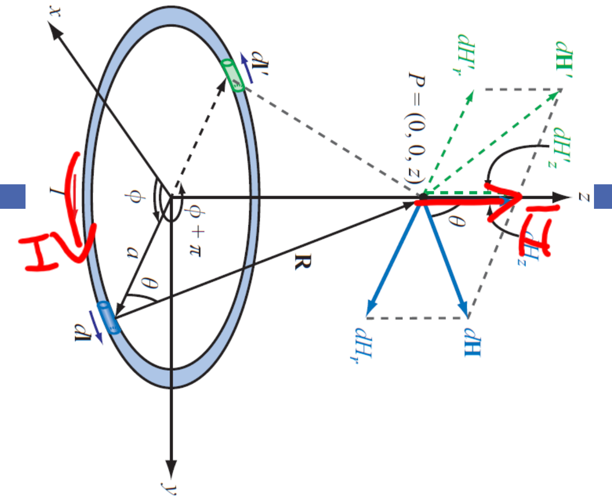
This is exactly the same expression as the one that we got by applying the Biot-Savart law with $\mathbf{B}=\mu_0\mathbf{H}$.



(b) Wire cross section



Magnetic Field of a Toroidal Coil



- A toroidal coil is a doughnut-shaped structure (called the core) on which a wire is wrapped in closely-spaced turns.
- Usually, the wire is tightly wound, forming approximately circular loops.
- Used to magnetically couple multiple circuits and to measure the magnetic properties of materials.

For a toroidal coil with *N* turns carrying a current I, find the magnetic field in all regions.

$r < a$, $a < r < b$, $r > b$

From Slide 4-23, we know that the magnetic field of a loop with CCW current points upwards along its axis.

From symmetry, $\vec{H}$ is uniform in $\hat{\Phi}$ dir.

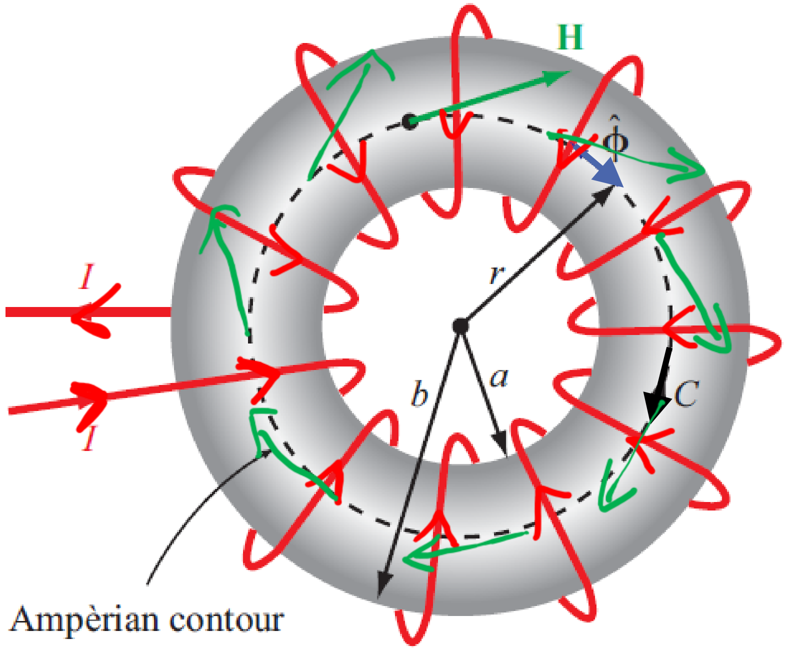


Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

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Magnetic Field of a Toroidal Coil

For $r < a$

- Choose a circular Ampèrian contour C_1 with center at the origin and radius $r < a$.

- There is no current flowing through the surface enclosed by C_1 Therefore,

$$\mathbf{H} = 0 \quad (\text{for } r < a)$$

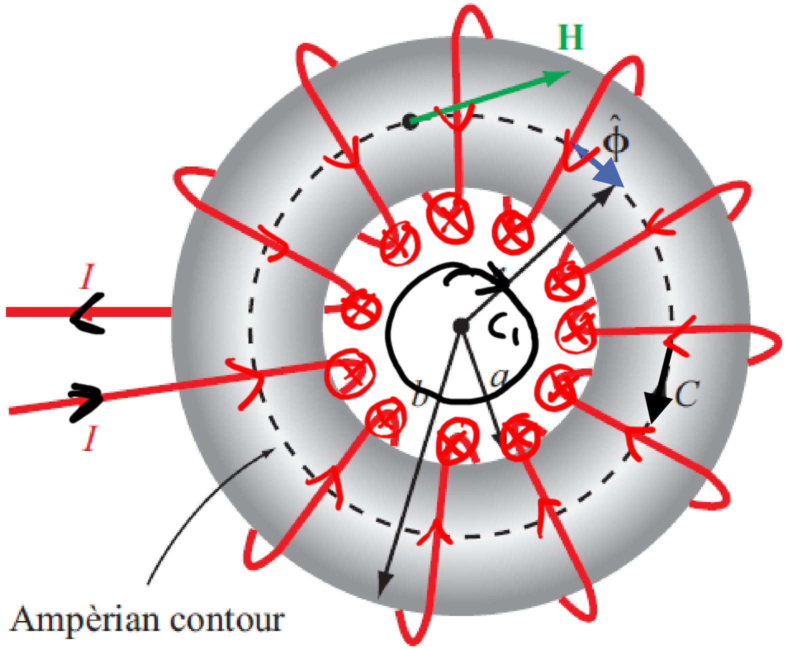

Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

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Magnetic Field of a Toroidal Coil

For $r > b$

- Choose a circular Ampèrian contour C_2 with center at the origin and radius $r > b$.
- The *net* current flowing through the surface enclosed by C_2 is zero, since there are *N* wires, each carrying current I that cross the surface inwards for a net inward current of NI, and a net outward current $-NI$. Therefore,

$$\mathbf{H} = 0 \quad (\text{for } r > b)$$

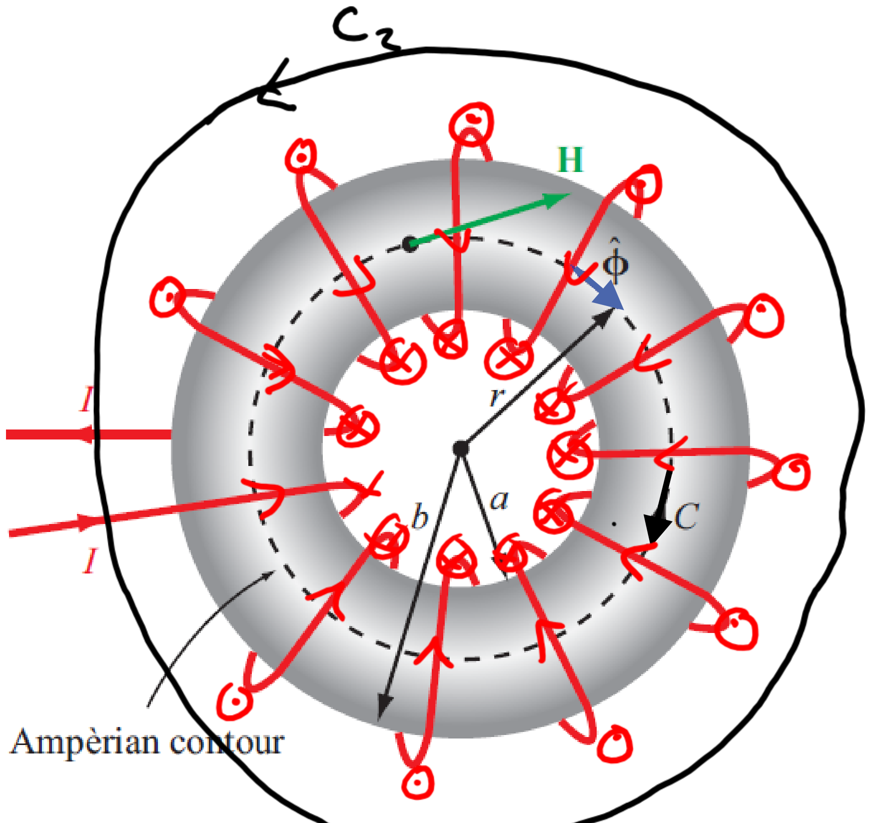

Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

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Magnetic Field of a Toroidal Coil

For $a < r < b$

Choose an Ampèrian contour C_3 such that:

- $\mathbf{H}$ is uniform along the entire contour.
- $\mathbf{H}$ is always either parallel or perpendicular to the contour.

We apply Ampère's law on the contour C_3 : $\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$

Ampère's law states that the line integral of $\mathbf{H}$ around the closed contour C_3 is equal to the total current crossing the surface bound by C_3, which in this case is NI.

$$\oint_c \boldsymbol{H} \cdot dl = \int_0^{2\pi} \hat{\boldsymbol{\phi}} H \cdot \hat{\boldsymbol{\phi}} r \, d\phi = 2\pi r H = NI \quad , \qquad H = \frac{NI}{2\pi r}$$

$$\boldsymbol{H} = \hat{\boldsymbol{\phi}} H = \hat{\boldsymbol{\phi}} \frac{NI}{2\pi r} \quad (\text{for } a < r < b)$$

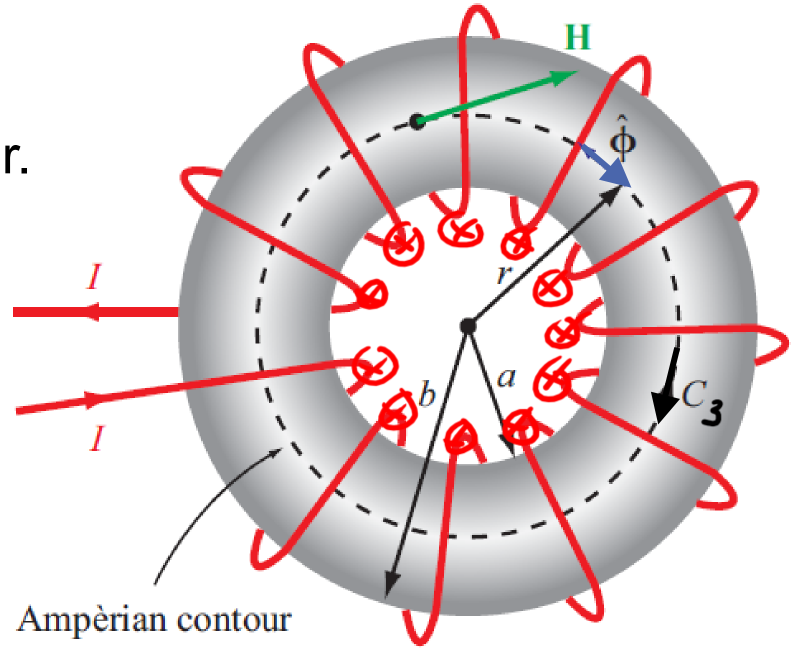

Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

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Example Ampere's Law on a Cylindrical Conductor

A cylindrical conductor has internal $\vec{H}$ given by

$$\vec{H} = \hat{\phi}\,\frac{2}{r}\left[1-(4r+1)e^{-4r}\right] \quad (A/m) \quad \text{for } r \leq a$$

What is the total I in the conductor?

1) Use integral form of Ampère's law.

$$\oint_C \vec{H}\cdot d\vec{\ell}\,\Big|_{r=a} = I$$

$$\int_0^{2\pi} \hat{\phi}\,\frac{2}{\cancel{r}}\underbrace{\left[1-(4r+1)e^{-4r}\right]}\cdot\hat{\phi}\,\cancel{r}\,d\phi\,\Big|_{r=a} = I$$

$$I = 4\pi\left[1-(4a+1)e^{-4a}\right] \quad (A)$$

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Example Ampere's Law on a Cylindrical Conductor

2) Use differential form of Ampere's law:

$$\vec{J} = \nabla \times \vec{H} = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) \quad (\text{only } \hat{z} \text{ component})$$

$$= \hat{z}\frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\frac{2}{\cancel{r}}\underbrace{\left[1-(4r+1)e^{-4r}\right]}\right)$$

$$= \hat{z}\frac{2}{r}\left[-4\frac{\partial}{\partial r}(re^{-4r}) - \frac{\partial}{\partial r}(e^{-4r})\right]$$

$$= \hat{z}\frac{2}{r}\left[-4(-4re^{-4r}+e^{-4r}) + 4e^{-4r}\right] = \hat{z}\frac{2}{\cancel{r}}\underbrace{(16\cancel{r}e^{-4r})}$$

$$= \hat{z}\,32e^{-4r}$$

$$I = \int_S \vec{J}\cdot d\vec{s}_z = \int_0^a (\hat{z}32e^{-4r})\cdot(\hat{z}2\pi r\,dr) = 64\pi\int_0^a re^{-4r}dr = 4\pi\int_0^a 16re^{-4r}dr$$

$$= 4\pi\left[1-(4r+1)e^{-4r}\right]\Big|_0^a = 4\pi\left[1-(4a+1)e^{-4a}\right] (A)$$

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Vector Magnetic Potential A

Electrostatics

$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon}$

$V = -\int \vec{E} \cdot d\vec{\ell}$

Electric scalar potential V is related to the electric field **E**.

$$\mathbf{E} = -\nabla V$$

$$\nabla^2 V = -\frac{\rho_V}{\varepsilon}$$

Poisson's scalar equation

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_V}{R'} \, d\mathcal{V}'$$

Solutions

Magnetostatics

From Gauss's law: $\nabla \cdot \mathbf{B} = 0$

Recalling the identity: $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

We define **B** in terms of the magnetic vector potential **A** as:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (\mathrm{Wb/m^2}) \text{ or (T)}$$

From Ampere's law: $\nabla \times \mathbf{B} = \mu \mathbf{J}$

$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J}$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}.$$

Poisson's vector equation

$\nabla^2 \vec{A} = \nabla (\cancel{\nabla \cdot \vec{A}})^0 - \nabla \times (\nabla \times \vec{A})$

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{\mathcal{V}'} \frac{\mathbf{J}}{R'} \, d\mathcal{V}' \qquad (\mathrm{Wb/m}).$$

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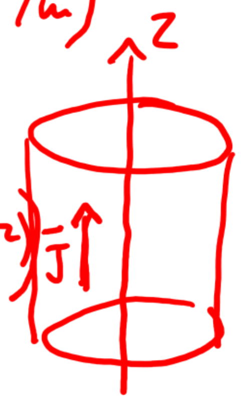
Example Finding H from A

A uniform current density inside cylinder is:

$$\vec{J} = \hat{z} J_o \ (A/m^2) \quad \text{gives rise to} \quad \vec{A} = -\hat{z}\frac{\mu_o J_o}{4}(x^2+y^2) \ (Wb/m)$$



a) Apply Poisson's vector eq. to confirm above.

$$\nabla^2\vec{A} = \hat{x}\cancel{\nabla^2 A_x}^{\,0} + \hat{y}\cancel{\nabla^2 A_y}^{\,0} + \hat{z}\nabla^2 A_z = \hat{z}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(-\frac{\mu_o J_o}{4}(x^2+y^2)\right)$$

$$= -\hat{z}\frac{\mu_o J_o}{\cancel{4}}(2\cancel{+}2) = -\hat{z}\mu_o J_o = -\mu_o\vec{J}$$

$\boxed{\nabla^2\vec{A} = -\mu_o J}$ Satisfies Poisson's Eq.

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Example Finding **H** from **A**

b) Use $\vec{A}$ to find $\vec{H}$

$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0}\nabla \times \vec{A} = \frac{1}{\mu_0}\left[\hat{x}\left(\frac{\partial A_z}{\partial y} - \cancel{\frac{\partial A_y}{\partial z}}^{0}\right) + \hat{y}\left(\cancel{\frac{\partial A_x}{\partial z}}^{0} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\cancel{\frac{\partial A_y}{\partial x}}^{0} - \cancel{\frac{\partial A_x}{\partial y}}^{0}\right)\right]$$

$$= \frac{1}{\mu_0}\left[\hat{x}\frac{\partial A_z}{\partial y} - \hat{y}\frac{\partial A_z}{\partial x}\right]$$

$$= \frac{1}{\mu_0}\left[\hat{x}\frac{\partial}{\partial y}\left(-\frac{\mu_0 J_0}{4}(x^2+y^2)\right) - \hat{y}\frac{\partial}{\partial x}\left(-\frac{\mu_0 J_0}{4}(x^2+y^2)\right)\right]$$

$$= -\hat{x}\frac{J_0}{2}y + \hat{y}\frac{J_0}{2}x \quad (A/m)$$

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Example Finding H from A

c) Use $\vec{J}$ with Ampère's law to find $\vec{H}$. Compare result with (b)

$$\oint_C \vec{H} \cdot d\vec{\ell} = I = \int \vec{J} \cdot d\vec{s}$$

LHS: $\int_0^{2\pi} (\hat{\phi} H_\phi) \cdot (\hat{\phi}\, r\, d\phi) = 2\pi r H_\phi$

RHS: $\int_S \vec{J} \cdot d\vec{s} = \int_S (\hat{z} J_0) \cdot (\hat{z}\, ds) = J_0 (\pi r^2)$

$$2\pi r H_\phi = J_0 (\pi r^2)$$

$$H_\phi = \frac{J_0 (\cancel{\pi} r^{\cancel{2}})}{2 \cancel{\pi} \cancel{r}} = \frac{J_0}{2} r \Rightarrow \vec{H} = \hat{\phi} \frac{J_0}{2} r \quad (A/m)$$

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Example Finding **H** from **A**

Convert from Cyl. to Cart. coord.

$$\hat{\phi} = \underline{-\hat{x}\sin\phi + \hat{y}\cos\phi} = -\hat{x}\frac{y}{\sqrt{x^2+y^2}} + \hat{y}\frac{x}{\sqrt{x^2+y^2}}, \quad r = \sqrt{x^2+y^2}$$

$$\Rightarrow \vec{H} = \left(-\hat{x}\frac{y}{\cancel{\sqrt{x^2+y^2}}} + \hat{y}\frac{x}{\cancel{\sqrt{x^2+y^2}}}\right)\left(\frac{J_0}{2}\cancel{\sqrt{x^2+y^2}}\right)$$

$$= -\hat{x}\,y\frac{J_0}{2} + \hat{y}\,x\frac{J_0}{2} \quad (A/m) \;//$$

Same as (b)

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Example Finding **H** from **A**

Alternatively, we can repeat (b) in cylindrical coordinates:

$$\vec{A} = \hat{z}\left(-\frac{\mu_0 J_0}{4}(x^2+y^2)\right)$$

But $x = r\cos\phi$, $y = r\sin\phi$

$x^2+y^2 = r^2\cos^2\phi + r^2\sin^2\phi = r^2$

$$\Rightarrow \vec{A} = \hat{z}\left(-\mu_0 \frac{J_0}{4} r^2\right)$$

$$\vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} = \frac{1}{\mu_0}\left[\hat{r}\,\underbrace{\frac{1}{r}\frac{\partial}{\partial\phi}\left(-\mu_0\frac{J_0}{4}r^2\right)}_{0} - \hat{\phi}\frac{\partial}{\partial r}\left(-\mu_0\frac{J_0}{4}r^2\right)\right]$$

$$= \frac{1}{\cancel{\mu_0}}\left[\hat{\phi}\left(\cancel{\mu_0}\frac{J_0}{4}(2r)\right)\right] = \hat{\phi}\,\frac{J_0}{2}r \quad (\text{A/m})$$

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Magnetic Properties of Materials

The magnetic behavior of a material is governed by the interaction of the magnetic dipole moments of its atoms ($\mathbf{m}_o$ & $\mathbf{m}_s$) with an external magnetic field. The nature of the behaviour depends on the crystalline structure of the material and is used as a basis for classifying materials as diamagnetic, paramagnetic or ferromagnetic.

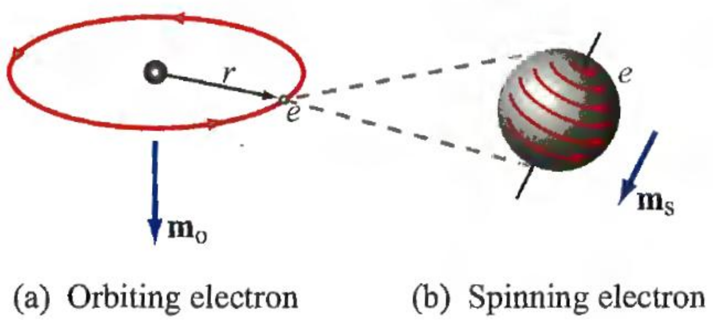

Figure 5-20 An electron generates (a) an orbital magnetic moment $\mathbf{m}_o$ as it rotates around the nucleus and (b) a spin magnetic moment $\mathbf{m}_s$ as it spins about its own axis.

In free-space: $\mathbf{B} = \mu_0 \mathbf{H}$

vector sum of all magnetic moments

In magnetic materials: $\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} = \mu_0 (\mathbf{H} + \mathbf{M})$

Similar to ($\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$)

Magnetization vector due to external magnetic field $\mathbf{M} = \chi_m \mathbf{H}$ χ_m = magnetic susceptibility

$$\mathbf{B} = \mu_0 (\mathbf{H} + \chi_m \mathbf{H}) = \mu_0 (1 + \chi_m) \mathbf{H},$$

$\mathbf{B} = \mu \mathbf{H},$ μ = magnetic permeability

$\mathbf{B} = \mu_0 \mu_r \mathbf{H}$ μ_r = relative permeability $= 1 + \chi_m$

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Magnetic Properties of Materials

Table 5-2: Properties of magnetic materials.

	Diamagnetism	Paramagnetism	Ferromagnetism
Permanent magnetic dipole moment	No	Yes, but weak	Yes, and strong
Primary magnetization mechanism	Electron orbital magnetic moment	Electron spin magnetic moment	Magnetized domains
Direction of induced magnetic field (relative to external field)	Opposite	Same	Hysteresis (see Fig. 5-22)
Common substances	Bismuth, copper, diamond, gold, lead, mercury, silver, silicon	Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten	Iron, nickel, cobalt
Typical value of χ_m **Typical value of μ_r**	$\approx -10^{-5}$ ≈ 1	$\approx 10^{-5}$ ≈ 1	$\lvert\chi_m\rvert \gg 1$ and hysteretic $\lvert\mu_r\rvert \gg 1$ and hysteretic

Thus, $\mu_r \simeq 1$ or $\mu \simeq \mu_0$ for diamagnetic and paramagnetic substances, which include dielectric materials and most metals. In contrast, $|\mu_r| \gg 1$ for ferromagnetic materials; $|\mu_r|$ of purified iron, for example, is on the order of 2×10^5.

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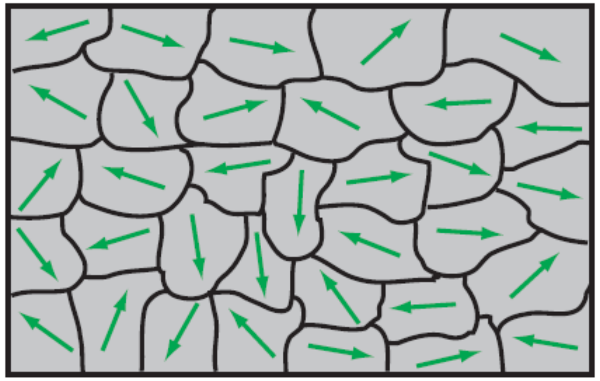
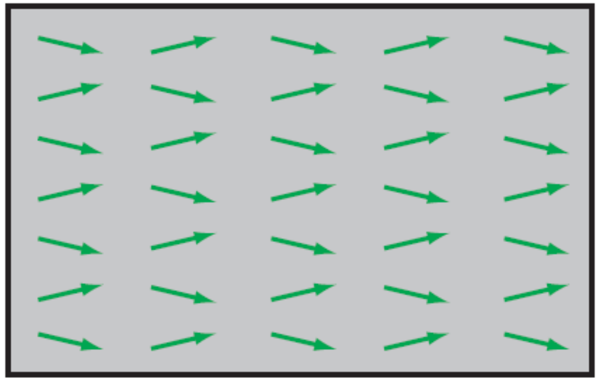
Magnetic Hysteresis



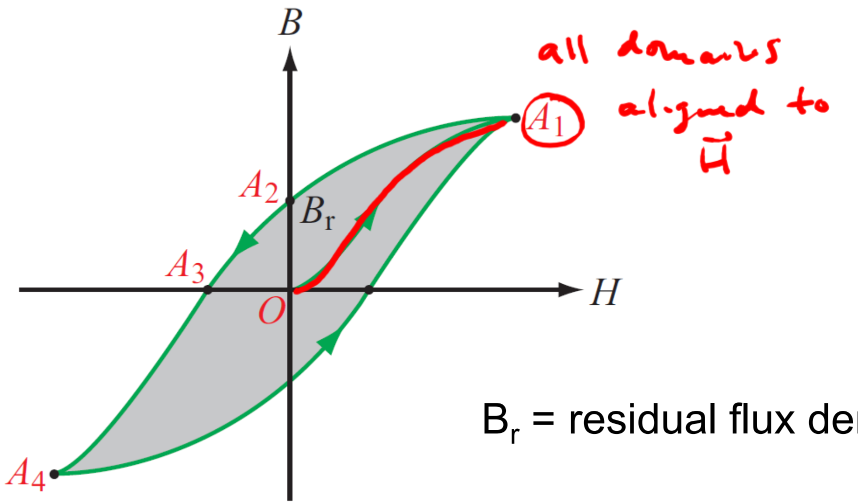
(a) Unmagnetized domains $\vec{H} = 0$



$\vec{H} \neq 0$

(b) Magnetized domains

- Ferromagnetic materials (iron, nickel, cobalt) exhibit unique magnetic properties because their magnetic moments align along the direction of an external H-field.
- Also, these materials remain partially magnetized even after the external H-field is removed.
- Because of these properties, ferromagnetic materials are used to make permanent magnets.



B_r = residual flux density

Figure 5-22: Typical hysteresis curve for a ferromagnetic material.

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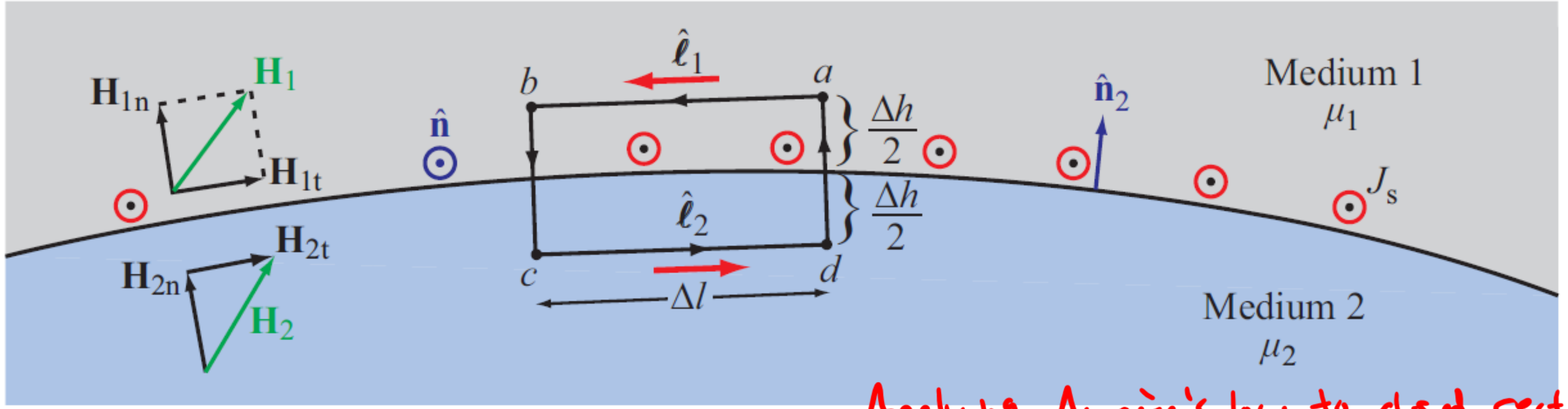
Magnetic Boundary Conditions



Previous electric boundary condition:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \rightarrow \boxed{D_{1n} - D_{2n} = \rho_s.} \qquad (5.78)$$

By analogy, application of Gauss' law for magnetism, as expressed by Eq. (5.44), leads to the conclusion that:

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \rightarrow \boxed{B_{1n} = B_{2n}.} \qquad (5.79)$$

The normal component of $\mathbf{B}$ is continuous across the boundary between the two adjacent media.

Applying Ampère's law to closed rectangle and letting Δh→0

$$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

The tangential components of $\mathbf{H}$ parallel to $\mathbf{J}_s$ are continuous across the interface, whereas those orthogonal to $\mathbf{J}_s$ are discontinuous by the amount of $\mathbf{J}_s$.

But, surface currents can exist only on the surfaces of perfect conductors and superconductors. Therefore, at the interface between media with finite conductivities, $\mathbf{J}_s = 0$ and

$$\boxed{H_{1t} = H_{2t}. \qquad (5.85)}$$

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Example Magnetic Boundary Conditions (Ulaby, Example 5-7)

No surface current exist on the boundary.

$\vec{B}_1 = \hat{x}2 + \hat{y}3$ (T)

Find $\vec{B}_2$ and evaluate for $\mu_2 = 2\mu_1$

Let's find $\hat{n}_2$ first.

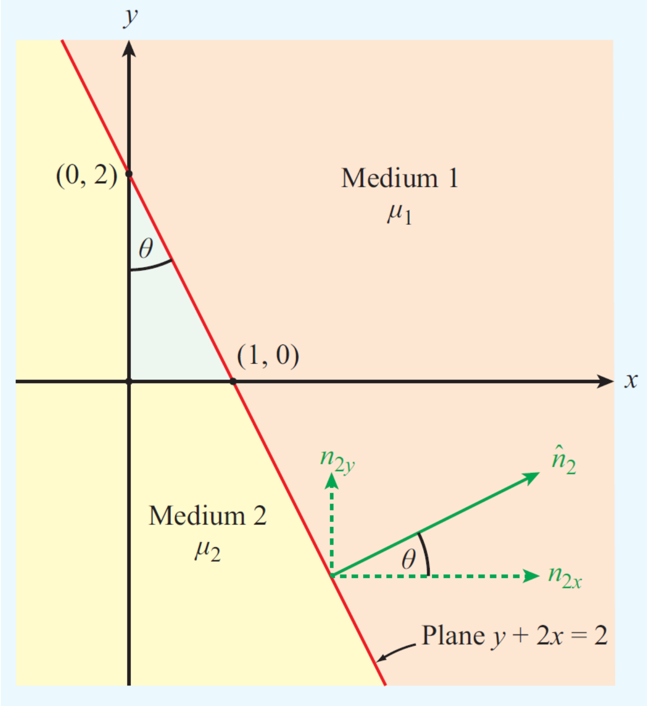
$\tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$

$\cos\theta = \frac{n_{2x}}{n_2} \Rightarrow n_{2x} = n_2 \cos\theta$

$\sin\theta = \frac{n_{2y}}{n_2} \Rightarrow n_{2y} = n_2 \sin\theta$

$\hat{n}_2 = \hat{x} n_{2x} + \hat{y} n_{2y} = \hat{x} n_2 \cos\theta + \hat{y} n_2 \sin\theta$

$= \hat{x}\cos\theta + \hat{y}\sin\theta = \hat{x}\,0.89 + \hat{y}\,0.45$



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Example Magnetic Boundary Conditions (Ulaby, Example 5-7)

$$\vec{B}_1 = \vec{B}_{1n} + \vec{B}_{1t}$$

$$\vec{B}_{1n} = B_{1n}\,\hat{n}_2 \Rightarrow B_{1n} = \hat{n}_2 \cdot \vec{B}_1 = (\hat{x}\cos\theta + \hat{y}\sin\theta) \cdot (2\hat{x} + 3\hat{y})$$

$$= 2\cos\theta + 3\sin\theta$$

$$\vec{B}_{1n} = B_{1n}\,\hat{n}_2 = \hat{x}\,(2\cos^2\theta + 3\sin\theta\cos\theta) + \hat{y}\,(2\sin\theta\cos\theta + 3\sin^2\theta)$$

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Example Magnetic Boundary Conditions (Ulaby, Example 5-7)

$$\vec{B}_{1t} = \vec{B}_1 - \vec{B}_{1n} = \hat{x}(2 - 2\cos^2\theta - 3\sin\theta\cos\theta) + \hat{y}(3 - 2\sin\theta\cos\theta - 3\sin^2\theta)$$

1st BC: $\vec{B}_{1n} = \vec{B}_{2n}$

2nd BC: $\vec{H}_{1t} = \vec{H}_{2t} \Rightarrow \frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2} \Rightarrow \vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t}$

$$\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} = \vec{B}_{1n} + \frac{\mu_2}{\mu_1}\vec{B}_{1t}$$

$$= \hat{x}\left[2\cos^2\theta + 3\sin\theta\cos\theta + \cancel{\frac{\mu_2}{\mu_1}}^{2}(2 - 2\cos^2\theta - 3\sin\theta\cos\theta)\right]$$

$$+ \hat{y}\left[2\sin\theta\cos\theta + 3\sin^2\theta + \cancel{\frac{\mu_2}{\mu_1}}^{2}(3 - 2\sin\theta\cos\theta - 3\sin^2\theta)\right]$$

$$\vec{B}_2 = 1.2\hat{x} + 4.6\hat{y} \ (\mathrm{T})$$

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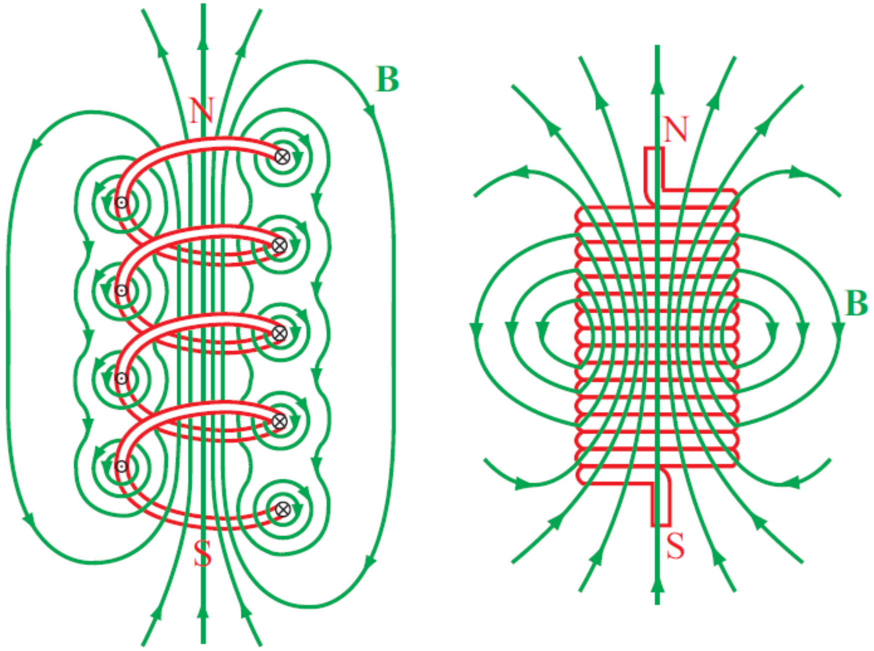
Solenoid

- An inductor is the magnetic analogue of an electric capacitor.
- A capacitor stores energy in the electric field between the two conductors.
- An inductor stores energy in the magnetic field around its current-carrying conductors.
- A typical inductor consists of multiple turns of wire helically coiled around a cylindrical core. This is called a solenoid.
- The magnetic fields of a solenoid resemble those of a permanent magnet.



(a) Loosely wound solenoid (b) Tightly wound solenoid

Inside the solenoid (Derivation found in Ulaby, Sec. 5-7.1 "Magnetic Field in a Solenoid")

$$\mathbf{B} \simeq \hat{\mathbf{z}}\mu \underbrace{n}_{N/l} I = \frac{\hat{\mathbf{z}}\mu N I}{l} \quad \text{(long solenoid with } l/a \gg 1\text{)}$$

At the centre the magnetic field is uniform

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Inductance

The magnetic flux Φ linking a surface S is defined as the total magnetic flux density that passes through the surface.

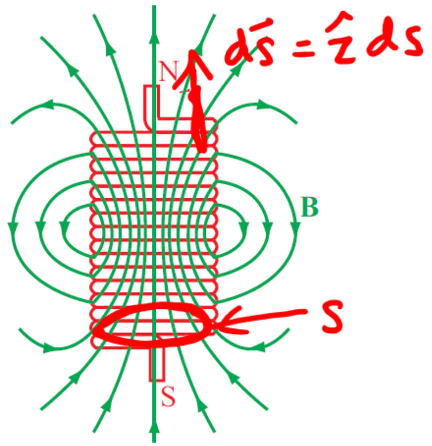
Magnetic Flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}.$$

For the solenoid, that has an approximately uniform magnetic field throughout its cross section, the flux linking a single loop is:

$$\Phi = \int_S \hat{\mathbf{z}} \left(\mu \frac{N}{l} I \right) \cdot \hat{\mathbf{z}}\, ds = \mu \frac{N}{l} I S$$

S is the cross sectional area of the loop.



The magnetic flux linkage Λ is defined as the total magnetic flux linking a given circuit or conducting structure. If the structure consists of a single conductor with multiple loops, as in the case of a solenoid, Λ is the flux linking all the loops of the structure. For a solenoid with N turns:

Flux Linkage (Total flux passing through all loops)

$$\Lambda = N\Phi = \mu \frac{N^2}{l} I S \qquad \text{(Wb)}$$

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Inductance

The inductance of a conducting structure is defined as the ratio of the magnetic flux linkage Λ to the current I flowing through the structure:

Inductance (Self-inductance)

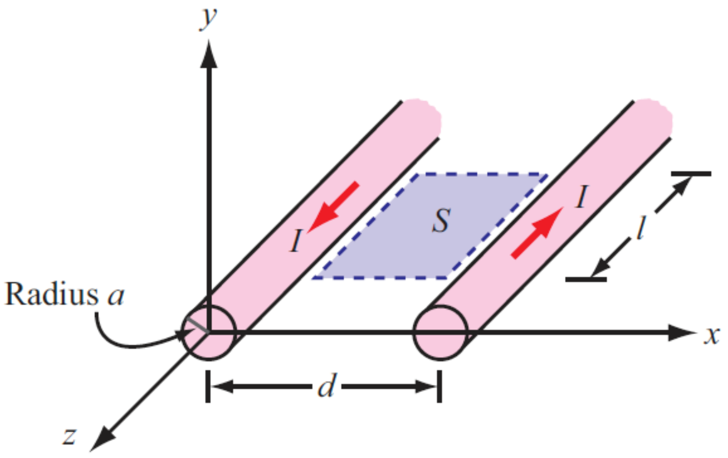
$$L = \frac{\Lambda}{I} \qquad \text{(H).}$$ or (Wb/A)

For the solenoid (single-conductor):

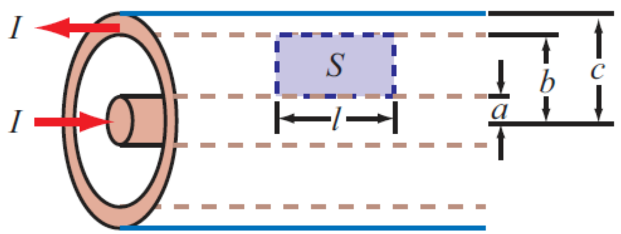
$$L = \mu \frac{N^2}{l} S \qquad \text{(solenoid)}, \qquad (5.95)$$

For two-conductor configurations, consider the flux Φ through the surface S between the conductors:

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}. \qquad (5.96)$$



(a) Parallel-wire transmission line



(b) Coaxial transmission line

Figure 5-27: To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area S between the conductors.

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Example: Inductance of a Coaxial Cable

The magnetic field in the region S between the two conductors is approximately

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r}$$

Total magnetic flux through S :

$$\Phi = \int_S \vec{B} \cdot d\vec{s}_\phi = \int_0^l \int_a^b (\hat{\phi} B) \cdot (\hat{\phi}\, dr\, dz) = l \int_a^b B\, dr$$

$$\Phi = l \int_a^b B\ dr = l \int_a^b \frac{\mu I}{2\pi r}\ dr = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\Phi}{lI} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right).$$

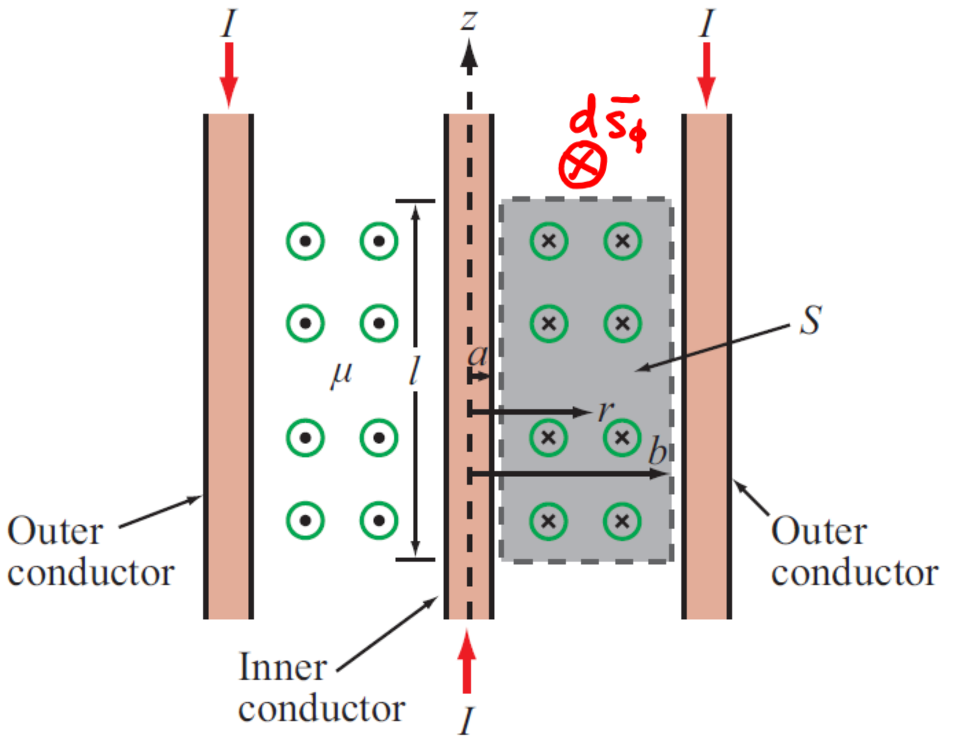

Figure 5-28: Cross-sectional view of coaxial transmission line (Example 5-7).

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Mutual Inductance

The magnetic coupling between two different conducting structures is described in terms of the mutual inductance between them.

Consider two multi-turn closed loops. Loop 1 is fed with a current I_1. The magnetic field B_1 generated by I_1 results in flux Φ_{12} through Loop 2:

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}, \qquad (5.100)$$

Total magnetic flux linkage through Loop 2:

$$\Lambda_{12} = N_2 \Phi_{12} = N_2 \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}. \qquad (5.101)$$

Mutual Inductance

$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s} \qquad \text{(H)}. \qquad (5.102)}$$

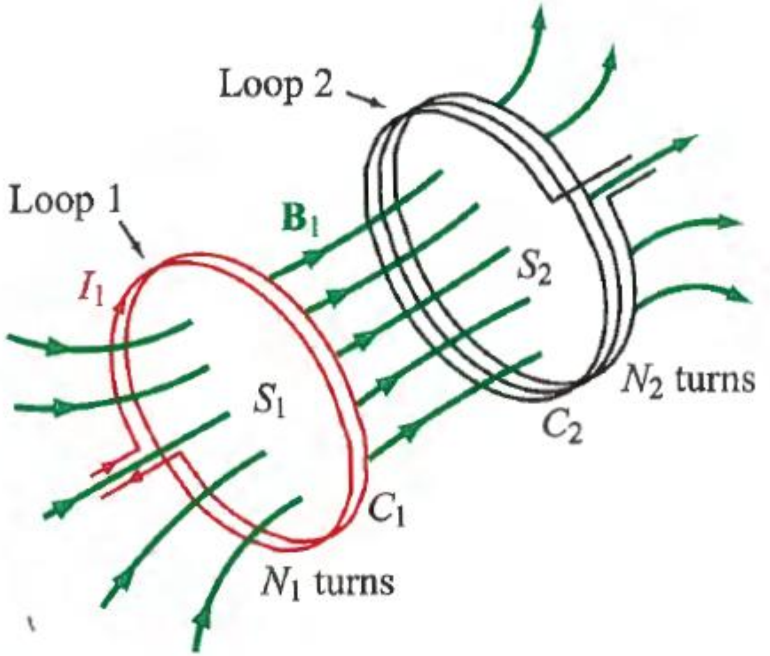

Figure 5-29: Magnetic field lines generated by current I_1 in loop 1 linking surface S_2 of loop 2.

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Example: Mutual Inductance Between Two Conductors

Determine the mutual inductance between the wire and the loop.

$$L_{12} = \frac{N_2}{I_1}\int_{S_2} \vec{B}\cdot d\vec{s} \qquad N_2 = 1$$

$$\vec{B}_1 = \hat{\phi}\frac{\mu I_1}{2\pi r} = -\hat{x}\frac{\mu_0 I_1}{2\pi y}$$

$$= \frac{1}{I_1}\int_{y=0.05}^{0.2}\int_{z=0.1}^{0.4}\left(-\hat{x}\frac{\mu_0 I_1}{2\pi y}\right)\cdot\left(-\hat{x}\,dx\,dy\right)$$

$$= \frac{\mu_0}{2\pi}\int_{0.05}^{0.2}\frac{1}{y}\,dy\int_{0.1}^{0.4}dz = \frac{0.3\mu_0}{2\pi}\ln\left(\frac{0.2}{0.05}\right)$$

$$= 83\,nH$$

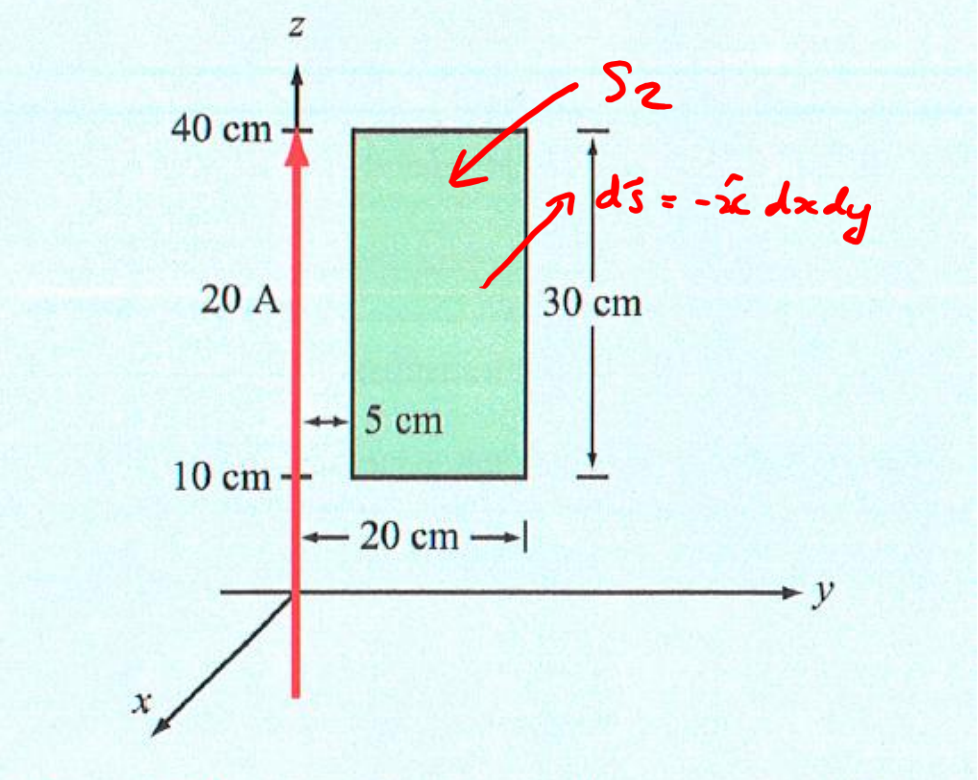

Figure 5-32 Mutual inductance between two conductors (Example 5-9).

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Magnetic Energy

Magnetic energy density (Valid for any inductor)

$$w_m = \frac{W_m}{\mathcal{V}} = \frac{1}{2}\mu H^2 \qquad (\text{J/m}^3).$$

Total energy to build up current in inductor to I

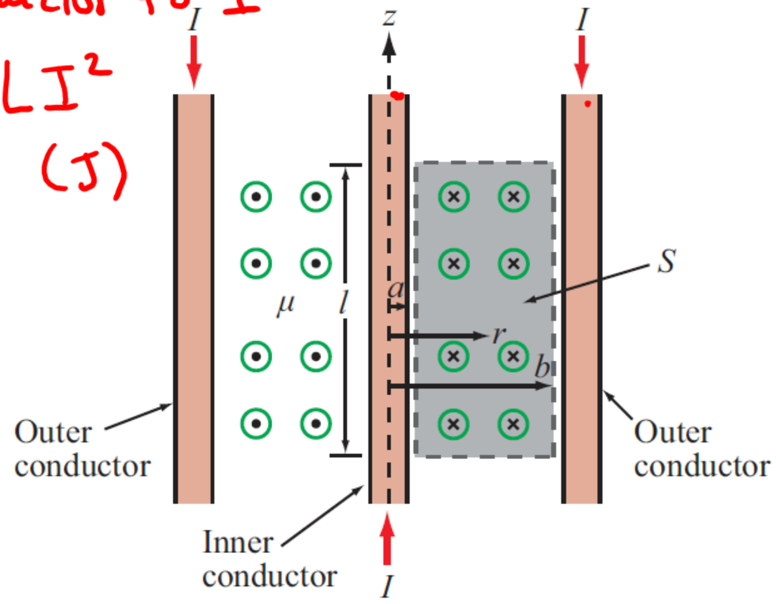
$W_m = \int p\,dt = \int iv\,dt = L\int_0^I i\,di = \frac{1}{2}LI^2$ (J)

power

$L\frac{di}{dt}$

$L = \frac{\mu N^2}{l}S$, $I = \frac{Bl}{\mu N}$, $\mathcal{V} = lS$ for solenoid.

Example 5-8: Magnetic Energy in a Coaxial Cable



The magnetic field in the insulating material is:

$$H = \frac{B}{\mu} = \frac{I}{2\pi r}$$

The magnetic energy stored in the coaxial cable is:

$$W_m = \frac{1}{2}\int_{\mathcal{V}} \mu H^2\, d\mathcal{V} = \frac{\mu I^2}{8\pi^2}\int_{\mathcal{V}} \frac{1}{r^2}\, d\mathcal{V}$$

$$W_m = \frac{\mu I^2}{8\pi^2}\int_a^b \frac{1}{r^2}\cdot 2\pi r l\, dr$$

$$\boxed{= \frac{\mu I^2 l}{4\pi}\ln\left(\frac{b}{a}\right)}$$

$$= \frac{1}{2}LI^2 \qquad (\text{J}),$$

Tech Brief: Inductive Sensors

LVDT can measure displacement with submillimeter precision

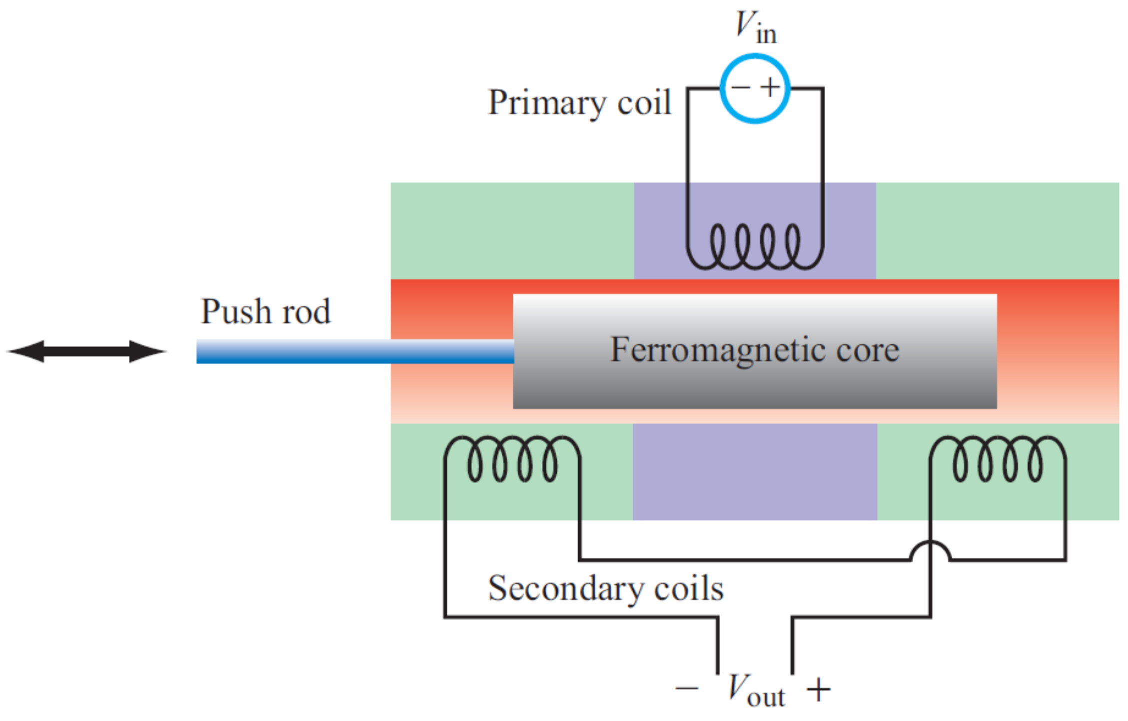


Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

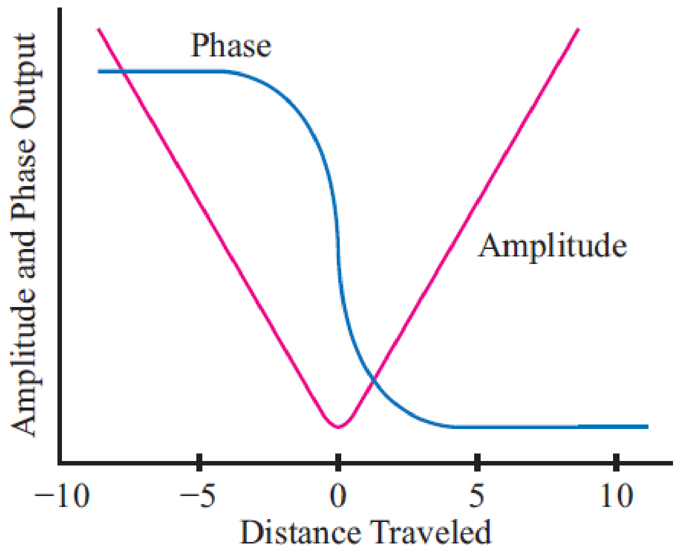


Figure TF11-2: Amplitude and phase responses as a function of the distance by which the magnetic core is moved away from the center position.

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Proximity Sensor

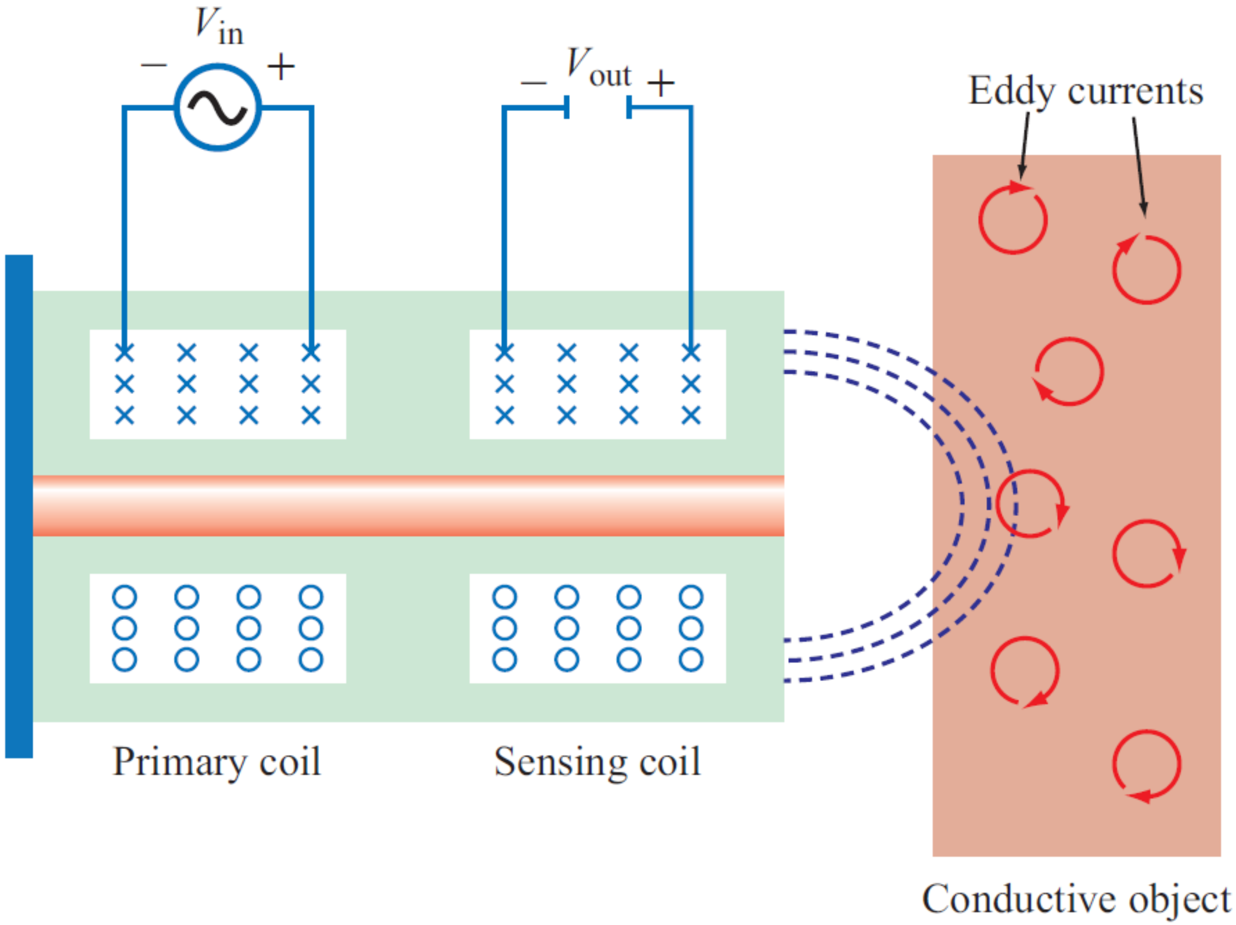

Figure TF11-5: Eddy-current proximity sensor.

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Summary

Chapter 5 Relationships

Maxwell's Magnetostatics Equations

Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \longleftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

Ampère's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Lorentz Force on Charge q

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Magnetic Force on Wire

$$\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}$$

Magnetic Torque on Loop

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N·m})$$

$$\mathbf{m} = \hat{\mathbf{n}} NIA \qquad (\text{A·m}^2)$$

Biot–Savart Law

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

Magnetic Field

Infinitely Long Wire $\quad \mathbf{B} = \hat{\boldsymbol{\phi}} \dfrac{\mu_0 I}{2\pi r} \quad (\text{Wb/m}^2)$

Circular Loop $\quad \mathbf{H} = \hat{\mathbf{z}} \dfrac{Ia^2}{2(a^2 + z^2)^{3/2}} \quad \text{(A/m)}$

Solenoid $\quad \mathbf{B} \simeq \hat{\mathbf{z}} \mu n I = \dfrac{\hat{\mathbf{z}} \mu N I}{l} \quad (\text{Wb/m}^2)$

Vector Magnetic Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (\text{Wb/m}^2)$$

Vector Poisson's Equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Inductance

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(H)}$$

Magnetic Energy Density

$$w_{\mathrm{m}} = \frac{1}{2} \mu H^2 \qquad (\text{J/m}^3)$$

References

Ulaby

- Sections: 5-1 to 5-8