Assignment 1

Wednesday, October 15, 2025 9:29 PM

Q1

Your music app chooses a tune that you like with probability p. The app's choices are independent of each other.

(a) What is the probability that the app chooses the second tune that you like on its fourth try?
(b) What is the probability that in four tries, the app will have chosen three good tunes?
(c) What is the probability that the app will have chosen less than four good tunes in four tries?
(d) What is the probability that the first good tune is the first chosen and the second is the fourth one chosen?
(e) What is the probability that the first good tune chosen is the fourth, given that the first two were not good?

a) $K=4$
$m=2$

$$P_K[K] = \binom{k-1}{m-1} p^m (1-p)^{k-m}$$

$$P(X=4) = \binom{4-1}{2-1} p^2 (1-p)^{4-2}$$

$$= \binom{3}{1} p^2 (1-p)^2$$

$$P(X=4) = 3p^2(1-p)^2$$

b) $n=4$
$K=3$

Binomial PMF:

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P(X=3) = \frac{4!}{3!(4-3)!} p^3 (1-p)^1$$

$$= \frac{4}{(4-3)!} p^3 (1-p)$$

$$= 4p^3(1-p)$$

c) Let A be the event of choosing less than 4 good tunes in 4 tries
A^c be the event of choosing exactly 4 good tunes

$$P(A) = 1 - P(A^c)$$

Using the binomal PMF for $n=4, k=4$:

$$P(A^c) = P(X=4) = \binom{4}{4} p^4 (1-p)^{4-4} = 1 \cdot p^4 (1-p)^0 = p^4$$

$\therefore$ Probability of choosing less than 4 tunes is $P(A) = 1 - p^4$

d) $P(\text{1st is good}) = p$
$P(\text{2nd is not good}) = 1-p$
$P(\text{3rd is not good}) = 1-p$
$P(\text{4th is good}) = p$

Probability in the sequence:

$$P(S,F,F,S) = p \cdot (1-p) \cdot (1-p) \cdot p = p^2(1-p)^2$$

is the product of all the possibilities

e) Conditional probability problem

Let A be the event that the first good tune is the fourth $(X=4)$
Let B be the event that the first 2 are not good $(X>2)$.
Finding $P(A|B)$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This event of $A \cap B$ is simply the event that the first good tune is the fourth. This sequence is (F,F,F,S).

$$P(A \cap B) = P(X=4) = (1-p)^3 p$$

The event B means that the first success occurs after the 2nd trial.
Prob of this is the prob of the sequence $(F,F) = (1-p)^2$

$$P(B) = (1-p)^2$$

$\therefore$ The conditional prob is: $P(A|B) = \dfrac{(1-p)^3 p}{(1-p)^2} = p(1-p)$

Q2 Three tribal elders win elections to lead the unstable region of North Vatisthisstan. Five identical assault rifles, a gift of the people of Sodabia, are airdropped among a meeting of the three leaders. The tribal leaders scamper to collect as many of the rifles as they each can carry, which is five.

(a) How many distinguishable distributions are there where at least one of the tribal leaders fails to collect any rifles?

Five identical assault rifles, a gift of the people of Sodabia, are airdropped among a meeting of the three leaders. The tribal leaders scamper to collect as many of the rifles as they each can carry, which is five.

(a) How many distinguishable distributions are there where at least one of the tribal leaders fails to collect any rifles?
(b) What is the probability that all tribal leaders collect at least one rifle?

a) Let S be total # of ways to distribute 5 identical rifles to 3 distinct leaders.
Treat this like a "stars and bars" problem where $n=5$ items and $k=3$ bins.
Total number of distributions is $\binom{n+k-1}{k-1}$

$$|S| = \binom{5+3-1}{3-1} = \binom{7}{2} = \frac{7!}{2!5!} = \frac{42}{2} = 21$$

Let A_i be the event that leader i gets no rifles. We want to find $|A_1 \cup A_2 \cup A_3|$.
Using principle of Inclusion - Exclusion:

$$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 + A_2 + A_3|$$

↳ $|A_1|$: # of ways to get 5 rifles to 2 leaders ($k=2, n=5$). $|A_1| = \binom{5+2-1}{2-1} = \binom{6}{1} = 6$
By symmetry $|A_1| = |A_2| = |A_3| = 6$

↳ $|A_1 \cap A_2|$: # of ways to give 5 rifles to 2 leaders: $|A_1 \cap A_2| = \binom{5}{0} = 1$. By symmetry this is 1 for any pair

↳ $|A_1 \cap A_2 \cap A_3| = 0$ as # of ways to give 5 rifles to 0 leaders

Sub in values: $|A_1 \cup A_2 \cup A_3| = (6+6+6) - (1+1+1) + 0 = 15$

$\therefore$ 15 distributions where at least one leader gets 0 rifles

b) Complement of event in (a): Let E be the event each leader gets at least one:

$$P(E) = 1 - P(E^c)$$

E^c is the event at least one leader gets no rifles, $|E^c| = 15$, $|S| = 21$

$$P(E^c) = \frac{|E^c|}{|S|} = \frac{15}{21} = \frac{5}{7}$$

$\therefore$ Probability is $P(E) = \frac{2}{7}$

Q3

A discrete random variable X has PMF:

$$P_K[k] = \begin{cases} k^2/30 & k = 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the sample space for K, S_K?
(b) What is the expected value of K, $E[K]$?
(c) What is the probability that $K = 1$ given that it is less than 4?
(d) In a sequence of independent draws of K, what is the probability that the first occurrence of the number 4 is on the third draw?
(e) In a sequence of 5 independent draws of K, what is the probability that 3 are ones?

a) Sample space is set of all possible values for which PMF ≥ 0. Given $P_K[K] = K^2/30$ for $K = 1,2,3,4$

$$\therefore S_K = \{1,2,3,4\}$$

b) Expected value: $E[K] = \sum_{K \in S_K} K \cdot P_K(K)$

$$E[K] = (1)\frac{1^2}{30} + 2\frac{2^2}{30} + 3\frac{3^2}{30} + 4\frac{4^2}{30}$$

$$= \frac{1}{30}(1 + 8 + 27 + 64)$$

$$= \frac{100}{30} = \frac{10}{3}$$

$= \frac{1}{30}(1+8+27+64)$

$= \frac{100}{30} = \frac{10}{3}$

c) Using formula for conditional property $P(A|B) = \frac{P(A\cap B)}{P(B)}$:

Let A be event $K=1$

Let B be event $K<4$, which means $K\in\{1,2,3\}$

The event $A\cap B$ is basically $k=1$:

$$P(K=1) = \frac{1^2}{30} = \frac{1}{30}$$

$$P(K<4) = P(K=1)+P(K=2)+P(K=3) = \frac{1^2}{30}+\frac{2^2}{30}+\frac{3^2}{30} = \frac{1+4+9}{30} = \frac{14}{30}$$

$$P(K=1|K<4) = \frac{P(K=1)}{P(K<4)} = \frac{1/30}{14/30} = \frac{1}{14}$$

d) This follows a geometric distribution where success is drawing number 4.

Let p be the prob of success:

$$p = P(K=4) = \frac{4^2}{30} = \frac{16}{30} = \frac{8}{15}$$

The PMF for a geometric random variable for the first success on trial y is $P(Y=y) = (1-p)^{y-1}p$. Here $y=3$:

$$P(Y=3) = \left(1-\frac{8}{15}\right)^{3-1}\left(\frac{8}{15}\right) = \left(\frac{7}{15}\right)^2\left(\frac{8}{15}\right)$$

$$P(Y=3) = \frac{49}{225}\cdot\frac{8}{15} = \frac{392}{3375}$$

e) This follows a binomial where success is drawing the number 1.

- # of trials, $n=5$
- # of successes, $k=3$
- Prob of success, $p = P(K=1) = \frac{1^2}{30} = \frac{1}{30}$

Using the binomial PMF, $P(X=k) = \binom{n}{k}p^k(1-p)^{n-k}$:

$$P(X=3) = \binom{5}{3}\left(\frac{1}{30}\right)^3\left(1-\frac{1}{30}\right)^{5-3}$$

$$P(X=3) = 10\left(\frac{1}{30}\right)^3\left(\frac{29}{30}\right)^2 = 10\cdot\frac{1}{27000}\cdot\frac{841}{900} = \frac{841}{243000}$$

Q4

Let X be a continuous random variable with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \qquad \text{for all } x\in\mathbb{R}.$$

and let $Y = \sqrt{|X|}$. Find $f_Y(y)$.

First we find the CDF of Y, $F_Y(y) = P(Y\le y)$. The range of Y is $y\ge 0$, so for $y<0$, $F_Y(y)=0$. For $y\ge 0$:

$$F_Y(y) = P(Y\le y) = P(\sqrt{|X|}\le y)$$

$$= P(|X|\le y^2)$$

$$= P(-y^2\le X\le y^2)$$

This probability can be expressed using CDF for X, which for a standard normal is denoted by $\Phi(x)$

$$P(-y^2 \le X \le y^2) = F_X(y^2) - F_X(-y^2) = \Phi(y^2) - \Phi(-y^2)$$

Using the property $\Phi(-x) = 1 - \Phi(x)$:

$$F_Y(y) = \Phi(y^2) - (1 - \Phi(y^2)) = 2\Phi(y^2) - 1$$

Now, we have PDF, $f_Y(y)$, by differentiating the CDF, $F_Y(y)$:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(2\Phi(y^2) - 1)$$

Using the chain rule and property: $\frac{d}{dx}\Phi(x) = f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$f_Y(y) = 2 \cdot f_x(y^2) \cdot \frac{d}{dy}(y^2)$$

$$f_Y(y) = 2 \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-(y^2)^2/2}\right) \cdot (2y)$$

$$f_Y(y) = \frac{4y}{\sqrt{2\pi}} e^{-y^4/2}$$

So the PDF for Y is:

$$f(y) \begin{cases} \frac{4y}{\sqrt{2\pi}} e^{-y^4/2} & y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Q5

Let X be a uniform random variable over the interval $(1-\theta, 1+\theta)$, where $0 < \theta < 1$ is a given parameter. Find a function of X, say $g(X)$, so that $E[g(X)] = \theta^2$.

Given $X \sim \text{Uniform}(1-\theta, 1+\theta)$, the PDF is $f_X(x) = \frac{1}{(1+\theta)-(1-\theta)} = \frac{1}{2\theta}$ for $x \in (1-\theta, 1+\theta)$

Trying a simple polynomial function centred around the mean of X. The mean of X is

$E[X] = \mu x = \frac{(1-\theta)+(1+\theta)}{2} = 1$. Using the Law of Unconscious Statistician (LOTUS):

$$E[g(X)] = E[c(X-1)^2] = c\,E[(X-1)^2]$$

We recognize that $E[(X-1)^2] = E[(X-\mu x)^2]$ is the definition of the variance of X, $Var(X)$.

The variance of a uniform distribution is $Var(X) = \frac{(b-a)^2}{12}$

$$Var(X) = \frac{((1+\theta)-(1-\theta))^2}{12} = \frac{(2\theta)^2}{12} = \frac{4\theta^2}{12} = \frac{\theta^2}{3}$$

So we get: $E[g(x)] = c \cdot Var(X) = c\frac{\theta^2}{3}$

We want this to be equal to θ^2:

$$c\frac{\theta^2}{3} = \theta^2$$

Solving for c, we find $c = 3$, Therefore a function that satisfies the condition is:

$$g(X) = 3(X-1)^2$$