

Signals and Systems I

Lecture 5

Last Lecture

- System Classification

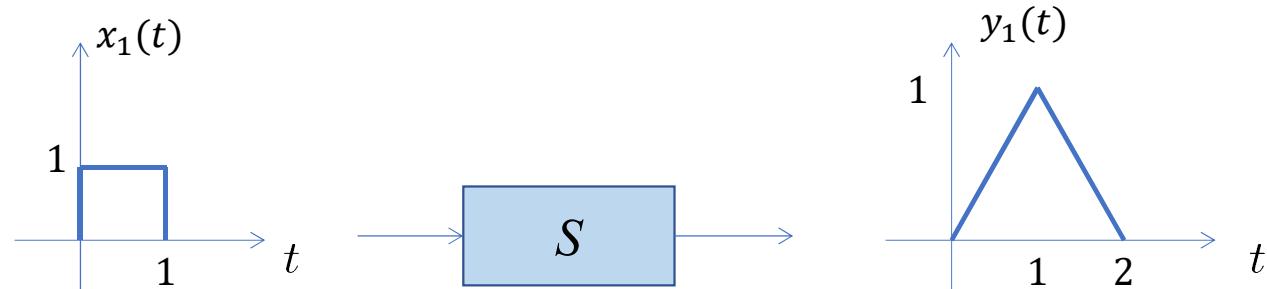
Today

- Linear Time-Invariant (LTI) Systems
- Impulse Response and its Importance for LTI Systems
- LTI Differential Equation (LTIDE) Systems
- Impulse Response of LTIDE Systems
- Convolution
- Initial Condition and LTI systems

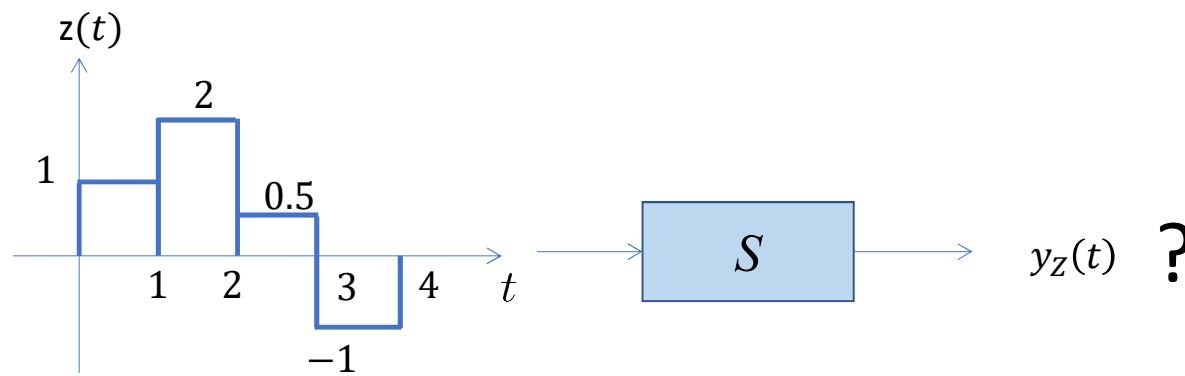
Why are Linear Time Invariant (LTI) Systems Important?

Linear Time Invariant (LTI) Systems are both Linear and Time Invariant (TI)!

Consider the following example:

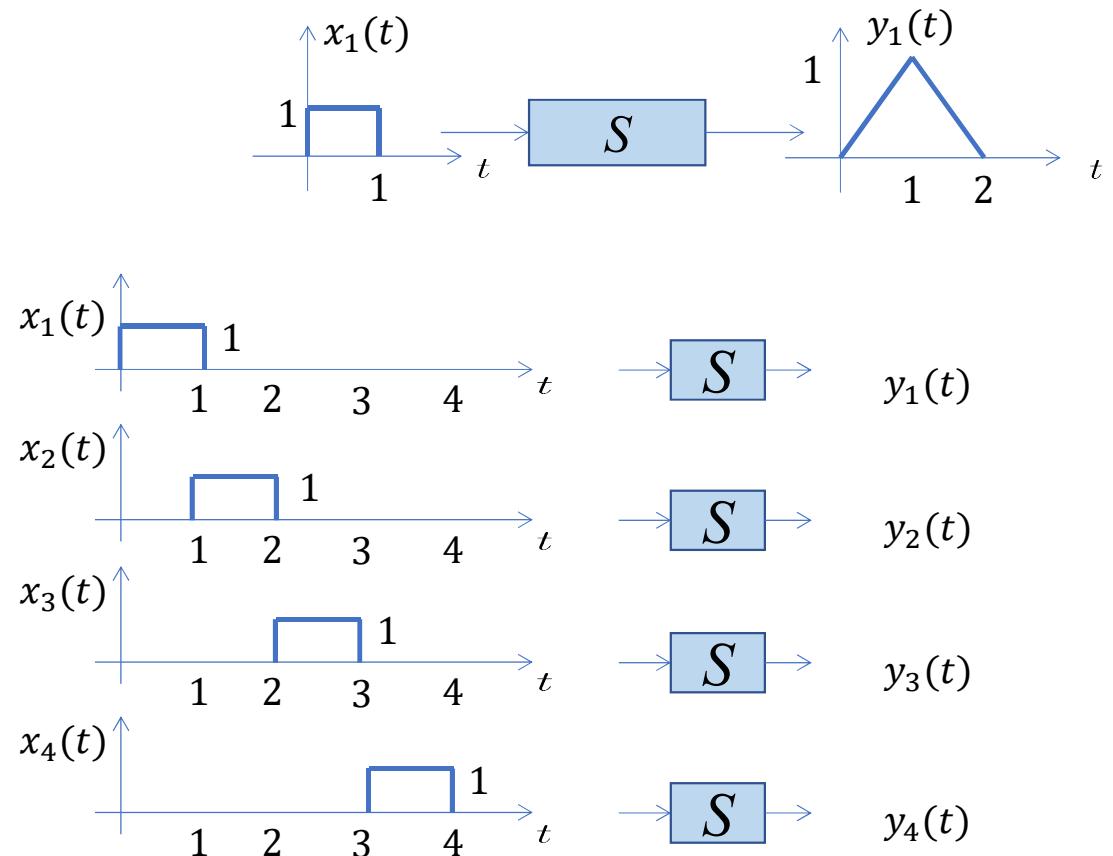
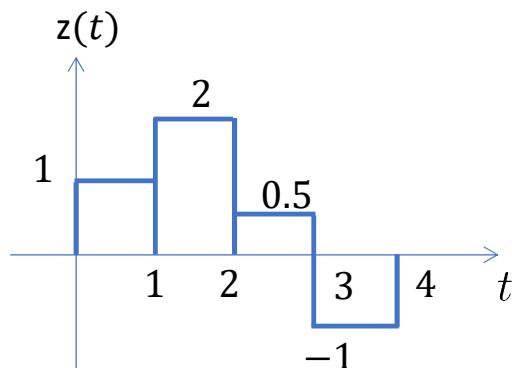


What can be said about the output of this system to $z(t)$.



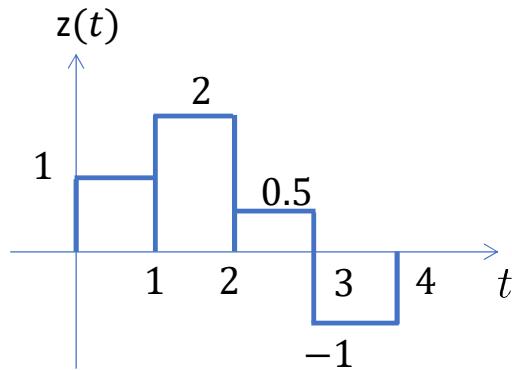
Why are Linear Time Invariant (LTI) Systems Important?

$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$



Why are Linear Time Invariant (LTI) Systems Important?

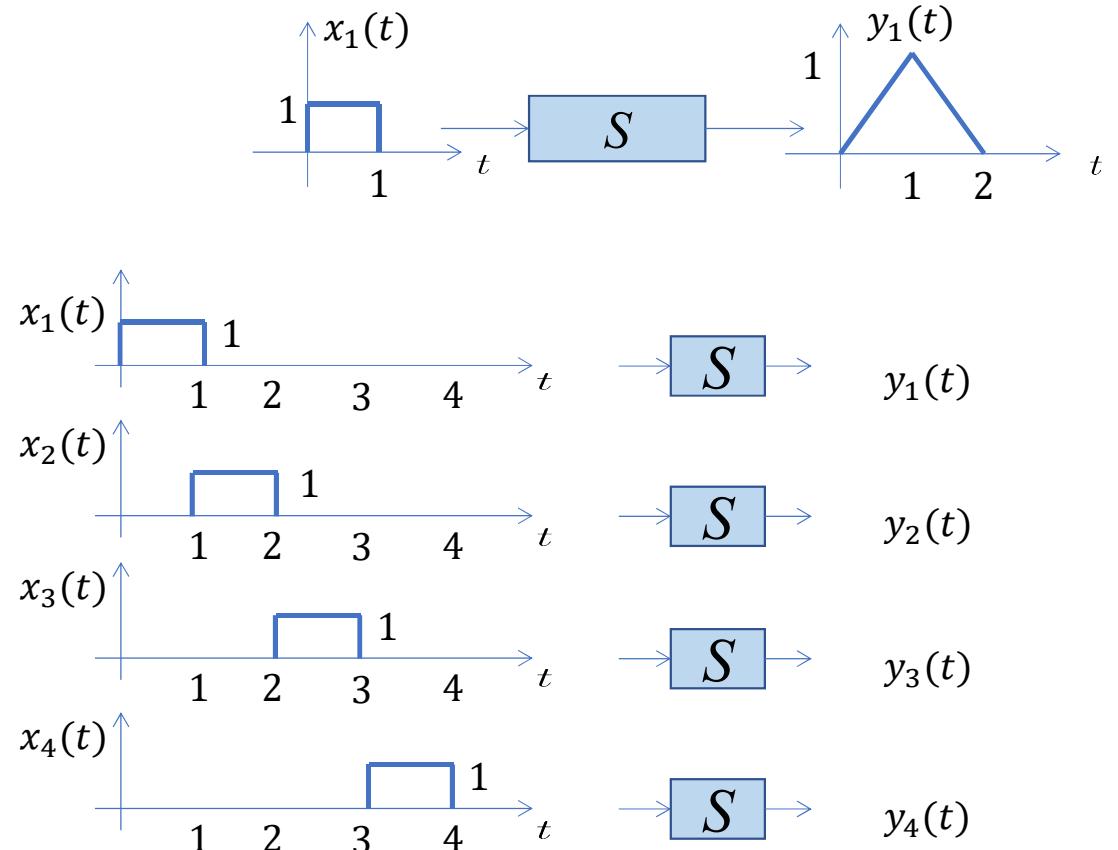
$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$



If the system is linear then:

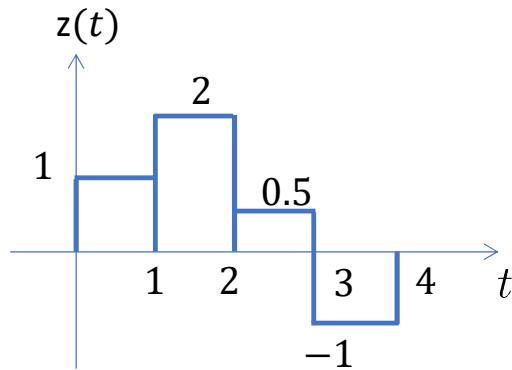
$$y_Z(t) = y_1(t) + 2y_2(t) + 0.5y_3(t) - y_4(t)$$

Can we find $y_2(t)$ and $y_3(t)$ and $y_4(t)$ only by using the available $y_1(t)$?



Why are Linear Time Invariant (LTI) Systems Important?

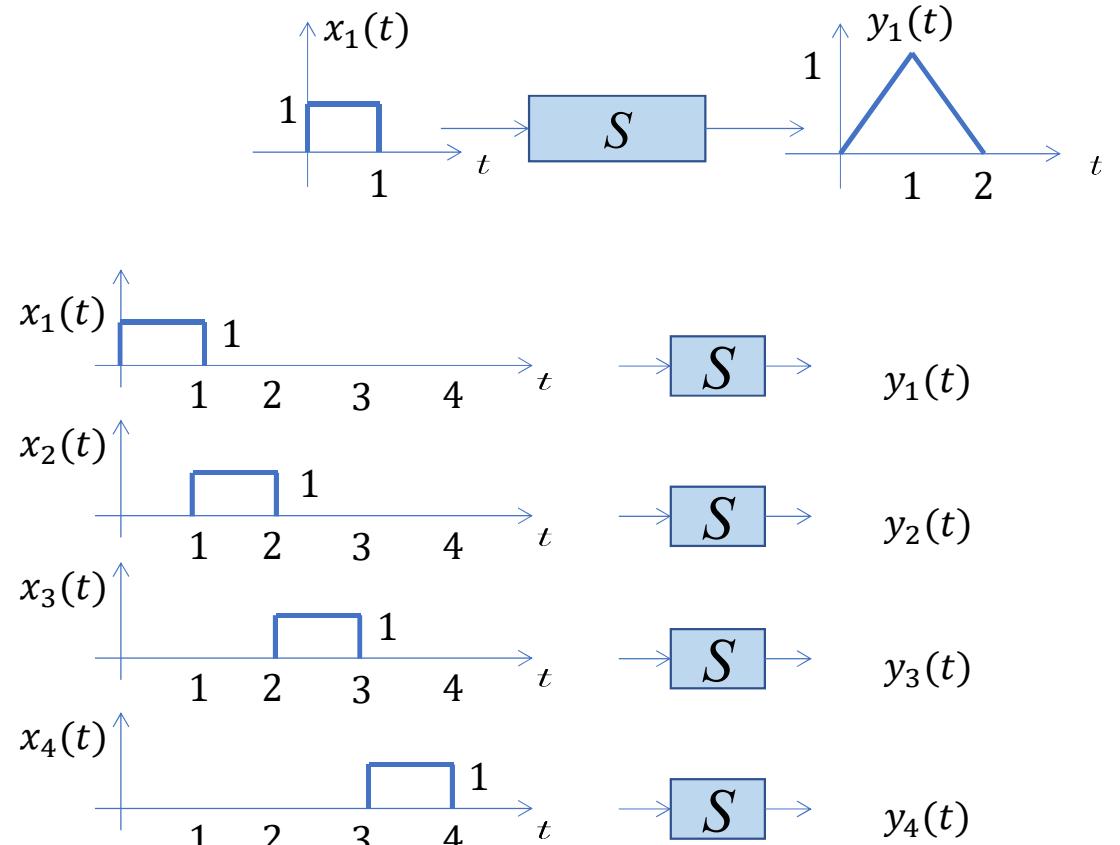
$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$



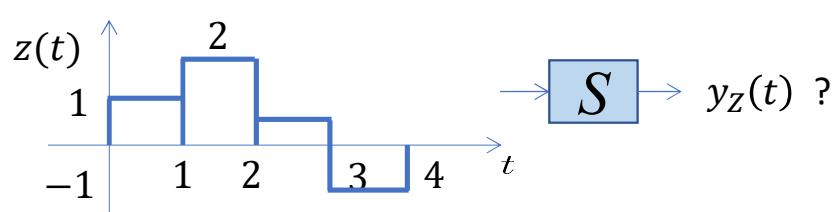
If the system is linear then:

$$y_Z(t) = y_1(t) + 2y_2(t) + 0.5y_3(t) - y_4(t)$$

Can we find $y_2(t)$ and $y_3(t)$ and $y_4(t)$ only by using the available $y_1(t)$? Only if the system is also TI!

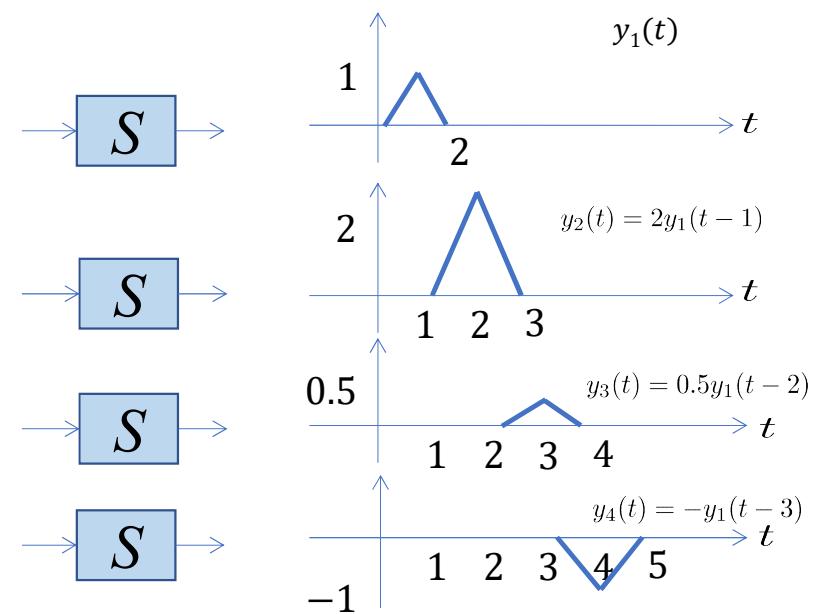
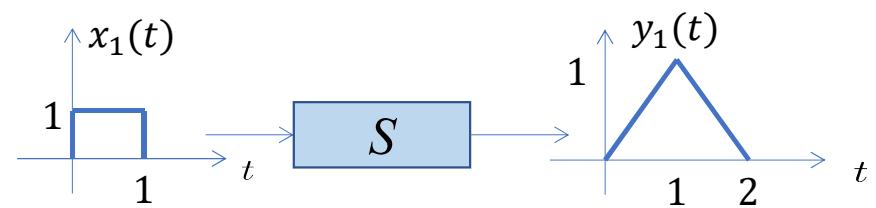
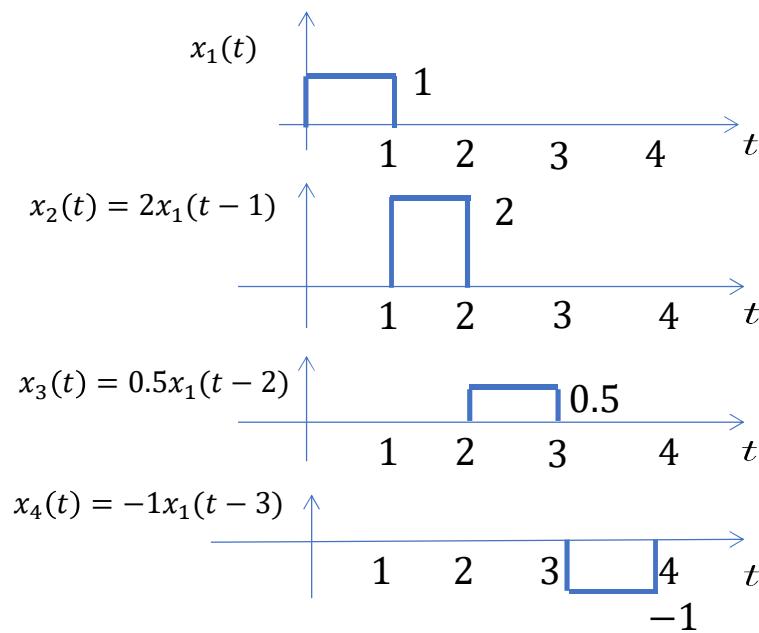


Why are Linear Time Invariant (LTI) Systems Important?

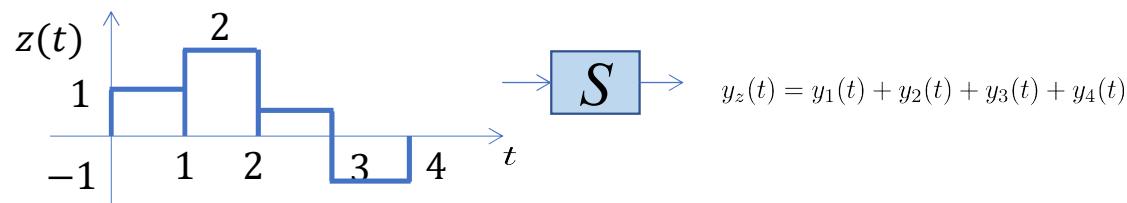


$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$

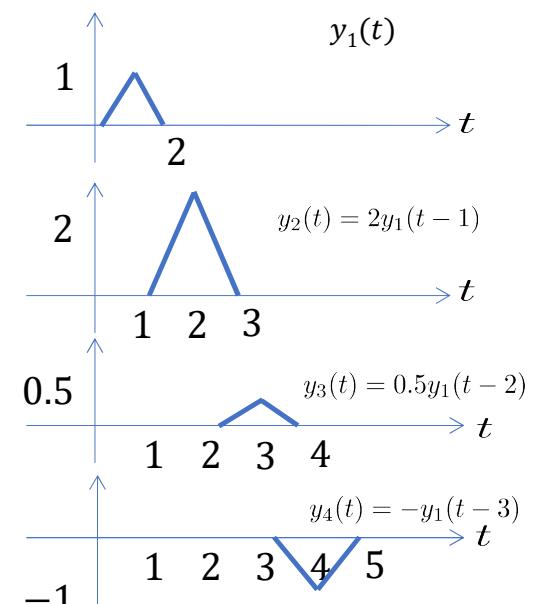
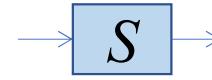
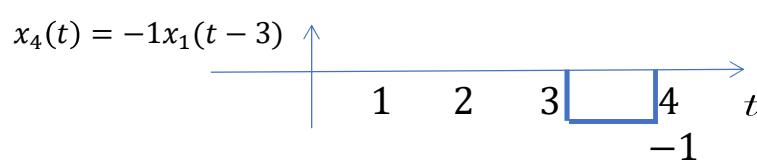
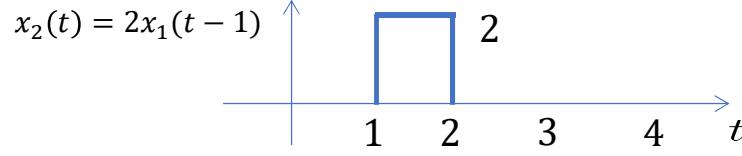
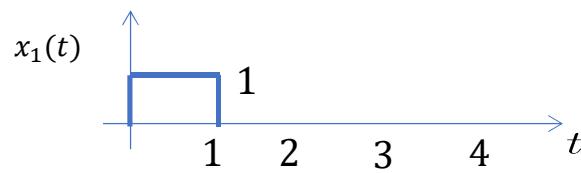
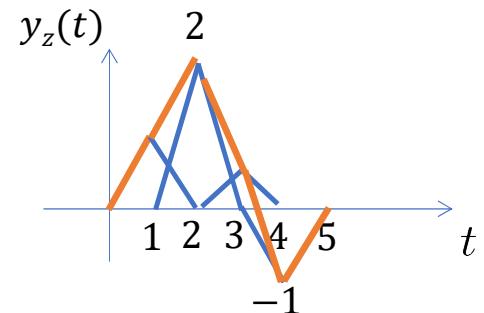
If the system is also TI, then we have:



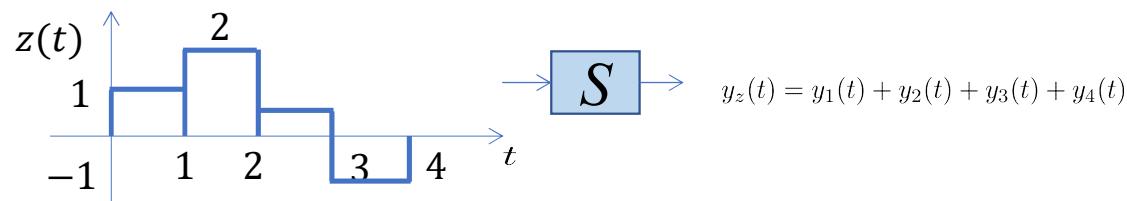
Why are Linear Time Invariant (LTI) Systems Important?



$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$

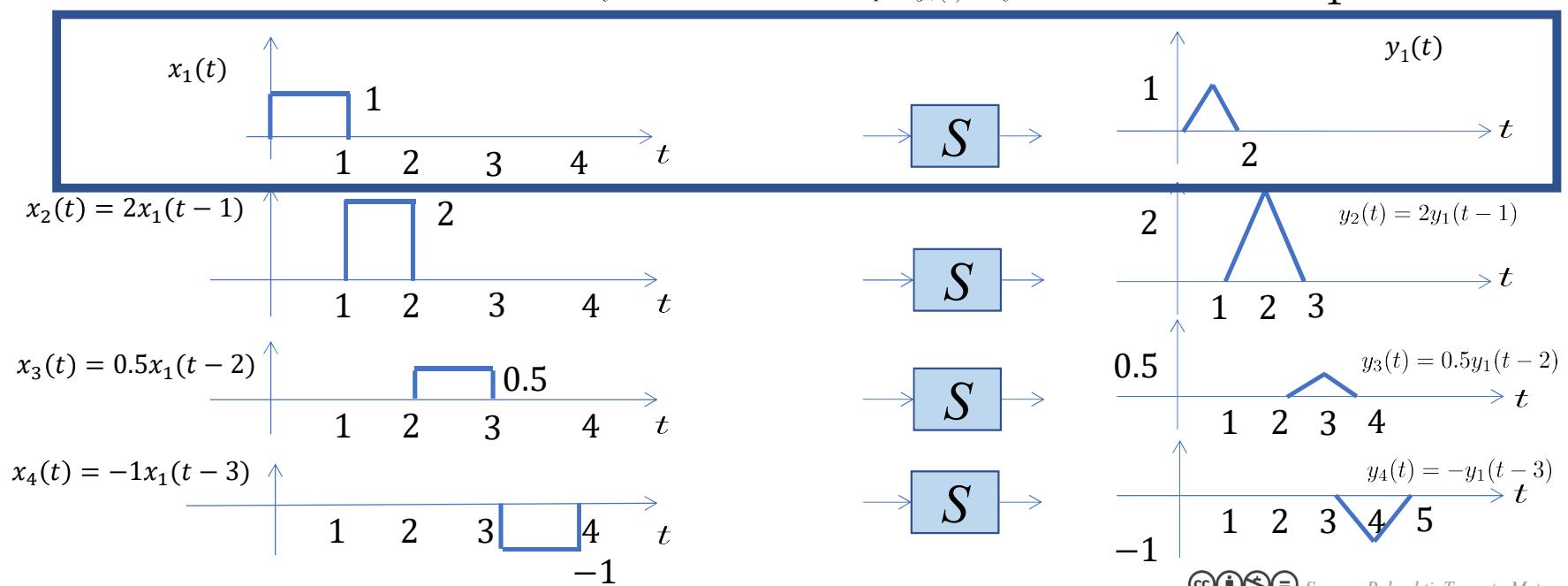
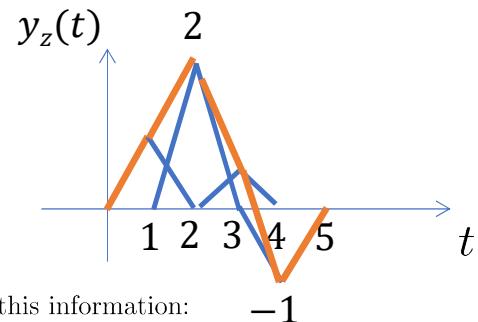


Why are Linear Time Invariant (LTI) Systems Important?



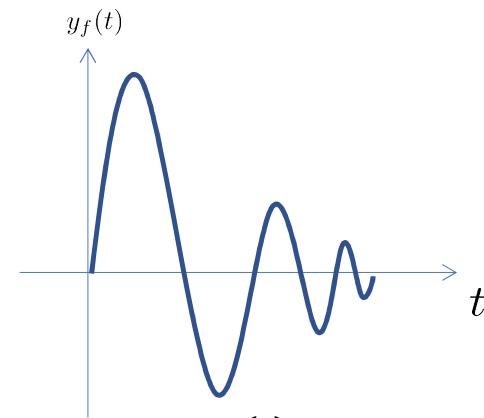
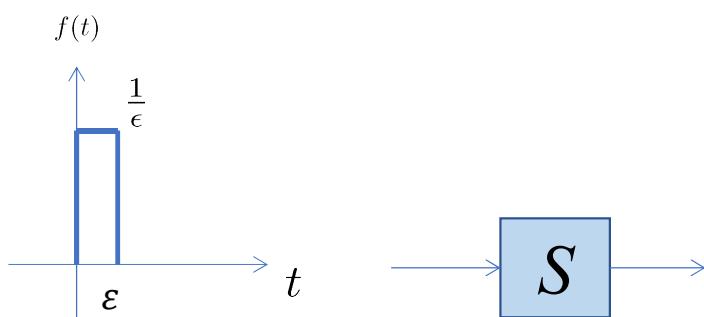
$$z(t) = x_1(t) + 2x_2(t) + 0.5x_3(t) - x_4(t)$$

Since the system is LTI we found output $y_z(t)$ only with this information:

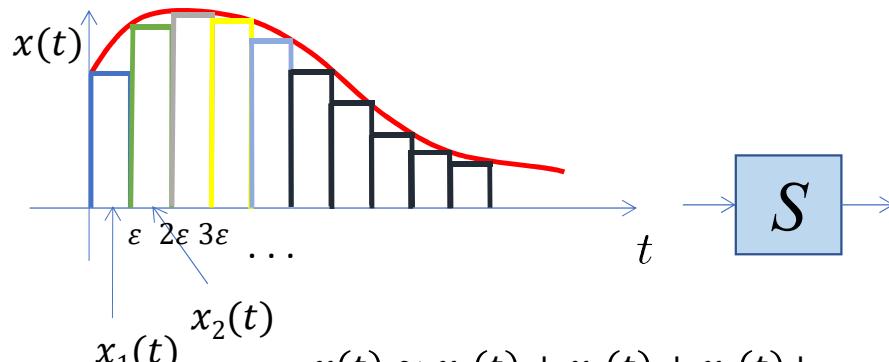


Why are Linear Time Invariant (LTI) Systems Important?

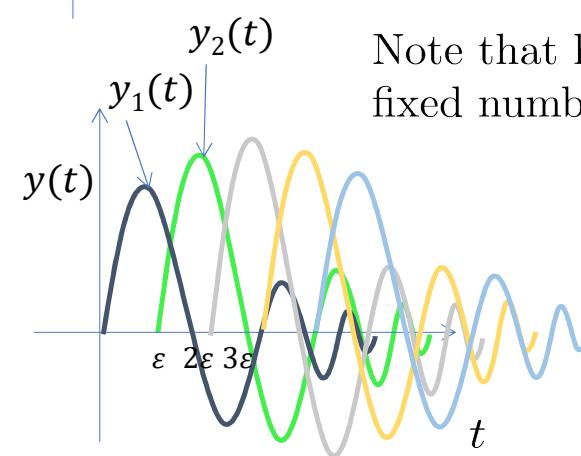
So for LTI system if we have the output of the system to $f(t)$



Then we can find the output of the system for ANY $x(t)$ using only $y_f(t)!!!$



$$\begin{aligned} x(t) &\approx x_1(t) + x_2(t) + x_3(t) + \dots \\ &\approx \sum_n ex(n\epsilon)f(t - n\epsilon) \end{aligned}$$

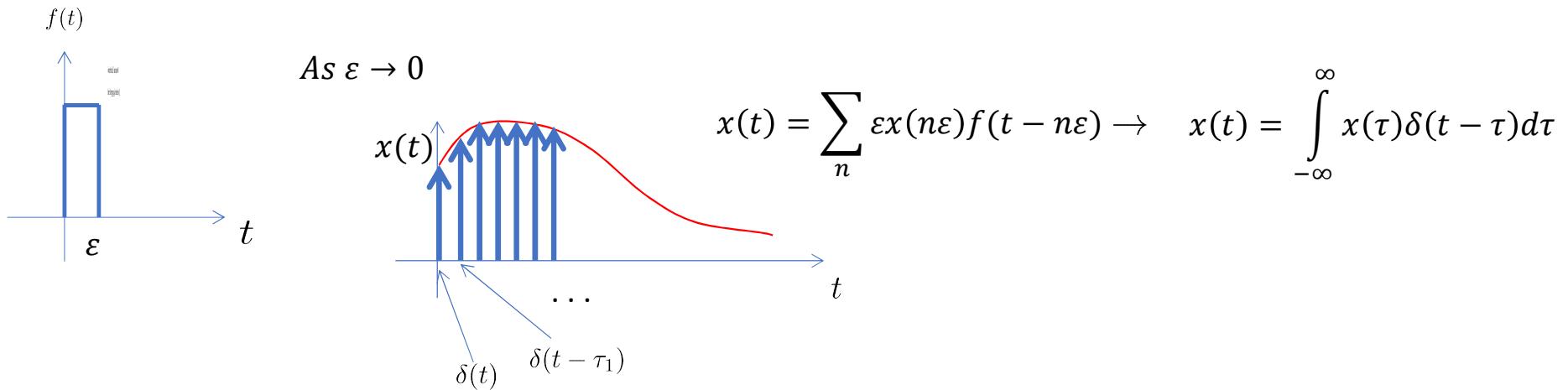


Note that here $ex(n\epsilon)$ are fixed numbers (coefficients)

$$\begin{aligned} y(t) &\approx y_1(t) + y_2(t) + \dots \\ &= \sum_n ex(n\epsilon)y_f(t - n\epsilon) \end{aligned}$$

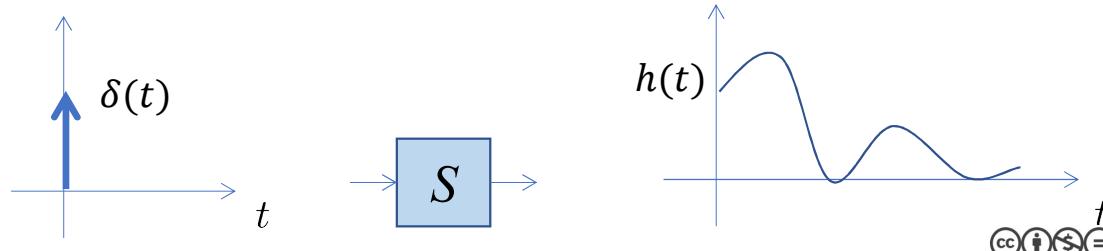
Linear Time Invariant (LTI) Systems and Impulse Response

As $\epsilon \rightarrow 0$, $f(t)$ becomes $\delta(t)$ and we have



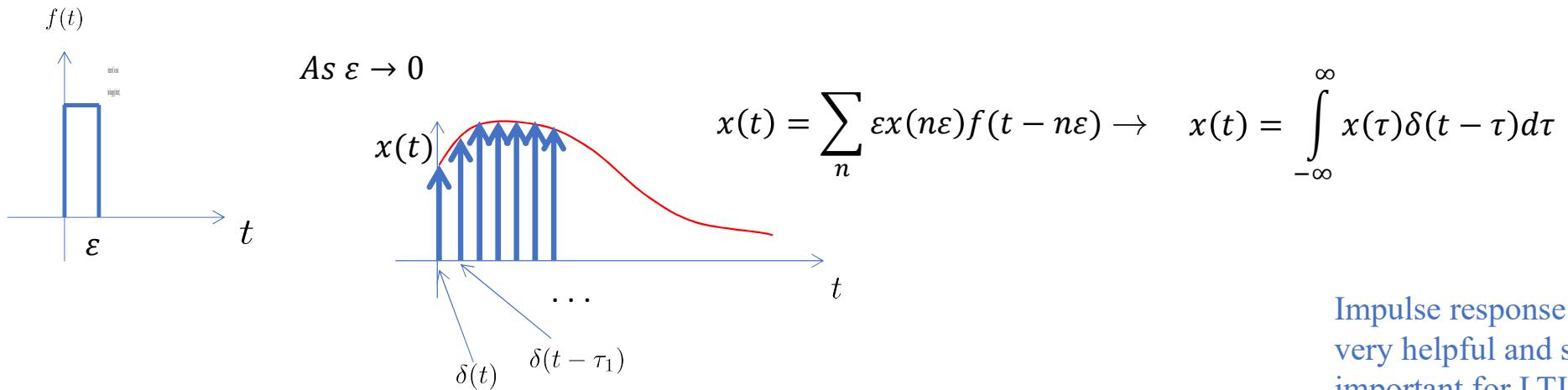
So if the output of linear system for all $\delta(t - T)$ is known , the output is available!

Definition: For **ALL** systems impulse response, $h(t)$, is the response of the system to $\delta(t)$!



Linear Time Invariant (LTI) Systems and Impulse Response

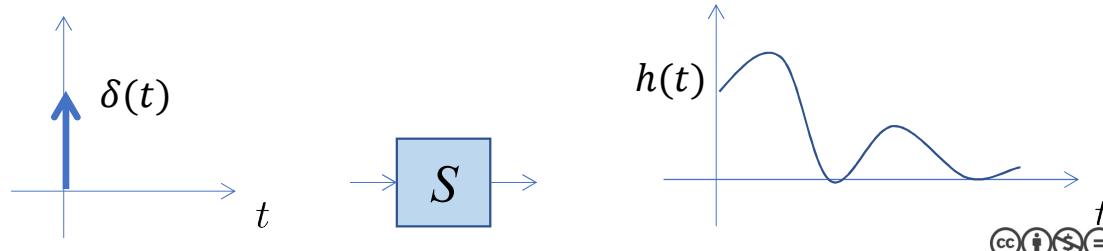
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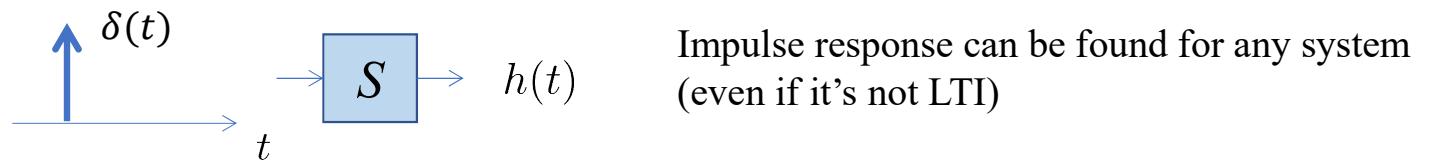
So if the output of linear system for all $\delta(t - T)$ is known , the output is available!

Impulse response is very helpful and super important for LTI systems! Why?

Definition: For **ALL** systems impulse response, $h(t)$, is the response of the system to $\delta(t)$!



LTI Systems, Impulse Response and Convolution



However if the system is TI then:

$$\delta(t - \tau) \rightarrow S \rightarrow h(t - \tau)$$

And if the system is also Linear, then we have:

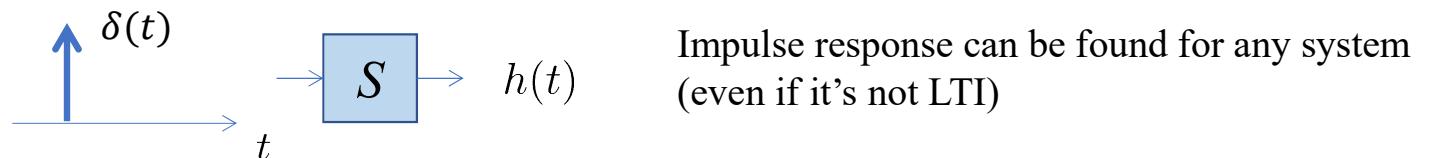
$$\sum x(\tau)\delta(t - \tau) \rightarrow S \rightarrow \sum x(\tau)h(t - \tau)$$

Here $x(\tau)$ s are numbers like a and b in the linear combination of signals

And therefore

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau \rightarrow S \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

LTI Systems, Impulse Response and Convolution



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$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau \rightarrow S \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

This operation is called Convolution

LTI systems are uniquely defined by their impulse response.

We can replace the LTI system with its impulse response that is a **signal!**



Linear Time Invariant Differential Equation (LTIDE) systems

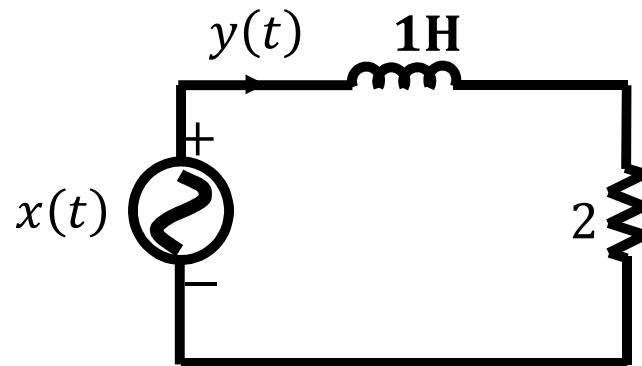
(important class of LTI systems)

Example:

$$x(t) = L \frac{dy}{dt} + Ry(t)$$

$$x(t) = \frac{dy}{dt} + 2y(t)$$

$$x(t) = (D + 2)y(t)$$



Example:

$$x(t) - \frac{2d^2x(t)}{dt^2} + 3 \frac{d^3x(t)}{dt^3} = y(t) + \frac{2dy}{dt}$$

$$x(t)(1 - 2D^2 + 3D^3) = (1 + 2D)y(t)$$

Linear combination of input and its higher order derivatives = Linear combination of output and its higher order derivatives



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General Form of Linear Time Invariant Differential Equation (LTIDE) systems

$$(D^N + a_1 D_{N-1} + \cdots + a_{N-1} D + a_N)y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_N)x(t)$$

N (highest derivative of output) is denoted as the order of the system.

For now we assume that $M \leq N$, (we discuss $M > N$ later).

Note that a_0 that is the coefficient associated with $D^N y(t)$ is one. If this is not the case first divide both sides so that a_0 is always one.

Example:

$$(D^2 + 5D + 6)y(t) = (3D^2 + D + 1)x(t)$$

$N = M = 2$ and $a_0 = 1, a_1 = 5, a_2 = 6, b_0 = 3, b_1 = 1, b_2 = 1$

Impulse Response of LTIDE systems

Output of the causal system to imput $x(t) = \delta(t)$ is denoted by $h(t)$:

$$(D^N + a_1 D_{N-1} + \cdots + a_{N-1} D + a_N)h(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_N)\delta(t)$$

N is the order of the system (represent the number of poles of the system)

If $M = N$ then

$$h(t) = b_0\delta(t) + (\text{charactrsitic mode term for } t > 0)$$

If $M < N$ then $b_0 = 0$,

$$h(t) = \text{charactristic mode term for } t > 0$$



Impulse Response of LTIDE systems

Example :

$$(D^2 + 5D + 6)y(t) = (3D^2 + D + 1)x(t)$$

$M = N = 2, b_0 = 3$:

$$h(t) = 3\delta(t) + (\text{char. mode term for } t > 0)$$

Note: we are assuming that the system is also causal (will discuss this later)

Char. Equation (reminder): $\lambda^N + a_1\lambda^{N-1} + \dots + a_n = 0$

Here: $\lambda^2 + 5\lambda + 6 = 0 \rightarrow \lambda_1 = -2, \lambda_2 = -3$

$$h(t) = 3\delta(t) + (c_1 e^{-2t} + c_2 e^{-3t})u(t)$$

The main challenge is now to find c_1 and c_2

Reminder: Char mode term (if there are no repeated roots for $t > 0$): $(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t})u(t)$

Char mode term (if there are no repeated roots) for $t > 0$: $(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t})u(t)$

Impulse Response of LTIDE systems

Example 1: Find the impulse response to the following LTIDE:

Input $x(t)$ and output $y(t)$

$$(D^2 + 5D + 6) y(t) = (D + 1) x(t)$$

We replace $x(t) = \delta(t)$ so $y(t) = h(t)$ (A kick to the system!)

$$(D^2 + 5D + 6) h(t) = (D + 1) \delta(t)$$

$$h'' + 5h' + 6h = \delta' + \delta$$

$$N = 2, M = 1, b_0 = 0$$

$$\lambda^2 + 5\lambda + 6 = 0 \rightarrow \lambda_1 = -2, \lambda_2 = -3 \rightarrow h(t) = (c_1 e^{-2t} + c_2 e^{-3t}) u(t)$$

$$h'(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t}) u(t) + (c_1 e^{-2t} + c_2 e^{-3t}) \delta(t)$$

$$h'(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t}) u(t) + (c_1 + c_2) \delta(t)$$

$$h''(t) = (4c_1 e^{-2t} + 9c_2 e^{-3t}) u(t) + (-2c_1 e^{-2t} - 3c_2 e^{-3t}) \delta(t) + (c_1 + c_2) \delta'(t)$$



Impulse Response of LTIDE systems

$$(D^2 + 5D + 6) h(t) = (D + 1) \delta(t)$$

So we build $(D^2 + 5D + 6)h(t)$ and set it equal to $(D + 1)\delta(t) = \delta'(t) + \delta(t)$.

$$6 \times h(t) = 6 \times [(c_1 e^{-2t} + c_2 e^{-3t})u(t)]$$

$$5 \times h'(t) = 5 \times [(-2c_1 e^{-2t} - 3c_2 e^{-3t})u(t) + (c_1 + c_2)\delta(t)]$$

$$1 \times h''(t) = 1 \times [(4c_1 e^{-2t} + 9c_2 e^{-3t})u(t) + (-2c_1 - 3c_2)\delta(t) + (c_1 + c_2)\delta'(t)]$$

$$6h(t) + 5h'(t) + h''(t) = 0 \times u(t) + (3c_1 + 2c_2)\delta(t) + (c_1 + c_2)\delta'(t)$$

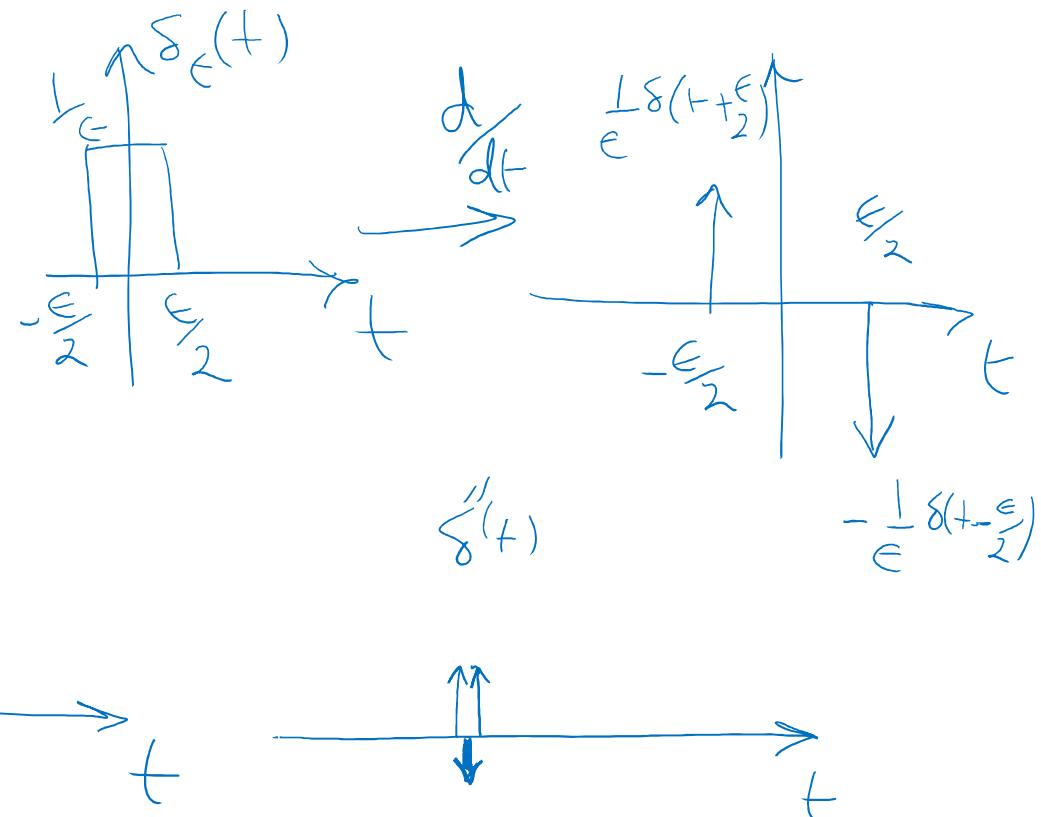
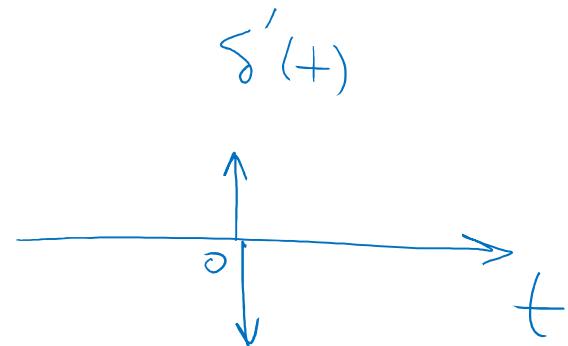
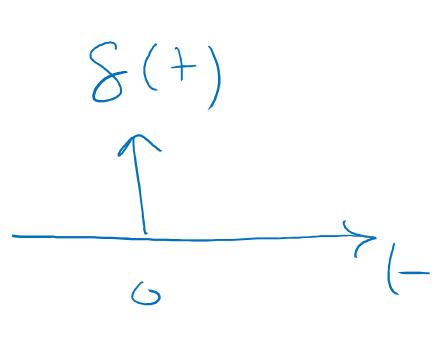
has to be the same as $= \delta(t) + \delta'(t)$

$$\begin{cases} 3c_1 + 2c_2 = 1 \\ c_1 + c_2 = 1 \end{cases} \rightarrow \begin{cases} c_1 = -1 \\ c_2 = 2 \end{cases} \rightarrow h(t) = (-e^{-2t} + 2e^{-3t})u(t)$$



Impulse Response of LTIDE systems

$\delta(t)$ and its derivatives ($\delta'(t), \dots$)



Impulse Response of LTIDE systems

Example 2: Find the impulse response to the following system:

$$(D^2 + 3D + 2)y(t) = Dx(t) \quad N = 2, M = 1, b_0 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0 \rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases} \rightarrow h(t) = (c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}) u(t) = (c_1 e^{-t} + c_2 e^{-2t}) u(t)$$

So we have to have:

$$h''(t) + 3h'(t) + 2h(t) = \delta'(t)$$

$$h(t) = (c_1 e^{-t} + c_2 e^{-2t}) u(t)$$

$$h'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) u(t) + (c_1 + c_2)\delta(t)$$

$$h''(t) = (c_1 e^{-t} + 4c_2 e^{-2t}) u(t) + (-c_1 - 2c_2)\delta(t) + (c_1 + c_2)\delta'(t)$$

We can write:

$$h''(t) + 3h'(t) + 2h(t) = \delta'(t)$$

$$0 \times u(t) + (2c_1 + c_2)\delta(t) + (c_1 + c_2)\delta'(t) = \delta'(t)$$

Impulse Response of LTIDE systems

Example 2: Find the impulse response to the following system:

$$(D^2 + 3D + 2)y(t) = Dx(t) \quad N = 2, M = 1, b_0 = 0$$

$$h''(t) + 3h'(t) + 2h(t) = \delta'(t)$$

$$h(t) = (c_1 e^{-t} + c_2 e^{-2t}) u(t)$$

$$h'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) u(t) + (c_1 + c_2) \delta(t)$$

$$h''(t) = (c_1 e^{-t} + 4c_2 e^{-2t}) u(t) + (-c_1 - 2c_2) \delta(t) + (c_1 + c_2) \delta'(t)$$

We can write:

$$h''(t) + 3h'(t) + 2h(t) = \delta'(t)$$

$$0 \times u(t) + (2c_1 + c_2) \delta(t) + (c_1 + c_2) \delta'(t) = \delta'(t)$$

$$\begin{cases} 2c_1 + c_2 = 0 \\ c_1 + c_2 = 1 \end{cases} \rightarrow c_1 = -1, c_2 = 2 \rightarrow h(t) = (-e^{-t} + 2e^{-2t}) u(t)$$

Validate your answer



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Impulse Response of LTIDE systems

$$(D^2 + 3D + 2)y(t) = Dx(t) \quad N = 2, M = 1, b_0 = 0$$

Here is how to validate your answer:

$$h(t) = (e^{-t} + 2e^{-2t}) u(t)$$

$$h'(t) = (e^{-t} - 4e^{-2t}) u(t) + \delta(t)$$

$$h''(t) = (-e^{-t} + 8e^{-2t}) u(t) + (-3)\delta(t) + \delta'(t)$$

$$2h(t) + 3h'(t) + h''(t) = 0 \times u(t) + (3 - 3)\delta(t) + \delta'(t)$$

Confirmed!



Impulse Response of LTIDE systems

Example 3: Find the impulse response to the following system:

$$(D + 2)y(t) = (3D + 5)x(t) \quad N = 1, M = 1, b_0 = 3$$

Solution:

$$x(t) = \delta(t) \rightarrow y(t) = h(t)$$

$$h(t) = b_0\delta(t) + (\text{Char. mode term for } t > 0)$$

$$\lambda + 2 = 0 \rightarrow \lambda = -2 \rightarrow h(t) = 3\delta(t) + ce^{-2t}u(t)$$

$$h'(t) = 3\delta'(t) - 2ce^{-2t}u(t) + c\delta(t)$$

$$h'(t) + 2h(t) = 3\delta'(t) + 5\delta(t)$$

$$3\delta'(t) + 0 \times u(t) + (c + 6)\delta = 3\delta'(t) + 5\delta(t)$$

$$c + 6 = 5 \rightarrow c = -1$$

$$h(t) = 3\delta(t) - e^{-2t}u(t)$$



Impulse Response of LTIDE systems

Example 4: Find the impulse response to the following system:

$$(D^2 + 2D + 1)y(t) = Dx(t) \quad N = 2, M = 1, b_0 = 0$$

Solution:

$$h'' + 2h' + h = \delta'(t)$$

Char. roots: $\lambda^2 + 2\lambda + 1 = 0 \rightarrow \lambda = -1, -1$ (repeated roots)

$$h(t) = (c_1 e^{-t} + t c_2 e^{-t}) u(t)$$

$$h'(t) = (-c_1 e^{-t} + (c_2 e^{-t} - t c_2 e^{-t})) u(t) + c_1 \delta(t)$$

$$h''(t) = (c_1 e^{-t} - c_2 e^{-t} - c_2 e^{-t} + t c_2 e^{-t}) u(t) + (-c_1 + c_2) \delta(t) + c_1 \delta'(t)$$

$$h''(t) + 2h'(t) + h(t) = \delta'(t)$$

$$0 \times u(t) + (c_1 + c_2) \delta(t) + c_1 \delta'(t) = \delta'(t)$$

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 = 1 \end{cases} \rightarrow \begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases} \rightarrow \boxed{h(t) = (1 - t)e^{-t}u(t)}$$



Impulse Response of LTIDE systems

Example 5: Find the impulse response to the following system:

$$(D^2 + 1)y(t) = 2x(t) \quad N = 2, M = 0, b_0 = 0$$

Solution:

Char. roots: $\lambda^2 + 1 = 0 \rightarrow \lambda = +j, -j$

$$h(t) = (c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}) u(t)$$

$$h(t) = (c_1 e^{jt} + c_2 e^{-jt}) u(t)$$

$$h'(t) = (jc_1 e^{jt} - jc_2 e^{-jt}) u(t) + (c_1 + c_2) \delta(t)$$

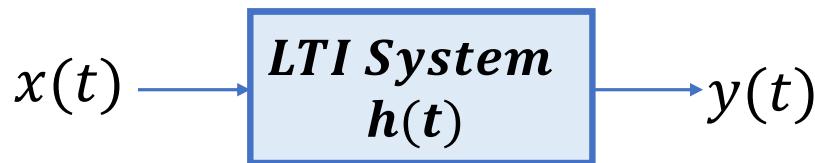
$$h''(t) = (-c_1 e^{jt} - c_2 e^{-jt}) u(t) + (jc_1 - jc_2) \delta(t) + (c_1 + c_2) \delta'(t)$$

$$h''(t) + h(t) = 2\delta(t)$$

$$h''(t) + h(t) = 0 \times u(t) + (jc_1 - jc_2) \delta(t) + (c_1 + c_2) \delta'(t) = 2\delta(t)$$

$$\begin{cases} jc_1 - jc_2 = 2 \\ c_1 + c_2 = 0 \end{cases} \rightarrow \begin{cases} c_1 = -j \\ c_2 = j \end{cases} \rightarrow \boxed{h(t) = (-je^{jt} + je^{-jt}) u(t) = 2\sin(t)u(t)}$$

Convolution



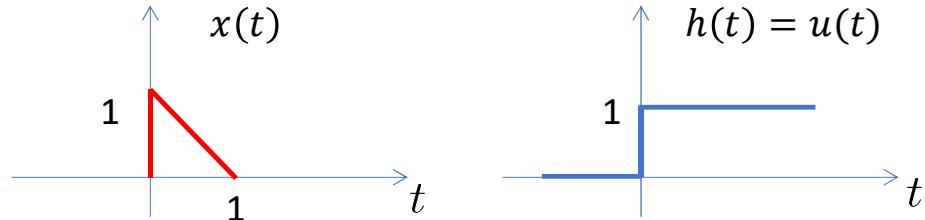
The output, $y(t)$, of a LTI system with impulse response $h(t)$ and input $x(t)$ can be calculated using convolution as follows:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

Note that in this equation for each value of t , variable τ is the independent variable inside the equation and t is the associated **delay**.

Convolution

Example: Find $y(t)$ output of the following system to $x(t)$



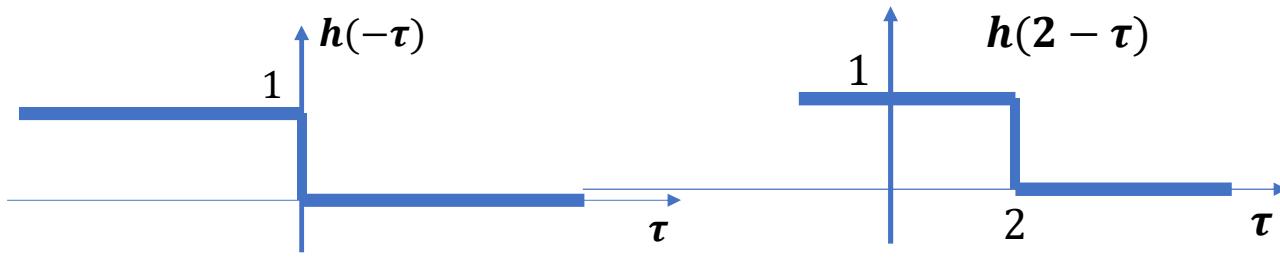
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

Solution:

First we need to build $h(t - \tau)$ (here t is the delay and τ is the IV)

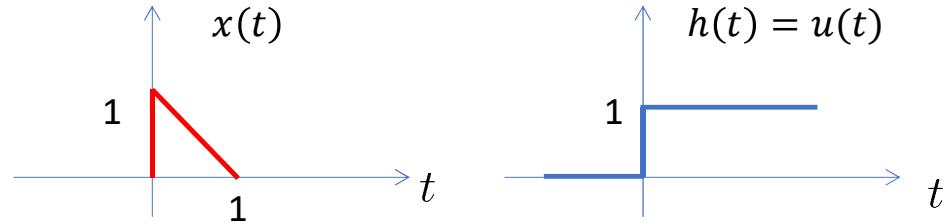
1- Flip $h(\tau)$ horizontally to build $h(-\tau)$.

2- two examples of $h(t - \tau)$ for $t = 2$ and $t = -2$.



Convolution

Example: Find $y(t)$ output of the following system to $x(t)$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

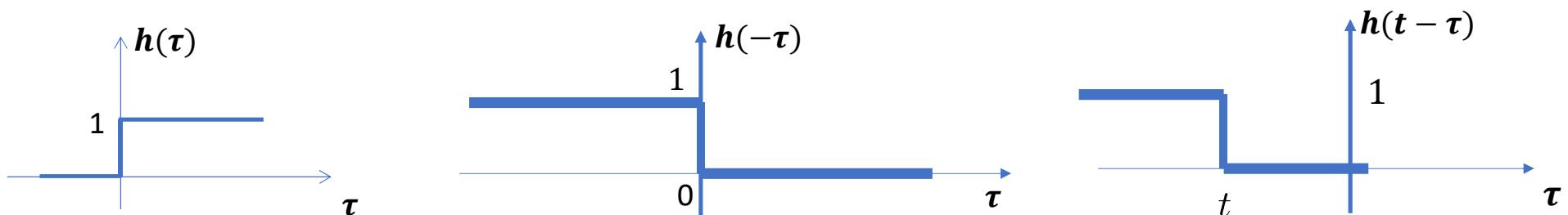
Solution:

First we need to build $h(t - \tau)$ (here t is the delay and τ is the independent variable (IV))

Step 1- Change the IV from t to τ .

Step 2- Flip $h(\tau)$ to build $h(-\tau)$.

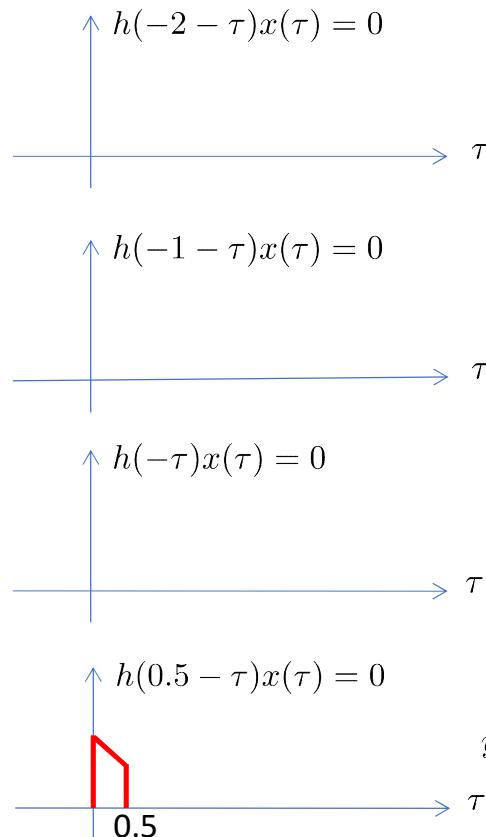
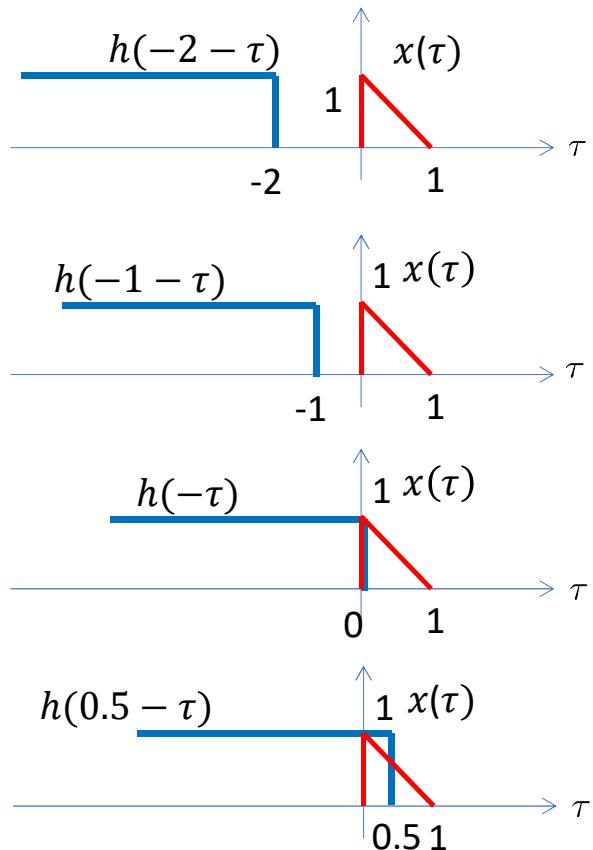
3- Move the value $h(-\tau)$ at zero to t to built $h(t - \tau)$



next step is to find $x(\tau)h(t - \tau)$ for different values of t and its integral:

Convolution

Finding $x(\tau)h(t - \tau)$ for different values of t and its integral:



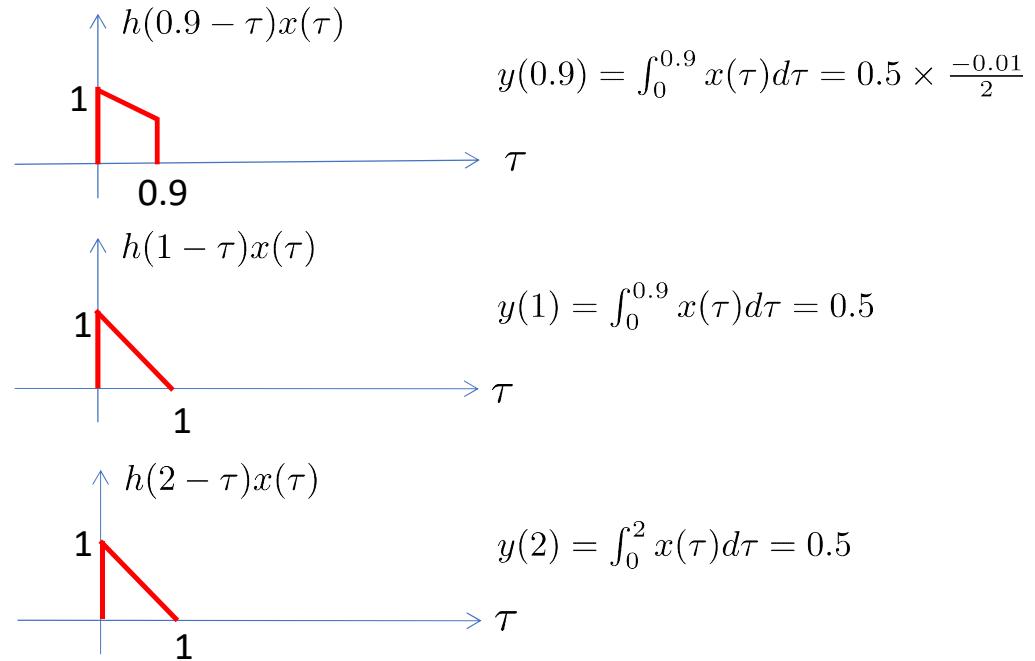
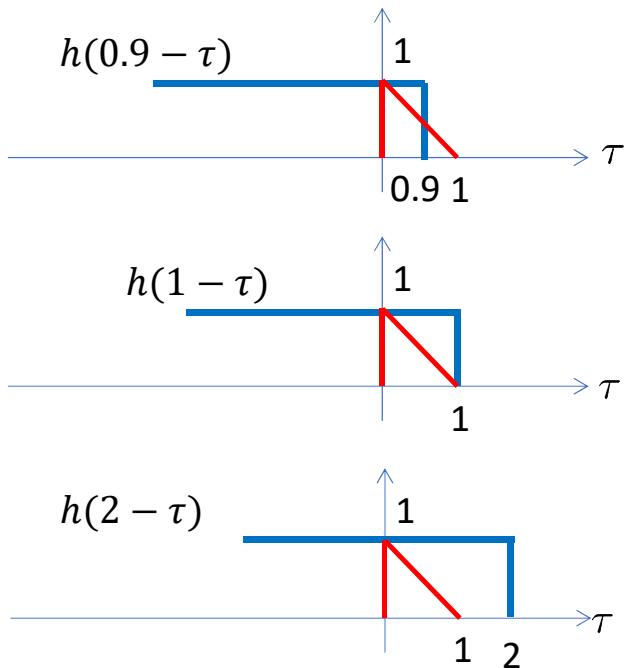
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

$$y(0.5) = \int_0^{0.5} x(\tau)d\tau = 0.25 \times \frac{3}{2}$$

Convolution

Finding $x(\tau)h(t - \tau)$ for different values of t and its integral (cont.):

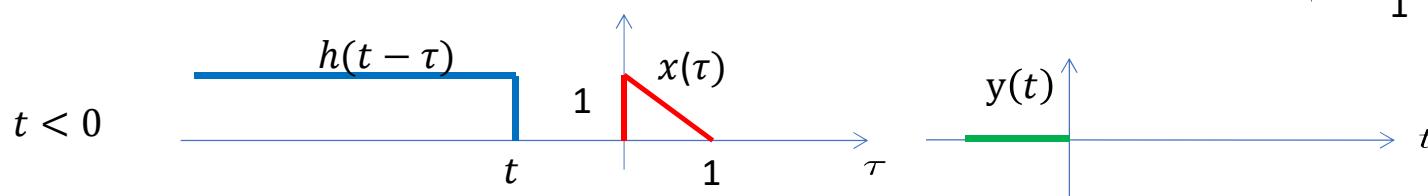
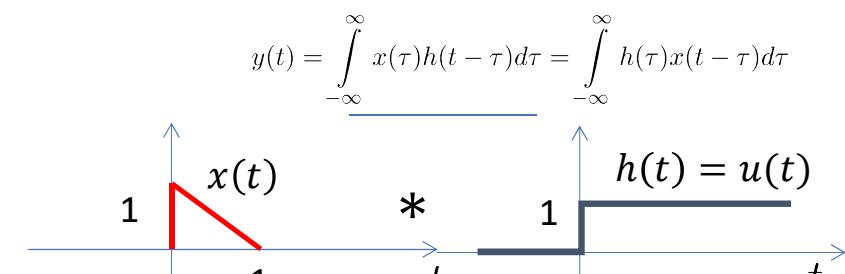
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \underline{\int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau}$$



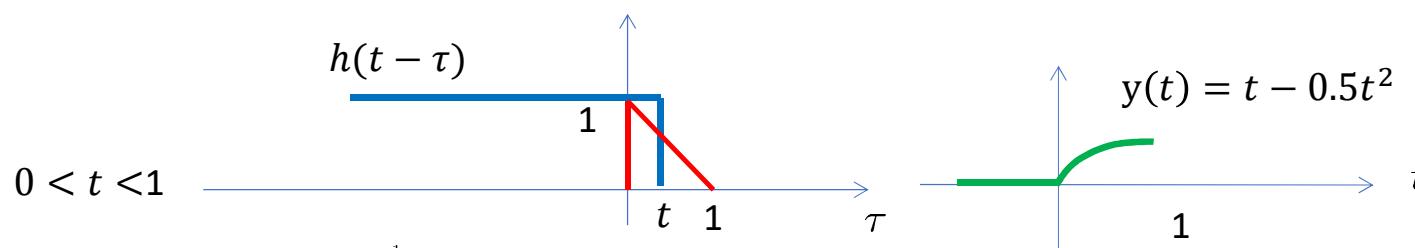
Convolution

Generalizing for all t :

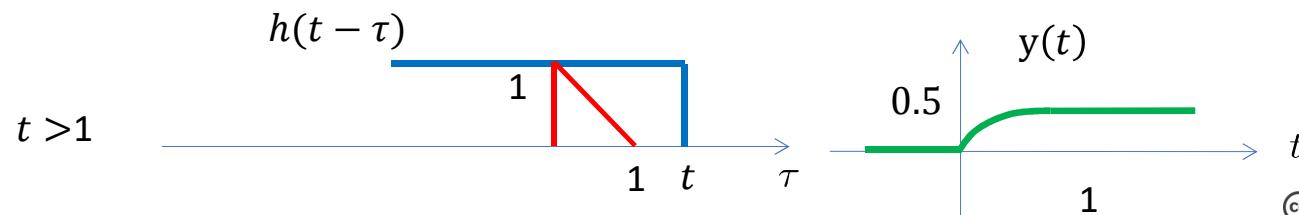
1- For $t < 0$ there is no overlap between $x(\tau)$ and $h(t - \tau)$. Therefore $x(\tau) \times h(t - \tau) = 0$ and $y(t) = 0$.



2- For $0 < t < 1$ there is a partial overlap: $y(t) = \int_0^t x(\tau)d\tau = \int_0^t (-\tau + 1)d\tau = (-\frac{\tau^2}{2} + \tau)|_0^t = t - \frac{t^2}{2}$

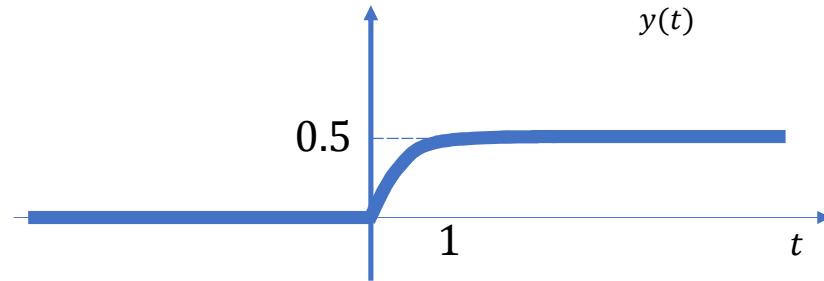


3- For $t > 1$ there is a full overlap: $y(t) = \int_0^1 x(\tau)d\tau = 0.5$



Convolution

Plot $y(t)$ as function of t



To check whether the function is concave or convex between 0 and 1 you can try finding the value at $t = .5$. Here this value is $3/8$ which is larger than $0.5/2$ and makes the function concave.

Now try the same problem with flipping $x(t)$, try to find the following integral:

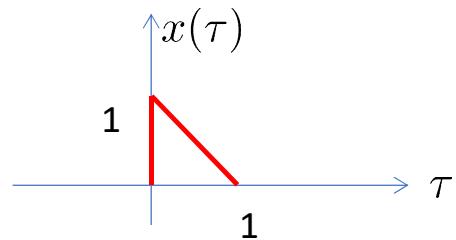
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

The same will be the same as what we have found previously.

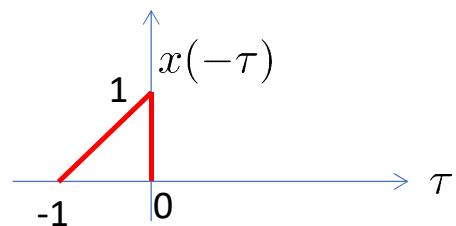
Convolution

First build $x(t - \tau)$

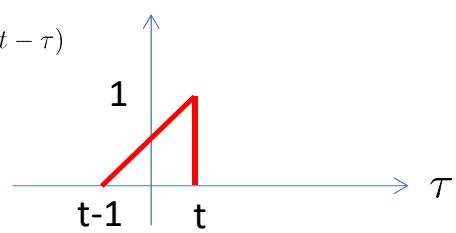
Step 1: Draw $x(\tau)$ by changing the independent variable from t to τ



Step 2: Flip $x(\tau)$ to show $x(-\tau)$



3- Move the value $x(-\tau)$ at zero to t to built $x(t - \tau)$

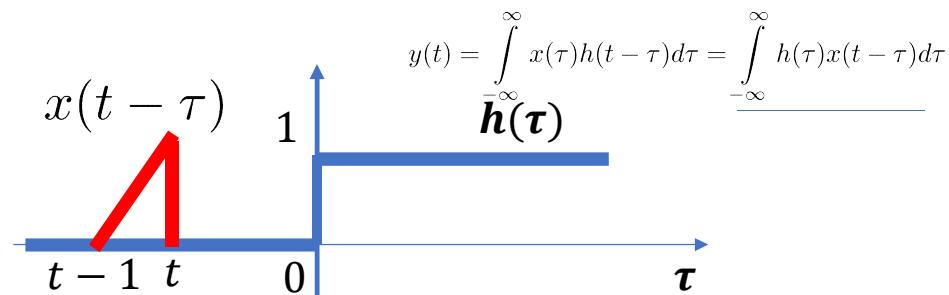


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \underline{\int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau}$$

Convolution

1- For $t < 0$ there is no overlap between $h(\tau)$ and $x(t - \tau)$:

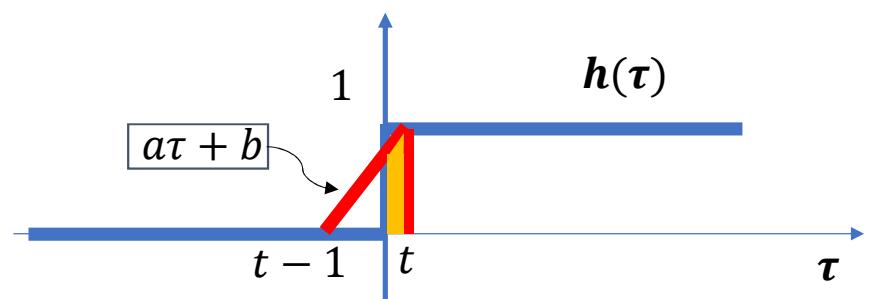
Therefore $y(t) = 0$



2- For $0 < t < 1$ there is a partial overlap between $h(\tau)$ and $x(t - \tau)$:

$$\begin{cases} at + b = 1 \\ a(t-1) + b = 0 \end{cases} \rightarrow \begin{cases} a = 1 \\ b = -t + 1 \end{cases} \rightarrow \text{The line equation: } \boxed{\tau + 1 - t}$$

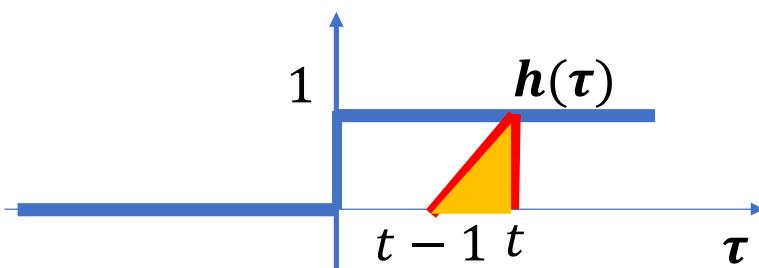
$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau = \int_0^t (\tau - t + 1)d\tau = [\tau^2/2 + (1-t)\tau]_0^t = t - t^2/2$$



$$3- \text{For } t > 1 \text{ there is a full overlap: } y(t) = \int_0^1 x(\tau)d\tau = 0.5$$

So the final answer is identical to what we had calculated before

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$



Convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \underline{\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

Example 2: Find the output of the system with $h(t) = u(t)$ for input $x(t) = e^{-2t}u(t)$

Solution:

First plot $h(t - \tau)$ through the three steps.

1- For $t < 0$ there is no overlap, therefore $y(t) = 0$.

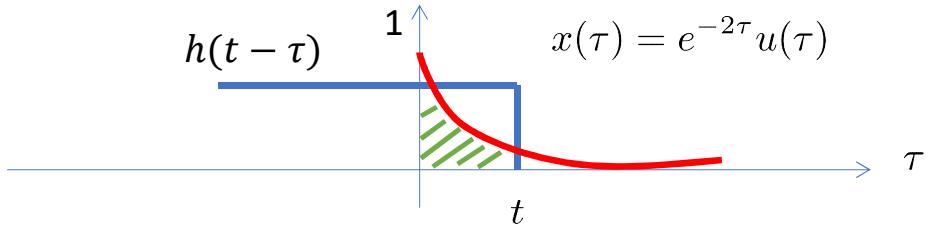
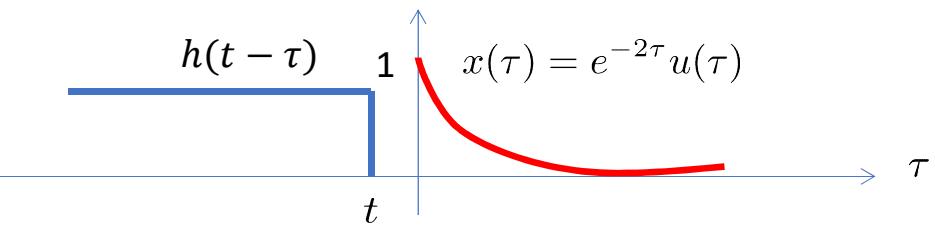
2- For $t > 0$ there is a partial overlap, and the convolution is equal to the area under the overlapped section.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^t e^{-2\tau}d\tau \\ &= \frac{e^{-2\tau}}{-2} \Big|_0^t = \frac{1}{2} - \frac{e^{-2t}}{2} \end{aligned}$$

So the final answer is

$$y(t) = \left[\left(\frac{1}{2} - \frac{e^{-2t}}{2} \right) u(t) \right]$$

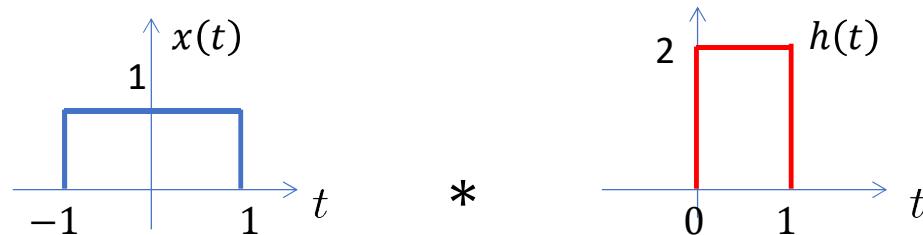
Try flipping and shifting $x(t)$ and verify that you get the same answer.



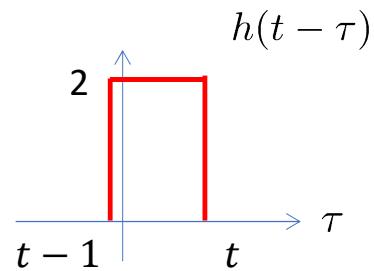
Convolution

Example 3: Find the output $y(t)$ for the following $h(t)$ and input $x(t)$.

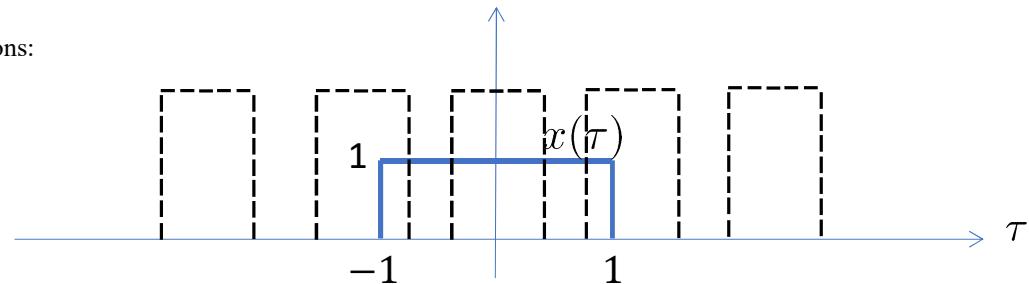
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$



First build $h(t - \tau)$

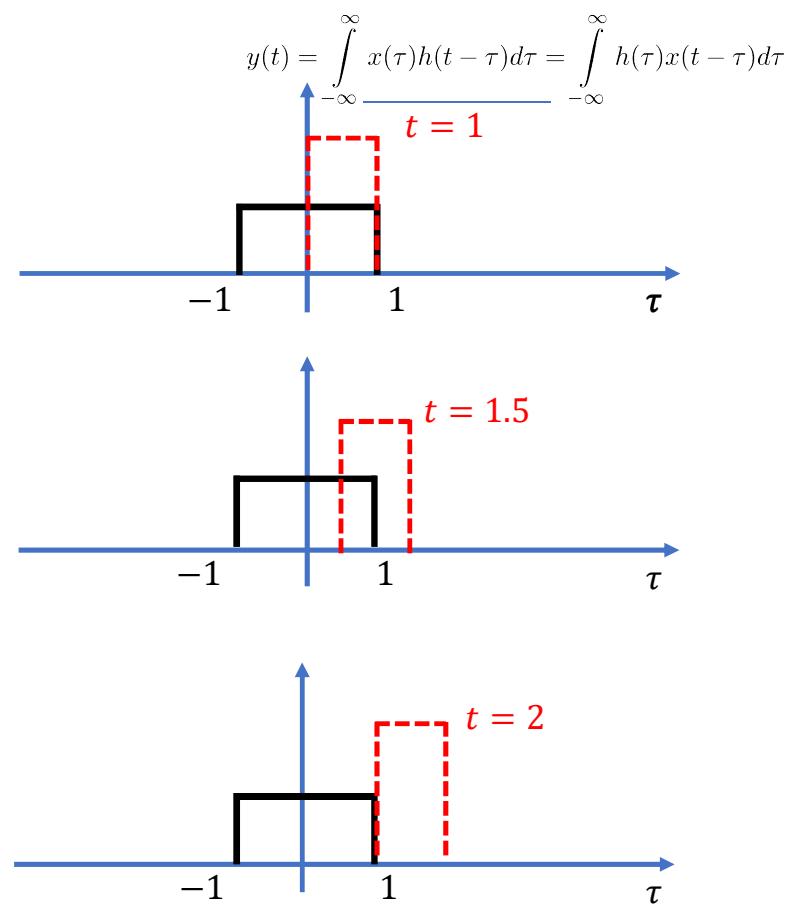
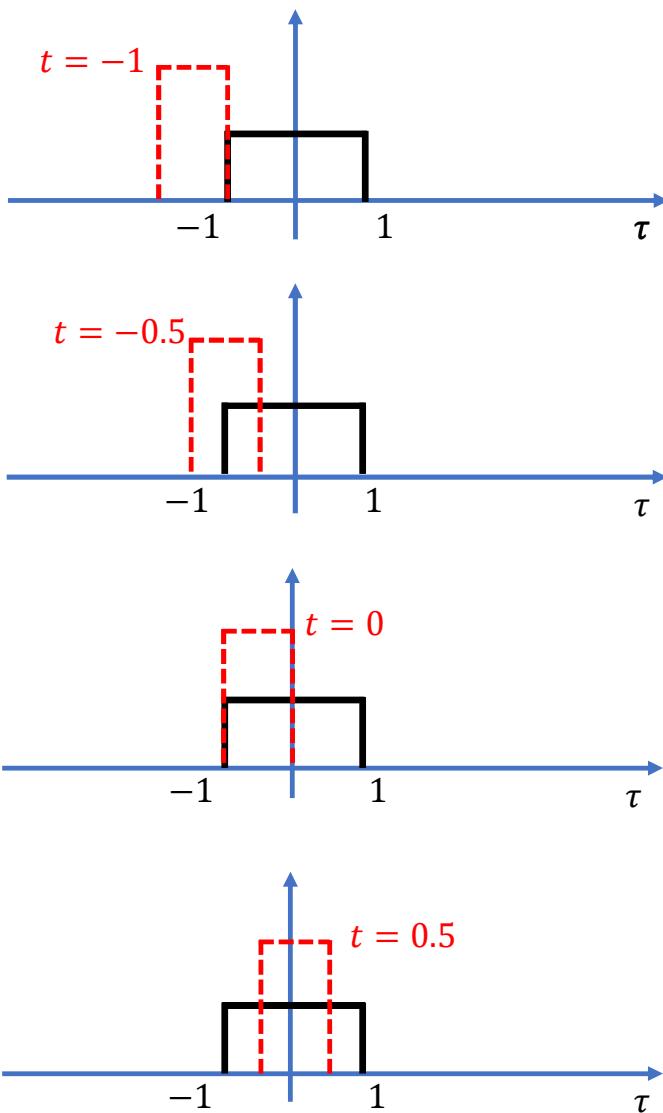


Based on value of t we have different overlapping sections:



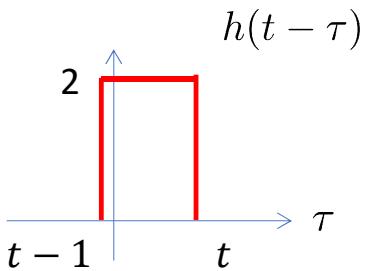
Find the different ranges of t

Convolution

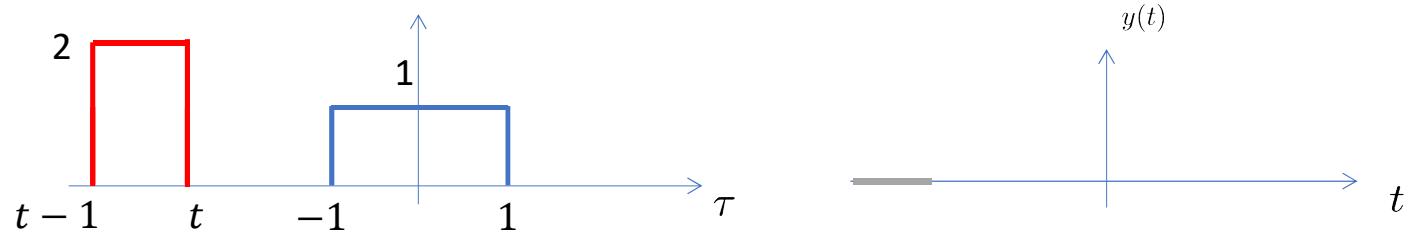


Convolution

1- For $t < -1$ there is no overlap and $y(t) = 0$.

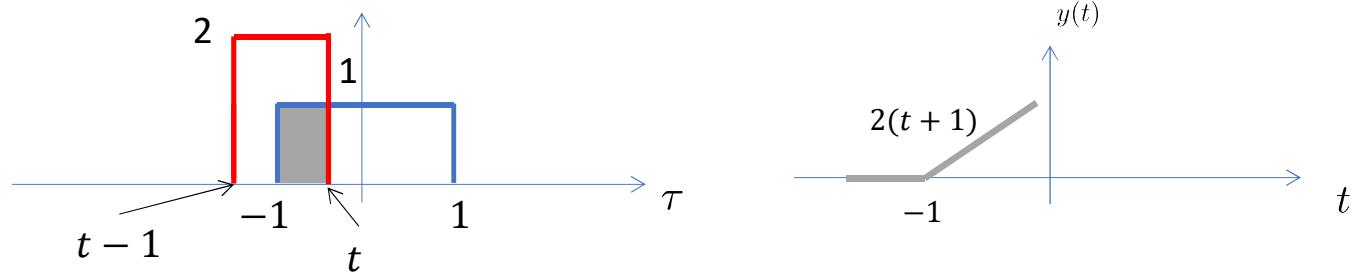


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$



2- For $-1 < t < 0$ there is a partial overlap

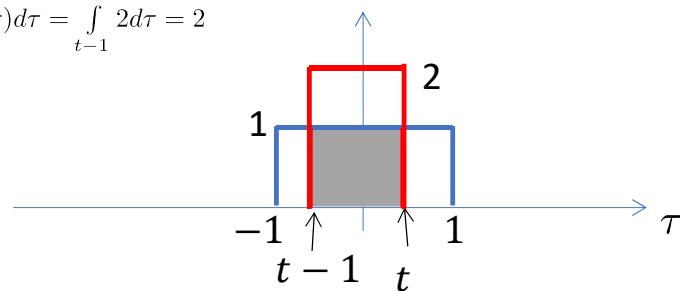
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-1}^t 2d\tau = 2(t + 1)$$



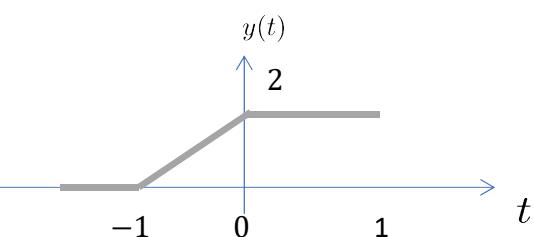
Convolution

3- For $0 < t < 1$ we have a full overlap

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{t-1}^t 2d\tau = 2$$

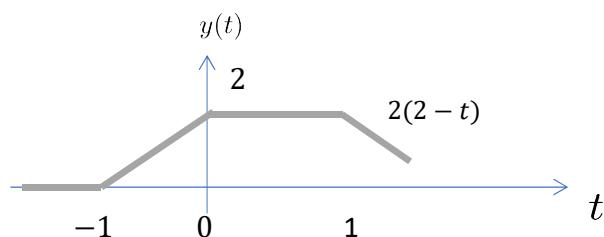
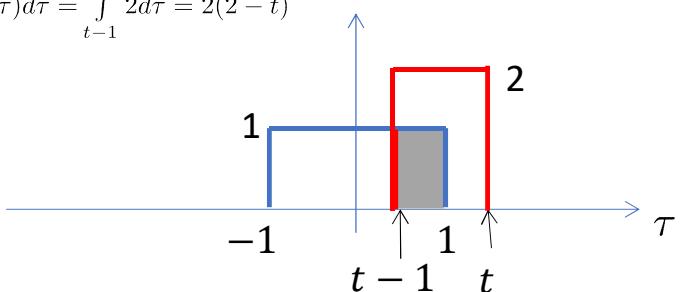


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$



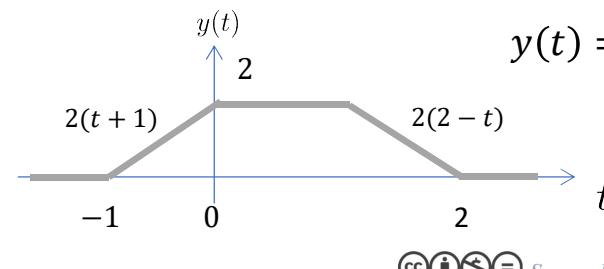
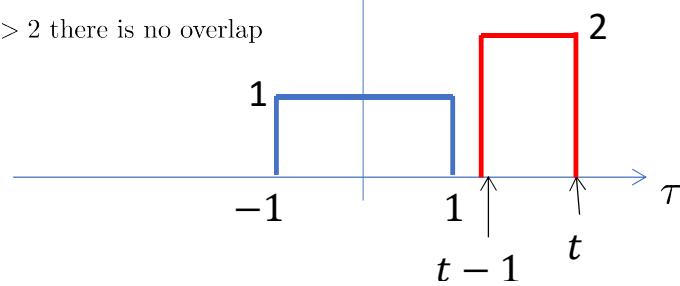
4- For $0 < t - 1 < 1 \rightarrow 1 < t < 2$ there is a partial overlap:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{t-1}^1 2d\tau = 2(2 - t)$$



5- For $t - 1 > 1 \rightarrow t > 2$ there is no overlap

$$y(t) = 0.$$

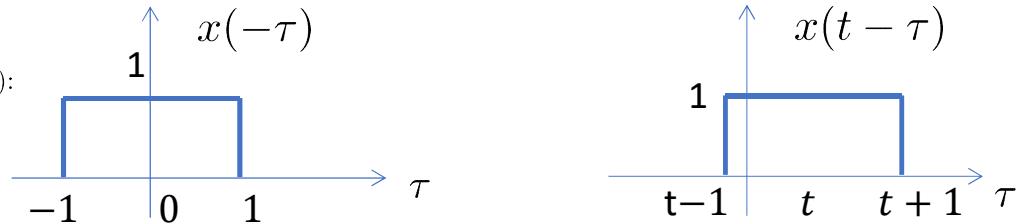


$$y(t) = x(t) * h(t)$$

Convolution

Try the same problem, this time by flipping and shifting $x(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau = h(t) * x(t)$$



Note: Width of convolution of two finite width signals with widths of T_1 and T_2 is always $T_1 + T_2$

In addition if one signal starts at t_1 and the other signal starts at t_2 , the convolution of two signals starts at $t_1 + t_2$.

Consequently, if the two signals end at t_3 and t_4 the convolution ends at $t_3 + t_4$.

Check for this example!

...and for all convolutions you solve

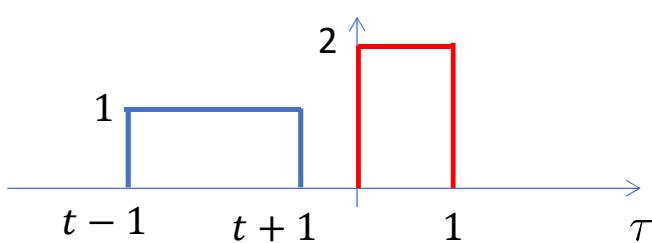


Soosan Beheshti, Toronto Metropolitan University

Convolution

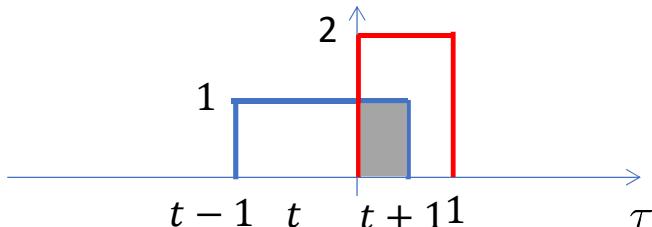
Sliding $x(t - \tau)$ over $h(t)$:

1- For $t + 1 < 0 \rightarrow t < -1$ there is no overlap, therefore, $y(t) = 0$.

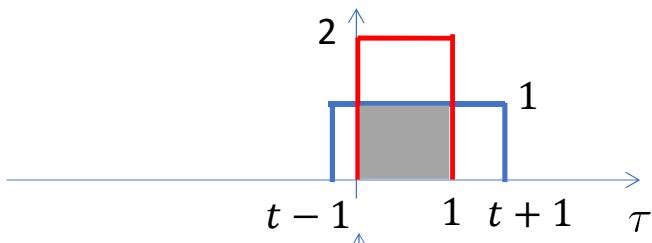


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

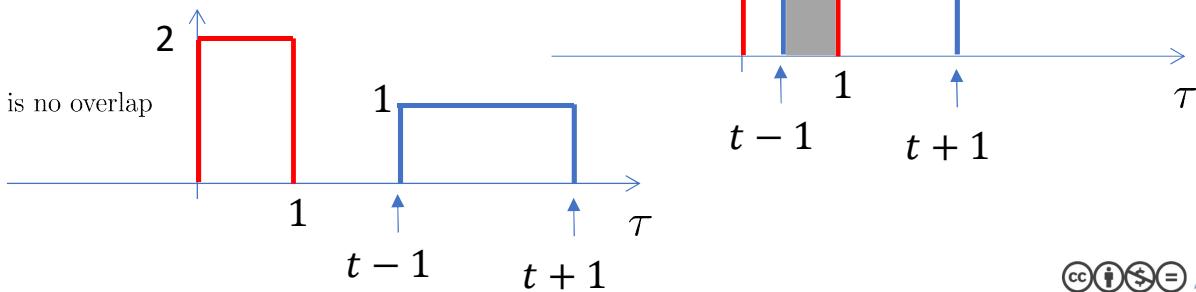
2- For $0 < t + 1 < 1 \rightarrow -1 < t < 0$ there is partial overlap



3- For $t + 1 > 1$ and $t - 1 < 0 \rightarrow 0 < t < 1$ there is a full overlap



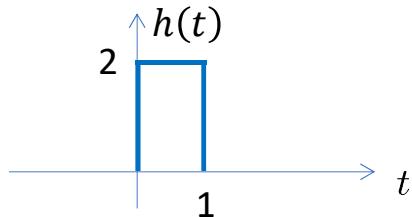
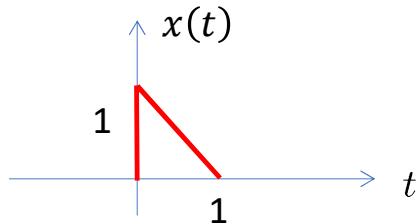
4- For $0 < t - 1 < 1 \rightarrow 1 < t < 2$ there is a partial overlap



5- For $t - 1 > 1 \rightarrow t > 2$ there is no overlap

Convolution

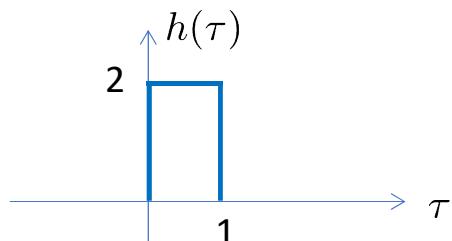
Example 4: Find the output of previous system $h(t)$ to the following input $x(t)$:



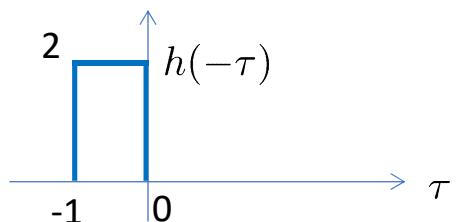
It is up to you to flip and shift $x(t)$ or $h(t)$. Start with the one that is easier!
For example here we start with $h(t)$ as it seems to be easier.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

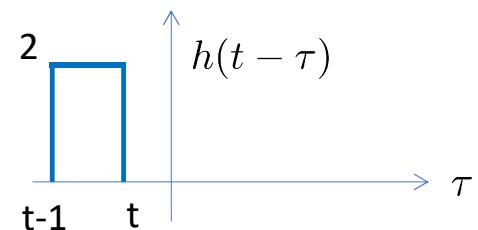
Step 1: Draw $h(\tau)$ by changing
the independent variable from t to τ



Step 2: Flip $h(\tau)$ to show $h(-\tau)$



3- Move the value $h(-\tau)$ at zero to t to build $h(t - \tau)$



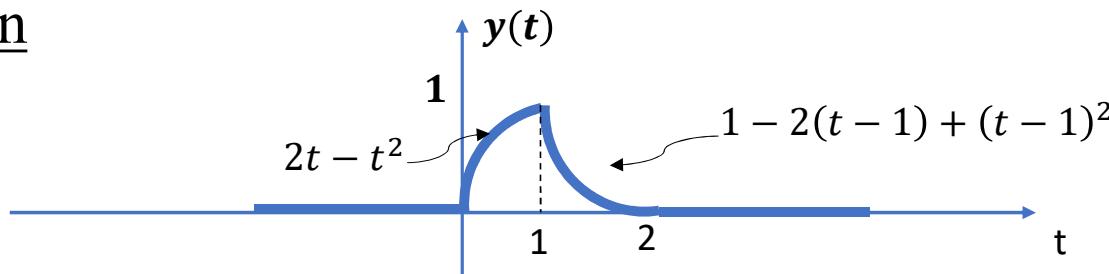
Convolution

Final answer:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

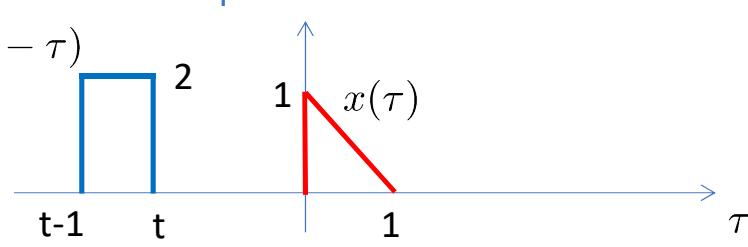
Solution:

1- For $t < 0$ there is no overlap and $y(t) = 0$



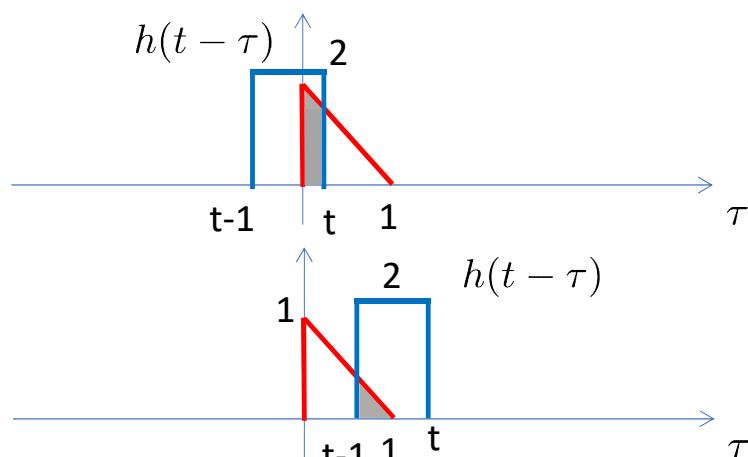
2- For $0 < t < 1$ there is a partial overlap and

$$y(t) = \int_0^t 2(-\tau + 1)d\tau = 2t - t^2$$

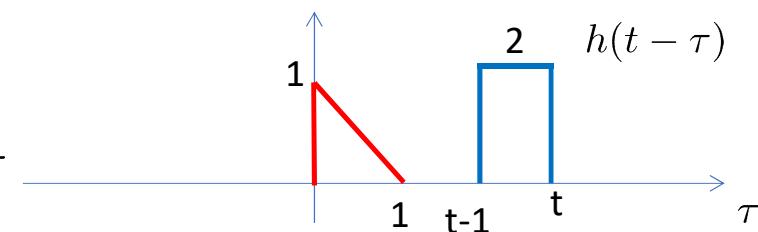


3- For $1 < t < 2$ there is a partial overlap and

$$y(t) = \int_{t-1}^1 2(-\tau + 1)d\tau = 1 - 2(t-1) + (t-1)^2$$



4- For $t > 2$ there is no overlap and $y(t) = 0$



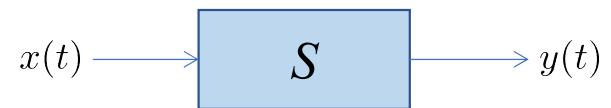
Try flipping and shifting $x(t)$ and verify that you get the same answer.

Initial Condition and LTIDE systems

$$(D^N + a_1 D^{N-1} + \dots + a_N) y(t) = (b_{N-M} D^M + b_{N-M-1} D^{M-1} + \dots + b_N) x(t)$$

The convolution answer is for causal LTI system **at rest or zero state (ZS)** system which has input $x(t)$ and impulse response $h(t)$. In this case it is assumed that $y(0)$ and derivatives of $y(t)$ up to order $N - 1$ at zero are zero:

$$y(0) = 0, y^{(1)}(0) = 0, \dots, y^{(N-1)}(0) = 0$$



The output of the system in this case is denoted as y_{zs} where ZS is for zero state:

$$y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$$

In presence of initial condition we have at least one nonzero values for $y(0)$ and its derivatives up to $N - 1$ th order. We find the output of the system for its initial conditions separately. This output is denoted by zero input response y_{zir} and you set the characteristic function $C(t)$ to the values provided for $y(0), y^{(1)}(0), \dots, y^{(N-1)}(0)$ at zero to find the coefficients. Using those coefficients, the system response after $t = 0$ is then:

$$y_{zir}(t) = C(t)u(t)$$

The final answer for having input and nonzero value for states (initial conditions) is

$$y(t) = y_{zs}(t) + y_{zir}(t)$$

For example for the case of non repeated roots Characteristic function is $C(t) = (c_1 e^{\lambda_1 t} + \dots + c_N e^{\lambda_N t})$