# Quiz-6 answers and solutions

Coursera. Stochastic Processes

December 30, 2020

1. Let $X_t = \cos(\omega t + \theta)$ be a stochastic process and $\theta \sim \mathrm{Unif}[0, 2\pi]$, $\omega = \pi/10$. Is this process ergodic? Is it stationary?

   **Answer:** It is ergodic and weakly stationary.

   **Solution:**

   Since the distribution of $X_t$ is symmetric, its mean is 0.

$$\begin{aligned} K(t,s) &= \mathbb{E}(cos(wt+\theta)cos(ws+\theta)) \\ &= \frac{1}{2}\mathbb{E}cos(w(t-s)) + \frac{1}{2}\mathbb{E}cos(w(t+s)+2\theta) \\ &= \gamma(t-s) + \mathbb{E}cos(w(t+s))\mathbb{E}cos(2\theta) - \mathbb{E}sin(w(t+s))\mathbb{E}sin(2\theta) \\ &= \gamma(t-s), \end{aligned}$$

   because the means of $cos(2\theta)$ and $sin(2\theta)$ are equal to 0. Consequently, this process is weakly stationary.

   To prove that it is also ergodic, we need to look at the following:

$$\begin{aligned} \frac{1}{T}\sum_{r=0}^{T}\gamma(r) &= \frac{1}{2T}(cos0 + \cdots + cos\frac{Tw}{10}) \\ &\le \frac{c_w}{2T} \to 0, \end{aligned}$$

   where $c_w$ depends on $w$. For instance, if $w = 10$, then $c_w \ge 5$, just $c_w = 5$ to have a sharp bound.

   Therefore, this process is ergodic.

2. Let $X_t = \varepsilon_t + \xi\cos(\pi t/12)$, $t = 1, 2, ...$, where $\xi, \varepsilon_1, \varepsilon_2, ...$ are i.i.d. standard normal random variables. Choose the correct statement.

   **Answer:** $X_t$ is not weakly stationary, but it is ergodic.

   **Solution:**

   The mean of the process is, obviously, nil, however, its covariance function is equal to: $K(t,s) = \mathbb{1}\{t=s\}\operatorname{Var}\xi_t + \operatorname{cov}(\xi\cos(\pi t/12), \xi\cos(\pi s/12)) = \mathbb{1}\{t-s=0\} + \cos(\pi t/12)\cos(\pi s/12)$, which cannot be presented as a function on $(t-s)$. Thus, it is not stationary.

$$\mathbb{E}\frac{1}{T}\sum_{t=0}^{T}(\varepsilon_t + \xi\cos(\pi t/12)) = 0$$

$$
\begin{aligned}
\operatorname{Var} \frac{1}{T} \sum_{t=0}^{T} (\varepsilon_t + \xi \cos(\pi t/12)) &= \frac{1}{T^2} \left( T + \sum_{t=1}^{T} \cos^2(\pi t/12) \right) \\
&= \frac{1}{T} + \frac{1}{T^2} \sum_{t=1}^{T} \cos^2(\pi t/12) \\
&\leq \frac{1}{T} + \frac{T}{T^2} = \frac{2}{T} \to 0
\end{aligned}
$$

as $T \to \infty$.

Therefore, the process is not stationary, but is ergodic, because $\mathbb{E}\frac{1}{T}\sum_{t=0}^{T} X_t \to$ *const*.

3. Assume that for a process $X_t$ it is known that $\mathbb{E}[X_t] = \alpha + \beta t$, $\operatorname{cov}(X_t, X_{t+h}) = e^{-h\lambda}$ for all $h \geq 0$, $t > 0$, and some constants $\lambda > 0$, $\alpha, \beta$. Is the process $Y_t = X_{t+1} - X_t$ stationary and ergodic?

   **Answer:** $Y_t$ is weakly stationary and ergodic.

   **Solution:**

   $\mathbb{E}[X_{t+1} - X_t] = \beta$ does not depend on time. And, $Y_t = X_{t+1} - X_t$, clearly, has an autocovariance function. Hence, it is weakly stationary. A strict stationarity is not the case, because, for instance, $Y_0 = X_1$ and $Y_{100} = X_{101} - X_{100}$ have different distribution laws:

$$
\operatorname{Var} X_1 = 1
$$

$$
\operatorname{Var}(X_{101} - X_{100}) = 1 + 1 - 2\operatorname{cov}(X_{101}; X_{100}) = 2 - 2e^{-\lambda}.
$$

   Additionally,

$$
\frac{1}{T}\sum_{t=0}^{T}(Y_t) = \frac{X_{T+1} - X_0}{T} \to const
$$

   as $T \to \infty$, so it is ergodic.

4. Let $X_t = \sigma W_t + ct$, where $W_t$ is Brownian motion, $\sigma$, $c > 0$. Choose the correct statements about this process:

   **Answer:** $X_t$ has continuous trajectories.

5. Consider the process $X_n = X_{n-1} + \xi_n$, $\quad X_0 = 0, \quad \xi_n \sim \mathcal{N}(0,1) \quad \forall n \in \mathbb{N}$. Choose the correct statements about this process.

   **Answer:** $\mathbb{E}\frac{1}{T}\sum_{t=1}^{T} X_t = 0$, $\operatorname{Var}\left(\frac{1}{T}\sum_{t=1}^{T} X_t\right) = \frac{(T+1)(2T+1)}{6T}$

   **Solution:**

   Assuming $T \geq 3$,

$$
\mathbb{E}\frac{1}{T}\sum_{t=1}^{T} X_t = \frac{1}{T}\mathbb{E}[\xi_1 + \xi_1 + \xi_2 + \ldots] = 0
$$

$$
\begin{aligned}
\operatorname{Var}\left(\frac{1}{T}\sum_{t=1}^{T} X_t\right) &= \frac{1}{T^2}\operatorname{Var}(\xi_1+\xi_1+\xi_2+\xi_1+\xi_2+\xi_3+\dots) \\
&= \frac{1}{T^2}\operatorname{Var}(T\xi_1+(T-1)\xi_2+(T-2)\xi_3+\dots+\xi_T) \\
&\overset{i.i.d.}{=} \frac{1}{T^2}(T^2\operatorname{Var}(\xi_1)+(T-1)^2\operatorname{Var}(\xi_2)+\dots) = \frac{1}{T^2}\sum_{t=1}^{T} t^2 \\
&= \frac{T(T+1)(2T+1)}{6T^2} = \frac{(T+1)(2T+1)}{6T} \underset{T\to\infty}{\to} \infty,
\end{aligned}
$$

$\Rightarrow$ the process is not ergodic.

6. Is it true that if two processes $X_t$ and $Y_t$ have the same covariance functions and $X_t$ is ergodic, then $Y_t$ is also ergodic?

   **Answer:** without any additional information — no

   **Solution:** The process $X_t$ is ergodic, if there exists such constant $c$ that $\frac{1}{T}\sum\limits_{t=1}^{T} X_t \overset{\mathbb{P}}{\to} c$ as $T \to \infty$, for which it is necessary that $\mathbb{E}\frac{1}{T}\sum\limits_{t=1}^{T} X_t \underset{T\to\infty}{\to}$ const and $\operatorname{Var}\left(\frac{1}{T}\sum\limits_{t=1}^{T} X_t\right) \underset{T\to\infty}{\to} 0$. While from ergodicity of $X_t$ the second property is fulfilled for $Y_t$, it is still possible that $\mathbb{E}\frac{1}{T}\sum\limits_{t=1}^{T} Y_t \to \infty$: for instance, take $X_t = \xi_t$ and $Y_t = t+\xi_t$, where $\xi_i$ are i.i.d. random variables for all $i = 1, 2, \dots$.

7. Let $X_t = e^{-t}W_{e^{2t}}$, where $W_t$ is a Brownian motion. Find the mean and the covariance function $K(t,s)$ of this process.

   **Answer:** $\mathbb{E}X_t = 0, \quad K(t,s) = e^{-|t-s|}$

   **Solution:**
$$
\mathbb{E}X_t = \mathbb{E}e^{-t}W_{e^{2t}} = e^{-t}\mathbb{E}W_{e^{2t}} = e^{-t}\cdot 0 = 0
$$

$$
\begin{aligned}
K(t,s) = \operatorname{cov}(e^{-t}W_{e^{2t}}, e^{-s}W_{e^{2s}}) &= e^{-(t+s)}\min\{e^{2t}, e^{2s}\} \\
&= \begin{cases} e^{-t+s}, & s \le t, \\ e^{-s+t}, & s > t \end{cases} \\
&= e^{-|t-s|}.
\end{aligned}
$$

8. Choose the correct statements about the process $X_t = e^{-t}W_{e^{2t}}$, where $W_t$ is a Brownian motion.

   **Answer:** $X_t$ is stationary, ergodic and continuous in the mean-squared sense

   **Solution:**
$$
\mathbb{E}X_t = 0, \Rightarrow \mathbb{E}\frac{1}{T}\sum_{t=1}^{T} X_t = 0
$$

$$
\begin{aligned}
K(t,s) = \operatorname{cov}(e^{-t}W_{e^{2t}}, e^{-s}W_{e^{2s}}) &= e^{-(t+s)}\min\{e^{2t}, e^{2s}\} \\
&= \begin{cases} e^{-t+s}, & s \leq t, \\ e^{-s+t}, & s > t \end{cases} \\
&= e^{-|t-s|}.
\end{aligned}
$$

Thus, the process is weakly stationary, and since $\gamma(t-s) = \gamma(u) = e^{-|u|} \underset{u\to\infty}{\to} 0$, it is also ergodic. Now, as $K(t,s)$ is continuous at $(t_0, t_0)$, $X_t$ is continuous in the mean-squared sense, and since $\nexists \gamma''(0)$, it is not stochastically differentiable.