Quiz-1 answers and solutions

Coursera. Stochastic Processes

March 11, 2021

1. Let ξ be a random variable with distribution function $F_\xi(\cdot)$ supported on $(0,\infty)$. Find the distribution function of finite dimensional distribution of the process $X_t = a(1+t)^\xi, \quad a,t>0$.

   Answer: $F_\xi\left(\min\left\{\frac{\log\frac{x_1}{a}}{\log(1+t_1)},\dots,\frac{\log\frac{x_n}{a}}{\log(1+t_n)}\right\}\right)$

   Solution:

$$
\begin{aligned}
F_{\vec{X}}(\vec{x}) &= \mathbb{P}\{X_{t_1}\le x_1,\dots,X_{t_n}\le x_n\}\\
&= \mathbb{P}\{a(1+t_1)^\xi \le x_1,\dots,a(1+t_n)^\xi\le x_n\}\\
&= \mathbb{P}\{\log a+\xi\log(1+t_1)\le \log x_1,\dots,\log a+\xi\log(1+t_n)\le\log x_n\}\\
&= \mathbb{P}\left\{\xi\le\frac{\log x_1-\log a}{\log(1+t_1)},\dots,\xi\le\frac{\log x_n-\log a}{\log(1+t_n)}\right\}\\
&= \mathbb{P}\left\{\xi\le\min\left\{\frac{\log\frac{x_1}{a}}{\log(1+t_1)},\dots,\frac{\log\frac{x_n}{a}}{\log(1+t_n)}\right\}\right\}\\
&= F_\xi\left(\min\left\{\frac{\log\frac{x_1}{a}}{\log(1+t_1)},\dots,\frac{\log\frac{x_n}{a}}{\log(1+t_n)}\right\}\right)
\end{aligned}
$$

2. Let ξ be a random variable with distribution function $F_\xi(\cdot)$ supported on $(0,\infty)$. Consider the processes $X_t=\log(\xi a t)$, $Y_t=\xi^{at}$ and $Z_t = at\log\xi$, $a,t>0$. Which of them have the same finite dimensional distributions?

   Answer: All the processes have different finite-dimensional distributions

   Solution: Let us calculate the finite dimensional distributions for each process.

   For the process X_t we have

$$
\begin{aligned}
F_{\vec{X}}(\vec{x}) &= \mathbb{P}\{X_{t_1}\le x_1,\dots X_{t_n}\le x_n\}\\
&= \mathbb{P}\{\log(\xi a t_1)\le x_1,\dots,\log(\xi a t_n)\le x_n\}\\
&= \mathbb{P}\left\{\xi a t_1\le e^{x_1},\dots,\xi a t_n\le e^{x_n}\right\}\\
&= \mathbb{P}\left\{\xi\le\frac{e^{x_1}}{at_1},\dots,\xi\le\frac{e^{x_n}}{at_n}\right\}\\
&= F_\xi\left(\min\left\{\frac{e^{x_1}}{at_1},\dots,\frac{e^{x_n}}{at_n}\right\}\right).
\end{aligned}
$$

For Y_t

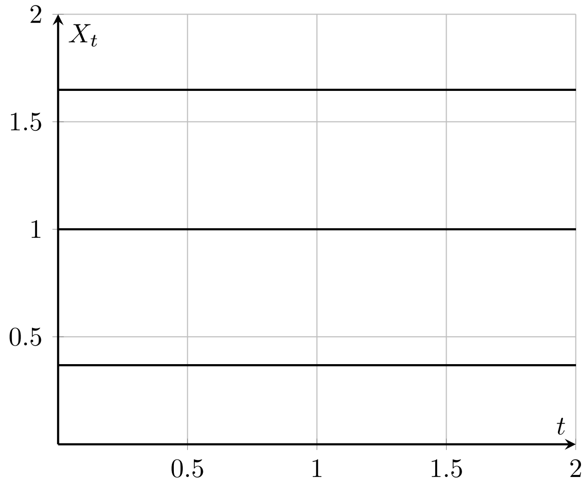
$$
\begin{aligned}
F_{\vec{Y}}(\vec{x}) &= \mathbb{P}\{Y_{t_1} \le x_1, \dots Y_{t_n} \le t_n\} \\
&= \mathbb{P}\{\xi^{at_1} \le x_1, \dots \xi^{at_n} \le x_n\} \\
&= \mathbb{P}\left\{\log\xi \le \frac{\log x_1}{at_1}, \dots \log\xi \le \frac{\log x_n}{at_n}\right\} \\
&= \mathbb{P}\left\{\xi \le \exp\left\{\frac{\log x_1}{at_1}\right\}, \dots, \xi \le \exp\left\{\frac{\log x_n}{at_n}\right\}\right\} \\
&= F_\xi\left(\min\left\{\exp\left\{\frac{\log x_1}{at_1}\right\}, \dots, \exp\left\{\frac{\log x_n}{at_n}\right\}\right\}\right),
\end{aligned}
$$

and for Z_t

$$
\begin{aligned}
F_{\vec{Z}}(\vec{x}) &= \mathbb{P}\{Z_{t_1} \le x_1, \dots Z_{t_n} \le t_n\} \\
&= \mathbb{P}\{at_1 \log\xi \le x_1, \dots at_n \log\xi \le x_n\} \\
&= \mathbb{P}\left\{\log\xi \le \frac{x_1}{at_1}, \dots, \log\xi \le \frac{x_n}{at_n}\right\} \\
&= \mathbb{P}\left\{\xi \le \exp\left\{\frac{x_1}{at_1}\right\}, \dots, \xi \le \exp\left\{\frac{x_1}{at_1}\right\}\right\} \\
&= F_\xi\left(\min\left\{\exp\left\{\frac{x_1}{at_1}\right\}, \dots, \exp\left\{\frac{x_n}{at_n}\right\}\right\}\right).
\end{aligned}
$$

3. Consider the process $X_t = e^{\xi}, \xi \sim \mathcal{N}(0,1)$. Which of the following pictures represent the possible trajectories of this process?

   Answer:

   Solution: Once ξ is fixed, $X_t = e^{\xi} = \text{const}\ \forall t > 0$.

4. Consider the renewal process $S_n = S_{n-1} + \xi_n, \quad n \in \mathbb{N}$, where ξ is a random variable taking values $k \in \{1,2,3\}$ with probabilities $p \in \{1/2, 1/4, 1/4\}$, respectively. Calculate $\mathbb{E}[N_t]$ at time moment $t = 2$.

   Answer: 1

Solution:

$$\begin{aligned}
\mathbb{E}[N_2] &= \sum_{k=0}^{\infty} k \cdot \mathbb{P}\{N_2 = k\} = 0 + 1 \cdot \mathbb{P}\{N_2 = 1\} + 2 \cdot \mathbb{P}\{N_2 = 2\} \\
&= 1 \cdot (\mathbb{P}\{\xi_1 = 1, \xi_2 = 2\} + \mathbb{P}\{\xi_1 = 1, \xi_2 = 3\} + \mathbb{P}\{\xi_1 = 2\}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + 2 \cdot \mathbb{P}\{\xi_1 = 1, \xi_2 = 1\} \\
&= 1/8 + 1/8 + 1/4 + 1/2 = 1.
\end{aligned}$$

5. Consider the renewal process $S_n = S_{n-1} + \xi_n, \quad n \in \mathbb{N}$ with $\xi \sim p_\xi(x) = \lambda^2 x e^{-\lambda x} \cdot \mathbb{I}\{x \geq 0\},\ \lambda > 0$. Find the mathematical expectation of the corresponding counting process N_t.

   Answer: $\dfrac{\lambda t}{2} - \dfrac{1}{4} + \dfrac{1}{4} e^{-2\lambda t}$

   Solution:

   1) $p(x) \to \mathcal{L}_p(s)$:

$$\begin{aligned}
\mathcal{L}_p(s) &= \int_0^\infty \lambda^2 x e^{-\lambda x} e^{-sx}\, dx = \lambda^2 \int_0^\infty x e^{-(\lambda+s)} x\, dx \\
&= \lambda^2 \left(-\frac{x e^{-(\lambda+s)x}}{\lambda+s}\Big|_0^\infty + \int_0^\infty \frac{e^{-(\lambda+s)}}{\lambda+s}\, dx \right) \\
&= \frac{\lambda^2}{(\lambda+s)^2};
\end{aligned}$$

   2) $\mathcal{L}_p(s) \to \mathcal{L}_U(s)$:

$$\begin{aligned}
\mathcal{L}_U(s) &= \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))} = \frac{\lambda^2}{(\lambda+s)^2} \left(s \cdot \frac{\lambda^2 + 2\lambda s + s^2 - \lambda^2}{(\lambda+s)^2} \right)^{-1} \\
&= \frac{\lambda^2}{s^2(s+2\lambda)};
\end{aligned}$$

   3) $\mathcal{L}_U(s) \to U(t)$:

$$\mathcal{L}_U(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2\lambda} = \frac{s^2(B+C) + s(A + 2B\lambda) + 2A\lambda}{s^2(s+2\lambda)}$$

$$\begin{cases} B + C = 0 \\ A + 2B\lambda = 0 \\ 2A\lambda = \lambda^2, \end{cases} \quad \Rightarrow A = \lambda/2, B = -1/4, C = 1/4,$$

$$\Rightarrow \mathcal{L}_U(s) = \frac{\lambda}{2s^2} - \frac{1}{4s} + \frac{1}{4(s+2\lambda)}$$

$$\Rightarrow U(t) = \frac{\lambda t}{2} - \frac{1}{4} + \frac{1}{4} e^{-2\lambda t} = \mathbb{E}[N_t].$$

6. Let ξ and η be two independent random variables with densities $p_\xi(x) = \lambda^2 x e^{-\lambda x}$ and $p_\eta(x) = \lambda e^{-\lambda x}$, $\lambda > 0$, $x \geq 0$. Find the density of $\xi + \eta$, $x \geq 0$.

   Answer: $p_{\xi+\eta}(x) = \dfrac{\lambda^3}{2} x^2 e^{-\lambda x}$

   Solution: Since ξ and η are independent, the density of their sum can be calculated by convolution:

$$\begin{aligned}
p_{\xi+\eta}(x) &= \int_{\mathbb{R}} p_\xi(x-y) p_\eta(y)\, dy \\
&= \int_{\mathbb{R}} \lambda^2 (x-y) e^{-\lambda(x-y)} \cdot \mathbb{I}_{\{x-y \geq 0\}} \cdot \lambda e^{-\lambda y} \cdot \mathbb{I}_{\{y \geq 0\}}\, dy \\
&= \lambda^3 \int_0^x (x-y) e^{-\lambda x}\, dy = \lambda^3 e^{-\lambda x} \left(x \int_0^x dy - \int_0^x y\, dy \right) \\
&= \frac{\lambda^3}{2} x^2 e^{-\lambda x}.
\end{aligned}$$

7. Let ξ and η be two random variables. It is known that η has a continuous symmetric distribution, that is, $\mathbb{P}\{\eta > x\} = \mathbb{P}\{\eta < -x\}$ for any $x > 0$. Find the probability that the trajectories of the process $X_t = (t^2+t-1)e^\xi \eta$ decrease for all $t \geq 0$.

   Answer: 1/2

   Solution:

$$\begin{aligned}
\mathbb{P}\left\{ \frac{d}{dt} X_t < 0 \quad \forall t \geq 0 \right\} &= \mathbb{P}\{2te^\xi \eta + e^\xi \eta < 0 \quad \forall t \geq 0\} \\
&= \mathbb{P}\{(2t+1)e^\xi \eta < 0 \quad \forall t \geq 0\} \\
&= \mathbb{P}\{\eta < 0\} = 1/2
\end{aligned}$$

   since $e^\xi > 0$ for any distribution of ξ.

8. The crashes of some mobile app occur according to the renewal process with increments having exponential distribution with parameter $\lambda = 7$. What is the asymptotic behaviour of the number of mobile application crashes in t months when $t \to \infty$?

   Answer: $\frac{N_t}{t} \underset{t\to\infty}{\to} 7$

   Solution: By the limit theorem for a renewal process $S_n = S_{n-1} + \xi_n, \quad \xi_1, \xi_2, \dots \sim$ i. i. d. $> 0, n \in \mathbb{N}$, if $\mathbb{E}\xi_1 = \mu < \infty$ we have that

$$\frac{N_t}{t} \underset{t\to\infty}{\to} \frac{1}{\mu} a.s.$$

   Thus, since $\mathbb{E}\xi_1 = 1/\lambda = 1/7$,

$$\frac{N_t}{t} \underset{t\to\infty}{\to} 7.$$

9. The calls to the call centre are received according to the renewal process with increments $\xi_n \sim \text{i. i. d.}\, exp(\lambda)$, $\lambda = 90\ \forall n \in \mathbb{N}$. The manager decides to open one more working place if with 95% confidence one can expect approximately 8250 calls to be received in $t = 90$ days. Should the manager open one more working place?

   Hint: the 0.975- and 0.025-quantiles of the standard normal distribution are $Z_{0.975} = -Z_{0.025} \approx 1.96$.

   Answer: Yes

   Solution: By the limit theorem for renewal processes, if $\mu = \mathbb{E}[\xi_1] < \infty, \sigma^2 = \operatorname{Var}\xi_1 < \infty$,

$$\frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \underset{t\to\infty}{\to} \mathcal{N}(0,1).$$

   Thus, one can construct the asymptotic confidence interval for N_t as

$$\mathbb{P}\left\{Z_{\alpha/2} \cdot \frac{\sigma\sqrt{t}}{\mu^{3/2}} + \frac{t}{\mu} \leq N_t \leq Z_{1-\alpha/2} \cdot \frac{\sigma\sqrt{t}}{\mu^{3/2}} + \frac{t}{\mu}\right\} \approx 1-\alpha,$$

   where Z_α is the α-quantile of standard normal distribution.
   Since $\mathbb{E}[\xi_1] = \sqrt{\operatorname{Var}\xi_1} = 1/\lambda = 1/90$, we arrive at

$$\mathbb{P}\left\{-1.96 \cdot \sqrt{\lambda t} + \lambda t \leq N_t \leq 1.96 \cdot \sqrt{\lambda t} + \lambda t\right\} \approx 0.95$$

$$\mathbb{P}\left\{-1.96 \cdot \sqrt{8100} + 8100 \leq N_t \leq 1.96 \cdot \sqrt{8100} + 8100\right\} \approx 0.95$$

$$\mathbb{P}\left\{7923.6 \leq N_t \leq 8276.4\right\} \approx 0.95$$

   As the confidence interval includes 8250, the manager should open one more working place.