Quiz-5 answers and solutions

Coursera. Stochastic Processes

December 30, 2020

1. Let W_t be a Brownian Motion and $h > 0$ be a fixed number. Find a covariance function of the process $X_t = W_{t+h} - W_t$.

   Answer: $K(t,s) = \begin{cases} h - |t-s|, & if |t-s| \le h \\ 0, & if |t-s| > h \end{cases}$

   Solution: $K(t,s) = \operatorname{cov}(W_{t+h} - W_t, W_{s+h} - W_s)$

$$= \begin{cases} 0, & if \quad t > s+h \\ \operatorname{cov}(W_{t+h} - W_{s+h} + W_{s+h} - W_t, W_{s+h} - W_s), & if \quad t \le s+h \end{cases}$$

$$= \begin{cases} 0, & if \quad t > s+h \\ \operatorname{cov}(W_{s+h} - W_t, W_{s+h} - W_t + W_t - W_s), & if \quad t \le s+h \end{cases}$$

$$= \begin{cases} 0, & if \quad t > s+h \\ \operatorname{Var}(W_{s+h} - W_t), & if \quad t \le s+h \end{cases}$$

$$= \begin{cases} 0, & if \quad t > s+h \\ s+h-t, & if \quad t \le s+h \end{cases}$$

2. Let X_t is a process with independent and stationary increments and h is a positive constant. Moreover, $\mathbb{E}X_t = 0$ and $\mathbb{E}X_t^2 < \infty$. Is $Y_t = X_{t+h} - X_t$ a wide-sense stationary process ?

   Answer: Yes

   Hint: If increments of the process X_t are stationary, then X_t is also stationary.

3. Let Y_n be a stochastic process which is defined as follows: $Y_{n+1} = \alpha Y_n + X_n$, $n = 0, 1, ...$ Assume $Y_0 = 0$, $|\alpha| < 1$ and X_n is a sequence of i.i.d. standard normal random variables for $n = 0, 1, 2, ...$ Determine whether Y_n is stationary and find its mean and variance:

   Answer: Y_n is non-stationary, $\mathbb{E}Y_n = 0$, $VarY_n = \dfrac{1-\alpha^{2n}}{1-\alpha^2}$

   Solution: The key for solution lies in the finding the covariance function:

$$
\begin{aligned}
K(t,s) &= Cov(Y_t;Y_s)\\
&= Cov(\alpha^{t-1}X_0+\cdots+\alpha^0X_{t-1};\alpha^{s-1}X_0+\cdots+\alpha^0X_{s-1})\\
&= \alpha^{t-1}\alpha^{s-1}+\alpha^{t-2}\alpha^{s-2}+\cdots+\alpha^{t-s+1}\alpha+\alpha^{t-s}\alpha\\
&= \alpha^{t-s}(\alpha^{2s-2}+\alpha^{2s-4}+\cdots+1)\\
&= \alpha^{t-s}\frac{1-\alpha^{2s}}{1-\alpha^2}.
\end{aligned}
$$

4. Consider the process $X_t = \cos\xi_t + \sin\xi_{t-1}$, where $\xi_t \sim Unif([0,2\pi])$ for all $t \in \mathbb{N}$. Choose the correct statements about this process.

   Answer: $\mathbb{E}X_t = 0$ and X_t is weakly stationary

   Solution:

$$
\mathbb{E}[\cos\xi_t+\sin\xi_{t-1}] = \mathbb{E}\cos\xi_t + \mathbb{E}\sin\xi_{t-1} = 0
$$

$$
\begin{aligned}
K(t,s) &= \operatorname{cov}(\cos\xi_t+\sin\xi_{t-1},\cos\xi_s+\sin\xi_{s-1})\\
&= \mathbb{E}[\cos\xi_t\cos\xi_s]+\mathbb{E}[\cos\xi_t\sin\xi_{s-1}]\\
&\qquad +\mathbb{E}[\sin\xi_{t-1}\cos\xi_s]+\mathbb{E}[\sin\xi_{t-1}\sin\xi_{s-1}]\\
&= (\mathbb{E}\cos^2\xi_t+\mathbb{E}\sin^2\xi_{t-1})\mathbb{I}\{t=s\}+(\mathbb{E}\cos\xi_t\sin\xi_{s-1})\mathbb{I}\{t-s=-1\}\\
&\qquad +(\mathbb{E}\sin\xi_{t-1}\cos\xi_s)\mathbb{I}\{t-s=1\}.
\end{aligned}
$$

   Since

$$
\mathbb{E}\sin\xi_t\cos\xi_t = \frac{1}{2}\mathbb{E}[\sin 2\xi_t] = 0
$$

   and $\mathbb{E}\cos^2\xi_t + \mathbb{E}\sin^2\xi_{t-1} = 1$, we get that $K(t,s) = 1\cdot\mathbb{I}\{t=s\}$, and the process is weakly stationary.

5. Consider the process

$$
X_t = \xi_{t+2} - \xi_{t+1} + \xi_t, \quad \xi \sim \text{i. i. d.}\,\mathcal{N}(0,1) \text{ for all } t = 0,1,2,\ldots.
$$

   Find the mean and the covariance function of this process and determine whether it is weakly stationary.

   Answer: X_t is weakly stationary.

$$
\mathbb{E}[X_t] = 0,
$$

$$
K(t,s) = \begin{cases} 3, & t=s,\\ -2, & |t-s|=1,\\ 1, & |t-s|=2,\\ 0, & |t-s|>2. \end{cases}
$$

   Solution:

$$
\mathbb{E}[X_t] = \mathbb{E}[\xi_{t+2}-\xi_{t+1}+\xi_t] = \mathbb{E}[\xi_{t+2}] - \mathbb{E}[\xi_{t+1}] + \mathbb{E}[\xi_t] = 0,
$$

$$\begin{aligned}\operatorname{cov}(X_t, X_s) &= \operatorname{cov}(\xi_{t+2} - \xi_{t+1} + \xi_t, \xi_{s+2} - \xi_{s+1} + \xi_s) \\ &= (1+1+1)\mathbb{I}\{t=s\} + (-1-1)\mathbb{I}\{|t-s|=1\} + \mathbb{I}\{|t-s|=2\} \\ &= \begin{cases} 3, & t=s, \\ -2, & |t-s|=1, \\ 1, & |t-s|=2, \\ 0, & |t-s|>2. \end{cases}\end{aligned}$$

Since $\mathbb{E}[X_t] = \text{const}$ and $K(t,s)$ depends only on the difference $t-s$, the process is weakly stationary.

6. Consider the process

$$X_t = \xi_{t+2} - \xi_{t+1} + \xi_t, \quad \xi_t \sim \text{i. i. d.} \, \mathcal{N}(0,1) \text{ for all } t = 0, 1, 2, \ldots.$$

Find the spectral density of this process.

Answer:

$$g(u) = \frac{1}{2\pi}(3 - 4\cos u + 2\cos 2u).$$

Solution: Since X_t is weakly stationary and has the autocovariance function

$$\gamma(x) = \begin{cases} 3, & x=0, \\ -2, & |x|=1, \\ 1, & |x|=2, \\ 0, & |x|>2, \end{cases}$$

the spectral density $g(u)$ can be calculated as

$$g(u) = \frac{1}{2\pi}\left(3 - 2e^{iu} - 2e^{-iu} + e^{2iu} + e^{-2iu}\right) = \frac{1}{2\pi}\left(3 - 4\cos u + 2\cos 2u\right).$$

7. Find the spectral density of the process

$$Y_t = X_t + X_{t-1} + X_{t-2},$$

where

$$X_t = \xi_{t+2} - \xi_{t+1} + \xi_t, \quad \xi_t \sim \text{i. i. d.} \, \mathcal{N}(0,1) \text{ for all } t = 0, 1, 2, \ldots.$$

Answer: $g_Y(u) = \frac{1}{2\pi}\left(9 + 12\cos 2u - 16\cos^2 u + 4\cos^2 2u\right)$.

Solution: Since Y_t is the linear filter for the process X_t, its spectral density can be calculated as

$$g_Y(u) = g_X(u)|\mathcal{F}[\rho](u)|^2,$$

where $\rho(h) = \mathbb{1}\{h=0\} + \mathbb{1}\{h=1\} + \mathbb{1}\{h=2\}$ and $g_X(u)$ is the spectral density of X_t. Thus,

$$\mathcal{F}[\rho](u) = 1 + e^{iu} + e^{2iu}$$

and

$$
\begin{aligned}
|\mathcal{F}[\rho](u)|^2 &= \mathcal{F}\bar{\mathcal{F}}(u) \\
&= (1+e^{iu}+e^{2iu})(1+e^{-iu}+e^{-2iu}) = 3+2(e^{iu}+e^{-iu})+(e^{2iu}+e^{-2iu}) \\
&= 3+4\cos u+2\cos 2u
\end{aligned}
$$

and

$$
\begin{aligned}
g_Y(u) &= \frac{1}{2\pi}\,(3-4\cos u+2\cos 2u)\,(3+4\cos u+2\cos 2u) \\
&= \frac{1}{2\pi}\left(9+12\cos 2u-16\cos^2 u+4\cos^2 2u\right).
\end{aligned}
$$