Quiz-3 answers and solutions

Coursera. Stochastic Processes

December 30, 2020

1. Consider the Markov chain with the one-step transition matrix

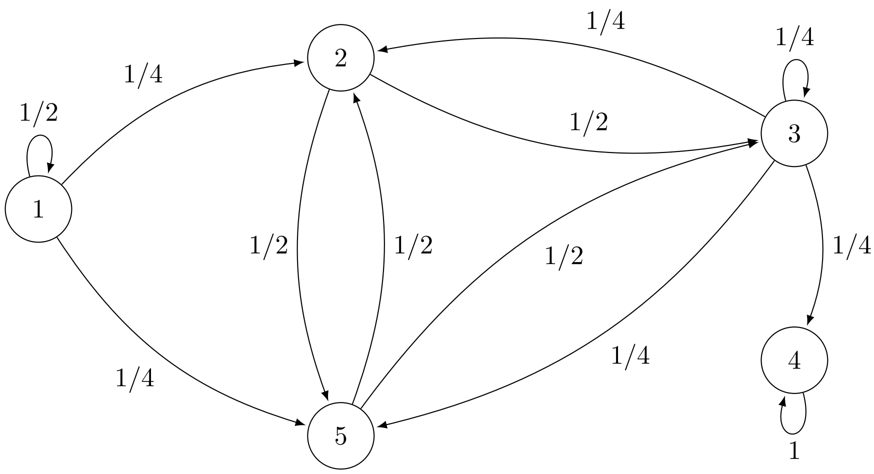
$$P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

   How many classes of equivalence does this chain have?

   Answer: 3

   Solution: The first class of equivalence is formed by the state 1, since it is not accessible from any other state except itself. Another equivalence class is the state 4 which does not give access to any other state. The last class of equivalence consists of states 2, 3 and 5, since all of them communicate.

2. The graphical representation of this chain has the following form



3. What is the maximum period of states in the Markov chain with the one-step transition matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}?$$

Answer: 1

Solution: States 1, 3 and 4 are aperiodic, since they are accessible from themselves. As the states 2 and 5 communicate with 3, they are also aperiodic.

4. How many states in the Markov chain with the one-step transition matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

are transient?

Answer: 4

Solution: The state 4 is accessible from any other state and gives access only to itself, thus, all the states except 4 are transient.

5. Find the stationary distribution of the Markov chain with the one-step transition matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

Answer: $\vec{\pi}^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Solution: The stationary distribution can be found as the solution to the equation $\vec{\pi}^* P = \vec{\pi}^*$. Let us denote $\vec{\pi}^* = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix}$. Then one should solve the system

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0. \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix},$$

from which

$$\begin{cases} \frac{\pi_1}{2} = \pi_1 \\ \frac{\pi_1}{4} + \frac{\pi_3}{4} + \frac{\pi_5}{2} = \pi_2 \\ \frac{\pi_2}{2} + \frac{\pi_3}{4} + \frac{\pi_5}{2} = \pi_3 \\ \frac{\pi_3}{4} + \pi_4 = \pi_4 \\ \frac{\pi_1}{4} + \frac{\pi_2}{2} + \frac{\pi_3}{4} = \pi_5 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1 \end{cases} \Leftrightarrow \begin{cases} -0.5\pi_1 = 0 \\ 0.25\pi_1 - \pi_2 + 0.25\pi_3 + 0.5\pi_5 = 0 \\ 0.5\pi_2 - 0.75\pi_3 + 0.5\pi_5 = 0 \\ 0.25\pi_3 = 0 \\ 0.25\pi_1 + 0.5\pi_2 + 0.25\pi_3 - \pi_5 = 0 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1, \end{cases}$$

$$\Rightarrow \vec{\pi}^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

6. Which adjustments to the one-step transition matrix

$$P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0. \end{pmatrix}$$

would make this chain ergodic?

Options:

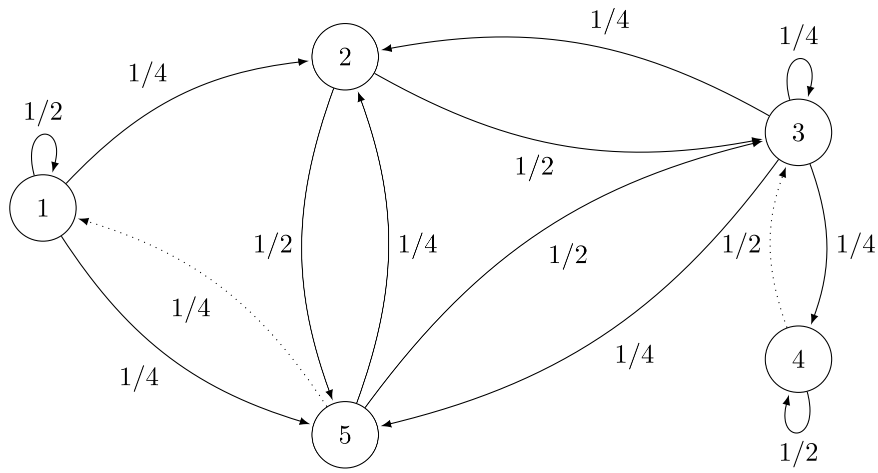
a) $P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$ b) $P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/2 & 0 & 0 \end{pmatrix}$

c) $P = \begin{pmatrix} 1/2 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 & 0 & 0 \end{pmatrix}$ d) None of above

e) The chain is already ergodic

Answer: c)

Solution: Since all the states are aperiodic, the chain will be ergodic if all of them communicate. The matrix in c) allows to connect the state 5 with the state 1 and the state 4 with the state 3, so there becomes one class of equivalence, and the chain is ergodic.



7. Assume that there is a series of integer numbers, in which numbers 1,2,..,9 appear randomly and independently of each other with equal probabilities. Let x_n be a quantity of different numbers in n first elements of the series. Find a stationary distribution of this chain.

Answer: $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Solution: Clearly, $x_1 = 1$. Then, with probability $\frac{1}{9}$ the second picked number will be the same as the first one giving $x_2 = 1$. With probability $\frac{8}{9}$ the second picked number will be different from the first one giving $x_2 = 1$. Therefore, the transition matrix is equal to:

$$P = \begin{pmatrix} \frac{1}{9} & \frac{8}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{9} & \frac{7}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{9} & \frac{6}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & \frac{5}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{9} & \frac{4}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{6}{9} & \frac{3}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{9} & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

8. At time moment $t = 0$ a six-dot side of a simple die faces upwards. Each time moment a die randomly flips on one of its sides. Find the $\vec{\pi}^2$ (the distribution of the Markov process after two flips). Note that the total number of dots on opposite sides of a simple die is equal to 7.

   Answer: $\vec{\pi}^2 = \vec{\pi}^1 \cdot P = \begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 \end{pmatrix}$

   Solution:

   Clearly, $\vec{\pi}^0 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$. To obtain the transition matrix we need to understand that a die can flip to 4 out of all 6 sides, i.e. it cannot flip to a current position and it cannot flip to an opposite side. Therefore,

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}$$

$$\vec{\pi}^1 = \vec{\pi}^0 \cdot P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}$$

$$\vec{\pi}^2 = \vec{\pi}^1 \cdot P = \begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 \end{pmatrix}$$

9. At time moment $t = 0$ a six-dot side of a simple die faces upwards. Each time moment a die randomly flips on one of its sides. Find all stationary distributions. Note that the total number of dots on opposite sides of a die is equal to 7.

   Answer: $\vec{\pi}^* = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$

   Solution: The stationary distribution $\vec{\pi}^*$ has the following feature:

   $\vec{\pi}^* \cdot P = \vec{\pi}^*$

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5\end{pmatrix}\begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5\end{pmatrix}$$

$$\begin{cases} \pi_2 + \pi_3 + \pi_4 + \pi_5 = 4\pi_1 = 4\pi_6 \\ \pi_1 + \pi_3 + \pi_4 + \pi_6 = 4\pi_2 = 4\pi_5 \\ \pi_1 + \pi_2 + \pi_5 + \pi_6 = 4\pi_3 = 4\pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1 \end{cases}$$

$\Rightarrow \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = \pi_6 = 1/6$

10. Jane and Peter participating in a chess championship. For Jane, the probabilities of wining, draw, and losing a game number t are (w, d, l). Peter is slightly more emotional. If he wins the current game, the probabilities of the win, draw, and lose in the next game are equal to $(w + \epsilon, d, l - \epsilon)$; if the current game ends in a draw, then the corresponding probabilities are (w, d, l); if Peter loses, the result of the next game is distributed as $(w - \epsilon, d, l + \epsilon)$. Find the condition which guarantees that the probability of the win is larger then the probability of the loose (separately for Peter and Jane). In other words, under which conditions it is better to be a slightly emotional?

    Answer: $l < w$.

    Solution: The transition matrices of these two players are:

    $$P_{Jane} = \begin{pmatrix} w & d & l \\ w & d & l \\ w & d & l \end{pmatrix}; \quad P_{Peter} = \begin{pmatrix} w+\epsilon & d & l-\epsilon \\ w & d & l \\ w-\epsilon & d & l+\epsilon \end{pmatrix}.$$

    Jane's stationary distribution is $(w \quad d \quad l)$. To calculate that for Peter we need to solve the following system of equations:

    $$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} w+\epsilon & d & l-\epsilon \\ w & d & l \\ w-\epsilon & d & l+\epsilon \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix}$$

    $$\begin{cases} w(x+y+z) + x\epsilon - z\epsilon = x \\ d(x+y+z) = y \\ l(x+y+z) - x\epsilon + z\epsilon = z \\ \mathbf{x+y+z=1} \end{cases} \qquad \Rightarrow x = \frac{w(1-\epsilon) - l\epsilon}{1-2\epsilon}$$

    $$(\vec{\pi}^*_{Jane})_{win} < (\vec{\pi}^*_{Peter})_{win}$$
    $$w < x$$
    $$w < \frac{w(1-\epsilon) - l\epsilon}{1-2\epsilon}$$
    $$l < w$$