Quiz-2 answers and solutions

Coursera. Stochastic Processes

December 30, 2020

1. Compute the mathematical expectation of a Poisson process N_t with intensity λ:

   Answer: λt

   Solution: This is the basic feature of the Poisson process. Keep in mind that $N_t \sim Pois(\lambda t)$

2. Find the probability generating function of a random variable with binomial distribution,

$$\mathbb{P}\{\xi = k\} = C_n^k p^k (1-p)^{n-k}, \quad k = 0, 1, .., n, \qquad p \in (0, 1) :$$

   Answer: $\varphi(u) = (up + (1-p))^n$

   Solution: $PGF = \varphi_\xi(u) = \mathbb{E}[u^\xi] = \sum_{k=0}^n u^k C_n^k p^k (1-p)^{n-k} = \sum_{k=0}^n C_n^k (up)^k (1-p)^{n-k} =$ (Newton binomial) $= (up + (1-p))^n$

3. Let N_t be a (homogeneous) Poisson process with intensity λ. Find the limit $\lim_{h\to 0} \mathbb{P}\{N_h = 0\}$:

   Answer: 1

   Solution: $\lim_{h\to 0} \mathbb{P}\{N_h = 0\} = \lim_{h\to 0} e^{-\lambda h} = 1$

4. Let N_t be a (homogeneous) Poisson process with intensity λ. Find the limit $\lim_{h\to 0} \mathbb{P}\{N_h = 1\}$:

   Answer: 0

   Solution: $\lim_{h\to 0} \mathbb{P}\{N_h = 3\} = \lim_{h\to 0} e^{-\lambda h} \frac{\lambda h}{1!} = 1 \cdot 0$

5. Let N_t be a (homogeneous) Poisson process with intensity λ. Find the limit $\lim_{h\to 0} \mathbb{P}\{N_h = 3\}$::

   Answer: 0

   Solution: $\lim_{h\to 0} \mathbb{P}\{N_h = 3\} = \lim_{h\to 0} e^{-\lambda h} \frac{(\lambda h)^3}{3!} = 1 \cdot 0$

6. 2 friends are chatting: one has a messaging speed equal to 3 messages per minute, another - 2 messages per minute. Assuming that for every person the process of writing the messages is modeled with Poisson process and these processes are independent, find the probability that there will be sent only 2 messages during the first minute:

   Answer: $e^{-5}\frac{25}{2}$

   Solution: $\mathbb{P}^* = \mathbb{P}(N_1^A = 2, N_1^B = 0) + \mathbb{P}(N_1^A = 0, N_1^B = 2) + \mathbb{P}(N_1^A = 1, N_1^B = 1) = e^{-5}(\frac{3^2}{2!}\frac{2^0}{0!} + \frac{3^0}{0!}\frac{2^2}{2!} + \frac{3}{1}\frac{2}{1}) = e^{-5}\frac{25}{2}$

7. Purchases in a shop are modelled by the homogeneous Poisson process: 30 purchases are made on average during an hour after the opening of the shop. Find the probability that the interval between k and $k+1$ purchases will be less than **or equal to** 4 minutes, given that the purchase number k was in the time moment s:

   Answer: $1 - e^{-2}$

   Solution:

$$\begin{aligned}\mathbb{P}(S_{k+1} - S_k \leq 4 | N_s = k) &= \mathbb{P}(N_{s+4} - N_s \geq 1 | N_s - N_0 = k) \\ &= \mathbb{P}(N_{s+4} - N_s > 0) \\ &= 1 - \mathbb{P}(N_{s+4} - N_s = 0) \\ &= 1 - \mathbb{P}(N_4 = 0) = 1 - e^{-2},\end{aligned}$$

   because $N_4 \sim Pois(4 \cdot 30/60)$

8. The amount of claims to an insurance company is modelled by the Poisson process, and the claim sizes are modelled by an exponential distribution. On average there are 100 claims per day, and the mean value of 1 claim is 5000 USD.

   Find the variance of the process X_t, which is equal to the total amount of claims till time t :

   Answer: $5t \times 10^9$

   Solution: $\xi \sim exp(\mu)$

   $\mathbb{E}\xi = \frac{1}{\mu} = 5 \times 10^3$

   $Var\xi = \frac{1}{\mu^2} = 25 \times 10^6$

   $\mathbb{E}\xi^2 = Var\xi + (\mathbb{E}\xi)^2 = 5 \times 10^7$

   $VarX_t = \lambda t \mathbb{E}\xi^2 = 100t \cdot 5 \times 10^7 = 5t \times 10^9$

9. A scientist wants to describe the death of cells in some organism by a non-homogeneous Poisson process. However, he is not sure which intensity function $\Lambda(t)$ to use. Thus, he decides to estimate the average amount of cells which are dead by some time moment t and choose such $\Lambda(t)$ that the theoretical mean of the corresponding process is as close as possible to the empirical value. Which of the following $\Lambda(t)$ would be the best choice, if the estimated number of dead cells at $t = 1$ is 8?

   Answer: $\Lambda(t) = (2t)^3$

   Solution: Since it is known that $\mathbb{E}N_t = \Lambda(t)$, we should choose $\Lambda(t)$ such that $\mathbb{E}N_1$ is as close as possible to 8. Plugging $t = 1$ into each of the functions above yields 9 for $9 \cdot t$, approximately 7.39 for e^{2t}, 7 for $7 \cdot t^2$ and 8 for $2 \cdot t^3$, from which the latter is the best choice.

10. The amount of claims to an insurance company is modelled by the Poisson process, and the claim sizes are modelled by an exponential distribution. On average there are 100 claims per day, and the mean value of 1 claim is 5000 USD.

    Find the probability that the process X_t, which is equal to the total amount of claims till time t, is equal to 0 at the moment t.

Answer: e^{-100t}

Solution: $\mathbb{P}(X_t = 0) = \mathbb{P}(N_t = 0) + \mathbb{P}(N_t > 0) \cdot \mathbb{P}(\sum_{k=1}^{N_t} \xi_k = 0) = e^{-100t} + 0 = e^{-100t}$

11. The speed camera records the cars that exceed the speed limit and transmits this information to the system. If the limit excess is smaller than 15 km/h, the issued fine is 30 Euro, otherwise it is equal to 50 Euro. The number of cars that exceed the limit is modelled by a Poisson process, and on average, every 10 minutes the limit is exceeded by 2 cars. Assuming that the fine is issued immediately after the speed limit excess is recorded, calculate the probability that the cumulative sum of all fines issued in the first 20 minutes of work ($t = 2$) is greater than 50 Euro.

    Answer: 0.91

    Solution: Let ξ_t be a random variable representing the value of fine issued at time moment t, that is,

$$\xi_t = \begin{cases} 30, & p \\ 50, & 1-p \end{cases} \qquad p \in (0,1), \quad t \geq 0.$$

    Then we need to calculate

$$\begin{aligned} \mathbb{P}\left\{\sum_{i=1}^{N_t} \xi_i > 50\right\} &= 1 - \mathbb{P}\left\{\sum_{i=1}^{N_t} \xi_i \leq 50\right\} \\ &= 1 - \mathbb{P}\{N_t \leq 1\} \\ &1 - \mathbb{P}\{N_t = 0\} - \mathbb{P}\{N_t = 1\} \\ &= 1 - e^{-\lambda t}\left(\frac{(\lambda t)^0}{0!} + \frac{(\lambda t)^1}{1!}\right) \\ &= 1 - e^{-\lambda t}(1 + \lambda t) \end{aligned}$$

    which is for $\lambda = 2$ and $t = 2$ is approximately equal to 0.91.

12. The amount of claims to an insurance company is modelled by the Poisson process, and the claim sizes are modelled by an exponential distribution. On average there are 100 claims per day, and the mean value of 1 claim is 5000 USD.

    Find the mean value of the process X_t, which is equal to the total amount of claims till time t:

    Answer: 500000

    Solution: According to the corollary about Compound Poisson processes: $\mathbb{E}(X_t) = \lambda t \cdot \mathbb{E}(\xi) = 100t \cdot 5000 = 500000$

13. Purchases in a shop are modelled with non-homogeneous Poisson process: $30t^{5/4}$ purchases are made on average during t hours after the opening of the shop. Find the probability that the interval between k and $k+1$ purchases will be less or equal than 2 minutes, given that the purchase number k was in the time moment s:

    Answer: $1 - e^{-30(s+1/30)^{5/4} + 30s^{5/4}}$

Solution: $\mathbb{P}(S_{k+1} - S_k \leq 2|N_s = k) = \mathbb{P}(N_{s+2} - N_s \geq 1|N_s = k) = \mathbb{P}(N_{s+2} - N_s \geq 1|N_s - N_0 = k) = \mathbb{P}(N_{s+2} - N_s \geq 1) = 1 - \mathbb{P}(N_{s+2} - N_s = 0) = 1 - e^{-30(s+1/30)^{5/4} + 30s^{5/4}}$

14. The number of downloads of an app in Google-Play are modelled by a non-homogeneous Poisson process with intensity $\Lambda(t) = t^{13/5}$, where t is measured in hours after app's commencement time. Find the probability that the time between the 1000^{th} and 1001^{st} downloads is less than or equal to 36 seconds (0.01 hour) given 1000^{th} download time being 14 hours after app's launch.

    Solution: $\mathbb{P}(S_{1001} - S_{1000} \leq 0.01|S_{1000} = 14) = \mathbb{P}(S_{1001} - S_{1000} \leq 0.01|N_{14} = 1000) = \mathbb{P}(N_{14.01} - N_{14} \geq 1|N_{14} - N_0 = 1000) = \mathbb{P}(N_{14.01} - N_{14} \geq 1) = 1 - \mathbb{P}(N_{14.01} - N_{14} = 0) = 1 - e^{-(\Lambda(14.01) - \Lambda(14))} \frac{(\Lambda(14.01) - \Lambda(14))^0}{0!} = 1 - e^{-14.01^{2.6} + 14^{2.6}} = 0.83.$

15. Find the probability generating function of the the random variable N_3 (where N_t is a homogeneous Poisson process) using the formula $PGF = \varphi_\alpha(u) = \mathbb{E}(u^\alpha)$:

    Answer: $e^{-3\lambda(1-u)}$

    Solution: $\mathbb{E}(u^{N_3}) = \sum_{k=0}^{\infty} u^k e^{-3\lambda} \frac{(3\lambda)^k}{k!} = e^{-3\lambda} \sum_{k=0}^{\infty} \frac{(3u\lambda)^k}{k!} = e^{-3\lambda} e^{3\lambda u} = e^{(-3\lambda(1-u))}$, because $\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^\alpha$

16. There is a speed limit on the street near the secondary school. To keep a lid on traffic violations, local administration decided to put a speed-register. If a car violates the speed limit, the register correctly identifies its ID number with probability 80%. Assume that the number of cars passing the school and violating the speed limit is modelled by the homogeneous Poisson process N_t with intensity equal to 20. Find the probability that during 2 hours after midday there will be 16 cars registered.

    Answer: 0.07%

    Solution: Denote the number of registered cars till time t by M_t. The probability of a correct identification of a car is equal to $p = 0.8$

    So, we need to calculate $\mathbb{P}(M_{2p.m.} - M_{12a.m.} = 16)$.

More generally,

$$
\begin{aligned}
\mathbb{P}(M_t - M_s = m) &= \mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{m}) \\
&\quad +\mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{m+1}) + \\
&\quad +\mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{m+2}) + \cdots \\
&= \sum_{n=m}^{\infty} \mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{n}) \\
&= \sum_{n=m}^{\infty} \mathbb{P}(M_t - M_s = m | N_t - N_s = \mathbf{n}) \cdot \mathbb{P}(N_t - N_s = \mathbf{n}) \\
&= \sum_{n=m}^{\infty} C_n^m p^m (1-p)^{n-m} \cdot e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!} \\
&= \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m} \cdot e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!} \\
&= \frac{p^m e^{-\lambda(t-s)}}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{(1-p)^n}{(1-p)^m} \cdot \frac{(\lambda(t-s))^n}{n!} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} (1-p)^n \cdot \frac{(\lambda(t-s))^n}{n!} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} \sum_{n=m}^{\infty} \frac{(\lambda(t-s)(1-p))^n}{(n-m)!} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} \sum_{\mathbf{k=0}}^{\infty} \frac{(\lambda(t-s)(1-p))^{\mathbf{k+m}}}{\mathbf{k!}} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} (\lambda(t-s)(1-p))^{\mathbf{k+m}} \sum_{\mathbf{k=0}}^{\infty} \frac{(\lambda(t-s)(1-p))^{\mathbf{k}}}{\mathbf{k!}} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} (\lambda(t-s)(1-p))^{\mathbf{k+m}} e^(\lambda(t-s)(1-p)) \\
&= \frac{(\lambda p(t-s))^m}{m!} e^{-\lambda p(t-s)}.
\end{aligned}
$$

Therefore,

$$
\mathbb{P}(M_{2p.m.} - M_{12a.m.} = 16) = \frac{(20 \cdot 0.8(2-0))^{16}}{16!} e^{-20 \cdot 0.8(2-0)} = 0.07\%
$$