Quiz-4 answers and solutions

Coursera. Stochastic Processes

December 30, 2020

1. Choose the functions $K(t,s)$, which can be covariance functions of some stochastic processes defined for $t \in [0,1]$.

   Options:

   a) $K(t,s) = 1 - \max\{t,s\}$ b) $K(t,s) = \max\{t,s\} - \min\{t,s\}$

   c) $K(t,s) = 1 + (t-s)^3$ d) $c > 0$

   Answer: a) $K(t,s) = 1 - \max\{t,s\}$, d) $c > 0$

   Solution:

   a) Clearly, this function is symmetric, since

   $$K(t,s) = 1 - \max\{t,s\} = 1 - \max\{s,t\} = K(s,t).$$

   Let us show that it is also positive semi-definite. Indeed, define $g_t(x) := \mathbb{I}\{x \in [t,1]\}$. Then

   $$1 - \max\{t,s\} = \int_0^\infty g_s(x) g_t(x)\, dx,$$

   from which

   $$\begin{aligned}\sum_{j=1}^n \sum_{k=1}^n u_j u_k K(t_j,t_k) &= \sum_{j=1}^n \sum_{k=1}^n u_j u_k \int_0^\infty g_{t_j}(x) g_{t_k}(x)\, dx \\ &= \int_0^\infty \sum_{j=1}^n \sum_{k=1}^n u_j u_k g_{t_j}(x) g_{t_k}(x)\, dx \\ &= \int_0^\infty \left(\sum_{j=1}^n u_j g_{t_j}(x)\right)\left(\sum_{k=1}^n u_j g_{t_k}(x)\right) dx \\ &= \int_0^\infty \left(\sum_{j=1}^n u_j g_{t_j}(x)\right)^2 dx \geq 0.\end{aligned}$$

   Thus, $K(t,s) = 1 - \max\{t,s\}$ can be a covariance function of some process defined for $t \in [0,1]$.

   b) This function cannot be a covariance function of any process. Assume the converse: let there exist some process X_t such that its covariance function $K(t,s) = \max\{t,s\} - \min\{t,s\}$. Then

   $$\operatorname{Var} X_t = K(t,t) = \max\{t,t\} - \min\{t,t\} = t - t = 0,$$

i.e., $X_t = const$ for all $t \in [0,1]$. However,

$$\operatorname{cov}(X_t, X_s) = \max\{t,s\} - \min\{t,s\} \neq const \quad \forall s \neq t, \quad (s,t) \in [0,1]^2,$$

which leads to the contradiction.

c) Since

$$K(t,s) = 1 + (t-s)^3 \neq 1 + (s-t)^3 = K(s,t),$$

the symmetry property is violated, and this function cannot be a covariance function of any process.

d) It can be seen that

$$K(t,s) = c = K(s,t)$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n} u_j u_k K(t_j,t_k) = \sum_{j=1}^{n}\sum_{k=1}^{n} u_j u_k c = c\left(\sum_{j=1}^{n} u_j\right)^2 \geq 0,$$

meaning that this function is symmetric and positive semi-definite. Thus, it can be a covariance function of some process. A corresponding example is given by $X_t = \xi$, where ξ is a random variable such that $\operatorname{Var}\xi = 1$.

2. Which of the following processes are Gaussian? (In the answers below W_t is a Brownian motion).

Options:

a) $X_t = W_t^2$

b) $X_t = a\xi_{t-1} + \xi_t, \quad \xi_i \sim i.i.d.\ \mathcal{N}(0,1) \quad \forall i = 0,1,\ldots,t, \quad t = 1,2,\ldots$

c) $X_t = W_{1-t} - W_1, \quad t \in [0,1]$

d) $X_t = W_t + \eta\xi, \quad \xi \sim \mathcal{N}(0,1),\ \eta = \begin{cases} -1, & 1/2 \\ 1, & 1/2 \end{cases}, \quad W_t, \xi, \eta$ are independent for all $t \geq 0$

Answer: b), c), d)

Options: The process X_t is Gaussian if and only if for any $t_1, \ldots, t_n$: $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector, that is, $\sum\limits_{k=1}^{n} \lambda_k X_{t_k} \sim \mathcal{N}, \forall \vec{\lambda}$.

a) Since $\mathbb{P}\{W_t^2 < 0\} = 0$, W_t^2 is not normally distributed, and this process is not Gaussian.

b) A finite dimensional distribution for this process has the form

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \quad = \quad (a\xi_{t_1-1} + \xi_{t_1}, a\xi_{t_2-1} + \xi_{t_2}, \ldots, a\xi_{t_n-1} + \xi_{t_n}).$$

Clearly, any linear combination of finite dimensional distributions of X_t can be represented as a sum of i.i.d. normally distributed random variables: for instance, if t_i are such that $t_i - 1 = t_{i-1}$ $\forall i = 1,2,\ldots,n$

$$\begin{aligned}
\sum_{k=1}^{n} \lambda_k X_{t_k} &= \lambda_1(a\xi_{t_1-1} + \xi_{t_1}) + \lambda_2(a\xi_{t_2-1} + \xi_{t_2}) + \cdots + \lambda_n(a\xi_{t_n-1} + \xi_{t_n}) \\
&= \lambda_1(a\xi_{t_1-1} + \xi_{t_1}) + \lambda_2(a\xi_{t_1} + \xi_{t_2}) + \cdots + \lambda_n(a\xi_{t_{n-1}} + \xi_{t_n}) \\
&= \lambda_1 a\xi_{t_1-1} + (\lambda_1 + a\lambda_2)\xi_{t_1} + (\lambda_2 + a\lambda_3)\xi_{t_1} + \cdots + \lambda_n\xi_{t_n} \sim \mathcal{N}.
\end{aligned}$$

c) This process is Gaussian as

$$\begin{aligned}\sum_{k=1}^{n}\lambda_k X_{t_k} &= \lambda_1 W_{1-t_1}-\lambda_1 W_1+\cdots+\lambda_n W_{1-t_n}-\lambda_n W_1\\ &= \sum_{k=1}^{n}\lambda_k W_{1-t_k}-W_1\sum_{k=1}^{n}\lambda_k\sim\mathcal{N}\end{aligned}$$

as a linear combination of the components of the finite dimensional distribution $(W_{1-t_1}, W_{1-t_2},\ldots W_{1-t_n}, W_1)$ of a Brownian motion which is a Gaussian process.

d) Let us show that $\eta\xi\sim\mathcal{N}(0,1)$. Indeed,

$$\begin{aligned}\mathbb{P}\{\eta\xi\le x\} &= \mathbb{P}\{-\xi\le x|\eta=-1\}\mathbb{P}\{\eta=-1\}+\mathbb{P}\{\xi\le x|\eta=1\}\mathbb{P}\{\eta=1\}\\ &= \frac{1}{2}(1-\mathbb{P}\{\xi>x\}+\mathbb{P}\{\xi\le x\})=\frac{1}{2}(1-1+\mathbb{P}\{\xi\le x\}+\mathbb{P}\{\xi\le x\})\\ &= \mathbb{P}\{\xi\le x\}.\end{aligned}$$

Thus

$$\sum_{k=1}^{n}\lambda_k X_{t_k}=\sum_{k=1}^{n}\lambda_k(W_{t_k}+\eta\xi)=\sum_{k=1}^{n}\tilde{\lambda}_k(W_{t_k}-W_{t_{k-1}})+\eta\xi\sum_{k=1}^{n}\lambda_k\sim\mathcal{N}$$

as a sum of i.i.d. normally distributed random variables.

3. Which of the following processes is a Brownian motion? (In the answers below W_t is a Brownian motion).

Options:

a) $X_t=W_t^2$

b) $X_t=a\xi_{t-1}+\xi_t,\quad \xi_i\sim i.i.d.\ \mathcal{N}(0,1)\quad \forall i=0,1,\ldots,t,\quad t=1,2,\ldots$

c) $X_t=W_{1-t}-W_1,\quad t\in[0,1]$

d) $X_t=W_t+\eta\xi,\quad \xi\sim\mathcal{N}(0,1),\ \eta=\begin{cases}-1, & 1/2\\ 1, & 1/2\end{cases},\quad W_t,\xi,\eta$ are independent for all $t\ge 0$

Answer: c) $X_t=W_{1-t}-W_1,\quad t\in[0,1]$

Solution: Since Brownian motion is a Gaussian process W_t such that $\mathbb{E}W_t=0$ and $\mathrm{cov}(W_t,W_s)=\min\{t,s\}$, we should only check the last two properties for the processes b)-d).

For b)

$$\mathbb{E}[X_t]=\mathbb{E}[a\xi_{t-1}+\xi_t]=a\mathbb{E}[\xi_{t-1}]+\mathbb{E}[\xi_t]=0,$$

but

$$\mathrm{cov}(X_t,X_s)=\mathrm{cov}(a\xi_{t-1}+\xi_t,a\xi_{s-1}+\xi_s)=(1+a^2)\mathbb{I}\{t=s\}+a\mathbb{I}\{|t-s|=1\},$$

thus, it is not a Brownian motion.

For c)

$$\mathbb{E}[W_{1-t}-W_1]=\mathbb{E}[W_{1-t}]-\mathbb{E}[W_1]=0$$

and

$$\begin{aligned}\operatorname{cov}(W_{1-t}-W_1, W_{1-s}-W_1) &= \operatorname{cov}(W_{1-t}, W_{1-s}) - \operatorname{cov}(W_1, W_{1-t}) \\ &\qquad - \operatorname{cov}(W_1, W_{1-s}) + \operatorname{cov}(W_1, W_1) \\ &= \min\{1-t, 1-s\} - (1-t) - (1-s) + 1 \\ &= t + s - \max\{t, s\} = \min\{t, s\}.\end{aligned}$$

Therefore, X_t is Brownian motion.

For d)

$$\mathbb{E}[W_t + \eta\xi] = \mathbb{E}[W_t] + E[\eta]E[\xi] = 0,$$

but

$$\begin{aligned}\operatorname{cov}(W_t + \eta\xi, W_s + \eta\xi) &= \operatorname{cov}(W_t, W_s) + \operatorname{cov}(W_t, \eta\xi) + \operatorname{cov}(W_s, \eta\xi) + \operatorname{cov}(\eta\xi, \eta\xi) \\ &= \min\{t, s\} + 0 + 0 + \mathbb{E}[(\eta\xi)^2] - (\mathbb{E}[\eta\xi])^2 \\ &\overset{\eta\xi\sim\mathcal{N}(0,1)}{=} \min\{t, s\} + 1 - 0 \\ &= \min\{t, s\} + 1 \neq \min\{t, s\}.\end{aligned}$$

So, X_t is not a Brownian motion.

4. Find the covariance function of the process $X_t = W_t - tW_1, \quad t \in [0, 1]$, where W_t is a Brownian motion.

   Answer: $\min\{t, s\} - ts$

   Solution:

$$\begin{aligned}\operatorname{cov}(X_t, X_s) &= \operatorname{cov}(W_t - tW_1, W_s - sW_1) \\ &= \operatorname{cov}(W_t, W_s) - t\operatorname{cov}(W_1, W_s) - s\operatorname{cov}(W_t, W_1) + ts\operatorname{cov}(W_1, W_1) \\ &= \min\{t, s\} - ts - st + ts = \min\{t, s\} - ts.\end{aligned}$$

5. Find the covariance function of the process $X_t = \xi W_t, \quad \xi \sim \mathcal{N}(1, 1)$ is independent of $W_t \quad \forall t, t \geq 0$.

   Answer: $2\min\{t, s\}$.

   Solution:

$$\begin{aligned}\operatorname{cov}(\xi W_t, \xi W_s) &= \mathbb{E}[\xi^2 W_t W_s] - \mathbb{E}[\xi W_t]\mathbb{E}[\xi W_s] \\ &= \mathbb{E}\xi^2\mathbb{E}[W_t W_s] - (\mathbb{E}[\xi])^2\mathbb{E}W_t\mathbb{E}W_s \\ &\overset{t\geq s}{=} (1 + 1^2)\mathbb{E}[(W_t - W_s + W_s)W_s] - 0^2 \cdot 0 \cdot 0 \\ &= 2(\mathbb{E}[(W_t - W_s)(W_s - W_0)] + \mathbb{E}[W_s^2]) \\ &= 2s = 2\min\{t, s\}.\end{aligned}$$

6. Harry is playing some game. At each given time moment t the number of points he gets is determined by the value of the process $X_t = \sigma W_t$, that is, he earns points if $X_t > 0$ and loses if $X_t < 0$. Every hour (at time moments $t = 1, 2, \ldots$) the results are automatically recorded, and if the sum of 3 results in a row is greater than 10, Harry gets a special prize. Find the probability that Harry gets this prize after 3 hours of playing (in

the answers below Φ is the distribution function of the standard normal distribution).

Answer: $1 - \Phi\left(\frac{10}{\sigma\sqrt{14}}\right)$

Solution: We need to find the probability that

$$\begin{aligned} \mathbb{P}\{X_1 + X_2 + X_3 \geq 10\} &= \mathbb{P}\{\sigma(W_1 + W_2 + W_3) \geq 10\} \\ &= 1 - \mathbb{P}\{W_1 + W_2 + W_3 \leq 10/\sigma\}. \end{aligned}$$

As

$$\mathbb{E}[W_1 + W_2 + W_3] = \mathbb{E}[W_1] + \mathbb{E}[W_2] + \mathbb{E}[W_3] = 0$$

and

$$\begin{aligned} \operatorname{Var}(W_1 + W_2 + W_3) &= \operatorname{Var} W_1 + \operatorname{Var} W_2 + \operatorname{Var} W_3 + 2\operatorname{cov}(W_1, W_2) \\ &\quad +2\operatorname{cov}(W_2, W_3) + 2\operatorname{cov}(W_1, W_3) \\ &= 1 + 2 + 3 + 1 \cdot 2 + 2 \cdot 2 + 1 \cdot 2 = 14, \end{aligned}$$

$$1 - \mathbb{P}\{W_1 + W_2 + W_3 \leq 10/\sigma\} = 1 - \Phi\left(\frac{10}{\sigma\sqrt{14}}\right).$$

7. Choose the correct statements about the process $X_t = \sigma W_t$, $\sigma > 0$:

Options:

a) $\operatorname{cov}(X_t, X_s) = \min\{t, s\}$

b) the Kolmogorov continuity theorem $\mathbb{E}[|X_t - X_s|^\alpha] \leq c|t - s|^{1+\beta}$ holds with $c = 3, \alpha = 4, \beta = 1$

c) X_t is not stationary

d) X_t is a Gaussian process

Answer: X_t is not stationary and is a Gaussian process.

Solution: Since X_t is a Brownian motion multiplied by a positive constant, it is also a Gaussian process. Now,

$$\operatorname{cov}(X_t, X_s) = \sigma^2 \operatorname{cov}(W_t, W_s) = \sigma^2 \min\{t, s\}$$

and since the covariance function depends not on the difference, but on the time moments t and s, X_t is not stationary. Finally,

$$\mathbb{E}[|X_t - X_s|^\alpha] = \mathbb{E}[|\sigma(W_t - W_s)|^\alpha] = \sigma^\alpha \mathbb{E}[|W_t - W_s|^\alpha],$$

therefore, the condition of the Kolmogorov continuity theorem

$$\mathbb{E}[|X_t - X_s|^\alpha] \leq c|t - s|^{1+\beta}$$

holds with $\alpha = 4, \beta = 1$ and $c = 3\sigma^4$.