Quiz-8 answers and solutions

Coursera. Stochastic Processes

August 3, 2021

1. Consider the process $X_t = W_t + \sum_{i=1}^{N_t} \xi_i$, where W_t is a Brownian motion, N_t is a Poisson process with intensity $\lambda > 0$, $\xi_i \sim i.i.d.\ Unif[0, \lambda]$, $i = 1, \ldots, n$, and ξ_i, W_t and N_t are independent. Compute the mean and covariance function of this process.

   Answer:
   $$\mathbb{E}[X_t] = \frac{\lambda^2 t}{2}, \quad K(t,s) = \min\{t,s\}\left(1 + \frac{\lambda^3}{3}\right)$$

   Solution:
   $$\mathbb{E}\left[X_t\right] = \mathbb{E}\left[W_t + \sum_{i=1}^{N_t} \xi_i\right] = \mathbb{E}W_t + E\left[\sum_{i=1}^{N_t} \xi_i\right] = 0 + \lambda t \mathbb{E}\xi_1 = \frac{\lambda^2 t}{2}.$$

   Now, since for any Lévy process L_t it is true that $K(t,s) = \operatorname{Var} L_{\min\{t,s\}}$ and $\operatorname{Var} L_t = t \operatorname{Var} L_1$,

   $$\begin{aligned} \operatorname{Var} X_1 = \operatorname{Var}(W_1 + \sum_{i=1}^{N_1} \xi_i) &= \operatorname{Var} W_1 + \operatorname{Var} \sum_{i=1}^{N_1} \xi_1 = 1 + \lambda \mathbb{E}\xi_1^2 \\ &= 1 + \lambda\left(\frac{\lambda^2}{12} + \frac{\lambda^2}{4}\right) = 1 + \frac{\lambda^3}{3} \end{aligned}$$

   and
   $$K(t,s) = \min\{t,s\}\left(1 + \frac{\lambda^3}{3}\right).$$

2. Consider the process $X_t = W_t + \sum_{i=1}^{N_t} \xi_i$, where W_t is a Brownian motion, N_t is a Poisson process with intensity $\lambda > 0$, $\xi_i \sim i.i.d.\ Unif[0, \lambda]$, $i = 1, \ldots, n$, and ξ_i, W_t and N_t are independent. Find the characteristic function $\phi_{X_t}(u)$ of this process.

   Answer:
   $$\phi_{X_t}(u) = \exp\left\{-\frac{tu^2}{2} + \lambda t\left(\frac{e^{iu\lambda} - 1}{iu\lambda} - 1\right)\right\}$$

Solution:

$$\begin{aligned}
\phi_{X_t}(u) &= \mathbb{E}[e^{iuX_t}] = \mathbb{E}\exp\left\{iu(W_t + \sum_{i=1}^{N_t}\xi_i)\right\} \\
&= \mathbb{E}\exp\{iuW_t\}\mathbb{E}\exp\left\{iu\sum_{i=1}^{N_t}\xi_i\right\} \\
&= \exp\left\{-\frac{tu^2}{2} - \lambda t + \lambda t\phi_{\xi_1}(u)\right\} \\
&= \exp\left\{-\frac{tu^2}{2} + \lambda t(\phi_{\xi_1}(u) - 1)\right\} \\
&= \exp\left\{-\frac{tu^2}{2} + \lambda t\left(\frac{e^{iu\lambda} - 1}{iu\lambda} - 1\right)\right\}.
\end{aligned}$$

3. Consider the process $X_t = W_t + \sum_{i=1}^{N_t}\xi_i$, where W_t is a Brownian motion, N_t is a Poisson process with intensity $\lambda > 0$, $\xi_i \sim i.i.d.\ Unif[0, \lambda]$, $i = 1, \ldots, n$, and ξ_i, W_t and N_t are independent. Find the Lévy triplet (μ, σ, ν) of this process under truncation function $h(x) = \mathbb{I}\{|x| < 1\}$ (in the answers below B is a Borel set).

   Answer: $(\mu, \sigma, \nu) = ((\min\{\lambda, 1\})^2/2, 1, \lambda\mathbb{P}\{\xi_1 \in B\})$

   Solution: The characteristic exponent $\psi(u)$ of this process has the form

$$\psi(u) = -\frac{u^2}{2} + \lambda\left(\frac{e^{iu\lambda} - 1}{iu\lambda} - 1\right).$$

   Recalling that by definition

$$\phi_{\xi_1}(u) = \frac{e^{iu\lambda} - 1}{iu\lambda} = \int_{\mathbb{R}} e^{iux}F_\xi(dx),$$

   where $F_\xi(x) = \mathbb{P}\{\xi \le x\}$, and

$$\int_{\mathbb{R}} F_\xi(dx) = 1,$$

we get that

$$\begin{aligned}
\psi(u) &= -\frac{u^2}{2} + \lambda\left(\frac{e^{iu\lambda}-1}{iu\lambda} - 1\right) = -\frac{u^2}{2} + \lambda \int_{\mathbb{R}} (e^{iux}-1)\, F_\xi(dx) \\
&= -\frac{u^2}{2} + \lambda \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x|<1\} + iux\mathbb{1}\{|x|<1\})\, F_\xi(dx) \\
&= \lambda \int_{\mathbb{R}} iux\mathbb{1}\{|x|<1\}\, F_\xi(dx) - \frac{u^2}{2} \\
&\qquad\qquad + \lambda \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x|<1\})\, F_\xi(dx) \\
&= \lambda \cdot \frac{iux^2}{2\lambda}\Big|_0^{\min\{\lambda,1\}} - \frac{u^2}{2} \\
&\qquad\qquad + \lambda \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x|<1\})\, F_\xi(dx) \\
&= iu \cdot \frac{(\min\{\lambda,1\})^2}{2} - \frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x|<1\})\, \nu(dx),
\end{aligned}$$

where $\nu(B) = \lambda\mathbb{P}\{\xi_1 \in B\}$ for any Borel set B. From this, the Lévy triplet (μ, σ, ν) has the form $((\min\{\lambda, 1\})^2/2, 1, \lambda\mathbb{P}\{\xi_1 \in B\})$.

4. Let X_t be a Levy process. What is the correct expression for $Var(X_t)$ in terms of characteristic exponent ψ?

   Answer: $Var(X_t) = -t\psi''(0)$

   Solution: According to the Lévy-Khinchine theorem, for any Lévy process a characteristic exponent is equal to:

$$\psi(u) = iu\mu - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}\{|x|<1\})\nu(dx)$$

$$\psi''(u) = -\sigma^2$$

$$Var(X_t) = \sigma^2 t = (-\psi''(u))t$$

5. Let X_t be a Lévy process. Assuming that $X_1 \sim N(0,1)$, find the mean and the variance of X_t:

   Answer: $\mathbb{E}[X_t] = 0$, $Var(X_t) = t$

   Solution:

   The characteristic exponent of X_1 is $\psi_{X_1}(u) = -\frac{1}{2}u^2$. Since X_t is a Lévy process, then $\psi_{X_t}(u) = -\frac{1}{2}u^2 t$. Hence, X_t is a Brownian motion. Consequently, $Var(X_t) = t$.

6. Let $T_a = \min\{s : B_s \geq a\}$, where B_s is a Brownian motion. Find the distribution function of the process T_a.

   Hint: $\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t) = \mathbb{P}(B_t - B_{T_a} > 0)$. It follows from the fact that for the Brownian motion and all other Lévy processes the increments are independent.

   Answer: $2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right)$

   Solution:

$$\begin{aligned}\mathbb{P}(B_t > a) &= \mathbb{P}(T_a < t, B_t > a) \\ &= \mathbb{P}(T_a < t)\mathbb{P}(B_t - a > 0 | T_a < t) \\ &= \mathbb{P}(T_a < t)\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t)\end{aligned}$$

   The conditional probability $\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t)$ is equal to the unconditional one, because the condition $(T_a < t)$ gives an information on the BM before T_a, which is, literally, the **time** by which BM has reached a . The increment $B_t - B_{T_a}$ is independent from that type of information.

   Thus, $\mathbb{P}(B_t - B_{T_a} > 0 | T_a < t) = \mathbb{P}(B_t - B_{T_a} > 0) = \mathbb{P}(B_{t-T_a} > 0) = 1/2$.

   Therefore,

$$\begin{aligned}\mathbb{P}(T_a < t) &= \frac{\mathbb{P}(B_t > a)}{\mathbb{P}(T_a < t, B_t > a)} \\ &= \frac{1 - \Phi(a/\sqrt{t})}{1/2} \\ &= 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right).\end{aligned}$$

7. Let L_t be a Lévy process. Choose the equality, which can serve as a proof of its infinite divisibility.

   Answer:

$$L_t = L_{t/n} + (L_{2t/n} - L_{t/n}) + \cdots + (L_t - L_{(n-1)t/n}), \qquad \forall t > 0,\ \forall n \in \mathbb{N}$$

   Solution: Since Lévy processes have stationary and independent increments, this equality shows that L_t can be represented as a sum of n independent identically distributed random variables.

8. Let X_t and Y_t be two independent Lévy processes. Choose the correct statements about the process $Z_t = X_t + Y_t$.

   Answer: Z_t is a Lévy process and $\psi_Z(u) = \psi_X(u) + \psi_Y(u)$

   Solution: Since X_t and Y_t are independent, the characteristic function $\phi_{Z_t}(u)$ of Z_t has the form

$$\phi_{Z_t}(u) = \phi_{X_t+Y_t}(u) = \phi_{X_t}(u)\phi_{Y_t}(u),$$

   and thus for any $n \geq 2$

$$\phi_{Z_t}^{1/n}(u) = (\phi_{X_t}(u)\phi_{Y_t}(u))^{1/n} = \phi_{X_t}^{1/n}(u)\phi_{Y_t}^{1/n}(u).$$

Since X_t and Y_t are independent Lévy processes, $\phi_{X_t}^{1/n}(u)$ and $\phi_{Y_t}^{1/n}(u)$ are characteristic functions of some random variables, and the right-hand side of the equation above is a characteristic function of their sum. Therefore, Z_t has an infinitely divisible distribution at any time moment t, from which we conclude that Z_t is a Lévy process. $Z_{t+1} - Z_t$ is weakly stationary since the increments of X_t are stationary, $\mathbb{E}[Z_{t+1} - Z_t] = 0$ and

$$\begin{aligned}
\operatorname{cov}(Z_{t+1} - Z_t, Z_{s+1} - Z_s) &\overset{t \geq s}{=} 0 \\
&\quad + \operatorname{cov}([Z_{t+1} - Z_{s+1}] + [Z_{s+1} - Z_s] - Z_t, Z_{s+1} - Z_s)\mathbb{1}\{t < s+1\} \\
&= \operatorname{Var}(Z_{s+1} - Z_t)\mathbb{1}\{t < s+1\} \\
&= (1 - t + s)\operatorname{Var}(X_1)\mathbb{1}\{t < s+1\},
\end{aligned}$$

from which the covariance function of $Z_{t+1} - Z_t$ has the form $K(t,s) = (1 - |t-s|)\mathbb{1}\{|t-s| < 1\}$.

Now, we get that

$$\begin{aligned}
\phi_Z(u) &= e^{t\psi_Z(u)} = \mathbb{E}[e^{iuZ_t}] = \mathbb{E}[e^{iu(X_t + Y_t)}] \\
&= \mathbb{E}[e^{iuX_t}]\mathbb{E}[e^{iuY_t}] = e^{t\psi_X(u)}e^{t\psi_Y(u)} = e^{t(\psi_X(u) + \psi_Y(u))},
\end{aligned}$$

that is, there exists a characteristic exponent $\psi_Z(u)$ and it is true that $\psi_Z(u) = \psi_X(u) + \psi_Y(u)$.

Lastly, since

$$K(t,s) = \min\{t,s\}\operatorname{Var} Z_1 = \operatorname{Var} Z_{\min\{t,s\}},$$

Z_t is not stochastically differentiable at any $t = t_0$ (recall week 6).