Quiz-7 answers and solutions

Coursera. Stochastic Processes

October 18, 2021

1. Find the mathematical expectation and the variance of the process $X_T = \int\limits_0^T \cos u \, dW_u$.

   Answer: $\mathbb{E}[X_T] = 0, \operatorname{Var} X_T = \frac{\sin(2T)}{4} + \frac{T}{2}$

   Solution: It is known that $\int\limits_0^T \cos u \, dW_u \sim N\left(0, \int\limits_0^T \cos^2 u \, du\right)$. Thus, $\mathbb{E}[X_T] = 0$ and

$$\begin{aligned} \operatorname{Var}\left(\int_0^T \cos u \, dW_u\right) &= \int_0^T \cos^2 u \, du = \frac{1}{2}\int_0^T (\cos(2u) + 1) \, du \\ &= \frac{1}{2}\left(\frac{\sin(2T)}{2} + T\right) = \frac{\sin(2T)}{4} + \frac{T}{2} \end{aligned}$$

2. Consider the process $X_t = \int\limits_0^t (W_u - uW_1) \, du$, where W_t is a Brownian motion. Find the expected value of this process (in the answers below $0 \le s < t \le 1$).

   Answer: 0

   Solution:

$$\mathbb{E}X_t = \mathbb{E}\int_0^t (W_u - uW_1) \, du = \int_0^t \mathbb{E}[W_u - uW_1] \, du = 0.$$

3. Consider the process $X_t = \int\limits_0^t (W_u - uW_1) \, du$, where W_t is a Brownian motion. Find the covariance function of this process (in the answers below $0 \le s < t \le 1$).

   Answer: $-\frac{s^3}{6} + \frac{s^2 t}{2} - \frac{s^2 t^2}{4}$

Solution: Let us denote $\widetilde{W}_t = W_t - tW_1$. Then

$$\begin{aligned}
\operatorname{cov}\left(\int_0^s \widetilde{W}_v\,dv, \int_0^t \widetilde{W}_u\,du\right) &= \int_0^s\int_0^t \operatorname{cov}(\widetilde{W}_v, \widetilde{W}_u)\,du\,dv \\
&= \int_0^s\int_0^t (\min\{u,v\} - uv)\,du\,dv \\
&= \int_0^s \left[\int_0^v \underbrace{\min\{u,v\}}_{=u}\,du + \int_v^t \underbrace{\min\{u,v\}}_{=v}\,du\right] dv - \int_0^s \frac{t^2 v}{2}\,dv \\
&= \int_0^s \left(-\frac{v^2}{2} + vt\right) dv - \frac{s^2t^2}{4} \\
&= -\frac{s^3}{6} + \frac{s^2 t}{2} - \frac{s^2t^2}{4}.
\end{aligned}$$

Note:

$$\begin{aligned}
\operatorname{cov}(\widetilde{W}_v, \widetilde{W}_u) &= \operatorname{cov}(W_v - vW_1, W_u - uW_1) \\
&= \operatorname{cov}(W_v, W_u) - v\operatorname{cov}(W_1, W_u) - u\operatorname{cov}(W_1, W_v) + vu\operatorname{Var}(W_1) \\
&= \min\{u,v\} - uv - uv + uv = \min\{u,v\} - uv.
\end{aligned}$$

4. Define the function

$$f(t) = \begin{cases} 0, & t \in [0,1) \\ t, & t \in [1,2) \\ 1, & t \in [2,3) \\ 0, & t \geq 3. \end{cases}$$

Compute the variance of the process $X_T = \int_0^T f(t)\,dW_t$.

Answer:

$$\operatorname{Var}\int_0^T f(t)\,dW_t = \begin{cases} 0, & T \in [0,1) \\ \dfrac{T^3-1}{3}, & T \in [1,2) \\ T + 1/3, & T \in [2,3) \\ 10/3, & T \geq 3. \end{cases}$$

Solution: As it is known, $\int_0^T f(t)\,dW_t \sim N\left(0, \int_0^T f^2(t)\,dt\right)$. Therefore, for $T \in [0,1)$ we have

$$\operatorname{Var}\int_0^T f(t)\,dW_t = 0,$$

for $T \in [1,2)$

$$\operatorname{Var}\int_0^T f(t)\,dW_t = \int_1^T t^2\,dt = \frac{T^3-1}{3}.$$

For $T \in [2,3)$

$$\operatorname{Var} \int_0^T f(t)\, dW_t = \int_1^2 t^2\, dt + \int_2^T dt = \frac{8-1}{3} + T - 2 = 1/3 + T,$$

and for $T \geq 3$

$$\operatorname{Var} \int_0^T f(t)\, dW_t = \int_1^2 t^2\, dt + \int_2^3 dt = 7/3 + 3 - 2 = 10/3.$$

5. Let

$$X_t = \begin{cases} \xi, & t \in [0,1) \\ \eta, & t \geq 1 \end{cases},$$

where ξ has an exponential distribution with parameter $\lambda > 0$ and η is uniformly distributed on $[0, A]$, $A > 0$, ξ and η are independent. Find the mean of the stochastic integral $\int_0^T X_t\, dt$.

Answer:

$$\mathbb{E}\left[\int_0^T X_t\, dt\right] = \begin{cases} T/\lambda, & T \in [0,1) \\ 1/\lambda + A(T-1)/2, & T \geq 1. \end{cases}$$

Solution: For $T \in [0,1)$ we have

$$\mathbb{E}\left[\int_0^T X_t\, dt\right] = \int_0^T \mathbb{E}[X_t]\, dt = \int_0^T \mathbb{E}[\xi]\, dt = \int_0^T \frac{1}{\lambda}\, dt = \frac{T}{\lambda}.$$

For $T \geq 1$

$$\begin{aligned} \mathbb{E}\left[\int_0^T X_t\, dt\right] = \int_0^T \mathbb{E}[X_t]\, dt = \int_0^1 \mathbb{E}[\xi]\, dt + \int_1^T \mathbb{E}[\eta]\, dt &= \frac{1}{\lambda} + \int_1^T \frac{A}{2}\, dt \\ &= \frac{1}{\lambda} + \frac{A(T-1)}{2}. \end{aligned}$$

6. Let

$$X_t = \begin{cases} \xi, & t \in [0,1) \\ \eta, & t \geq 1 \end{cases},$$

where ξ has an exponential distribution with parameter $\lambda > 0$ and η is uniformly distributed on $[0, A]$, $A > 0$, ξ and η are independent. Find the variance of the stochastic integral $\int_0^T X_t\, dt$.

Answer:

$$\operatorname{Var} \int_0^T X_t\, dt = \begin{cases} T^2/\lambda^2, & T \in [0,1) \\ 1/\lambda^2 + A^2(T-1)^2/12, & T \geq 1. \end{cases}$$

Solution: The variance of this stochastic integral can be calculated as

$$\operatorname{Var}\int\limits_0^T X_t\,dt = \int_0^T\int_0^T K(t,s)\,ds\,dt.$$

Therefore for $T \in [0,1)$ we have

$$\operatorname{Var}\int\limits_0^T X_t\,dt = \int_0^T\int_0^T \operatorname{cov}(\xi,\xi)\,ds\,dt = \int_0^T\int_0^T \frac{1}{\lambda^2}\,ds\,dt = \frac{T^2}{\lambda^2},$$

while for $T \geq 1$

$$\begin{aligned}\operatorname{Var}\int\limits_0^T X_t\,dt &= \int_0^1\int_0^1 \operatorname{cov}(\xi,\xi)\,ds\,dt + \int_1^T\int_1^T \operatorname{cov}(\eta,\eta)\,ds\,dt \\ &= \frac{1}{\lambda^2} + \int_1^T\int_1^T \frac{A^2}{12}\,ds\,dt \\ &= \frac{1}{\lambda^2} + \frac{A^2(T-1)^2}{12}.\end{aligned}$$

7. Compute the variance of the stochastic integral $\int_0^T W_t\,dW_t$, where W_t is a Brownian motion.

Answer: $\dfrac{T^2}{2}$

Solution:

$f(t,x) = W_t^2/2, \quad f_2'(t,x) = W_t, \quad f_1'(t,x) = 0, \quad f_{2,2}''(t,x) = 1.$

$$\frac{1}{2}W_T^2 = 0 + 0 + \int_0^T W_t\,dW_t + \frac{1}{2}\int_0^T \sigma_s^2\,ds,$$

where $\sigma_s^2 = 1$ which can be derived by applying the definition of the Itô process to the Brownian motion. From that equation we obtain:

$$\int_0^T W_t dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T$$

$$\begin{aligned}\mathbb{V}ar\left(\int_0^T W_t\,dW_t\right) &= \mathbb{V}ar\left(\frac{1}{2}W_T^2\right) \\ &= \frac{1}{4}(\mathbb{E}W_T^4 - (\mathbb{E}W_T^2)^2) \\ &= \frac{1}{4}(\mathbb{E}W_T^4 - (\mathbb{E}W_T^2)^2) \\ &= \frac{1}{4}(T^2\mathbb{E}N(0;1)^4 - T^2(\mathbb{E}N(0;1)^2)^2) \\ &= \frac{1}{4}(3T^2 - T^2).\end{aligned}$$

8. Find the equivalent expression for the process $X_t = \int\limits_0^t \frac{1}{1+W_s}\, dW_s$, where W_t is a Brownian motion.

Answer:

$$\int\limits_0^t \frac{1}{1+W_s}\, dW_s = \log(1+W_t) + \frac{1}{2}\int\limits_0^t \frac{1}{(1+W_s)^2}\, ds.$$

Solution: From the application of the Itô formula to the integral $\int\limits_0^t \frac{1}{1+W_s}\, dW_s$ we get that $g(t,x) = \frac{1}{1+x}$, $f(t,x) = \log(1+x)$, $f_1'(t,x) = 0$, $g_2'(t,x) = -\frac{1}{(1+x)^2}$. Thus,

$$\log(1+W_t) = 0 + 0 + \int\limits_0^t \frac{1}{1+W_s}\, dW_s - \frac{1}{2}\int\limits_0^t \frac{1}{(1+W_s)^2}\, ds,$$

$$\Rightarrow \int\limits_0^t \frac{1}{1+W_s}\, dW_s = \log(1+W_t) + \frac{1}{2}\int\limits_0^t \frac{1}{(1+W_s)^2}\, ds.$$

9. Choose the process X_t which satisfies the following property:

$$X_t = X_0 + \int\limits_0^t X_s dW_s + \int\limits_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}}\, ds. \tag{1}$$

Answer: $X_t = e^{W_t}\sqrt{1+t}$

Solution:

Itô formula:

$$f(t,W_t) = f(0,0) + \int\limits_0^t f_1'(s,W_s)\, ds + \int\limits_0^t g(s,W_s)\, dW_s + \frac{1}{2}\int\limits_0^t g_2'(s,W_s)\, ds,$$

where $g = f_2'$.

Let $X_t = a \cdot e^{W_t}\sqrt{1+t}$ with $a \neq 0$. Then $f(t,x) = g_2'(t,x) = g(t,x) = a \cdot e^x\sqrt{1+t}$ and $f_1'(t,x) = \frac{a \cdot e^x}{2\sqrt{1+t}}$.

Therefore,

$$\begin{aligned}
\int\limits_0^t X_s\, dW_s &= \int\limits_0^t a \cdot e^{W_s}\sqrt{1+s}\, dW_s \\
&= a \cdot \left[e^{W_t}\sqrt{1+t} - 1 - \int\limits_0^t \left(\frac{1}{2}\frac{e^{W_s}}{\sqrt{1+s}} + \frac{1}{2}e^{W_s}\sqrt{1+s} \right) ds \right] \\
&= a \cdot \left[e^{W_t}\sqrt{1+t} - 1 - \int\limits_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}}\, ds \right].
\end{aligned}$$

Substituting the result into (1) gives

$$a \cdot e^{W_t}\sqrt{1+t} = a + a \cdot e^{W_t}\sqrt{1+t} - a - a\int\limits_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}}\,ds + \int\limits_0^t \frac{e^{W_s}(2+s)}{2\sqrt{1+s}}\,ds$$

which is true if and only if $a = 1$.