ST501 Group Project

Our Group

7/20/2018

Introduction

Part 1 Convergences

A

$$f_Y(y) = \frac{1}{2b} e^{-(\frac{|y-\mu|}{b})}$$

Given: $\mu = 0$ and $b = 5$

$$E(Y) = \int_{-\infty}^{\infty} y * f(y)dy$$

$$= \int_{-\infty}^{0} y * \frac{1}{10} e^{\frac{y}{5}} + \int_{0}^{\infty} y * \frac{1}{10} e^{\frac{-y}{5}}$$

$$= \frac{1}{10} \left[\int_{-\infty}^{0} y e^{\frac{y}{5}} + \int_{0}^{\infty} y * e^{\frac{-y}{5}} \right]$$

$$= \frac{1}{10} \left[\left[(5y - 25) e^{\frac{y}{5}} \right]_{-\infty}^{0} + \left[(-5y - 25) e^{\frac{-y}{5}} \right]_{0}^{\infty} \right]$$

$$= \frac{1}{10} \left[-25 + 25 \right]$$

$$E(Y) = 0 \blacksquare$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 * f(y)dy$$

$$= \frac{1}{10} \left[\int_{-\infty}^{0} y^2 e^{\frac{y}{5}} + \int_{0}^{\infty} y^2 * e^{\frac{-y}{5}} \right]$$

$$= \frac{1}{10} * \left[250 + 250 \right]$$

$$E(Y^2) = 50 \quad \blacksquare$$

Therefor $E(Y^2)$ exists.

Thus:

$E(Y) = 0$, $E(Y^2) = 50$ then $Var(Y) = 50$ by the variance computing formula.

$$L = \frac{1}{n} \sum_{i=1}^{n} Y_i^2$$

By the Law of Large Numbers:

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \quad \underset{\rightarrow}{p} \; E(Y_i)$$

By generalisation of the Law of Large Numbers:

$$L = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 \quad \underset{\rightarrow}{p} \; E(Y_i^2) = 50$$

Therefore:

$$L \quad \underset{\rightarrow}{p} \quad 50 \;\blacksquare$$

B

By the continuity theorem, $K = \sqrt{L} \quad \underset{\rightarrow}{p} \; \sqrt{50}$

Thus: $K \quad \underset{\rightarrow}{p} \; \sqrt{50}$.

C

Derrive the CDF of the double exponential distribution.

$$f_Y(y) = \frac{1}{2b} * e^{-(\frac{|y-\mu|)}{b})}$$

This yields two cases for the CDF:

$$F_Y(y) = \begin{cases} \int_{-\infty}^{t} f_Y(Y)dt & \text{for } y \leq \mu \\ \int_{t}^{\infty} f_Y(Y)dt & \text{for } y \geq \mu \end{cases}$$

Solving for these leaves us with the following:

For

$$y \leq \mu$$

$$\int_{-\infty}^{y} f_Y(Y)dt$$

$$\int_{-\infty}^{y} \frac{1}{2b} * e^{-(\frac{t-\mu}{b})}dt$$

We know that in this case we are solving for when the exponent is positive.

$$\int_{-\infty}^{y} \frac{1}{2b} * e^{(\frac{|t-\mu|)}{b})}dt$$

$$\frac{1}{2b} * e^{(\frac{t-\mu}{b})} * b|_{-\infty}^{y} = \frac{1}{2}e^{(\frac{t-\mu}{b})} - 0 = \frac{1}{2}e^{(\frac{t-\mu}{b})}$$

For the case where $y \geq \mu$

$$\int_{-\infty}^{y} f_Y(Y)dt$$

$$\int_{-\infty}^{y} \frac{1}{2b} * e^{-(\frac{t-\mu}{b})}dt$$

This must be further split into:

$$\int_{-\infty}^{\mu} \frac{1}{2b} * e^{-(\frac{t-\mu}{b})}dt + \int_{\mu}^{y} \frac{1}{2b} * e^{-(\frac{t-\mu}{b})}dt$$

$$\frac{1}{2b} * e^{(\frac{-t+\mu}{b})} * -b|_{-\infty}^{\mu} + \frac{1}{2b} * e^{(\frac{-t+\mu}{b})} * -b|_{\mu}^{y}$$

$$\frac{1}{2} - \frac{1}{2}e^{\frac{\mu-y}{b}} + \frac{1}{2}$$

Thus for this case:

$$1 - \frac{1}{2}e^{\frac{\mu-y}{b}}$$

In conclusion the CDF for the double exponential distribution is:

$$F_Y(y) = \begin{cases} \frac{1}{2}e^{(\frac{y-\mu}{b})} & \text{for } y \leq \mu \\ 1 - \frac{1}{2}e^{\frac{\mu-y}{b}} & \text{for } y \geq \mu \end{cases}$$

D and E

First to generate these random values we need to create the inverse functions.

For the case when:

$$y < \mu$$

$$u = \frac{1}{2}e^{\frac{y-\mu}{b}}$$

$$b * log(2u) = y - \mu$$

Thus

$$y = \mu + b * log(2u)$$

for

$$u < 0.5$$

$$y < \mu$$

For the case when

$$y \geq \mu$$

$$u = 1 - \frac{1}{2}e^{-\frac{y-\mu}{b}}$$
$$-b * log(2 - 2u) = y - \mu$$

Thus

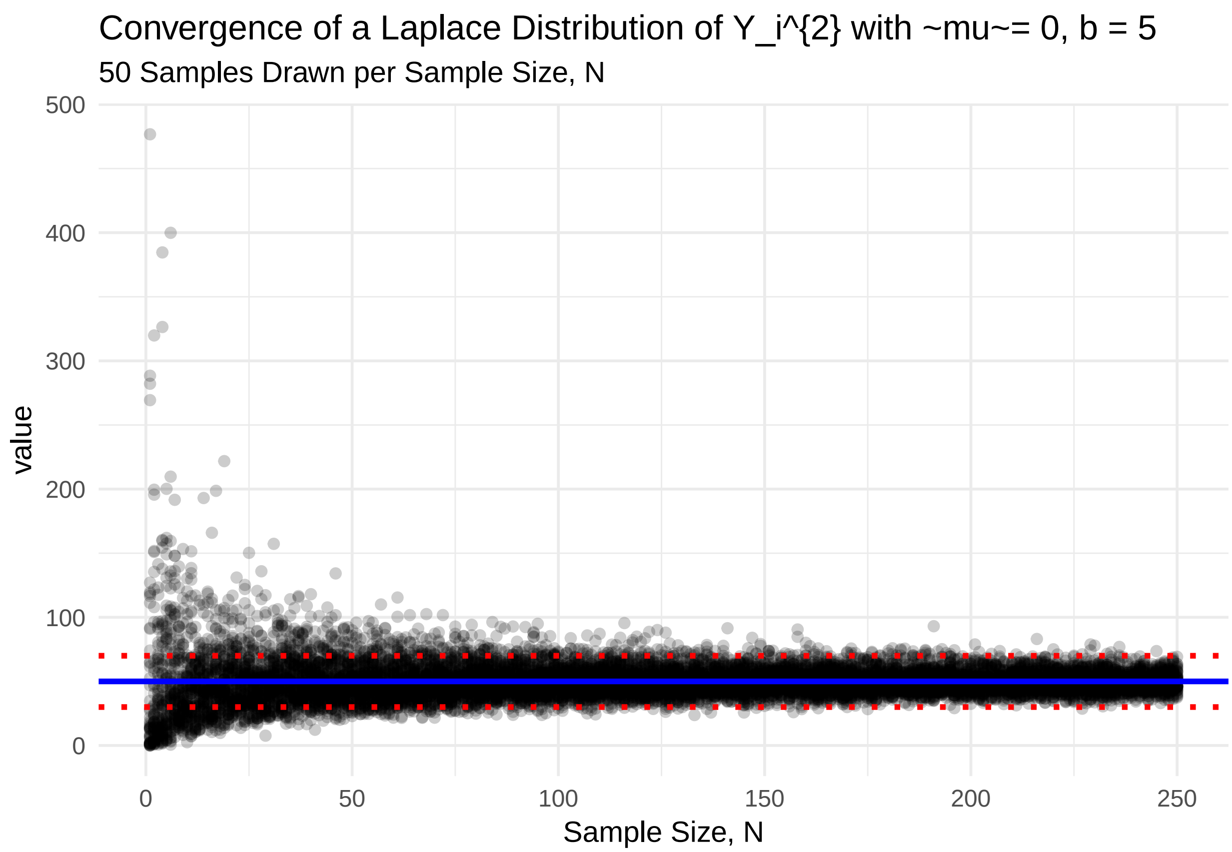
$$\mu - b * log(2 - 2u) = y$$

For $u > 0.5$, $x \geq \mu$

Using these inverse functions we can now do simulations.

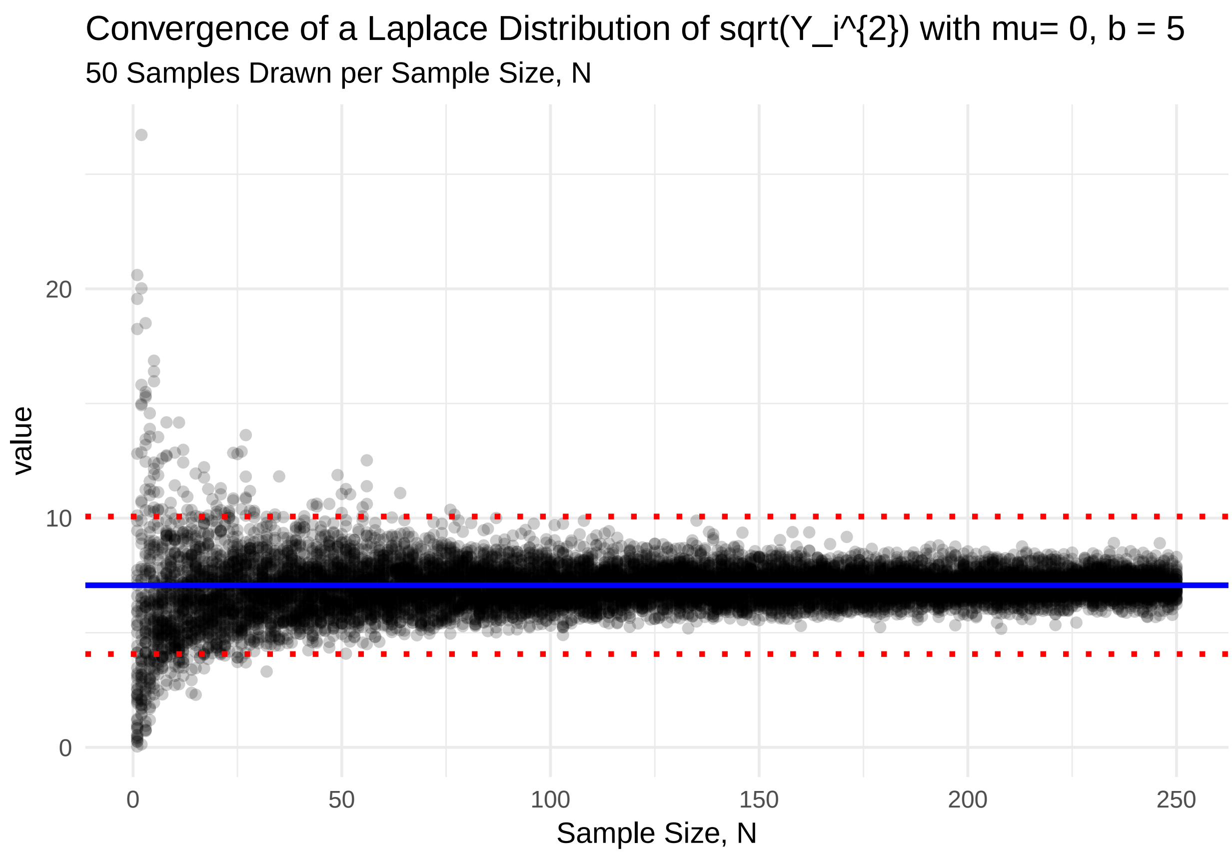
```
set.seed(336)


rdoublex <- function(u, mu = 0, b = 5){
  if( u < 0.5){
    out_come_1 <- mu + b*log(2*u)
    out_come_1
  } else{
     out_come_2 <- mu - b*log(2-2*u)
     out_come_2
  }

}



output_matrix <- matrix(0, nrow= 250, ncol = 50)

# Basic Function
for( j in 1:50){
  for(i in 1:250){
    random_value <- runif(i, 0, 1)
    output_matrix[i, j]<- mean(vapply(random_value, FUN = rdoublex, double(1)))
  }
}

p1 <- output_matrix %>%
  as.data.frame() %>%
  mutate(N = 1:250) %>%
  gather(replication, value, -N) %>%
  ggplot(aes(N, value))+
  geom_point(alpha=1/5)+
  theme_minimal()+
  labs(
    title = "Convergence of a Laplace Distribution with mu = 0, b = 5",
    subtitle = "50 Samples Drawn per Sample Size, N",
    x = "Sample Size, N"
  )

#L
```

```r
output_matrix <- matrix(0, nrow= 250, ncol = 50)

# Basic Function
for( j in 1:50){
  for(i in 1:250){
    random_value <- runif(i, 0, 1)
    output_matrix[i, j]<- mean(vapply(random_value, FUN = rdoublex, double(1))^2)
  }
}

limit_one <- 20
p2 <- output_matrix %>%
  as.data.frame() %>%
  mutate(N = 1:250) %>%
  gather(replication, value, -N) %>%
  ggplot(aes(N, value))+
  geom_point(alpha=1/5)+
  theme_minimal()+
  labs(
    title = "Convergence of a Laplace Distribution of Y_i^{2} with ~mu~= 0, b = 5",
    subtitle = "50 Samples Drawn per Sample Size, N",
    x = "Sample Size, N"
  )+
  geom_hline(yintercept = 50, size = 1, color = "blue")+
  geom_hline(yintercept = 50 + limit_one, linetype = "dotted",
             color = "red", size = 1)+
  geom_hline(yintercept = 50 - limit_one,
             linetype = "dotted", color = "red", size = 1)
p2
```




```
#K

output_matrix <- matrix(0, nrow= 250, ncol = 50)

# Basic Function
for( j in 1:50){
  for(i in 1:250){
    random_value <- runif(i, 0, 1)
    output_matrix[i, j]<- sqrt(mean(vapply(random_value, FUN = rdoublex, double(1))^2))
  }
}

limit_two <- 3
p3 <- output_matrix %>%
  as.data.frame() %>%
  mutate(N = 1:250) %>%
  gather(replication, value, -N) %>%
  ggplot(aes(N, value))+
  geom_point(alpha=1/5)+
  theme_minimal()+
  labs(
    title = "Convergence of a Laplace Distribution of sqrt(Y_i^{2}) with mu= 0, b = 5",
    subtitle = "50 Samples Drawn per Sample Size, N",
    x = "Sample Size, N"
  )+
  geom_hline(yintercept = 7.07, size = 1, color = "blue")+
```

```r
  geom_hline(yintercept = 7.07 + limit_two,
             linetype = "dotted", color = "red", size = 1)+
  geom_hline(yintercept = 7.07 - limit_two,
             linetype = "dotted", color = "red", size = 1)
p3
```



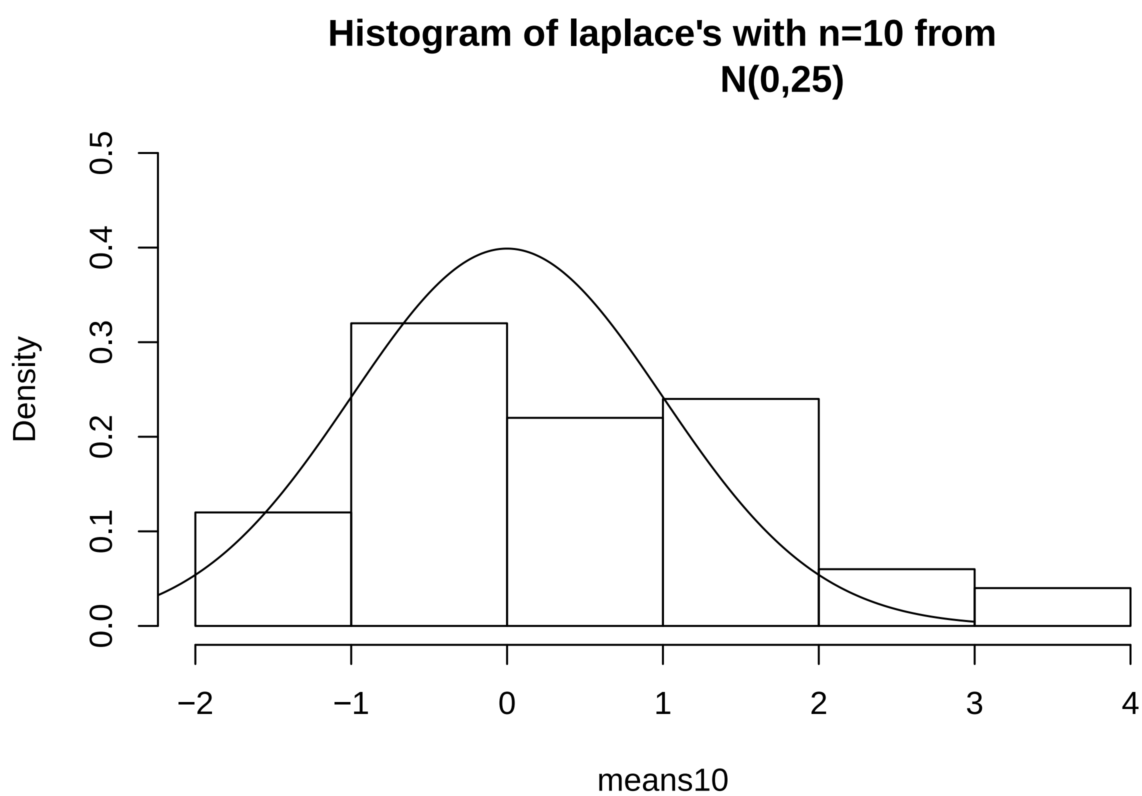
These plots show that as the sample size increases the values we see approach the expected values. This is graphically showing the convergence of these functions to their expected values as the sample size increases.

Part 2

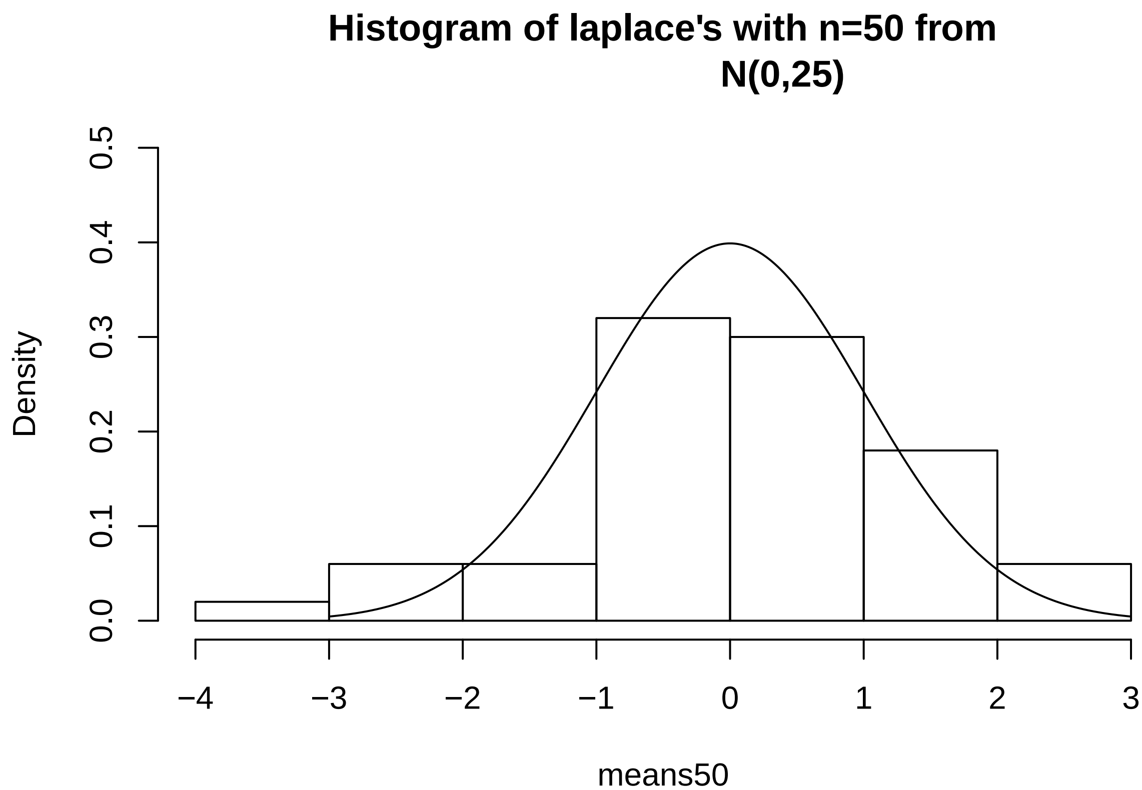
2

```r
#Simulate 10000 samples of size 10,50,100,250 from Laplace distribution
n<-c(10,50,100,250)
N<-50
mu<-0
beta<-5

rlaplace <- function(n = 1, mu = 0, beta = 5) {
  q <- runif(n)
  ifelse(q < 0.5, beta * log(2 * q) + mu, -beta * log(2 * (1 - q)) + mu)
}
```

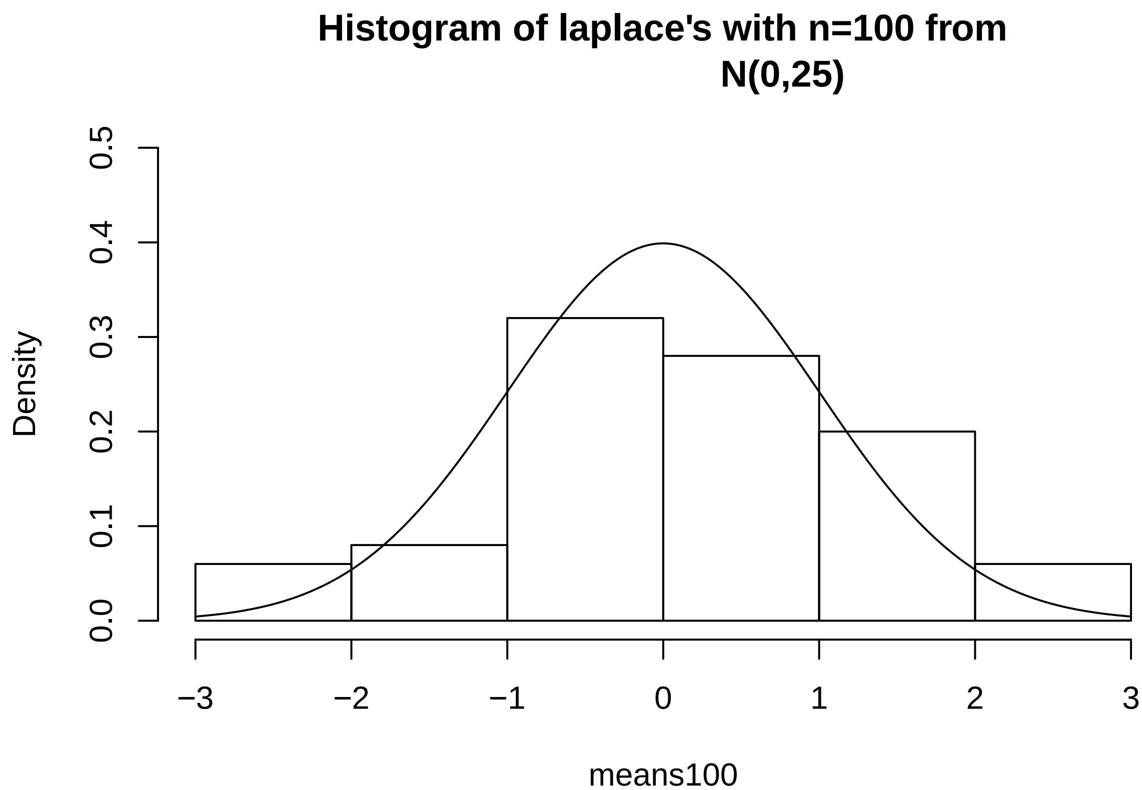
```
#list to save data values in
data<-list()
for(i in 1:length(n)){data[[i]]<-matrix(0,nrow=N,ncol=n[i])}
#Create the data
#loop over sample sizes
for (j in 1:length(n)){
  #loop over data sets
  for (i in 1:N){
    data[[j]][i,]<-rlaplace(n=n[j],0,5)
    }
}
#calculate the z statistic for each sample
means10<-apply(X=data[[1]],FUN=function(data){(mean(data)-mu)/(beta/sqrt(n[1]))},MARGIN=1)
means50<-apply(X=data[[2]],FUN=function(data){(mean(data)-mu)/(beta/sqrt(n[2]))},MARGIN=1)
means100<-apply(X=data[[3]],FUN=function(data){(mean(data)-mu)/(beta/sqrt(n[3]))},MARGIN=1)
means250<-apply(X=data[[3]],FUN=function(data){(mean(data)-mu)/(beta/sqrt(n[4]))},MARGIN=1)

# Make Histog
hist(means10,main=paste("Histogram of laplace's with n=",n[1]," from
                       N(",mu,",",beta^2,")",sep=""), ylim=c(0,.5), prob=T)
lines(seq(from=-3,to=3,by=0.01),dnorm(seq(from=-3,to=3,by=0.01)))
```



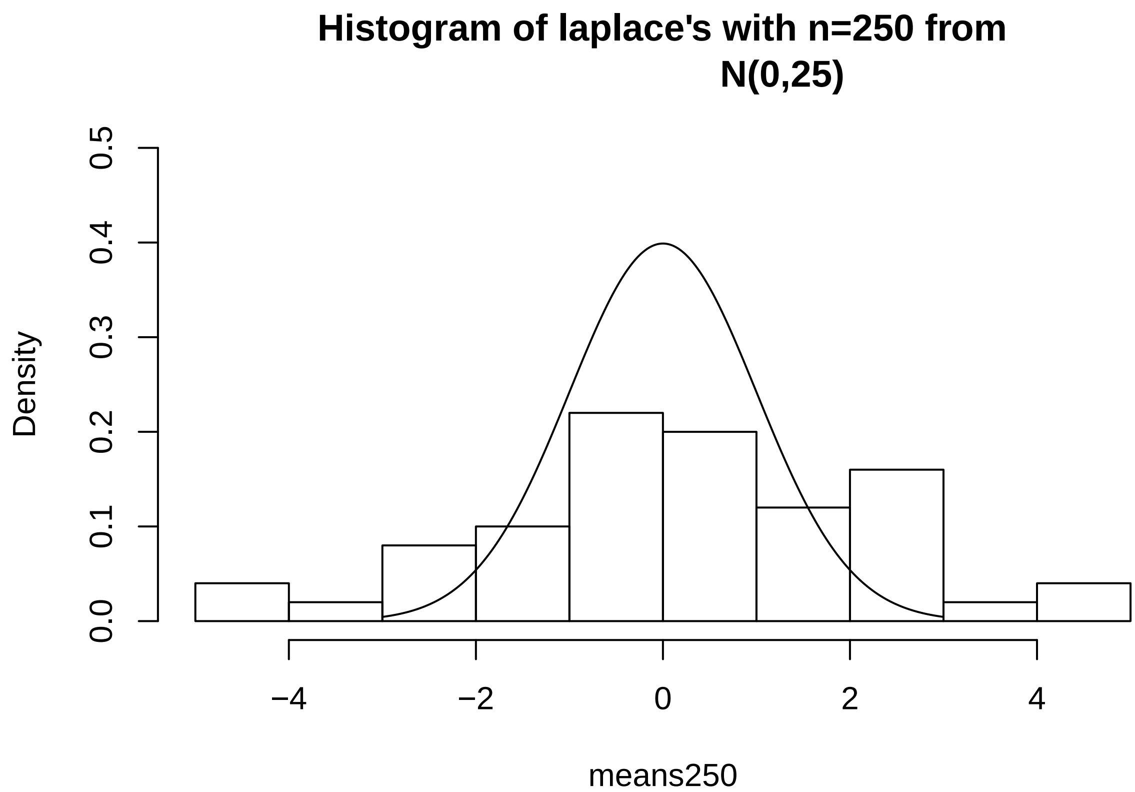

```
hist(means50,main=paste("Histogram of laplace's with n=",n[2]," from
                       N(",mu,",",beta^2,")",sep=""),prob=T,ylim=c(0,.5),)
lines(seq(from=-3,to=3,by=0.01),dnorm(seq(from=-3,to=3,by=0.01)))
```




```r
hist(means100,main=paste("Histogram of laplace's with n=",n[3]," from
                         N(",mu,",",beta^2,")",sep=""),prob=T,ylim=c(0,.5),)
lines(seq(from=-3,to=3,by=0.01),dnorm(seq(from=-3,to=3,by=0.01)))
```




```
hist(means250,main=paste("Histogram of laplace's with n=",n[4]," from
                          N(",mu,",",beta^2,")",sep=""),prob=T,ylim=c(0,.5),)
lines(seq(from=-3,to=3,by=0.01),dnorm(seq(from=-3,to=3,by=0.01)))
```




```r
par(mfrow= c(2,2))
```

3

$$Var(L) = Var(\frac{1}{n}\sum_{i=1}^{2} Y_i^2$$

$$= \frac{1}{n^2}\sum_{i=1}^{2} Var(Y_i^2)$$

$$= \frac{1}{n^2}\sum_{i=1}^{2}\Big[E(Y_i^2)^2 - (E(Y_i^2))^2\Big]$$

Given that for kurtosis:

$$k(x) = E\Big(\frac{x-\mu}{\sigma}\Big)^4 = \frac{E((x-\mu)^4)}{(E(x-\mu)^2)^2}$$

And given that the kurtosis of this place is 6:

$$6 = \frac{E((x-\mu)^4)}{(E(x-\mu)^2)^2}$$

Thus

$$E(Y_i^4) = 6(E(Y_i^2))^2$$

$$Var(L) = \frac{1}{n^2}\sum_{i=1}^{n}\left(6*(E(Y_i^2))^2 - ((E(Y_i^2))^2\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(5*(E(Y_i^2))^2)$$

And

$$\mu = E(L) = E(\frac{1}{n}\sum_{i=1}^{n}Y_i^2$$

by the central limit theorem

$$\frac{L-\mu}{\frac{\sqrt{Var(L)}}{\sqrt{n}}} \xrightarrow{d} N(0,1) \blacksquare$$

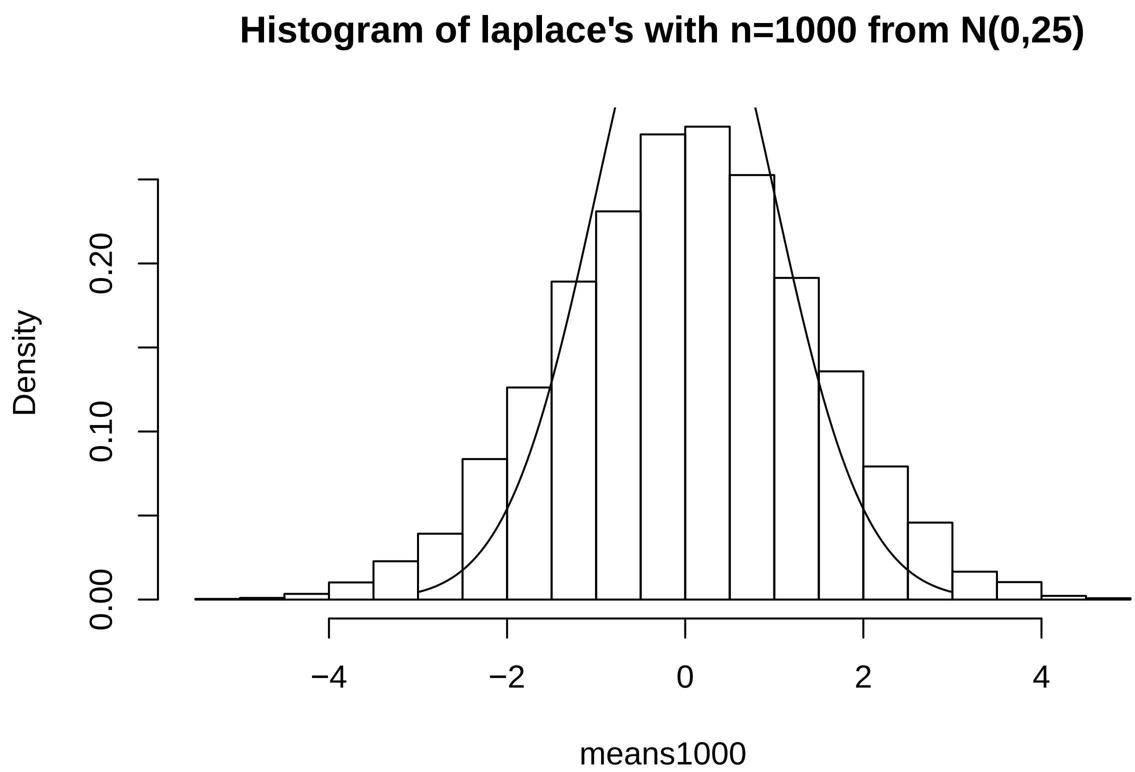
4

5

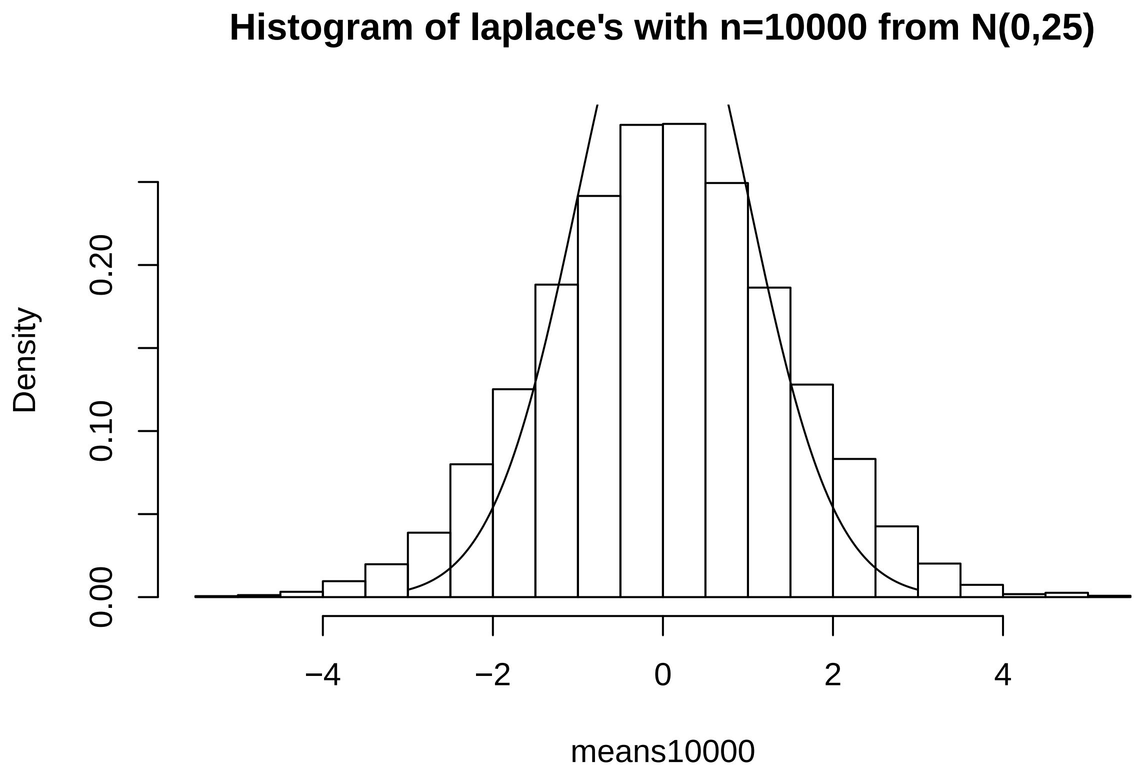
6

```
#Simulate 10,000 samples of size 1000 and 10,000 from Laplace distribution

n<-c(1000, 10000)
N<-10000
mu<-0
beta<-5

#list to save data values in

data<-list()
for(i in 1:length(n)){data[[i]]<-matrix(0,nrow=N,ncol=n[i])}
#Create the data
#loop over sample sizes
for (j in 1:length(n)){

  #loop over data sets
  for (i in 1:N){data[[j]][i,]<-rlaplace(n=n[j],0,beta = beta)}
}

#calculate the z statistic for each sample

means1000<-apply(X=data[[1]],FUN=function(data){(mean(data)-mu)/(beta/sqrt(n[1]))},
                 MARGIN=1)
means10000<-apply(X=data[[2]],FUN=function(data){(mean(data)-mu)/(beta/sqrt(n[2]))},
                  MARGIN=1)
```

```r
hist(means1000,main=paste("Histogram of laplace's with n=",n[1],
                          " from N(",mu,",",beta^2,")",sep=""),prob=T)
lines(seq(from=-3,to=3,by=0.01),dnorm(seq(from=-3,to=3,by=0.01)))
```




```r
hist(means10000,main=paste("Histogram of laplace's with n=",n[2],
                           " from N(",mu,",",beta^2,")",sep=""),prob=T)
lines(seq(from=-3,to=3,by=0.01),dnorm(seq(from=-3,to=3,by=0.01)))
```




```r
par(mfrow= c(2,2))
```

It appears that with n=30 there is inadequate coverage in the tails. This indicates that n=30 is not a sufficient sample size.