Understanding diffusion models (1): DDPM

$$x_t = \alpha_t x_{t-1} + \beta_t \varepsilon_t \, , \; \varepsilon_t \sim Normal\left(0, I\right)$$

We have $\alpha_t, \beta_t > 0$ and $\alpha_t^2 + \beta_t^2 = 1$ (an extra constraint which makes the subsequent calculation easier and which is reasonable because if you add more noise, the left part of the data is less).

$$\begin{aligned} x_t &= \alpha_t x_{t-1} + \beta_t \varepsilon_t \\ &= \alpha_t (\alpha_{t-1} x_{t-2} + \beta_{t-1} \varepsilon_{t-1}) + \beta_t \varepsilon_t \\ &= \ldots \\ &= (\alpha_t \ldots \alpha_1) x_0 + (\alpha_t \ldots \alpha_2) \beta_1 \varepsilon_1 + (\alpha_t \ldots \alpha_3) \beta_2 \varepsilon_2 + \ldots + \alpha_t \beta_{t-1} \varepsilon_{t-1} + \beta_t \varepsilon_t \end{aligned} \tag{1}$$

We assume $\varepsilon_{all} = (\alpha_t \ldots \alpha_2) \beta_1 \varepsilon_1 + (\alpha_t \ldots \alpha_3) \beta_2 \varepsilon_2 + \ldots + \alpha_t \beta_{t-1} \varepsilon_{t-1} + \beta_t \varepsilon_t$. ε_{all} is the sum of multiple independent normal variables, and therefore it also follows normal distribution, with the mean of 0, and the variance of $(\alpha_t \ldots \alpha_2)^2 \beta_1^2 + (\alpha_t \ldots \alpha_3)^2 \beta_2^2 + \ldots + \alpha_t^2 \beta_{t-1}^2 + \beta_t^2$.

With the assumption of $\alpha_t^2 + \beta_t^2 = 1$, see equation (1), the sum of square of coefficients $(\alpha_t \ldots \alpha_1)^2 + (\alpha_t \ldots \alpha_2)^2 \beta_1^2 + (\alpha_t \ldots \alpha_3)^2 \beta_2^2 + \ldots + \alpha_t^2 \beta_{t-1}^2 + \beta_t^2$ needs to be calculated.

$$\begin{aligned} &(\alpha_t \ldots \alpha_1)^2 + (\alpha_t \ldots \alpha_2)^2 \beta_1^2 \\ =& (\alpha_t \ldots \alpha_2)^2 \alpha_1^2 + (\alpha_t \ldots \alpha_2)^2 \beta_1^2 \\ =& (\alpha_t \ldots \alpha_2)^2 (\alpha_1^2 + \beta_1^2) \\ =& (\alpha_t \ldots \alpha_2)^2 \end{aligned}$$

$$\begin{aligned} &(\alpha_t \ldots \alpha_1)^2 + (\alpha_t \ldots \alpha_2)^2 \beta_1^2 + (\alpha_t \ldots \alpha_3)^2 \beta_2^2 \\ =& (\alpha_t \ldots \alpha_2)^2 + (\alpha_t \ldots \alpha_3)^2 \beta_2^2 \\ =& (\alpha_t \ldots \alpha_3)^2 \alpha_2^2 + (\alpha_t \ldots \alpha_3)^2 \beta_2^2 \\ =& (\alpha_t \ldots \alpha_3)^2 (\alpha_2^2 + \beta_2^2) \\ =& (\alpha_t \ldots \alpha_3)^2 \end{aligned}$$

Therefore,

$$(\alpha_t \ldots \alpha_1)^2 + (\alpha_t \ldots \alpha_2)^2 \beta_1^2 + (\alpha_t \ldots \alpha_3)^2 \beta_2^2 + \ldots + \alpha_t^2 \beta_{t-1}^2 = \alpha_t^2$$

Now we have

$$
\begin{aligned}
&(\alpha_t \dots \alpha_1)^2 + (\alpha_t \dots \alpha_2)^2 \beta_1^2 + (\alpha_t \dots \alpha_3)^2 \beta_2^2 + \dots + \alpha_t^2 \beta_{t-1}^2 + \beta_t^2 \\
&= \alpha_t^2 + \beta_t^2 \\
&= 1
\end{aligned}
$$