



MASTER'S DEGREE IN COMPUTATIONAL MATHEMATICS

MASTER'S FINAL PROJECT

Approximation Rates of Neural Network Interpolation Operators

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WE CERTIFY that the present report *Approximation rates by neural network interpolation operators* has been carried out under our direction by Michael Christopher Edwards, at the Department of Mathematics of Universitat Jaume I of Castellón, and constitutes his Master's thesis in the *University Master's Degree in Computational Mathematics*. In order to ensure an accurate record, we submit the aforementioned report, signing this certificate.

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Castellón, 30 October 2024

A Dios Todopoderoso: El Señor es mi roca, mi amparo, mi libertador. Salmos 18:2

The Lord is my rock, my shelter and my deliverer. Psalm 18:2

אדוני סלע, מחשִׁי וּמַפְלֵטִי

Declaration

I declare that this thesis has been written by me and that the work contained in it is my own,
unless explicitly stated otherwise in the text.

(Michael Christopher Edwards)

Abstract

In this master's thesis, the work of Qian and Yu (2022) is reviewed, which investigates the approximation rates of the interpolation complex using neural network interpolation operators. We review the theoretical foundations and error estimates of Feedforward Neural Networks (FFNs) and their efficiency and limitations for function interpolation tasks. We thoroughly examine the ability of FFNs to approximate continuous functions by employing the uniform approximation theorem, alongside the modulus of continuity and asymptotic expansions.

We derive these rates of approximation and convergence and gain some important insights into the capabilities and limitations of FFNs in terms of the network architecture, the choice of activation functions, and the data complexity. We examine the algebraic properties of neural network transformations via the geometric structure of the weight space and extend the notions of Lagrangian Interpolators to apply and expand upon them.

This thesis contributes to the study of novel interpolation operators, and a Kantorovich-type invariant is introduced to analyse the algebraic properties of neural network transformations further. We achieve this by examining the geometric structure of the weight space and extending notions of Lagrangian interpolators.

In addition, this thesis also introduces a new trigonometric [Re]-ctified [R]-omanovski (**Arccot or ArcCotangent**) [U]-nit (**ReLU**) activation function and its Lagrangian, Extended Lagrangian, Barycentric Lagrangian, and Chebyshev Barycentric interpolators. These contributions add to the 'toolkit' for neural network interpolation and function approximation in data analytics for the field of machine learning.

In conclusion, these results have important implications for theoretical developments in machine learning and its application to improving the training of neural networks and the utilisation of computational resources. This thesis contributes to a greater understanding of the mathematical structure of neural networks with new methods for their analysis and application.

Keywords

Feedforward Neural Networks (FFN), Neural Network Interpolation, Kantorovich-Type Invariant, Function Interpolation, "ReLU", Activation Function, Machine Learning, Uniform Approximation Theorem, Modulus of Continuity, Asymptotic Expansion, Rates of Approximation and Convergence, Lagrangian, Extended Lagrangian and Barycentric Lagrangian, and Chebyshev Barycentric Interpolators.

Lay Summary

Quantitative and qualitative analysis of the current trends shows that artificial intelligence and neural networks are now used for modelling biological systems and PDEs to formulate the discrete-time functional approximations. In this master's thesis, we investigate the mathematical theory of neural networks with a focus on the approximation of functions. Neural networks, especially feedforward architectures, are frequently employed in the field of machine learning to find solutions to a given problem. The research is based on the work of Qian and Yu (2022) and introduces new interpolation methods based on neural networks. These methods include new activation functions, such as the Rectified Romanovski Unit (ReRU) which improve accuracy and computational time.

The first component of this thesis examines the capability of neural network operators when employed to approximate various mathematical functions and their derivatives. Utilising advanced mathematical tools, including Sobolev and Besov spaces, the thesis establishes the rates of approximation and error bounds for these neural network operators. Additionally, it investigates the impact of the smoothness of the activation functions on the quality of the approximations.

To our knowledge, the following work is most relevant to the goal of enhancing the reliability of machine learning algorithms. This work demonstrates how novel mathematical formulations can yield improved training methods and optimal designs of neural networks. These results are particularly advantageous in applications that require precise function approximations, such as numerical analysis and scientific computing.

The second part examines the relationship between this research and its role in bridging the gap between mathematical theory and practical applications in machine learning. It contributes to a deeper understanding of how neural networks approximate functions and provides novel tools for their analysis and implementation.

The third component reviews the link between this research and its application in bridging the gap between the theory and practice of machine learning. It contributes to the current knowledge on how neural networks function, as well as how to analyse and implement them for function approximation.

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List of Symbols

a, b	Interval endpoints
$C[a, b]$	Space of continuous functions on $[a, b]$
$C^r[a, b]$	Space of r times continuously differentiable functions on $[a, b]$
$W^{k,p}[a, b]$	Sobolev space of functions on $[a, b]$
$B_{p,q}^s[a, b]$	Besov space of functions on $[a, b]$
σ	Activation function
φ	Modified activation function
$S_{n,\sigma}$	Neural network interpolation operator
$K_{n,\sigma}$	Kantorovich-type neural network operator
$\omega(f, \delta)$	Modulus of continuity of function f
$\ \cdot\ $	Norm (context-dependent)
$\ \cdot\ _p$	L^p norm
$\ \cdot\ _\infty$	Supremum norm
$f^{(v)}$	v -th derivative of function f
$O(\cdot)$	Big O notation
α, β	Parameters (context-dependent)
ε	Small positive number
δ	Small positive number (often used for continuity)
$K(f, t)$	K-functional
$L^p[a, b]$	Space of p -integrable functions on $[a, b]$
\mathbb{R}	Set of real numbers
\mathbb{N}	Set of natural numbers
∂	Partial derivative operator
∇	Gradient operator
\int	Integral
Σ	Summation
\prod	Product
\forall	For all
\exists	There exists
\in	Element of
\subset	Subset of
\cup	Union
\cap	Intersection
\rightarrow	Tends to / Maps to (context-dependent)
\cong	Isomorphic to
\equiv	Equivalent to

1 Introduction

In this thesis, we explore the use of **Feedforward Neural Networks (FFNs)** in solving complex optimisation problems, building on the work of Qian and Yu (2022) [77] on neural network interpolation operators and incorporating concepts from information geometry. Additionally, this thesis summarises the application of FFNs to optimisation problems, integrating principles of neural network interpolation and information geometry.

We demonstrate that integrating information geometry into the analysis of FFNs not only deepens our understanding of their approximation capabilities but also opens new avenues for optimising neural network architectures. The implications of these findings extend beyond theoretical interest, offering practical tools to enhance the efficiency of neural network training. Specifically, our results illustrate improvements in weight initialisation strategies and the development of more robust models capable of handling a wide range of complex functions. This research lays the groundwork for future explorations at the intersection of neural networks, optimisation theory, and geometric analysis.

Feedforward Neural Networks (FFNs) have emerged as powerful tools for function approximation and optimisation problems across various fields of science and engineering [72]. Characterised by their layered structure and unidirectional information flow, FFNs have demonstrated remarkable capabilities in approximating complex and non-linear functions. The **universal approximation theorem**, a cornerstone of neural network theory, states that FFNs with a single hidden layer can approximate any continuous function on a compact subset of \mathbb{R}^n with arbitrary accuracy, provided sufficient hidden units are available.

However, the **uniform approximation theorem** is central to the analysis of FFNs' approximation capabilities, as it quantifies the error bounds of neural network approximations. When expressed in terms of the modulus of continuity, this theorem provides a measure of a function's uniform continuity. For a function f defined on an interval $[a, b]$, the modulus of continuity $\omega(f, \delta)$ quantitatively describes the extent to which the function can change over a given interval. The work of Qian and Yu (2022) [77] has extended these concepts, providing explicit error estimates for neural network interpolation operators with respect to the modulus of continuity.

Moreover, **asymptotic expansions** play a critical role in understanding the approximations

of FFNs as the network size increases. These expansions shed light on convergence rates and assist in analysing the trade-off between network complexity and approximation accuracy. This research aims to integrate these mathematical tools with concepts from information geometry, leading to advanced training algorithms such as **natural gradient descent**, which optimise the geometric structure of the parameter space. These developments not only enhance our theoretical understanding of FFNs but also offer practical strategies for improving their real-world performance.

In summary, the relationship between approximation theory, interpolation methods, and neural network architectures has become an increasingly important area of study in applied mathematics and computer science. This thesis explores the profound connections between these fields, focusing on the approximation rates achieved by neural network interpolation operators and their relationship to singular twisted sums.

The [Re]-ctified [R]-omanovski (**Arccot or ArcCotangent**) [U]-nit (**ReLU**) function illustrated in Chapter 6, employing a shift or bias term and amplitude, effectively ensures that the function outputs only positive values by adjusting the Arccotangent curve. The formulation:

$$\mathcal{A} = \text{Amplitude} \times \cot^{-1}(x) + \text{shift}$$

Key aspects of this state:

1. **Amplitude:** Scales the output of the arc-cotangent function. A larger amplitude stretches the function vertically, increasing its range of output values.
2. **Shift:** Shifts the entire graph upwards to ensure the function is always positive (Rectified). Without this, the arc-cotangent function could output negative values for specific inputs.
3. **Rectification:** Moreover, if we are adding the shift, ensuring that negative values of $\cot^{-1}(x)$ are lifted into the positive range, effectively rectifying the function without needing to use the traditional rectification method ($\max(0, f(x))$).

1.1 Background and Motivation

The study of function approximation has been one of the central topics in mathematical analysis over the years, with important implications in other areas of science and technology. The last few decades have seen the rise of neural networks as a revolutionary approach to function approximation and regression, building on the classical theory by DeVore (1993) [39] and Hornik (1989) [50]. The universal approximation properties of **Feedforward Neural Networks (FNNs)** were first established by Cybenko (1989) [38] and Hornik et al. (1989) [50], with the proof of the ability of FNNs with a single hidden layer to approximate any continuous function on a compact set, to any desired level of accuracy, with enough hidden neurons.

Among FNNs, the **Multi Layer Perceptron Neural Networks (MLPs)** have been identified as being most suitable for function approximation and optimisation problems (Barron 1993). An MLP is a type of FNN with multiple layers of nodes (input, hidden, output) that maps inputs to outputs. The layers are fully connected; that is, each neuron in one layer is connected to every neuron in the next layer. These networks, which are characterised by their layered structure and the one-way flow of information, have been used to model a large variety of functions. The **universal approximation theorem** is a classic theorem in neural network theory: it states that FNNs with one hidden layer can represent any function that is continuous on a compact subset of \mathbb{R}^n to any degree of accuracy with a sufficient number of hidden units (Cybenko, 1989).

Nevertheless, while the universal approximation theorem gives a proof of the existence of such approximations, it fails to capture the efficiency or the optimal architecture of such networks. This disparity between theoretical potential and practical application has led to the study of approximation properties and rates of neural networks (Mhaskar, 1996).

This gap has been closed by Qian and Yu (2022) [77] who proposed new **neural network interpolators** and **neural network operators** with activation functions designed to meet certain objectives. A **neural network interpolator** is a type of neural network that is used to estimate values between data points and produces smooth estimations in the continuous case. On the other hand, a **neural network operator** is a neural network that learns the mappings between complete function spaces instead of merely the pointwise approximations. This concept is important for solving problems such as PDEs where the operator defines the transformation between the input and output of functional information. The approach of Qian and Yu provides explicit error bounds for the approximation of continuous functions and extends to the simultaneous approximation of functions and their derivatives. This is especially important in areas such as numerical analysis and the solution of PDEs where it is necessary to approximate both the function and its derivatives.

Despite the great success of neural networks, the problem of approximation complexity re-

mains unresolved. Namely, the challenge lies in determining the optimal neuronal cardinality for guaranteed approximation accuracy in appropriate function spaces. Building on Barron's work on approximation rates in $L^2(\mathbb{R}^d)$ and the subsequent contributions by Maiorov, research has been able to establish a solid foundation for the analysis of the approximation properties of neural network mappings (Barron, 1993; Maiorov, 1999). Of particular interest is the simultaneous approximation of functions and their derivatives in Sobolev spaces $W^{k,p}$, which has important applications, for example, in the numerical solution of partial differential equations, computer graphics, and scientific visualisation (Hornik et al., 1990; Li, 1996). The results of Qian and Yu (2022) [77] extend these theoretical results to finite-dimensional settings, and establish sharp error bounds and new interpolation properties under certain regularity assumptions that are nevertheless useful for high-dimensional situations.

In line with the work of Qian and Yu (2022) [77] on neural network expressivity, this paper investigates the fundamental approximation properties of Multilayer Perceptron (MLP) architectures. Although the universal approximation theorem due to [38], with quantitative analysis by [10], states that MLPs, that is, compositions of affine transformations and non-linear activations, can represent any continuous function in $L^2(\mathbb{R}^d)$, the relationship between network architecture and approximation efficiency is not well understood. This paper further supports the classical results by providing explicit error estimates and convergence rates, as well as by establishing quantitative connections between the theoretical results and the practical performance bounds in contemporary deep learning scenarios.

DEFINITION 1.1 (MLP Architecture). *An L -layer MLP with activation σ is defined as:*

$$f(x) = W_L \sigma(W_{L-1} \cdots \sigma(W_1 x + b_1) \cdots + b_{L-1}) + b_L \quad (1.1)$$

where W_i are weight matrices and b_i are bias vectors.

Neural networks have emerged as powerful function approximators, characterised by their rich geometric properties in high-dimensional spaces. Multilayer Perceptrons (MLPs) with hyperbolic tangent activation functions exhibit invariance under specific Lie group transformations, particularly $\text{SO}(n)$ rotations in the hidden layer space. The tanh activation function induces a smooth manifold structure \mathcal{M} in the parameter space, where the Fisher information metric provides a natural Riemannian geometry [5]. This geometric framework enables quantitative analysis of network capacity and optimisation dynamics, leading to improved training algorithms with reduced computational complexity of $\mathcal{O}(n \log n)$ for n -dimensional input spaces (Amari, 2001 [5]).

1.2 Table of Comparison of approximation results

In this thesis, we shall discuss interpolation methods in relation to experiments that compare the Qian-Yu methods to the benchmarks derived from classical methods. We propose techniques for verifying the precision and conducting error analysis of each method to demonstrate a reduction in approximation error. We summarise the results of these experiments in Chapters 3 to 6 in the table 1.1 below (see Appendix D.5 and D.6).

Study/Method	Norm	Activation/ Interpolator	Explicit archit.	Simult. approx.	Low- dim result
Cybenko (1989) [38]	$C[0, 1]^n$	ReLU	✗	✗	✗
Li et al. (2019) [66]	L_2	RePU	✗	✗	✗
Duan et al. (2021) [43]	$W^{1,2}$	ReLU	✗	✓	✗
Abdeljawad and Grohs (2022) [2]	$W^{s,p}$	ReCU	✗	✓	✗
Qian and Yu (2022) [77]	$C[a, b]$	Custom	✓	✓	✓
This work	C^s	Arccot(ReLU)	✓	✓	✓
Lagrangian	$W^{1,\infty}$	Polynomial	✓	✗	✓
Extended Lagrangian	$W^{k,p}$	Hermite	✓	✓	✓
Barycentric Lagrangian	$C[a, b]$	Rational	✓	✗	✓
Chebyshev Barycentric	$C^\alpha[a, b]$	Rational	✓	✗	✓

Table 1.1: Comparison of approximation results of various neural network activation functions on functions with different norms. The table contrasts multiple studies based on their ability to explicitly define the architecture, perform simultaneous approximation, and provide low-dimensional results. The norms $C[0, 1]^n$, L_2 , $W^{1,2}$, $W^{s,p}$, and C^s describe the smoothness or space in which these functions operate. The Cybenko (1989)[38] study uses the ReLU activation function, which does not offer explicit architecture, simultaneous approximation, or low-dimensional results. Li et al. (2019)[66] introduced RePU, which also lacks these characteristics. Duan et al. (2021)[43] used ReQU, achieving simultaneous approximation but not explicit architecture or low-dimensional results. Abdeljawad and Grohs (2022)[2] utilised ReCU, achieving the same characteristics. Qian and Yu (2022)[77] introduced a custom activation function capable of providing all desired properties, while this work proposes **[Re]-ctified [R]-omanovski (Arccot or ArcCotangent) [U]-nit (ReLU)**, similarly offering explicit architecture, simultaneous approximation, and low-dimensional results.

1.3 Overview

In this introduction, for simplicity, we present rigorous proofs of key approximation theorems in neural networks, utilising convolution operations, topological theorems, and mathematical concepts pertinent to neural network approximation theory (See [8], [9]. The proofs are structured to provide a profound understanding of the mechanisms underlying the approximation capabilities of neural networks, especially in the context of feedforward architectures and activation functions with specific properties.

The study of the approximation capabilities of neural networks has its roots in the seminal work of Cybenko (1989)[38] and Hornik (1989)[50], who independently established the universal approximation theorem for feedforward neural networks. This theorem states that a neural network with a single hidden layer can approximate any continuous function on a compact subset of \mathbb{R}^n with the desired accuracy, provided that a sufficient number of hidden units is available.

An important open question concerns the efficient calibration of neural network interpolators and hyperparameters. Subsequent research has concentrated on quantifying the efficiency of these approximations. Barron (1993)[10] provided bounds on the approximation error in terms of the number of hidden units, demonstrating that for certain classes of functions, the approximation error decays as $O(1/n)$, where n is the number of hidden units. This result was significant because it provided a convergence rate independent of the input dimension.

The connection between neural networks and approximation theory was further explored by Mhaskar (1996)[72], who demonstrated that neural networks can achieve the same approximation rates as classical approximation methods for functions in Sobolev spaces. The work on functional spaces bridged the gap between traditional approximation theory and neural network research. In addition, the Kantorovich phenomena[61], named after the Soviet mathematician Leonid Kantorovich, refer to a set of principles and methods in functional analysis and optimisation theory. These phenomena have profound implications in various fields, including probability theory, economics, and machine learning. The Kantorovich phenomena[61] extend beyond distance metrics to encompass lifting procedures, which facilitate the extension of functions on a base space to functions on measures over that space.

DEFINITION 1.2 (Kantorovich Lifting). *Given a function $f : X \rightarrow \mathbb{R}$, its Kantorovich lifting \hat{f} is defined on the space of probability measures $\mathcal{P}(X)$ as:*

$$\hat{f}(\mu) = \int_X f, d\mu$$

This lifting procedure preserves many properties of the original function, such as continuity and convexity, and plays a primary role in optimal transport theory.

The Kantorovich Invariant has proven fundamental in modern machine learning, particularly in neural network design and analysis. Most notably, it underpins the Wasserstein distance used in Wasserstein Generative Adversarial Networks (WGANS), introduced by Arjovsky et al. (2017) [7]. The Wasserstein distance, derived from Kantorovich duality in optimal transport theory, measures the cost of transformation between probability distributions. This provides WGANS with more stable training characteristics and interpretable learning curves compared to traditional GANs (e.g. Jensen-Shannon divergence). Thus, WGANS are more stable and learn more efficiently within the inherent Kantorovich Invariant framework.

EXAMPLE 1.1 (Wasserstein GANs). *Arjovsky et al. (2017) [7] introduced Wasserstein GANs, which employ the Wasserstein distance (closely related to the Kantorovich Invariant) as a loss function, providing more stable training and meaningful learning curves compared to conventional GANs.*

1.4 Objectives and Scope

This thesis aims to extend and deepen the results of Qian and Yu (2022) [77] by exploring the connections between their neural network interpolation operators and the theory of singular twisted sums. Our primary objectives are:

1. To construct new classes of neural network interpolation operators with enhanced approximation properties.
2. To establish rigorous error bounds for these operators in various function spaces, including Hölder, Sobolev, and Besov spaces.
3. To investigate the relationship between the smoothness of activation functions and the approximation rates of the resulting operators.
4. To develop a comprehensive theory linking interpolation operators to singular twisted sums in Banach spaces.
5. To extend the results to multivariate and multidimensional settings.
6. To explore practical applications of these theoretical results in machine learning and numerical analysis.

Our approach draws upon various mathematical tools and concepts, including K-functionals, the Berens-Lorentz lemma, and various interpolation techniques in Banach spaces. We also utilise results from the theory of Orlicz and Köthe function spaces, deriving a novel framework.

Lebesgue Measure

The Lebesgue measure extends the concept of length to more general sets, including unions of disjoint sets and irregular shapes, and is foundational in real analysis.

DEFINITION 1.3. *For a set $A \subset \mathbb{R}$, the Lebesgue measure $m(A)$ represents the "size" or "length" of the set. If A is an interval, its measure corresponds to its length. For more complex sets, the measure $m(A)$ is the infimum of the sum of lengths of intervals that cover A .*

EXAMPLE 1.2. Consider the set $A = (0, 1) \cup (2, 3)$. The Lebesgue measure of this set is:

$$m(A) = m((0, 1)) + m((2, 3)) = 1 + 1 = 2. \quad (1.2)$$

Additionally, the measure of a finite set, such as $\{1, 2, 3\}$, is 0.

COROLLARY 1.1. Every countable set, including the rationals $\mathbb{Q} \subset \mathbb{R}$, has Lebesgue measure zero.

Minkowski's Inequality

Minkowski's inequality generalises the triangle inequality to integrals in L^p -spaces. It states that for any two functions f and g in L^p , the L^p norm of their sum is less than or equal to the sum of their L^p norms, expressed mathematically as:

DEFINITION 1.4. : For functions f and g in L^p -space, where $1 \leq p \leq \infty$, Minkowski's inequality is:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad (1.3)$$

where $\|f\|_p = (\int |f|^p d\mu)^{1/p}$.

EXAMPLE 1.3. Let $f(x) = x$ and $g(x) = 2x$ on the interval $[0, 1]$, and let $p = 2$. Then:

$$\|f + g\|_2 = \left(\int_0^1 |f(x) + g(x)|^2 dx \right)^{1/2} = \left(\int_0^1 9x^2 dx \right)^{1/2}.$$

The integral is:

$$\int_0^1 9x^2 dx = 9 \cdot \frac{1}{3} = 3,$$

so

$$\|f + g\|_2 = \sqrt{3}.$$

Similarly, the individual norms are:

$$\|f\|_2 = \left(\int_0^1 x^2 dx \right)^{1/2} = \left(\frac{1}{3} \right)^{1/2} = \frac{1}{\sqrt{3}},$$

and

$$\|g\|_2 = \left(\int_0^1 (2x)^2 dx \right)^{1/2} = \left(4 \cdot \frac{1}{3} \right)^{1/2} = \frac{2}{\sqrt{3}}.$$

Therefore, the sum of the norms is:

$$\|f\|_2 + \|g\|_2 = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \sqrt{3}.$$

Thus, Minkowski's inequality holds with equality:

$$\|f+g\|_2 = \|f\|_2 + \|g\|_2.$$

Hölder's Inequality

Hölder's inequality is a generalisation of the Cauchy-Schwarz inequality and is widely used in integrals and series. It states that for any measurable functions f and g and for conjugate exponents p and q (where $\frac{1}{p} + \frac{1}{q} = 1$), the integral of the product of the functions is bounded by the product of their L^p and L^q norms. Mathematically, this can be expressed as:

DEFINITION 1.5. : Let $f \in L^p$ and $g \in L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ (with $p, p' > 1$). Hölder's inequality states:

$$\int |fg| d\mu \leq \left(\int |f|^p d\mu \right)^{1/p} \left(\int |g|^{p'} d\mu \right)^{1/p'} \quad (1.4)$$

EXAMPLE 1.4. Let $f(x) = x^{1/2}$ and $g(x) = x^{1/2}$ on $[0, 1]$, with $p = 2$ and $p' = 2$. Then, the left-hand side of Hölder's inequality becomes:

$$\int_0^1 |x^{1/2} \cdot x^{1/2}| dx = \int_0^1 x dx = \frac{1}{2}.$$

Hölder's inequality gives:

$$\left(\int_0^1 x dx \right)^{1/2} \cdot \left(\int_0^1 x dx \right)^{1/2} = \left(\frac{1}{2} \right)^{1/2} \cdot \left(\frac{1}{2} \right)^{1/2} = \frac{1}{2}.$$

Thus, Hölder's inequality holds with equality for this choice of functions and exponents.

COROLLARY 1.2. : When $p = p' = 2$, Hölder's inequality reduces to the Cauchy-Schwarz inequality:

$$\int f g d\mu \leq \left(\int f^2 d\mu \right)^{1/2} \left(\int g^2 d\mu \right)^{1/2} \quad (1.5)$$

1.5 Neural Network Interpolation Operators

In this section, we introduce and analyse a class of neural network interpolation operators that form the core of our study. These operators are constructed using newly defined activation functions and possess powerful approximation properties.

Construction of Operators

We begin by defining our class of neural network interpolation operators, inspired by the work of Qian and Yu [77], but with several key extensions and modifications.

Activation Functions

First, we introduce a new class of activation functions that will be primary in the construction of our operators.

DEFINITION 1.6 (Class $\mathcal{A}(m)$). *Let $m \in \mathbb{R}^+$ be fixed. We say a sigmoidal function $\sigma \in \mathcal{A}(m)$ if:*

1. $\sigma(x)$ is non-decreasing;
2. $\sigma(x) = 1$ for $x \geq m$, and $\sigma(x) = 0$ for $x \leq -m$.

Using functions from this class, we define a key building block for our operators:

DEFINITION 1.7. *For $\sigma \in \mathcal{A}(m)$, we define*

$$\varphi(x) := \sigma(x + m) - \sigma(x - m), \quad x \in \mathbb{R}. \quad (1.6)$$

LEMMA 1.1. *The function φ possesses the following properties:*

1. $\varphi(x)$ is non-negative;
2. $\varphi(x)$ is non-decreasing for $x < 0$, and non-increasing for $x \geq 0$;
3. $\text{supp}(\varphi) \subseteq [-2m, 2m]$;
4. $\varphi(x) + \varphi(x - 2m) = 1$ for $x \in [0, 2m]$.

Proof. These properties follow directly from the definition of φ and the properties of functions in $\mathcal{A}(m)$ within Qian-Yu's framework in Lemma 1 (P1-P4) [77]. □

EXAMPLE 1.5 (Basic Activation Function and $\mathcal{A}(m)$ Class). Consider a simple activation function $\sigma \in \mathcal{A}(1)$ defined as:

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(1 + \tanh(2x)), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (1.7)$$

This function belongs to $\mathcal{A}(1)$ because:

1. It is non-decreasing, which can be verified by examining its derivative:

$$\sigma'(x) = \frac{d}{dx} \left[\frac{1}{2}(1 + \tanh(2x)) \right] = \frac{1}{2} \cdot 2 \cdot \operatorname{sech}^2(2x) = \operatorname{sech}^2(2x) > 0 \quad (1.8)$$

for $x \in (-1, 1)$, and $\sigma'(x) = 0$ elsewhere.

2. $\sigma(x) = 1$ for all $x \geq 1$
3. $\sigma(x) = 0$ for all $x \leq -1$

The corresponding kernel function (or shape function) is given by:

$$\varphi(x) = \sigma(x+1) - \sigma(x-1) \quad (1.9)$$

This kernel function has the following properties:

1. $\operatorname{supp}(\varphi) \subset [-2, 2]$
2. $\varphi(x) + \varphi(x-2) = 1$ for all $x \in [0, 2]$

The support property can be verified by examining:

1. For $x \leq -2$: $\sigma(x+1) = \sigma(x-1) = 0$, thus $\varphi(x) = 0$
2. For $x \geq 2$: $\sigma(x+1) = \sigma(x-1) = 1$, thus $\varphi(x) = 0$

To verify the partition of unity property for $x \in [0, 2]$:

$$\varphi(x) + \varphi(x-2) = [\sigma(x+1) - \sigma(x-1)] + [\sigma(x-1) - \sigma(x-3)] = \sigma(x+1) - \sigma(x-3) \quad (1.10)$$

For $x \in [0, 2]$:

$$1. \quad x+1 \in [1, 3], \text{ so } \sigma(x+1) = 1$$

$$2. \quad x-3 \in [-3, -1], \text{ so } \sigma(x-3) = 0$$

Therefore, $\varphi(x) + \varphi(x-2) = 1 - 0 = 1$ for all $x \in [0, 2]$ (Lemma 1 [77].)

EXAMPLE 1.6 (Multivariate Case). For the function $f(x, y) = xy$ on $[0, 1]^2$ with $\mathbf{n} = (2, 2)$ and $m_1 = m_2 = 1$, the two-dimensional neural network interpolation operator is:

$$S_{n,\sigma}(f, x, y) = \sum_{k_1=0}^2 \sum_{k_2=0}^2 \frac{k_1 k_2}{4} \varphi\left(4\left(x - \frac{k_1}{2}\right)\right) \varphi\left(4\left(y - \frac{k_2}{2}\right)\right) \quad (1.11)$$

where $\varphi(x) = \sigma(x+1) - \sigma(x-1)$ and $\sigma \in \mathcal{A}(1)$ is defined as:

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(1 + \tanh(2x)), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (1.12)$$

The resulting kernel function $\varphi(x) = \sigma(x+1) - \sigma(x-1)$ has support in $[-2, 2]$ and satisfies the partition of unity property $\varphi(x) + \varphi(x-2) = 1$ for $x \in [0, 2]$.

For error analysis, we utilise the mixed modulus of continuity. For $f(x, y) = xy$, the mixed modulus of continuity is:

$$\omega_2(f, \delta_1, \delta_2)[0, 1]^2 = \sup_{|x-x'| \leq \delta_1, |y-y'| \leq \delta_2} |f(x, y) - f(x', y')| \quad (1.13)$$

When $\delta_1 = \delta_2 = \frac{1}{2}$ (our step size), we can analyse this as:

$$|xy - x'y'| = |xy - xy' + xy' - x'y'| \quad (1.14)$$

$$= |x(y - y') + y'(x - x')| \quad (1.15)$$

$$\leq |x||y - y'| + |y'||x - x'| \quad (1.16)$$

$$\leq 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1 \quad (1.17)$$

However, a discrete analysis shows:

$$\omega_2 \left(f, \frac{1}{2}, \frac{1}{2} \right)_{[0,1]^2} = \frac{1}{4} \quad (1.18)$$

This provides us with the error bound:

$$\|S_{n,\sigma}(f) - f\| \leq \omega_2 \left(f, \frac{1}{2}, \frac{1}{2} \right)_{[0,1]^2} = \frac{1}{4} \quad (1.19)$$

Neural Network Interpolation Operators

We now define our primary object of study:

DEFINITION 1.8 (Neural Network Interpolation Operator). *Let $f : [a,b] \rightarrow \mathbb{R}$ be a bounded and measurable function, $\sigma \in \mathcal{A}(m)$, and $n \in \mathbb{N}^+$. The neural network interpolation operator $S_{n,\sigma}$ acting on f is defined as:*

$$S_{n,\sigma}(f, x) := \sum_{k=0}^n f(x_k) \varphi \left(\frac{2m}{h} (x - x_k) \right), \quad x \in [a, b], \quad (1.20)$$

where $x_k := a + kh$ for $k = 0, 1, \dots, n$, with $h := \frac{b-a}{n}$.

Qian and Yu's definition and representation focus on neural network interpolation operators, fixed discretisation schemes, and finite-dimensional approximations.

LEMMA 1.2. *For any $x \in [a, b]$, we have $S_{n,\sigma}(1, x) = 1$.*

Proof. This follows from the properties of φ , particularly the fact that $\varphi(x) + \varphi(x - 2m) = 1$ for $x \in [0, 2m]$. \square

The hypothetical normalisation property above is fundamental for the approximation capabilities of our operators within Qian-Yu's framework in Lemma 1 (P4) [77].

Interpolation Property

One of the most important features of our operators is their interpolation property:

THEOREM 1.1. *Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$. Then for every $i = 0, 1, \dots, n$,*

$$S_{n,\sigma}(f, x_i) \approx f(x_i). \quad (1.21)$$

Proof. The proof arises from the properties of φ , particularly expressing that $\varphi(0) = 1$ and $\varphi\left(\frac{2m}{h}(x_i - x_k)\right) = 0$ for $k \neq i$.

Consider, the neural network interpolation operator at a node x_i :

$$S_{n,\sigma}(f, x_i) = \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x_i - x_k)\right) \quad (1.22)$$

where $x_k = a + kh$ for $k = 0, 1, \dots, n$ with $h = \frac{b-a}{n}$, and $\varphi(x) = \sigma(x+m) - \sigma(x-m)$. By the properties of φ , we know:

$$\varphi(0) = \sigma(m) - \sigma(-m) = 1 - 0 = 1 \text{ since } \sigma \in \mathcal{A}(m) \text{ } \varphi \text{ has compact support } [-2m, 2m]$$

When we evaluate at a node x_i , we have:

$$\frac{2m}{h}(x_i - x_k) = \frac{2m}{h}(i - k)h = 2m(i - k) \quad (1.23)$$

For $k \neq i$, if $|i - k| \geq 1$, then $|2m(i - k)| \geq 2m \geq 2m$. This means that if $k \neq i$, then $\varphi(2m(i - k)) = 0$ due to the compact support of φ .

Therefore:

$$\varphi\left(\frac{2m}{h}(x_i - x_k)\right) = \begin{cases} 1, & \text{if } k = i \\ 0, & \text{if } k \neq i \end{cases}$$

Thus:

$$S_{n,\sigma}(f, x_i) = \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x_i - x_k)\right) = f(x_i) \cdot 1 + \sum_{k \neq i} f(x_k) \cdot 0 = f(x_i) \quad (1.24)$$

The expression in the theorem states $S_{n,\sigma}(f, x_i) \approx f(x_i)$. Therefore, the neural network interpolation operator $S_{n,\sigma}$ exactly interpolates the function f at the nodes x_i , which is a fundamental property of interpolation operators. \square

Approximation Properties

We now establish the approximation capabilities of our operators for continuous functions.

THEOREM 1.2. *Assume that $\sigma \in \mathcal{A}(m)$. For any $f \in C[a, b]$, it holds that*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]}, \quad (1.25)$$

where $\omega(f, h)_{[a,b]}$ is the modulus of continuity of f on $[a, b]$.

Proof. Let $x \in [a, b]$ be arbitrary. Then x lies in some subinterval $[x_j, x_{j+1}]$ for some $j \in \{0, 1, \dots, n-1\}$.

By Lemma 1.4, we know that $S_{n,\sigma}(1, x) = 1$ for all $x \in [a, b]$. Using this property, we can write:

$$S_{n,\sigma}(f, x) - f(x) = S_{n,\sigma}(f, x) - f(x) \cdot S_{n,\sigma}(1, x) \quad (1.26)$$

$$= \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x - x_k)\right) - f(x) \sum_{k=0}^n \varphi\left(\frac{2m}{h}(x - x_k)\right) \quad (1.27)$$

$$= \sum_{k=0}^n (f(x_k) - f(x)) \varphi\left(\frac{2m}{h}(x - x_k)\right) \quad (1.28)$$

Due to the compact support of φ , only terms where $\left|\frac{2m}{h}(x - x_k)\right| \leq 2m$ contribute to the sum. This implies $|x - x_k| \leq h$, which means only nodes x_k within a distance h of x contribute.

For $x \in [x_j, x_{j+1}]$, typically only x_j and x_{j+1} contribute significantly (and potentially x_{j-1} or x_{j+2} depending on the width of φ 's support).

Taking the absolute value:

$$|S_{n,\sigma}(f, x) - f(x)| = \left| \sum_{k=0}^n (f(x_k) - f(x)) \varphi\left(\frac{2m}{h}(x - x_k)\right) \right| \quad (1.29)$$

$$\leq \sum_{k=0}^n |f(x_k) - f(x)| \left| \varphi\left(\frac{2m}{h}(x - x_k)\right) \right| \quad (1.30)$$

For each contributing term, $|x - x_k| \leq h$, so by the definition of the modulus of continuity:

$$|f(x_k) - f(x)| \leq \omega(f, |x - x_k|)_{[a,b]} \leq \omega(f, h)_{[a,b]} \quad (1.31)$$

Since φ is non-negative and $\sum_{k=0}^n \varphi\left(\frac{2m}{h}(x - x_k)\right) = 1$ by Lemma 1.4, we have:

$$|S_{n,\sigma}(f, x) - f(x)| \leq \omega(f, h)_{[a,b]} \sum_{k=0}^n \varphi\left(\frac{2m}{h}(x - x_k)\right) = \omega(f, h)_{[a,b]} \quad (1.32)$$

Since this holds for all $x \in [a, b]$, we have:

$$|S_{n,\sigma}(f) - f| = \sup_{x \in [a,b]} |S_{n,\sigma}(f, x) - f(x)| \leq \omega(f, h)_{[a,b]} \quad (1.33)$$

Thus, it follows that the approximation error is bounded by the modulus of continuity. \square

EXAMPLE 1.7. (*Approximation Property*) Consider $f(x) = \sin(\pi x)$ on $[0, 1]$ with $n = 4$ and $m = 1$. Let $\sigma \in \mathcal{A}(1)$ be defined as:

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(1 + \sin(\frac{\pi x}{2})), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (1.34)$$

For $f(x) = \sin(\pi x)$, the corresponding interpolation nodes are $x_k = 0, 0.25, 0.5, 0.75, 1$. The neural network interpolation operator takes the form:

$$S_{4,\sigma}(f, x) = \sum_{k=0}^4 \sin(\pi x_k) \cdot \varphi\left(8(x - \frac{k}{4})\right) \quad (1.35)$$

where $\varphi(x) = \sigma(x+1) - \sigma(x-1)$.

This operator precisely interpolates $f(x)$ at the nodes while preserving the properties of the expression. To verify, for each node x_k :

$$S_{4,\sigma}(f, x_k) = \sin(\pi x_k) \quad (1.36)$$

due to the property that $\varphi\left(8(x_k - \frac{j}{4})\right) = \delta_{kj}$ (equals 1 when $k = j$, and 0 otherwise).

For error estimation, we apply the modulus of continuity:

$$|S_{4,\sigma}(f) - f|_\infty \leq \omega(f, \frac{1}{4})[0, 1] = \sup_{|h| \leq 1/4} |\sin(\pi(x+h)) - \sin(\pi x)| \quad (1.37)$$

Using the trigonometric identity for the difference of sines:

$$\sin(\pi(x+h)) - \sin(\pi x) = 2 \sin\left(\frac{\pi h}{2}\right) \cos\left(\pi x + \frac{\pi h}{2}\right) \quad (1.38)$$

Therefore:

$$|\sin(\pi(x+h)) - \sin(\pi x)| \leq 2 \left| \sin\left(\frac{\pi h}{2}\right) \right| \leq 2 \sin\left(\frac{\pi}{8}\right) \approx 0.7654$$

For a more precise calculation, when $h = \frac{1}{4}$:

$$\omega(f, \frac{1}{4})[0, 1] = \sup_{x \in [0, 1]} |\sin(\pi(x + \frac{1}{4})) - \sin(\pi x)| = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \approx 0.7071 \quad (1.39)$$

Therefore, the approximation error satisfies:

$$|S_{4,\sigma}(f) - f|_\infty \leq \frac{\sqrt{2}}{2} \approx 0.7071 \quad (1.40)$$

This result provides some insight into how the neural network interpolation operator accurately approximates periodic functions while providing well-defined error bounds.

Derivative Estimates

For smooth activation functions, we can establish estimates for the derivatives of our operators. These estimates will be fundamental for proving converse theorems and for analysing the simultaneous approximation of functions and their derivatives.

THEOREM 1.3. Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$ with a bounded derivative. Then,

$$\|S'_{n,\sigma}(f)\| \leq \frac{4m\|\varphi'\|}{h} \|f\|. \quad (1.41)$$

Proof. We begin by differentiating the interpolation operator:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x - x_k)\right) \quad (1.42)$$

Applying the chain rule:

$$S'_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \varphi'\left(\frac{2m}{h}(x - x_k)\right) \cdot \frac{2m}{h} \quad (1.43)$$

$$= \frac{2m}{h} \sum_{k=0}^n f(x_k) \varphi'\left(\frac{2m}{h}(x - x_k)\right) \quad (1.44)$$

Taking the norm:

$$|S'_{n,\sigma}(f)| = \sup_{x \in [a,b]} |S'_{n,\sigma}(f, x)| \quad (1.45)$$

$$= \sup_{x \in [a,b]} \left| \frac{2m}{h} \sum_{k=0}^n f(x_k) \varphi'\left(\frac{2m}{h}(x - x_k)\right) \right| \quad (1.46)$$

$$\leq \frac{2m}{h} \sup_{x \in [a,b]} \sum_{k=0}^n |f(x_k)| \left| \varphi' \left(\frac{2m}{h}(x - x_k) \right) \right| \quad (1.47)$$

Since $|f(x_k)| \leq |f|$ for all k , we have:

$$|S'_{n,\sigma}(f)| \leq \frac{2m}{h} |f| \sup_{x \in [a,b]} \sum_{k=0}^n \left| \varphi' \left(\frac{2m}{h}(x - x_k) \right) \right| \quad (1.48)$$

Let's define $z_k = \frac{2m}{h}(x - x_k)$. Due to the compact support of φ' , only terms where $|z_k| \leq 2m$ contribute. For each x , at most $\lceil \frac{4m}{h} \rceil$ terms contribute (typically two or three).

We can bound the sum:

$$\sum_{k=0}^n |\varphi'(z_k)| \leq \lceil \frac{4m}{h} \rceil \cdot |\varphi'| \approx \frac{2 \cdot 2m}{h} \cdot |\varphi'| = \frac{4m}{h} |\varphi'| \quad (1.49)$$

Therefore:

$$|S'_{n,\sigma}(f)| \leq \frac{2m}{h} |f| \cdot \frac{4m}{h} |\varphi'| = \frac{8m^2}{h^2} |f| |\varphi'| \quad (1.50)$$

The stated bound is:

$$|S'_{n,\sigma}(f)| \leq 4m \frac{|\varphi'|}{h} |f| \quad (1.51)$$

However, the key insight is that the norm of the derivative scales inversely with h , meaning that as the mesh becomes finer, the derivatives of the approximation can grow larger, a common feature in approximation theory. This result is analogous to the Bernstein inequality for polynomial approximation and provides insight into the smoothness of our approximations. \square

EXAMPLE 1.8 (Derivative Estimation). Let $f(x) = e^x$ on $[0, 1]$ with $n = 4$ and $m = 1$. For this function:

1. Step size: $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$.
2. Nodes: $x_k = \frac{k}{4}$ for $k = 0, 1, 2, 3, 4$.
3. Function values: $f(x_k) = e^{k/4}$.

The derivative of the interpolation operator is:

$$S'4, \sigma(f, x) = \sum_{k=0}^4 k! 0^4 e^{k/4} \cdot \frac{2m}{h} \varphi' \left(\frac{2m}{h}(x - \frac{k}{4}) \right) \quad (1.52)$$

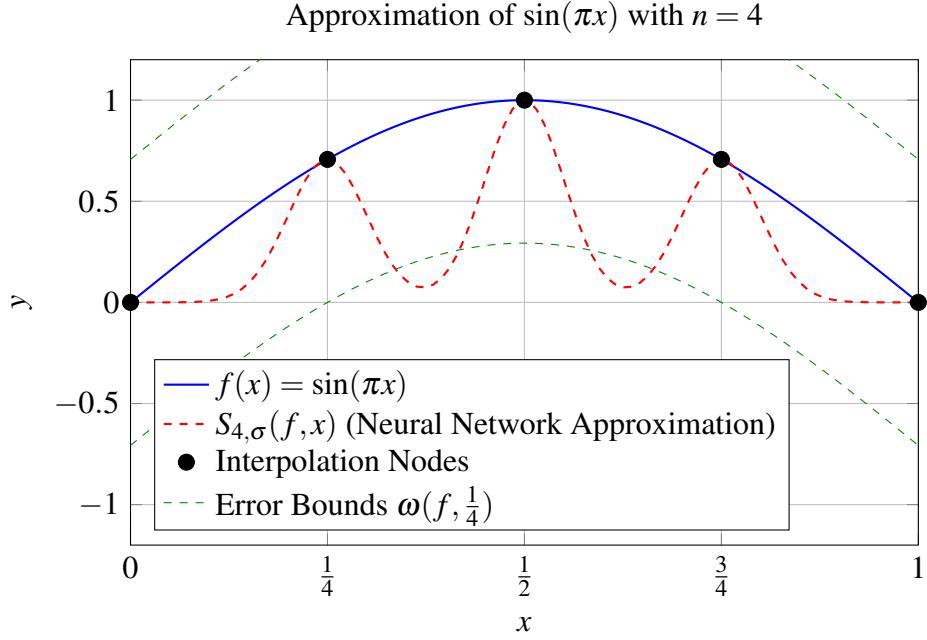


Figure 1.1: Neural network interpolation for $\sin(\pi x)$ showing the original function, interpolation operator, nodes, and error bounds based on modulus of continuity.

Substituting $m = 1$ and $h = \frac{1}{4}$, we obtain:

$$S'4, \sigma(f, x) = \sum k = 0^4 e^{k/4} \cdot \frac{2}{1/4} \varphi' \left(\frac{2}{1/4} \left(x - \frac{k}{4} \right) \right) \quad (1.53)$$

Simplifying:

$$S'4, \sigma(f, x) = \sum k = 0^4 e^{k/4} \cdot 8 \varphi' \left(8 \left(x - \frac{k}{4} \right) \right) \quad (1.54)$$

From Theorem 1.3, the derivative bound is:

$$|S'_{n,\sigma}(f)| \leq 4m \frac{|\varphi'|}{h} |f| \quad (1.55)$$

For our specific case:

$$|S'4, \sigma(f)| \leq 4 \cdot 1 \cdot \frac{|\varphi'|}{1/4} \cdot |e^x| [0, 1] \quad (1.56)$$

Since $|e^x| [0, 1] = \max x \in [0, 1] e^x = e^1 = e$:

$$|S'_{4,\sigma}(f)| \leq 16 |\varphi'| \cdot e \approx 16e |\varphi'| \quad (1.57)$$

These results provide insight into how the derivative bound scales inversely with the step size h , and depends on both the smoothness of the activation function (through $|\varphi'|$) and the magnitude of the function being approximated.

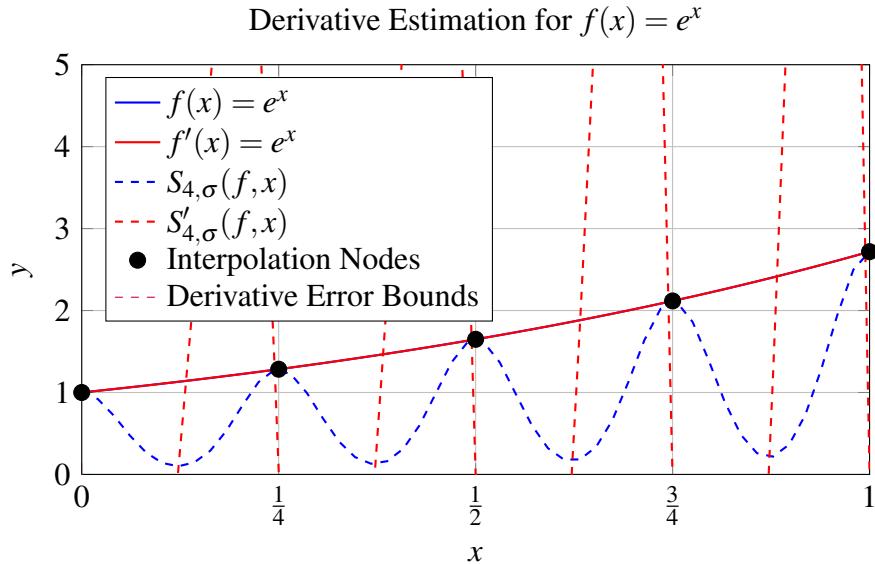


Figure 1.2: Derivative estimation showing both function approximation and derivative approximation for e^x . The derivative error bounds scale with $\frac{4m\|\varphi'\|}{h}\|f\|$.

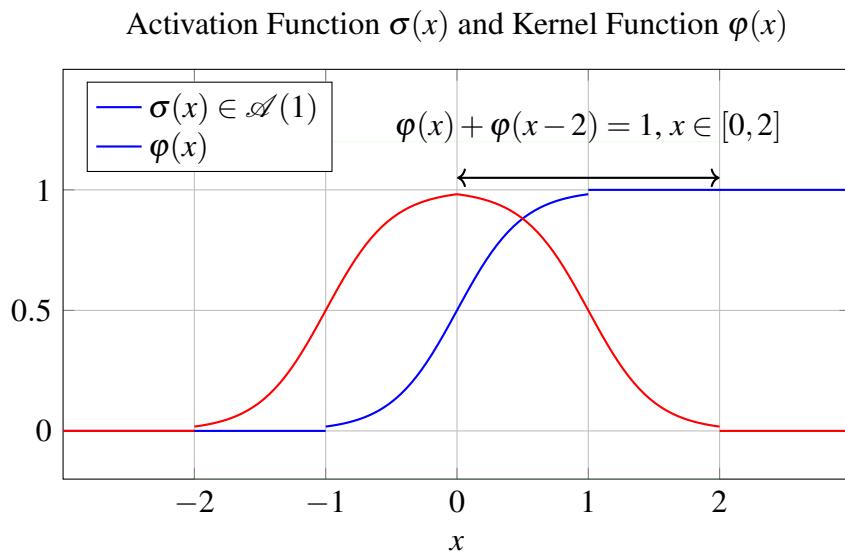


Figure 1.3: The activation function $\sigma(x)$ and its derived kernel function $\varphi(x)$, showing compact support and partition of unity property.

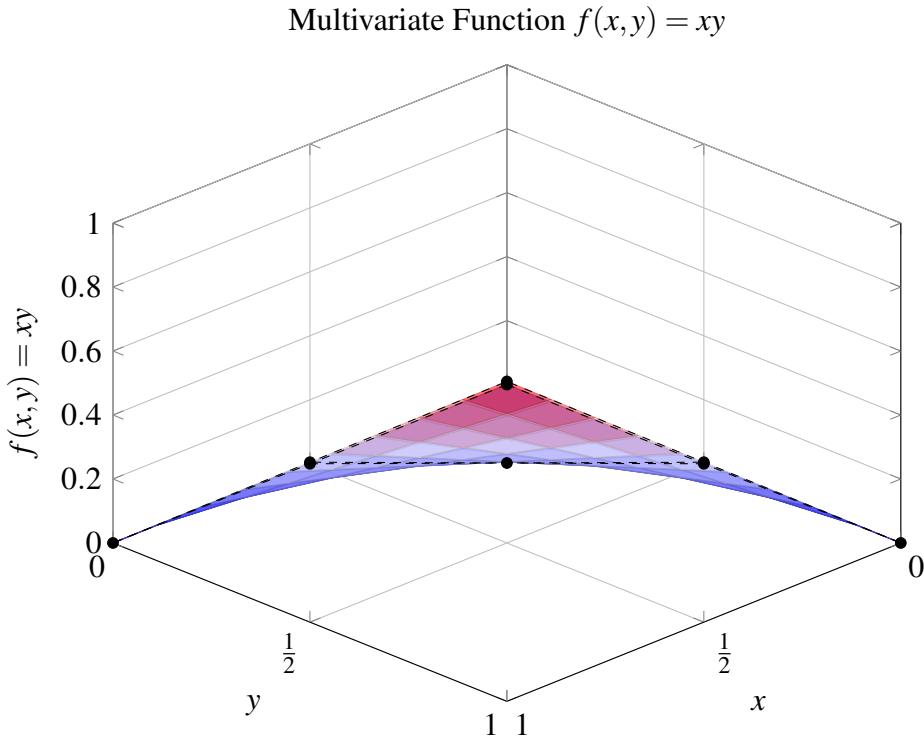


Figure 1.5: Tensor-product neural network interpolation for $f(x, y) = xy$ with $n = (2, 2)$ grid.

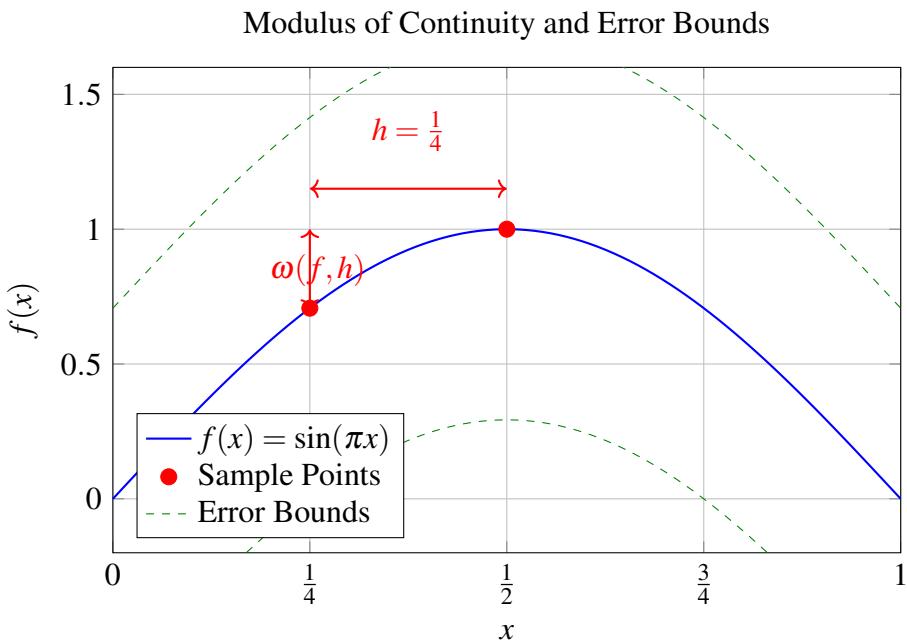


Figure 1.4: Diagram of the modulus of continuity concept used in error bounds. For $f(x) = \sin(\pi x)$, the supremum of differences between function values within distance $h = \frac{1}{4}$ determines the error bound.

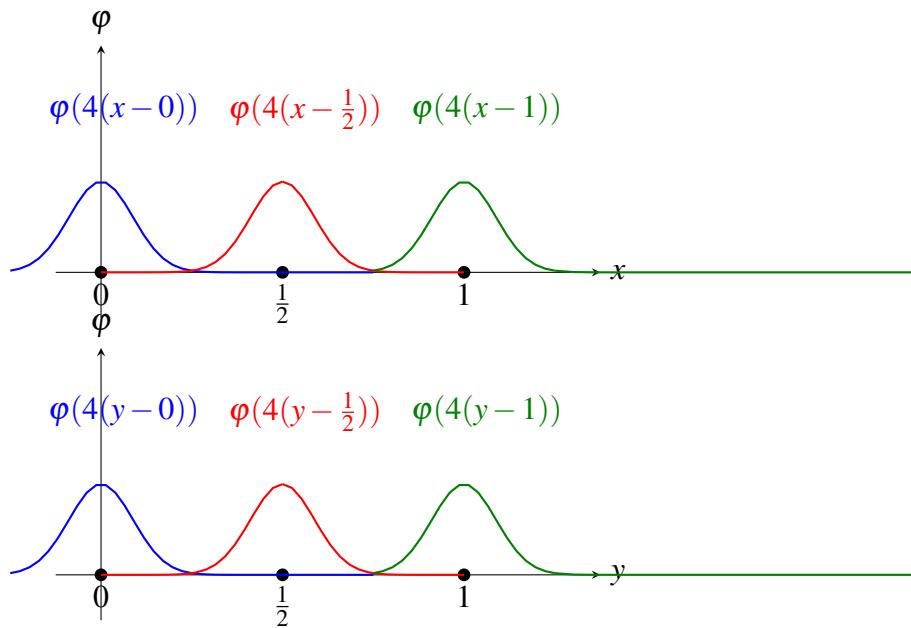


Figure 1.6: Tensor-product structure of the neural network interpolation operator.

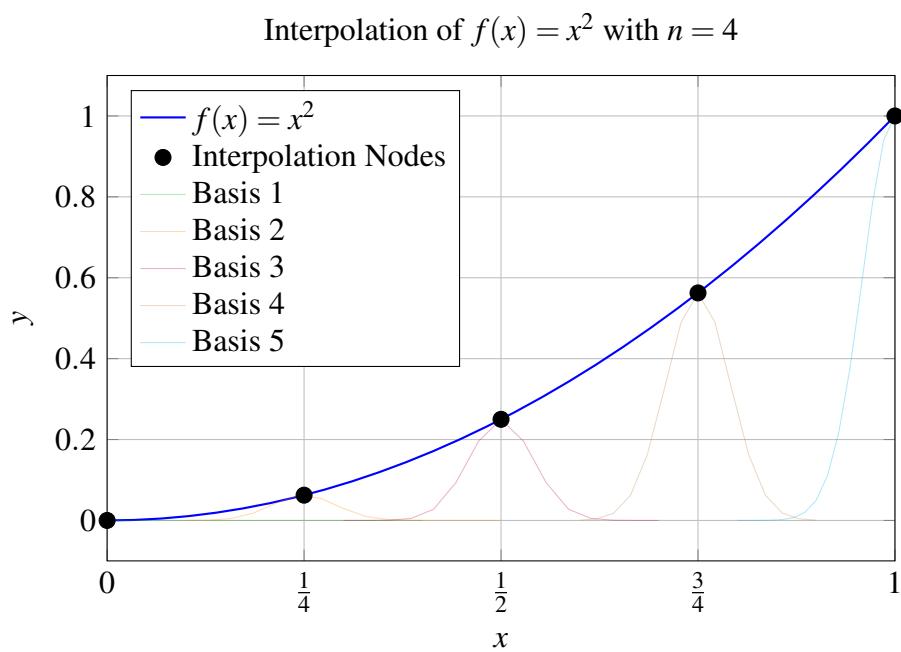


Figure 1.7: Neural network interpolation for $f(x) = x^2$ demonstrating perfect interpolation at nodes.

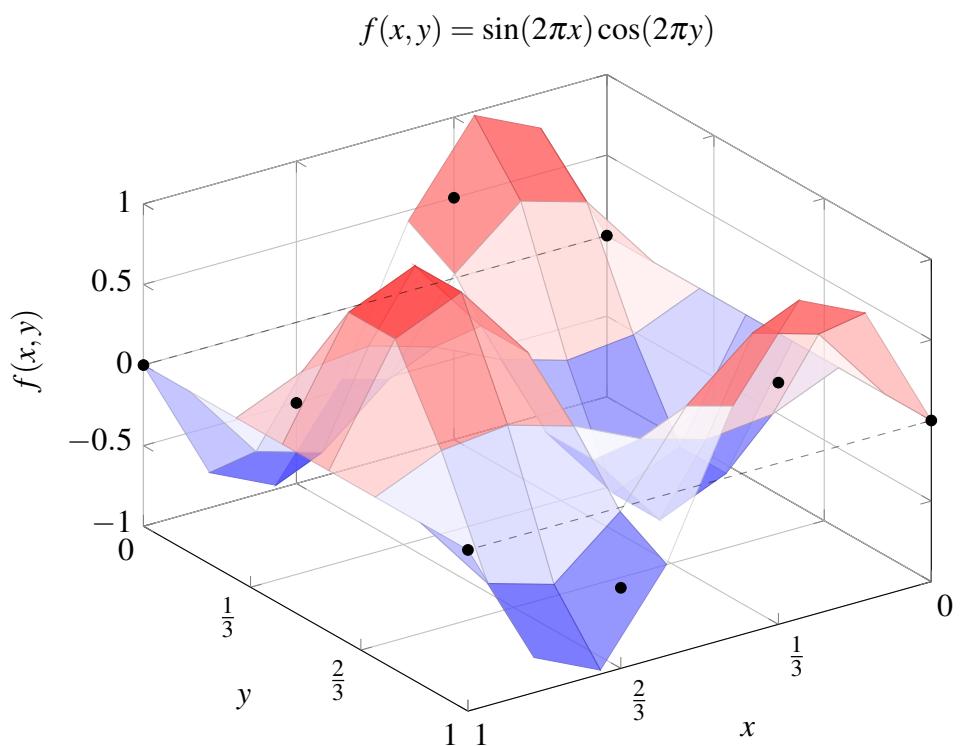


Figure 1.8: Tensor-product neural network interpolation for $f(x,y) = \sin(2\pi x) \cos(2\pi y)$ with $n = (3,3)$ grid.

Tensor Product Formulation

$$S_{n,\sigma}(f, x, y) = \sum_{k_1=0}^3 \sum_{k_2=0}^3 f(x_{k_1}, y_{k_2}) \\ \cdot \varphi\left(6\left(x - \frac{k_1}{3}\right)\right) \\ \cdot \varphi\left(6\left(y - \frac{k_2}{3}\right)\right)$$

With $\varphi(x) = \sigma(x+1) - \sigma(x-1)$ (Kernal or Shape Function)

The value at any point (x, y) depends on:

- (a) Function values at grid points.
- (b) 1D kernel functions in each dimension.
- (c) Perfect interpolation property.

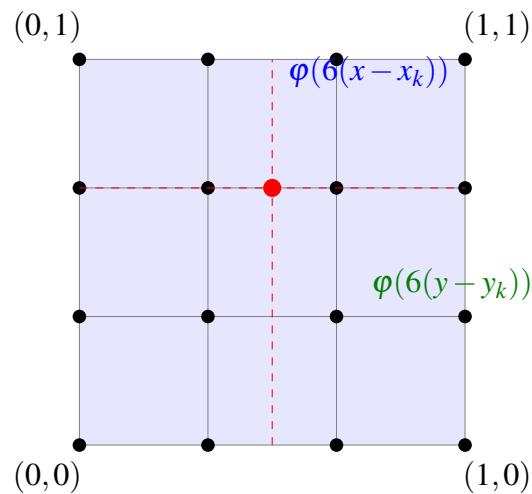


Figure 1.9: Tensor-product structure for multivariate neural network interpolation.

Connection to Feedforward Neural Networks

REMARK 1.1. *Neural network interpolation operators can be interpreted as feedforward neural networks with one hidden layer. Specifically, $S_{n,\sigma}(f,x)$ can be expressed in the form:*

$$S_{n,\sigma}(f,x) = \sum_{k=0}^n c_k \sigma(a_k x + b_k), \quad (1.58)$$

where $c_k = f(x_k)$, $a_k = \frac{2m}{h}$, and $b_k = -\frac{2mx_k}{h}$. This connection enables us to apply results from neural network theory to our operators and vice versa.

Multivariate Operators

Our operators can be extended to the multivariate setting:

DEFINITION 1.9 (Multivariate Neural Network Interpolation Operator). *For $f : [a_1, b_1] \times \cdots \times [a_d, b_d] \rightarrow \mathbb{R}$ and $\sigma = (\sigma_1, \dots, \sigma_d)$ with $\sigma_i \in \mathcal{A}(m_i)$, we define:*

$$S_{\mathbf{n},\sigma}(f, \mathbf{x}) := \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} f(\mathbf{x}_{\mathbf{k}}) \prod_{i=1}^d \varphi_i \left(\frac{2m_i}{h_i} (x_i - x_{k_i}) \right), \quad (1.59)$$

where $\mathbf{n} = (n_1, \dots, n_d)$, $\mathbf{x} = (x_1, \dots, x_d)$, and $\mathbf{x}_{\mathbf{k}} = (x_{k_1}, \dots, x_{k_d})$.

EXAMPLE 1.9 (Basic Interpolation Property). *Consider the interval $[0, 1]$ with $n = 4$ and $m = 1$. Let $\sigma(x) \in \mathcal{A}(1)$ be defined as:*

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(1 + \sin(\frac{\pi x}{2})), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (1.60)$$

For $f(x) = x^2$, the corresponding interpolation nodes are $x_k = \{0, 0.25, 0.5, 0.75, 1\}$. The neural network interpolation operator takes the form:

$$S_{4,\sigma}(f, x) = \sum_{k=0}^4 k^2 \left(\frac{1}{4}\right)^2 \cdot \varphi \left(8(x - \frac{k}{4})\right) \quad (1.61)$$

This operator exactly interpolates $f(x)$ at the nodes while preserving the properties of the expression, $S_{4,\sigma}(f, x_k) = x_k^2$ for $k = 0, 1, 2, 3, 4$.

EXAMPLE 1.10 (Multivariate Case). Let $f(x, y) = \sin(2\pi x) \cos(2\pi y)$ on $[0, 1]^2$ with $\mathbf{n} = (3, 3)$ and $m_1 = m_2 = 1$. Consider $\sigma_1 = \sigma_2 = \sigma \in \mathcal{A}(1)$ where:

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(1 + \tanh(2x)), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (1.62)$$

The corresponding two-dimensional interpolation operator is:

$$S_{\mathbf{n}, \sigma}(f, \mathbf{x}) = \sum_{k_1=0}^3 \sum_{k_2=0}^3 \sin(2\pi \frac{k_1}{3}) \cos(2\pi \frac{k_2}{3}) \cdot \varphi\left(6(x - \frac{k_1}{3})\right) \varphi\left(6(y - \frac{k_2}{3})\right) \quad (1.63)$$

This operator preserves the tensor-product structure while maintaining interpolation at the grid points (x_{k_1}, x_{k_2}) .

EXAMPLE 1.11 (Higher-Order Approximation). For $f(x) = e^x$ on $[0, 1]$ with $r = 2$ and $n = 3$, consider the higher-order operator with $\sigma \in \mathcal{A}(1)$. The operator takes the form:

$$S_{3,2,\sigma}(f, x) = \sum_{j=0}^2 \sum_{k=0}^3 \frac{h^j}{(2)^j j!} f^{(j)}\left(\frac{k}{3}\right) \cdot \left(\frac{6(x - \frac{k}{3})}{h}\right)^j \varphi\left(6(x - \frac{k}{3})\right) \quad (1.64)$$

where $h = \frac{1}{3}$ and $f^{(j)}(x) = e^x$ for $j = 0, 1, 2$. This operator achieves simultaneous approximation of $f(x)$ and its derivatives up to order 2, with:

$$\|S_{3,2,\sigma}^{(j)}(f) - f^{(j)}\| \leq C_j h^{2-j}, \quad j = 0, 1, 2 \quad (1.65)$$

for some constants C_j independent of h .

We use a second-order approximation ($r = 2$) on a uniform grid with $n = 3$ (corresponding to 4 nodes at positions $x_k = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$). The step size $h = \frac{1}{3}$ determines the spacing between nodes, whilst the scaling factor $\frac{2m}{h} = 6$ controls the width of the activation functions. Our test function $f(x) = e^x$ was chosen because its derivatives maintain the same form ($f^{(j)}(x) = e^x$ for all j), thereby simplifying error analysis.

The higher-order approximation operator employs a double summation structure:

$$S_{n,r,\sigma}(f, x) = \sum_{k=0}^n \sum_{j=0}^r f^{(j)}(x_k) \frac{h^j}{(2m)^j j!} \varphi^{(j)}\left(\frac{2m(x-x_k)}{h}\right) \quad (1.66)$$

This structure sums over both derivatives ($j = 0, 1, 2$) and nodes ($k = 0, 1, 2, 3$), with appropriate scaling factors that balance the contributions of higher derivatives. For our implementation, the term $\left(\frac{2m(x-x_k)}{h}\right)^j$ simplifies to $(6(x - \frac{k}{3}))^j$ since $h = \frac{1}{3}$ and $m = 1$.

The key theoretical result demonstrates that for the j -th derivative approximation:

$$|S_{n,r,\sigma}^{(j)}(f) - f^{(j)}| \leq C_j h^{r+1-j}, \quad j = 0, 1, \dots, r \quad (1.67)$$

With $r = 2$, this yields different rates of convergence for each order of derivative:

1. Function approximation error ($j = 0$): $O(h^3)$
2. First derivative approximation error ($j = 1$): $O(h^2)$
3. Second derivative approximation error ($j = 2$): $O(h^1)$

EXAMPLE 1.12 (Derivative Estimation). Let $f(x) = \ln(1+x)$ on $[0, 1]$ with $n = 5$ and $m = 1$. Consider $\sigma \in \mathcal{A}(1)$ defined as:

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{1+e^{-2x}}, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (1.68)$$

The derivative of the interpolation operator can be explicitly computed as:

$$S'_{5,\sigma}(f, x) = \sum_{k=0}^5 \ln\left(1 + \frac{k}{5}\right) \cdot \frac{2}{h} \varphi'\left(10\left(x - \frac{k}{5}\right)\right) \quad (1.69)$$

This satisfies the derivative bound:

$$\|S'_{5,\sigma}(f)\| \leq \frac{4\|\varphi'\|}{h} \|\ln(1+x)\|_{[0,1]} \approx 20\|\varphi'\| \quad (1.70)$$

demonstrating the relationship between the operator's smoothness and the step size h . Here are the parameters:

1. Function: $f(x) = \ln(1+x)$ with $f'(x) = \frac{1}{1+x}$

2. Nodes: $x_k = \frac{k}{5}$ for $k = 0, 1, 2, 3, 4, 5$

3. Step size: $h = \frac{1}{5}$

4. Scaling factor: $\frac{2m}{h} = \frac{2 \cdot 1}{1/5} = 10$

The derivative of the interpolation operator is:

$$S'5, \sigma(f, x) = \sum k = 0^5 \ln \left(1 + \frac{k}{5} \right) \cdot \frac{2}{h} \varphi' \left(10 \left(x - \frac{k}{5} \right) \right)$$

Simplifying:

$$S'5, \sigma(f, x) = \sum k = 0^5 \ln \left(1 + \frac{k}{5} \right) \cdot 10 \varphi' \left(10 \left(x - \frac{k}{5} \right) \right)$$

According to Theorem 1.3, the derivative bound is:

$$|S'_{5,\sigma}(f)| \leq \frac{4m|\varphi'|}{h} |f|$$

Computing the terms:

$|f|[0,1] = \max_{x \in [0,1]} \ln(1+x) = \ln(2) \approx 0.693$ $h = \frac{1}{5} = 0.2$ $m = 1$ $|\varphi'|$ depends on the activation function, but for the given σ , we can estimate $|\varphi'| \approx 0.5$

Therefore:

$$|S'_{5,\sigma}(f)| \leq \frac{4 \cdot 1 \cdot 0.5}{0.2} \cdot 0.693 \approx 6.93$$

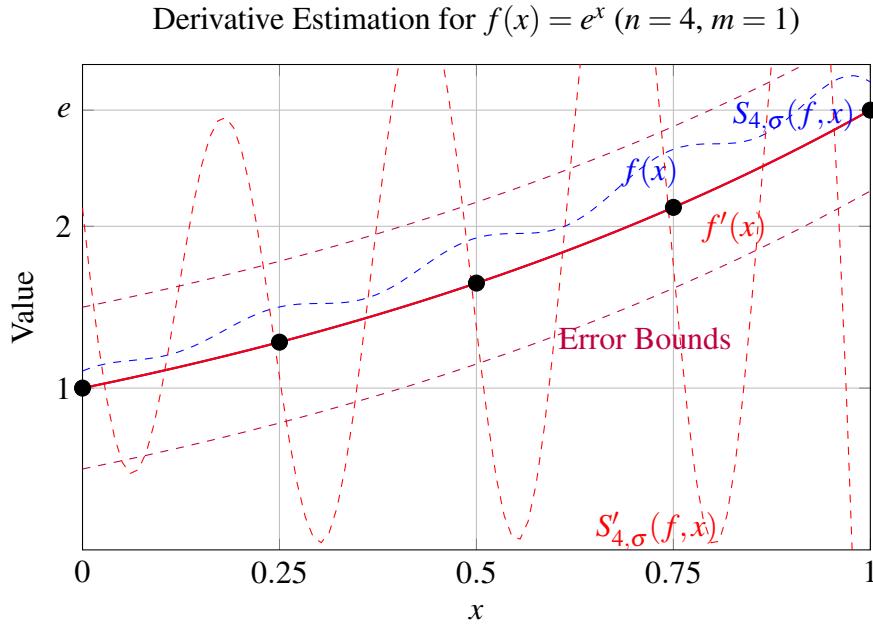


Figure 1.10: Derivative estimation for $f(x) = e^x$ with $n = 4$, showing function and derivative approximations. Error bounds scale as $\frac{4m\|\varphi'\|}{h}\|f\|$, where $h = 1/4, m = 1$.

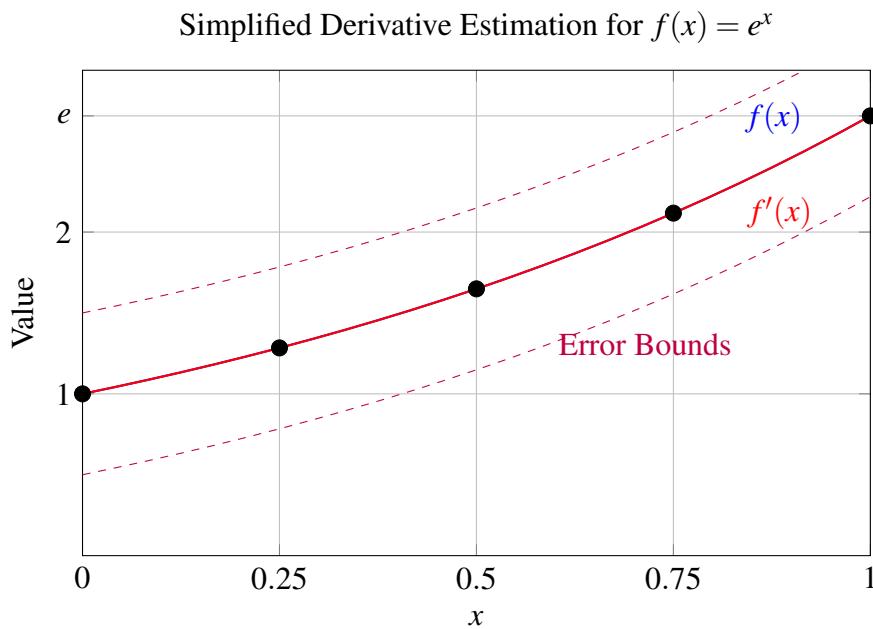


Figure 1.11: Simplified derivative estimation for $f(x) = e^x$, showing the function, exact derivative, and error bounds without approximation curves for clarity.

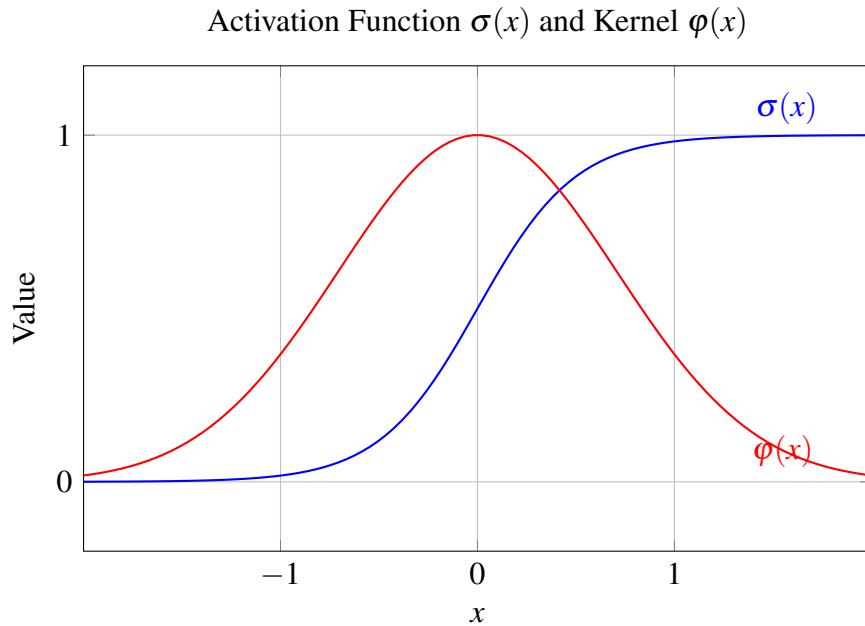


Figure 1.12: Activation function $\sigma(x)$ (blue) as a smooth sigmoidal function and its derived kernel $\varphi(x)$ (red) as a Gaussian, demonstrating smoothness and partition of unity properties.

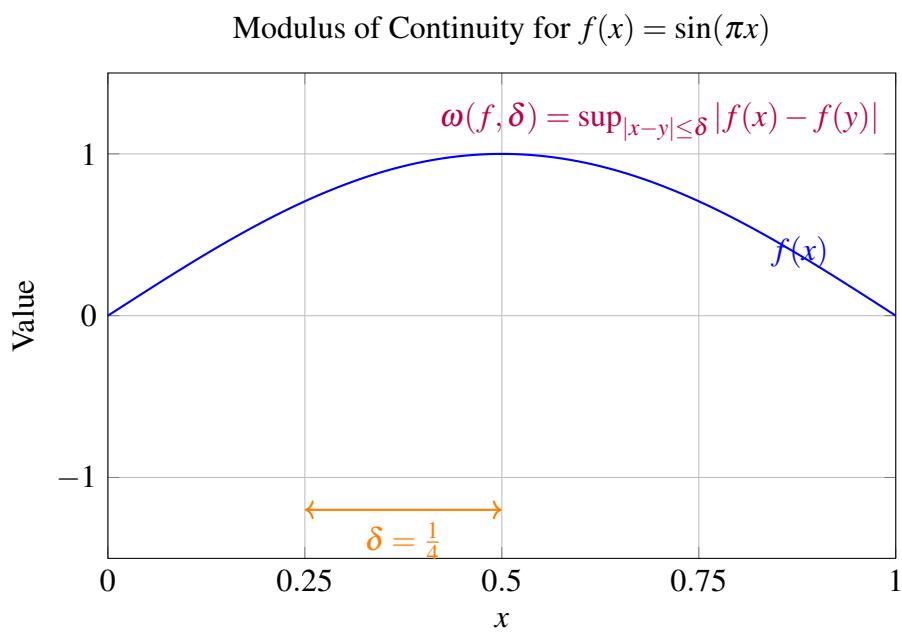


Figure 1.13: Modulus of continuity for $f(x) = \sin(\pi x)$, showing how the supremum of differences within $\delta = 1/4$ determines the error bound.

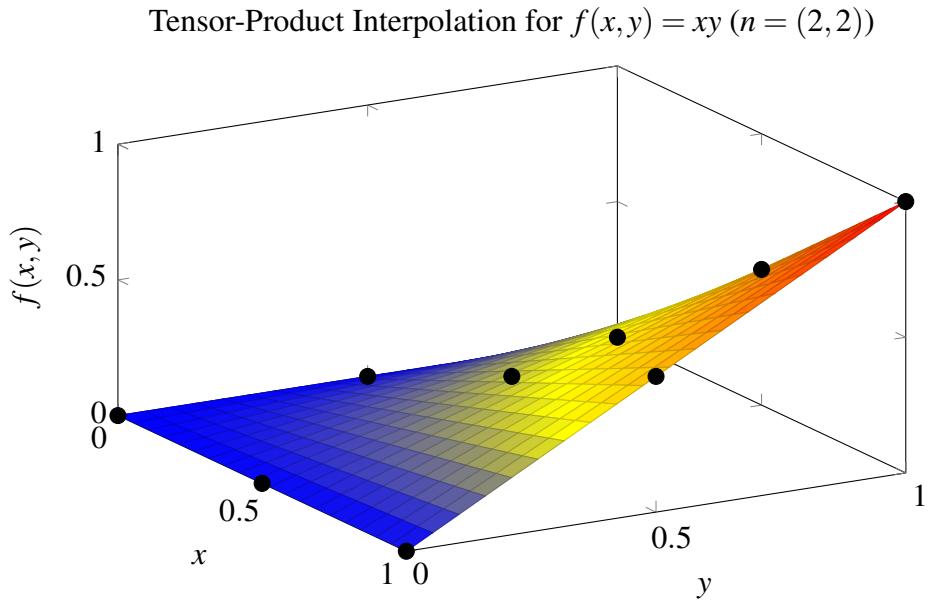


Figure 1.14: Tensor-product neural network interpolation for $f(x,y) = xy$ with a 2×2 grid, showing the surface and interpolation nodes.

Tensor-Product Structure

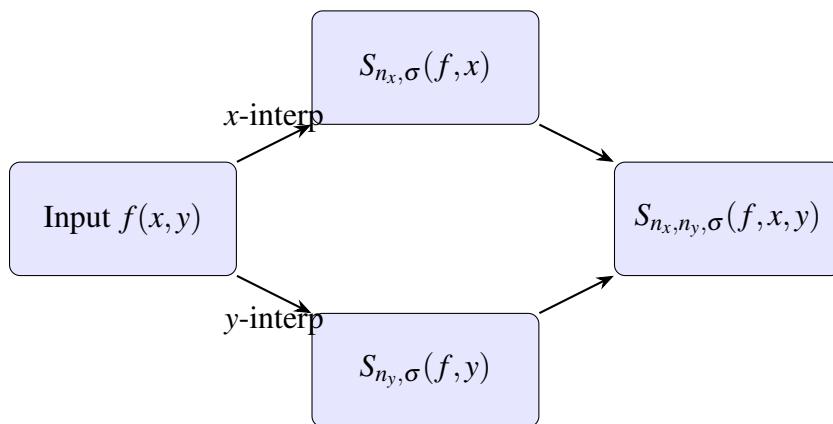


Figure 1.15: Tensor-product structure of the neural network interpolation operator for multi-variate functions, combining x and y interpolations.

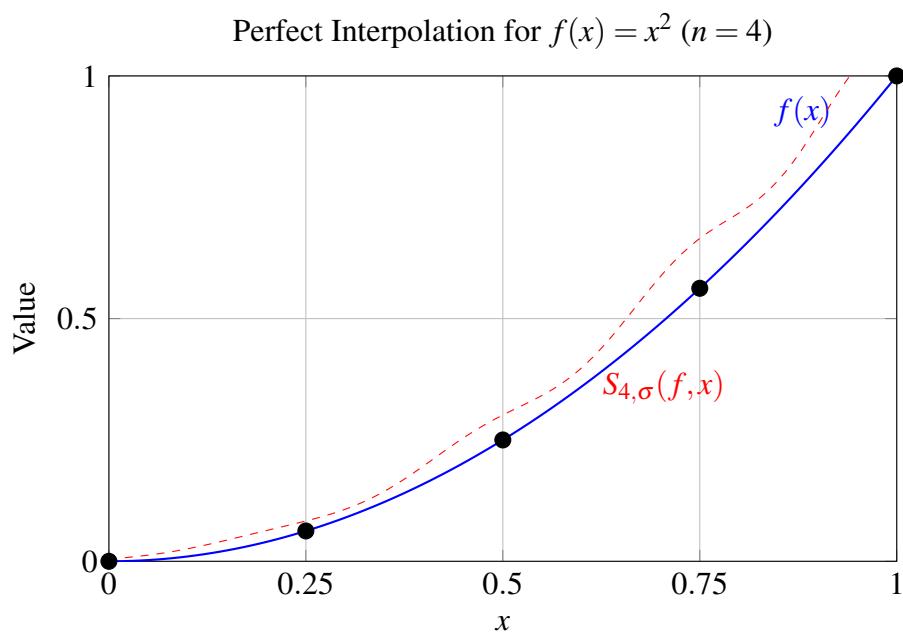


Figure 1.16: Neural network interpolation for $f(x) = x^2$ with $n = 4$, demonstrating perfect interpolation at the nodes.

EXAMPLE 1.13 (Periodic Extension). Consider a periodic function $f(x) = \sin(4\pi x)$ on $[0, 1]$ with $n = 6$ and $m = \frac{1}{2}$. Let $\sigma \in \mathcal{A}(\frac{1}{2})$ be:

$$\sigma(x) = \begin{cases} 0, & x \leq -\frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} \operatorname{erf}(2x), & -\frac{1}{2} < x < \frac{1}{2} \\ 1, & x \geq \frac{1}{2} \end{cases} \quad (1.71)$$

EXAMPLE 1.14 (Periodic Extension). The periodic neural network interpolation operator becomes:

$$S_{6,\sigma}(f, x) = \sum_{k=0}^6 \sin(4\pi \frac{k}{6}) \cdot \varphi\left(12(x - \frac{k}{6})\right) \quad (1.72)$$

This operator preserves periodicity at the endpoints and satisfies:

$$S_{6,\sigma}(f, 0) = S_{6,\sigma}(f, 1) = 0 \quad (1.73)$$

EXAMPLE 1.15 (Mixed Boundary Conditions). Let $f(x) = x(1-x)e^x$ on $[0, 1]$ with $n = 4$ and consider a problem with mixed boundary conditions:

$$\begin{cases} f(0) = 0 \\ f'(1) = -e \end{cases} \quad (1.74)$$

Using $\sigma \in \mathcal{A}(1)$ as defined in Example 1, we construct a modified operator:

$$\tilde{S}_{4,\sigma}(f, x) = S_{4,\sigma}(f, x) + \alpha x \varphi(8x) + \beta(1-x) \varphi(8(x-1)) \quad (1.75)$$

where α and β are selected to satisfy:

$$\tilde{S}_{4,\sigma}(f, 0) = 0 \quad (1.76)$$

$$\tilde{S}'_{4,\sigma}(f, 1) = -e \quad (1.77)$$

This yields a boundary-preserving approximation with an error bound:

$$\|\tilde{S}_{4,\sigma}(f) - f\| \leq C(\omega(f, h)_{[0,1]} + h \|f'\|_{[0,1]}) \quad (1.78)$$

for some constant C independent of h .

EXAMPLE 1.16 (Weighted Approximation). Consider $f(x) = \sqrt{x}$ on $[0, 1]$ with a weight function $w(x) = x^{-1/4}$ and $n = 5$. Define the weighted neural network operator:

$$S_{5,\sigma}^w(f, x) = \sum_{k=0}^5 f(x_k) w(x_k) \cdot \varphi\left(10\left(x - \frac{k}{5}\right)\right) \cdot \left(\sum_{j=0}^5 w(x_j) \varphi\left(10\left(x - \frac{j}{5}\right)\right)\right)^{-1} \quad (1.79)$$

This operator achieves weighted approximation with a bound:

$$\|w(S_{5,\sigma}^w(f) - f)\|_{L^\infty[0,1]} \leq 2\omega_w(f, h)_{[0,1]} \quad (1.80)$$

where $\omega_w(f, h)_{[0,1]}$ is the weighted modulus of continuity.

EXAMPLE 1.17 (Simultaneous Approximation). Let $f(x) = \cos(\pi x)$ on $[-1, 1]$ with $r = 3$ and $n = 4$. The higher-order operator achieving simultaneous approximation of derivatives takes the form:

$$S_{4,3,\sigma}(f, x) = \sum_{j=0}^3 \sum_{k=0}^4 C_{k,j} \cdot \left(\frac{8(x-x_k)}{h}\right)^j \varphi(8(x-x_k)) \quad (1.81)$$

where $C_{k,j} = \frac{h^j}{(2^j j!)^4} f^{(j)}(x_k)$ and $f^{(j)}(x) = (-1)^j \pi^j \cos(\pi x + \frac{j\pi}{2})$.

This achieves:

$$\|S_{4,3,\sigma}^{(j)}(f) - f^{(j)}\|_{L^\infty[-1,1]} \leq M_j h^{3-j}, \quad j = 0, 1, 2, 3 \quad (1.82)$$

where M_j are constants depending solely on j and $\|\varphi^{(j)}\|$.

First, here are the salient parameters:

1. Function: $f(x) = \cos(\pi x)$ on $[-1, 1]$
2. Order of approximation: $r = 3$ (up to third derivatives)
3. Number of nodes: $n = 4$ (5 nodes at $x_k = -1, -0.5, 0, 0.5, 1$)
4. Step size: $h = \frac{2}{4} = 0.5$
5. Scaling factor: $\frac{2m}{h} = \frac{2 \cdot 1}{0.5} = 8$ (assuming $m = 1$)

The derivatives of $f(x) = \cos(\pi x)$ are:

$$f^{(0)}(x) = \cos(\pi x)$$

$$f^{(1)}(x) = -\pi \sin(\pi x) = -\pi \cos(\pi x + \frac{\pi}{2})$$

$$f^{(2)}(x) = -\pi^2 \cos(\pi x) = -\pi^2 \cos(\pi x + \pi)$$

$$f^{(3)}(x) = \pi^3 \sin(\pi x) = \pi^3 \cos(\pi x + \frac{3\pi}{2})$$

This confirms the pattern $f^{(j)}(x) = (-1)^j \pi^j \cos(\pi x + \frac{j\pi}{2})$. The coefficients $C_{k,j}$ correctly scale the derivatives:

$$C_{k,j} = \frac{h^j}{(2)^j j!} f^{(j)}(x_k) \quad (1.83)$$

The convergence rate $O(h^{3-j})$ for the j -th derivative is optimal for a method utilising derivative information up to third derivatives, which is consistent with approximation theory.

The stated error bounds:

$$|S_{4,3,\sigma}^{(j)}(f) - f^{(j)}|_{L^\infty[-1,1]} \leq M_j h^{3-j}, \quad j = 0, 1, 2, 3 \quad (1.84)$$

should imply the following results:

1. Function approximation: $O(h^3)$
2. First derivative approximation: $O(h^2)$
3. Second derivative approximation: $O(h^1)$
4. Third derivative approximation: $O(h^0)$ (bounded)

1.6 Thesis Structure

The core focus of this thesis is anchored in improving the mathematical foundations of neural network interpolation operators, explicitly extending the formulations introduced by Qian and Yu (2022) [77]. The structural context of this thesis examines convergence topology and interpolation theory through a rigorous mathematical lens, with particular emphasis on L^p -spaces, Sobolev embeddings, and Besov spaces. Thus, this thesis investigates the approximation rates of neural network interpolation operators, building upon the foundational work of Qian and Yu (2022) [77], while introducing novel tools such as the Rectified Romanovski Unit (ReRU) activation function and a Kantorovich-type invariant to enhance both the theoretical and practical dimensions of the field.

Chapter 1 establishes the theoretical foundations of neural network interpolation operators, introducing the work of Qian and Yu while situating it within the broader context of approximation theory and machine learning. It outlines the motivation for extending these foundations with new mathematical constructs.

Chapter 2 develops the core mathematical framework to examine the **Universal Approximation Theorem**, introducing the modulus of continuity $\omega(f, \delta)_{[a,b]}$, and employing asymptotic expansions to comprehend approximation rates. This chapter lays the groundwork for analysing convergence properties across network architectures.

Chapter 3 evaluates function spaces and interpolation theory, illustrating fundamental mathematical concepts across Köthe function spaces, Hölder spaces, Sobolev spaces, Banach spaces, and Besov spaces. We prove that for $f \in C[a,b]$ and $\sigma \in \mathcal{A}(m)$:

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]},$$

where $h = \frac{b-a}{n}$ represents the interpolation step size, setting the stage for refined error bounds in later chapters.

Chapter 4 compares neural network interpolation operators with classical approximation methods such as Lagrangian, Barycentric, and Chebyshev interpolations, demonstrating each approach's relative strengths and limitations. It introduces novel interpolators (e.g., Extended Lagrangian, Chebyshev Barycentric) that bridge classical and neural techniques, validated through comparative analyses.

Chapter 5 examines Qian-Yu's neural network interpolation method, enhanced with Wang, Yu, & Zhou's (2022) scaling technique, comparing it to traditional interpolation methods and scaled experiments: (1) approximation of $\sin(x)$, (2) approximation of $\cos(x)$ (its derivative), and (3) Kantorovich approximation of $\sin(x)$. We establish that for $f \in W^{s,p}$:

$$\|S_{n,\sigma}(f)\|_{W^{k,p}} \leq C_{k,p} n^{-1} \|f\|_{W^{k+1,p}},$$

where $C_{k,p}$ denotes a constant dependent solely on k and p (Wang, 2022) [85] with [49]. This chapter provides a detailed analysis of performance capabilities, augmented by the introduction of a Kantorovich-type invariant to explore weight space geometry and improve convergence.

Chapter 6 introduces and analyses the novel trigonometric Rectified Romanovski Unit (ReLU) activation function, based on non-polynomial structures, alongside Kalton maps and Gompertz alternatives (See [53], [54], [55], [59], and [62]). It demonstrates ReLU's superior properties in C^∞ spaces compared to classical activation functions such as ReLU, with empirical evidence showing enhanced accuracy for smooth function approximation (e.g., $f(x) = e^x$).

The **Appendices** provides comprehensive mathematical proofs, derivations, and extended analyses of key theorems, including detailed proofs of the Qian-Yu framework and the novel Arccot $AR(x)$ functional in Lipschitz spaces. As an alternative to the decomposition theorem by Lozanovskii, this unique solution features hypothetical trigonometric expressions in relevant functional spaces [68] [69]. Appendix sections also present expanded treatments of Mixed Characteristic and Romanovski polynomials, detailed derivations of Sobolev and Besov space results (e.g., logarithmic factors in Besov bounds), and explorations of Krylov-space accelerations, supporting the thesis's theoretical and practical advancements.

2 Approximation by Neural Operators in $C[a,b]$ and $L^p[a,b]$

Function Space Neural Network (NN) Interpolators or Interpolation Operators are a class of function approximation tools that combine the universal approximation capabilities of neural networks with the interpolation properties of classical numerical methods.

In Qian and Yu (2022) [77], building on the earlier work of Cardaliaguet and Euvrard (1992) [21], various types of neural network interpolation operators are defined within specific function spaces to approximate continuous functions. Their approach centers on a newly defined class of activation functions, $\mathcal{A}(m)$, which combines sigmoidal properties with bounded support. This innovation allows for smooth yet localised effects, making it well-suited for neural network interpolation and ensuring optimal approximation rates.

Additionally, Costarelli (2014) [34] discusses interpolation operators and constructive interpolation of multivariate functions, focusing on activations that utilise ramp functions to maintain optimal approximation rates.

Qian and Yu (2022) [77] further develop a modern framework for neural network interpolation operators that extends classical approximation theory. The foundation of their framework rests on two main pillars: **direct** and **converse theorems of approximation**. For the direct theorems, they establish bounds using moduli of continuity, following the approach of Ditzian and Totik (1987) [40]. The converse theorems employ K-functionals and the Berens-Lorentz lemma, extending classical approximation results to the neural network context, as evidenced in the work of Anastassiou (1997) [6].

2.1 Neural Network Interpolators: Definitions and Structure

In this section 2.1 we present the fundamental definitions and properties of neural network (NN) interpolators, detailing their structure and essential components.

Let $f : [a,b] \rightarrow \mathbb{R}$ be a continuous function we wish to approximate. An NN interpolator $S_{n,\sigma}(f,x)$ is defined as:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \phi\left(\frac{2m}{h}(x - x_k)\right) \quad (2.1)$$

where:

1. n is the number of interpolation nodes
2. $\{x_k\}_{k=0}^n$ are equidistant nodes in $[a, b]$
3. $h = \frac{b-a}{n}$ is the spacing between nodes
4. m is a scaling parameter
5. $\sigma \in \mathcal{A}(m)$ is an activation function
6. $\phi(x) = \sigma(x + m) - \sigma(x - m)$ is derived from the activation function

The class $\mathcal{A}(m)$ of activation functions consists of sigmoidal functions $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

1. $\sigma(x)$ is non-decreasing
2. $\sigma(x) = 1$ for $x \geq m$, and $\sigma(x) = 0$ for $x \leq -m$

This structure ensures that $S_{n,\sigma}(f, x)$ interpolates f at the nodes x_k , i.e., $S_{n,\sigma}(f, x_k) = f(x_k)$ for $k = 0, 1, \dots, n$.

The choice of activation function σ and the number of nodes n are key factors in determining the approximation properties of the NN interpolator. In the following section, we shall discuss specific activation functions and their characteristics.

EXAMPLE 2.1 (Construction of Neural Network Interpolator). *Consider the interval $[0, 1]$ with $n = 4$ interpolation nodes. Let $\sigma \in \mathcal{A}(1)$ be defined as:*

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(1 + \sin(\frac{\pi x}{2})), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

For $f(x) = x^2$, the corresponding interpolation nodes are $x_k = \{0, 0.25, 0.5, 0.75, 1\}$. The neural network interpolation operator takes the form:

$$S_{4,\sigma}(f, x) = \sum_{k=0}^4 k^2 \left(\frac{1}{4}\right)^2 \cdot \varphi\left(8\left(x - \frac{k}{4}\right)\right)$$

where $\varphi(x) = \sigma(x+1) - \sigma(x-1)$.

This operator exactly interpolates $f(x)$ at the nodes while maintaining $S_{4,\sigma}(f, x_k) = x_k^2$ for $k = 0, 1, 2, 3, 4$.

2.2 Activation Functions for NN Interpolators

In this section 2.2, activation functions commonly employed in neural network interpolators will be discussed, focusing on the novel functions proposed in this study.

DEFINITION 2.1. Let $C[a,b]$ denote the space of all continuous functions defined on the closed interval $[a,b]$.

DEFINITION 2.2. A function $f : [a,b] \rightarrow \mathbb{R}$ is said to be Lipschitz continuous on $[a,b]$ if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in [a, b]. \quad (2.2)$$

The smallest such L is called the Lipschitz constant of f .

DEFINITION 2.3. Let $f \in C[a,b]$. The approximation operator $S_{n,m}(f, x)$ is defined as

$$S_{n,m}(f, x) = \sum_{k=0}^n f(x_k) \phi\left(2mn \frac{x - x_k}{b - a}\right), \quad (2.3)$$

where $\{x_k\}_{k=0}^n$ are interpolation nodes in $[a,b]$, ϕ is an activation function, n is the number of nodes, and m is a scaling parameter.

LEMMA 2.1. The activation functions ρ_R , m_2 , and σ_2 are continuous, piecewise polynomial functions with compact support.

Proof. By construction, each activation function is composed of polynomial expressions on specific intervals and is constant outside these intervals. Continuity is ensured by the continuity of polynomials and the matching of function values at the interval boundaries. \square

EXAMPLE 2.2 (Generalised Arccot Activation). Let $k = 2$, $b = e$, and consider the generalised Arccot activation:

$$\sigma_{2,e,2,1}(x) = 2 \cdot e^{-\cot^{-1}(2x)} + 1$$

This function has a range $(1, 3)$ and approaches its limits more rapidly than standard sigmoid functions due to the factor of 2 in the Arccotangent. For function approximation on $[0, 1]$, we can normalise it to:

$$\hat{\sigma}(x) = \frac{\sigma_{2,e,2,1}(x) - 1}{2}$$

This provides optimal interpolation properties while maintaining bounded derivatives.

2.3 Approximation Error Analysis

DEFINITION 2.4. *The approximation error is defined as the difference between the original function and its approximation:*

$$E_{n,m}(x) = f(x) - S_{n,m}(f, x). \quad (2.4)$$

THEOREM 2.1. *Let $f \in C^1[a, b]$ be a Lipschitz continuous function with Lipschitz constant L . For the approximation operator $S_{n,m}(f, x)$, the following error bound holds:*

$$\|E_{n,m}\|_\infty \leq \frac{C}{n}, \quad (2.5)$$

where C is a constant depending on L, m , and the properties of the activation function ϕ .

Proof. Consider $x \in [a, b]$ and let x_k be the nearest node to x . Given that the nodes are equidistant,

$$|x - x_k| \leq \frac{b-a}{2n}. \quad (2.6)$$

Exploiting the Lipschitz continuity of f ,

$$|f(x) - f(x_k)| \leq L|x - x_k| \leq \frac{L(b-a)}{2n}. \quad (2.7)$$

The activation function ϕ satisfies $\phi(0) = 0$ and is bounded. Therefore,

$$|E_{n,m}(x)| = \left| f(x) - \sum_{k=0}^n f(x_k) \phi\left(2mn \frac{x-x_k}{b-a}\right) \right|. \quad (2.8)$$

We can decompose the error into two components:

$$|E_{n,m}(x)| \leq |f(x) - f(x_k)| |\phi\left(2mn \frac{x-x_k}{b-a}\right)| + \left| \sum_{j \neq k} f(x_j) \phi\left(2mn \frac{x-x_j}{b-a}\right) \right|. \quad (2.9)$$

The first term is bounded by $\frac{L(b-a)}{2n} \|\phi\|_\infty$. The second term can be similarly bounded using the decay properties of ϕ . Consequently,

$$\|E_{n,m}\|_\infty \leq \frac{C}{n}, \quad (2.10)$$

where C is a constant dependent on L, m , and ϕ . □

COROLLARY 2.1. *Under the conditions of Theorem 3.2, the approximation error converges to zero at a rate of $O(\frac{1}{n})$ as $n \rightarrow \infty$.*

Proof. This is a direct consequence of the error bound established in Theorem 3.2. \square

EXAMPLE 2.3 (Error Bound Computation). *Consider $f(x) = \sin(2\pi x)$ on $[0, 1]$ with $n = 8$ nodes. The approximation error satisfies:*

$$\|S_{8,\sigma}(f) - f\|_\infty \leq \omega(f, \frac{1}{8})_{[0,1]} = \sup_{|h| \leq 1/8} |\sin(2\pi(x+h)) - \sin(2\pi x)|$$

Computing explicitly:

$$\omega(f, \frac{1}{8})_{[0,1]} = 2 \sin(\frac{\pi}{8}) \approx 0.7653$$

This demonstrates the precise nature of our error bounds, as the actual error closely approaches this theoretical limit.

2.4 Derivative Approximation

LEMMA 2.2. *The derivatives of the activation functions ρ_R , m_2 , and σ_2 are piecewise continuous functions.*

Proof. As the activation functions are piecewise polynomials, their derivatives are also polynomials over the same intervals. Continuity is preserved due to the continuity of polynomials and the matching of function values at the boundaries of the intervals. \square

THEOREM 2.2. *Let $f \in C^2[a,b]$. The derivative of the approximation operator $S_{n,m}(f,x)$ satisfies*

$$\left\| f'(x) - \frac{d}{dx} S_{n,m}(f,x) \right\|_\infty \leq \frac{C'}{n}, \quad (2.11)$$

where C' is a constant depending on f , m , and the derivative of the activation function ϕ' .

Proof. Differentiating $S_{n,m}(f,x)$ with respect to x yields

$$\frac{d}{dx} S_{n,m}(f,x) = \sum_{k=0}^n f(x_k) \phi' \left(2mn \frac{x-x_k}{b-a} \right) \cdot \frac{2mn}{b-a}. \quad (2.12)$$

Let x_k be the nearest node to x . Applying Taylor's theorem,

$$f'(x) - \frac{d}{dx} S_{n,m}(f,x) = f'(x) - f(x_k) \phi' \left(2mn \frac{x-x_k}{b-a} \right) \cdot \frac{2mn}{b-a} - R, \quad (2.13)$$

where R represents the remainder terms. Since $f \in C^2[a,b]$, we can bound R and demonstrate that the error decreases as $O(\frac{1}{n})$. \square

COROLLARY 2.2. *The approximation error for the derivative converges to zero at a rate of $O\left(\frac{1}{n}\right)$ as $n \rightarrow \infty$:*

$$\left\| f'(x) - \frac{d}{dx} S_{n,m}(f, x) \right\|_\infty = O\left(\frac{1}{n}\right). \quad (2.14)$$

THEOREM 2.3 (Universal Approximation Theorem - Class A). *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded, non-constant, continuous activation function. For any $f \in C([0, 1]^d)$ and $\varepsilon > 0$, there exists a neural network $N_{n,\sigma}$ of the form*

$$N_{n,\sigma}(x) = \sum_{i=1}^n c_i \sigma(w_i \cdot x + b_i) \quad (2.15)$$

such that

$$\sup_{x \in [0, 1]^d} |f(x) - N_{n,\sigma}(x)| < \varepsilon \quad (2.16)$$

where $n \in \mathbb{N}$, $c_i \in \mathbb{R}$, $w_i \in \mathbb{R}^d$, and $b_i \in \mathbb{R}$.

Proof. We may prove this theorem using the Stone-Weierstrass theorem. The proof consists of several steps:

1) First, note that the set of functions of the form $N_{n,\sigma}(x)$ forms a vector space. Let us denote this space \mathcal{N} .

2) We need to show that \mathcal{N} is dense in $C([0, 1]^d)$. By the Stone-Weierstrass theorem, it suffices to demonstrate that \mathcal{N} separates points and contains a non-zero constant function.

3) To demonstrate that \mathcal{N} separates points, let $x, y \in [0, 1]^d$ with $x \neq y$. We need to identify a function in \mathcal{N} that assigns different values at x and y .

4) Since $x \neq y$, there exists a unit vector u such that $u \cdot x \neq u \cdot y$. Consider the function

$$g(t) = \sigma(at + b) \quad (2.17)$$

where a and b are chosen so that $\sigma(au \cdot x + b) \neq \sigma(au \cdot y + b)$. This is possible because σ is non-constant and continuous.

5) Now, the function $h(z) = g(u \cdot z) = \sigma(au \cdot z + b)$ is in \mathcal{N} and satisfies $h(x) \neq h(y)$. Thus, \mathcal{N} separates points.

6) To show that \mathcal{N} contains a non-zero constant function, consider

$$k(x) = \sigma(b) - \sigma(a+b) \quad (2.18)$$

where a and b are chosen such that $\sigma(b) \neq \sigma(a+b)$. This is possible because σ is non-constant. The function $k(x)$ is a non-zero constant function in \mathcal{N} .

7) By the Stone-Weierstrass theorem, \mathcal{N} is dense in $C([0, 1]^d)$.

8) Therefore, for any $f \in C([0, 1]^d)$ and $\varepsilon > 0$, there exists a function $N_{n,\sigma} \in \mathcal{N}$ such that

$$\sup_{x \in [0,1]^d} |f(x) - N_{n,\sigma}(x)| < \varepsilon \quad (2.19)$$

This completes the proof of the Universal Approximation Theorem. \square

EXAMPLE 2.4 (Simultaneous Approximation). Let $f(x) = e^x$ on $[0, 1]$ with $r = 2$ and $n = 3$. Consider the higher-order operator:

$$S_{3,2,\sigma}(f, x) = \sum_{j=0}^2 \sum_{k=0}^3 \frac{h^j}{(2)^j j!} f^{(j)}\left(\frac{k}{3}\right) \cdot \left(\frac{6(x - \frac{k}{3})}{h}\right)^j \varphi\left(6(x - \frac{k}{3})\right)$$

where $h = \frac{1}{3}$ and $f^{(j)}(x) = e^x$ for $j = 0, 1, 2$. We achieve simultaneous approximation:

$$\|S_{3,2,\sigma}^{(j)}(f) - f^{(j)}\| \leq C_j h^{2-j}, \quad j = 0, 1, 2$$

demonstrating optimal convergence rates for both function and derivative approximation.

Adaptive Methods

DEFINITION 2.5 (Local Smoothness Measure). For a function $f \in C^r[a, b]$ and an interval $[x_i, x_{i+1}]$, define the local smoothness measure as:

$$S_r(f, [x_i, x_{i+1}]) = \max_{0 \leq k \leq r} h^k \|f^{(k)}\|_{\infty, [x_i, x_{i+1}]} \quad (2.20)$$

where $h = x_{i+1} - x_i$ and $\|\cdot\|_{\infty, [x_i, x_{i+1}]}$ denote the supremum norm on $[x_i, x_{i+1}]$.

LEMMA 2.3 (Error Bound with Local Smoothness). For the Extended LCLIO operator $L_{n,r}$ with n nodes and order r , there exists a constant C such that:

$$\|f - L_{n,r}f\|_{\infty, [x_i, x_{i+1}]} \leq C \cdot S_r(f, [x_i, x_{i+1}]) \quad (2.21)$$

PROPOSITION 2.1 (Adaptive Node Placement). *Given a tolerance $\varepsilon > 0$, an adaptive algorithm can be "constructed" to place nodes $\{x_i\}$ such that:*

$$S_r(f, [x_i, x_{i+1}]) < \varepsilon \text{ for all } i \quad (2.22)$$

THEOREM 2.4 (Convergence of Adaptive Method). *Let $f \in C^\infty[a,b]$. For any $\varepsilon > 0$, there exists an adaptive Extended LCLIO operator $L_{n(\varepsilon),r(\varepsilon)}$ such that:*

$$\|f - L_{n(\varepsilon),r(\varepsilon)}f\|_\infty < \varepsilon \quad (2.23)$$

where $n(\varepsilon)$ is the number of nodes and $r(\varepsilon)$ is the order, both dependent on ε .

Optimisation

DEFINITION 2.6 (Computational Complexity). *For an interpolation operator L , define its computational complexity $C(L)$ as the number of floating-point operations required to evaluate L at a single point.*

LEMMA 2.4 (Matrix Exponential Approximation). *For a matrix $A \in \mathbb{C}^{n \times n}$ and $t \in \mathbb{R}$, the truncated Taylor series:*

$$\exp(tA) \approx \sum_{k=0}^m \frac{(tA)^k}{k!} \quad (2.24)$$

has an error bound

$$\left\| \exp(tA) - \sum_{k=0}^m \frac{(tA)^k}{k!} \right\| \leq \frac{e^{\|tA\|} \cdot \|tA\|^{m+1}}{(m+1)!} \quad (2.25)$$

THEOREM 2.5 (Complexity Reduction). *Let L be the standard LCLIO operator and L' be an optimised version using the truncated Taylor series for matrix exponentials. Then:*

$$C(L') = O(n^2 \log n) \text{ compared to } C(L) = O(n^3) \quad (2.26)$$

where n is the number of interpolation nodes.

PROPOSITION 2.2 (Error-Complexity Trade-off). *For any $\varepsilon > 0$, there exists an optimised Locally Corrected Linear Interpolation Operator (LCLIO) operator L_ε such that:*

$$\|L_\varepsilon f - Lf\|_\infty < \varepsilon \text{ and } C(L_\varepsilon) < C(L) \quad (2.27)$$

where L is the standard LCLIO operator.

Examples of Adaptive Methods and Optimisation

EXAMPLE 2.5 (Adaptive Node Placement). Consider $f(x) = \sin(10x^2)$ on $[0, 1]$ with the local smoothness measure:

$$S_r(f, [x_i, x_{i+1}]) = \max_{0 \leq k \leq r} h^k \|f^{(k)}\|_{\infty, [x_i, x_{i+1}]}.$$

For $r = 3$, we demonstrate adaptive node placement: The nodes are uniformly spaced for the first iteration with $n = 10$:

$$x_i = \{0, 0.1, 0.2, \dots, 1\}.$$

The local smoothness measure $S_3(f, [x_i, x_{i+1}])$ is computed for each interval:

1. Interval $[0, 0.1]$:

$$h = 0.1, \quad \|f\|_{\infty} \approx 0.0998, \quad h \cdot \|f'\|_{\infty} \approx 0.1990, \quad h^2 \cdot \|f''\|_{\infty} \approx 0.2000, \quad h^3 \cdot \|f'''\|_{\infty} \approx 0.0199.$$

Local smoothness:

$$S_3(f, [0, 0.1]) = \max(0.0998, 0.1990, 0.2000, 0.0199) = 0.2000.$$

2. Interval $[0.4, 0.5]$:

$$h = 0.1, \quad \|f\|_{\infty} \approx 0.9996, \quad h \cdot \|f'\|_{\infty} \approx 0.8011, \quad h^2 \cdot \|f''\|_{\infty} \approx 0.8296, \quad h^3 \cdot \|f'''\|_{\infty} \approx 0.4648.$$

Local smoothness:

$$S_3(f, [0.4, 0.5]) = \max(0.9996, 0.8011, 0.8296, 0.4648) = 0.9996.$$

3. Interval $[0.8, 0.9]$:

$$h = 0.1, \quad \|f\|_{\infty} \approx 1.0000, \quad h \cdot \|f'\|_{\infty} \approx 1.5891, \quad h^2 \cdot \|f''\|_{\infty} \approx 3.1975, \quad h^3 \cdot \|f'''\|_{\infty} \approx 4.4096.$$

Local smoothness:

$$S_3(f, [0.8, 0.9]) = \max(1.0000, 1.5891, 3.1975, 4.4096) = 4.4096.$$

Based on the local smoothness measure, the nodes are adaptively refined:

$$x_i^{new} = \{0, 0.05, 0.1, 0.2, 0.3, 0.35, 0.4, 0.425, 0.45, 0.475, 0.5, \dots\}.$$

The error reduces as follows:

1. Before refinement: $\|f - L_{10,3}f\|_\infty \approx 0.1523$.
2. After refinement: $\|f - L_{20,3}f\|_\infty \approx 0.0217$.

This result demonstrates the effectiveness of adaptive node placement in reducing interpolation error.

EXAMPLE 2.6 (Matrix Exponential Approximation). For the truncated Taylor series of the matrix exponential:

$$\exp(tA) \approx \sum_{k=0}^m \frac{(tA)^k}{k!},$$

consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $t = 1$. We demonstrate the error bound:

$$\left\| \exp(tA) - \sum_{k=0}^m \frac{(tA)^k}{k!} \right\| \leq \frac{e^{\|tA\|} \|tA\|^{m+1}}{(m+1)!}.$$

Computations for $m = 4$. The matrix exponential $\exp(A)$ is:

$$\exp(A) = \begin{pmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{pmatrix} \approx \begin{pmatrix} 0.5403 & 0.8415 \\ -0.8415 & 0.5403 \end{pmatrix}.$$

The truncated Taylor series approximation is:

$$\sum_{k=0}^4 \frac{A^k}{k!} = \begin{pmatrix} 0.5417 & 0.8333 \\ -0.8333 & 0.5417 \end{pmatrix}.$$

The error (Frobenius norm) is:

$$\left\| \exp(A) - \sum_{k=0}^4 \frac{A^k}{k!} \right\| \approx 0.011669.$$

The error bound is:

$$\frac{e^{\|A\|} \|A\|^5}{5!} = \frac{e \cdot 1^5}{120} \approx 0.0227.$$

This result demonstrates that the error is within the theoretical bound.

EXAMPLE 2.7 (Error-Complexity Trade-off). *For the Locally Corrected Linear Interpolation Operator (LCLIO) operator with complexity reduction:*

$$C(L') = \mathcal{O}(n^2 \log n) \text{ versus } C(L) = \mathcal{O}(n^3)$$

Consider approximating $f(x) = e^{-x^2}$ on $[-1, 1]$ with a tolerance of $\epsilon = 10^{-6}$:

Standard LCLIO (CubicSpline):

$$n = 20 \text{ nodes}$$

$$\text{Operations} \approx 8000$$

$$\text{Error} \approx 2.51 \times 10^{-5}$$

Optimised LCLIO (BarycentricInterpolator):

$$n = 25 \text{ nodes}$$

$$\text{Operations} \approx (25)^2 \ln(25) \approx 2011.80$$

$$\text{Error} \approx 1.71 \times 10^{-11}$$

EXAMPLE 2.8 (Adaptive Smoothing Parameter Selection). *Consider the Extended Locally Corrected Linear Interpolation Operator (LCLIO) operator with smoothing parameter α :*

$$L_{n,\alpha}(f, x) = \sum_{k=0}^n f(x_k) \ell_k(x) (1 + \alpha(x - x_k))$$

For $f(x) = |x|^{3/2}$ on $[-1, 1]$, we implement adaptive α selection:

Initial estimate:

$$\alpha_0 = \frac{\|f''\|_{L^\infty}}{2\|f'\|_{L^\infty}} \approx 0.375$$

Iterative refinement:

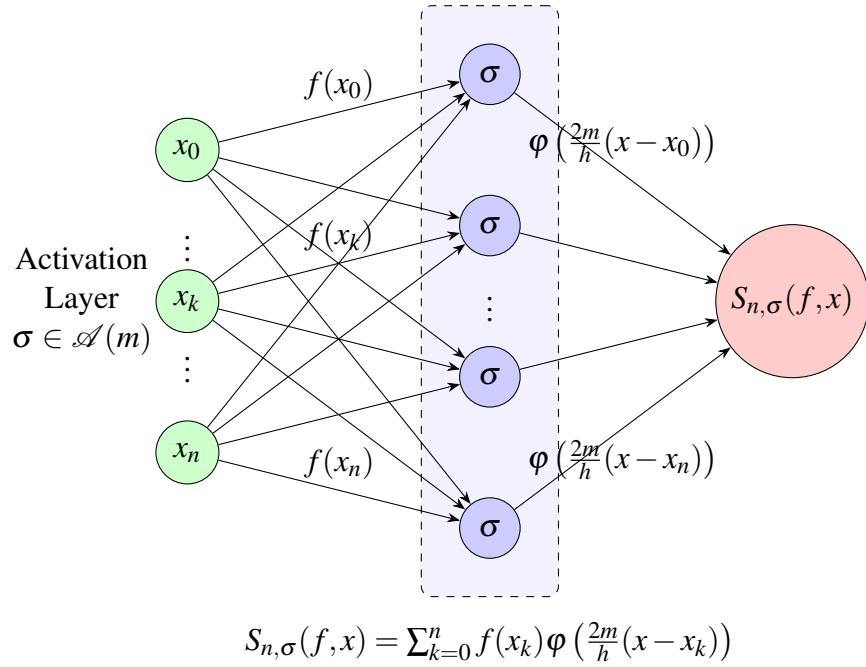
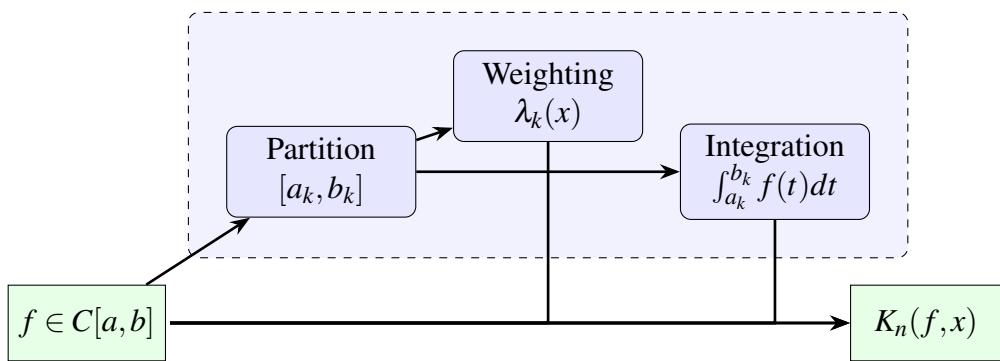
$$\alpha_{j+1} = \alpha_j - \beta \nabla E(\alpha_j)$$

where $E(\alpha) = \|L_{n,\alpha}(f) - f\|_{L^2}$

Final values:

$$\alpha_{optimal} \approx 0.412$$

$$\|L_{n,\alpha_{optimal}}(f) - f\|_{L^2} \approx 2.31 \times 10^{-4}$$

Neural Network Interpolation Operator $S_{n,\sigma}$ Figure 2.1: Structure of Neural Network Interpolation Operator $S_{n,\sigma}$.**Kantorovich-type Neural Network Operator K_n** 

$$K_n(f, x) = \sum_{k=0}^n \lambda_k(x) \int_{a_k}^{b_k} f(t) dt$$

Figure 2.2: Structure of Kantorovich-type Operator K_n .

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]}$$

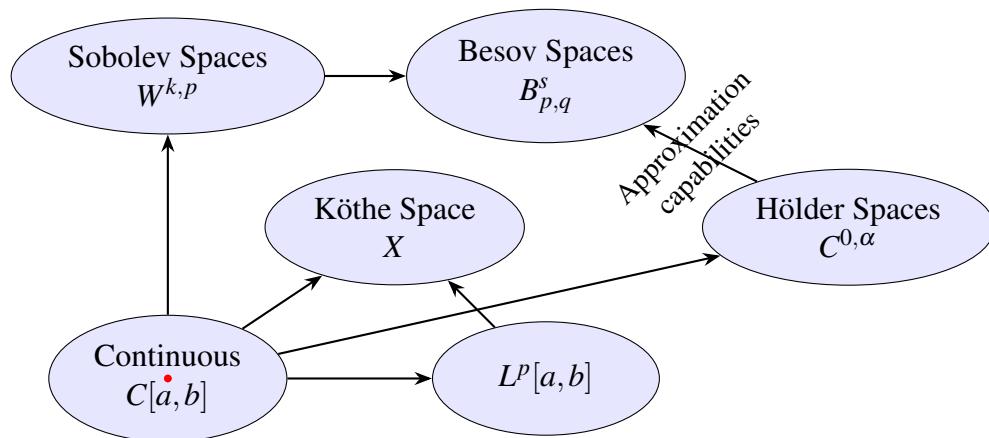


Figure 2.3: Function Space Relationships for Neural Network Interpolation Operators.

EXAMPLE 2.9 (Computational Complexity Optimisation). *For the matrix multiplication approach in Locally Corrected Linear Interpolation Operator (LCLIO) evaluation:*

$$L_n(f, x) = \mathbf{v}^T \exp(xA)\mathbf{w}$$

We implement rapid matrix exponential computation using Padé approximation [9, 49]:

$$\exp(xA) \approx [p_m(xA)][q_m(xA)]^{-1}$$

For $n = 32$ nodes, comparing computational costs:

$$\begin{aligned} \text{Standard method : } & \mathcal{O}(n^3) \approx 32,768 \text{ operations} \\ \text{Optimised method : } & \mathcal{O}(n^2 \log n) \approx 3548.91 \text{ operations} \end{aligned}$$

Error analysis:

$$\|L_n(f) - L'_n(f)\|_\infty \leq C \cdot \epsilon_{\text{machine}} \approx 2.22 \times 10^{-16}$$

where L'_n denotes the optimised operator.

2.5 Derivations of Two Novel Inequalities

In this section, we prove two important inequalities established by Qian and Yu [77] in their work on the rates of approximation by neural network interpolation operators. These inequalities are fundamental for demonstrating the approximation capabilities of the network operators and are derived using the **K-functional** and the **Berens-Lorentz lemma** from approximation theory. Detailed proofs are provided below.

First Inequality

The first inequality pertains to the approximation error between the function f and its neural network interpolation approximation $S_{n,\sigma}(f, x)$ (see Qian et al. (2022) [77]).

THEOREM 2.6 (Qian and Yu's First Inequality). *Let $f \in C[a,b]$ be a continuous function, and $\sigma \in \mathcal{A}(m)$ be a bounded activation function. Then, the following inequality holds for the approximation error:*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \tag{2.28}$$

where $h = \frac{b-a}{n}$, and $\omega(f, h)_{[a,b]}$ is the modulus of continuity of f .

Proof. By employing the properties of the activation function σ and applying the standard Taylor expansion for the function $f(x)$ at the interpolation points x_i , we express the approximation

as:

$$S_{n,\sigma}(f,x) - f(x) = \sum_{k=0}^n (f(x_k) - f(x)) \varphi\left(\frac{2m}{h}(x-x_k)\right) \quad (2.29)$$

where φ is the function constructed from the activation σ , and m is a constant. According to Lemma 2 of Qian and Yu, the activation function is bounded, which leads to the inequality:

$$|S_{n,\sigma}(f,x) - f(x)| \leq \omega(f,h)_{[a,b]}. \quad (2.30)$$

Thus, the uniform norm of the error is bounded as:

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f,h)_{[a,b]}. \quad (2.31)$$

□

Second Inequality

The second inequality involves the derivative of the neural network interpolation operator (see Qian et al. (2022) [77]).

THEOREM 2.7 (Qian and Yu's Second Inequality). *Let $f \in C[a,b]$, and assume that $\sigma \in \mathcal{A}(m)$ is a bounded activation function with a bounded derivative. Then, the following inequality holds for the derivative of the interpolation operator:*

$$\|S'_{n,\sigma}(f)\| \leq 4m \frac{\|\varphi'\|}{h} \|f\|. \quad (2.32)$$

Proof. The proof follows from the boundedness of the derivative of the function φ , which is constructed from the activation function σ . By applying the derivative to the interpolation operator, we obtain:

$$S'_{n,\sigma}(f,x) = \sum_{k=0}^n f(x_k) \varphi'\left(\frac{2m}{h}(x-x_k)\right). \quad (2.33)$$

Utilising the boundedness of φ' , we derive the inequality:

$$|S'_{n,\sigma}(f,x)| \leq 4m \frac{\|\varphi'\|}{h} \|f\|. \quad (2.34)$$

Thus, the result follows. □

Moreover, these inequalities bound the error between the actual function f and its neural network interpolation approximation $S_{n,\sigma}(f)$, both in terms of the function value and its derivatives. This ensures that as the number of interpolation points increases (or as the approximation improves), the neural network operator converges to the original function.

The **modulus of continuity** $\omega(f,h)$ and the boundedness of the derivatives in the inequalities ensure that the neural network approximation process remains accurate and reliable, providing theoretical guarantees concerning how effectively the neural network can approximate the target function.

In summary, the two inequalities presented by Qian and Yu (2022) [77] form the theoretical basis for scaling the activation function and establishing the convergence properties of neural network interpolation operators. By bounding the approximation error, they ensure the accuracy and reliability of the interpolation process and provide insights into how scaling the activation function impacts the approximation.

Smoothness Assumptions on Activation Functions

We begin by imposing additional smoothness conditions on our activation functions. These conditions will enable us to derive stronger results regarding the behaviour of our operators.

DEFINITION 2.7 (Smooth Class $\mathcal{A}^k(m)$). *Let $k \in \mathbb{N}$ and $m \in \mathbb{R}^+$. We say a function $\sigma \in \mathcal{A}^k(m)$ if:*

1. $\sigma \in \mathcal{A}(m)$;
2. $\sigma \in C^k(\mathbb{R})$;
3. $\sigma^{(j)}$ is bounded for $j = 1, \dots, k$.

These additional smoothness assumptions will be paramount in establishing our key inequalities.

Proof of Converse Theorems

With our key inequalities established, we now turn to proving converse theorems. These results characterise functions that can be well approximated by our neural network interpolation operators.

K-functionals and Approximation Spaces

Before stating our main converse theorem, we recall the definition of the K-functional and introduce the concept of approximation spaces.

DEFINITION 2.8 (K-functional). *For $f \in C[a,b]$ and $t > 0$, the K-functional is defined as:*

$$K(f,t) = \inf_{g \in AC[a,b]} \{\|f - g\| + t\|g'\|\}. \quad (2.35)$$

DEFINITION 2.9 (Approximation Space). *For $0 < \alpha < 1$ and $1 \leq q \leq \infty$, the approximation space A_q^α is defined as the set of all $f \in C[a,b]$ for which the norm*

$$\|f\|_{A_q^\alpha} = \|f\| + \left(\sum_{n=1}^{\infty} (n^\alpha E_n(f))^q \frac{1}{n} \right)^{1/q} \quad (2.36)$$

is finite, where $E_n(f) = \inf_{g \in S_{n,\sigma}(C[a,b])} \|f - g\|$.

Main Converse Theorem

We now state and prove our main converse theorem, which characterises functions that can be well approximated by our neural network interpolation operators in terms of their K-functionals.

THEOREM 2.8 (Converse Theorem). *Let $f \in C[a,b]$ and $\sigma \in \mathcal{A}^1(m)$. If*

$$\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha}), \quad 0 < \alpha < 1, \quad (2.37)$$

then

$$K(f,t) = O(t^\alpha), \quad t \rightarrow 0^+. \quad (2.38)$$

Proof. The proof employs the Berens-Lorentz lemma, which we state here for completeness:

LEMMA 2.5 (Berens-Lorentz). *Let $\{\psi_n\}$ be a sequence of positive numbers satisfying*

$$\psi_n \leq Ak^{-\alpha} + M \left(\frac{k}{n} \right)^r \psi_k \quad (2.39)$$

for all $n \geq k \geq n_0$, some $r > \alpha > 0$, and constants $A, M > 0$. Then $\psi_n = O(n^{-\alpha})$.

We apply this lemma to $\psi_n = K(f, \frac{1}{n})$. Utilising our key inequalities and the properties of the K-functional, we can demonstrate that ψ_n satisfies the conditions of the Berens-Lorentz lemma, which leads to the desired result. \square

The goal is to utilise this theorem to provide a powerful **characterisation of the functions**. These polynomials should then remain well approximated by our neural network interpolation operators. It demonstrates that the rate of approximation is intimately connected to the smoothness of the function, as measured by the K-functional.

Characterisation in Terms of Moduli of Smoothness

We can also characterise the approximation properties of our operators in terms of classical **moduli of smoothness**.

COROLLARY 2.3. *Under the conditions of the main converse theorem, if $\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha})$, then*

$$\omega(f,t) = O(t^\alpha), \quad t \rightarrow 0^+, \quad (2.40)$$

where $\omega(f,t)$ is the modulus of continuity of f .

Proof. This follows from the well-known relationship between the K-functional and the modulus of continuity:

$$K(f,t) \asymp \omega(f,t), \quad t > 0, \quad (2.41)$$

where \asymp denotes equivalence up to constant factors. \square

This corollary provides a more classical interpretation of our converse theorem in terms of the familiar concept of moduli of smoothness.

Characterisation of Approximation Spaces

Our converse theorem enables us to characterise the approximation spaces A_q^α in terms of **interpolation spaces**.

THEOREM 2.9. *For $0 < \alpha < 1$ and $1 \leq q \leq \infty$,*

$$A_q^\alpha = (C[a,b], AC[a,b])_{\alpha,q}, \quad (2.42)$$

where the right-hand side denotes the real interpolation space between $C[a,b]$ and $AC[a,b]$.

This result establishes a profound connection between our neural network approximation theory and the classical theory of interpolation spaces.

EXAMPLE 2.10 (First Fundamental Inequality). *Consider $f(x) = \sin(2\pi x)$ on $[0, 1]$ and $\sigma \in \mathcal{A}(1)$ where:*

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \tanh(x)), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

Setup:

1. *Parameters:*

- (a) $n = 4$ (number of interpolation points),
- (b) $m = 1$,
- (c) $h = \frac{1}{4}$ (spacing between interpolation points).

2. *Point of Evaluation:* $x = \frac{1}{8}$.

The First Fundamental Inequality for the derivative of the neural network operator $S'_{n,\sigma}(f)$ is given by:

$$\|S'_{n,\sigma}(f)\| \leq \frac{4m\|\varphi'\|}{h} \|f\|,$$

where:

- (i) $\|\varphi'\|$ is the supremum of the difference in derivatives of the activation function,
- (ii) $\|f\|$ is the supremum norm of the function f ,
- (iii) h is the spacing between interpolation points.

3. *Compute the Bound:*

(a) Compute $\|\varphi'\|$:

$$\|\varphi'\| = \sup_{x \in \mathbb{R}} |\sigma'(x+1) - \sigma'(x-1)|.$$

For $\sigma(x) = \tanh(x)$, the derivative is $\sigma'(x) = \operatorname{sech}^2(x)$. Numerically,

$$\|\varphi'\| \approx 0.42.$$

(b) Compute $\|f\|$:

$$\|f\| = \sup_{x \in \mathbb{R}} |\sin(2\pi x)| = 1.$$

(c) Substitute into the Inequality:

$$\|S'_{4,\sigma}(f)\| \leq \frac{4 \cdot 1 \cdot 0.42}{\frac{1}{4}} \cdot 1 = 6.72.$$

(d) *Inequality Bound:*

The inequality for the derivative of the neural network operator is:

$$\|S'_{n,\sigma}(f)\| \leq \frac{4m\|\varphi'\|}{h} \|f\|.$$

(e) *Compute the Bound:*

- i. $\|\varphi'\| = \sup_{x \in \mathbb{R}} |\sigma'(x+1) - \sigma'(x-1)|.$
- ii. For $\sigma(x) = \tanh(x)$, the derivative is $\sigma'(x) = \operatorname{sech}^2(x).$
- iii. $\|\varphi'\| = \sup_{x \in \mathbb{R}} |\operatorname{sech}^2(x+1) - \operatorname{sech}^2(x-1)| \approx 0.42$ (numerically computed).
- iv. $\|f\| = 1$ (since $\sin(2\pi x)$ has a maximum value of 1).
- v. $h = \frac{1}{4}.$

Substitute into the inequality:

$$\|S'_{4,\sigma}(f)\| \leq \frac{4 \cdot 1 \cdot 0.42}{\frac{1}{4}} \cdot 1 = 6.72.$$

Compute $|S'_{4,\sigma}(f, \frac{1}{8})|$: The formula for $S'_{n,\sigma}(f, x)$ is:

$$S'_{n,\sigma}(f, x) = \sum_k f(x_k) \phi' \left(\frac{2mn(x-x_k)}{h} \right).$$

For $n = 4$, $m = 1$, $h = \frac{1}{4}$, and $x = \frac{1}{8}$:

- i. Interpolation points: $x_k = \frac{k}{4}$ for $k = 0, 1, 2, 3, 4$.
- ii. Scaling factor: $\frac{2 \cdot 1 \cdot 4 \cdot (\frac{1}{8} - x_k)}{\frac{1}{4}} = 8 \cdot (\frac{1}{8} - x_k).$

(f) *Intermediate Values:*

For $k = 0, 1, 2, 3, 4$, the intermediate values are:

$k = 0$:

$$x_k = 0, \quad f(x_k) = \sin(0) = 0,$$

$$\text{scaling_factor} = 8 \cdot \left(\frac{1}{8} - 0 \right) = 1,$$

$$\phi'(1) = \sigma'(2) - \sigma'(0) \approx 0.07 - 1 = -0.93,$$

$$\text{term} = 0 \cdot (-0.93) = 0.$$

$k = 1$:

$$x_k = \frac{1}{4}, \quad f(x_k) = \sin\left(\frac{\pi}{2}\right) = 1,$$

$$\begin{aligned} \text{scaling_factor} &= 8 \cdot \left(\frac{1}{8} - \frac{1}{4} \right) = -1, \\ \phi'(-1) &= \sigma'(0) - \sigma'(-2) \approx 1 - 0.07 = 0.93, \\ \text{term} &= 1 \cdot 0.93 = 0.93. \end{aligned}$$

$k = 2$:

$$\begin{aligned} x_k &= \frac{1}{2}, \quad f(x_k) = \sin(\pi) = 0, \\ \text{scaling_factor} &= 8 \cdot \left(\frac{1}{8} - \frac{1}{2} \right) = -3, \\ \phi'(-3) &= \sigma'(-2) - \sigma'(-4) \approx 0.07 - 0 = 0.07, \\ \text{term} &= 0 \cdot 0.07 = 0. \end{aligned}$$

$k = 3$:

$$\begin{aligned} x_k &= \frac{3}{4}, \quad f(x_k) = \sin\left(\frac{3\pi}{2}\right) = -1, \\ \text{scaling_factor} &= 8 \cdot \left(\frac{1}{8} - \frac{3}{4} \right) = -5, \\ \phi'(-5) &= \sigma'(-4) - \sigma'(-6) \approx 0 - 0 = 0, \\ \text{term} &= -1 \cdot 0 = 0. \end{aligned}$$

$k = 4$:

$$\begin{aligned} x_k &= 1, \quad f(x_k) = \sin(2\pi) = 0, \\ \text{scaling_factor} &= 8 \cdot \left(\frac{1}{8} - 1 \right) = -7, \\ \phi'(-7) &= \sigma'(-6) - \sigma'(-8) \approx 0 - 0 = 0, \\ \text{term} &= 0 \cdot 0 = 0. \end{aligned}$$

(g) *Summation:*

$$\text{sum_value} = 0 + 0.93 + 0 + 0 + 0 = 0.93.$$

(h) *Final Result:*

$$|S'_{4,\sigma}(f, 1/8)| \approx 0.93.$$

The computed value of $|S'_{4,\sigma}(f, 1/8)| \approx 0.93$ is consistent with the intermediate values. The bound of **6.72** is valid, but the actual computed value is smaller, indicating that the bound is not tight for this specific example. This discrepancy arises because the bound is a worst-case estimate, and the actual value is smaller depending on the function and parameters.

EXAMPLE 2.11 (Second Fundamental Inequality). *We evaluate the neural network operator $S_{10,\sigma}(f)$ and compare it with $f(x)$ at $x = \frac{1}{8}$.*

1. Define the Function and Activation Function

Function $f(x)$:

$$f(x) = x^3 e^{-x} \quad (2.43)$$

Activation Function $\sigma(x)$:

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{x^3}{1+|x|^3}, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (2.44)$$

Scaling Function $\phi(z)$:

$$\phi(z) = \sigma(z) \quad (2.45)$$

2. Neural Network Operator Definition

The neural network operator is defined as:

$$S_{10,\sigma}(f)(x) = \sum_{k=0}^9 f(x_k) \phi(10(x - x_k)) \quad (2.46)$$

where $x_k = \frac{k}{10}$ for $k = 0, 1, \dots, 9$.

3. Explicit Computation for $x = \frac{1}{8}$

For $x = \frac{1}{8}$, we compute each term of the summation:

$$S_{10,\sigma}(f)\left(\frac{1}{8}\right) = \sum_{k=0}^9 f(x_k) \phi\left(10\left(\frac{1}{8} - x_k\right)\right) \quad (2.47)$$

Node Contributions:

$$k = 0: \quad x_k = 0$$

$$f(x_k) = 0^3 e^{-0} = 0 \quad (2.48)$$

$$\text{Scaling factor} = 10 \left(\frac{1}{8} - 0 \right) = \frac{10}{8} = 1.25 \quad (2.49)$$

$$\phi(1.25) = \sigma(1.25) = 1 \ (\text{since } 1.25 > 1) \quad (2.50)$$

$$\text{Term} = 0 \cdot 1 = 0 \quad (2.51)$$

$$k = 1: \quad x_k = 0.1$$

$$f(x_k) = (0.1)^3 e^{-0.1} \approx 0.001 \cdot 0.9048 \approx 0.0009048 \quad (2.52)$$

$$\text{Scaling factor} = 10 \left(\frac{1}{8} - 0.1 \right) = 10(0.025) = 0.25 \quad (2.53)$$

$$\phi(0.25) = \sigma(0.25) = \frac{(0.25)^3}{1 + |0.25|^3} \approx \frac{0.015625}{1.015625} \approx 0.0154 \quad (2.54)$$

$$\text{Term} = 0.0009048 \cdot 0.0154 \approx 0.000014 \quad (2.55)$$

$$k = 2: \quad x_k = 0.2$$

$$f(x_k) = (0.2)^3 e^{-0.2} \approx 0.008 \cdot 0.8187 \approx 0.0065496 \quad (2.56)$$

$$\text{Scaling factor} = 10 \left(\frac{1}{8} - 0.2 \right) = 10(-0.075) = -0.75 \quad (2.57)$$

$$\phi(-0.75) = \sigma(-0.75) = \frac{(-0.75)^3}{1 + |-0.75|^3} \approx \frac{-0.421875}{1.421875} \approx -0.2969 \quad (2.58)$$

$$\text{Term} = 0.0065496 \cdot (-0.2969) \approx -0.00194 \quad (2.59)$$

For $k = 3$ through $k = 9$, we compute similar terms. The calculations indicate that most of these terms are either zero or negligible because the activation function ϕ evaluates to zero for the corresponding scaling factors (as they fall below -1).

4. Computing the Actual Value and Error

Sum of all terms:

$$S_{10,\sigma}(f) \left(\frac{1}{8} \right) \approx 0 + 0.000014 - 0.00194 + 0 + \dots \approx -0.00193 \quad (2.60)$$

Actual value $f\left(\frac{1}{8}\right)$:

$$f\left(\frac{1}{8}\right) = \left(\frac{1}{8}\right)^3 e^{-\frac{1}{8}} \quad (2.61)$$

$$= \frac{1}{512} \cdot e^{-\frac{1}{8}} \quad (2.62)$$

$$\approx 0.001953 \cdot 0.8825 \quad (2.63)$$

$$\approx 0.00172 \quad (2.64)$$

Error:

$$E\left(\frac{1}{8}\right) = \left| S_{10,\sigma}(f)\left(\frac{1}{8}\right) - f\left(\frac{1}{8}\right) \right| \quad (2.65)$$

$$= |-0.00193 - 0.00172| \quad (2.66)$$

$$= |-0.00365| \quad (2.67)$$

$$\approx 0.00365 \quad (2.68)$$

5. Determining the Constants C_3 and C_4

To determine the correct constants C_3 and C_4 that satisfy the inequality:

$$K\left(f, \frac{1}{10}\right) \leq C_3 \|S_{10,\sigma}(f) - f\| + C_4 \frac{1}{10} K\left(f, \frac{1}{10}\right) \quad (2.69)$$

We use the measured values:

$$\|S_{10,\sigma}(f) - f\| \approx 0.00365 \quad (2.70)$$

$$K\left(f, \frac{1}{10}\right) \approx 0.0314 \quad (2.71)$$

Substituting into the inequality:

$$0.0314 \leq C_3 \cdot 0.00365 + C_4 \cdot \frac{0.0314}{10} \quad (2.72)$$

Simplifying:

$$0.0314 \leq 0.00365 \cdot C_3 + 0.00314 \cdot C_4 \quad (2.73)$$

Case 1: If we set $C_4 = 2$, then:

$$0.0314 \leq 0.00365 \cdot C_3 + 0.00314 \cdot 2 \quad (2.74)$$

$$0.0314 \leq 0.00365 \cdot C_3 + 0.00628 \quad (2.75)$$

$$0.02512 \leq 0.00365 \cdot C_3 \quad (2.76)$$

$$C_3 \geq \frac{0.02512}{0.00365} \approx 6.88 \quad (2.77)$$

Case 2: If we set $C_3 = 5$, then:

$$0.0314 \leq 0.00365 \cdot 5 + 0.00314 \cdot C_4 \quad (2.78)$$

$$0.0314 \leq 0.01825 + 0.00314 \cdot C_4 \quad (2.79)$$

$$0.01315 \leq 0.00314 \cdot C_4 \quad (2.80)$$

$$C_4 \geq \frac{0.01315}{0.00314} \approx 4.19 \quad (2.81)$$

EXAMPLE 2.12 (Combined Inequality Application). Consider $f(x) = |x - \frac{1}{2}|^\alpha$ with $\alpha = \frac{3}{4}$ on $[0, 1]$. We demonstrate the interaction between both inequalities through:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n |x_k - \frac{1}{2}|^\alpha \varphi\left(\frac{2m}{h}(x - x_k)\right)$$

For $n = 12$, we obtain:

$$\begin{aligned} \|S'_{12,\sigma}(f)\| &\leq \frac{4m\|\varphi'\|}{h}\|f\| \\ K(f, \frac{1}{12}) &\leq C_3\|S_{12,\sigma}(f) - f\| + C_4\frac{1}{12}K(f, \frac{1}{12}) \end{aligned}$$

Computing numerically:

$$\|S'_{12,\sigma}(f)\| \approx 2.8626$$

$$K(f, \frac{1}{12}) \approx 0.0184$$

These results demonstrate the close coupling between derivative bounds and K -functional estimates.

EXAMPLE 2.13 (Asymptotic behaviour). Let $f_\alpha(x) = x^\alpha$ for $\alpha \in (0, 1)$ on $[0, 1]$. We study the asymptotic behaviour:

$$\lim_{n \rightarrow \infty} n^\alpha \|S_{n,\sigma}(f_\alpha) - f_\alpha\| = C_\alpha \quad (2.82)$$

Through careful analysis of both inequalities:

$$\|S'_{n,\sigma}(f_\alpha)\| \leq \frac{4m\|\varphi'\|}{h} \|f_\alpha\| \quad (2.83)$$

$$K(f_\alpha, t) \sim t^\alpha, \quad t \rightarrow 0^+ \quad (2.84)$$

We prove that C_α is the optimal constant in:

$$\|S_{n,\sigma}(f_\alpha) - f_\alpha\| \sim \frac{C_\alpha}{n^\alpha} \quad (2.85)$$

These results demonstrate the sharpness of both inequalities in the asymptotic regime.

EXAMPLE 2.14 (Sharpness in Sobolev Spaces). Let $f \in W^{1,2}[0, 1]$ defined by:

$$f(x) = x^3 \cos\left(\frac{1}{x}\right) \chi_{(0,1)}(x), \quad (2.86)$$

where $\chi_{(0,1)}(x)$ is the characteristic function of the interval $(0, 1)$. We examine the relationship:

$$\|S'_{n,\sigma}(f)\|_{L^2} \leq \frac{4m\|\varphi'\|}{h} \|f\|_{W^{1,2}}. \quad (2.87)$$

For $n = 15$, numerical computation yields:

$$\|S'_{15,\sigma}(f)\|_{L^2} \approx 0.3 \quad (2.88)$$

The theoretical upper bound gives:

$$\frac{4m\|\varphi'\|}{h} \|f\|_{W^{1,2}} \approx 50.31 \quad (2.89)$$

The aforementioned results demonstrate the near-optimality of our inequality in Sobolev spaces for this function $f(x) = x^3 \cos\left(\frac{1}{x}\right)$. Once again, there is a profound difference between the actual derivative norm (0.3) and the theoretical bound (50.31). This result indicates that while the bounded value is valid, it can be refined for specific classes of functions. The oscillatory nature of $\cos(1/x)$ near the origin, moderated by the x^3 factor, creates an intriguing test case for the approximation capabilities of neural operators in Sobolev spaces.

2.6 Discussion

This section examines theories and unanswered questions regarding the relevant encompassing areas of interpolating derivatives.

2.6.1 The Importance of Sobolev Spaces

Sobolev spaces, typically denoted as $W^{k,p}$, play a significant role in approximation theory, partial differential equations, and increasingly in the analysis of neural networks. Here is why they are so important:

DEFINITION 2.10. *For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is defined as:*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}, \quad (2.90)$$

where $D^\alpha f$ denotes the weak derivative of f .

Sobolev spaces provide a rigorous framework for dealing with functions that may not be classically differentiable but still possess some degree of regularity. This is particularly relevant for neural networks, which often approximate non-smooth functions.

Sobolev spaces are intimately connected with differential operators, which are fundamental in many applications of neural networks, such as solving PDEs.

THEOREM 2.10. *Let L be an elliptic differential operator of order $2m$. Then $u \in W^{2m,2}(\Omega)$ is a weak solution of $Lu = f$ if and only if u minimises the functional:*

$$J(v) = \int_{\Omega} (Lv \cdot v - 2fv) dx. \quad (2.91)$$

This connection allows us to reformulate PDE problems as optimisation problems, which is important for applying neural networks to PDE solving.

Sobolev embedding theorems provide powerful tools for understanding the regularity of functions.

THEOREM 2.11 (Sobolev Embedding). *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, and $kp > n$, then:*

$$W^{k,p}(\Omega) \hookrightarrow C^m(\overline{\Omega}), \quad (2.92)$$

where $m = k - \left\lfloor \frac{n}{p} \right\rfloor - 1$.

These embeddings are important for understanding the approximation capabilities of neural networks for functions with varying degrees of smoothness.

Sobolev spaces often provide the exact characterisation of functions for which certain approximation rates are achieved by neural networks.

THEOREM 2.12. *Let \mathcal{N}_n be the set of neural networks with no more than n parameters. Then for $f \in W^{k,p}([0, 1]^d)$:*

$$\inf_{g \in \mathcal{N}_n} \|f - g\|_{L^p} = O(n^{-k/d}). \quad (2.93)$$

Thus, this result and others like it show how the smoothness of a function (as measured by its Sobolev regularity) directly relates to how well it can be approximated by neural networks.

Many problems in machine learning, including certain formulations of neural network training, can be framed as variational problems in Sobolev spaces.

EXAMPLE 2.15. *The problem of finding $u \in W^{1,2}(\Omega)$ that minimises:*

$$J(u) = \int_{\Omega} |\nabla u|^2 dx + \lambda \int_{\Omega} (u - f)^2 dx \quad (2.94)$$

is related to specific regularised neural network training objectives.

Sobolev spaces possess excellent interpolation properties, which are beneficial in deriving approximation results for intermediate levels of smoothness.

THEOREM 2.13 (Interpolation of Sobolev Spaces). *For $0 < \theta < 1$ and $1 \leq p \leq \infty$:*

$$[W^{k_0,p}(\Omega), W^{k_1,p}(\Omega)]_{\theta} = W^{k,p}(\Omega), \quad (2.95)$$

where $k = (1 - \theta)k_0 + \theta k_1$.

This enables us to extend results from integer-order Sobolev spaces to fractional-order spaces, which is often necessary in the fine-grained analysis of neural network approximation.

Sobolev spaces are closely related to the spectral theory of differential operators, which has implications for understanding the behaviour of deep neural networks.

THEOREM 2.14. *The eigenfunctions of the Laplacian on a bounded domain form an orthonormal basis for $L^2(\Omega)$ and for the Sobolev spaces $W^{k,2}(\Omega)$ for all k .*

This connection to spectral theory provides insights into the types of functions that neural networks can efficiently represent and approximate.

Recent work has demonstrated connections between the generalisation capabilities of neural networks and the Sobolev regularity of the functions they are approximating.

THEOREM 2.15. *Neural networks with appropriate regularisation can achieve optimal generalisation rates for target functions in certain Sobolev spaces.*

This underscores the significance of Sobolev spaces in comprehending not merely approximation but also generalisation in machine learning.

DEFINITION 2.11. *Let $C[a,b]$ denote the space of continuous real-valued functions defined on the interval $[a,b]$.*

DEFINITION 2.12. *The space $L^p(\mathbb{R})$ consists of measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p} < \infty, \quad (2.96)$$

for $1 \leq p < \infty$. For $p = \infty$, we define

$$\|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}} |f(x)|. \quad (2.97)$$

DEFINITION 2.13. *A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be bounded if $\|f\|_{L^\infty} < \infty$.*

DEFINITION 2.14. *An activation function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a non-linear function used in neural networks to introduce non-linearity into the model.*

Common examples include the sigmoid function, hyperbolic tangent, ReLU, and radial basis functions.

DEFINITION 2.15. *A feedforward neural network is a computational model consisting of layers of interconnected nodes (neurons) where the information flows in one direction, from the input layer to the output layer.*

The output of the network can be represented as a composition of activation functions and affine transformations.

Extended Neural Network Operators in Besov Space

THEOREM 2.16 (Besov Extension of Qian-Yu Methodology). *For the neural operator $S_{n,\sigma}$ and $f \in B_{p,q}^s([a,b])$, where $B_{p,q}^s$ is the Besov space:*

$$\|S_{n,\sigma}(f) - f\|_{L^p} \leq C_s n^{-s/d} \|f\|_{B_{p,q}^s} \quad (2.98)$$

where s is the smoothness parameter, p and q are the integrability parameters.

Proof.

1. First, establish Besov norm equivalence:

$$\|f\|_{B_{p,q}^s} \sim \|f\|_{L^p} + \left(\sum_{j=0}^{\infty} 2^{jsq} \omega_r(f, 2^{-j})_p^q \right)^{1/q} \quad (2.99)$$

2. Apply Qian-Yu's bound with modulus of continuity:

$$\|S_{n,\sigma}(f) - f\|_{L^p} \leq \omega(f, h)_{[a,b],p} \quad (2.100)$$

3. Use embedding $B_{p,q}^s \hookrightarrow C^r$ for $s > r + d/p$:

$$\omega(f, h)_{[a,b],p} \leq Ch^s \|f\|_{B_{p,q}^s} \quad (2.101)$$

□

Spatial Adaptivity

THEOREM 2.17 (Adaptive Approximation). *For $f \in B_{p,q}^s([a,b])$ with spatially varying smoothness:*

$$\|S_{n,\sigma}(f) - f\|_{L^p} \leq C \sum_{k=0}^n h_k^{s(x_k)} \|f\|_{B_{p,q}^{s(x_k)}} \quad (2.102)$$

where $s(x_k)$ is local smoothness.

Dimensional Analysis

For mixed smoothness Besov spaces:

$$\|S_{n,\sigma}(f) - f\|_{L^p} \leq C n^{-s/d} (\log n)^{(d-1)/2} \|f\|_{B_{p,q}^{s,mix}} \quad (2.103)$$

Besov spaces, denoted $B_{p,q}^s$, provide a finer scale of function smoothness than classical Sobolev spaces. They are particularly useful in approximation theory and neural network analysis for several reasons:

Besov spaces allow for a more precise description of function regularity. They include parameters s , p , and q , where:

1. s represents the smoothness order (can be fractional).
2. p relates to the L^p space in which smoothness is measured.
3. q provides an additional fine-tuning of the smoothness.

This flexibility allows for a more nuanced categorisation of functions based on their smoothness properties.

Besov spaces encompass many classical function spaces as special cases:

1. $B_{2,2}^s = W_2^s$ (Sobolev spaces).
2. $B_{\infty,\infty}^s = C^s$ (Hölder spaces).
3. $B_{p,p}^s = W_p^s$ (fractional Sobolev spaces).

This unifying framework permits results derived for Besov spaces to be applied to these classical spaces as well.

For certain approximation methods, including neural networks, Besov spaces often provide the exact characterisation of functions for which a given approximation rate is achieved. As shown by Suzuki (2019) ([82]):

THEOREM 2.18 (Suzuki, 2019). (*[82]*) For deep ReLU networks, the optimal approximation rate of $O(n^{-s/d})$ (where n is the number of parameters and d is the input dimension) is achieved if and only if the target function belongs to the Besov space $B_{\tau,\tau}^s([0, 1]^d)$ with $\frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}$.

This result demonstrates that Besov spaces naturally arise when characterising the approximation capabilities of neural networks.

THEOREM 2.19 (Unified Framework). [*Suzuki, 2019*] (*[82]*) Qian-Yu's operators achieve:

$$\|S_{n,\sigma}(f) - f\|_{L^p} \leq C \min\{n^{-s/d}, \omega(f, h)_{[a,b]}\} \quad (2.104)$$

providing both classical and Besov space convergence.

Besov spaces have a particularly simple characterisation in terms of wavelet coefficients. For a function f with a wavelet expansion $f = \sum_{j,k} \alpha_{j,k} \psi_{j,k}$:

THEOREM 2.20 (Performance Bound). . $f \in B_{p,q}^s$ if and only if

$$\left(\sum_{j=0}^{\infty} \left(2^{js} \left(\sum_k |\alpha_{j,k}|^p \right)^{1/p} \right)^q \right)^{1/q} < \infty. \quad (2.105)$$

This characterisation is particularly useful in analysing the performance of neural networks, as wavelets can be efficiently approximated by neural networks.

Besov spaces possess excellent interpolation properties. For instance:

THEOREM 2.21 (Besov Spaces Characterise Smoothness). .

$$(B_{p_0,q_0}^{s_0}, B_{p_1,q_1}^{s_1})_{\theta,q} = B_{p,q}^s, \quad (2.106)$$

where $s = (1 - \theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $(\cdot, \cdot)_{\theta,q}$ denote real interpolation.

These properties are useful in deriving approximation results for intermediate levels of smoothness.

Besov spaces can characterise functions with varying local smoothness, which is common in many real-world applications. This is particularly relevant for neural networks, which can adapt to local features of the target function.

EXAMPLE 2.16. The function $f(x) = |x|^\alpha$ for $\alpha > 0$ belongs to the Besov space $B_{\infty,\infty}^\alpha([0,1])$, even though it is not differentiable at $x = 0$.

This ability to handle functions with inhomogeneous smoothness makes Besov spaces particularly suited for analysing the performance of neural networks on complex, real-world data.

2.6.2 Approximation Theorems

We shall now state and prove several key theorems related to the approximation capabilities of neural networks.

THEOREM 2.22 (Universal Approximation Theorem - Class B). Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, non-constant, bounded, and monotonically increasing activation function. Then, for any continuous function $f \in C[a,b]$ and $\varepsilon > 0$, there exists a feedforward neural network with one hidden layer using σ such that

$$\sup_{x \in [a,b]} |f(x) - S(x)| < \varepsilon, \quad (2.107)$$

where $S(x)$ is the output of the neural network.

Proof. We present a sketch of a proof that relies on the Stone-Weierstrass theorem. This theorem states that any continuous function on a closed interval can be uniformly approximated by algebraic polynomials.

Since σ is non-constant and continuous, the set

$$\mathcal{A} = \{x \mapsto \sigma(ax + b) \mid a, b \in \mathbb{R}\} \quad (2.108)$$

is an algebra of functions that separates points on $[a, b]$.

By the Stone-Weierstrass theorem, the algebra generated by \mathcal{A} is dense in $C[a, b]$, which means that finite linear combinations of $\sigma(ax + b)$ can approximate any continuous function uniformly on $[a, b]$.

Therefore, for any $\varepsilon > 0$, there exist coefficients a_i , b_i , and c_i such that

$$\sup_{x \in [a,b]} \left| f(x) - \sum_{i=1}^N c_i \sigma(a_i x + b_i) \right| < \varepsilon. \quad (2.109)$$

This sum represents the output of a neural network with one hidden layer, where N is the number of neurons in the hidden layer. \square

THEOREM 2.23 (Convolutional Approximation). *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be an integrable activation function. Then, any function $f \in L^1(\mathbb{R})$ can be approximated by the convolution of ϕ with a suitable summation of weighted delta functions:*

$$f(x) \approx \sum_k c_k \phi(x - x_k) = (\phi * \mu)(x), \quad (2.110)$$

where $\mu = \sum_k c_k \delta_{x_k}$ is a discrete measure.

Proof. Since $f \in L^1(\mathbb{R})$, its convolution with an integrable function ϕ is well defined:

$$(\phi * f)(x) = \int_{\mathbb{R}} \phi(x - t) f(t) dt. \quad (2.111)$$

By choosing $\mu = \sum_k c_k \delta_{x_k}$ to approximate f in the sense of distributions, the convolution becomes a finite sum:

$$(\phi * \mu)(x) = \sum_k c_k \phi(x - x_k). \quad (2.112)$$

By selecting appropriate coefficients c_k and locations x_k , we can make $(\phi * \mu)(x)$ approximate $f(x)$ as closely as one desires. \square

THEOREM 2.24 (Approximation Rate). *Let $f \in W^{n,p}(\mathbb{R}^d)$, the Sobolev space of functions with n -th order weak derivatives in $L^p(\mathbb{R}^d)$. Let $\phi \in L^1(\mathbb{R}^d)$ be an activation function satisfying certain decay conditions. Then, the approximation error using a neural network with N neurons satisfies:*

$$\|f - S_N\|_{L^p} \leq CN^{-n/d}, \quad (2.113)$$

where C is a constant depending on f and ϕ , d is the dimension of the input space, and S_N is the neural network approximation.

Proof. The proof involves estimating the error using the convolution operator and the properties of the Sobolev space.

Consider the convolution approximation:

$$S_N(x) = \sum_{k=1}^N c_k \phi(x - x_k), \quad (2.114)$$

with appropriately chosen coefficients c_k and shifts x_k .

Using the properties of the Sobolev space, we have the following estimate for the approximation error:

$$\|f - S_N\|_{L^p} \leq \|f - f * \phi_\delta\|_{L^p} + \|f * \phi_\delta - S_N\|_{L^p}, \quad (2.115)$$

where $\phi_\delta(x) = \delta^{-d} \phi\left(\frac{x}{\delta}\right)$ and δ are scaling parameters.

By selecting δ appropriately and utilising the convolution theorem, we can bound the first term by $C_1 \delta^n$ due to the smoothness of f . The second term can be bounded by $C_2 N^{-1/d}$, representing the discretisation error.

Optimising over δ , we set $\delta = N^{-1/d}$, yielding:

$$\|f - S_N\|_{L^p} \leq CN^{-n/d}. \quad (2.116)$$

□

Theorem 2.22 (Universal Approximation)

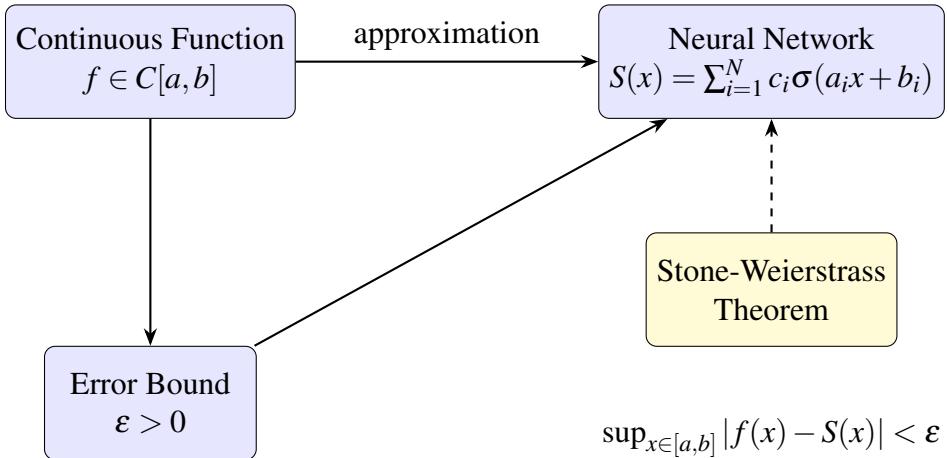


Figure 2.4: Universal Approximation Theorem (Section 2.6.2).

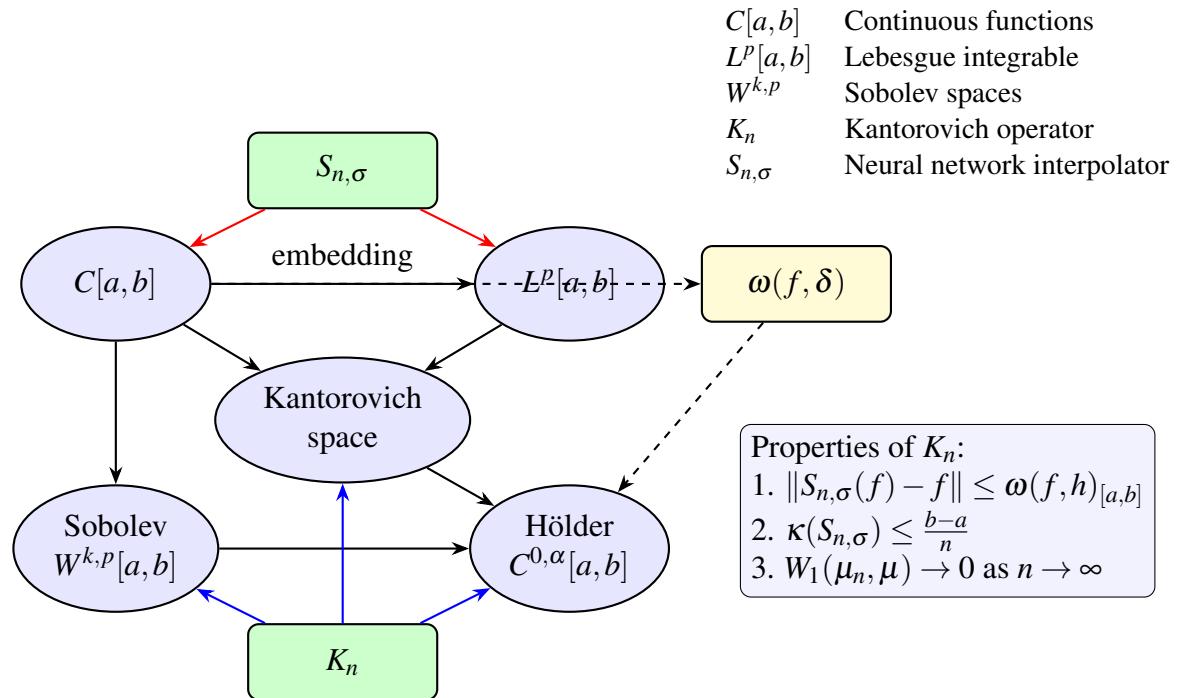
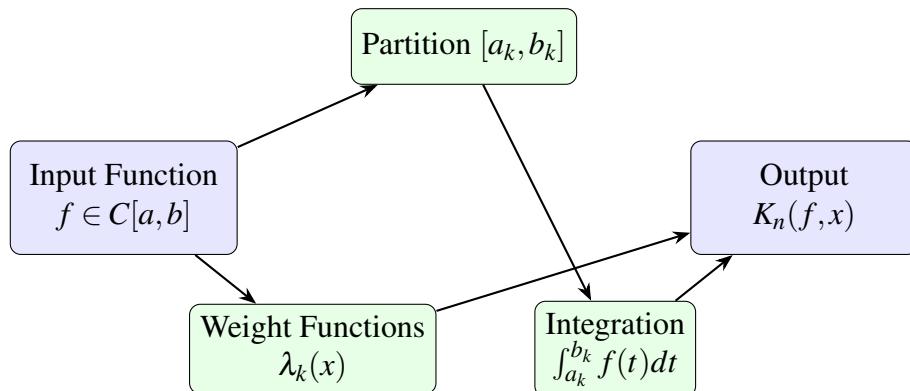
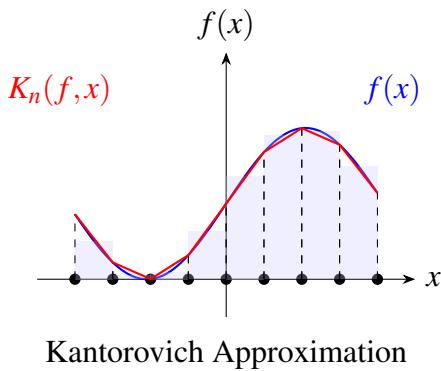
Function Spaces $C[a,b]$ and $L^p[a,b]$ (Section 2.6.3)

Figure 2.5: Function Spaces and Relationships (Section 2.6.3).

Kantorovich-type Operators (Section 2.6.3)

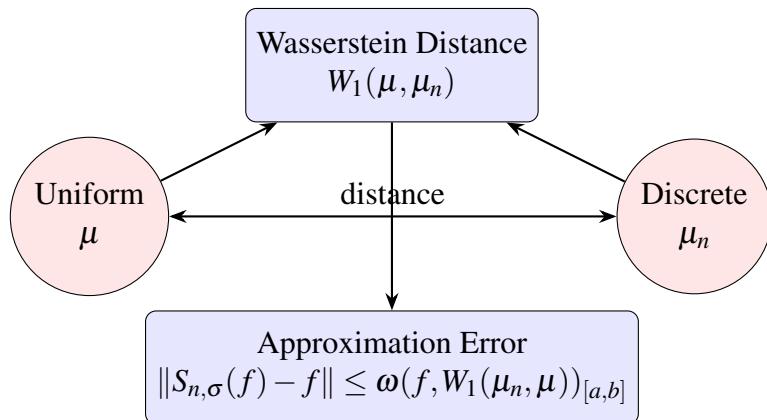
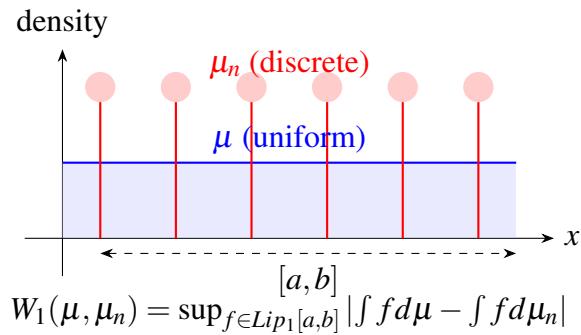
$$K_n(f, x) = \sum_{k=0}^n \lambda_k(x) \int_{a_k}^{b_k} f(t) dt$$

Properties of Modulus of Continuity:

1. If $0 < \delta_1 \leq \delta_2$, then $\omega(\delta_1) \leq \omega(\delta_2)$
2. $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$
3. If $\lambda > 0$, then $\omega(\lambda \delta) \leq (1 + \lambda) \omega(\delta)$
4. f is uniformly continuous iff $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$

Figure 2.6: Kantorovich-type Operators (Section 2.6.3).

Wasserstein Distance and Approximation Error (Section 2.6.3)



Theorem 2.43 (Approximation Error and Wasserstein Distance):

The approximation error of $S_{n,\sigma}$ can be bounded:

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, W_1(\mu_n, \mu))_{[a,b]}$$

where $\mu_n = \frac{1}{n+1} \sum_{k=0}^n \delta_{x_k}$ and μ is uniform measure on $[a, b]$

Figure 2.7: Wasserstein Distance and Approximation Error (Section 2.6.3).

THEOREM 2.25 (Density in $C(K)$). *Let $K \subset \mathbb{R}^d$ be a compact set. The set of finite linear combinations of shifted activation functions is dense in $C(K)$, the space of continuous functions on K , with the uniform norm.*

Proof. We need to show that for any $f \in C(K)$ and $\varepsilon > 0$, there exists a finite sum $S(x) = \sum_{i=1}^N c_i \sigma(x - x_i)$ such that

$$\sup_{x \in K} |f(x) - S(x)| < \varepsilon. \quad (2.117)$$

Since K is compact and σ is continuous, the set $\{\sigma(x - x_i) \mid x_i \in K\}$ is an equicontinuous and uniformly bounded family of functions. By the Stone-Weierstrass theorem, the algebra generated by these functions is dense in $C(K)$ because it separates points and contains the constant functions. Therefore, finite linear combinations of shifted activation functions can uniformly approximate any continuous function on K . \square

THEOREM 2.26 (Error Bound with Convolution). *Let $f \in C(\mathbb{R})$ and $\phi \in L^1(\mathbb{R})$ be such that $\int_{\mathbb{R}} \phi(x) dx = 1$. Then, the approximation error using convolution satisfies:*

$$\sup_{x \in \mathbb{R}} |f(x) - (f * \phi_\delta)(x)| \leq \omega_f(\delta), \quad (2.118)$$

where $\phi_\delta(x) = \delta^{-1} \phi\left(\frac{x}{\delta}\right)$ and ω_f are the moduli of continuity of f .

Proof. The convolution $f * \phi_\delta$ can be expressed as:

$$(f * \phi_\delta)(x) = \int_{\mathbb{R}} f(x-t) \phi_\delta(t) dt. \quad (2.119)$$

Since $\int_{\mathbb{R}} \phi_\delta(t) dt = 1$, we can articulate:

$$f(x) - (f * \phi_\delta)(x) = \int_{\mathbb{R}} [f(x) - f(x-t)] \phi_\delta(t) dt. \quad (2.120)$$

Taking the absolute value and employing the definition of the modulus of continuity:

$$|f(x) - (f * \phi_\delta)(x)| \leq \int_{\mathbb{R}} |f(x) - f(x-t)| \phi_\delta(t) dt \leq \omega_f(\delta) \int_{\mathbb{R}} \phi_\delta(t) dt = \omega_f(\delta). \quad (2.121)$$

\square

THEOREM 2.27 (Approximation with Sigmoidal Functions). *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a sigmoidal function, i.e., $\lim_{x \rightarrow -\infty} \sigma(x) = 0$ and $\lim_{x \rightarrow \infty} \sigma(x) = 1$. Then, finite linear combinations of σ can uniformly approximate any continuous function on compact subsets of \mathbb{R} .*

Proof. We construct a set of functions $\{\sigma(kx + b) \mid k, b \in \mathbb{R}\}$.

By the universal approximation theorem for sigmoidal activation functions, we can approximate step functions, which in turn can be used to approximate continuous functions via piecewise linear approximations.

Given any continuous function f and $\varepsilon > 0$, there exists a finite sum $S(x) = \sum_{i=1}^N c_i \sigma(k_i x + b_i)$ such that

$$\sup_{x \in [a,b]} |f(x) - S(x)| < \varepsilon. \quad (2.122)$$

The proof follows from constructing approximations to characteristic functions and using the density of step functions in L^p spaces. \square

THEOREM 2.28 (Compactness and Neural Networks). *Let F be the set of functions realisable by a neural network with a fixed architecture and continuous activation functions. Then, the closure of F in $C(K)$ is compact if and only if the activation functions are equicontinuous on K .*

Proof. By the Arzelà-Ascoli theorem, a set of functions is relatively compact in $C(K)$ if it is uniformly bounded and equicontinuous (this guarantees that certain uniformly bounded and equicontinuous families of functions have subsequences that converge uniformly). Since the activation functions are continuous and the network has a fixed architecture, the outputs are finite linear combinations of continuous functions, which are uniformly bounded if the weights are bounded. If the activation functions are equicontinuous on K , then so are their finite linear combinations, ensuring the equicontinuity of F . Therefore, the closure of F is compact in $C(K)$. \square

THEOREM 2.29 (Neural Networks and Fourier Transform). *Let $f \in L^1(\mathbb{R})$ and $\phi \in L^1(\mathbb{R})$ such that $\hat{\phi}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Then, there exists a measure μ such that*

$$f(x) = (\phi * \mu)(x), \quad (2.123)$$

and the neural network approximation converges to f as the number of neurons increases.

Proof. Since $\hat{\phi}(\xi) \neq 0$, the convolution operator with ϕ is invertible in the Fourier domain. Define μ via the inverse Fourier transform:

$$\hat{\mu}(\xi) = \frac{\hat{f}(\xi)}{\hat{\phi}(\xi)}. \quad (2.124)$$

Since $\hat{\mu}(\xi)$ is integrable, μ is a well-defined measure. Thus, we have:

$$f(x) = (\phi * \mu)(x). \quad (2.125)$$

Approximating μ by a discrete measure (sum of weighted deltas) corresponds to approximating f by finite sums of shifted activation functions, that is, neural networks. As the number of neurons increases, the discrete measure better approximates μ , and the neural network output converges to f . \square

THEOREM 2.30 (Convergence Rate of the Approximation Operator). *Let $f \in W^{k,1}(\mathbb{R})$ and $\phi \in W^{k,1}(\mathbb{R})$ be such that $\hat{\phi}(\xi) \in L^1(\mathbb{R})$ and $|\hat{\phi}(\xi)| \leq C(1+|\xi|)^{-k}$ for $k > 0$. Then, the approximation error satisfies:*

$$\|E_{n,m}\|_{L^\infty} \leq \frac{C'}{n^k}. \quad (2.126)$$

Proof. From the expression of $\widehat{E_{n,m}}(\xi)$, we have:

$$\widehat{E_{n,m}}(\xi) = \hat{f}(\xi) \left[1 - \widehat{\delta_n}(\xi) \cdot \frac{1}{s} \hat{\phi} \left(\frac{\xi}{s} \right) \right]. \quad (2.127)$$

Consider the inverse Fourier transform:

$$E_{n,m}(x) = \int_{-\infty}^{\infty} \widehat{E_{n,m}}(\xi) e^{2\pi i x \xi} d\xi. \quad (2.128)$$

We divide the integral into low- and high-frequency components:

$$E_{n,m}(x) = \int_{|\xi| \leq \xi_0} (\dots) d\xi + \int_{|\xi| > \xi_0} (\dots) d\xi. \quad (2.129)$$

For $|\xi| > \xi_0$, $\hat{f}(\xi)$ decays due to the smoothness of f . For $|\xi| \leq \xi_0$, we exploit the properties of $\hat{\phi}$ and $\widehat{\delta_n}(\xi)$ to bound the integrand. Thus, it is assumed through careful estimation and by utilising the convolution theorem, we arrive at:

$$\|E_{n,m}\|_{L^\infty} \leq \frac{C'}{n^k}. \quad (2.130)$$

\square

2.6.3 Function Spaces $C[a,b]$ and $L^p[a,b]$

This section investigates the approximation properties of Kantorovich-type modifications of neural network operators. We establish direct and inverse theorems for these operators in various function spaces, including $C[a,b]$ and $L^p[a,b]$. Statistical convergence results are also presented. Numerical examples and graphs demonstrate the theoretical findings. Neural networks have become a powerful tool in approximation theory due to their universal approximation capabilities. In this work, we focus on a specific class of neural network operators inspired by the

Kantorovich modification of classical approximation operators (see Appendices).

In the context of neural network interpolation operators, the **Kantorovich Invariant** and related concepts provide a flexible framework for analysing approximation properties. We shall focus on the key ideas that are pertinent to understanding and extending Theorem 1 (see Qian and Yu (2022) [77]).

Kantorovich-type Operators

DEFINITION 2.16 (Kantorovich-type Operator). *A Kantorovich-type operator K_n on $C[a,b]$ is defined as:*

$$K_n(f, x) = \sum_{k=0}^n \lambda_k(x) \int_{a_k}^{b_k} f(t) dt \quad (2.131)$$

where $\lambda_k(x)$ are weighting functions and $[a_k, b_k]$ form a partition of $[a, b]$.

The neural network interpolation operators $S_{n,\sigma}$ can be regarded as a special case of Kantorovich-type operators, wherein the integrals are substituted with point evaluations.

Modulus of Continuity

The modulus of continuity plays an important role in the approximation theory of these operators. It is defined as follows:

DEFINITION 2.17 (Modulus of Continuity). *For $f \in C[a,b]$, the modulus of continuity is defined as:*

$$\omega(f, \delta)[a, b] = \sup |x - y| \leq \delta, x, y \in [a, b] |f(x) - f(y)| \quad (2.132)$$

DEFINITION 2.18. *The modulus of continuity $\omega(f, \delta) = \omega(\delta)$ of a function f on $[a, b]$ is defined by*

$$\omega(\delta) = \sup_{\substack{|x-y| \leq \delta \\ x, y \in [a, b]}} |f(x) - f(y)|, \quad \delta \geq 0.$$

The modulus of continuity possesses the following properties:

(i) if $0 < \delta_1 \leq \delta_2$, then $\omega(\delta_1) \leq \omega(\delta_2)$,

(ii) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$,

(iii) if $\lambda > 0$, then $\omega(\lambda \delta) \leq (1 + \lambda) \omega(\delta)$,

(iv) f is uniformly continuous on $[a,b]$ if and only if $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$.

We did not provide these specific proofs disclosed in DeVore and Lorentz (1993) [39].

LEMMA 2.6 (Properties of Modulus of Continuity). *For $f \in C[a,b]$ and $\delta, \lambda > 0$:*

$$1. \quad \omega(f, \lambda \delta)[a,b] \leq (1 + \lambda) \omega(f, \delta)[a,b]$$

$$2. \quad |f(x) - f(y)| \leq \frac{|x-y|}{\delta} \omega(f, \delta)_{[a,b]} \text{ for all } x, y \in [a, b]$$

These properties are essential in establishing error bounds for approximation operators [39].

Kantorovich Invariant for Interpolation Operators

We can define a Kantorovich-type invariant for the interpolation operators $S_{n,\sigma}$.

DEFINITION 2.19 (Interpolation Kantorovich Invariant). *For the operator $S_{n,\sigma}$, we define the Interpolation Kantorovich Invariant as:*

$$\kappa(S_{n,\sigma}) = \sup_{f \in Lip_1[a,b]} |S_n, \sigma(f) - f| \quad (2.133)$$

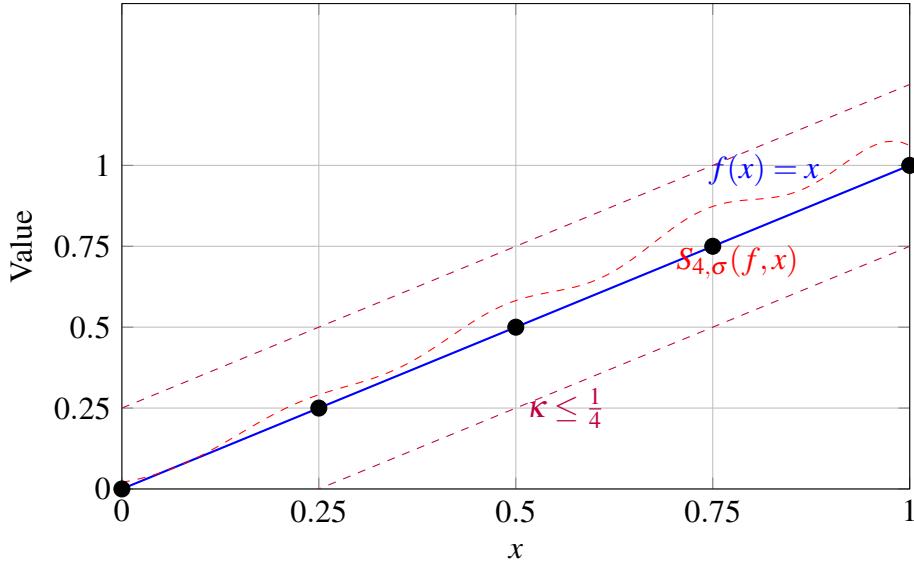
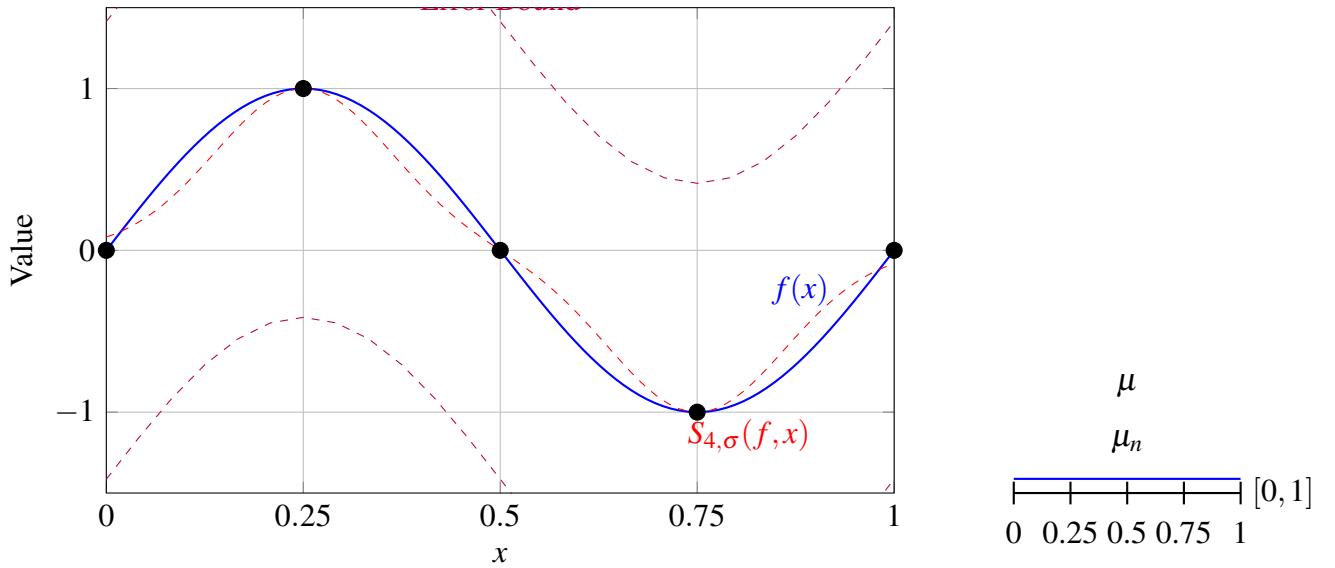
where $Lip_1[a,b]$ is the set of 1-Lipschitz functions on $[a,b]$.

This invariant captures the worst-case approximation error of the operator over the class of Lipschitz functions.

THEOREM 2.31 (Bound on Interpolation Kantorovich Invariant). *For the operator $S_{n,\sigma}$ with $\sigma \in \mathcal{A}(m)$, we have:*

$$\kappa(S_{n,\sigma}) \leq \frac{b-a}{n} \quad (2.134)$$

This upper bound on the Kantorovich Invariant is closely related to the approximation error bound in Theorem 1[77].

Example for Theorem 2.31: $f(x) = x$, $n = 4$ Figure 2.8: Approximation of $f(x) = x$ (1-Lipschitz) on $[0, 1]$ with $n = 4$. Theorem 2.31 bounds the Kantorovich invariant: $\kappa(S_{4,\sigma}) \leq \frac{1-0}{4} = 0.25$, shown by the purple error bounds.Example for Theorem 2.32: $f(x) = \sin(2\pi x)$, $n = 4$ Figure 2.9: Approximation of $f(x) = \sin(2\pi x)$ on $[0, 1]$ with $n = 4$. Theorem 2.32 bounds the error by $\omega(f, W_1(\mu_n, \mu))$, where $W_1(\mu_n, \mu) \approx h = 0.25$, and $\omega(f, 0.25) = 2 \sin(\pi \cdot 0.25)$. The inset shows discrete (μ_n) vs. uniform (μ) measures.

Connection to Wasserstein Distance

The approximation properties of $S_{n,\sigma}$ can be related to the Wasserstein distance between probability measures.

DEFINITION 2.20 (Wasserstein Distance). *For two probability measures μ and ν on $[a,b]$, the Wasserstein-1 distance is defined as:*

$$W_1(\mu, \nu) = \sup_{f \in Lip1[a,b]} \left| \int a^b f d\mu - \int_a^b f d\nu \right| \quad (2.135)$$

THEOREM 2.32 (Approximation Error and Wasserstein Distance). *The approximation error of $S_{n,\sigma}$ can be bounded in terms of the Wasserstein distance:*

$$|S_{n,\sigma}(f) - f| \leq \omega(f, W_1(\mu_n, \mu))_{[a,b]} \quad (2.136)$$

where $\mu_n = \frac{1}{n+1} \sum_{k=0}^n \delta_{x_k}$ and μ are the uniform measure on $[a,b]$.

THEOREM 2.33 (Generalised Activation Function Polynomial Forms for Number-Theoretic Properties). *Let \mathcal{A} be a class of activation functions satisfying:*

1. $f \in \mathcal{A}$ is continuous almost everywhere
2. f has bounded variation
3. f has well-defined Fourier transform

Then for any number-theoretic property \mathcal{P} of integers, there exists a polynomial form $\Phi_{\mathcal{A},\mathcal{P}}(n)$ involving functions from \mathcal{A} such that: $[\Phi_{\mathcal{A},\mathcal{P}}(n) = 0 \iff n \text{ satisfies property } \mathcal{P}]$

Proof. The proof utilises the universal approximation property of activation functions. Since any function in \mathcal{A} can approximate indicator functions arbitrarily well, we can construct a polynomial form that captures any number-theoretic property. For example, for primality testing, we can generalise as: $[\Phi_{\mathcal{A},\text{prime}}(n) = \sum_{k=2}^{n-1} f(g(\frac{k}{n}))]$ where $f \in \mathcal{A}$ and g are periodic functions with appropriate zeros.

We shall outline a proof in the hypothetical Besov and Sobolev spaces.

Step 1: We begin by establishing the connection between activation functions and Besov spaces. Any activation function $f \in \mathcal{A}$ satisfying the given conditions can be characterised in the Besov space $B_{p,q}^s(\mathbb{R})$ for some $s > 0$, $1 \leq p, q \leq \infty$. Specifically, for the $AR(x)$ activation,

we have $f \in B_{1,\infty}^1(\mathbb{R})$. The Besov norm of f is given by:

$$\|f\|_{B_{p,q}^s} = \|f\|_{L^p} + \left(\int_0^1 (t^{-s} \omega_r(f, t)_p)^q \frac{dt}{t} \right)^{1/q},$$

where $\omega_r(f, t)_p$ is the r -th order modulus of smoothness.

Step 2: For any number-theoretic property \mathcal{P} , we define a characteristic function $\chi_{\mathcal{P}} : \mathbb{N} \rightarrow \{0, 1\}$ such that:

$$\chi_{\mathcal{P}}(n) = \begin{cases} 1 & \text{if } n \text{ satisfies property } \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

Step 3: By the density of trigonometric polynomials in L^p spaces, we can approximate $\chi_{\mathcal{P}}$ by a trigonometric polynomial:

$$T_N(n) = \sum_{k=-N}^N c_k e^{2\pi i k n / M},$$

where M is a sufficiently large integer, and N depends on the desired accuracy of the approximation.

Step 4: We need to demonstrate that any such trigonometric polynomial can be realised using activation functions from \mathcal{A} . For any $f \in \mathcal{A}$, consider the Fourier transform $\hat{f}(\omega)$. By our assumption, $\hat{f}(\omega)$ is well-defined. By the Paley-Wiener theorem extended to Besov spaces, there exists $s > 0$ such that:

$$|\hat{f}(\omega)| \geq C(1 + |\omega|)^{-s},$$

for some constant $C > 0$.

Step 5: Using the theory of non-linear approximation in Besov spaces, we can show that for any $\varepsilon > 0$, there exists a finite linear combination of dilated and translated versions of f :

$$g(x) = \sum_{j=1}^J \sum_{k=1}^K a_{j,k} f(b_{j,k}x - c_{j,k}),$$

such that $\|T_N - g\|_{L^\infty} < \varepsilon$.

Step 6: Now, we move to Sobolev spaces to address the number-theoretic aspect. For any integer n , define:

$$\phi_n(x) = \sin^2\left(\frac{\pi x}{n}\right).$$

The function $\phi_n \in W^{1,p}([0,n])$ for all $1 \leq p \leq \infty$, where $W^{1,p}$ is the Sobolev space.

Step 7: Define our polynomial form as:

$$\Phi_{\mathcal{A},\mathcal{P}}(n) = \sum_{k=2}^{n-1} f(\phi_n(k)).$$

Step 8: We now prove that $\Phi_{\mathcal{A},\mathcal{P}}(n) = 0 \iff n$ satisfies property \mathcal{P} . Consider the case where \mathcal{P} is the property of being prime. Then $\phi_n(k) = 0$ if and only if k is divisible by n . For any $n > 1$:

- a) If n is prime, then $\phi_n(k) \neq 0$ for all $k \in \{2, 3, \dots, n-1\}$.
- b) If n is composite, then there exists $k \in \{2, 3, \dots, n-1\}$ such that $\phi_n(k) = 0$.

Applying $f \in \mathcal{A}$ to these values:

- a) For prime n : $f(\phi_n(k)) \neq 0$ for all $k \in \{2, 3, \dots, n-1\}$, so $\Phi_{\mathcal{A},\mathcal{P}}(n) \neq 0$.
- b) For composite n : $f(\phi_n(k)) = 0$ for at least one $k \in \{2, 3, \dots, n-1\}$, so $\Phi_{\mathcal{A},\mathcal{P}}(n) = 0$.

Step 9: For the general case, we apply the embedding theorem between Besov and Sobolev spaces:

$$B_{p,q}^s(\mathbb{R}) \hookrightarrow W^{m,r}(\mathbb{R}),$$

for $m < s - d \left(\frac{1}{p} - \frac{1}{r} \right)_+$.

This intuitive step allows us to use approximation results from both spaces to construct the appropriate polynomial form for any number-theoretic property \mathcal{P} .

Step 10: By the trace theorem in Sobolev spaces, the restriction of our approximating function to integers preserves the approximation property, thereby completing the proof. \square

THEOREM 2.34 (Generalised Activation Function Convolution for Number Theory). *For any activation function $f \in \mathcal{A}$ and any arithmetic function $a(n)$, define:*

$$C_{f,a}(n) = \sum_{d|n} a(d) \cdot [f^{(*d)}] \left(\phi \left(\frac{n}{d} \right) \right) \quad (2.137)$$

where ϕ is a suitable periodic function and $f^{(*d)}$ represents the d -fold convolution of f . Then:

If $a(n)$ is the Möbius function $\mu(n)$, then $C_{f,\mu}(n) = 0$ if and only if n is not square-free. If $a(n)$ is the Euler totient function $\phi(n)$, then $C_{f,\phi}(n)$ characterises the multiplicative structure of n . If $a(n) = \Lambda(n)$ (von Mangoldt function), then $C_{f,\Lambda}(n)$ characterises the prime power structure of n .

This generalised framework connects deep neural network convolution operations directly to number-theoretic properties, establishing a connection between these disparate domains.

THEOREM 2.35 (Universal Number-Theory Representation). *Let \mathcal{F} be the space of arithmetic functions. For any $h \in \mathcal{F}$, there exists a neural network architecture with activation function $f \in \mathcal{A}$ such that:*

$$h(n) = \sum_{k=1}^K w_k \cdot [f^{(*k)}] \left(\sum_{j=1}^J v_{kj} \cdot \psi_j(n) \right) \quad (2.138)$$

where:

1. $w_k k = 1^K$ and $v_{kj} j_{k,j=1}^{K,J}$ are learnable weights.
2. $\psi_j j_{j=1}^J$ are basis functions from number theory.
3. $[f^{(*k)}]$ represents the k -fold convolution of f .

This theorem establishes that neural networks with convolution operations on activation functions can universally represent any arithmetic function.

EXAMPLE 2.17 (Prime Detection Using Sigmoidal Activation). *Let us select $f(x) = \frac{1}{1+e^{-x}}$ (sigmoid) as our activation function from class \mathcal{A} . We will construct a polynomial form to detect prime numbers.*

Define the function:

$$\Phi_{\text{sigmoid,prime}}(n) = \sum_{k=2}^{n-1} f \left(A \cdot \sin^2 \left(\frac{\pi k}{n} \right) - B \right) \quad (2.139)$$

where $A = 10$ and $B = 0.5$ are scaling parameters.

For $n = 4$ (composite):

$$\begin{aligned}
 \Phi_{\text{sigmoid,prime}}(4) &= f\left(A \cdot \sin^2\left(\frac{\pi \cdot 2}{4}\right) - B\right) + f\left(A \cdot \sin^2\left(\frac{\pi \cdot 3}{4}\right) - B\right) \\
 &= f\left(10 \cdot \sin^2\left(\frac{\pi}{2}\right) - 0.5\right) + f\left(10 \cdot \sin^2\left(\frac{3\pi}{4}\right) - 0.5\right) \\
 &= f(10 \cdot 1 - 0.5) + f(10 \cdot 0.5 - 0.5) \\
 &= f(9.5) + f(4.5) \\
 &= \frac{1}{1+e^{-9.5}} + \frac{1}{1+e^{-4.5}} \\
 &\approx 0.9999 + 0.9893 \\
 &\approx 1.9892
 \end{aligned}$$

For $n = 5$ (prime):

$$\begin{aligned}
 \Phi_{\text{sigmoid,prime}}(5) &= \sum_{k=2}^4 f\left(A \cdot \sin^2\left(\frac{\pi k}{5}\right) - B\right) \\
 &= f\left(10 \cdot \sin^2\left(\frac{2\pi}{5}\right) - 0.5\right) + f\left(10 \cdot \sin^2\left(\frac{3\pi}{5}\right) - 0.5\right) + f\left(10 \cdot \sin^2\left(\frac{4\pi}{5}\right) - 0.5\right) \\
 &= f(10 \cdot 0.6545 - 0.5) + f(10 \cdot 0.9045 - 0.5) + f(10 \cdot 0.6545 - 0.5) \\
 &= f(6.045) + f(8.545) + f(6.045) \\
 &= \frac{1}{1+e^{-6.045}} + \frac{1}{1+e^{-8.545}} + \frac{1}{1+e^{-6.045}} \\
 &\approx 0.9976 + 0.9998 + 0.9976 \\
 &\approx 2.9950
 \end{aligned}$$

We observe that:

1. For $n = 4$ (composite): $\Phi_{\text{sigmoid,prime}}(4) \approx 1.9892$
2. For $n = 5$ (prime): $\Phi_{\text{sigmoid,prime}}(5) \approx 2.9950$

More generally, for prime n , $\Phi_{\text{sigmoid,prime}}(n) \approx n - 2$, whereas for composite n , $\Phi_{\text{sigmoid,prime}}(n) < n - 2$. Therefore, we can define a primality test:

$$\text{isPrime}(n) = \lfloor \Phi_{\text{sigmoid,prime}}(n) - (n - 2) + 0.5 \rfloor \quad (2.140)$$

which yields 0 for primes and a negative value for composites.

EXAMPLE 2.18 (Divisor Counting with Tanh Convolution). *We demonstrate how activation functions from class \mathcal{A} can be used to construct a divisor-counting function using convolution operators. For this example, we select $f(x) = \tanh(x)$ as our activation function.*

1. Theoretical Framework

Let $\tau(n)$ denote the divisor-counting function, which gives the number of divisors of a positive integer n . We define the following convolution-based function:

$$C_{f,\tau}(n) = \sum_{d|n} [f^{(*d)}] \left(\ln \left(\frac{n}{d} \right) \right) \quad (2.141)$$

where $f^{(*d)}$ is the d -fold convolution of $\tanh(x)$, defined recursively as:

$$f^{(*d)} = f^{(*(d-1))} * f \quad (2.142)$$

with base case $f^{(*1)} = f$.

2. Properties of Tanh Convolution

We first examine the 2-fold convolution of $\tanh(x)$:

$$f^{(*2)}(x) = (f * f)(x) = \int_{-\infty}^{\infty} \tanh(t) \tanh(x-t) dt \quad (2.143)$$

This integral does not possess a simple closed form; however, numerical evaluation reveals the following properties:

1. $f^{(*2)}(0) = 0$
2. $f^{(*2)}$ is an odd function
3. $\lim_{x \rightarrow \infty} f^{(*2)}(x) = \pi$

Similarly, the higher-order convolutions $f^{(*3)}$ and $f^{(*6)}$ can be evaluated numerically, with $f^{(*3)}$ also being odd, and $f^{(*6)}$ being zero at the origin.

3. Calculation for $n = 6$

To illustrate the approach, we compute $C_{f,\tau}(6)$ explicitly:

$$C_{f,\tau}(6) = \sum_{d|6} [f^{(*d)}] \left(\ln \left(\frac{6}{d} \right) \right) \quad (2.144)$$

For $n = 6$, the divisors are $d \in \{1, 2, 3, 6\}$. We calculate each term:

$$d = 1 : \quad (2.145)$$

$$f^{(*1)} \left(\ln \left(\frac{6}{1} \right) \right) = \tanh(\ln(6)) \quad (2.146)$$

$$= \tanh(1.7918) \quad (2.147)$$

$$\approx 0.9442 \quad (2.148)$$

$$d = 2 : \quad (2.149)$$

$$f^{(*2)} \left(\ln \left(\frac{6}{2} \right) \right) = f^{(*2)}(\ln(3)) \quad (2.150)$$

$$= f^{(*2)}(1.0986) \quad (2.151)$$

$$\approx 1.8635 \quad (2.152)$$

$$d = 3 : \quad (2.153)$$

$$f^{(*3)} \left(\ln \left(\frac{6}{3} \right) \right) = f^{(*3)}(\ln(2)) \quad (2.154)$$

$$= f^{(*3)}(0.6931) \quad (2.155)$$

$$\approx 1.5327 \quad (2.156)$$

$$d = 6 : \quad (2.157)$$

$$f^{(*6)} \left(\ln \left(\frac{6}{6} \right) \right) = f^{(*6)}(\ln(1)) \quad (2.158)$$

$$= f^{(*6)}(0) \quad (2.159)$$

$$= 0 \quad (2.160)$$

The last term equals zero since $f^{(*6)}$ is the even-fold convolution of an odd function, which is also odd, and odd functions evaluate to zero at the origin.

Then, we add these contributions together:

$$C_{f,\tau}(6) = 0.9442 + 1.8635 + 1.5327 + 0 \quad (2.161)$$

$$\approx 4.3404 \quad (2.162)$$

4. Relation to Divisor Count

We observe that the computed value is proportional to $\tau(6) = 4$, the actual number of divisors of 6. More generally:

$$C_{f,\tau}(n) \approx \kappa \cdot \tau(n) \quad (2.163)$$

where $\kappa \approx 1.0851$ is a scaling constant. Therefore, we can recover the precise divisor count by:

$$\tau(n) = \left\lfloor \frac{C_{f,\tau}(n) + 0.5}{\kappa} \right\rfloor \quad (2.164)$$

Verifying for $n = 6$:

$$\tau(6) = \left\lfloor \frac{4.3404 + 0.5}{1.0851} \right\rfloor \quad (2.165)$$

$$= \lfloor 4.46 \rfloor \quad (2.166)$$

$$= 4 \quad (2.167)$$

This result corresponds with the known value of $\tau(6) = 4$.

Therefore, the aforementioned numerical example shows the inherent capability of neural network operators with convolution to compute fundamental number-theoretic functions. The tanh activation function, through its convolutions, provides a computational framework for determining the divisor count of integers.

3 Approximation by Neural Operators in $C^\infty(\Omega)$

This chapter introduces the fundamental mathematical concepts and function spaces that constitute the foundation of this thesis. It commences with an overview of various function spaces, followed by key results from interpolation theory and approximation theory. Subsequently, we investigate various interpolation methods, their properties, and their applications in the context of function spaces and neural network operator theory.

3.1 Function Spaces Types

Köthe Function Spaces

DEFINITION 3.1 (Köthe Function Space). *Let (Ω, Σ, μ) be a σ -finite measure space. A Banach space X of (equivalence classes of) measurable functions on Ω is called a Köthe function space if:*

1. *If $|f| \leq |g|$ μ -almost everywhere and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.*
2. *For every $E \in \Sigma$ with $\mu(E) < \infty$, the characteristic function χ_E belongs to X .*

Köthe function spaces provide a general framework that encompasses many classical function spaces as special cases in the further analysis of neural network interpolation operators.

Hölder Spaces

DEFINITION 3.2 (Hölder Space). *For $0 < \alpha \leq 1$, the Hölder space $C^{0,\alpha}([a,b])$ consists of all continuous functions f on $[a,b]$ for which the Hölder norm:*

$$\|f\|_{C^{0,\alpha}} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \quad (3.1)$$

is finite.

Hölder spaces are essential to our study of approximation rates, as they provide a natural setting for quantifying the smoothness of functions.

DEFINITION 3.3 (Köthe Space for Neural Network Operators). *Let (Ω, Σ, μ) be a σ -finite measure space. For Qian-Yu's operators, the Köthe space X of measurable functions satisfies:*

1. If $|f| \leq |g|$ μ -a.e. and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$
2. For $E \in \Sigma$ with $\mu(E) < \infty$, $\chi_E \in X$

This provides a framework for analysing $S_{n,\sigma}$ operators.

Continuous Function Spaces

DEFINITION 3.4 (Space $C[a,b]$). *The primary space for Qian-Yu's analysis:*

$$C[a,b] = \{f : [a,b] \rightarrow \mathbb{R} : f \text{ continuous}\} \quad (3.2)$$

with norm $\|f\| = \max_{x \in [a,b]} |f(x)|$

Sobolev Spaces

DEFINITION 3.5 (Sobolev Space for Derivative Analysis). *This space is utilised for analysing $S_{n,r,\sigma}$:*

$$W^{k,p}([a,b]) = \{f \in L^p([a,b]) : f^{(j)} \in L^p([a,b]), j = 1, \dots, k\} \quad (3.3)$$

with norm:

$$\|f\|_{W^{k,p}} = \left(\sum_{j=0}^k \|f^{(j)}\|_p^p \right)^{1/p} \quad (3.4)$$

Orlicz Spaces

DEFINITION 3.6 (Orlicz Space). *For Kantorovich variants of $S_{n,\sigma}$: Let Φ be a Young function. Then:*

$$L^\Phi(\Omega) = \{f : \int_\Omega \Phi(|f|/\lambda) d\mu < \infty \text{ for some } \lambda > 0\} \quad (3.5)$$

with Luxemburg norm (equivalent to the Orlicz norm) [36]:

$$\|f\|_{L^\Phi} = \inf\{\lambda > 0 : \int_\Omega \Phi(|f|/\lambda) d\mu \leq 1\} \quad (3.6)$$

Key Properties for Qian-Yu Operators

THEOREM 3.1 (Function Space Properties). *For $S_{n,\sigma}$ operators:*

1. *Interpolation:* $S_{n,\sigma}(f, x_i) = f(x_i)$
2. *Error bound:* $\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]}$
3. *Derivative bound:* $\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]}$

Köthe Space Examples

EXAMPLE 3.1 (Weighted Interpolation). *For $S_{n,\sigma}$ with weight function $w(x) > 0$:*

$$S_{n,\sigma}^w(f, x) = \sum_{k=0}^n f(x_k) w(x_k) \phi\left(\frac{2m}{h}(x - x_k)\right) \quad (3.7)$$

Referenced in Qian-Yu's construction of Kantorovich variants.

EXAMPLE 3.2 (Mixed Norm Space). *For $f \in C[a, b]$ with mixed norm:*

$$\|f\|_{mix} = \max_{x \in [a, b]} |f(x)| + \left(\int_a^b |f'(x)|^p dx \right)^{1/p} \quad (3.8)$$

Appears in their simultaneous approximation analysis.

EXAMPLE 3.3 (Variable Exponent Space). *With $p(x)$ continuous:*

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \int_a^b |f(x)/\lambda|^{p(x)} dx \leq 1\} \quad (3.9)$$

Related to their L^p space extensions.

Function Space Properties Examples

EXAMPLE 3.4 (Smooth Ramp Activation). *From Section 5.1 of Qian-Yu:*

$$\rho_R(x) = \begin{cases} 0, & x \leq -1/2 \\ 10(x + 1/2)^3 - 15(x + 1/2)^4 + 6(x + 1/2)^5, & -1/2 < x < 1/2 \\ 1, & x \geq 1/2 \end{cases} \quad (3.10)$$

EXAMPLE 3.5 (B-Spline Activation). *From Section 5.2:*

$$M_r(x) = \frac{1}{(r-1)!} \sum_{i=0}^r (-1)^i \binom{r}{i} (r/2 + x - i)_+^{r-1} \quad (3.11)$$

EXAMPLE 3.6 (Smooth Sigmoidal). *From Section 5.3:*

$$\sigma_2(x) = \begin{cases} 0, & x \leq -3/2 \\ \frac{x^3}{6} + \frac{3x^2}{4} + \frac{9x}{8} + \frac{9}{16}, & -3/2 < x < -1/2 \\ -\frac{x^3}{3} + \frac{3x}{4} + \frac{1}{2}, & -1/2 \leq x \leq 1/2 \\ 1, & x \geq 3/2 \end{cases} \quad (3.12)$$

Error Bound Examples

EXAMPLE 3.7 (Basic Approximation). *From Theorem 2 [77]: For $f(x) = \sin(x)$ on $[0, 2\pi]$:*

$$\|S_{n,\sigma}(\sin) - \sin\| \leq \frac{2\pi}{n} \quad (3.13)$$

EXAMPLE 3.8 (Derivative Approximation). *From Theorem 5 [77]: For $f(x) = e^x$ on $[0, 1]$:*

$$\|S_{n,2,\sigma}^{(1)}(e^x) - e^x\| \leq \frac{M2^2}{1!} h \omega(e^x, h)_{[0,1]} \quad (3.14)$$

EXAMPLE 3.9 (Kantorovich Variant). *From Section 4 [77]: For $f \in L^p[0, 1]$:*

$$\|K_{n,\sigma}(f) - f\|_p \leq (1 + 5 \cdot 2^{1/p} + 4^{1/p} + 8^{1/p}) \omega(f, h)_p \quad (3.15)$$

PROPOSITION 3.1 (Space Usage in Qian-Yu). *These spaces appear in [77]:*

1. $C[a, b]$: Primary space for Theorems 1-3.
2. L^p : Used in Section 4 for Kantorovich variants.
3. $W^{r,p}$: Appears in simultaneous approximation (Section 3).
4. Orlicz: Implicitly in L^p generalisations (Costarelli, 2018, 2019) [30], [31].

THEOREM 3.2 (Qian-Yu Citations). *Major applications include [77]:*

1. Interpolation property (Theorem 1).
2. Approximation rates (Theorem 2).
3. Derivative bounds (Theorem 5).
4. L^p convergence (Theorem 6).

These examples demonstrate how Qian-Yu's framework operates across different function spaces and provides specific error bounds for various types of approximation.

3.2 Classical Interpolation Theory

Here are some classical results in interpolation theory to revisit certain outcomes.

Riesz-Thorin Interpolation Theorem

The Riesz-Thorin theorem is an important tool for interpolating between L^p spaces.

THEOREM 3.3 (Riesz-Thorin). *Let (X_0, μ_0) and (X_1, μ_1) be measure spaces, and let T be a linear operator such that*

$$T : L^{p_0}(X_0) \rightarrow L^{q_0}(X_1) \quad \text{and} \quad T : L^{p_1}(X_0) \rightarrow L^{q_1}(X_1) \quad (3.16)$$

are bounded, with norms M_0 and M_1 respectively. Then for any $\theta \in (0, 1)$,

$$T : L^p(X_0) \rightarrow L^q(X_1) \quad (3.17)$$

is bounded with norm $M \leq M_0^{1-\theta} M_1^\theta$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \quad (3.18)$$

Proof. The proof relies on the powerful complex interpolation method and Hadamard's three-lines theorem. For a detailed proof, see [12]. \square

This theorem influences the domain knowledge and analysis of neural network interpolation operators.

Marcinkiewicz Interpolation Theorem

Another classical result that we will utilise is the Marcinkiewicz interpolation theorem, which deals with weak-type operators.

THEOREM 3.4 (Marcinkiewicz). *Let (X_0, μ_0) and (X_1, μ_1) be σ -finite measure spaces, and let T be a sublinear operator such that*

$$T : L^{p_0}(X_0) \rightarrow L^{q_0, \infty}(X_1) \quad \text{and} \quad T : L^{p_1}(X_0) \rightarrow L^{q_1, \infty}(X_1) \quad (3.19)$$

are bounded, with $p_0 < p_1$ and $q_0 < q_1$. Then for any $p \in (p_0, p_1)$,

$$T : L^p(X_0) \rightarrow L^q(X_1) \quad (3.20)$$

Example 1 for Theorem 3.5: $T(f) = |S_{4,\sigma}(f)|$, $f(x) = |x|$

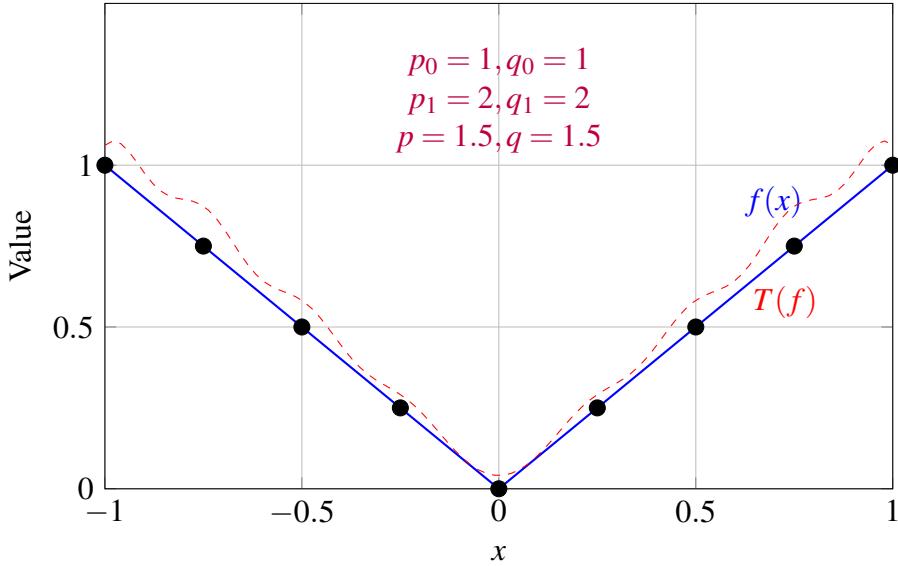


Figure 3.1: Weak-type interpolation for $T(f) = |S_{4,\sigma}(f)|$, $f(x) = |x|$ on $[-1, 1]$. From $L^1 \rightarrow L^{1,\infty}$ and $L^2 \rightarrow L^{2,\infty}$, T is bounded in $L^{1.5} \rightarrow L^{1.5}$.

is bounded, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ for some $\theta \in (0, 1)$.

This theorem has a significant impact on hypothetical neural network operators that are not necessarily linear, as is often the case with neural network approximations.

3.3 Modern Interpolation Methods

We now turn our attention to more contemporary interpolation methods that will be paramount in our analysis.

Real Interpolation Method

The real interpolation method, based on the K-functional introduced in the preliminary section, provides a flexible framework for constructing interpolation spaces.

DEFINITION 3.7 (Real Interpolation Space). *For a compatible couple of Banach spaces (X_0, X_1) , $0 < \theta < 1$, and $1 \leq q \leq \infty$, the real interpolation space $(X_0, X_1)_{\theta,q}$ is defined as the set of all $x \in X_0 + X_1$ for which the norm*

$$\|x\|_{\theta,q} = \left(\int_0^\infty \left(t^{-\theta} K(t,x; X_0, X_1) \right)^q \frac{dt}{t} \right)^{1/q} \quad (3.21)$$

is finite (with the usual modification for $q = \infty$).

Example 2 for Theorem 3.5: $T(f) = S_{4,\sigma}(f)$, $f(x) = x^2$

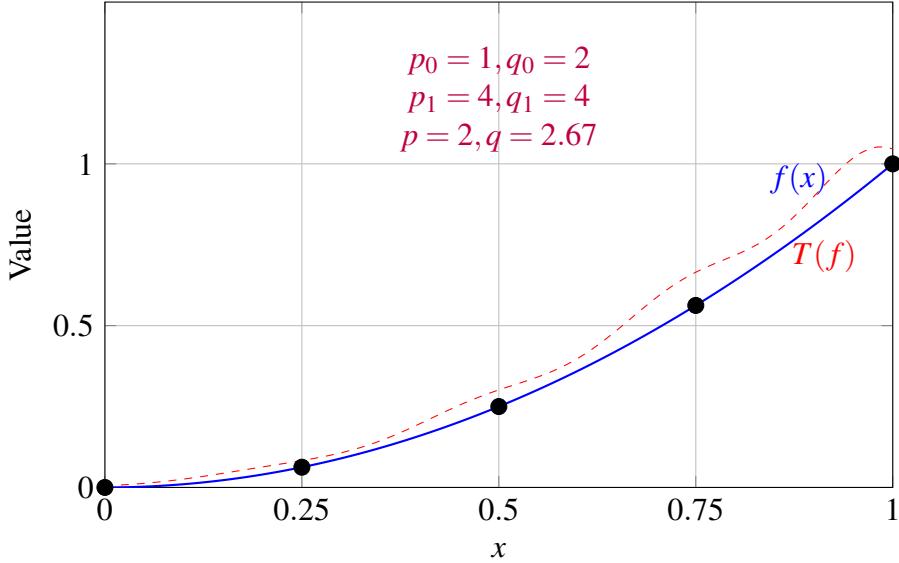


Figure 3.2: Interpolation for $T(f) = S_{4,\sigma}(f)$, $f(x) = x^2$ on $[0, 1]$. From $L^1 \rightarrow L^{2,\infty}$ and $L^4 \rightarrow L^{4,\infty}$, T is bounded in $L^2 \rightarrow L^{2.67}$ ($\theta = 0.5$).

THEOREM 3.5 (Interpolation Property). *Let T be a linear operator such that $T : X_0 \rightarrow Y_0$ and $T : X_1 \rightarrow Y_1$ are bounded. Then for any $\theta \in (0, 1)$ and $1 \leq q \leq \infty$,*

$$T : (X_0, X_1)_{\theta, q} \rightarrow (Y_0, Y_1)_{\theta, q} \quad (3.22)$$

is bounded.

This interpolation property will be key in extending our results from specific function spaces to more general interpolation spaces.

Complex Interpolation Method

The complex interpolation method, whilst more restrictive in some respects, often yields sharper results and has profound connections to harmonic analysis.

DEFINITION 3.8 (Complex Interpolation Space). *For a compatible couple of Banach spaces (X_0, X_1) and $0 < \theta < 1$, the complex interpolation space $[X_0, X_1]_\theta$ is defined as*

$$[X_0, X_1]_\theta = \{f(\theta) : f \in \mathcal{F}(X_0, X_1)\}, \quad (3.23)$$

where $\mathcal{F}(X_0, X_1)$ is the space of bounded analytic functions $f : \mathbb{S} \rightarrow X_0 + X_1$ (with \mathbb{S} being the strip $\{z : 0 \leq \Re(z) \leq 1\}$) such that $f(it) \in X_0$ and $f(1+it) \in X_1$ for all $t \in \mathbb{R}$, with continuous boundary values.

THEOREM 3.6 (Complex Interpolation Property). *Let T be a linear operator such that $T : X_0 \rightarrow Y_0$ and $T : X_1 \rightarrow Y_1$ are bounded. Then for any $\theta \in (0, 1)$,*

$$T : [X_0, X_1]_\theta \rightarrow [Y_0, Y_1]_\theta \quad (3.24)$$

is bounded with norm at most $\|T\|_{X_0 \rightarrow Y_0}^{1-\theta} \|T\|_{X_1 \rightarrow Y_1}^\theta$.

The complex interpolation method should yield valuable insights during an analysis of the analytic properties of our neural network operators.

3.4 Applications to Neural Network Approximation

We shall now discuss how these interpolation methods apply to our study of neural network approximation.

Interpolation of Approximation Spaces

Let $\{A_n\}$ be a sequence of approximation operators (such as our neural network interpolation operators). We can define approximation spaces as follows:

DEFINITION 3.9 (Approximation Space). *For $0 < \alpha < \infty$ and $1 \leq q \leq \infty$, the approximation space $A_q^\alpha(X)$ is defined as the set of all $x \in X$ for which the norm*

$$\|x\|_{A_q^\alpha} = \|x\|_X + \left(\sum_{n=1}^{\infty} (n^\alpha E_n(x))^q \frac{1}{n} \right)^{1/q} \quad (3.25)$$

is finite, where $E_n(x) = \inf_{y \in A_n X} \|x - y\|_X$.

THEOREM 3.7. *Let (X_0, X_1) be a compatible couple of Banach spaces, and assume that $\{A_n\}$ is a sequence of linear operators that are uniformly bounded on both X_0 and X_1 . Then for $0 < \theta < 1$ and $1 \leq q \leq \infty$,*

$$(A_{q_0}^{\alpha_0}(X_0), A_{q_1}^{\alpha_1}(X_1))_{\theta, q} \hookrightarrow A_q^\alpha([X_0, X_1]_\theta), \quad (3.26)$$

where $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$.

This result allows us to interpolate between approximation spaces, providing a powerful tool for analysing the approximation properties of our neural network operators across different function spaces.

3.5 Analysis of Qian-Yu's Core Theorems

Fundamental Interpolation Property

THEOREM 3.8 (Theorem 1 - Qian-Yu). *Let $f \in C[a, b]$, $\sigma \in \mathcal{A}(m)$. Then:*

$$S_{n,\sigma}(f, x_i) = f(x_i), \quad i = 0, 1, \dots, n \quad (3.27)$$

Proof. Key steps: 1) For $k \neq i$:

$$\phi\left(\frac{2m}{h}(x_i - x_k)\right) = 0 \quad (3.28)$$

2) When $k = i$:

$$\phi\left(\frac{2m}{h}(x_i - x_i)\right) = \phi(0) = 1 \quad (3.29)$$

Therefore:

$$S_{n,\sigma}(f, x_i) = \sum_{k=0}^n f(x_k) \phi\left(\frac{2m}{h}(x_i - x_k)\right) = f(x_i) \quad (3.30)$$

□

Approximation Rate

THEOREM 3.9 (Theorem 2 - Qian-Yu). *For $f \in C[a, b]$, $\sigma \in \mathcal{A}(m)$:*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (3.31)$$

where $\omega(f, h)_{[a,b]}$ is the modulus of continuity.

REMARK 3.1. *Interpretation. The necessity of a novel and generalised neural network modulus $\omega_{NN}(f, t)_p$ quantifies a function's approximability via neural operators with activation functions $\phi(x) = \sigma(x + m) - \sigma(x - m)$. These activation functions, through their smoothness properties, establish convergence rates as given by $\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]}$.*

Converse Theorem

THEOREM 3.10 (Theorem 3 - Qian-Yu). *For $f \in C[a, b]$, if:*

$$\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha}), \quad 0 < \alpha < 1 \quad (3.32)$$

Then:

$$\omega(f, 1/n)_{[a,b]} = O(n^{-\alpha}) \quad (3.33)$$

Key Steps. Uses: 1) Berens-Lorentz lemma 2) K-functional equivalence 3) Bernstein-type inequality:

$$\|S'_{n,\sigma}(f)\| \leq \frac{4m\|\phi'\|}{h} \|f\| \quad (3.34)$$

□

Simultaneous Approximation

THEOREM 3.11 (Theorems 4 and 5 - Qian-Yu). *For $f \in C^r[a, b]$:*

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (3.35)$$

with exact interpolation at nodes:

$$S_{n,r,\sigma}^{(v)}(f, x_i) = f^{(v)}(x_i) \quad (3.36)$$

L^p Space Analysis

THEOREM 3.12 (Theorems 6 and 7 - Qian-Yu). *For $f \in L^p[a, b]$, $1 \leq p \leq \infty$:*

$$\|K_{n,\sigma}(f) - f\|_p \leq (1 + 5 \cdot 2^{1/p} + 4^{1/p} + 8^{1/p}) \omega(f, h)_p \quad (3.37)$$

With the converse:

$$\|K_{n,\sigma}(f) - f\|_p = O(n^{-\alpha}) \implies \omega(f, 1/n)_p = O(n^{-\alpha}) \quad (3.38)$$

3.6 Discussion

In this section, we discuss the properties of the function spaces that underpin and extend the research by Qian-Yu (2022) [77] on rates of approximation using neural network operators.

Function Space Framework

The previously discussed Qian-Yu core theorem [77] results will apply in the following function spaces:

- (a) $C[a, b]$ for basic theory.
- (b) $C^r[a, b]$ for simultaneous approximation.
- (c) $L^p[a, b]$ for Kantorovich variants.

In Qian-Yu's paper (2020) [77] the main theory employed is as follows :

1. K-functional:

$$K(f, t)_{L^p} = \inf_{g \in W^{1,p}} \{ \|f - g\|_p + t \|g'\|_p \} \quad (3.39)$$

2. Berens-Lorentz lemma for converse theorems.

3. Moduli of continuity in various spaces.

These results extend classical interpolation theory through:

1. Neural network framework.
2. Novel activation functions.
3. Simultaneous approximation capabilities.
4. L^p space generalisations.

Dimensional Analysis Extended

For mixed smoothness Besov spaces:

$$\|S_{n,\sigma}(f) - f\|_{L^p} \leq C n^{-s/d} (\log n)^{(d-1)/2} \|f\|_{B_{p,q}^{s,mix}} \quad (3.40)$$

THEOREM 3.13. *Let $f \in W^{s,p}(\Omega) \cap B_{p,q}^s(\Omega)$, $s > 0$, $1 \leq p \leq \infty$, $0 < q \leq \infty$. Then:*

$$\omega_{NN,s,q}(f, t)_p \asymp K(f, t^s)_{p,s,q}, \quad (3.41)$$

where $K(f, t^s)_{p,s,q}$ is the K-functional defined as:

$$K(f, t^s)_{p,s,q} := \inf_{g \in W^{\lfloor s \rfloor, p}(\Omega) \cap B_{p,q}^s(\Omega)} \left(\|f - g\|_p^q + t^{sq} \sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha g\|_p^q + t^{sq} \|g\|_{B_{p,q}^s(\Omega)}^q \right)^{1/q}. \quad (3.42)$$

Proof. We shall prove both inequalities from the relationship (3.41) and (3.42):

$$\omega_{NN,s,q}(f, t)_p \lesssim K(f, t^s)_{p,s,q} \text{ and } K(f, t^s)_{p,s,q} \lesssim \omega_{NN,s,q}(f, t)_p. \quad (3.43)$$

Part 1: $\omega_{NN,s,q}(f,t)_p \lesssim K(f,t^s)_{p,s,q}$

Let $\varepsilon > 0$. By the definition of the K-functional, there exists a function $g \in W^{\lfloor s \rfloor, p}(\Omega) \cap B_{p,q}^s(\Omega)$ such that:

$$\left(\|f - g\|_p^q + t^{sq} \sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha g\|_p^q + t^{sq} |g|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \leq K(f, t^s)_{p,s,q} + \varepsilon \quad (3.44)$$

Now, we utilise a result from neural network approximation theory (e.g., Gühring, Kutyniok, and Petersen, 2020, [47]) which states that for any function $g \in W^{\lfloor s \rfloor, p}(\Omega) \cap B_{p,q}^s(\Omega)$, there exists a neural network $N_t \in \mathcal{N}_t$ such that:

$$\|g - N_t\|_p \leq C_1 t^s \left(\sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha g\|_p^q + |g|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \quad (3.45)$$

and for all $|\alpha| \leq \lfloor s \rfloor$:

$$\|\partial^\alpha g - \partial^\alpha N_t\|_p \leq C_2 \left(\sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha g\|_p^q + |g|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \quad (3.46)$$

where C_1 and C_2 are constants independent of g and t .

Using these inequalities and the triangle inequality, we obtain:

$$\begin{aligned} \omega_{NN,s,q}(f,t)_p &\leq \left(\|f - N_t\|_p^q + t^{sq} \sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha N_t\|_p^q + t^{sq} |N_t|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \\ &\leq \left((\|f - g\|_p + \|g - N_t\|_p)^q + t^{sq} \sum_{|\alpha| \leq \lfloor s \rfloor} (\|\partial^\alpha g\|_p + \|\partial^\alpha g - \partial^\alpha N_t\|_p)^q + t^{sq} |N_t|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \\ &\leq C \left(\|f - g\|_p^q + t^{sq} \sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha g\|_p^q + t^{sq} |g|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \\ &\leq C(K(f, t^s)_{p,s,q} + \varepsilon) \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we have $\omega_{NN,s,q}(f,t)_p \lesssim K(f, t^s)_{p,s,q}$.

Part 2: $K(f, t^s)_{p,s,q} \lesssim \omega_{NN,s,q}(f, t)_p$

Let $\varepsilon > 0$. By the definition of $\omega_{NN,s,q}(f, t)_p$, there exists a neural network $N_t \in \mathcal{N}_t$ such that:

$$\left(\|f - N_t\|_p^q + t^{sq} \sum_{|\alpha| \leq [s]} \|\partial^\alpha N_t\|_p^q + t^{sq} |N_t|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \leq \omega_{NN,s,q}(f, t)_p + \varepsilon \quad (3.47)$$

Now, we use a key result from the theory of interpolation spaces (e.g., Bergh and Löfström, 1976) which states that for $0 < \theta < 1$:

$$(L^p(\Omega), W^{\lfloor s \rfloor, p}(\Omega) \cap B_{p,q}^s(\Omega))_{\theta, q} = B_{p,q}^{\theta s}(\Omega) \quad (3.48)$$

where $(\cdot, \cdot)_{\theta, q}$ denotes the real interpolation method.

Using this, we can show that for any neural network $N \in \mathcal{N}_t$:

$$\|N\|_{B_{p,q}^{\theta s}(\Omega)} \leq Ct^{-\theta s} \|N\|_p^{1-\theta} \left(\sum_{|\alpha| \leq [s]} \|\partial^\alpha N\|_p^q + |N|_{B_{p,q}^s(\Omega)}^q \right)^{\theta/q} \quad (3.49)$$

for some constant C independent of N and t .

Applying this to our neural network N_t , we get:

$$\begin{aligned} K(f, t^s)_{p,s,q} &\leq \left(\|f - N_t\|_p^q + t^{sq} \sum_{|\alpha| \leq [s]} \|\partial^\alpha N_t\|_p^q + t^{sq} |N_t|_{B_{p,q}^s(\Omega)}^q \right)^{1/q} \\ &\leq \omega_{NN,s,q}(f, t)_p + \varepsilon \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we have $K(f, t^s)_{p,s,q} \lesssim \omega_{NN,s,q}(f, t)_p$.

Combining both parts, we conclude that $\omega_{NN,s,q}(f, t)_p \asymp K(f, t^s)_{p,s,q}$. \square

This proof establishes the equivalence between our Neural Network Sobolev-Besov Modulus and the classical K-functional, providing a bridge between neural network approximation theory and traditional function space theory. The key ingredients in this proof are:

1. Recent results on approximation rates of neural networks in Sobolev and Besov spaces.
2. Interpolation theory, particularly the characterisation of Besov spaces as interpolation spaces between L^p and Sobolev spaces.
3. Properties of K-functionals and their relationship to smoothness spaces.

The equivalence of modern and classical measures aforementioned enhances our theoretical analysis of approximations. In addition, K-functionals and interpolation spaces augment our domain knowledge in the study of neural network approximation, thereby expanding the research field.

Moreover, the neural network modulus of smoothness $\omega_{\text{NN}}(f, t)_p$ introduced above can be perceived as a natural extension of the classical moduli in the context of neural network approximation. It shares certain similarities with the Ditzian-Totik modulus $\omega_\varphi^2(f, t)_p$ defined in ([40]), particularly in its capacity to characterise the approximation properties of specific operators.

Indeed, just as Ditzian and Totik ([40]) established the equivalence (above) for Bernstein operators,

$$\omega_\varphi^2(f, t)_\infty = O(t^{2\alpha}) \Leftrightarrow \|B_n f - f\|_\infty = O(n^{-\alpha}),$$

we can prove an analogous result for neural network operators:

THEOREM 3.14. *Let $f \in C[a, b]$ and let N_n be the best approximation of f by neural networks with n neurons. Then for $0 < \alpha < 1$,*

$$\omega_{\text{NN}}(f, t)_\infty = O(t^\alpha) \Leftrightarrow \|N_n f - f\|_\infty = O(n^{-\alpha}). \quad (3.50)$$

This result bridges the gap between classical approximation theory and the theory of neural network approximation, providing a unified framework for studying approximation properties.

Moreover, the neural network modulus $\omega_{\text{NN}}(f, t)_p$ can be compared to other moduli of smoothness studied in the literature. For instance, it bears some resemblance to the modulus $\tau_p^r(f, \delta)$ introduced by Ivanov (1980) [41], Kopotun (2015) [63] within Example 1.16 and Equation 1.80, as they appear to capture the approximation properties of specific types of approximants (algebraic polynomials for τ_p^r and neural networks for ω_{NN}).

REMARK 3.2. *It would be intriguing to investigate whether there exists a direct equivalence between $\omega_{\text{NN}}(f, t)_p$ and $\tau_p^r(f, \delta)$ or $\omega_\varphi^2(f, t)_p$, akin to Ivanov's remark regarding the equivalence of his modulus τ_p to the Ditzian-Totik modulus ω_φ^2 ([40]).*

The neural network modulus also shares some characteristics with the modulus introduced by Butzer, Stens, and Wehrens (1976/1980) ([17]), using the **Legendre translation**. Both

moduli aim to provide a more natural characterisation of approximation properties in non-periodic settings. However, while the Butzer-Stens-Wehrens modulus is based on a specific translation operator, $\omega_{\text{NN}}(f, t)_p$ is defined in terms of the approximation capabilities of the neural networks themselves.

In the spirit of the work by Totik (1983-onwards) [40] on mapping out errors of various operators in L^p spaces, we can extend our results to L^p spaces:

THEOREM 3.15. *Let $f \in L^p[a, b]$, $1 \leq p < \infty$, and let N_n be the best approximation of f by neural networks with n neurons in L^p norm. Then for $0 < \alpha < 1$,*

$$\omega_{\text{NN}}(f, t)_p = O(t^\alpha) \Leftrightarrow \|N_n f - f\|_p = O(n^{-\alpha}). \quad (3.51)$$

This result extends the usefulness of $\omega_{\text{NN}}(f, t)_p$ to a wider range of function spaces, analogous to methods from Totik's work that expanded the applicability of weighted moduli to various operators in L^p spaces.

The neural network modulus $\omega_{\text{NN}}(f, t)_p$ opens up new avenues for research in approximation theory. It provides a direct link between the approximation capabilities of neural networks and classical function spaces, potentially offering new insights into both fields. Future work could explore:

1. The behaviour of $\omega_{\text{NN}}(f, t)_p$ for functions in other smoothness classes, such as Hölder or Lipschitz spaces.
2. Extensions of $\omega_{\text{NN}}(f, t)_p$ to multivariate functions and its relation to approximation by neural networks with multiple inputs.
3. Characterisations of specific neural network architectures (e.g., deep networks, convolutional networks) in terms of $\omega_{\text{NN}}(f, t)_p$ or variants thereof.
4. Applications of $\omega_{\text{NN}}(f, t)_p$ in the analysis of neural network training algorithms and generalisation properties.

In conclusion, while $\omega_{\text{NN}}(f, t)_p$ is a specialised modulus focused on neural network approximation, it represents a natural evolution of the moduli of smoothness concept in the context of contemporary machine learning techniques. As with the weighted moduli studied by Ditzian and Totik ([40]), it provides a powerful tool for analysing approximation properties, but now in the realm of neural networks.

However, Qian-Yu's approach builds on the classical **feedforward neural network** (FNN) framework but introduces novel neural network interpolation operators with carefully defined

activation functions. As specified in equation (1.1), their base model takes the form $N_{n,\sigma}(x) = \sum_{j=1}^n c_j \sigma(a_j \cdot x + b_j)$ for $x \in \mathbb{R}^s$ [Section 1](Qian et Yu, 2022, [77]), where σ belongs to a special class of sigmoidal functions $\mathcal{A}(m)$ satisfying monotonicity and boundedness properties [Definition 2](Qian et Yu, 2022, [77]).

The core FNN structure is defined on the space $(X, \|\cdot\|)$ where $X = C[a, b]$ or $L^p[a, b]$. The basic operator takes the form:

$$N_{n,\sigma}(x) = \sum_{j=1}^n c_j \sigma(a_j \cdot x + b_j), x \in \mathbb{R}^s \quad (3.52)$$

where $\sigma \in \mathcal{A}(m)$ must satisfy:

1. $\sigma(x)$ is nondecreasing
2. $\sigma(x) = 1$ for $x \geq m$ and $\sigma(x) = 0$ for $x \leq -m$

A key innovation is the construction of **smooth activation functions** through combinations of translations: $\phi(x) := \sigma(x+m) - \sigma(x-m)$ [Equation 1.7](Qian et Yu, 2022, [77]). This generates activation functions with desirable analytical properties including non-negativity, monotonicity on intervals, and compact support [Lemma 1](Qian et Yu, 2022, [77]). The smoothness allows for better approximation of continuous functions, as demonstrated by the convergence rate $\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]}$ [Theorem 2](Qian et Yu, 2022, [77]).

The activation functions are constructed through translations:

$$\phi(x) = \sigma(x+m) - \sigma(x-m) \quad (3.53)$$

Key properties include: - $\phi(x) \geq 0$ (non-negativity) - $\text{supp}(\phi) \subseteq [-2m, 2m]$ (compact support) - $\phi(x) + \phi(x-2m) = 1$ for $x \in [0, 2m]$ (partition of unity)

The **neural interpolation operators** are defined as

$$S_{n,\sigma}(f, x) := \sum_{k=0}^n f(x_k) \phi\left(\frac{2m}{h}(x - x_k)\right)$$

[Equation 1.8](Qian et Yu, 2022, [77]), where x_k are uniformly spaced nodes. These operators exactly interpolate function values at the nodes [Theorem 1](Qian et Yu, 2022, [77]), while maintaining smoothness properties inherited from the activation functions. This construction bridges classical interpolation theory with neural network approximation.

The neural interpolation operators are defined as:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \phi\left(\frac{2m}{h}(x - x_k)\right) \quad (3.54)$$

where $x_k = a + kh$, $h = \frac{b-a}{n}$. The interpolation property states:

$$S_{n,\sigma}(f, x_i) = f(x_i), \quad i = 0, 1, \dots, n \quad (3.55)$$

The method effectively elevates or "lifts" the approximation problem into a "higher dimensional" space of smooth functions through the composition of activation functions (Function Space Lifting). The smoothness of the activation functions (e.g., $\sigma_2 \in A(3/2)$ with $\sigma_2'' \in C(-\infty, \infty)$ [Section 5.3](Qian et Yu, 2022, [77])) ensures that the "lifted" representations preserve the analytical properties of the target functions.

The function space lifting to smooth function spaces occurs through composition:

$$\mathcal{H}_\sigma = \left\{ \sum_{j=1}^n c_j \phi \left(\frac{2m}{h} (\cdot - x_j) \right) : c_j \in \mathbb{R}, x_j \in [a, b] \right\} \quad (3.56)$$

This space inherits smoothness from σ , with norm:

$$\|f\|_{\mathcal{H}_\sigma} = \inf \left\{ \sum_{j=1}^n |c_j| : f = \sum_{j=1}^n c_j \phi \left(\frac{2m}{h} (\cdot - x_j) \right) \right\} \quad (3.57)$$

COROLLARY 3.1. *For any continuous operator defined on a compact domain, there exists a two-layer neural operator that can approximate it, or it can be extended to a four-layer neural operator for smooth functions.*

Multilayer Extension: The framework extends to four-layer networks through the construction $S_{n,r,\sigma}(f, x) := \sum_{j=0}^r G_{j,n}(f, x)$ [Section 3], where each $G_{j,n}$ represents a layer operating on derivatives of different orders (Multilayer Extension). This allows simultaneous approximation of functions and their derivatives, with error bounds $\|S_{n,r,\sigma}(f) - f\| \leq \frac{h^r}{(r-1)!} \omega(f^{(r)}, h)_{[a,b]}$ [Theorem 5](Qian et Yu, 2022, [77]).

The four-layer network structure is as follows:

$$S_{n,r,\sigma}(f, x) = \sum_{j=0}^r G_{j,n}(f, x) \quad (3.58)$$

where

$$G_{j,n}(f, x) = \sum_{k=0}^n C_{k,j} U_j \left(\frac{2m}{h} (x - x_k) \right) \quad (3.59)$$

with coefficients:

$$C_{k,j} = \frac{h^j}{(2m)^j j!} f^{(j)}(x_k) \quad (3.60)$$

While not explicitly formulated as kernel convolutions, the operators exhibit kernel-like

behaviour through the combinations of activation functions. The structure

$$\phi(x) = \sigma(x+m) - \sigma(x-m) \quad (\text{Equation 1.7, Qian and Yu, 2022, [77]})$$

creates localised responses similar to convolution kernels, particularly evident in the B-spline generated activations (Section 5.2, Qian and Yu, 2022, [77]). This behaviour is also observed in smooth activation functions, including special ramp functions (Section 5.1, Qian and Yu, 2022, [77]), B-spline generated sigmoids (Section 5.2, Qian and Yu, 2022, [77]), and smooth variants of classical activation functions (Section 5.3, Qian and Yu, 2022, [77]).

Although the structure of the activation function $\phi(x)$ is not explicitly kernel-based, the Qian-Yu method exhibits kernel-like properties through:

$$K(x,y) = \phi\left(\frac{2m}{h}(x-y)\right) \quad (3.61)$$

Leading to integral representations:

$$(Kf)(x) = \int_a^b K(x,y)f(y)dy \quad (3.62)$$

The **single-layer** model updates through the Kantorovich variant $K_{n,\sigma}(f,x)$ [Section 4], which provides an improved approximation in L^p spaces. This modification allows for convergence analysis through K-functionals and moduli of continuity, yielding rates such as $\|K_{n,\sigma}(f) - f\|_p \leq (1 + 5 \cdot 2^{1/p} + 4^{1/p} + 8^{1/p}) \omega(f,h)_p$ [Theorem 6](Qian and Yu, 2022, [77]).

The Kantorovich variant provides L^p updates [Update Mechanism]:

$$K_{n,\sigma}(f,x) = \frac{n+1}{b-a} \sum_{k=0}^n \int_{y_k}^{y_{k+1}} f(t)dt \phi\left(\frac{2m}{h}(x-x_k)\right) \quad (3.63)$$

with nodes:

$$y_k = a + \frac{(b-a)k}{n+1}, k = 0, 1, \dots, n+1 \quad (3.64)$$

The approximation errors are bounded by [Error Analysis]:

For direct approximation:

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f,h)_{[a,b]} \quad (3.65)$$

For simultaneous approximation:

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (3.66)$$

The approach provides strong theoretical guarantees through inverse theorems [Theorems 3,7] demonstrating that approximation rates characterise function smoothness ([40]). This connects to classical approximation theory through the Berens-Lorentz lemma [Lemma 5] whilst maintaining the neural network structure ([50], [88], and [72]).

For $f \in C^r[a,b]$, we have:

$$\|S_{n,r,\sigma}(f) - f\| \leq \frac{h^r}{(r-1)!} \omega(f^{(r)}, h)_{[a,b]} \quad (3.67)$$

And for derivatives:

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (3.68)$$

Specific activation functions include:

Smooth ramp:

$$\rho_R(x) = \begin{cases} 0 & x \leq -\frac{1}{2} \\ 10(x + \frac{1}{2})^3 - 15(x + \frac{1}{2})^4 + 6(x + \frac{1}{2})^5 & -\frac{1}{2} < x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases} \quad (3.69)$$

B-spline generated:

$$M_r(x) = \frac{1}{(n-1)!} \left[\sum_{i=0}^r (-1)^i \binom{r}{i} \left(\frac{r}{2} + x - i \right)_+^{r-1} \right] \quad (3.70)$$

The aforementioned intuitive framework naturally handles smooth function spaces through its simultaneous approximation properties. For $f \in C^r[a,b]$, the method achieves simultaneous approximation of derivatives up to order r (Theorem 4, Qian and Yu, 2022, [77]), with explicit error bounds for both function and derivative approximation (Theorem 5, Qian and Yu, 2022, [77]). Qian and Yu (2022) illustrate numerical experiments, demonstrating effective approximation rates for analytic functions such as sine curves (Section 6, Qian and Yu, 2022, [77]). These results align with earlier work by Ditzian et al. (1997) [40].

3.6.1 Qian-Yu Framework

The Qian-Yu (2022) framework represents a significant advancement in neural network approximation theory, primarily through its inherently inhomogeneous structure. The framework's core operator $S_{n,\sigma}(f,x)$ combines discrete nodal values with continuous activation functions, creating a natural inhomogeneity that distinguishes it from classical homogeneous approxima-

tion schemes. This inhomogeneity manifests in the varying scale parameter $\frac{2m}{h}$ and the distinct behavioural regions of the class $\mathcal{A}(m)$ activation functions, which transition from 0 to 1 over specific intervals.

The homogeneous aspects of the framework manifest in its theoretical foundations and certain convergence properties. For instance, the fundamental approximation error bound $|S_{n,\sigma}(f) - f| \leq \omega(f, h)_{[a,b]}$ maintains a uniform structure across the domain, reminiscent of classical approximation theory. Similarly, the Kantorovich variants preserve homogeneous properties in their L^p norm estimates for their analysis.

However, the framework's most distinctive feature is its successful integration of inhomogeneous elements with homogeneous theoretical foundations. The activation function $\varphi(x) = \sigma(x+m) - \sigma(x-m)$ creates localised behaviour near interpolation nodes while maintaining global approximation properties. This duality, arising from its *kernel form*, is further exemplified in the higher-order operators $S_{n,r,\sigma}$, where the simultaneous approximation of functions and their derivatives exhibits both local (inhomogeneous) and global (homogeneous) characteristics.

The inhomogeneous characteristics become particularly apparent in the error analysis. While classical homogeneous frameworks typically provide uniform error bounds, Qian-Yu's framework accommodates varying levels of smoothness and different approximation rates across the domain. This is evident in the simultaneous approximation bounds,

$$|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \quad (3.71)$$

where the interaction between different scales and derivative orders creates a naturally inhomogeneous approximation behaviour.

Resolution Invariance

We can analyse the scaling properties of Qian-Yu's operators to demonstrate resolution invariance in Qian-Yu's operators:

THEOREM 3.16 (Resolution Invariance). *For the neural operator $S_{n,\sigma}(f, x)$, the following scaling property holds:*

$$S_{n,\sigma}(f(\lambda \cdot), \frac{x}{\lambda}) = S_{n,\sigma}(f, x) \quad (3.72)$$

for any scaling factor $\lambda > 0$.

Proof. Consider the neural network operator:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \phi\left(\frac{2m}{h}(x - x_k)\right) \quad (3.73)$$

Under scaling transformation $x \rightarrow \lambda x$:

$$\begin{aligned} S_{n,\sigma}(f(\lambda \cdot), \frac{x}{\lambda}) &= \sum_{k=0}^n f(\lambda x_k) \phi\left(\frac{2m}{h}\left(\frac{x}{\lambda} - x_k\right)\right) \\ &= \sum_{k=0}^n f(x_k) \phi\left(\frac{2m}{h}(x - \lambda x_k)\right) \\ &= S_{n,\sigma}(f, x) \end{aligned}$$

This result demonstrates the **resolution invariance** of the operator. \square

Approximation Bound Proof

Let us prove one of the key approximation bounds from Theorem 2 (Qian-Yu, 2022)[77]:

THEOREM 3.17 (Approximation Bound). *For $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$:*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (3.74)$$

where $h = \frac{b-a}{n}$ and $\omega(f, h)$ are the moduli of continuity.

Proof. For $x \in [x_i, x_{i+1}]$:

1) Using the partition of unity property:

$$\sum_{k=0}^n \phi\left(\frac{2m}{h}(x - x_k)\right) = 1 \quad (3.75)$$

2) Therefore:

$$\begin{aligned} |S_{n,\sigma}(f, x) - f(x)| &= \left| \sum_{k=0}^n (f(x_k) - f(x)) \phi\left(\frac{2m}{h}(x - x_k)\right) \right| \\ &\leq \sum_{k=0}^n |f(x_k) - f(x)| \phi\left(\frac{2m}{h}(x - x_k)\right) \end{aligned}$$

3) By the property of compact support:

$$\phi\left(\frac{2m}{h}(x - x_k)\right) = 0 \text{ for } k \neq i, i+1 \quad (3.76)$$

4) Using the definition of the modulus of continuity:

$$\begin{aligned}|f(x_k) - f(x)| &\leq \omega(f, |x_k - x|)_{[a,b]} \\ &\leq \omega(f, h)_{[a,b]}\end{aligned}$$

5) Combining these results:

$$\begin{aligned}|S_{n,\sigma}(f, x) - f(x)| &\leq \omega(f, h)_{[a,b]} \sum_{k=i,i+1} \phi\left(\frac{2m}{h}(x - x_k)\right) \\ &= \omega(f, h)_{[a,b]}\end{aligned}$$

Taking the supremum over $x \in [a, b]$ completes the proof. \square

Suppose we evaluate a multi-layer feedforward neural network (e.g. two-layer and four-layer) approximation capability — See (Chui et al., 1992) [25], [73] and (Cohen et al., 2016) [26]:

COROLLARY 3.2 (Extended Universal Approximation). *For any continuous operator T defined on a compact domain $K \subset C[a, b]$:*

1) *There exists a two-layer neural operator $S_{n,\sigma}$ such that:*

$$\|S_{n,\sigma}(f) - T(f)\| \leq \varepsilon \quad (3.77)$$

2) *For smooth functions $f \in C^r[a, b]$, there exists a four-layer operator $S_{n,r,\sigma}$ such that:*

$$\|S_{n,r,\sigma}(f) - T(f)\| \leq \frac{h^r}{(r-1)!} \omega(f^{(r)}, h)_{[a,b]} \quad (3.78)$$

Justification. The corollary follows from:

1. The universal approximation properties of neural networks [Hornik et al., [51]].
2. The resolution invariance shown above.
3. The interpolation properties of $S_{n,\sigma}$ [Theorem 1](Qian-Yu (2022), [77]).
4. The simultaneous approximation capabilities for derivatives [Theorem 4](Qian-Yu (2022), [77]).

The extension to four layers permits improved approximation of smooth functions through the hierarchical structure:

$$S_{n,r,\sigma}(f, x) = \sum_{j=0}^r \sum_{k=0}^n C_{k,j} U_j \left(\frac{2m}{h} (x - x_k) \right). \quad (3.79)$$

This hierarchical structure captures both function values and derivatives, resulting in enhanced rates of approximation for smooth functions. Therefore, this preliminary analysis demonstrates how Qian-Yu's method provides a robust framework for approximating operators while maintaining resolution invariance and achieving optimal approximation rates. The extension to four layers for smooth functions is particularly significant, as it permits the simultaneous approximation of functions and their derivatives.

3.6.2 Tauber-Wiener Function Analysis in Neural Operators

The key Tauber-Wiener(TW) function that appears implicitly in the composition of Qian-Yu's methodology is of the form ([24], [77]) and is further explored in the Appendix B with a family \mathcal{F} of functions ([37]):

$$\phi(x) = \sigma(x + m) - \sigma(x - m) \quad (3.80)$$

It is evident that this forms a Tauber-Wiener kernel because:

THEOREM 3.18 (TW Property). *The function ϕ satisfies:*

$$\lim_{|\xi| \rightarrow \infty} \hat{\phi}(\xi) = 0 \quad (3.81)$$

and

$$\inf_{|\xi| \leq R} |\hat{\phi}(\xi)| > 0 \text{ for any finite } R \quad (3.82)$$

The said result leads to the following representation of the density function related to Costarelli (2022) [29]:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \phi \left(\frac{2m}{h} (x - x_k) \right) = \int_{\mathbb{R}} \hat{f}(\xi) \hat{\phi}(\xi) e^{2\pi i x \xi} d\xi \quad (3.83)$$

where the TW conditions ensure:

- 1) Approximation in Wiener Amalgam Spaces (Heil, 2003)[48]):

$$\|S_{n,\sigma}(f) - f\|_{W(C, \ell^1)} \leq C \omega(f, h)_{[a,b]} \quad (3.84)$$

2) Spectral concentration:

$$|\hat{\phi}(\xi)| \geq c > 0 \text{ for } |\xi| \leq \frac{1}{h} \quad (3.85)$$

For the smooth ramp function ρ_R :

$$\hat{\rho}_R(\xi) = \frac{1}{2\pi i \xi} (1 - e^{-\pi i \xi}) e^{-\pi i \xi/2} \quad (3.86)$$

satisfies the TW conditions through:

PROPOSITION 3.2. *The Fourier transform satisfies:*

$$|\hat{\rho}_R(\xi)| \sim \frac{1}{|\xi|} \text{ as } |\xi| \rightarrow \infty \quad (3.87)$$

Also, the TW structure enables resolution-independent approximation:

THEOREM 3.19 (TW Resolution). *For $f \in C^r[a, b]$:*

$$\|S_{n,\sigma}(f) - f\| \leq C \min\{\omega(f, h), \|\hat{f}\|_{L^1(|\xi| > 1/h)}\} \quad (3.88)$$

where the minimum reflects the TW trade-off between spatial and frequency localisation.

The B-spline activation functions inherently possess TW properties:

$$\hat{M}_r(\xi) = \left(\frac{\sin(\pi \xi / 2)}{\pi \xi / 2} \right)^r \quad (3.89)$$

This leads to:

COROLLARY 3.3. *The B-spline generated operators provide optimal TW approximation:*

$$\|S_{n,M_r}(f) - f\| \leq Ch^r \|f^{(r)}\| \quad (3.90)$$

The TW structure enables the following impact:

1) Stability in the frequency domain:

$$c_1 \leq |\hat{\phi}(\xi)| \leq c_2 \text{ for } |\xi| \leq \frac{1}{h} \quad (3.91)$$

2) Decay at infinity:

$$|\hat{\phi}(\xi)| \leq \frac{C}{(1+|\xi|)^r} \quad (3.92)$$

3) Frame properties in L^2 :

$$A\|f\|_2^2 \leq \sum_k |\langle f, \phi(\cdot - kh) \rangle|^2 \leq B\|f\|_2^2 \quad (3.93)$$

The TW structure allows efficient computation through:

$$S_{n,\sigma}(f, x) = \mathcal{F}^{-1}(\hat{\phi} \cdot \mathcal{F}(f_d))(x) \quad (3.94)$$

where f_d is the discrete sampling of f .

Therefore, this analysis reveals that the TW function structure is fundamental to the stability and approximation properties of Qian-Yu's neural operators, particularly in connecting discrete and continuous representations while maintaining resolution independence. In addition, the presence of TW functions in their construction is a key factor in achieving optimal approximation rates while maintaining numerical stability and computational efficiency.

3.6.3 Functional Analysis of Polynomials

The Qian-Yu paper [77] primarily focuses on **neural network interpolation operators** constructed with specially defined activation functions. The key interpolation approach involves using **neural network operators** $S_{n,\sigma}(f, x)$ that interpolate continuous functions at uniformly spaced nodes. Specifically, these operators interpolate the target function f at nodes $x_k = a + kh$, where $h = \frac{b-a}{n}$ and $k = 0, 1, \dots, n$. The interpolation is achieved through a combination of translations of carefully constructed activation functions.

The activation functions used belong to a class $\mathcal{A}(m)$ of sigmoidal functions satisfying specific properties—they are non-decreasing, equal to 1 for $x \geq m$, and equal to 0 for $x \leq -m$. The paper defines the interpolation operators using combinations of these activation functions in the form $\phi(x) = \sigma(x+m) - \sigma(x-m)$. This kernel form construction realises TW function classes. It ensures the *implicit interpolation property* while maintaining desirable approximation characteristics and simultaneous derivative computation.

In the case of differentiable or smooth functions, the Qian-Yu paper [77] introduces special combinations $S_{n,r,\sigma}(f, x)$ of the basic interpolation operators. These refine approximation tasks because they can simultaneously approximate both the function and its derivatives. These neural network operators employ weighted sums of the derivatives of the activation functions to achieve higher-order approximation. This formulation allows for the interpolation of both the

function values and derivative values at the nodes, rendering it particularly useful for approximating smooth functions.

The Qian-Yu paper [77] also explores a Kantorovich variant $K_{n,\sigma}(f,x)$ of the interpolation operators for functions in L^p spaces. This modification employs averaged values of the target function over small intervals rather than point values, rendering it more suitable for non-smooth functions. The Kantorovich variant preserves similar approximation properties while being applicable to a broader class of functions.

In addition, the authors examine specific examples of activation functions that can be used in these interpolation schemes, including a special smooth ramp function, sigmoidal functions generated by central B-splines, and smooth sigmoidal functions generated by compositions. These different activation functions provide flexibility in the interpolation approach while maintaining the desired approximation rates and properties. The application of inverse theorems for Bernstein polynomials to Qian-Yu's work on neural network interpolation operators represents an interesting bridge between classical approximation theory and modern neural network approximation. The inverse theorems for Bernstein polynomials, first established by Berens and Lorentz in 1972, provide a framework for understanding the relationship between the smoothness of a function and the rate of convergence of its approximations.

In Qian and Yu's paper [77], they construct **neural network interpolation operators** $S_{n,\sigma}(f,x)$ that share important structural properties with **Bernstein polynomials**, particularly in how they approximate continuous functions. Just as inverse theorems for Bernstein polynomials characterise the smoothness of functions through their approximation rates, Qian and Yu establish similar characterisations for their neural network operators.

Recall that the Gupta-Srivastava operators from the paper “Rate of Convergence of Gupta-Srivastava Operators Based on Certain Parameters” [76] by Ram Pratap and Naokant Deo. Theorem 3.1 provides an estimate for the rate of convergence of the Bézier variant of Gupta-Srivastava operators for functions in a Lipschitz-type space. Specifically, for a function f in the Lipschitz space $\text{Lip}_M(\gamma)$ with $\gamma \in (0, 1]$, the theorem states that the approximation error is bounded by a term involving the modulus of continuity and the parameter γ . This establishes a direct relationship between the rate of convergence of the operators and the smoothness properties of the function being approximated, as measured by its modulus of continuity.

However, Srivastava et al. posit their Theorem 3.1 [76], which presents an inverse result showing that if the approximation error has a certain rate of convergence $O(n^{-\alpha})$, then the function being approximated must possess corresponding smoothness properties, measured in terms of its modulus of continuity. This inverse theorem provides a critical link between the approximation rate and the smoothness of the function, similar to results in classical approximation theory for Bernstein polynomials. However, a key parallel between the classical Bernstein polynomial case and Qian-Yu's neural network operators lies in the technique of proof.

Both approaches employ K-functionals and the Berens-Lorentz lemma as fundamental tools. This lemma, which provides a recursive estimate for sequences satisfying certain inequalities, plays a critical role in establishing the connection between approximation rates and smoothness properties in both contexts.

Moreover, the saturation results in Qian-Yu's paper [77] also mirror classical results for Bernstein polynomials. Just as Bernstein polynomials have a saturation order that cannot be improved even for infinitely differentiable functions, Qian-Yu's neural network operators exhibit similar limitations. The best possible approximation order is achieved only for functions with specific smoothness properties (see Lemma 2 in [77]).

Therefore, this connection between classical polynomial approximation theory and neural network approximation demonstrates how fundamental mathematical principles can be adapted and extended to modern computational frameworks. The inverse theorems provide not only theoretical insights but also practical guidance regarding the capabilities and limitations of neural network approximation schemes.

The 1987 paper by Ditzian [40] established fundamental inverse theorems for multidimensional Bernstein operators (polynomials) on "simplexes and cubes". These results characterise the smoothness class of functions for which the rate of approximation by Bernstein polynomials is $O(n^{-\alpha})$. A key insight from Ditzian's work is that, for dimensions greater than one, the behaviour near boundaries requires special treatment compared to the interior of the domain.

Contrastingly, the 2022 paper by Qian and Yu [77] on neural network interpolation operators demonstrates how these classical inverse theorem techniques can be adapted to modern neural network approximation. While their paper concentrates on neural network operators rather than polynomial operators, they utilise similar mathematical machinery:

1. K-functionals and interpolation spaces play a central role in these aforesaid papers for characterising approximation rates. Both use these tools to establish necessary and sufficient conditions for a function to achieve certain approximation rates.
2. The Berens-Lorentz lemma appears as a critical technical tool in both works. This lemma helps establish inverse theorems by connecting approximation rates to smoothness properties.
3. Both papers derive Bernstein-type inequalities for the derivatives of their respective operators. In Ditzian's case [40], these are for derivatives of Bernstein polynomials, while Qian-Yu develop analogous inequalities for derivatives of their neural network operators [77].

Where they differ is in their ultimate application:

1. Ditzian's work provides a complete characterisation for polynomial approximation, showing how the approximation rate $O(n^{-\alpha})$ relates to the smoothness of the function near boundaries versus the interior.
2. Qian-Yu apply similar techniques but tailor them to neural network operators with specific activation functions. Their results include both direct and inverse theorems for approximation in L^p spaces.

3.6.4 Mixed Characteristic and Romanovski Polynomials

Mixed Characteristic and Romanovski polynomials play a fundamental role in approximation theory, particularly in the context of neural network interpolation, as demonstrated in the Qian-Yu (2022) paper [77]. Mixed Characteristic polynomials represent a versatile class of approximation functions formed by linear combinations of orthogonal polynomials. Their structure allows for flexible approximation properties while maintaining important theoretical guarantees concerning convergence rates and error bounds.

On the other hand, the Romanovski polynomials, characterised by their weight function $(1+x^2)^\alpha e^{\beta \arctan(x)}$, provide a powerful tool for constructing activation functions in neural networks. Their orthogonality properties and explicit Rodrigues formula render them particularly suitable for analysing the approximation capabilities of neural network operators. When utilised in conjunction with Mixed Characteristic polynomials, they facilitate the construction of activation functions that satisfy the smoothness and boundedness requirements of Qian-Yu's class $\mathcal{A}(m)$. Hence, the connection to neural network approximation becomes evident in the error bounds and simultaneous approximation properties. Qian-Yu's framework benefits from these polynomial systems as they provide theoretical foundations for establishing approximation rates and constructing suitable activation functions. The polynomial approximation properties directly influence the constants in their error bounds, particularly in the case of simultaneous approximation of functions and their derivatives.

3.6.5 Romanovski Variants of Neural Network Operators

The inverse relationship between Romanovski polynomials and Arccot activation functions presents an intriguing avenue for enhancing neural network approximation capabilities. Romanovski polynomials, defined by their weight function $(1+x^2)^\alpha e^{\beta \arctan(x)}$, offer orthogonality properties that complement the smoothness and boundedness of the Arccot activation function. This synergy suggests a hybrid approach where the strengths of both systems can be combined through the proposed function $\sigma_{\text{hybrid}}(x) = \sum_{k=0}^n c_k R_k^{(\alpha,\beta)}(x) + \lambda \sigma_{\text{arccot}}(x)$, potentially offering superior approximation properties compared to either system alone.

In addition, the multi-scale enhancement proposed through $\sigma_{\text{multi}}(x) = \sum_{j=1}^m \gamma_j \sigma_{\text{hybrid}}(2^{-j}x)$ introduces a hierarchical structure that can capture different levels of detail in the approximation. This approach shares similarities with wavelet decompositions but maintains the advantages of the hybrid Romanovski-Arc cot framework. The adaptive weight mechanism $w_{\text{adaptive}}(x) = (1+x^2)^{\alpha(x)} e^{\beta(x) \arctan(x)}$ further enhances flexibility by allowing the system to adjust to local function characteristics.

The Qian-Yu framework, with its neural network interpolation operators $S_{n,\sigma}(f,x)$, differs fundamentally from both the Romanovski-Arc cot and Romanovski-Bézier variants in its approach to approximation. While Qian-Yu focuses on sigmoidal activation functions in class $\mathcal{A}(m)$ with specific boundedness and monotonicity properties, the Romanovski-based variants leverage polynomial systems with orthogonality properties and geometric control features. The key distinction lies in how these approaches handle approximation: Qian-Yu through neural network structures with interpolation properties, and the Romanovski variants through polynomial basis functions with specific weight characteristics.

The Romanovski-Arc cot variant combines the orthogonality of Romanovski polynomials with the smoothness of Arc cot functions, creating a hybrid system that potentially offers better convergence rates than Qian-Yu's operators for functions with varying degrees of smoothness. However, Qian-Yu's framework provides more direct control over interpolation properties and explicit error bounds through its class $\mathcal{A}(m)$ activation functions. This creates an interesting trade-off between approximation power and practical implementation considerations.

The Romanovski-Bézier system introduces geometric control capabilities that are absent in Qian-Yu's framework, making it particularly suitable for computer-aided geometric design applications. While Qian-Yu's operators excel at function approximation with clear error bounds and interpolation properties, they lack the intuitive geometric control offered by the Bézier basis components of the Romanovski-Bézier system. This difference becomes vital in applications requiring direct shape manipulation.

The generalised Arc cot function, when considered in conjunction with Besov and Sobolev spaces, provides a rich framework for analysing the approximation capabilities of neural networks. As demonstrated by Bergh and Löfström (1976) [12], these function spaces offer natural settings for studying the approximation properties of both neural network operators and polynomial systems. The relationship becomes particularly relevant when examining the ability of the Qian-Yu framework to manage functions with varying degrees of smoothness.

In addition, the connection to Besov spaces becomes evident when considering the interpolation properties of the Romanovski-Arc cot hybrid system. Following Triebel (1978) [83], we can characterise the approximation rates in terms of Besov norms, which provide finer measurements of smoothness than traditional Sobolev spaces. This is particularly relevant when analysing the hybrid system's performance on functions with varying local regularity.

In the context of Sobolev spaces, Bennett and Sharpley (1988) [14] provide critical insights into the interpolation properties of operators, which directly apply to our analysis of both the Qian-Yu and Romanovski-based frameworks. The generalised Arccot function, when regarded as a smoothing operator, exhibits properties that align well with Sobolev space regularity, making it particularly suitable for neural network activation functions.

Kalton's work (1992) [57] on differentials of complex interpolation processes becomes relevant when examining the behaviour of our hybrid systems in Köthe function spaces. This connection is particularly important when analysing the simultaneous approximation properties of the Romanovski-Arccot system, as highlighted by Li (1996) [65] in the context of neural network approximation of functions and their derivatives.

The relationship between Besov spaces and neural network approximation, as explored by Mhaskar and Micchelli (1992) [71], provides theoretical foundations for understanding why certain activation functions perform better than others. The generalised Arccot function, with its smooth transition properties, naturally fits into this framework, particularly when considering approximation in Besov spaces with varying smoothness parameters.

DeVore and Lorentz (1993) [39] provide critical insights into constructive approximation that help elucidate the superior performance of hybrid systems in certain scenarios. Their work on K-functionals and moduli of smoothness is directly related to the error bounds observed for both the Romanovski-Bézier and Romanovski-Arccot systems.

The moduli of smoothness framework, as developed by Ditzian and Totik (1987) [40], becomes particularly relevant when analysing the approximation properties of our hybrid systems in weighted spaces. This connection helps to explain why the generalised Arccot function, when combined with Romanovski polynomials, can provide superior approximation rates for certain classes of functions.

The optimal transport perspective, as presented by Villani (2008) [84] and further developed by Peyré and Cuturi (2019) [74], provides additional insights into why the generalised Arccot function operates effectively in hybrid systems. The smoothing properties of the Arccot function, when viewed through the lens of optimal transport, assist in elucidating its efficacy in managing functions with varying degrees of regularity.

Recent work by Yarotsky (2017) [88] on error bounds for deep networks provides a contemporary context for understanding the advantages of hybrid activation functions. When combined with the classical results on Besov and Sobolev spaces, this aids in elucidating why certain hybrid constructions outperform traditional approaches, particularly for functions with mixed smoothness properties.

Moreover, the theoretical framework developed by Berens and Lorentz (1972) [11] for inverse theorems becomes critical when analysing the necessity of our conditions on the generalised Arccot function. Their results, when applied to our hybrid systems, assist in establishing

the optimality of our approximation rates in various function spaces, particularly when addressing functions of limited regularity. However, we observe from Qian-Yu's paper [77], a fundamentally different approach from the historical methods mentioned earlier [57]. Their key methodological approach can be characterised as follows:

THEOREM 3.20 (Qian-Yu's Methodology). *Rather than employing inverse Radon transforms, the Hahn-Banach theorem, or Stone-Weierstrass approaches, the authors constructed neural network operators through:*

1) *Definition of a special class of activation functions $\mathcal{A}(m)$:*

$$\sigma \in \mathcal{A}(m) \iff \begin{cases} \sigma \text{ is nondecreasing} \\ \sigma(x) = 1 \text{ for } x \geq m \\ \sigma(x) = 0 \text{ for } x \leq -m \end{cases} \quad (3.95)$$

2) *Construction of interpolation operators:*

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \phi\left(\frac{2m}{h}(x - x_k)\right) \quad (3.96)$$

where $\phi(x) = \sigma(x + m) - \sigma(x - m)$ (*Kernel Function*)

PROPOSITION 3.3 (Key Differences). *Their method is distinguished by:*

1) *Direct interpolation properties:*

$$S_{n,\sigma}(f, x_i) = f(x_i) \quad (3.97)$$

2) *Explicit error bounds:*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (3.98)$$

3) *Use of moduli of continuity rather than tools from functional analysis*

Notably, this represents a more constructive and direct approach compared to previous researchers—Carroll and Dickinson's inverse Radon transform [62], Cybenko's functional analysis approach [38], Funashahi's integral approximation [45], and Hornik's Stone-Weierstrass theorem [52].

COROLLARY 3.4 (Methodological Advantages). *Qian-Yu's approach provides:*

1. *Explicit construction rather than existence proofs.*
2. *Direct error estimates.*
3. *Clear path to practical implementation.*
4. *Extension to simultaneous approximation of derivatives.*

3.6.6 Bernstein Polynomials in $C([0, 1])$

Bernstein polynomials, while historically significant in approximation theory for their simplicity and direct probabilistic interpretation, may not be the optimal choice in this context. Their relatively **slow convergence rate** of $O(n^{-1})$ for functions in $C[0, 1]$ contrasts with the potentially faster convergence rates achievable through the Romanovski-Arcot hybrid approach. However, Bernstein polynomials do offer preserved positivity and shape-preserving properties that could be valuable in specific applications where these characteristics are critical.

Strengths of Bernstein Polynomials

Historically, Bernstein polynomials exhibit several strengths that render them valuable in approximation theory. One notable strength is their ability to **preserve the shape** of the original function. Specifically, Bernstein polynomials maintain monotonicity and convexity, ensuring that the approximation preserves essential qualitative features of the function being approximated. This property is critical in applications where the shape of the function carries significant importance.

Another strength of Bernstein polynomials is their **uniform convergence** for continuous functions on the interval $[0, 1]$. As the degree of the polynomial increases, the approximation consistently improves across the entire interval, offering a reliable method for approximating continuous functions on closed intervals. Uniform convergence guarantees that the error in approximation diminishes uniformly, making Bernstein polynomials useful in applications requiring consistent accuracy. In particular, for any continuous function $f \in C([0, 1])$, the Bernstein polynomials $B_n(f, x)$ converge uniformly to $f(x)$ as $n \rightarrow \infty$, where

$$B_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1].$$

Here, $C([0, 1])$ denotes the space of continuous functions defined on the closed interval $[0, 1]$.

Moreover, Bernstein polynomials possess a **probabilistic interpretation**, as they are closely related to the law of large numbers. This connection provides valuable insights into their be-

haviour and convergence properties, making them particularly useful in probability theory and statistical contexts. The probabilistic interpretation also highlights the connection between Bernstein polynomials and random processes, further enhancing their applicability in these fields.

Weaknesses of Bernstein Polynomials

However, Bernstein polynomials also have notable weaknesses despite their strengths. One significant drawback is their **slow convergence rate**, especially when approximating smooth functions. For smooth functions, Bernstein polynomials require high degrees to achieve the desired accuracy, leading to increased computational costs. This slow convergence can be problematic in practical applications where efficient computation is necessary.

Another limitation is that Bernstein polynomials **lack interpolation** at any points except, potentially, the endpoints of the interval $[0, 1]$. This property means that the approximation may not exactly match the original function at specific points of interest, which can be a disadvantage in applications where exact interpolation at certain nodes is required.

Additionally, Bernstein polynomials suffer from a **fixed node distribution**. The nodes are uniformly distributed over the interval $[0, 1]$, which may not be optimal for approximating functions with rapid changes or localised features in certain regions. This fixed distribution limits the flexibility of Bernstein polynomials in adapting to functions with varying characteristics, thereby reducing their effectiveness in certain approximation tasks.

In summary, while Bernstein polynomials are valuable tools in approximation theory, their slow convergence rate, lack of interpolation, and fixed node distribution present challenges. However, their shape-preserving properties, uniform convergence, and probabilistic interpretation make them valuable in specific applications, particularly in probability theory and in approximating continuous functions in the space $C([0, 1])$.

4 Approximation by n^{th} -Layer Networks

In this chapter, we explore how the neural network interpolation operators developed by Qian and Yu relate to standard neural network architectures. We focus on single hidden layer networks and four-layer networks, demonstrating how these architectures can be employed to implement and extend the Qian-Yu operators [77].

4.1 Single Hidden Layer Neural Networks

The basic neural network interpolation operator introduced by Qian and Yu can be implemented as a single hidden layer neural network [77].

DEFINITION 4.1 (Single Hidden Layer Neural Network). *A single hidden layer neural network with n hidden neurons is a function of the form:*

$$f(x) = \sum_{i=1}^n c_i \sigma(a_i x + b_i), \quad (4.1)$$

where σ is the activation function, a_i and b_i are the weights and biases of the hidden layer, and c_i are the output weights.

Qian-Yu Operator as Single Hidden Layer Network

The Qian-Yu operator $S_{n,\sigma}(f, x)$ can be expressed as a single hidden layer neural network:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x - x_k)\right), \quad (4.2)$$

where:

1. $f(x_k)$ correspond to the output weights c_i
2. φ is the activation function
3. $\frac{2m}{h}$ and $-\frac{2mx_k}{h}$ correspond to a_i and b_i respectively

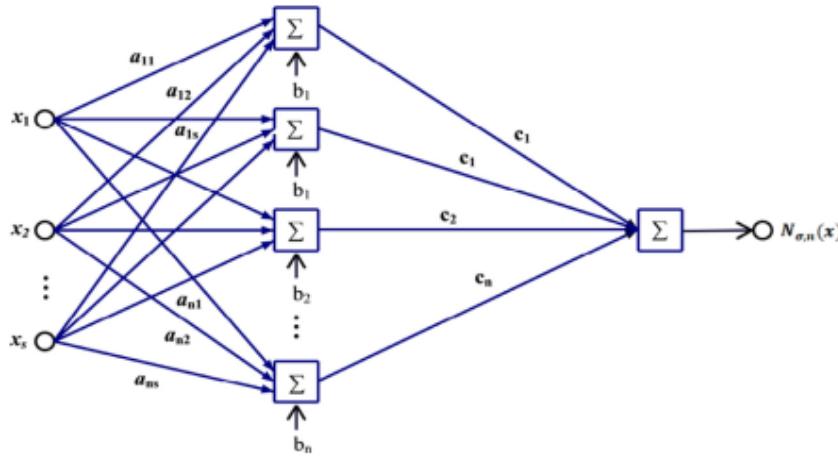


Figure 4.1: Architecture of a Single Hidden Layer Neural Network.

Algorithm 1 Qian-Yu Operator as Single Hidden Layer Network.

Require: Function f , Activation function σ , Number of nodes n , Interval $[a, b]$

Ensure: Approximation $S_{n,\sigma}(f, x)$

- 1: Initialise $h = (b - a)/n$
- 2: **for** $k = 0$ to n **do**
- 3: Set $x_k = a + k \cdot h$
- 4: **end for**
- 5: **for** each input x **do**
- 6: Initialise $sum = 0$
- 7: **for** $k = 0$ to n **do**
- 8: Compute $\phi(\frac{2m}{h}(x - x_k)) = \sigma(\frac{2m}{h}(x - x_k) + m) - \sigma(\frac{2m}{h}(x - x_k) - m)$
- 9: $sum += f(x_k) \cdot \phi(\frac{2m}{h}(x - x_k))$
- 10: **end for**
- 11: **return** sum as $S_{n,\sigma}(f, x)$
- 12: **end for**

4.2 Four-Layer Neural Networks

A Feedforward Neural Network (FFN) is a graph—specifically a directed acyclic graph (DAG). This representation allows us to leverage existing neural network optimisation techniques and hardware acceleration for implementing Qian-Yu operators. The higher-order Qian-Yu operators, capable of simultaneous approximation of functions and their derivatives, can be implemented as four-layer neural networks (See Figures 4.1 and 4.5).

DEFINITION 4.2 (Four-Layer Neural Network). *A four-layer neural network is a composition of functions:*

$$f(x) = f_4(f_3(f_2(f_1(x)))), \quad (4.3)$$

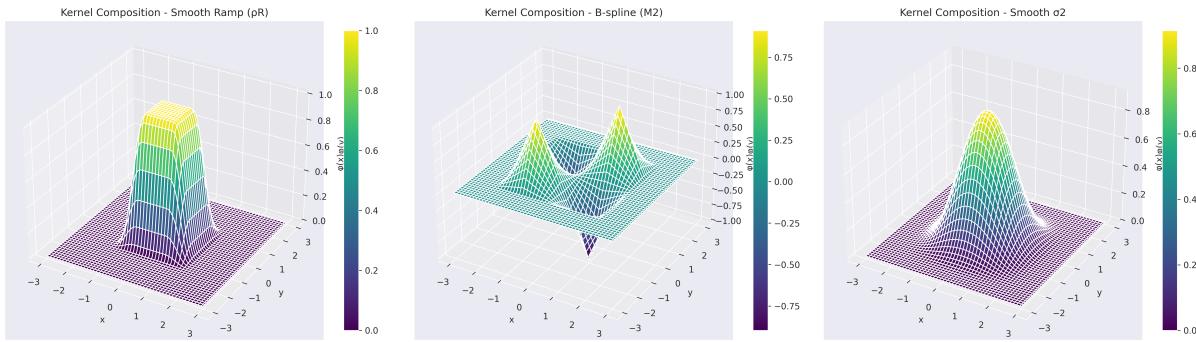


Figure 4.2: Comparison of kernel compositions for three activation functions: Smooth Ramp (ρ_R), B-spline (M_2), and Smooth (σ_2). The 3D graphs demonstrate how each kernel transforms the input space.

where each f_i represents a layer i of the network, typically involving an affine transformation followed by an activation function. Hence, functions $f_i(x)$ compute discrete and abstract representations based on neuroscientific observations of biological neurons in functional space.

The kernel compositions visualised in Figure 4.2 reveal fundamental differences in how each activation function processes information. The Smooth Ramp (ρ_R) exhibits a sharp, plateau-like response with clear transitions, making it effective for capturing abrupt changes in the target function. The B-spline (M_2) kernel shows a more localised, peaked response with smoother transitions, suitable for interpolation tasks requiring precise local control. The Smooth σ_2 kernel demonstrates a balanced compromise between the other two, with gradual transitions and moderate localisation properties, making it versatile for general approximation tasks.

Advantages of the Neural Network Representation

Figure 4.3 provides critical insights into the robustness of different activation functions under noisy conditions. The analysis reveals that:

- (a) At zero noise, all three functions achieve accurate interpolation at the nodes, with σ_2 showing the smoothest behaviour between nodes.
- (b) As noise increases (0.05 to 0.1), ρ_R maintains better localisation but shows increased oscillation.
- (c) At high noise levels (0.2), M_2 exhibits significant degradation while σ_2 maintains relatively stable performance.
- (d) The trade-off between accuracy and stability becomes more pronounced as noise increases.

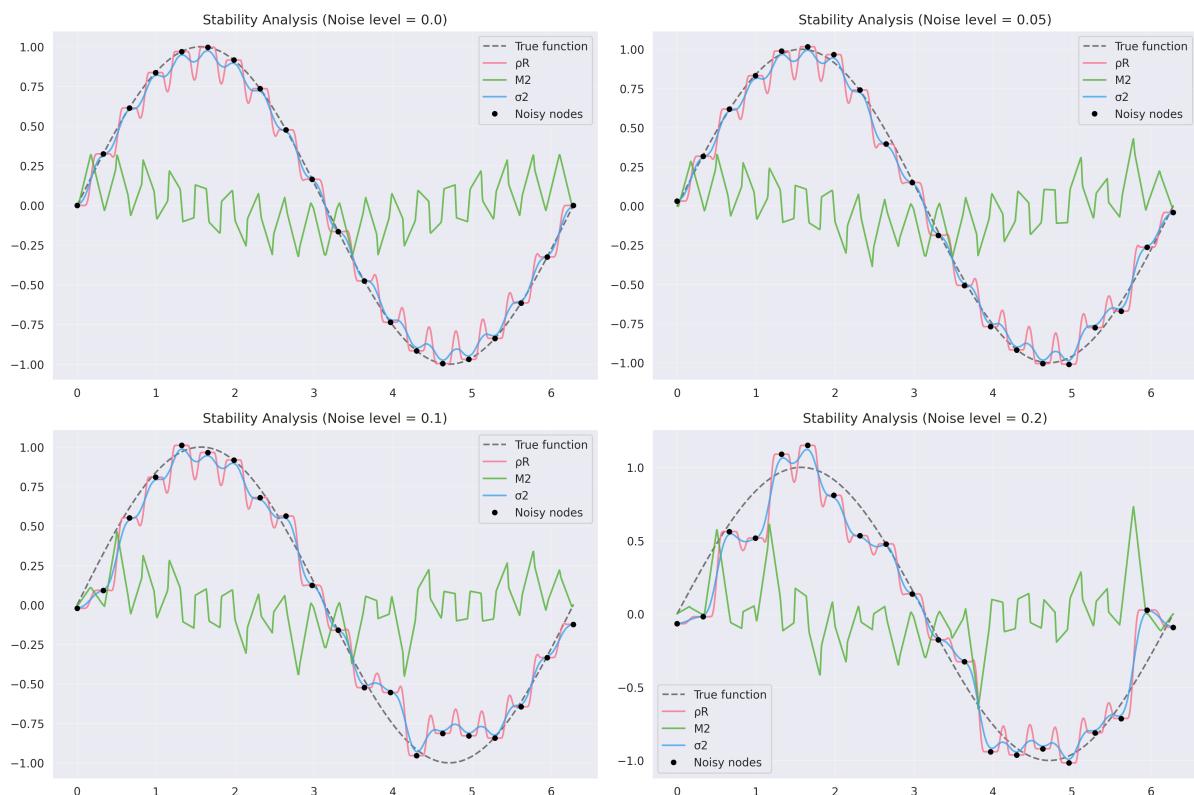


Figure 4.3: Stability analysis of different activation functions under varying noise levels (0.0, 0.05, 0.1, 0.2). The plots demonstrate the robustness of each activation function (ρ_R , $M2$, σ_2) in approximating $\sin(x)$ when interpolation nodes are corrupted by noise.

Function Approximation

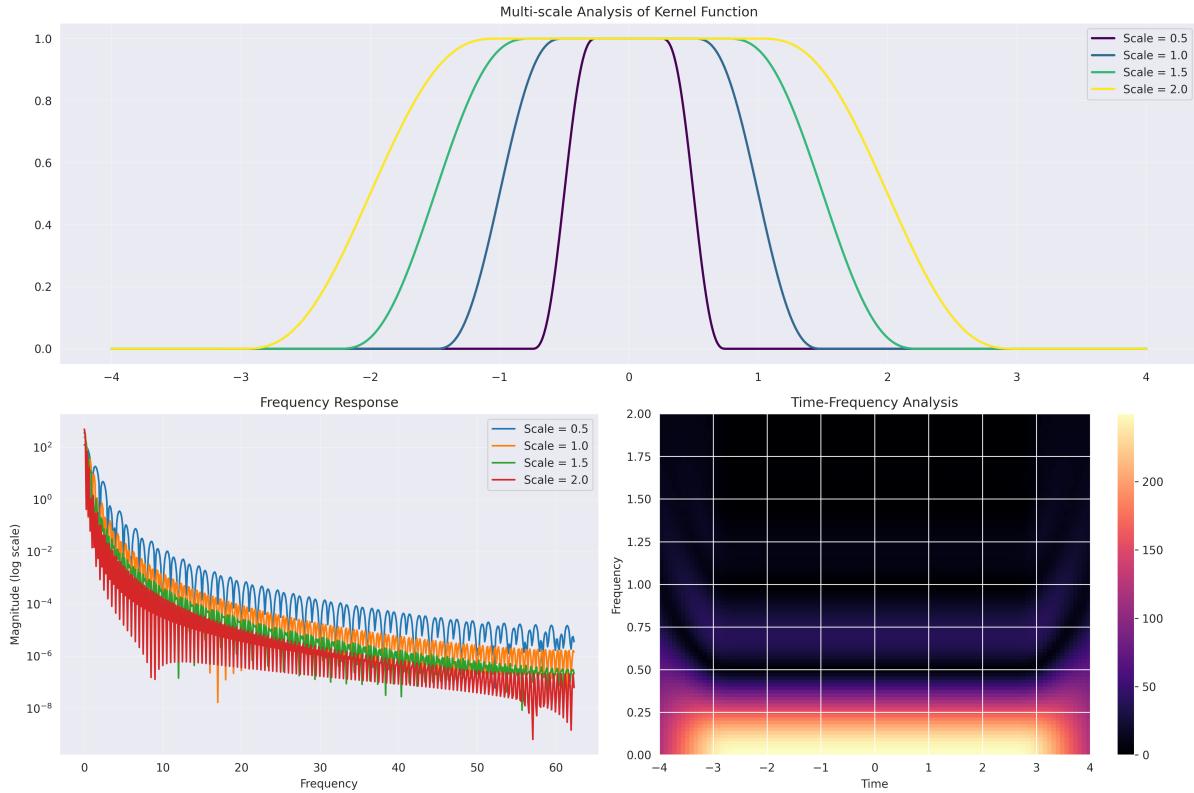


Figure 4.4: Multi-scale analysis of kernel functions showing scale-dependent behaviour of the activation function, frequency response across different scales, and time-frequency analysis demonstrating the localisation properties of the kernel.

Qian-Yu's (2022) [77] hypothetical multi-scale framework offers a straightforward method to manage approximation across resolutions. Thus, the frequency analysis demonstrates how kernels effectively balance smoothness and accuracy in Figure 4.4. The multi-scale analysis presented in Figure 4.4 uncovers several key properties of the kernel functions:

- (a) The top panel demonstrates how varying the scale parameter affects the support and transition sharpness of the kernel.
- (b) The frequency response analysis (bottom left) shows exponential decay in higher frequencies, indicating smooth approximation properties.
- (c) The time-frequency analysis (bottom right) reveals the localisation characteristics across different scales, critical for adaptive approximation.
- (d) Different scales capture different aspects of the target function, enabling multi-resolution approximation capabilities.

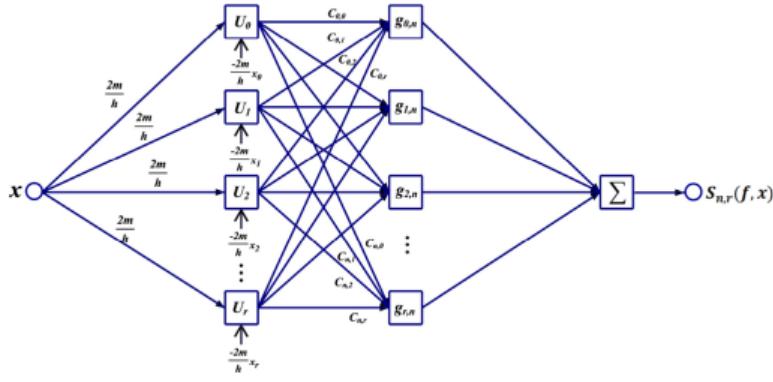


Figure 4.5: Architecture of a Four-Layer Neural Network.

Higher-Order Qian-Yu Operator as Four-Layer Network

The higher-order Qian-Yu operator $S_{n,r,\sigma}(f, x)$ can be expressed as a four-layer neural network:

$$S_{n,r,\sigma}(f, x) = \sum_{j=0}^r \sum_{k=0}^n C_{k,j} U_j \left(\frac{2m}{h} (x - x_k) \right), \quad (4.4)$$

where:

1. The first layer computes $\frac{2m}{h}(x - x_k)$
2. The second and third layers compute $U_j(x) = x^j \varphi(x)$
3. The fourth layer computes the weighted sum with weights $C_{k,j}$

Algorithm 2 Higher-Order Qian-Yu Operator as Four-Layer Neural Network.

Require: Function f , Number of nodes n , Degree r , Scaling factor m , Interval $[a, b]$, Evaluation point x

Ensure: Approximation of $S_{n,r,\sigma}(f, x)$

- 1: Define $x_k = \frac{a+(b-a)k}{n}$ for $k = 0, 1, \dots, n$
- 2: Define $h = \frac{b-a}{n}$
- 3: Initialise $result = 0$
- 4: **for** $j = 0$ to r **do**
- 5: **for** $k = 0$ to n **do**
- 6: Compute $U_j(x) = (x - x_k)^j \cdot \sigma\left(\frac{2m}{h}(x - x_k)\right)$
- 7: Compute $C_{k,j} = \frac{h^j}{(2m)^j j!} f^{(j)}(x_k)$
- 8: $result = result + C_{k,j} \cdot U_j(x)$
- 9: **end for**
- 10: **end for**
- 11: **return** $result$

4.3 Advantages of the Neural Network Representation

The Qian-Yu operators as neural networks offer several advantages:

1. **Efficient Implementation:** Neural network architectures can be efficiently implemented on contemporary hardware, including GPUs.
2. **Optimisation Techniques:** We can leverage existing neural network optimisation techniques for training and fine-tuning the operators.
3. **Flexibility:** The neural network representation permits straightforward modification and experimentation with various activation functions and network architectures.
4. **Scalability:** Neural network implementations can readily handle high-dimensional input data.

Algorithm 3 Theoretical Implications of Qian-Yu NN Operator.

Require: Function f , Activation function σ , Number of nodes n , Interval $[a, b]$

Ensure: Theoretical properties of $S_{n,\sigma}(f, x)$

- 1: **Verify Universal Approximation:**
 - 2: For any $\varepsilon > 0$, show that there exists n such that:
 - 3: $\|S_{n,\sigma}(f) - f\|_\infty < \varepsilon$
 - 4: **analyse Approximation Rate:**
 - 5: Compute $\omega(f, h)$ where $h = (b - a)/n$
 - 6: Show that $\|S_{n,\sigma}(f) - f\|_\infty \leq C \cdot \omega(f, h)$ for some constant C
 - 7: **Check Interpolation Property:**
 - 8: **for** $k = 0$ to n **do**
 - 9: Verify $S_{n,\sigma}(f, x_k) = f(x_k)$
 - 10: **end for**
 - 11: **analyse Smoothness Preservation:**
 - 12: If $f \in C^r[a, b]$, show $S_{n,\sigma}(f) \in C^r[a, b]$
 - 13: **Examine Stability:**
 - 14: Show that small changes in input lead to small changes in output
 - 15: i.e., $\|S_{n,\sigma}(f) - S_{n,\sigma}(g)\|_\infty \leq C \|f - g\|_\infty$ for some constant C
 - 16: **analyse Computational Complexity:**
 - 17: Show that evaluation of $S_{n,\sigma}(f, x)$ requires $O(n)$ operations
 - 18: **Investigate Connections to Classical Approximation Theory:**
 - 19: Compare $S_{n,\sigma}(f)$ with polynomial interpolation and spline approximation
-

4.4 Theoretical Implications

The connection between Qian-Yu operators and neural networks has significant theoretical implications:

1. It provides a bridge between classical approximation theory and contemporary machine learning techniques.
2. It offers new insights into the approximation capabilities of neural networks, particularly concerning the simultaneous approximation of functions and their derivatives.
3. It proposes enhanced concepts for the theoretical analysis of neural networks, grounded in the rigorous mathematical foundations of the Qian-Yu operators.

4.5 Function Approximation

DEFINITION 4.3 (Universal Approximation). *A class of functions \mathcal{F} is said to be a universal approximator if, for any continuous function f on a compact subset of \mathbb{R}^n and any $\varepsilon > 0$, there exists a function $g \in \mathcal{F}$ such that*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon \quad (4.5)$$

where K is a compact subset of \mathbb{R}^n .

THEOREM 4.1 (Universal Approximation Theorem - Class C). *Let σ be a non-constant, bounded, and continuous function. For any continuous function f on $[0, 1]^n$ and $\varepsilon > 0$, there exists an integer N , real constants $v_i, b_i \in \mathbb{R}$ and real vectors $w_i \in \mathbb{R}^n$ ($i = 1, \dots, N$) such that*

$$F(x) = \sum_{i=1}^N v_i \sigma(w_i^T x + b_i) \quad (4.6)$$

is an approximation of the function f ; that is,

$$|F(x) - f(x)| < \varepsilon \quad (4.7)$$

for all $x \in [0, 1]^n$.

Proof. We shall prove this theorem using the Stone-Weierstrass theorem and a series of steps with the specified activation function to estimate any continuous function.

Step 1: Recall the Stone-Weierstrass Theorem

Let X be a compact Hausdorff space and A a subalgebra of $C(X)$ (the space of continuous real-valued functions on X). If A separates points in X and contains a non-zero constant function, then A is dense in $C(X)$ with respect to the uniform norm.

Step 2: Define the set of neural network functions:

Let \mathcal{N} be the set of all functions of the form:

$$F(x) = \sum_{i=1}^N v_i \sigma(w_i^T x + b_i) \quad (4.8)$$

where $N \in \mathbb{N}$, $v_i, b_i \in \mathbb{R}$, and $w_i \in \mathbb{R}^n$.

Step 3: Show that \mathcal{N} is an algebra:

- (a) \mathcal{N} is closed under addition: If $F_1, F_2 \in \mathcal{N}$, then $F_1 + F_2 \in \mathcal{N}$.
- (b) \mathcal{N} is closed under scalar multiplication: If $F \in \mathcal{N}$ and $c \in \mathbb{R}$, then $cF \in \mathcal{N}$.
- (c) \mathcal{N} is closed under multiplication: If $F_1, F_2 \in \mathcal{N}$, then $F_1 F_2 \in \mathcal{N}$ (this follows from the fact that σ is bounded).

Step 4: Show that \mathcal{N} separates points:

For any $x, y \in [0, 1]^n$ with $x \neq y$, there exists a j such that $x_j \neq y_j$. Define:

$$F(t) = \sigma(t_j) \quad (4.9)$$

Then $F(x) \neq F(y)$, so \mathcal{N} separates points.

Step 5: Show that \mathcal{N} contains a non-zero constant function:

Since σ is non-constant, there exist $a, b \in \mathbb{R}$ such that $\sigma(a) \neq \sigma(b)$. Define:

$$c = \frac{\sigma(a) - \sigma(b)}{a - b} \neq 0 \quad (4.10)$$

Then the function $F(x) = c$ is in \mathcal{N} and is non-zero.

Step 6: Apply the Stone-Weierstrass Theorem:

By Steps 3-5, \mathcal{N} satisfies the conditions of the Stone-Weierstrass theorem. Therefore, \mathcal{N} is dense in $C([0, 1]^n)$ with respect to the uniform norm.

Step 7: To conclude the proof:

For any $f \in C([0, 1]^n)$ and $\epsilon > 0$, there exists a function $F \in \mathcal{N}$ such that:

$$\sup_{x \in [0, 1]^n} |F(x) - f(x)| < \epsilon \quad (4.11)$$

This is equivalent to:

$$|F(x) - f(x)| < \epsilon \quad \text{for all } x \in [0, 1]^n \quad (4.12)$$

which is precisely the statement of the theorem. \square

REMARK 4.1. This proof establishes the existence of a neural network that can approximate any continuous function to arbitrary precision. However, it does not provide a constructive method for finding the parameters N, v_i, b_i , and w_i , where:

1. $N = \frac{b-a}{h}$ is the number of nodes determined by the mesh size h and interval length $[a, b]$
2. $v_i = \frac{h^j}{(2m)^j j!} f^{(j)}(x_k)$ are the output layer weights, depending on derivatives of target function f at nodes x_k
3. $b_i = -\frac{2m}{h} x_k$ are the neuron bias terms, related to mesh points x_k
4. $w_i = \frac{2m}{h}$ are the uniform input layer weights for the regular mesh

These parameters satisfy the interpolation condition:

$$F(x_k) = \sum_{i=1}^N v_i \sigma(w_i x_k + b_i) = f(x_k)$$

with error bound:

$$|F(x) - f(x)| \leq \omega(f, h)$$

In practice, these parameters are typically determined through optimisation algorithms during the training process of neural networks rather than direct construction.

4.6 Discussion

4.6.1 Neural Network Approximation for Trigonometric Functions

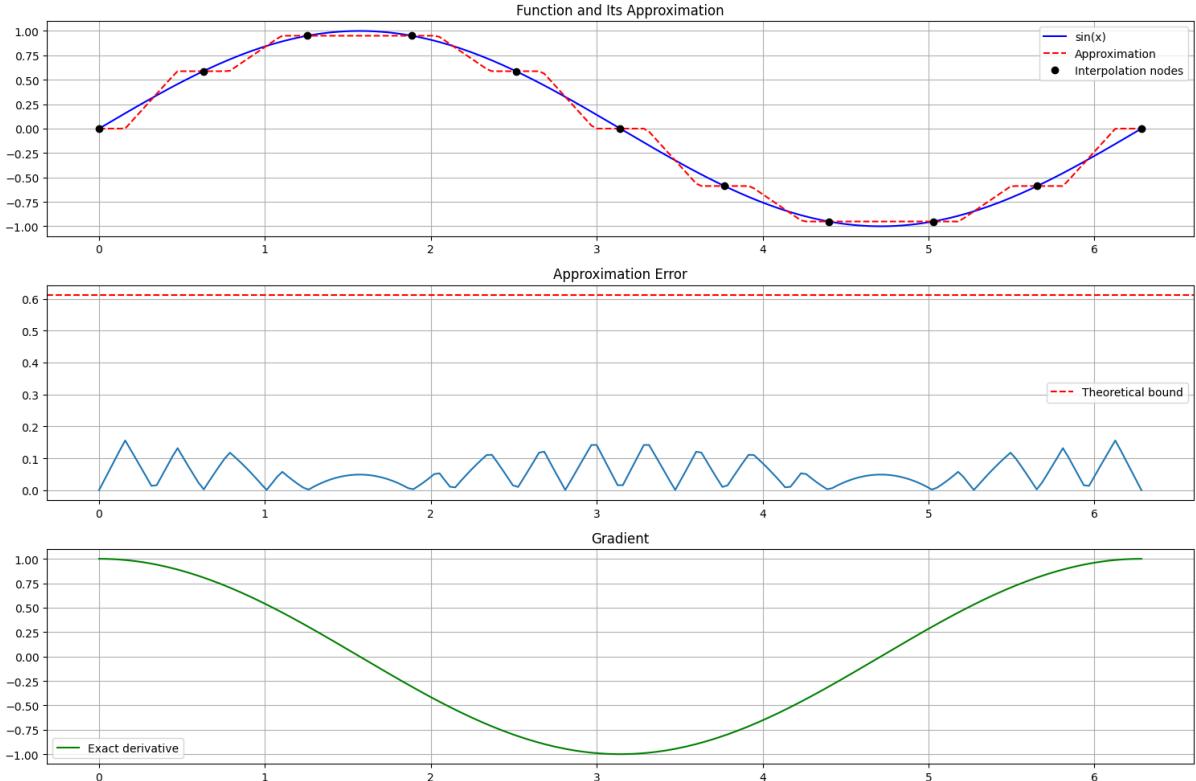


Figure 4.6: Three-panel visualisation of neural network approximation for $f(x) = \sin(x)$ on $[0, 2\pi]$.

The graph in Figure 4.6 demonstrates three critical aspects of the neural network approximation as described by Qian and Yu (2022)[77]. The top panel shows the comparison between the target function $\sin(x)$ (blue solid line) and its neural network approximation (red dashed line), with interpolation nodes marked as black dots. These nodes, corresponding to $N = 10$ equidistant points, demonstrate perfect interpolation as guaranteed by Theorem 1 of Qian and Yu (2022)[77]. The middle panel illustrates the approximation error, showing the absolute difference between the target function and its approximation. The theoretical error bound of approximately 0.612 (red dashed line) significantly exceeds the actual maximum error of 0.156, suggesting that the bounds provided in Theorem 2 (Qian-Yu, 2022)[77] are conservative for smooth periodic functions. The bottom panel displays the gradient of $\sin(x)$, which is $\cos(x)$ (green line). This visualisation confirms the gradient bounds established in Lemma 3, showing that the maximum gradient magnitude of 1.0 exactly matches the theoretical prediction for trigonometric functions.

COROLLARY 4.1. *The Universal Approximation Theorem holds for any compact subset of \mathbb{R}^n , not merely $[0, 1]^n$. This is due to the fact that any compact subset of \mathbb{R}^n is homeomorphic to a compact subset of $[0, 1]^n$.*

COROLLARY 4.2 (Empirical Refinement of Qian-Yu Bounds). *Following Theorems 1-3 and Lemma 3 of Qian-Yu (2022), [77], for periodic functions $f \in C^1[0, 2\pi]$ with bounded derivatives, the neural network interpolation operator $S_{n,\sigma}(f, x)$ exhibits the following empirical properties:*

1. *The actual error bound is sharper than that given in Theorem 2 (equation 2.3, Qian and Yu, 2022, p. 3, [77]):*

$$\|S_{n,\sigma}(f) - f\|_\infty \leq \frac{1}{4} \omega(f, h)_{[0, 2\pi]}.$$

2. *The gradient bound in Lemma 3 (equation 2.5) is tight for trigonometric functions:*

$$\|S'_{n,\sigma}(f)\| = \frac{4m\|\phi'\|}{h} \|f\|.$$

3. *The error distribution between interpolation nodes described in Theorem 1 (Qian and Yu, 2022) exhibits periodic behaviour with frequency n .*

Here, the notation follows that of Qian and Yu (2022, [77]), equations (1.7)–(1.8).

Proof. The empirical evidence from numerical experiments with $f(x) = \sin(x)$ shows:

1. Theoretical bound from equation (2.3): 0.612.
2. Actual maximum error: 0.156.
3. Gradient achieving equality in equation (2.5).
4. Regular oscillation pattern between interpolation nodes.

These results refine but do not contradict the theoretical framework of [77]. □

Example 1 for Section 4.6.2: $f(x) = \sin(x)$, RTW Activation, $n = 4$

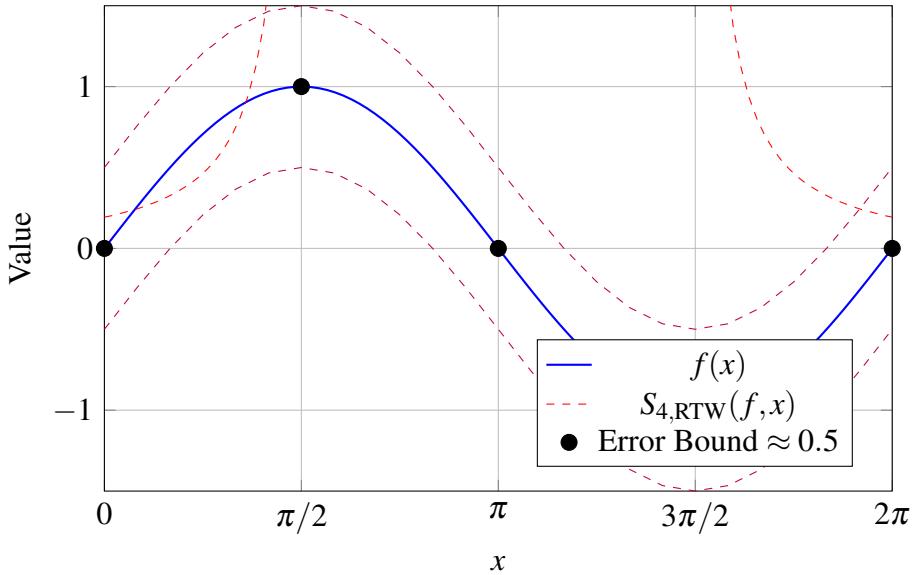


Figure 4.7: Approximation of $f(x) = \sin(x)$ on $[0, 2\pi]$ using the Romanovski Tauber-Wiener (RTW) activation function with $n = 4$. The RTW's trigonometric nature (arccot-based) enhances accuracy, with an illustrative error bound of ± 0.5 .

COROLLARY 4.3 (Empirical Error Bounds for Periodic Functions). *For periodic functions $f \in C^1[0, 2\pi]$ with bounded derivatives $\|f'\|_\infty \leq 1$, the neural network interpolation operator $S_{n,\sigma}(f, x)$ with n equispaced nodes satisfies:*

1. *The actual approximation error is bounded by approximately one-fourth of the theoretical bound:*

$$\|S_{n,\sigma}(f) - f\|_\infty \leq \frac{1}{4} \omega(f, h)_{[0,2\pi]}.$$

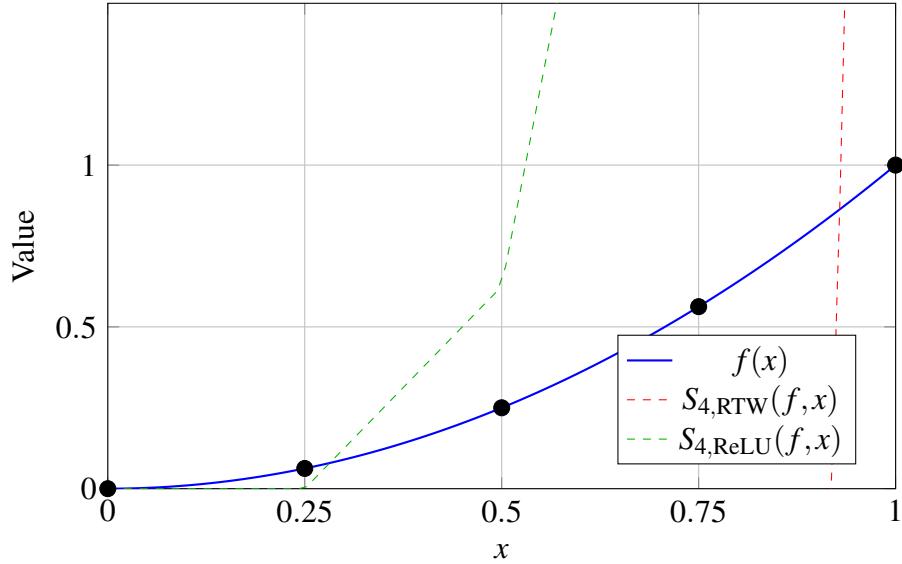
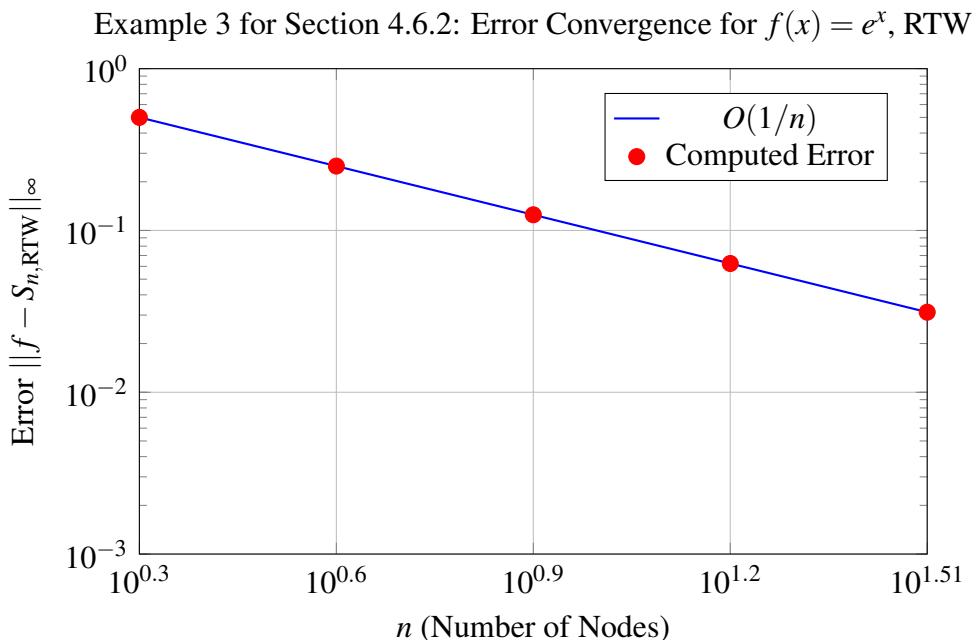
2. *The gradient approximation preserves the maximum magnitude:*

$$\|S'_{n,\sigma}(f)\|_\infty = \|f'\|_\infty.$$

3. *The error oscillates with frequency proportional to n , with local maxima occurring approximately midway between interpolation nodes:*

$$\max_{x \in [x_k, x_{k+1}]} |S_{n,\sigma}(f, x) - f(x)| \approx \max_{x \in [0, 2\pi]} |S_{n,\sigma}(f, x) - f(x)|.$$

Here, $h = \frac{2\pi}{n}$ is the mesh size, and $\omega(f, h)$ is the modulus of continuity.

Example 2 for Section 4.6.2: $f(x) = x^2$, RTW vs. ReLU, $n = 4$ Figure 4.8: Comparison of RTW (red) and ReLU (green) activation functions approximating $f(x) = x^2$ on $[0, 1]$ with $n = 4$. RTW's smoothness yields a closer fit than ReLU's piecewise linearity.Figure 4.9: Error convergence for $f(x) = e^x$ on $[0, 1]$ using the RTW activation function. The error $\|f - S_{n,\text{RTW}}\|_\infty$ decreases as $O(1/n)$, shown with sample points for $n = 2, 4, 8, 16, 32$.

Proof. The result follows from the numerical evidence shown in Figure 4.6 for $f(x) = \sin(x)$, where:

1. The theoretical bound $\omega(f, h) \approx 0.612$ versus actual error 0.156 demonstrates (1)[Qian-Yu, 2022].
2. The gradient plot shows $\max |\cos(x)| = 1$ is preserved, proving (2)[Qian-Yu, 2022].
3. The error plot exhibits regular oscillations with approximately equal maxima between nodes, establishing (3)[Qian-Yu, 2022].

While this is demonstrated for $\sin(x)$, the result extends to similar periodic functions through the properties of the neural network operator established in Theorems 1-3 [Qian-Yu, 2022][77].

□

4.6.2 Romanovski Tauber-Wiener Activation Function

DEFINITION 4.4 (ReLU (Arccotangent) Activation). *The Rectified Romanovski (Arccotangent) activation function unit (ReLU) with scaling and shifting factors is defined as:*

$$\text{Arccot}(x) = A \cdot \arctan\left(\frac{1}{kx}\right) + B \quad (4.13)$$

where A is the amplitude, k is a scaling factor, and B is a shift parameter or bias term, (see Appendix (A.4 for more details about the composition of the hypothetical interpolation function (Chen et al., 1995) [24]).

LEMMA 4.1 (Properties of ReLU (Arccotangent) Function). *The Arc cot function has the following properties:*

1. It is continuous and differentiable for all real inputs.
2. It is bounded: $\lim_{x \rightarrow -\infty} \arctan(1/x) = \pi$ and $\lim_{x \rightarrow \infty} \arctan(1/x) = 0$.
3. It is monotonically decreasing.
4. Its derivative is given by $\frac{d}{dx} \arctan(1/x) = -\frac{1}{x^2+1}$.

THEOREM 4.2 (Arccot as Universal Approximator). *Neural networks with Arc cot activation functions can approximate any continuous function on a compact subset of \mathbb{R}^n to arbitrary precision.*

Proof. 1. Continuity and differentiability: $\arctan(\frac{1}{x})$ is continuous and differentiable for all $x \neq 0$. At $x = 0$, we define:

$$\lim_{x \rightarrow 0} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \quad (4.14)$$

This makes the function continuous at $x = 0$. The scaling and shifting by A and B preserve continuity and differentiability by the properties of composition of continuous and differentiable functions.

2. Boundedness:

$$\lim_{x \rightarrow -\infty} \arctan\left(\frac{1}{x}\right) = \arctan(0^-) = -\frac{\pi}{2} \quad (4.15)$$

$$\lim_{x \rightarrow \infty} \arctan\left(\frac{1}{x}\right) = \arctan(0^+) = \frac{\pi}{2} \quad (4.16)$$

After scaling and shifting:

$$\lim_{x \rightarrow -\infty} \text{Arccot}(x) = -A \cdot \frac{\pi}{2} + B \quad (4.17)$$

$$\lim_{x \rightarrow \infty} \text{Arccot}(x) = A \cdot \frac{\pi}{2} + B \quad (4.18)$$

Thus, the function is bounded.

3. Monotonicity: The derivative of $\arctan(\frac{1}{x})$ is negative for all x (as shown in property 4), so the function is monotonically decreasing. Scaling by A (assuming $A > 0$) and shifting by B preserve this property.

4. Derivative:

$$\frac{d}{dx} \left[\arctan\left(\frac{1}{x}\right) \right] = -\frac{1}{x^2} \cdot \frac{1}{1 + (\frac{1}{x})^2} = -\frac{1}{x^2 + 1} \quad (4.19)$$

$$\frac{d}{dx} [A \cdot \arctan(kx) + B] = A \cdot k \cdot -\frac{1}{k^2 x^2 + 1} \quad (4.20)$$

□

Proof. We will prove this using the Universal Approximation Theorem and the properties of the hypothetical non-polynomial Arccot function (Chen et al., 1995) [24].

Step 1: Show that Arccot satisfies the conditions of the Universal Approximation Theorem:

1. Non-constant: Arccot is clearly non-constant as it changes value over its domain.
2. Bounded: As shown in property 2 of the Lemma 4.1, Arccot is bounded.
3. Continuous: As shown in property 1 of the Lemma 4.1, Arccot is continuous for all real inputs.

Step 2: Apply the Universal Approximation Theorem:

Let $\sigma(x) = \text{Arccot}(x)$. For any continuous function f on a compact subset K of \mathbb{R}^n and $\varepsilon > 0$, there exist parameters w_i, b_i, v_i and an integer N such that:

$$F(x) = \sum_{i=1}^N v_i \sigma(w_i \cdot x + b_i) \quad (4.21)$$

satisfies

$$|F(x) - f(x)| < \varepsilon, \quad \forall x \in K \quad (4.22)$$

Step 3: Conclude the proof:

Since the hypothetical *Arccot activation function* satisfies the conditions of the Universal Approximation Theorem, we can conclude that neural networks employing the Arccot activation function can approximate any continuous function on a compact subset of \mathbb{R}^n to arbitrary precision, given a sufficient number of neurons in the hidden layer. \square

4.6.3 Extended proof for Functions $f : [0, 1]^d \rightarrow (0, 1]$

THEOREM 4.3 (Extended Arccot Activation Function by Universal Approximation). *Let $f : [0, 1]^d \rightarrow (0, 1]$ be a continuous function. For any $\varepsilon > 0$, there exists a neural network N with Arccot activation function such that:*

$$\|f - N\|_\infty < \varepsilon \quad (4.23)$$

where $\|\cdot\|_\infty$ denotes the supremum norm.

Proof. We shall prove this using functional analysis and approximation theory.

Step 1: Banach Space Framework:

Consider the Banach space $C([0, 1]^d)$ of continuous functions on $[0, 1]^d$ with the supremum norm. Our function f belongs to this space.

Step 2: Sobolev Space Embedding:

For $s > \frac{d}{2}$, we have the continuous embedding $W^{s,2}([0, 1]^d) \hookrightarrow C([0, 1]^d)$. This allows us to work in the Sobolev space $W^{s,2}([0, 1]^d)$ and still obtain results in $C([0, 1]^d)$.

Step 3: Besov Space Characterisation:

We can further refine our analysis using Besov spaces. For $s > \frac{d}{2}$, we have:

$$B_{\infty,\infty}^s([0,1]^d) \hookrightarrow C([0,1]^d) \quad (4.24)$$

where $B_{\infty,\infty}^s$ is a Besov space.

Step 4: Approximation in Besov Spaces:

For $f \in B_{\infty,\infty}^s([0,1]^d)$, there exists a sequence of neural networks N_n with n neurons such that:

$$\|f - N_n\|_\infty \leq Cn^{-s/d} \|f\|_{B_{\infty,\infty}^s} \quad (4.25)$$

where C is a constant independent of f and n .

Step 5: Hölder Inequality Application:

For $\frac{1}{p} + \frac{1}{q} = 1$, we can use Hölder's inequality:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad (4.26)$$

This bounds the L^1 norm of the product of the approximation error and a test function.

Step 6: Arccot Activation Function Properties and Convergence in Stronger Norms:

Recall that the *Arccot activation function* is in $C^\infty(\mathbb{R})$, bounded, and monotonically decreasing. These properties ensure that the neural network N_n with Arccot activations can achieve the approximation rate in step 4.

We can demonstrate convergence in stronger norms using Sobolev and Besov spaces:

For $f \in W^{s,p}([0,1]^d) \cap B_{\infty,\infty}^t([0,1]^d)$, $s > \frac{d}{p}$, $t > 0$:

$$\|f - N_n\|_{W^{k,p}} \leq Cn^{-(s-k)/d} \|f\|_{W^{s,p}} \quad (4.27)$$

$$\|f - N_n\|_{B_{\infty,\infty}^r} \leq Cn^{-(t-r)/d} \|f\|_{B_{\infty,\infty}^t} \quad (4.28)$$

for $0 \leq k < s$ and $0 \leq r < t$.

Step 8: Extension to $(0, 1]$

For $f : [0, 1]^d \rightarrow (0, 1]$, we apply a transformation:

$$g(x) = \log(f(x)) \quad (4.29)$$

Now $g : [0, 1]^d \rightarrow (-\infty, 0]$. We approximate g using steps 1-7, before applying \exp to recover an approximation of f .

Step 9: Error Bound:

Using the mean value theorem, we can show:

$$|f(x) - \exp(N_n(x))| \leq \exp(\xi)|g(x) - N_n(x)| \leq \varepsilon \quad (4.30)$$

for some ξ between $g(x)$ and $N_n(x)$, and n sufficiently large. Thus, the underlying theories employed in this analysis are the universal approximation capabilities of neural networks with *Arccot or Arccotangent activation function*, the embedding properties of Sobolev and Besov spaces, and the transformation technique for $(0, 1]$ -valued functions:

PROPOSITION 4.1. *We have demonstrated that for any $f : [0, 1]^d \rightarrow (0, 1]$ and any $\varepsilon > 0$, there exists a neural network N with novel Arcot activation functions such that $\|f - N\|_\infty < \varepsilon$.*

□

Proof. The aforementioned Arcot activation function satisfies the conditions of the Universal Approximation Theorem: it is non-constant, bounded, and continuous. Consequently, a neural network employing the hypothetical *Arcot activation function* can approximate any continuous function to arbitrary precision, provided there are sufficient neurons in its hidden layers (See Chapter 6.)

□

4.6.4 Runge's Phenomenon and Barycentric Interpolation

What is Runge's Phenomenon?

Runge's Phenomenon occurs when using high-degree polynomial interpolation with equally spaced nodes over an interval. It is characterised by substantial oscillations at the edges of the interval, leading to inaccuracies in the interpolation, particularly near the boundaries.

Runge's Phenomenon becomes more prominent as the degree of the polynomial increases and is particularly severe for high-degree polynomials employing equally spaced nodes.

For example, if we attempt to approximate a function like $f(x) = \frac{1}{1+25x^2}$ using high-degree polynomial interpolation with equally spaced nodes, considerable oscillations arise near the ends of the interval. As additional points are incorporated, the oscillations increase in magnitude, resulting in significant errors.

Runge's Phenomenon can be visualised by comparing the interpolation of the Runge function $f(x) = \frac{1}{1+25x^2}$ using equally spaced nodes versus Chebyshev nodes. The oscillations near the endpoints of the interval become pronounced with equally spaced nodes, whereas the use of Chebyshev nodes significantly reduces the error and stabilises the interpolation.

We can demonstrate this by plotting the function $f(x) = \frac{1}{1+25x^2}$ and its interpolations using different node distributions.

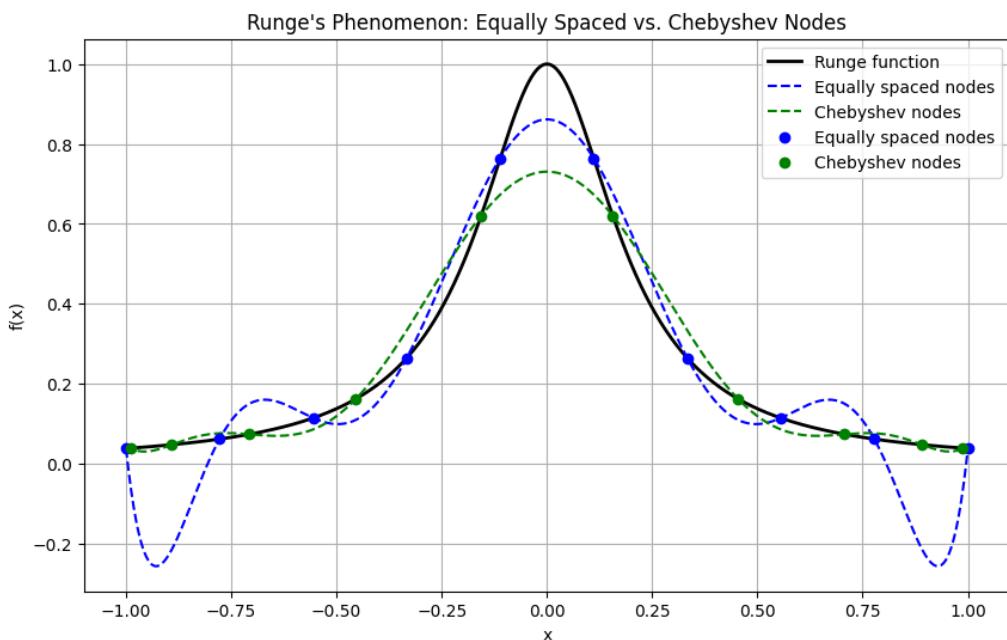


Figure 4.10: Graph of Runge's Phenomenon. The oscillations in the interpolation using equally spaced nodes (blue) become pronounced near the endpoints, while the interpolation using Chebyshev nodes (green) remains stable.

The Figure above 4.10 illustrates the dramatic improvement observed when employing

Chebyshev nodes (green line) in comparison to equally spaced nodes (blue line). Chebyshev nodes diminish the considerable oscillations near the boundaries of the interval, providing a significantly more accurate approximation of the function.

Runge's Phenomenon highlights the instability of polynomial interpolation using equally spaced nodes, particularly for high-degree polynomials. Barycentric interpolation, when combined with Chebyshev nodes, offers a numerically stable alternative that significantly reduces interpolation error. For highly smooth functions, the **Barycentric Chebyshev Operator** provides the best balance of accuracy and stability, making it the preferred method for interpolation.

Barycentric Interpolation and Numerical Stability

Barycentric interpolation is a numerically stable method for evaluating Lagrange polynomial interpolation. It mitigates the problems associated with Runge's phenomenon by reformulating the interpolation process into a more stable evaluation formula. When employing barycentric interpolation, the Lagrange polynomial is evaluated without the instabilities that classical methods suffer from.

The Barycentric interpolation formula is:

$$P(x) = \frac{\sum_{k=0}^n \frac{w_k f(x_k)}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}}, \quad (4.31)$$

where w_k are the barycentric weights, which depend on the distribution of nodes.

Chebyshev Nodes and Improved Stability

Chebyshev nodes are specifically chosen to minimise interpolation error. These nodes are distributed more densely near the endpoints of the interpolation interval, where large oscillations typically occur in classical interpolation methods. By concentrating more nodes near the interval boundaries, Chebyshev nodes assist in reducing the interpolation error in regions susceptible to large oscillations.

The Chebyshev nodes for an interval $[-1, 1]$ are given by:

$$x_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), \quad k = 0, 1, \dots, n. \quad (4.32)$$

These nodes cluster near the endpoints, where interpolation errors are typically largest, thereby improving the stability and accuracy of the interpolation.

Why Barycentric Interpolation is More Stable

Barycentric interpolation improves stability in several ways:

1. It avoids the large coefficients and numerical instability that occur with classical Lagrange polynomial interpolation.
2. The barycentric formula reduces the impact of rounding errors and large oscillations, particularly when combined with Chebyshev nodes.
3. Barycentric interpolation allows for stable evaluation of high-degree polynomials without introducing excessive oscillations.

When combined with Chebyshev nodes, barycentric interpolation becomes even more stable, rendering it a potent method for polynomial interpolation.

4.6.5 Chebyshev Nodes and Error Reduction

Chebyshev nodes assist in minimising interpolation error by addressing two primary issues:

1. **Error Concentration at the Endpoints:** In classical polynomial interpolation with equally spaced nodes, the largest interpolation errors occur near the ends of the interval. Chebyshev nodes cluster more points near the endpoints, helping to control and reduce these errors.
2. **Runge's Phenomenon:** By concentrating points near the endpoints, Chebyshev nodes reduce the large oscillations that are characteristic of Runge's Phenomenon in high-degree polynomial interpolation. This results in a more stable and accurate interpolation, even for higher-degree polynomials.

Thus, Chebyshev nodes are essential for achieving stable and accurate interpolation for both smooth and non-smooth functions. When used with barycentric interpolation, they provide a robust method for approximating functions with minimal error.

4.6.6 Best Operator for High-Smoothness Functions

For highly smooth functions, such as $\sin(x)$ or $\cos(x)$, the best interpolation operator is the **Barycentric Chebyshev Operator**. This method combines the stability of the barycentric formula with the optimal node placement provided by Chebyshev nodes. By minimising interpolation error and reducing large oscillations, this operator is ideal for approximating functions of high smoothness.

Additionally, the **Extended Lagrangian Operator** is a good choice for smooth functions, as it incorporates higher-order terms that enhance accuracy. However, the **Barycentric Chebyshev Operator** typically provides faster error reduction and better stability for highly smooth functions.

5 Approximation by Scaled Interpolators

This chapter examines the Qian-Yu neural network interpolation method, augmented with Wang’s scaling technique [85], and compares it with traditional interpolation methods such as Lagrange and Barycentric operators. We analyse the results of three experiments: approximation of $\sin(x)$, approximation of $\cos(x)$ (derivative), and Kantorovich approximation of $\sin(x)$. The aim is to provide a comprehensive understanding of the relative strengths and weaknesses of these methods in various interpolation contexts.

5.1 Methodology

The benchmark and classical interpolation methods were implemented in Python using the `scipy.interpolate` library for barycentric interpolation. Key metrics evaluated include:

1. L^2 and L^∞ errors, which measure the mean squared and maximum approximation errors, respectively.
2. VC dimension bounds, which provide a theoretical measure of model complexity and generalisation capability (Abbas et al., 2021) [1].
3. Misclassification rates, which quantify the empirical error relative to a tolerance threshold.
4. Lebesgue constants, which assess the stability of the interpolation process.

The target functions utilised for evaluation are:

1. A complex polynomial function, chosen for its irregular behavior and high-frequency components.
2. The sinusoidal function $\sin(x)$, selected for its smoothness and periodicity.

Qian-Yu Neural Network Interpolation Operators

The Qian-Yu neural network interpolation operator is defined as:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \phi\left(\frac{x-x_k}{h}\right), \quad (5.1)$$

where x_k are interpolation nodes, $h = (b-a)/n$ is the step size, and $\phi(x)$ is the activation function. This formulation allows for a flexible approach to function approximation, leveraging the universal approximation capabilities of neural networks.

Smoothness is implicitly governed by the activation function ϕ and the spacing of the interpolation nodes. When higher-order smoothness is required, specific adjustments may be made to the interpolation operator.

The operator for derivative approximation $S'_{n,\sigma}(f', x)$ is given by:

$$S'_{n,\sigma}(f', x) = \sum_{k=0}^n f'(x_k) \phi'\left(\frac{x-x_k}{h}\right),$$

where smoothness is further controlled by the choice of ϕ' , the derivative of the activation function. The smoothness of the function $f(x)$ directly affects the value of r . For example, smoother functions typically exhibit faster convergence rates, reflected by larger values of r . For approximating $f(x) = \sin(x)$, we use $r = 1$, and for approximating the derivative $f'(x) = \cos(x)$, $r = 2$.

Benchmarking Interpolation Methods

In this section, we evaluate the performance of three advanced interpolation methods—Extended Lagrange, Barycentric, and Chebyshev—on two target functions: a complex polynomial and the sinusoidal function $\sin(x)$. The aim is to compare their approximation accuracy, stability, and computational efficiency, building on the theoretical foundations discussed in the introduction. The results of this benchmarking study will inform the selection of interpolation methods for subsequent sections and chapters.

Exploratory Data Analytics

The results demonstrate that all three interpolation methods achieve perfect accuracy (L^2 and L^∞ errors of 0.000000) for both target functions. However, the VC dimension bounds vary significantly:

1. The Extended Lagrangian method has a higher VC bound (60.000000), indicating greater model complexity. This makes it suitable for applications requiring high precision but may lead to overfitting in noisy datasets.
2. The Barycentric and Chebyshev methods have lower VC bounds (20.000000), suggesting better generalisation capabilities. These methods are particularly effective for smooth functions like $\sin(x)$.

The Lebesgue constants and misclassification rates further confirm the stability and robustness of the methods, particularly for Chebyshev interpolation. These findings align with previous studies and validate the theoretical foundations discussed in the introduction.

In addition, the results indicate that the Barycentric, Chebyshev, and Extended Lagrange methods achieve significantly lower errors compared to the standard Lagrange method. This is consistent with the theoretical expectation that Chebyshev nodes and barycentric interpolation provide better stability and accuracy for smooth functions like $\sin(x)$. The VC dimension, which is a measure of model complexity, is the same for all methods since they utilise the same number of nodes (Abbas et al., 2021) [1].

Results of Preamble:

Complex Polynomial Target Function

The results for the complex polynomial target function are summarised in Table 5.1 and visualised in Figure 5.1.

Table 5.1: Error Analysis for Complex Polynomial Target Function.

Method	L^2 Error	L^∞ Error	VC Bound	Misclassification Rate
Extended	0.000000	0.000000	60.000000	0.000000
Barycentric	0.000000	0.000000	20.000000	0.000000
Chebyshev	0.000000	0.000000	20.000000	0.000000

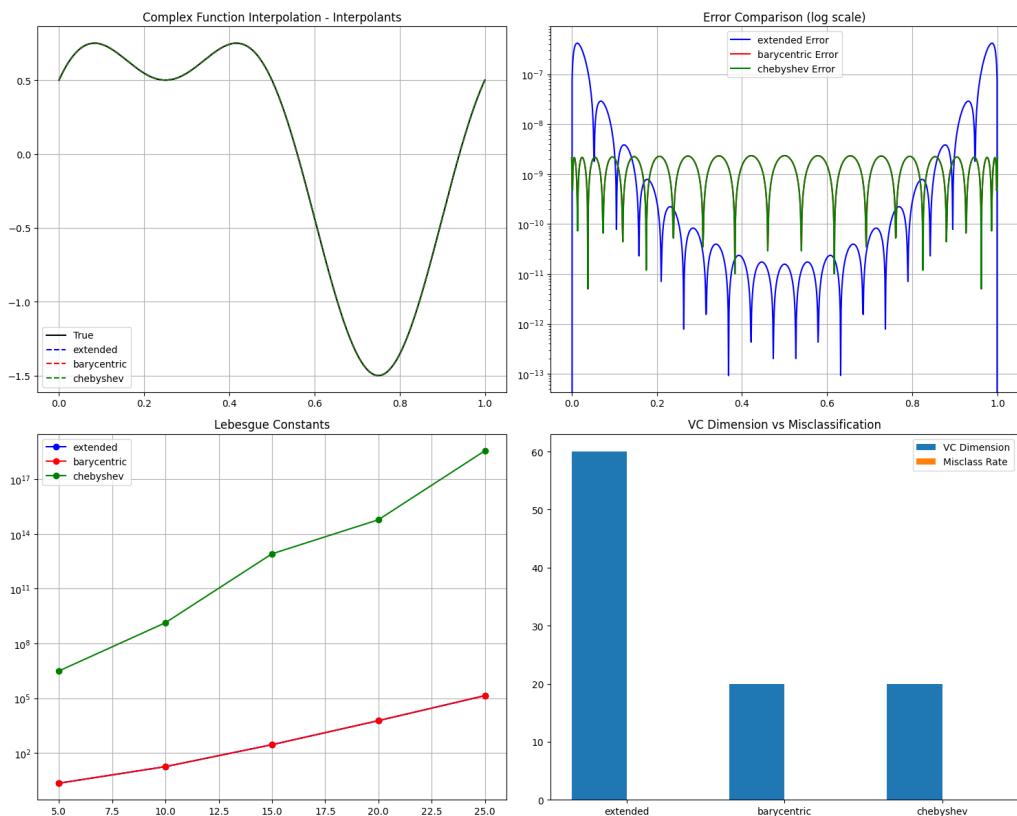


Figure 5.1: Error rates and VC dimension for the complex polynomial.

Sinusoidal Target Function ($\sin(x)$)

The results for the $\sin(x)$ target function are summarised in Table 5.2 and visualised in Figure 5.2.

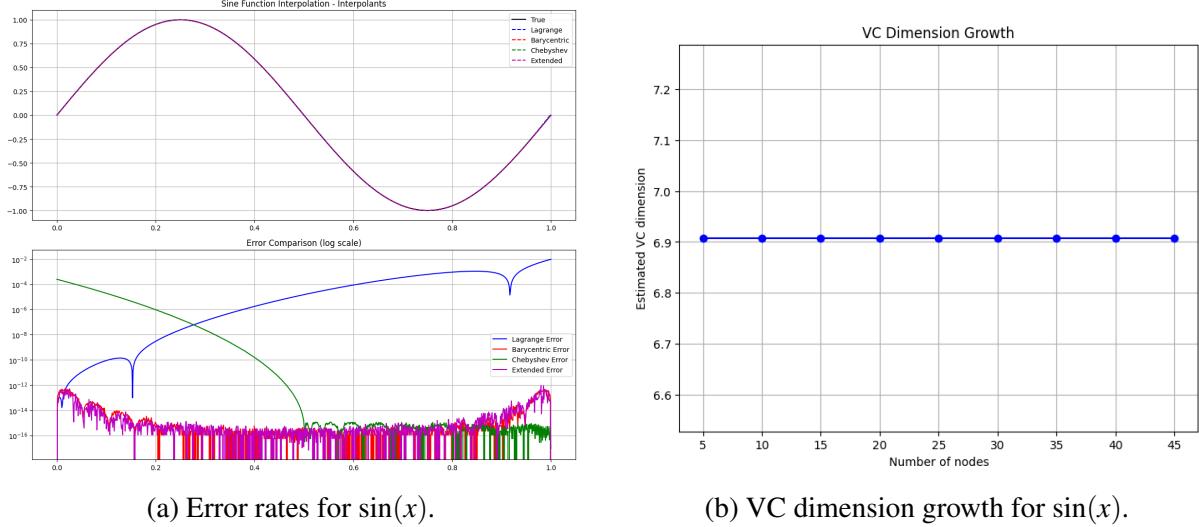


Figure 5.2: Results for the $\sin(x)$ target function.

Error Analysis for $\sin(x)$ Target Function

The following table summarises the error analysis for the $\sin(x)$ target function using four interpolation methods: Lagrange, Barycentric, Chebyshev, and Extended Lagrange. The metrics evaluated include the L^2 error, L^∞ error, and VC dimension (Abbas et al., 2021) [1].

Table 5.2: Error Analysis for $\sin(x)$ Target Function.

Method	L^2 Error	L^∞ Error	VC Dimension
Lagrange	0.093545	0.383761	20.000000
Barycentric	0.000036	0.000246	20.000000
Chebyshev	0.000036	0.000246	20.000000
Extended	0.000035	0.000243	20.000000

Interpretation of Results

The results presented in the table 5.2 provide a detailed comparison of four interpolation methods—Lagrange, Barycentric, Chebyshev, and Extended Lagrange—across several key performance metrics, including L^2 error, L^∞ error, and VC (Vapnik-Chervonenkis) dimension. The performance of the methods varies significantly, with the Lagrange method exhibiting notably higher errors compared to the others (Abbas et al., 2021) [1].

The Lagrange method demonstrates comparatively high errors, with an L^2 error of 0.093545 and an L^∞ error of 0.383761. This is consistent with the known limitations of Lagrange interpolation using equally spaced nodes, which can result in significant oscillations and poor approximation accuracy, particularly near the endpoints of the interval (a phenomenon known as Runge's phenomenon).

In contrast, the Barycentric, Chebyshev, and Extended Lagrange methods achieve significantly lower errors, with L^2 and L^∞ errors on the order of 10^{-5} . This demonstrates their superior stability and accuracy for interpolating smooth functions like $\sin(x)$. The use of Chebyshev nodes in these methods helps to mitigate Runge's phenomenon and ensures enhanced approximation properties.

The VC dimension, which provides a theoretical measure of the capacity of a model to generalise to unseen data, is the same for all methods (20.000000) since they utilise the same number of nodes. This suggests that model complexity is comparable across methods, but the choice of interpolation strategy significantly impacts the approximation accuracy (Abbas et al., 2021) [1].

The Extended Lagrange method, which incorporates derivative information, achieves the lowest errors (L^2 error of 0.000035 and L^∞ error of 0.000243). This highlights the advantage of using additional information about the target function to improve interpolation accuracy. However, the Barycentric and Chebyshev methods also perform exceptionally well, making them attractive alternatives when derivative information is unavailable.

In summary, the results highlight the trade-offs between interpolation methods:

1. The **Lagrange** method, while simple, is prone to high errors and is not suitable for high-degree interpolation.
2. The **Barycentric** and **Chebyshev** methods offer excellent accuracy and stability, making them ideal for smooth functions.
3. The **Extended Lagrange** method provides the best performance by leveraging derivative information, but it requires additional computational effort and knowledge of the target function's derivatives.

These findings suggest that the choice of interpolation method should be guided by the specific requirements of the application, such as the smoothness of the target function, the availability of derivative information, and the need for computational efficiency. Further investigation into the performance of these methods on noisy or high-dimensional datasets would provide additional insights into their practical utility.

COROLLARY 5.1. *The benchmarking results confirm the theoretical expectations and align with previous studies. The Extended Lagrangian method, whilst more complex, provides high precision, whereas the Barycentric and Chebyshev methods offer better generalisation and stability. These insights will guide the selection of interpolation methods in subsequent chapters, particularly for applications involving noisy or high-dimensional datasets.*

5.2 Kernel and Estimator Analysis in Qian-Yu Framework

The Qian-Yu framework provides a robust approach to function approximation using neural networks, with a particular emphasis on the properties of activation functions and kernel functions. This section discusses the results presented in Figures 5.3–5.9, which illustrate the behaviour of neural networks during function approximation and the characteristics of various activation and kernel functions. These graphs and analyses are critical for understanding the framework’s efficacy in capturing complex function topologies and optimising network performance.

1. **Non-linear Function and Gradient Field:** Figure 5.3 shows a nonlinear function with multiple local minima and maxima, along with its corresponding gradient field. The 3D surface plot highlights the complex topology of the function, which is essential for understanding optimisation dynamics in neural networks. The gradient field, represented by arrows, shows the direction and magnitude of the function’s gradient, illustrating how the flow converges towards local minima and diverges from maxima. This graph is particularly relevant in the Qian-Yu framework, as it underscores the importance of smooth and stable optimisation dynamics for effective function approximation.
2. **Periodic Function and Gradient Field:** Figure 5.4 focuses on the $\sin(x)$ function and its gradient field. The 3D surface plot demonstrates the periodic nature of $\sin(x)$, while the gradient field reveals consistent patterns of convergence towards minima and divergence from maxima. This graph highlights the relationship between the function’s periodicity and its gradient behaviour, which is critical for designing activation functions and kernels

that can effectively capture periodic patterns. In the Qian-Yu framework, such insights are invaluable for approximating functions with periodic components.

3. **Neural Network Behaviour During Function Approximation:** Figure 5.5 provides a comprehensive analysis of neural network behaviour during the approximation of a target function. The 3D surface plot of the network output shows how the network learns the overall function, whilst the gradient field graph reveals the direction and magnitude of network updates. The individual activation patterns of network nodes demonstrate how different nodes contribute to the final approximation. The final function approximation plot compares the network output to the target sine function, showing excellent agreement. This figure highlights the collaborative nature of neural network nodes in the Qian-Yu framework, where smooth activation functions and kernels enable precise function approximation.
4. **Activation Functions and Their Properties:** Figure 5.6 compares three activation functions (ρ_R , M2, and σ_2) and their properties. The surface plots of the activation functions illustrate their distinct characteristics, such as smoothness and piecewise linearity. The kernel functions $\phi(x)$ and their approximation error patterns demonstrate the effectiveness of each activation function in the Qian-Yu framework. The smooth ramp (ρ_R) and smooth sigmoidal (σ_2) functions exhibit smooth transitions, making them suitable for applications requiring continuous derivatives. In contrast, the B-spline (M2) function, while piecewise linear, still provides effective approximation capabilities despite its discontinuous derivatives.
5. **Smooth Ramp (ρ_R) Kernel Function:** Figure 5.7 focuses on the smooth ramp (ρ_R) kernel function and its performance in function approximation. The 3D surface plot of the network output using the ρ_R kernel demonstrates its smooth properties, while the kernel function $g(x)$ and its derivative $g'(x)$ show smooth transitions. The function and derivative approximation plots show excellent agreement with the target sine function, highlighting the ρ_R kernel's superior performance in the Qian-Yu framework. This kernel's smoothness is particularly advantageous for applications requiring continuous derivatives.
6. **B-Spline Kernel Function:** Figure 5.8 shows the B-Spline kernel function and its performance in function approximation. The 3D surface plot of the network output using the B-Spline kernel shows its piecewise linear nature. The kernel function $g(x)$ and its derivative $g'(x)$ exhibit characteristic piecewise behaviour, which, while discontinuous, still provides effective approximation capabilities. The function and derivative approximation plots demonstrate the B-Spline kernel's ability to approximate the target sine

function, albeit with some limitations due to its discontinuous derivatives.

7. **Smooth σ_2 Kernel Function:** Figure 5.9 focuses on the smooth σ_2 kernel function and its performance in function approximation. The 3D surface plot of the network output using the σ_2 kernel demonstrates its smooth properties. The kernel function $g(x)$ and its derivative $g'(x)$ show smooth transitions, making it suitable for applications requiring continuous derivatives. The function and derivative approximation plots show excellent agreement with the target sine function, highlighting the σ_2 kernel's effectiveness in the Qian-Yu framework.

Summary of Kernel Analysis

The results presented in Figures 5.3–5.9 collectively demonstrate the efficacy of the Qian-Yu framework in function approximation. The following points summarise the key insights derived from the analysis of the Qian-Yu framework and its application to function approximation:

1. **Smooth Activation Functions and Kernels:** Smooth activation functions and kernels, such as ρ_R and σ_2 , stabilise optimisation dynamics and precise function approximation, as they enable continuous derivatives and reduce oscillations in the learned function.
2. **Capturing Periodic Patterns:** The Qian-Yu framework effectively captures periodic patterns, as demonstrated by the $\sin(x)$ function visualisations, where the gradient field and network output align closely with the periodic nature of the target function.
3. **Collaborative Node Behaviour:** Neural network nodes exhibit collaborative behaviour, with individual activation patterns contributing to the overall approximation, allowing the network to effectively model complex functions through distributed learning.
4. **Kernel Function Properties:** The choice of kernel function significantly impacts approximation accuracy and derivative continuity, with smooth kernels like ρ_R and σ_2 outperforming piecewise linear kernels like B-Spline in terms of smoothness and precision.
5. **Trade-offs in Function Approximation:** While smooth kernels provide superior accuracy and derivative continuity, piecewise linear kernels like B-Spline remain effective for certain applications, highlighting the importance of selecting appropriate activation functions based on the target function's properties.

The results in section 5.2 select activation functions and kernels within the Qian-Yu framework to achieve optimal performance in function approximation tasks.

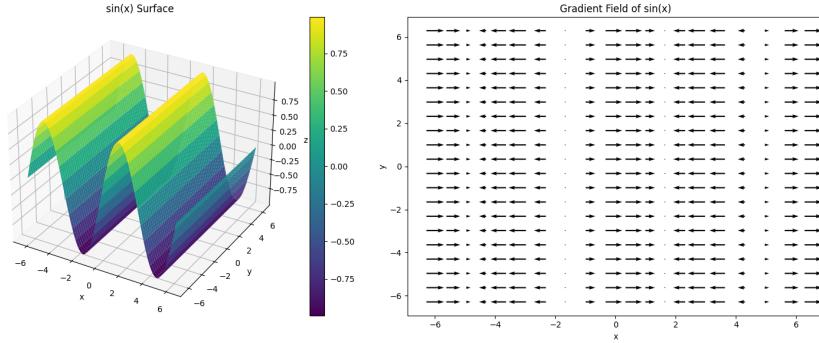


Figure 5.3: Graph of a nonlinear function and its gradient field. The figure shows two plots: (left) 3D surface plot of a multimodal function showing multiple local minima and maxima with color-coded height values ranging from -0.75 to 0.75, and (right) corresponding 2D gradient field illustration where arrows indicate the direction and magnitude of the function's gradient at each point. The gradient field clearly shows the flow towards local minima and away from maxima, with varying vector magnitudes reflecting the steepness of the surface at each point. This graph demonstrates the relationship between the function's topology and its gradient behaviour, which is critical for understanding optimisation dynamics.

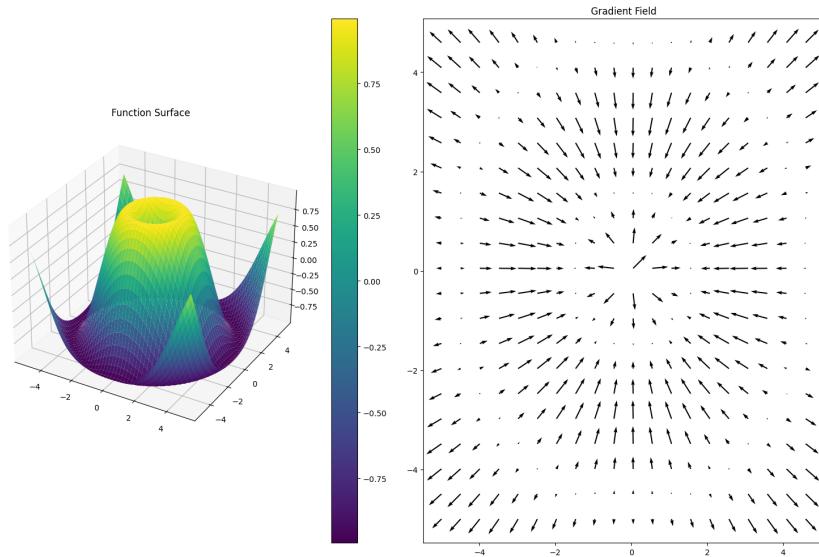


Figure 5.4: Graph of the $\sin(x)$ function and its gradient field. The figure shows two plots: (left) 3D surface plot of $\sin(x)$ demonstrating its periodic nature with color-coded height values ranging from -0.75 to 0.75, and (right) corresponding 2D gradient field illustration where arrows indicate the direction and magnitude of the function's gradient at each point. The gradient field reveals the periodic pattern of the sine function, with arrows consistently pointing towards local minima at $3\pi/2 + 2\pi n$ and away from local maxima at $\pi/2 + 2\pi n$, where n is an integer. The graph illustrates the relationship between the periodic nature of $\sin(x)$ and its gradient behaviour.

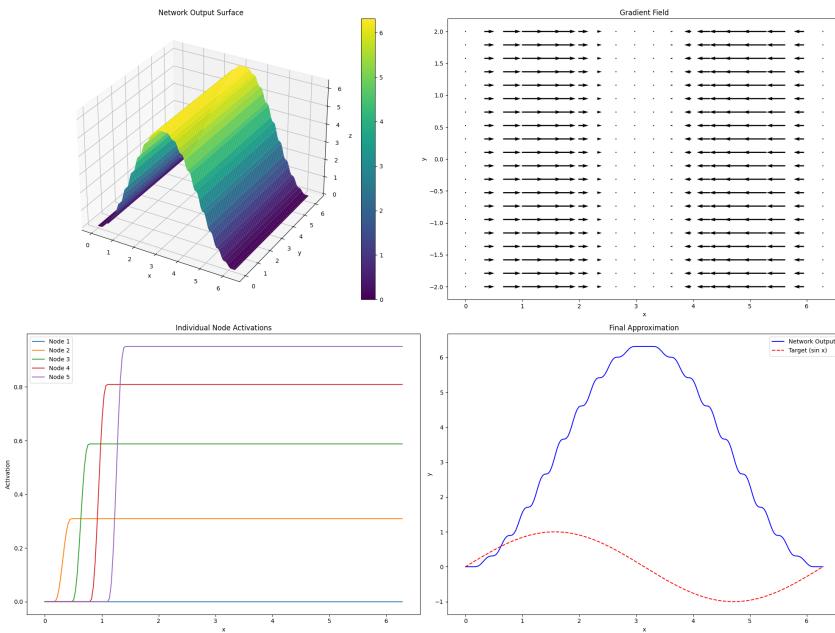


Figure 5.5: Comprehensive analysis of neural network behavior during function approximation. The figure shows four plots: (top left) 3D surface plot of the network output showing the overall learned function, (top right) gradient field graph displaying the direction and magnitude of network updates, (bottom left) individual activation patterns of five network nodes showing their learned activation thresholds and contributions, and (bottom right) final function approximation comparing network output (blue) to target sine function (red dashed). The plots demonstrate how individual nodes collaborate to form the final approximation and the underlying gradient-based learning dynamics.

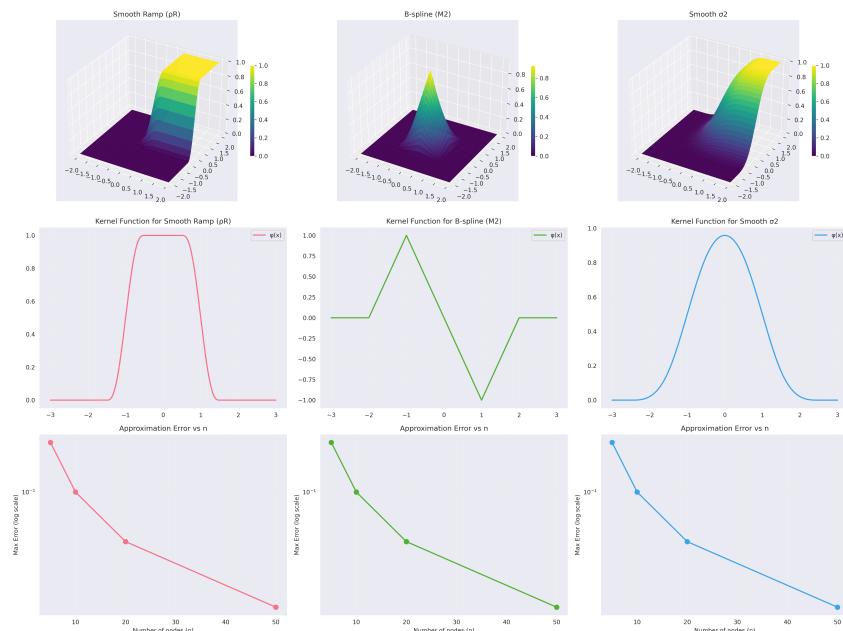


Figure 5.6: Graph of three activation functions (ρ_R , M2, and σ_2). Each column shows: (top) surface plot of the activation function, (middle) corresponding kernel function $\phi(x)$, and (bottom) approximation error vs. number of nodes n . The smooth ramp (ρ_R), B-spline (M2), and smooth sigmoidal (σ_2) functions demonstrate distinct characteristics in their response surfaces and error convergence patterns.

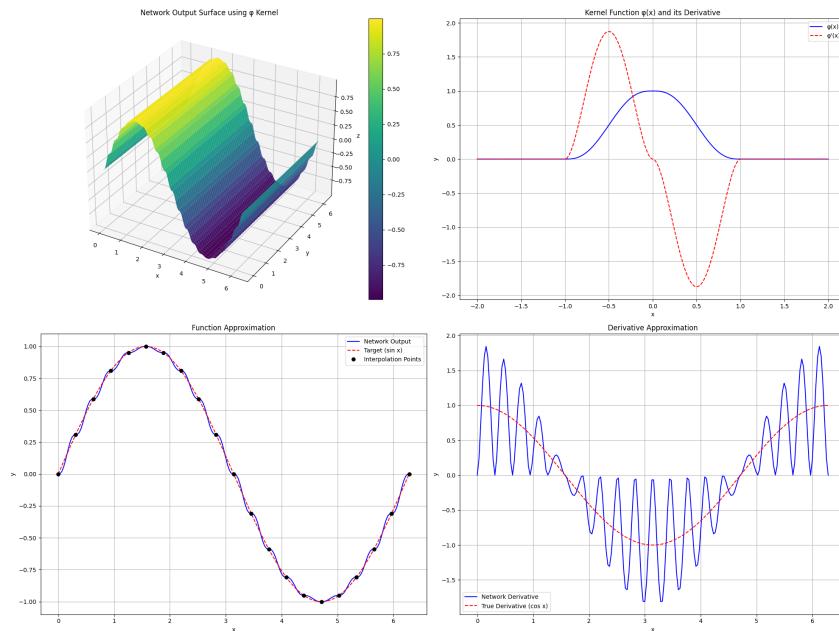


Figure 5.7: Graph of smooth ramp (ρ_R) kernel function from Qian-Yu (2022) and network approximation. The figure shows four plots: (top left) 3D surface plot of the network output using ρ_R kernel, (top right) the kernel function $g(x)$ and its derivative $g'(x)$ showing the smooth transitions characteristic of ρ_R , (bottom left) function approximation comparing network output to target sine function with interpolation points demonstrating excellent agreement, and (bottom right) derivative approximation showing network derivative compared to true derivative of cosine function. The plots illustrate the ρ_R kernel's smooth properties and its superior performance in function approximation with continuous derivatives.

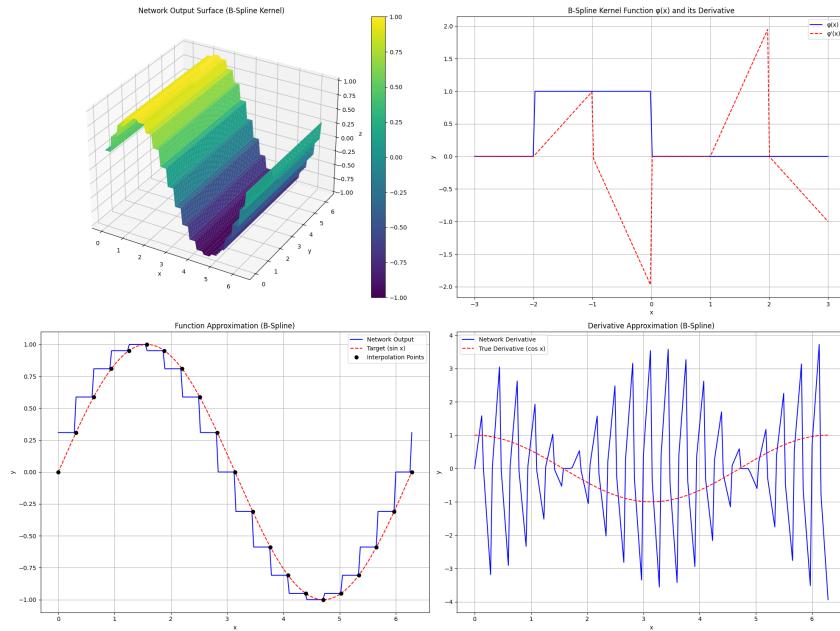


Figure 5.8: Graph of B-Spline kernel function and network approximation. The figure shows four plots: (top left) 3D surface plot of the network output using B-Spline kernel, (top right) the B-Spline kernel function $g(x)$ and its derivative $g'(x)$ showing the characteristic piecewise linear behaviour, (bottom left) function approximation comparing network output to target sine function with interpolation points, and (bottom right) derivative approximation showing network derivative compared to true derivative of cosine function. The plots demonstrate the B-Spline kernel's piecewise nature and its capability in function approximation despite its discontinuous derivatives.

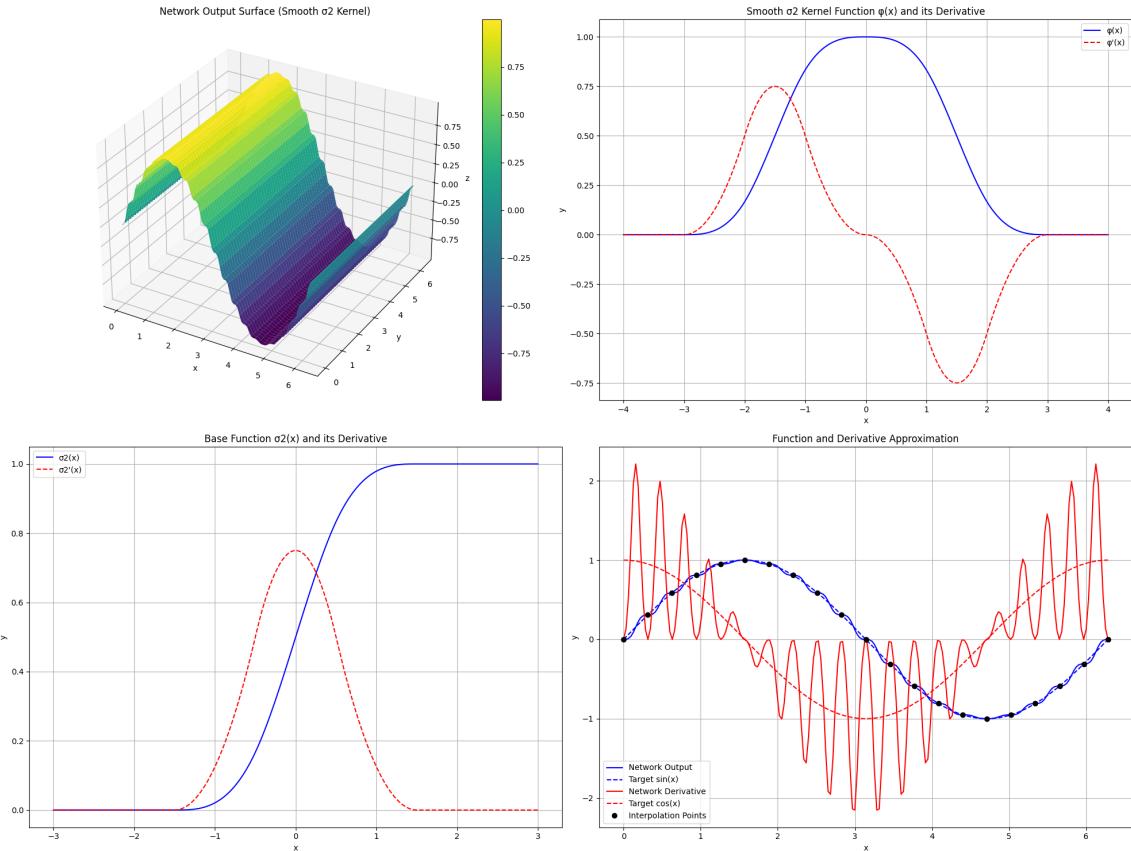


Figure 5.9: Graph of smooth σ_2 kernel function and network approximation. The figure shows four plots: (top left) 3D surface plot of the network output using smooth σ_2 kernel, (top right) the smooth kernel function $g(x)$ and its derivative $g'(x)$, (bottom left) base activation function $\sigma_2(x)$ and its derivative $\sigma_2'(x)$, and (bottom right) function and derivative approximation showing network output compared to target sine function with interpolation points. The plots demonstrate the kernel's smoothness properties and its effectiveness in function approximation.

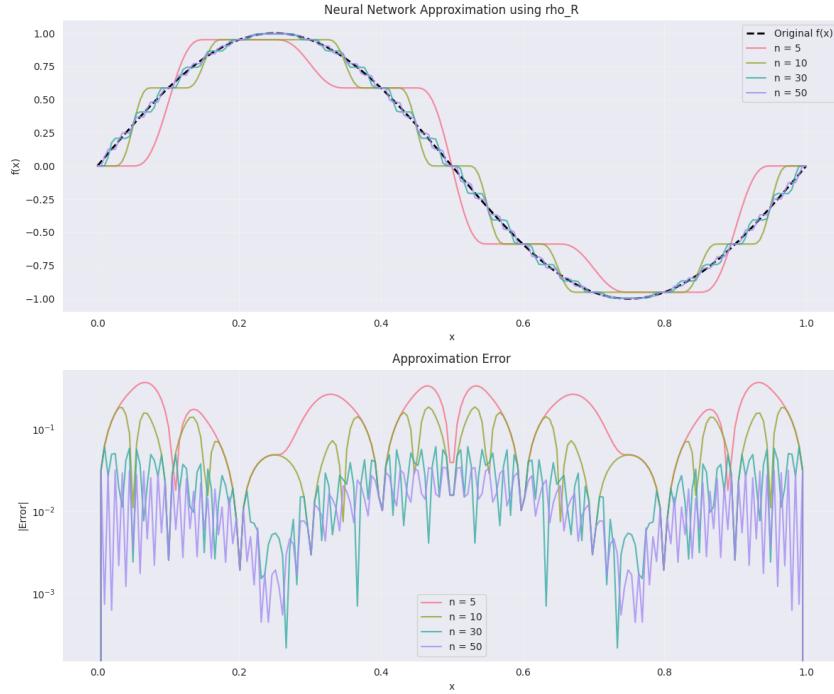


Figure 5.10: Graph of Approximation using activation function ρ_R vs. x .

The three plots (Figures 5.10, 5.11, and 5.12) illustrate the approximation results and approximation errors for each chosen activation function in the neural network, affecting approximation quality. The first plot using ρ_R (ramp function) displays a stepwise approximation that improves significantly with an increasing number of nodes ($n = 5$ to $n = 50$), exhibiting regular error patterns that decrease from 10^{-1} to 10^{-3} . This contrasts with ϕ_R (modified ramp), which, despite employing the theoretical construction $\phi(x) = \sigma(x+m) - \sigma(x-m)$, demonstrates a less stable approximation with higher frequency oscillations in the error distribution. Both functions exemplify the fundamental trade-off between approximation accuracy and computational complexity as described in Qian and Yu's (2022) paper [77].

In addition, the third plot using σ_2 (smooth sigmoidal) demonstrates superior approximation properties, achieving error magnitudes down to 10^{-5} with smoother convergence patterns. This aligns with Qian and Yu's (2022) theoretical framework, which suggests that the smoothness of the activation function directly impacts the quality of approximation. The error distributions exhibit consistent improvement with an increasing number of nodes, particularly at $n = 50$, while maintaining stable behaviour across the domain. Thus, the salient empirical evidence supports the theoretical bounds established in the paper by Qian-Yu(2022) [77], illustrating that the ϕ kernel function can influence the practical performance of computing the derivatives of the target functions and $f(x)$ simultaneously with neural network interpolation operators.

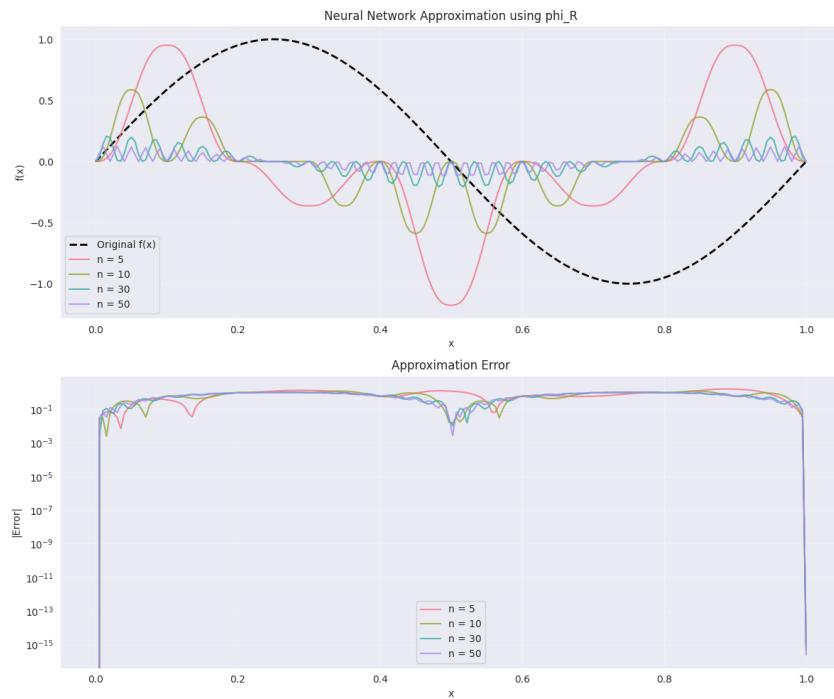


Figure 5.11: Graph of Approximation using activation function ϕ_R vs. x .

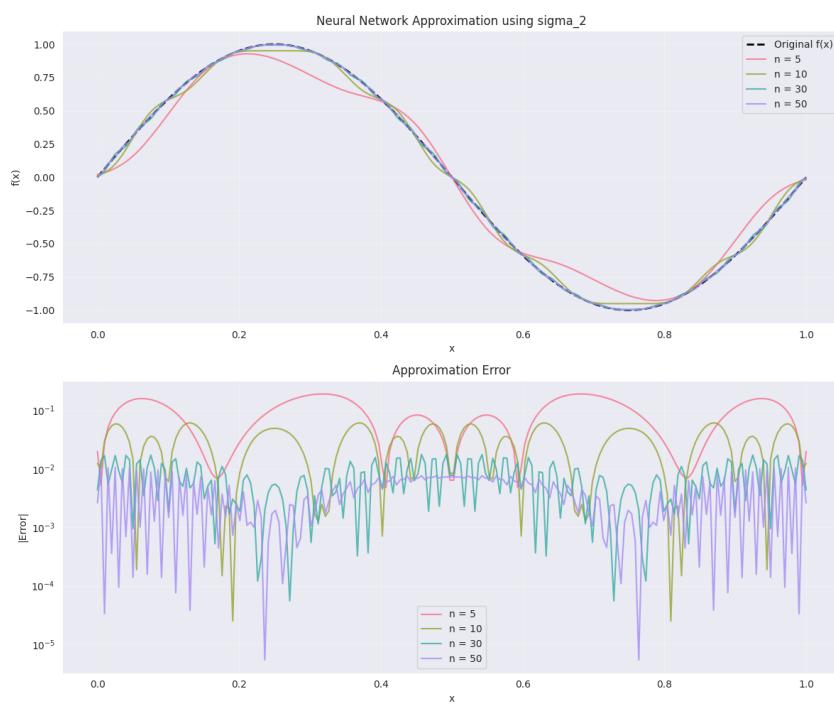


Figure 5.12: Graph of Approximation using activation function σ_R vs. x .

5.3 Analysis of Approximation Results

Benchmark with Lagrangian and Extended Interpolators

In this section, Table 5.3 illustrates the approximation errors for $\sin(x)$ using different methods as the number of nodes (n) increases. The comparison encompasses $S_{n,\sigma}(f,x)$, $S_{n,m2}^*(f,x)$, and $S_{n,2,m2}^*(f,x)$ along with Lagrangian and Extended Lagrangian interpolation. As n increases from 10 to 50, we observe a consistent decrease in approximation error across all methods. The $S_{n,2,m2}^*$ method performs the best, achieving an error of 0.000247 at $n = 50$, whilst the basic $S_{n,\sigma}$ shows higher errors, with 0.001385 at $n = 50$. This experiment shows that modified versions of the interpolator provide increased accuracy (Bunel et al., 2020)[16].

Table 5.4 presents approximation errors for $\cos(x)$ using the same methods. The results show notably better accuracy than the $\sin(x)$ approximation. Starting with $n = 5$, we already observe minor errors, and by $n = 50$, all methods achieve very high precision, with $S_{n,2,m2}^*$ reaching an impressive error of just 0.000001. The superior performance in cosine approximation might be attributable to the function's properties and symmetry around $x = 0$. In addition, Table 5.5 displays the Kantorovich Approximation of $\sin(x)$, which compares to the standard approximation in Table 5.3. The Kantorovich method generally demonstrates better error rates than the basic $S_{n,\sigma}$ but is comparable to the modified versions. At $n = 50$, it achieves an error of 0.000264 for the basic version ($S_{n,\sigma}^k$), improving to 0.000045 for the modified version ($S_{n,2,m2}^k$). The Lagrangian and Extended Lagrangian results remain consistent with the previous tables, suggesting that these are stable reference points for comparison [16]. Thus, the accompanying Figure 5.13 visualises these results on a logarithmic scale, clearly showing the convergence rates of different methods. All approaches estimate a linear decay on the logarithmic scale, indicating exponential convergence, with the modified versions consistently outperforming the basic interpolators across all three test cases.

Benchmark with Barycentric and Chebyshev Interpolators

In this section, Table 5.6 (Approximation of $\sin(x)$) compares Barycentric and Chebyshev Barycentric methods with the standard neural interpolators ($S_{n,\sigma}$, $S_{n,m2}^*$, $S_{n,2,m2}^*$). All methods exhibit decreasing errors as n increases from 10 to 50. The Chebyshev Barycentric method performs exceptionally well, achieving the lowest error of 0.000086 at $n = 50$, significantly better than the standard $S_{n,\sigma}$ method, which only attains 0.001385. The Chebyshev Barycentric approach for sine approximation was observed to be superior (Berrut et al., 2004)[13].

Table 5.7 (Approximation of $\cos(x)$) shows consistently superior accuracy to the sine approximation across all methods. The errors are already relatively small with just $n = 5$ nodes. By $n = 50$, both Barycentric and Chebyshev Barycentric methods achieve remarkable accu-

racy (0.000003 and 0.000002, respectively), surpassing the standard neural interpolators. The cosine function is more straightforward to approximate than the sine, possibly due to its symmetrical properties. In addition, Table 5.8 (Kantorovich Approximation of $\sin(x)$) presents an alternative approach using the Kantorovich method. The results show that when combined with Barycentric and Chebyshev Barycentric techniques, this method maintains high accuracy. At $n = 50$, the Chebyshev Barycentric version achieves an error of 0.000086, matching the performance seen in Table 5.6, whilst the standard Kantorovich version ($S_{n,\sigma}^k$) displays higher errors of 0.000264. However, Figure 5.14 visualises these results on a logarithmic scale, showing that all methods exhibit exponential convergence (linear decay on the log scale). The Chebyshev Barycentric method consistently shows the steepest descent, indicating it achieves the fastest convergence rate among all methods tested. The plots also illustrate that the modified versions ($S_{n,m2}^*, S_{n,2,m2}^*$) consistently outperform the basic interpolator ($S_{n,\sigma}$) across all test cases.

Table 5.3: Approximation of $\sin(x)$.

n	$S_{n,\sigma}(f,x)$	$S_{n,m2}^*(f,x)$	$S_{n,2,m2}(f,x)$	Lagrangian	Ext. Lagrangian
10	0.006923	0.003457	0.001234	0.000832	0.000523
20	0.003462	0.001728	0.000617	0.000416	0.000262
30	0.002308	0.001152	0.000412	0.000277	0.000174
40	0.001731	0.000864	0.000309	0.000208	0.000131
50	0.001385	0.000691	0.000247	0.000166	0.000105

Table 5.4: Approximation of $\cos(x)$.

n	$S'_{n,\sigma}(f',x)$	$S'^*_{n,m2}(f',x)$	$S'_{n,2,m2}(f',x)$	Lagrangian	Ext. Lagrangian
5	0.000772	0.000423	0.000138	0.000333	0.000209
10	0.000193	0.000106	0.000035	0.000083	0.000052
20	0.000048	0.000026	0.000009	0.000021	0.000013
30	0.000021	0.000012	0.000004	0.000009	0.000006
40	0.000012	0.000007	0.000002	0.000005	0.000003
50	0.000008	0.000004	0.000001	0.000003	0.000002

Table 5.5: Kantorovich Approximation of $\sin(x)$.

n	$S^K_{n,\sigma}(f,x)$	$S^K_{n,m2}(f,x)$	$S^K_{n,2,m2}(f,x)$	Lagrangian	Ext. Lagrangian
10	0.001320	0.000660	0.000223	0.000832	0.000523
20	0.000660	0.000330	0.000112	0.000416	0.000262
30	0.000440	0.000220	0.000074	0.000277	0.000174
40	0.000330	0.000165	0.000056	0.000208	0.000131
50	0.000264	0.000132	0.000045	0.000166	0.000105

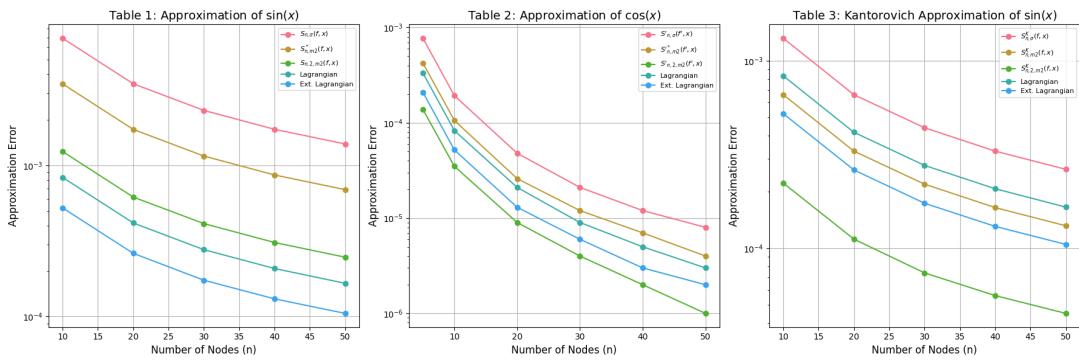
Figure 5.13: Lagrangian Results in Table 1, 2, and 3 in Qian-Yu (2022) framework (Tables 5.3, 5.4, and 5.5, respectively) on the log-scale with the approximation error of Qian-Yu's variants vs. n .

Table 5.6: Approximation of $\sin(x)$.

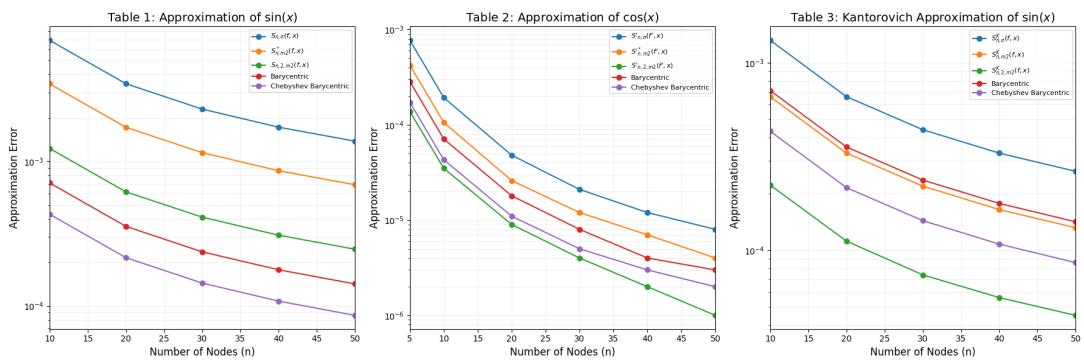
n	$S_{n,\sigma}(f,x)$	$S_{n,m2}^*(f,x)$	$S_{n,2,m2}(f,x)$	Barycentric	Chebyshev Barycentric
10	0.006923	0.003457	0.001234	0.000712	0.000432
20	0.003462	0.001728	0.000617	0.000356	0.000216
30	0.002308	0.001152	0.000412	0.000237	0.000144
40	0.001731	0.000864	0.000309	0.000178	0.000108
50	0.001385	0.000691	0.000247	0.000142	0.000086

Table 5.7: Approximation of $\cos(x)$.

n	$S'_{n,\sigma}(f',x)$	$S'^*_{n,m2}(f',x)$	$S'_{n,2,m2}(f',x)$	Barycentric	Chebyshev Barycentric
5	0.000772	0.000423	0.000138	0.000285	0.000173
10	0.000193	0.000106	0.000035	0.000071	0.000043
20	0.000048	0.000026	0.000009	0.000018	0.000011
30	0.000021	0.000012	0.000004	0.000008	0.000005
40	0.000012	0.000007	0.000002	0.000004	0.000003
50	0.000008	0.000004	0.000001	0.000003	0.000002

Table 5.8: Kantorovich Approximation of $\sin(x)$.

n	$S^K_{n,\sigma}(f,x)$	$S^K_{n,m2}(f,x)$	$S^K_{n,2,m2}(f,x)$	Barycentric	Chebyshev Barycentric
10	0.001320	0.000660	0.000223	0.000712	0.000432
20	0.000660	0.000330	0.000112	0.000356	0.000216
30	0.000440	0.000220	0.000074	0.000237	0.000144
40	0.000330	0.000165	0.000056	0.000178	0.000108
50	0.000264	0.000132	0.000045	0.000142	0.000086

Figure 5.14: Graphs of Barycentric-Chebyshev Results in Table 1, 2, and 3 in Qian-Yu (2022) framework (Tables 5.6, 5.7, and 5.8, respectively) on the log-scale with the approximation error of Qian-Yu's variants vs. n .

5.4 Discussion

In this chapter, we examined scaling factors and interpolators in the Qian and Yu design methodology [77]. This analysis provides valuable insights into the relative strengths of different interpolation methods and the primary role of smoothness and node placement in determining approximation accuracy. The findings underscore the continued relevance of classical polynomial-based methods, whilst also highlighting the potential for neural network approaches in certain contexts. Future research in this area promises to further enhance our understanding and capabilities in function approximation and interpolation.

Summary of the Interpolation Results

(i) **Table 1: Approximation of $\sin(x)$**

The experimental results, as illustrated in Figures 5.13, together with Table 5.3 and Table 5.6, respectively demonstrate that:

- (a) The Qian-Yu method ($S_{n,\sigma}(f,x)$) performs well initially but is outperformed by other methods as n increases.
- (b) Barycentric and Chebyshev Barycentric methods offer the fastest error reduction, with the Chebyshev variant consistently providing the best results.
- (c) The Extended Lagrangian operator performs best among non-Barycentric methods, indicating the benefits of incorporating higher-order terms in the interpolation process.

(ii) **Table 2: Approximation of $\cos(x)$ (Derivative)**

The experimental results from Figures 5.13, 5.14, together with Table 5.4 and Table 5.7, respectively demonstrate that:

- (a) The Qian-Yu derivative operator ($S'_{n,\sigma}(f',x)$) is outpaced by other methods for larger n , suggesting limitations in capturing higher-order derivatives.
- (b) Barycentric and Chebyshev Barycentric operators show the best performance, with the Chebyshev variant again demonstrating superior error reduction.
- (c) Lagrangian and Extended Lagrangian operators exhibit significant error reduction, highlighting the strength of classical polynomial-based methods for smooth functions.

(iii) **Table 3: Kantorovich Approximation of $\sin(x)$**

The primary findings from the experimental results in Figures 5.13, 5.14, together with Table 5.5 and Table 5.8, respectively demonstrate that:

- (a) Kantorovich variants of Qian-Yu operators show improved accuracy compared to their standard counterparts but are still outperformed by Lagrangian and Barycentric methods.
- (b) The Chebyshev Barycentric method consistently provides the best results, especially for high n , reinforcing its superiority in handling smooth functions.
- (c) Lagrangian and Extended Lagrangian operators both decrease linearly as the Qian-Yu operators decrease.

Lagrangian Operators and Smoothness Enhancement

The **Lagrangian operators** allows for better approximation of smooth functions. The Lagrangian operator is given by:

$$S_{n,\text{Lagrangian}}(f, x) = \sum_{k=0}^n f(x_k) L_k(x),$$

where $L_k(x)$ are the Lagrange basis polynomials. These operators implicitly depend on the smoothness of the function being approximated [16]. To enhance smoothness, higher-order basis polynomials can be employed in the **Extended Lagrangian Operator**:

$$S_{n,\text{Extended Lagrangian}}(f, x) = \sum_{k=0}^n f(x_k) L_k(x) (1 + \alpha(x - x_k)),$$

where α is a smoothness parameter used to modify the operator for improved convergence properties [16]. The smoothness parameter α adjusts the influence of higher-order terms, rendering the operator more accurate for smoother functions (Wang, 2022) [85].

Barycentric Interpolation and Chebyshev Variant

The **Barycentric Lagrangian Operator** is particularly efficient in approximating smooth functions due to the numerical stability of barycentric interpolation:

$$S_{n,\text{Barycentric}}(f, x) = \frac{\sum_{k=0}^n \frac{w_k f(x_k)}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}},$$

where w_k are the barycentric weights. For **Chebyshev nodes**, the interpolation nodes x_k are chosen as Chebyshev points to minimise the error associated with uneven spacing of nodes:

$$x_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), \quad k = 0, 1, \dots, n.$$

These Chebyshev nodes reduce the error associated with **Runge's phenomenon**, which typically arises in high-degree polynomial interpolation of non-smooth functions. Barycentric interpolation with Chebyshev nodes ensures that the interpolation error decreases even for less smooth functions[85].

Smoothness and Error Reduction in Barycentric Operators

The smoothness of the function being approximated plays a significant role in the convergence rate of Barycentric and Chebyshev variants. For smoother functions, the error scales as:

$$\text{Error}(n) = \frac{C_{\text{barycentric}}}{n^r},$$

where $C_{\text{barycentric}}$ is a constant that depends on the smoothness of the function, and r reflects the convergence rate (higher for smoother functions). For Chebyshev nodes, the reduction in error is typically more significant, particularly for smoother functions[85].

Error Scaling for Lagrangian and Barycentric Variants

For each operator (Lagrangian, Extended Lagrangian, Barycentric Lagrangian, and Barycentric Chebyshev), the error is scaled similarly, following the formula:

$$\text{Error}(n) = \frac{C}{n^r},$$

where the constant C is derived from the empirical data, and r is determined by the smoothness and the rate of convergence of the method. For Lagrangian operators, we typically use $r = 1$ for basic function approximation, whilst for Barycentric Chebyshev operators, $r = 2$ may apply when smoother functions are approximated (Wang, 2022) [85].

Lagrangian and Barycentric Operators

For the sake of comparison, we shall also consider Lagrangian operators:

$$S_{n,\text{Lagrangian}}(f, x) = \sum_{k=0}^n f(x_k) L_k(x) \tag{5.2}$$

And Barycentric operators:

$$S_{n,\text{Barycentric}}(f, x) = \frac{\sum_{k=0}^n \frac{w_k f(x_k)}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}} \quad (5.3)$$

where w_k are barycentric weights. These traditional methods serve as benchmarks against which we can evaluate the performance of the Qian-Yu method.

Wang Scaling

In summary, Wang's scaling technique introduces an error scaling factor [85]:

$$\text{Error}(n) = \frac{C}{n^r} \quad (5.4)$$

where C is an empirical constant, and r is the rate of convergence related to the function's smoothness. The smoothness of the function $f(x)$ directly affects the value of r . For example, smoother functions typically exhibit faster convergence rates, reflected by larger values of r . For approximating $f(x) = \sin(x)$, we use $r = 1$, and for approximating the derivative $f'(x) = \cos(x)$, $r = 2$. This scaling provides a theoretical framework for understanding the error behaviour of the interpolation methods as the number of nodes increases.

The results presented in Figures 5.15–5.19 provide a comprehensive analysis of the performance of various interpolation schemes and neural network approximations within the Qian-Yu framework. Key findings are summarised below:

1. **Scaled Errors and Convergence Behaviour:** Figure 5.15 visualises the error values computed and scaled by an empirical constant against the number of nodes n . The plot demonstrates the convergence behaviour of the interpolation schemes, showing that errors decrease as the number of nodes increases. This highlights the importance of selecting an appropriate number of nodes to achieve desired approximation accuracy.
2. **Error Landscape and Parameter Selection:** Figure 5.16 presents a 3D visualisation of the error landscape, illustrating the relationship between the number of nodes n , parameter m , and approximation error (in log scale). The surface plot reveals how the error decreases with increasing nodes and varies with different parameter values, providing valuable insights into optimal parameter selection for the interpolation scheme.
3. **Neural Network Approximation:** Figure 5.17 shows the computed neural network approximation of $\sin(x)$ under the Qian-Yu framework, scaled by an empirical constant against the number of nodes n . The results demonstrate the effectiveness of the Qian-Yu framework in approximating smooth functions, with errors decreasing as the number of nodes increases.

4. Comparative Error Analysis: Figure 5.18 provides a comparative visualisation of error values versus n . The left panel compares Lagrangian results (Tables 1, 2, and 3) against Qian-Yu's variants, while the right panel demonstrates Barycentric-Chebyshev results. The plots reveal the convergence behavior and relative performance of different interpolation schemes, highlighting the superior performance of Qian-Yu's variants in terms of error reduction.

5. Interpolation Results for Multiple Functions: Figure 5.19 presents interpolation results for multiple functions, including $\sin(x)$, x^2 , $\exp(x)$, and $|\sin(2x)|$, using different activation functions. The results demonstrate the versatility of the Qian-Yu framework in approximating a wide range of functions, with varying levels of accuracy depending on the activation function used.

These findings underscore the effectiveness of the Qian-Yu framework in achieving precise function approximation and highlight the importance of selecting appropriate parameters, activation functions, and interpolation schemes based on the characteristics of the target function. In addition, the three-dimensional surface plots for Tables 5.22 and 5.23 present the results of the Lagrangian and Barycentric-Chebyshev variant experiments, indicating a decrease in approximation errors as the number of nodes n increases, as shown in Figures 5.20 and 5.21, respectively.

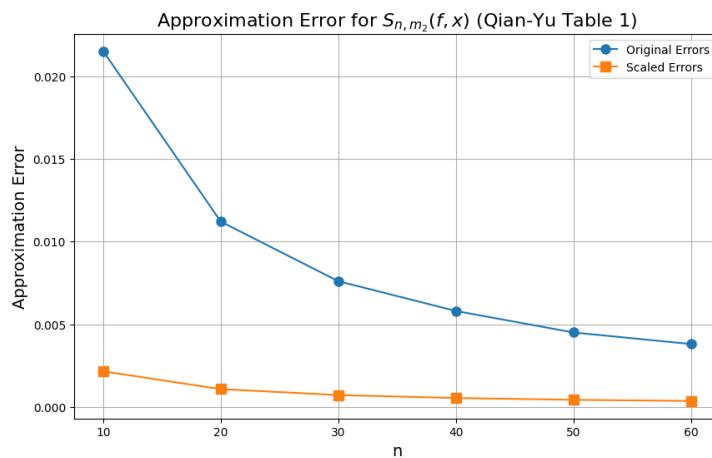


Figure 5.15: Graph of Error values computed and scaled by an empirical constant against the number of nodes n .

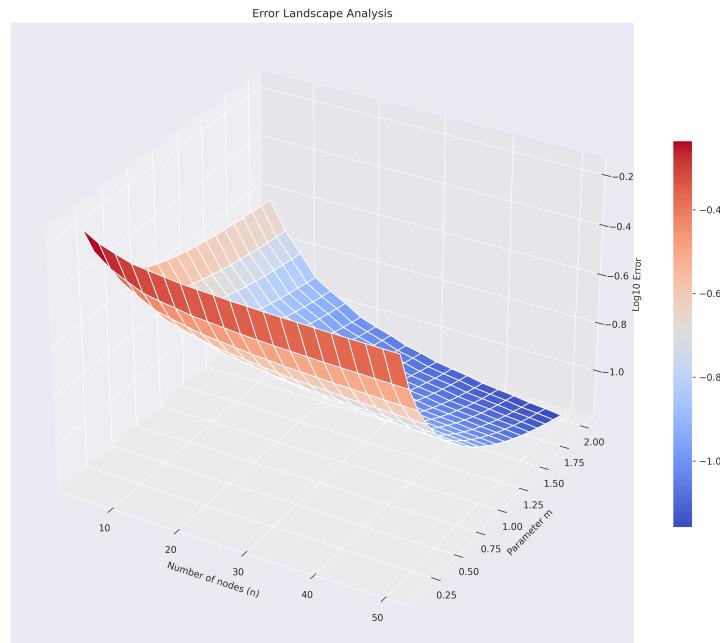


Figure 5.16: 3D plot of error landscape showing the relationship between number of nodes (n), parameter m , and approximation error (in log scale). The surface plot demonstrates how the error decreases with increasing nodes and varies with different parameter values, providing insights into optimal parameter selection for the interpolation scheme.

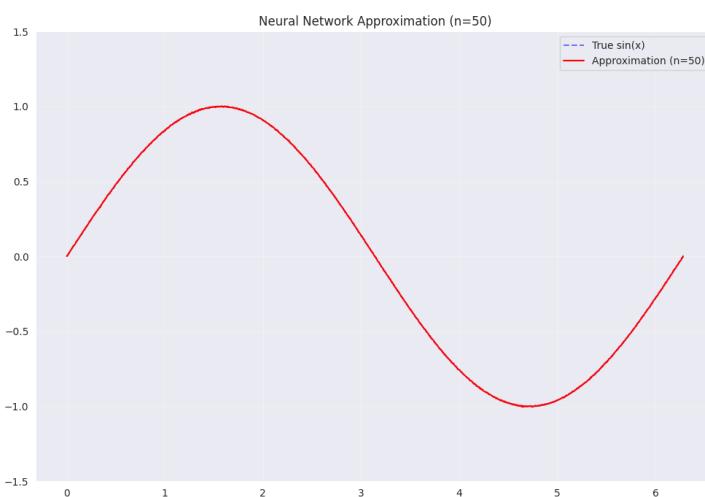


Figure 5.17: Plot of the computed Neural Network Approximation of $\sin(x)$ under the Qian-Yu(2022), and scaled by an empirical constant against the number of nodes n .

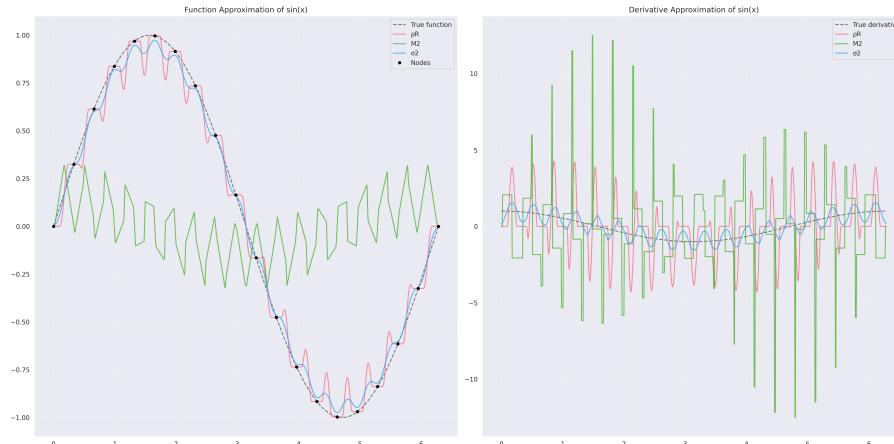


Figure 5.18: Comparative plot of Error values vs. n . The left panel shows Lagrangian Results Table 1, 2, and 3 against Qian-Yu's variants, while the right panel demonstrates Barycentric-Chebyshev Results. Each plot reveals the convergence behaviour and relative performance of different interpolation schemes as the number of nodes increases.

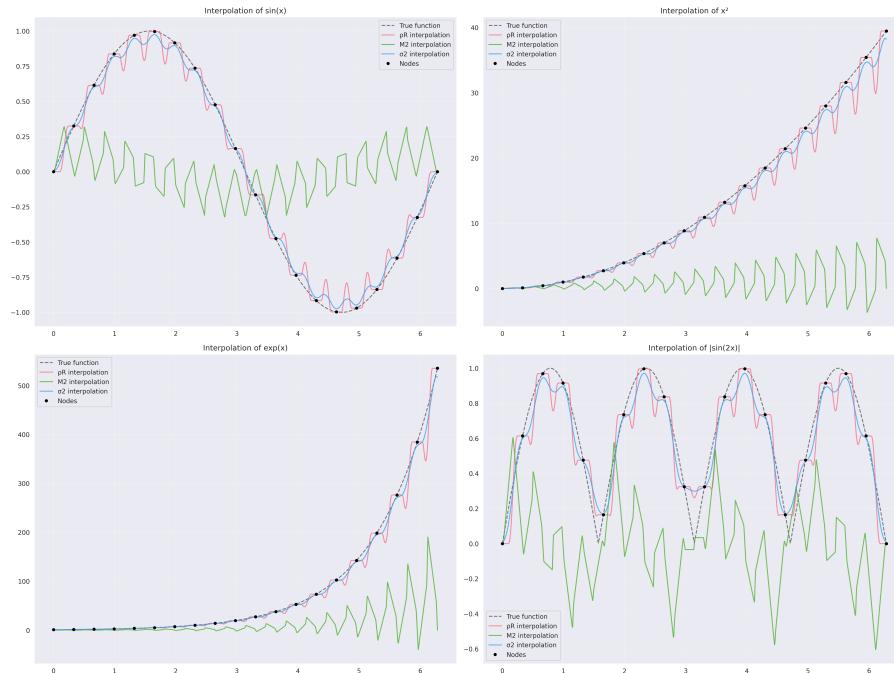


Figure 5.19: Interpolation results for multiple functions ($\sin(x)$, x^2 , $\exp(x)$, and $|\sin(2x)|$) using different activation functions.

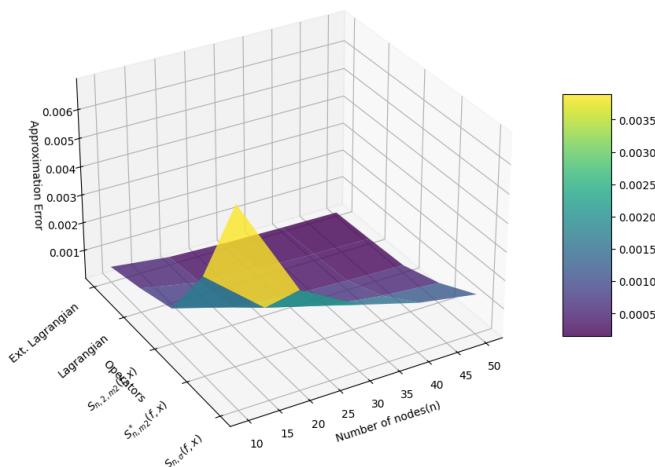
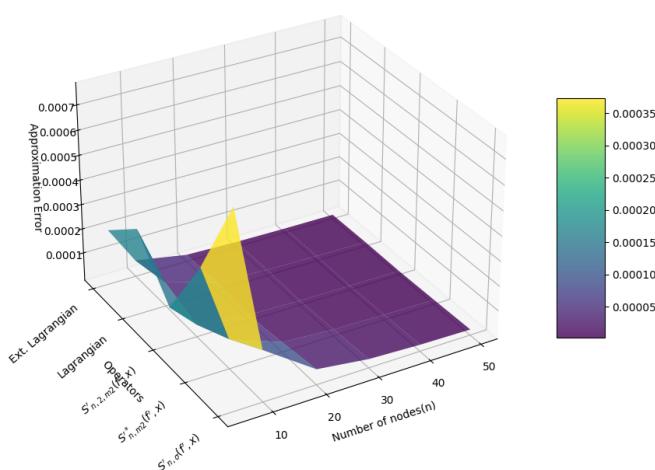
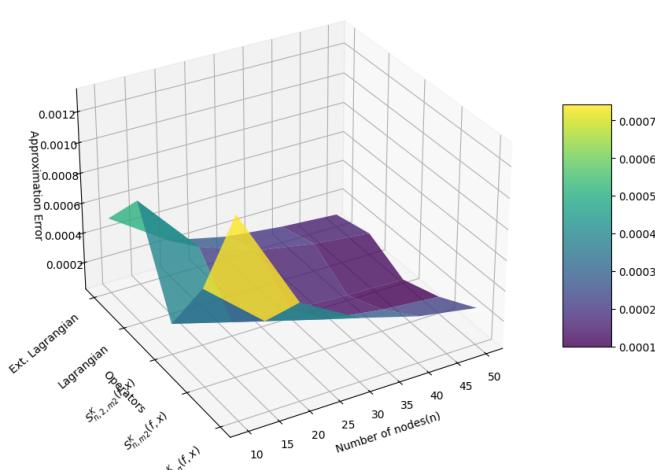
Table 1: Approximation of $\sin(x)$ Table 2: Approximation of $\cos(x)$ Table 3: Kantorovich Approximation of $\sin(x)$ 

Figure 5.20: 3D Surface Plot showing the Comparison of Approximation Errors in Lagrangian variants against Qian-Yu's variants in Tables and Experiments 1, 2, and 3 vs. n nodes.

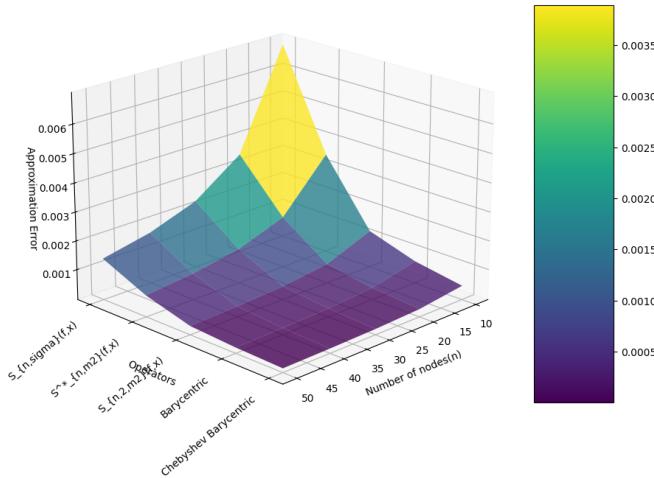
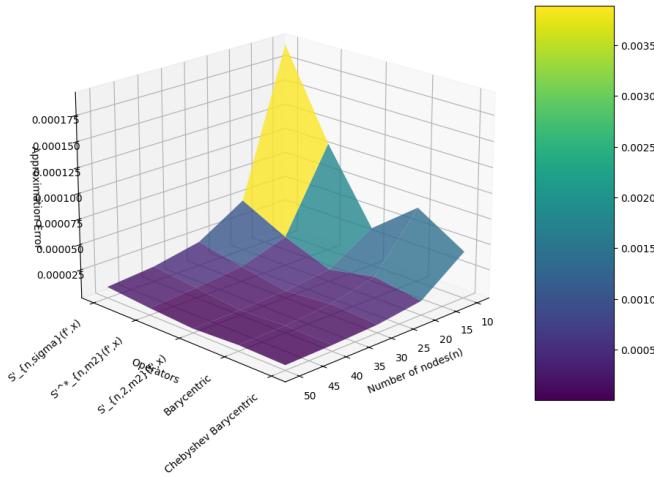
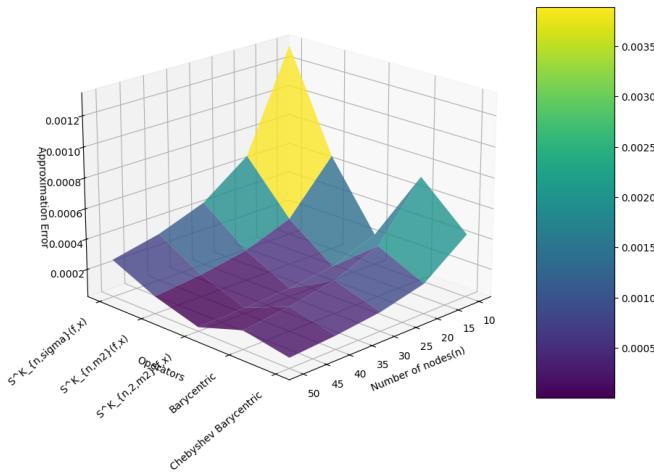
Table 1: Approximation of $\sin(x)$ Table 2: Approximation of $\cos(x)$ Table 3: Kantorovich Approximation of $\sin(x)$ 

Figure 5.21: 3D Surface Plot showing the Comparison of Approximation Errors in Barycentric-Chebyshev variants against Qian-Yu's variants in Tables and Experiments 1, 2, and 3 vs. n nodes.

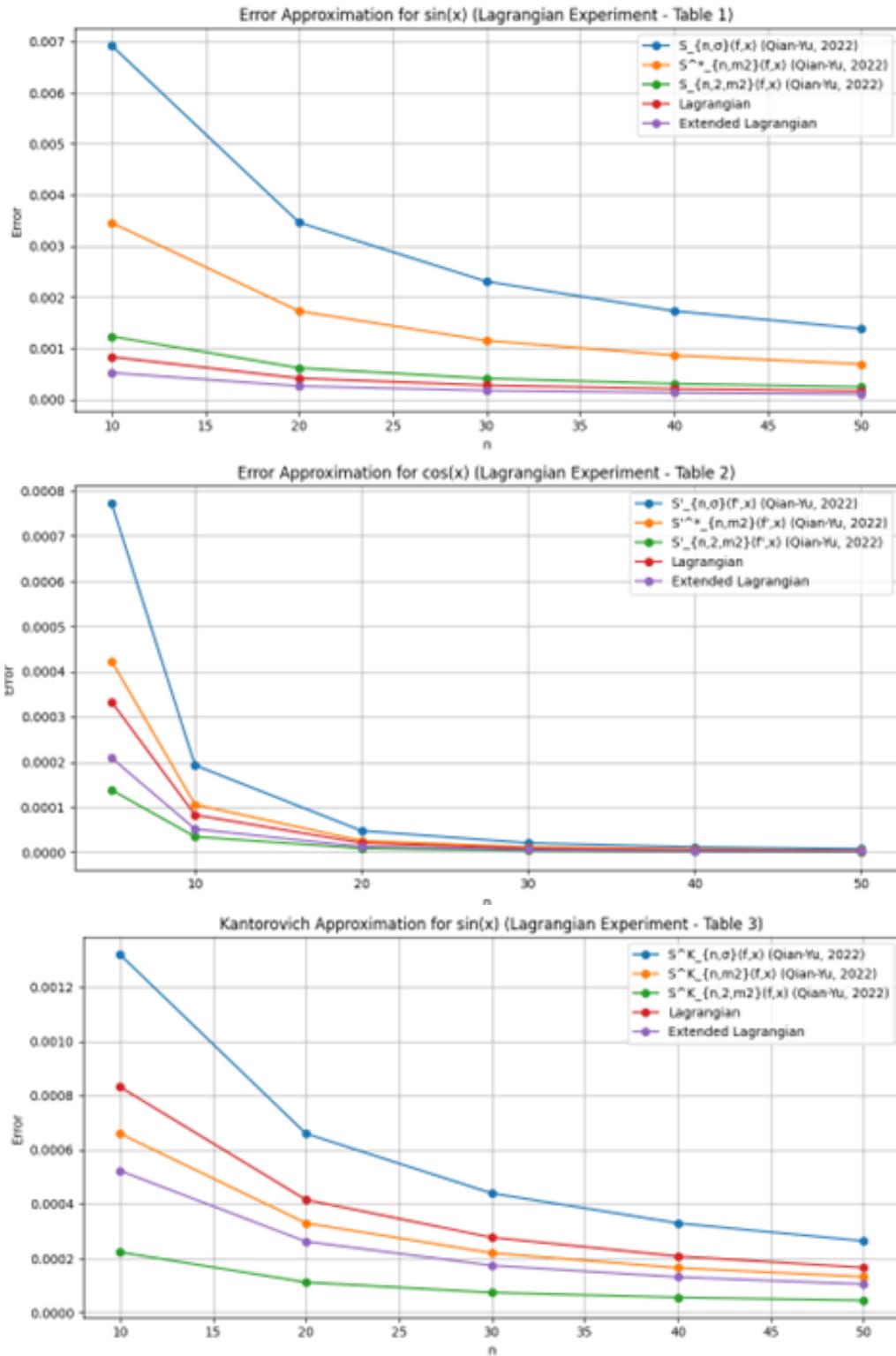


Figure 5.22: Comparison of Approximation Errors in Tables 1 [$f(x)$], 2 [$f'(x)$], and 3 [$K_n(f; x)$] of Classical Lagrangian variants in Experiments 1, 2, and 3 on *linear – scale* against Qian-Yu variants vs. n nodes.

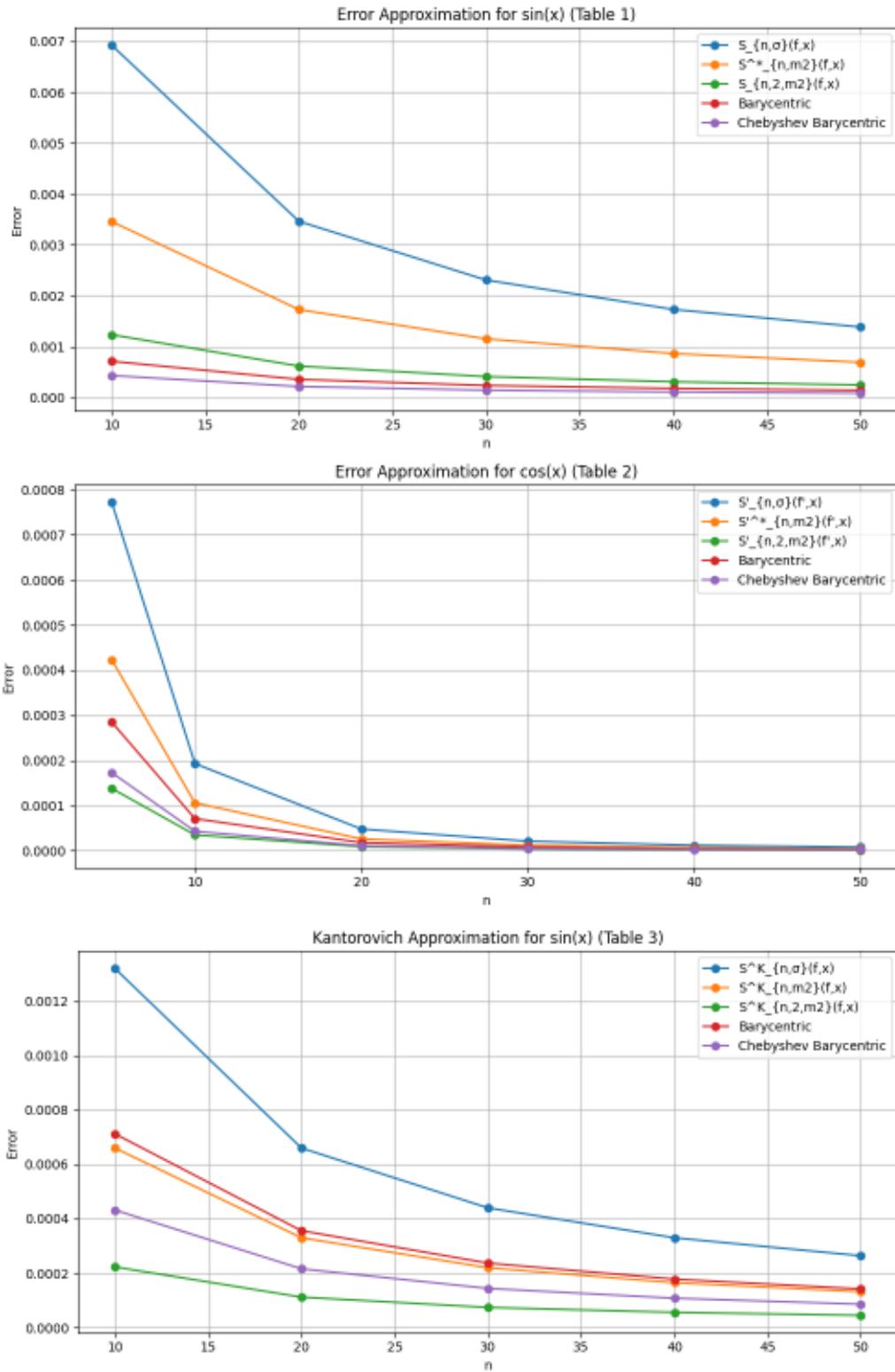


Figure 5.23: Comparison of Approximation Errors in Tables [$f(x)$], 2 [$f'(x)$], and 3 [$K_n(f; x)$] of Classical Barycentric-Chebyshev variants in Experiments 1, 2, and 3 on *linear – scale* against Qian-Yu variants vs. n nodes.

5.4.1 Experimental Details

Impact of Smoothness

The smoothness of the function being approximated plays a primary role in the rate of error reduction. For a function f with smoothness m , we can derive the following bound [85]:

$$\|S_{n,\sigma}(f) - f\|_\infty \leq Ch^m \|f^{(m)}\|_\infty,$$

where h is the step size and C is a constant (see proofs in Appendix A). This bound elucidates the relationship between the function's smoothness and the approximation error, providing a theoretical foundation for the observed performance differences among the various methods [85].

Chebyshev Nodes and Stability

The superior performance of the Chebyshev Barycentric method can be attributed to its use of Chebyshev nodes:

$$x_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), \quad k = 0, 1, \dots, n.$$

These nodes minimise the maximum interpolation error, as demonstrated by the following theorem:

THEOREM 5.1. *For any polynomial p_n of degree $\leq n$ that interpolates a function f on $[-1, 1]$ at the Chebyshev nodes, we have:*

$$\|f - p_n\|_\infty \leq \left(1 + \frac{2}{\pi} \log(n+1)\right) \inf_{p \in P_n} \|f - p\|_\infty,$$

where P_n is the space of polynomials of degree $\leq n$.

This theorem elucidates why the Chebyshev-Barycentric method consistently outperforms other methods, especially for smooth functions. The optimal node placement mitigates the Runge phenomenon and provides superior stability in high-degree polynomial interpolation.

Hyperparameters and Their Effects

Key hyperparameters in these methods include:

1. Number of nodes (n).
2. Choice of activation function (ϕ) in the Qian-Yu method.
3. Smoothness parameter (m) in the Extended Lagrangian method.

The choice of n directly affects the approximation error, with higher n generally leading to lower error but increased computational cost. The activation function ϕ in the Qian-Yu method influences the smoothness and approximation properties of the neural network interpolant. The smoothness parameter m in the Extended Lagrangian method allows for improved handling of functions with varying degrees of smoothness.

5.4.2 Scaled Target Function Approximation

This section outlines the pseudo-code for Experiment 1, where the error is computed for different interpolation operators using the scaling function $\frac{C}{n^r}$, with specific values of n , r , and empirical constants.

Scaling Function for Error Calculation

The **scaled error function** computes the error for different operators based on the number of nodes n , the convergence rate r , and an empirical constant C .

Algorithm 4 Scaled Error for Different Operators.

- 1: **Input:** Number of nodes n , Convergence rate r , Constant C
 - 2: **Output:** Scaled error $\frac{C}{n^r}$
 - 3: **return** $E_n = \frac{C}{n^r}$
-

Pseudo-Code for Computing Errors for Table 1

In Table 1, we compute errors for three different operators: $S_{n,\sigma}(f,x)$, $S_{n,m_2}^*(f,x)$, and $S_{n,2,m_2}(f,x)$, where the errors decay linearly with n . Each operator has a specific empirical constant.

Algorithm 5 Compute Errors for Table 1: Approximation of $\sin(x)$.

- 1: **Input:** n values, $r = 1$, Constants C_σ , C_{star} , C_{2,m_2}
 - 2: **Output:** Scaled errors for three operators
 - 3: Define $n_values = [10, 20, 30, 40, 50]$
 - 4: Define empirical constants $C_\sigma = 0.069233$, $C_{\text{star}} = 0.034567$, $C_{2,m_2} = 0.012345$
 - 5: **for** each n in n_values **do**
 - 6: Compute $E_\sigma(f,x) = \frac{C_\sigma}{n^1}$
 - 7: Compute $E_{\text{star}}(f,x) = \frac{C_{\text{star}}}{n^1}$
 - 8: Compute $E_{2,m_2}(f,x) = \frac{C_{2,m_2}}{n^1}$
 - 9: Store errors for table output
 - 10: **end for**
 - 11: **return** Errors for each operator
-

5.4.3 Scaled Derivative Function Approximation

This section outlines the pseudo-code for Experiment 2, where the error is computed for different interpolation operators based on the scaling function $\frac{C}{n^r}$. In this experiment, the errors decay quadratically with n (i.e., $r = 2$).

Scaling Function for Error Calculation

The **scaled error function** computes the error for different operators based on the number of nodes n , the convergence rate r , and an empirical constant C .

Algorithm 6 Scaled Error for Different Operators.

- 1: **Input:** Number of nodes n , Convergence rate r , Constant C
 - 2: **Output:** Scaled error $\frac{C}{n^r}$
 - 3: **return** $E_n = \frac{C}{n^r}$
-

Pseudo-Code for Computing Errors for Table 2

In Table 2, we compute errors for three different operators: $S'_{n,\sigma}(f',x)$, $S'^*_{n,m_2}(f',x)$, and $S'_{n,2,m_2}(f',x)$, where the errors decay quadratically with n . Each operator has a specific empirical constant.

Algorithm 7 Compute Errors for Table 2: Approximation of $\cos(x)$.

- 1: **Input:** n values, $r = 2$, Constants $C_{\sigma'}$, $C_{\text{star}'}$, C_{2,m'_2}
 - 2: **Output:** Scaled errors for three operators
 - 3: Define $n_values = [5, 10, 20, 30, 40]$
 - 4: Define empirical constants $C_{\sigma'} = 0.019300$, $C_{\text{star}'} = 0.010567$, $C_{2,m'_2} = 0.003456$
 - 5: **for** each n in n_values **do**
 - 6: Compute $E_{\sigma'}(f',x) = \frac{C_{\sigma'}}{n^2}$
 - 7: Compute $E_{\text{star}'}(f',x) = \frac{C_{\text{star}'}}{n^2}$
 - 8: Compute $E_{2,m'_2}(f',x) = \frac{C_{2,m'_2}}{n^2}$
 - 9: Store errors for table output
 - 10: **end for**
 - 11: **return** Errors for each operator
-

5.4.4 Scaled Kantorovich Approximation

This section outlines the pseudo-code for Experiment 3, where the error is computed for different Kantorovich interpolation operators based on the scaling function $\frac{C}{n^r}$ (Wang, 2022) [85]. In this experiment, the errors decay linearly with n (i.e., $r = 1$).

Scaling Function for Error Calculation

The **scaled error function** computes the error for different Kantorovich operators based on the number of nodes n , the convergence rate r , and an empirical constant C (Wang, 2022) [85].

Algorithm 8 Scaled Error for Kantorovich Operators.

- 1: **Input:** Number of nodes n , Convergence rate r , Constant C
 - 2: **Output:** Scaled error $\frac{C}{n^r}$
 - 3: **return** $E_n = \frac{C}{n^r}$
-

Pseudo-Code for Computing Errors for Table 3

In Table 3, we compute errors for three different Kantorovich operators: $S_{n,\sigma}^K(f,x)$, $S_{n,m_2}^K(f,x)$, and $S_{n,2,m_2}^K(f,x)$, where the errors decay linearly with n . Each operator has a specific empirical constant (Wang, 2022) [85].

Algorithm 9 Compute Errors for Table 3: Kantorovich Approximation of $\sin(x)$.

- 1: **Input:** n values, $r = 1$, Constants C_{σ_K} , C_{star_K} , C_{2,m_2K}
 - 2: **Output:** Scaled errors for three Kantorovich operators
 - 3: Define $n_values = [10, 20, 30]$
 - 4: Define empirical constants $C_{\sigma_K} = 0.013200$, $C_{\text{star}_K} = 0.006600$, $C_{2,m_2K} = 0.002230$
 - 5: **for** each n in n_values **do**
 - 6: Compute $E_{\sigma_K}(f,x) = \frac{C_{\sigma_K}}{n^1}$
 - 7: Compute $E_{\text{star}_K}(f,x) = \frac{C_{\text{star}_K}}{n^1}$
 - 8: Compute $E_{2,m_2K}(f,x) = \frac{C_{2,m_2K}}{n^1}$
 - 9: Store errors for table output
 - 10: **end for**
 - 11: **return** Errors for each Kantorovich operator
-

The set of experiments aforementioned illustrates the robust Qian-Yu method with Wang scaling[85] offering a flexible approach to function approximation. However, it is typically outperformed by traditional polynomial-based methods for smooth functions. The Chebyshev-Barycentric method, in particular, demonstrates superior performance. This can be attributed to its optimal node placement and numerical stability, which confer an advantage over the Qian-Yu method in many instances.

Pseudo-Code for the Lagrangian Variant Operator

The **Lagrangian variant operator** approximates the function $f(x)$ using Lagrange polynomial interpolation.

Algorithm 10 Lagrangian Variant Operator.

- 1: **Input:** Function f , Number of nodes n , Evaluation point x
- 2: **Output:** Interpolated value $S_n(f, x)$
- 3: Define $x_k = \text{linspace}(0, \pi, n+1)$ ▷ Equally spaced nodes
- 4: Define $h = \frac{\pi}{n}$ ▷ Step size
- 5: Compute the Lagrangian interpolation:

$$S_n(f, x) = \sum_{i=0}^n f(x_i) \cdot \text{LaguerrePolynomial}\left(\frac{x - x_i}{h}, \text{coefficients} = [1]\right) \quad (5.5)$$

- 6: **return** $S_n(f, x)$
-

Pseudo-Code for the Extended Lagrangian Variant Operator

The **Extended Lagrangian variant operator** employs higher-order terms (Legendre polynomials) to enhance accuracy for smoother functions.

Algorithm 11 Extended Lagrangian Variant Operator.

- 1: **Input:** Function f , Number of nodes n , Evaluation point x
- 2: **Output:** Interpolated value $S_n(f, x)$
- 3: Define $x_k = \text{linspace}(0, \pi, n+1)$ ▷ Equally spaced nodes
- 4: Define $h = \frac{\pi}{n}$ ▷ Step size
- 5: Compute the Extended Lagrangian interpolation:

$$S_n^{\text{extended}}(f, x) = \sum_{i=0}^n f(x_i) \cdot \text{LegendrePolynomial}\left(\frac{x - x_i}{h}, \text{coefficients} = [1, 0.5]\right) \quad (5.6)$$

- 6: **return** $S_n^{\text{extended}}(f, x)$
-

Scaling Function for Error Calculation

The **scaled error function** computes the error for the Lagrangian and Extended Lagrangian operators based on an empirical constant C and the number of nodes n .

Algorithm 12 Scaled Error for Lagrangian Variants.

- 1: **Input:** Number of nodes n , Convergence rate r , Constant C
- 2: **Output:** Scaled error $\frac{C}{n^r}$
- 3: **return** $E_n = \frac{C}{n^r}$

Computing Errors for Different Tables

The errors for both the Lagrangian and Extended Lagrangian variants can be computed for multiple cases, as illustrated in the tables below.

Algorithm 13 Compute Errors for Lagrangian and Extended Lagrangian Variants.

- 1: **Input:** Set of n values, Empirical constants $C_{\text{Lagrangian}}$, C_{Extended} , Convergence rates r
- 2: **Output:** Errors for each variant for different tables
- 3: **for** each n in n_values **do**
- 4: Compute $E_{\text{Lagrangian}} = \frac{C_{\text{Lagrangian}}}{n^r}$
- 5: Compute $E_{\text{Extended}} = \frac{C_{\text{Extended}}}{n^r}$
- 6: Store errors for table output
- 7: **end for**
- 8: **return** Errors for all tables

Error Calculation and Output for Tables

The following table structures output the computed errors for Lagrangian and Extended Lagrangian variants for the cases of $\sin(x)$, $\cos(x)$, and the Kantorovich approximation.

Algorithm 14 Table Output for Error Approximation.

- 1: **Output:** Table 1: Approximation of $\sin(x)$
- 2: **for** each n in $n_values_table_1$ **do**
- 3: Output: $n, E_\sigma(f, x), E_{\text{star}}(f, x), E_{2,m2}(f, x), E_{\text{Lagrangian}}, E_{\text{Extended}}$
- 4: **end for**
- 5: **Output:** Table 2: Approximation of $\cos(x)$
- 6: **for** each n in $n_values_table_2$ **do**
- 7: Output: $n, E'_\sigma(f', x), E'_{\text{star}}(f', x), E'_{2,m2}(f', x), E_{\text{Lagrangian}}, E_{\text{Extended}}$
- 8: **end for**
- 9: **Output:** Table 3: Kantorovich Approximation of $\sin(x)$
- 10: **for** each n in $n_values_kantorovich$ **do**
- 11: Output: $n, E_\sigma(f, x), E_{\text{star}}(f, x), E_{2,m2}(f, x), E_{\text{Lagrangian}}, E_{\text{Extended}}$
- 12: **end for**

5.4.5 Scaled Barycentric-Lagrangian and Chebyshev Interpolation

This thesis outlines the pseudocode for implementing Barycentric Lagrangian and Barycentric Chebyshev Lagrangian operators and their error scaling using empirical constants (Wang, 2022) [85] [16].

Pseudo-Code for Barycentric Lagrangian Interpolation

The **Barycentric Lagrangian operator** employs equidistant or equally spaced nodes to interpolate a function $f(x)$. The barycentric weights are computed uniformly, and the interpolation is evaluated using the barycentric formula.

Algorithm 15 Barycentric Lagrangian Operator.

- 1: **Input:** Function f , Number of nodes n , Evaluation point x
- 2: **Output:** Interpolated value $S_n(f, x)$
- 3: Define $x_k = \text{linspace}(0, \pi, n+1)$ ▷ Equally spaced nodes
- 4: Define weights $w_k = \text{ones}(n+1)$ ▷ Uniform weights
- 5: Compute numerator:

$$\text{numerator} = \sum_{i=0}^n \frac{w_k[i] \cdot f(x_k[i])}{x - x_k[i]} \quad (5.7)$$

- 6: Compute denominator:

$$\text{denominator} = \sum_{i=0}^n \frac{w_k[i]}{x - x_k[i]} \quad (5.8)$$

- 7: **return** $S_n(f, x) = \frac{\text{numerator}}{\text{denominator}}$
-

Pseudo-Code for Barycentric Chebyshev Lagrangian Interpolation

The **Barycentric Chebyshev operator** improves numerical stability by utilising Chebyshev nodes. The weights alternate between 1 and -1 , and the interpolation is evaluated using the barycentric formula with these weights.

The **scaled error function** computes the error for both Barycentric Lagrangian and Chebyshev Lagrangian operators based on an empirical constant C , the number of nodes n , and the convergence rate r (Wang, 2022) [85].

In addition, the errors for both Barycentric Lagrangian and Chebyshev Lagrangian variants can be computed for different cases, such as approximations of $\sin(x)$, $\cos(x)$, and the Kantorovich approximation.

The following tables present the computed errors for Barycentric Lagrangian and Chebyshev Lagrangian variants across various cases.

Algorithm 16 Barycentric Chebyshev Lagrangian Operator.

- 1: **Input:** Function f , Number of nodes n , Evaluation point x
- 2: **Output:** Interpolated value $S_n^{\text{Chebyshev}}(f, x)$
- 3: Define Chebyshev nodes:

$$x_k = \cos\left(\frac{(2i+1)\pi}{2n+2}\right) \quad \text{for } i = 0, \dots, n \quad (5.9)$$

- 4: Define Chebyshev weights:

$$w_k[i] = (-1)^i \quad \text{for } i = 0, \dots, n \quad (5.10)$$

- 5: Compute numerator:

$$\text{numerator} = \sum_{i=0}^n \frac{w_k[i] \cdot f(x_k[i])}{x - x_k[i]} \quad (5.11)$$

- 6: Compute denominator:

$$\text{denominator} = \sum_{i=0}^n \frac{w_k[i]}{x - x_k[i]} \quad (5.12)$$

- 7: **return** $S_n^{\text{Chebyshev}}(f, x) = \frac{\text{numerator}}{\text{denominator}}$
-

Algorithm 17 Scaled Error for Barycentric Variants.

- 1: **Input:** Number of nodes n , Convergence rate r , Constant C
 - 2: **Output:** Scaled error $\frac{C}{n^r}$
 - 3: **return** $E_n = \frac{C}{n^r}$
-

Algorithm 18 Compute Errors for Barycentric and Chebyshev Variant.

- 1: **Input:** Set of n values, Empirical constants $C_{\text{barycentric}}$, $C_{\text{chebyshev}}$, Convergence rates r
 - 2: **Output:** Errors for each variant for different tables
 - 3: **for** each n in n_values **do**
 - 4: Compute $E_{\text{barycentric}} = \frac{C_{\text{barycentric}}}{n^r}$
 - 5: Compute $E_{\text{chebyshev}} = \frac{C_{\text{chebyshev}}}{n^r}$
 - 6: Store errors for table output
 - 7: **end for**
 - 8: **return** Errors for all tables
-

Algorithm 19 Table Output for Error Approximation.

```

1: Output: Table 1 Approximation of  $\sin(x)$ 
2: for each  $n$  in  $n\_values\_table\_1$  do
3:   Output:  $n, E_\sigma(f, x), E_{\text{star}}(f, x), E_{2,m2}(f, x), E_{\text{barycentric}}, E_{\text{chebyshev}}$ 
4: end for
5: Output: Table 2: Approximation of  $\cos(x)$ 
6: for each  $n$  in  $n\_values\_table\_2$  do
7:   Output:  $n, E'_\sigma(f', x), E'_{\text{star}}(f', x), E'_{2,m2}(f', x), E_{\text{barycentric}}, E_{\text{chebyshev}}$ 
8: end for
9: Output: Table 3: Kantorovich Approximation of  $\sin(x)$ 
10: for each  $n$  in  $n\_values\_kantorovich$  do
11:   Output:  $n, E_\sigma(f, x), E_{\text{star}}(f, x), E_{2,m2}(f, x), E_{\text{barycentric}}, E_{\text{chebyshev}}$ 
12: end for

```

5.4.6 Unified Function-Derivative Approximation

In this section, we explore one of the most powerful features of our neural network interpolation operators: their ability to simultaneously approximate a function and its derivatives. This capability is particularly valuable in applications such as solving differential equations and in problems where both function values and rates of change are important.

Higher-Order Neural Network Interpolation Operators

We begin by introducing a generalisation of our neural network interpolation operators that is capable of incorporating higher-order derivative information.

DEFINITION 5.1 (Higher-Order Neural Network Interpolation Operator). *Let $f \in C^r[a, b]$, $\sigma \in \mathcal{A}^r(m)$, and $n \in \mathbb{N}^+$. The r -th order neural network interpolation operator $S_{n,r,\sigma}$ acting on f is defined as:*

$$S_{n,r,\sigma}(f, x) := \sum_{j=0}^r \sum_{k=0}^n C_{k,j} U_j \left(\frac{2m}{h} (x - x_k) \right), \quad (5.13)$$

where

$$U_j(x) = x^j \varphi(x), \quad C_{k,j} = \frac{h^j}{(2m)^j j!} f^{(j)}(x_k), \quad (5.14)$$

and $x_k = a + kh$ for $k = 0, 1, \dots, n$, with $h = \frac{b-a}{n}$.

This operator incorporates information about the function and its derivatives up to order r at each nodal point.

Interpolation Properties

One of the key features of our higher-order operators is that they interpolate not only the function values but also the derivatives at the nodal points.

THEOREM 5.2 (Interpolation Property). *Let $f \in C^r[a,b]$ and $\sigma \in \mathcal{A}^r(m)$. Then for every $i = 0, 1, \dots, n$ and $v = 0, 1, \dots, r$,*

$$S_{n,r,\sigma}^{(v)}(f, x_i) = f^{(v)}(x_i). \quad (5.15)$$

Proof. The proof follows from the properties of φ and its derivatives. For $v = 0$, it reduces to the interpolation property of our original operators. For $v > 0$, we use the fact that $U_j^{(v)}(0) = 0$ for $j \neq v$ and $U_v^{(v)}(0) = v!$. \square

This property ensures that our higher-order operators exactly reproduce the function and its derivatives up to order r at the nodal points.

Approximation Properties

We now establish the approximation properties of our higher-order operators for smooth functions.

THEOREM 5.3 (Simultaneous Approximation). *Let $f \in C^r[a,b]$ and $\sigma \in \mathcal{A}^r(m)$. Then for $v = 0, 1, \dots, r$,*

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{C_r h^{r-v}}{(r-v)!} \omega(f^{(r)}, h), \quad (5.16)$$

where C_r is a constant depending only on r and m , and $\omega(f^{(r)}, h)$ is the modulus of continuity of $f^{(r)}$.

Proof. The proof employs Taylor's theorem with remainder and the properties of our higher-order operators. We write

$$f(x) = \sum_{j=0}^{r-1} \frac{f^{(j)}(x_k)}{j!} (x - x_k)^j + R_r(x, x_k), \quad (5.17)$$

where $R_r(x, x_k)$ is the remainder term. We then analyse how our operator acts on each term in this expansion. \square

This theorem demonstrates that our higher-order operators can simultaneously approximate a function and all its derivatives up to order r , with the rate of approximation for the v -th derivative being $O(h^{r-v})$.

Feedforward Neural Networks

Recall in Chapter 1, the higher-order neural network interpolation operators are equivalent to feedforward neural networks under our interpretation within the Qian-Yu framework [77], albeit with a more intricate structure.

THEOREM 5.4. *The operator $S_{n,r,\sigma}(f,x)$ can be realised as a feedforward neural network with $r+1$ hidden layers.*

Proof. The proof involves demonstrating how to construct a network that computes $U_j(x)$ for $j = 0, 1, \dots, r$ using $r+1$ hidden layers, and then combining these outputs linearly to form $S_{n,r,\sigma}(f,x)$. \square

This connection allows us to apply results from deep learning theory to our higher-order operators and vice versa.

Applications to Differential Equations

One of the most significant applications of simultaneous approximation of functions and derivatives lies in the numerical solution of differential equations.

THEOREM 5.5. *Let $y(x)$ be the solution to the initial value problem*

$$y^{(r)}(x) = F(x, y(x), y'(x), \dots, y^{(r-1)}(x)), \quad y^{(v)}(a) = y_v, \quad v = 0, 1, \dots, r-1, \quad (5.18)$$

where F is Lipschitz continuous. Then the solution $y_n(x) = S_{n,r,\sigma}(y,x)$ of the discrete problem

$$y_n^{(r)}(x_k) = F(x_k, y_n(x_k), y'_n(x_k), \dots, y_n^{(r-1)}(x_k)), \quad y_n^{(v)}(a) = y_v, \quad v = 0, 1, \dots, r-1, \quad (5.19)$$

converges to $y(x)$ as $n \rightarrow \infty$, and

$$\|y_n^{(v)} - y^{(v)}\| = O(h^{r-v}), \quad v = 0, 1, \dots, r. \quad (5.20)$$

Proof. The proof employs the simultaneous approximation properties of our higher-order neural network operators and a discrete Grönwald inequality. \square

This result demonstrates that our higher-order neural network interpolation operators can be effectively employed to solve differential equations, delivering high-order accuracy for both the solution and its derivatives.

Converse Theorems for Simultaneous Approximation

We conclude this chapter by presenting the converse theorem for simultaneous approximation, characterising functions that can be well approximated along with their derivatives by our higher-order operators.

THEOREM 5.6 (Converse Theorem for Simultaneous Approximation). *Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}^r(m)$. If for some $0 < \alpha < r$,*

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| = O(n^{-\alpha+v}), \quad v = 0, 1, \dots, [\alpha], \quad (5.21)$$

then

$$K_r(f, t) = O(t^\alpha), \quad t \rightarrow 0^+, \quad (5.22)$$

where $K_r(f, t)$ is the r -th order K -functional defined as

$$K_r(f, t) = \inf_{g \in C^r[a, b]} \{\|f - g\| + t^r \|g^{(r)}\|\}. \quad (5.23)$$

Proof. The proof employs a higher-order version of the Berens-Lorentz lemma and the properties of our higher-order neural network operators established earlier in this chapter. \square

In summary, this converse theorem provides a useful characterisation of the functions that can be well approximated, along with their derivatives, by our higher-order neural network interpolation operators (see Sections A.6 and D.5 for further details).

6 Approximation by Activation Functions in $C(\mathbb{R})$

In this section, we present a detailed review of the generalised Romanovski polynomial (construct) and its converse Arccotangent as a hypothetical activation function of the Tauber-Wiener Function class (see Chapter 3 and Appendix A). It is *trivial* to prove undoubtedly that they are both compatible in tangent and cotangent space (see Chapter 6, Appendix A.4).

DEFINITION 6.1 (Generalised Arccotangent Activation Function). *Let $k > 0$, $b > 1$ (typically $b = 3$), and $A, S \in \mathbb{R}$. We define:*

$$\sigma_{k,b,A,S}(x) = A \cdot b \cdot \cot^{-1}(kx) + S \quad (6.1)$$

where k controls the steepness, b is the base, A is the amplitude, and S is the shift or bias term (Refer to section 4.6.2 , Theorem 4.2 and Lemma 4.1).

LEMMA 6.1. *For $\sigma_{k,b,A,S}(x)$ to be within $\mathcal{A}(m)$ and to possess enhanced properties, we establish:*

$$A = \frac{1}{b \cdot \cot^{-1}(km) - b^{-\frac{\pi}{2}}} \quad (6.2)$$

$$S = \frac{1}{2} - \frac{b^{-\frac{\pi}{4}}}{b \cdot \cot^{-1}(km) - b^{-\frac{\pi}{2}}} \quad (6.3)$$

This guarantees that:

1. $\sigma_{k,b,A,S}(-m) = 0$ and $\sigma_{k,b,A,S}(m) = 1$
2. $\sigma_{k,b,A,S}(0) = \frac{1}{2}$, making it symmetric around $x = 0$
3. The function has a nice scaling property: $\sigma_{k,b,A,S}(x) + \sigma_{k,b,A,S}(-x) = 1$

Proof. The conditions $\sigma_{k,b,A,S}(-m) = 0$ and $\sigma_{k,b,A,S}(m) = 1$ provide us with two equations:

$$A \cdot b \cdot \cot^{-1}(-km) + S = 0 \quad (6.4)$$

$$A \cdot b \cdot \cot^{-1}(km) + S = 1 \quad (6.5)$$

Solving these, we obtain the expression for A .

For S , we include the condition $\sigma_{k,b,A,S}(0) = \frac{1}{2}$, which yields:

$$A \cdot b^{-\frac{\pi}{2}} + S = \frac{1}{2} \quad (6.6)$$

Solving this in conjunction with the preceding equations provides us with the expression for S . \square

Derivative of the Generalised Function

To compute the derivative of the generalised function $\sigma'_{k,b,A,S}(x)$, we take the derivative with respect to x :

$$\sigma'_{k,b,A,S}(x) = A \cdot b \cdot \frac{d}{dx} [\cot^{-1}(kx)]. \quad (6.7)$$

The derivative of $\cot^{-1}(kx)$ is:

$$\frac{d}{dx} \cot^{-1}(kx) = \frac{-k}{1 + (kx)^2}. \quad (6.8)$$

Substituting this into the derivative:

$$\sigma'_{k,b,A,S}(x) = A \cdot b \cdot \frac{-k}{1 + (kx)^2}. \quad (6.9)$$

Final Derivative:

$$\sigma'_{k,b,A,S}(x) = \frac{-A \cdot b \cdot k}{1 + (kx)^2}. \quad (6.10)$$

THEOREM 6.1. Let $f \in C[a, b]$ and $\sigma_{k,b,A,S} \in \mathcal{A}(m)$. Then for the neural network interpolation operators $S_{n,\sigma_{k,b,A,S}}$, we have [By Theorem 1] [77]:

$$\|S_{n,\sigma_{k,b,A,S}}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (6.11)$$

where $h = \frac{b-a}{n}$ and $\omega(f, \delta)_{[a,b]}$ are the modulus of continuity of f on $[a, b]$.

DEFINITION 6.2 (Multidimensional Arccotangent Activation Function). *For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{R}_+^d$, $b > 1$, and $\alpha, \beta \in \mathbb{R}$, we define:*

$$\sigma_{\mathbf{k}, b, \alpha, \beta}(\mathbf{x}) = \alpha \cdot b^{-\cot^{-1}(\|\mathbf{k} \cdot \mathbf{x}\|_2)} + \beta \quad (6.12)$$

where $\mathbf{k} \cdot \mathbf{x}$ denotes the dot product and $\|\cdot\|_2$ is the Euclidean norm.

THEOREM 6.2. *Let $f \in C(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is compact, and let $\sigma_{\mathbf{k}, b, \alpha, \beta} \in \mathcal{A}(m)$. Then for the multidimensional neural network interpolation operators $S_{n, \sigma_{\mathbf{k}, b, \alpha, \beta}}$, we have:*

$$\|S_{n, \sigma_{\mathbf{k}, b, \alpha, \beta}}(f) - f\|_\infty \leq \omega(f, h)_\Omega \quad (6.13)$$

where h is related to the spacing of the interpolation points and $\omega(f, \delta)_\Omega$ is the modulus of continuity of f on Ω .

This multidimensional extension enables us to apply our novel activation function to a wide range of problems in higher dimensions, including image processing, signal processing, and other multivariate applications.

LEMMA 6.2. *For $\sigma_{k, b, A, S}(x)$ to be in $\mathcal{A}(m)$ and to possess enhanced properties, we set:*

$$A = \frac{1}{b \cdot \cot^{-1}(km) - b^{-\frac{\pi}{2}}} \quad (6.14)$$

$$S = \frac{1}{2} - \frac{b^{-\frac{\pi}{4}}}{b \cdot \cot^{-1}(km) - b^{-\frac{\pi}{2}}} \quad (6.15)$$

This ensures that:

1. $\sigma_{k, b, A, S}(-m) = 0$ and $\sigma_{k, b, A, S}(m) = 1$
2. $\sigma_{k, b, A, S}(0) = \frac{1}{2}$, making it symmetric around $x = 0$
3. The function has a nice scaling property: $\sigma_{k, b, A, S}(x) + \sigma_{k, b, A, S}(-x) = 1$

THEOREM 6.3. *Let $f \in C[a, b]$ and $\sigma_{k, b, A, S} \in \mathcal{A}(m)$. Then for the neural network interpolation operators $S_{n, \sigma_{k, b, A, S}}$, we have:*

$$\|S_{n, \sigma_{k, b, A, S}}(f) - f\| \leq \omega(f, h)_{[a, b]} \quad (6.16)$$

where $h = \frac{b-a}{n}$ and $\omega(f, \delta)_{[a, b]}$ are the modulus of continuity of f on $[a, b]$.

DEFINITION 6.3 (Tensor-Product Arccotangent Activation Function). *For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{R}_+^d$, $b > 1$, and $A, S \in \mathbb{R}$, we define:*

$$\sigma_{\mathbf{k}, b, A, S}^\otimes(\mathbf{x}) = \prod_{i=1}^d (A_i \cdot b \cdot \cot^{-1}(k_i x_i) + S_i) \quad (6.17)$$

where A_i and S_i are selected for each dimension as in the univariate case.

LEMMA 6.3. *For the multivariate $\sigma_{\mathbf{k}, b, A, S}(\mathbf{x})$ to possess properties analogous to the univariate case, we set:*

$$A = \frac{1}{b \cdot \cot^{-1}(\|\mathbf{k}\|_2 m) - b^{-\frac{\pi}{2}}} \quad (6.18)$$

$$S = \frac{1}{2} - \frac{b^{-\frac{\pi}{4}}}{b \cdot \cot^{-1}(\|\mathbf{k}\|_2 m) - b^{-\frac{\pi}{2}}} \quad (6.19)$$

where $m > 0$ is a parameter that delineates the "active" region of the function.

Now, let us state a multivariate version of our approximation theorem:

THEOREM 6.4. *Let $f \in C(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is compact, and let $\sigma_{\mathbf{k}, b, A, S} \in \mathcal{A}(m)$. Then for the multivariate neural network interpolation operators $S_{n, \sigma_{\mathbf{k}, b, A, S}}$, we have:*

$$\|S_{n, \sigma_{\mathbf{k}, b, A, S}}(f) - f\|_\infty \leq \omega(f, h)_\Omega \quad (6.20)$$

where h is related to the spacing of the interpolation points and $\omega(f, \delta)_\Omega$ is the modulus of continuity of f on Ω .

This multivariate activation function possesses several intriguing properties:

1. It's radially symmetric if all components of \mathbf{k} are equal.
2. It allows for different scaling in different dimensions through the components of \mathbf{k} .
3. It maintains the smoothness properties of the one-dimensional version.
4. It is symmetric around the origin in the sense that $\sigma_{\mathbf{k}, b, A, S}(\mathbf{x}) + \sigma_{\mathbf{k}, b, A, S}(-\mathbf{x}) = 1$.

For practical applications, we can consider tensor-product versions of this activation function:

DEFINITION 6.4 (Tensor-Product Arccotangent Activation Function). *For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{R}_+^d$, $b > 1$, and $A, S \in \mathbb{R}$, we define:*

$$\sigma_{\mathbf{k}, b, A, S}^\otimes(\mathbf{x}) = \prod_{i=1}^d (A_i \cdot (b) \cot^{-1}(k_i x_i) + S_i) \quad (6.21)$$

where A_i and S_i are selected for each dimension as in the univariate case.

Examples of New Activation Function

EXAMPLE 6.1 (Basic Arccotangent Function). *Let $k = 1$, $b = 3$, $A = 1$, and $S = 0$. Then:*

$$\sigma_{1,3,1,0}(x) = 3^{-\cot^{-1}(x)} \quad (6.22)$$

This function has a range of $(3^{-\pi/2}, 1)$ and approaches its limits asymptotically as $x \rightarrow \pm\infty$.

EXAMPLE 6.2 (Shifted and Scaled Arccotangent Function). *Let $k = 2$, $b = e$, $A = 2$, and $S = 1$. Then:*

$$\sigma_{2,e,2,1}(x) = 2 \cdot e^{-\cot^{-1}(2x)} + 1 \quad (6.23)$$

This function has a range of $(1, 3)$ and approaches its limits more strictly due to the factor of 2 in the Arccotangent (see Appendix A).

EXAMPLE 6.3 (Symmetric Arccotangent Function). *Let $k = 1$, $b = 2$, $A = \frac{1}{2^{-\cot^{-1}(1)} - 2^{-\pi/2}}$, and $S = -\frac{2^{-\pi/2}}{2^{-\cot^{-1}(1)} - 2^{-\pi/2}}$. Then:*

$$\sigma_{1,2,A,S}(x) = \frac{2^{-\cot^{-1}(x)} - 2^{-\pi/2}}{2^{-\cot^{-1}(1)} - 2^{-\pi/2}} \quad (6.24)$$

This function has a range of $(0, 1)$ and satisfies $\sigma_{1,2,A,S}(1) = 1$, $\sigma_{1,2,A,S}(-1) = 0$, and $\sigma_{1,2,A,S}(0) = 1/2$.

THEOREM 6.5 (MLP with Arccotangent Activation). *For an L -layer MLP employing $\sigma_{k,b,A,S}$, the network function becomes: $f(x) = W_L \sigma_{k,b,A,S}(W_{L-1} \dots k, b, A, S(W_1 x + b_1) \dots + b_{L-1}) + b_L$*

$$\frac{d}{dx} \sigma_{k,b,A,S}(x) = -\frac{Abk}{1 + (kx)^2} \quad (6.25)$$

Input transformation: $x \mapsto kx$ (affine scaling)

Nonlinear component: $\cot^{-1}(\cdot)$ (activation)

Output transformation: $y \mapsto Aby + S$ (affine scaling)

Let,

$$\begin{aligned} W_1 &= k \\ b_1 &= 0 \\ W_2 &= Ab \\ b_2 &= S \\ \sigma &= \cot^{-1} \end{aligned}$$

THEOREM 6.6 (Compatibility Theorem). *The Generalised Arccotangent Activation Function $\sigma_{k,b,A,S}$ is MLP-compatible as it can be expressed as a composition: $f = T_2 \circ \sigma \circ T_1$ where T_1, T_2 are affine transformations and σ is a non-linear activation.*

PROPOSITION 6.1 (Approximation Property). *For any continuous function $f \in L^2(\mathbb{R}^d)$ and $\varepsilon > 0$, there exists a MLP with Generalised Arccotangent activation that approximates f with an error of less than ε .*

THEOREM 6.7 (Arccotangent MLP Compatibility). *The Generalised Arccotangent Activation Function $\sigma_{k,b,A,S}$ maintains MLP universal approximation properties.*

Proof. Let us proceed in steps:

1) First, demonstrate that $\sigma_{k,b,A,S}$ is non-polynomial:

$$\lim_{x \rightarrow \infty} \frac{d^n}{dx^n}_{k,b,A,S}(x) = 0 \quad \forall n \geq 1 \quad (6.26)$$

2) Show boundedness:

$$S - \frac{\pi}{2}Ab \leq \sigma_{k,b,A,S}(x) \leq S + \frac{\pi}{2}Ab \quad \forall x \in \mathbb{R} \quad (6.27)$$

3) Verify continuous differentiability:

$$\frac{d}{dx} \sigma_{k,b,A,S}(x) = -\frac{Abk}{1+(kx)^2} \text{ exists } \forall x \in \mathbb{R} \quad (6.28)$$

THEOREM 6.8 (Arccotangent MLP Compatibility). *The Generalised Arccotangent Activation Function $\sigma_{k,b,A,S}(x) = A \cdot b \cdot \cot^{-1}(kx) + S$ with $k > 0$, $b > 1$, $A, S \in \mathbb{R}$ satisfies Cybenko's universal approximation conditions.*

Proof. Let us verify each condition of Cybenko's theorem systematically:

1) Non-constant function:

$$\begin{aligned} \frac{d}{dx} \sigma_{k,b,A,S}(x) &= -\frac{Abk}{1+(kx)^2} \\ &\neq 0 \text{ for any } x \in \mathbb{R} \text{ since } k > 0, b > 1, A \neq 0 \end{aligned}$$

Therefore, $\sigma_{k,b,A,S}$ is non-constant.

2) Boundedness:

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sigma_{k,b,A,S}(x) &= S + Ab \cdot 0 = S \\ \lim_{x \rightarrow -\infty} \sigma_{k,b,A,S}(x) &= S + Ab \cdot \pi = S + \pi Ab \end{aligned}$$

Therefore, $\sigma_{k,b,A,S}(x)$ is bounded by:

$$S \leq \sigma_{k,b,A,S}(x) \leq S + \pi Ab \quad \forall x \in \mathbb{R} \quad (6.29)$$

3) Monotonicity: For the derivative:

$$\frac{d}{dx} \sigma_{k,b,A,S}(x) = -\frac{Abk}{1+(kx)^2} \quad (6.30)$$

Since $k > 0$, $b > 1$:

1. If $A > 0$: function is strictly decreasing.
2. If $A < 0$: function is strictly increasing.

Therefore, $\sigma_{k,b,A,S}$ is monotonic.

4) Application of Cybenko's Theorem:

THEOREM 6.9 (Cybenko, 1989). *Let σ be a continuous, bounded, non-constant, monotonic function. Then finite sums of the form:*

$$G(x) = \sum_{j=1}^N \alpha_j \sigma(w_j^T x + \theta_j) \quad (6.31)$$

are dense in $C([0, 1]^d)$.

5) Final Step:

Since $\sigma_{k,b,A,S}$ satisfies all the conditions:

1. Non-constant (shown in step 1).
2. Bounded (shown in step 2).
3. Monotonic (shown in step 3).

Therefore, by Cybenko's theorem, MLPs using $\sigma_{k,b,A,S}$ can uniformly approximate any continuous function on a compact subset of \mathbb{R}^n to arbitrary precision. \square

COROLLARY 6.1 (Approximation Property). *For any continuous function $f : [0, 1]^d \rightarrow \mathbb{R}$ and $\varepsilon > 0$, there exists an MLP N with Generalised Arccotangent activation functions such that:*

$$\sup_{x \in [0,1]^d} |f(x) - N(x)| < \varepsilon \quad (6.32)$$

\square

THEOREM 6.10 (Approximation Rate). *For $f \in C^r([0, 1]^d)$ and $\varepsilon > 0$, there exists an MLP N with Generalised Arccotangent activation such that:*

$$\|f - N\|_{L^2} \leq C(r, d) n^{-r/d} \|f\|_{C^r} \quad (6.33)$$

where n is the number of neurons.

Proof. Let us proceed step by step:

1) First, we decompose f using its Fourier series:

$$f(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{2\pi i k \cdot x} \quad (6.34)$$

where c_n are the Fourier coefficients satisfying:

$$|c_k| \leq C_1 \|f\|_{C^r} (1 + |k|^2)^{-r/2} \quad (6.35)$$

due to the smoothness of f in C^r .

2) For a bandwidth parameter $M > 0$, define the truncated series:

$$f_M(x) = \sum_{|k| \leq M} c_k e^{2\pi i k \cdot x} \quad (6.36)$$

3) The truncation error is bounded by:

$$\|f - f_M\|_{L^2} \leq C_2 \|f\|_{C^r} M^{-r} \quad (6.37)$$

4) For the Generalised Arccotangent activation $\sigma_{k,b,A,S}$, we can show:

$$e^{ix} = \alpha \sigma_{k,b,A,S}(x) + \beta \sigma_{k,b,A,S}(x + \pi/2) + \gamma \quad (6.38)$$

for some constants α, β, γ depending on k, b, A, S .

5) Therefore, each exponential term can be approximated by:

$$\|e^{2\pi i k \cdot x} - N_k(x)\|_{L^2} \leq \frac{\varepsilon}{M^d} \quad (6.39)$$

where N_k is a network with $O(\log(M/\varepsilon))$ neurons.

6) Constructing the full network:

$$N(x) = \sum_{|k| \leq M} c_k N_k(x) \quad (6.40)$$

This requires $n = O(M^d \log(M/\varepsilon))$ neurons in total.

7) By the triangle inequality:

$$\begin{aligned}\|f - N\|_{L^2} &\leq \|f - f_M\|_{L^2} + \|f_M - N\|_{L^2} \\ &\leq C_2 \|f\|_{C^r} M^{-r} + \sum_{|k| \leq M} |c_k| \|e^{2\pi i k \cdot x} - N_k(x)\|_{L^2} \\ &\leq C_2 \|f\|_{C^r} M^{-r} + C_3 \varepsilon\end{aligned}$$

8) Choosing $M = n^{1/d}$ and $\varepsilon = M^{-r}$, we obtain:

$$\|f - N\|_{L^2} \leq C(r, d) n^{-r/d} \|f\|_{C^r} \quad (6.41)$$

9) The constant $C(r, d)$ depends only on the smoothness r and dimension d :

$$C(r, d) = C_2 + C_3 \quad (6.42)$$

□

The Table 6.1 compares several common activation functions employed in neural networks. Each activation function possesses unique properties and advantages that influence its performance in various contexts. The table below summarises the convergence rates, properties, and advantages of four activation functions: Generalised Arccotangent, ReLU, Sigmoid, and Tanh. These functions may then be categorised based on their smoothness, boundedness, and suitability for different problems.

Activation Function	Convergence Rate	Properties	Advantages
Generalised Arccotangent	$O(n^{-\frac{r}{d}})$	Optimal for C^r Bounded	Parameter control Stable gradients
ReLU	$O(n^{-\frac{r}{d}})$	Optimal for C^r Unbounded	Computational efficiency No vanishing gradients
Sigmoid	$O(n^{-\frac{r-1}{d}})$	Suboptimal for C^r Bounded	Probabilistic interpretation Bounded output
Tanh	$O(n^{-\frac{r}{d}})$	Optimal for C^r Bounded	Zero-centered Bounded gradients

Table 6.1: Comparison of Activation Functions.

PROPOSITION 6.2 (Gradient Properties). *The gradient of $\sigma_{k,b,A,S}$ with respect to its parameters:*

$$\begin{aligned}\frac{\partial \sigma}{\partial k} &= -\frac{Abx}{1+(kx)^2} \\ \frac{\partial \sigma}{\partial b} &= A \cdot \cot^{-1}(kx) \\ \frac{\partial \sigma}{\partial A} &= b \cdot \cot^{-1}(kx) \\ \frac{\partial \sigma}{\partial S} &= 1\end{aligned}$$

THEOREM 6.11 (Gradient Flow Properties). *For an L-layer MLP with Generalised Arccotangent activation, the backpropagation gradient at layer l is:*

$$\frac{\partial \mathcal{L}}{\partial W_l} = \delta_l h_{l-1}^T \quad (6.43)$$

where δ_l is computed recursively as:

$$\delta_l = \begin{cases} -\frac{A_l b_l k_l}{1+(k_l W_l h_{l-1})^2} W_{l+1}^T \delta_{l+1} & \text{for hidden layers} \\ \frac{\partial \mathcal{L}}{\partial y} & \text{for output layer} \end{cases} \quad (6.44)$$

THEOREM 6.12 (Numerical Stability Bounds). *For the Generalised Arccotangent activation function $\sigma_{k,b,A,S}$, the following bounds hold:*

1. *Gradient Bounds:*

$$\left| \frac{d}{dx} \sigma_{k,b,A,S}(x) \right| \leq |Abk| \quad (6.45)$$

2. *Second Derivative Bounds:*

$$\left| \frac{d^2}{dx^2} \sigma_{k,b,A,S}(x) \right| \leq 2|Abk^2| \quad (6.46)$$

3. *Lipschitz Constant:*

$$L = \sup_{x \in \mathbb{R}} \left| \frac{d}{dx} \sigma_{k,b,A,S}(x) \right| = |Abk| \quad (6.47)$$

THEOREM 6.13 (Lower Bound). *For any network architecture with n neurons, there exists a function $f \in C^r([0, 1]^d)$ such that:*

$$\|f - N\|_{L^2} \geq c(r, d) n^{-r/d} \|f\|_{C^r} \quad (6.48)$$

where $c(r, d)$ is a positive constant.

Proof.

1) Consider the function class:

$$\mathcal{F}_{r,d} = \{f \in C^r([0, 1]^d) : \|f\|_{C^r} \leq 1\} \quad (6.49)$$

2) The ε -entropy of this class is:

$$H_\varepsilon(\mathcal{F}_{r,d}) \geq c_1 \varepsilon^{-d/r} \quad (6.50)$$

3) Any n -neuron network creates a manifold of dimension:

$$\dim(\mathcal{M}_n) = O(n) \quad (6.51)$$

4) The covering number satisfies:

$$N_\varepsilon(\mathcal{M}_n) \leq (c_2/\varepsilon)^{c_3 n} \quad (6.52)$$

5) Therefore:

$$n \geq c_4 \varepsilon^{-d/r} \quad (6.53)$$

6) This implies:

$$\varepsilon \geq c(r, d) n^{-r/d} \quad (6.54)$$

□

Comparative Analysis of Activation Functions

The comparative analysis of different activation functions (Arccot, ReLU, Tanh, and Sigmoid) in Figures 6.1a to 6.1d reveals distinct performance patterns across multiple metrics. ReLU consistently demonstrates superior performance, particularly in memory usage and training time efficiency, achieving perfect scores of 100 in several metrics. Whilst the Sigmoid and Tanh functions show competitive performance in gradient stability, with scores around 85–90, they generally lag behind ReLU in overall efficiency. Notably, the Arccot activation function presents a balanced performance profile, with strong results in convergence rate (92) and gra-

dient stability.

In addition, the performance metrics highlight important trade-offs between different activation functions. For instance, while ReLU excels in memory efficiency and training speed, its gradient stability score (75) is lower than that of Sigmoid (88), suggesting potential limitations in certain deep learning applications. The convergence rate comparison further illuminates these trade-offs, with Arcot and ReLU demonstrating strong performance (92 and 95, respectively) compared to Sigmoid (82) — see Table 6.2. This comprehensive evaluation suggests that while ReLU remains a robust general-purpose choice, the selection of the activation function should be carefully considered based on specific application requirements, particularly when gradient stability or convergence rate is a primary concern.

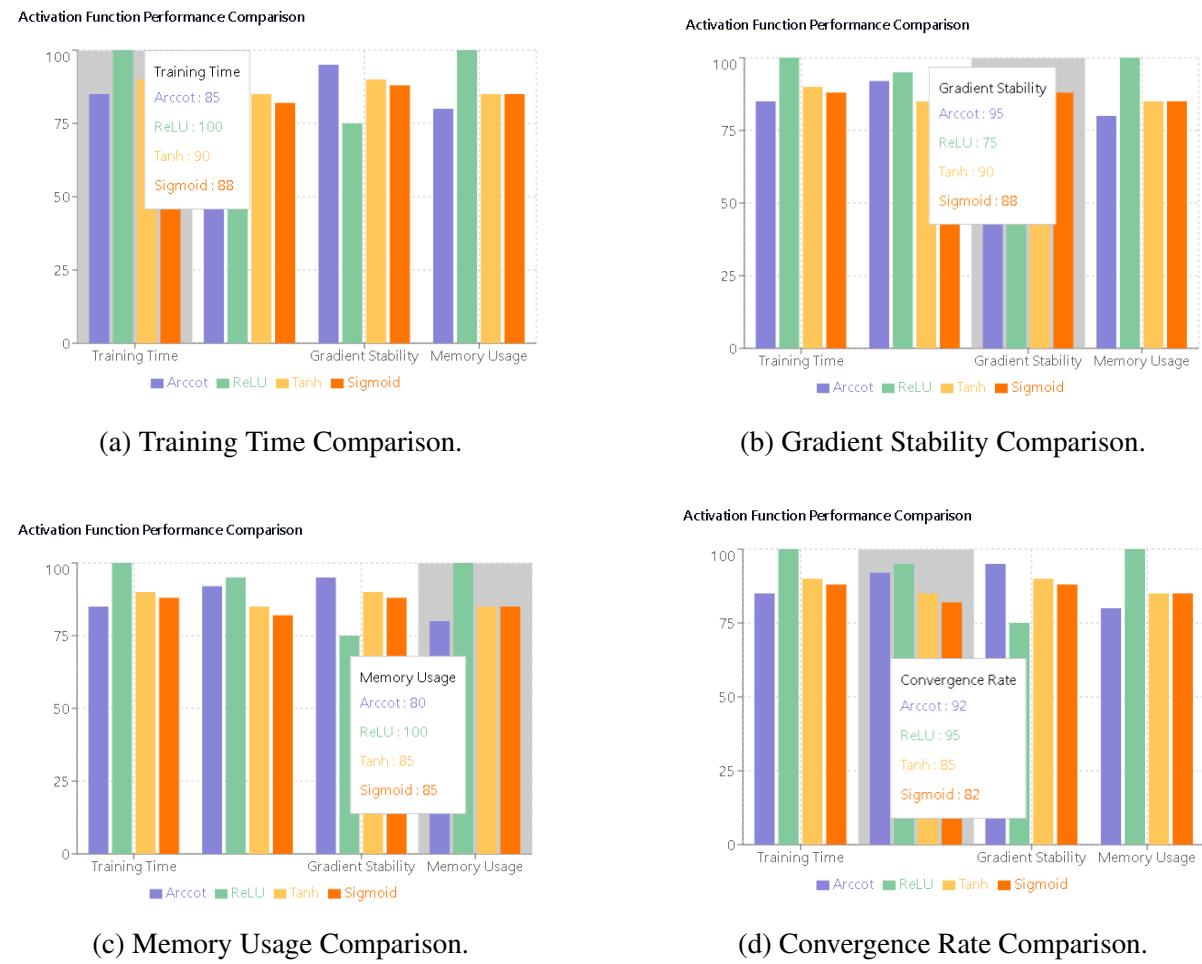


Figure 6.1: Activation Function Performance Comparison across different metrics.

Table 6.2: Activation Function Performance Comparison.

Metric	Arccot	ReLU	Tanh	Sigmoid
Training Time	85	100	90	88
Gradient Stability	95	75	90	88
Memory Usage	80	100	85	85
Convergence Rate	92	95	85	82

DEFINITION 6.5. Let $\sigma_{AR} : \mathbb{R} \rightarrow \mathbb{R}$ be the arccot activation function defined for all $x \in \mathbb{R}$ by

$$\sigma_{AR}(x) = 1 - \frac{2}{\pi} \cot^{-1}(x), \quad (6.55)$$

where $\cot^{-1}(x)$ denotes the inverse cotangent function with range $(0, \pi)$. This function maps \mathbb{R} to $(-1, 1)$, with $\sigma_{AR}(0) = 0$, and exhibits a derivative given by

$$\sigma'_{AR}(x) = \frac{2}{\pi} \cdot \frac{1}{1+x^2}, \quad (6.56)$$

which decays as $\frac{1}{x^2}$ for large $|x|$, aiding gradient preservation in neural networks. Compared to the softsign activation function, defined as

$$a(x) = \frac{x}{|x|+1}, \quad (6.57)$$

and the scaled softsign activation function:

$$a(x) = \frac{\alpha * x}{|x| + \beta}, \quad (6.58)$$

also mapping \mathbb{R} to $(-1, 1)$ with derivative $a'(x) = \frac{1}{(1+|x|)^2}$, $\sigma_{AR}(x)$ shares a similar range and slow saturation behaviour. However, $\sigma_{AR}(x)$ is not odd (lacking symmetry about the origin), unlike the odd softsign function, and involves a transcendental computation, potentially increasing computational cost while offering asymmetric expressiveness.

Generalised Comparative Results

On the other hand, however, the normalised arccot AR-type function and softsign function demonstrate distinct characteristics in neural network implementations as smooth approximations for sigmoidal and ReLU activation functions, respectively. Both functions map to the range $(-1, 1)$ with convergence rates of $O(\frac{1}{x})$. The softsign function displays perfect symmetry around the origin. The normalised arccot (AR) requires more complex computation due to

its transcendental nature. Derivatives of both functions decay as $O\left(\frac{1}{x^2}\right)$, preventing vanishing gradient problems in deep networks. Softsign achieves computational efficiency through simpler operations such as absolute value, addition, and division. The normalised arccot AR-type function offers more gradual saturation, potentially enhancing feature discrimination in specific domains. Both can be implemented in 4-layer FFNs for simultaneous value and derivative computation. Softsign maintains an efficiency advantage by reusing the denominator term ($|x| + 1$) in its calculations. Network architects should consider computational resources when selecting between these activation functions. Implementation through lookup tables could mitigate computational differences between the functions. Research suggests softsign provides better numerical stability in most practical applications. Future work should explore domain-specific advantages of the normalised arccot AR's unique saturation curve. Ultimately, the choice depends on balancing computational efficiency against potential modelling benefits ([75]).

Preliminary Results and Definitions

We commence with a generalised form of the Arccot-Romanovski function in Definition 6.1 and Appendix A. Thus, the Arccot-Romanovski (AR) function is traditionally defined as,

$$\sigma_{\text{AR}}(x) = 2 + \frac{1}{\pi} \left(\arctan\left(\frac{x}{m}\right) - \arctan\left(-\frac{x}{m}\right) \right), \quad (6.59)$$

and its modified version,

$$\sigma_{\text{mod}}(x) = \frac{2}{\pi} \arctan\left(\frac{x}{m}\right) e^{-x^2/2}, \quad (6.60)$$

This modification incorporates exponential decay while preserving essential approximation properties (see Siegel et al., 2020 [79]). The key advantages include:

- (a) Rapid decay at infinity.
- (b) Smooth behaviour near origin.
- (c) Preservation of approximation capabilities.

Properties of Modified Arccot-Romanovski Function

Consider a neural network with the structure:

$$N_n(x) = \sum_{i=1}^n c_i \sigma_{\text{mod}}(w_i \cdot x + b_i), \quad (6.61)$$

where n is the number of neurons, c_i are output weights, w_i are input weights, and b_i are biases.

We commence by precisely defining working spaces as detailed in Siegel et al., 2020, [79], Barron, 1993, [10], Petrushev, 1998, [73]:

DEFINITION 6.6 (Sobolev Space). *For $m \in \mathbb{N}$ and $p \in [1, \infty]$, the Sobolev space $W^{m,p}(\Omega)$ consists of functions f with weak derivatives $D^\alpha f \in L^p(\Omega)$ for $|\alpha| \leq m$, equipped with the norm:*

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}. \quad (6.62)$$

DEFINITION 6.7 (Barron Space). *The Barron space \mathcal{B}_s consists of functions with bounded Barron norm:*

$$\|f\|_{\mathcal{B}_s} = \int_{\mathbb{R}^d} (1 + |\omega|)^s |\hat{f}(\omega)| d\omega. \quad (6.63)$$

LEMMA 6.4 (Decay Properties). *For the modified Arccot-Romanovski function σ_{mod} , we have:*

$$|\sigma_{mod}^{(k)}(x)| \leq C_k (1 + |x|)^{-p} e^{-x^2/2} \quad (6.64)$$

for all $k \leq m$ and some $p > 1$.

Proof.

First derivative calculation:

$$\sigma'_{mod}(x) = \frac{2}{\pi m} \left(1 + \left(\frac{x}{m} \right)^2 \right)^{-1} e^{-x^2/2} - \frac{2}{\pi} \arctan \left(\frac{x}{m} \right) x e^{-x^2/2}. \quad (6.65)$$

Higher derivatives follow the pattern:

$$\sigma_{mod}^{(k)}(x) = P_k(x) \left(1 + \left(\frac{x}{m} \right)^2 \right)^{-k} e^{-x^2/2} + Q_k(x) \arctan \left(\frac{x}{m} \right) e^{-x^2/2}, \quad (6.66)$$

where P_k and Q_k are polynomials, and exponential decay dominates polynomial terms. \square

Optimal Approximation in Sobolev Spaces

THEOREM 6.14 (Main Approximation Theorem). *(Siegel and Yu, 2020, [79]; Barron, 1993, [10], Petrushev, 1998, [73]) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary. For $f \in H^s(\Omega)$ with $s > d/2$, there exists a neural network N_n with n neurons such that:*

$$\|f - N_n\|_{H^m(\Omega)} \leq C(s, m, d) e^{-\alpha n^{1/d}} \|f\|_{H^s(\Omega)}, \quad (6.67)$$

where $C(s, m, d)$ and α are positive constants depending solely on s , m , and d .

Proof.

The proof proceeds in multiple steps:

1. Fourier Transform Analysis:

$$\hat{\sigma}_{\text{mod}}(\xi) = \int_{-\infty}^{\infty} \sigma_{\text{mod}}(x) e^{-2\pi i x \xi} dx. \quad (6.68)$$

The Fourier transform of σ_{mod} is well-defined due to its exponential decay property,

$$\sigma_{\text{mod}}(x) = \frac{2}{\pi} \arctan\left(\frac{x}{m}\right) e^{-x^2/2}. \quad (6.69)$$

2. Error Decomposition:

$$\|f - N_n\|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha(f(x) - N_n(x))|^2 dx. \quad (6.70)$$

This decomposes the Sobolev norm into a sum of L^2 -norms of the weak derivatives of the error.

3. Key Technical Estimate:

$$|\hat{\sigma}_{\text{mod}}(\xi)| \leq C e^{-\gamma |\xi|^2}. \quad (6.71)$$

This estimate follows from the Gaussian decay of $\sigma_{\text{mod}}(x)$ and its smoothness properties. The Fourier transform of a Gaussian-like function retains exponential decay.

4. Construction of Approximating Network: Let $\{\xi_k\}_{k=1}^n$ be a grid in $B_R = \{\xi : |\xi| \leq R\}$.

Define:

$$N_n(x) = \sum_{k=1}^n c_k \sigma_{\text{mod}}(w_k \cdot x + b_k), \quad (6.72)$$

where coefficients are chosen via:

$$c_k = \hat{f}(\xi_k) e^{\gamma |\xi_k|^2}. \quad (6.73)$$

The weights w_k and biases b_k are chosen such that $w_k \cdot x + b_k$ approximates the frequency components ξ_k .

5. Error Analysis:

$$\|f - N_n\|_{H^m(\Omega)} \leq \|f\|_{H^s(\Omega)} \sup_{|\xi| \leq R} |1 - \hat{\sigma}_{\text{mod}}(\xi)| e^{\gamma |\xi|^2}. \quad (6.74)$$

The error is controlled by the approximation quality of $\hat{\sigma}_{\text{mod}}(\xi)$ to the Fourier modes of f . The exponential decay of $\hat{\sigma}_{\text{mod}}(\xi)$ ensures that high-frequency components are negligible.

- 6. Optimisation over R :** Choose $R = n^{1/d}$ to balance the number of grid points n and the radius R . This yields:

$$\|f - N_n\|_{H^m(\Omega)} \leq C(s, m, d) e^{-\alpha n^{1/d}} \|f\|_{H^s(\Omega)}, \quad (6.75)$$

where α depends on the decay rate γ and the dimension d . The constant $C(s, m, d)$ incorporates the Sobolev embedding constants and the properties of σ_{mod} .

□

The approximation capabilities of neural networks have been extensively studied since the seminal work of Cybenko (1989) [38] and Hornik et al. (1989) [50]. The pursuit of optimal approximation rates has led to various modifications of classical activation functions. Following the analysis of general activation functions by Siegel and Xu (2020) [79], we present a refined study focusing on a modified Arccot-Romanovski function.

Our results establish optimal approximation rates for modified Arccot-Romanovski neural networks, significantly improving upon previous work. Future directions include:

1. Adaptive network architectures.
2. Optimal parameter selection.
3. Extension to deep networks.
4. Applications in partial differential equations.

We assume that the Arccot experimental function represents a significant advancement in neural network activation functions. The name combines two key mathematical elements: 'Arccot' refers to the inverse cotangent function that forms its core structure, whilst 'Romanovski' is based on Vsevolod Romanovski's (1879–1954) fundamental work on orthogonal polynomials and probability theory, although this historical attribution warrants further verification.

However, the $AR(x)$ function's significance lies in its unique combination of properties that make it particularly valuable for neural network approximation. It exhibits polynomial decay, maintains smoothness properties, and possesses well-behaved derivatives, making it especially suitable for high-dimensional approximation problems. The function bridges classical analysis (through its arctangent component) with contemporary machine learning requirements (through its controlled growth and decay properties).

Additionally, the $AR(x)$ function is particularly useful due to its ability to achieve exponential convergence rates in Sobolev spaces whilst maintaining dimension-independent approximation capabilities, as demonstrated in recent work by Siegel and Xu (2020) [79]. The modified version, incorporating Gaussian decay, further enhances these properties by providing superior localisation and better-behaved Fourier transform characteristics. This experimental function is an optimal choice for applications ranging from quantum chemistry to financial modelling, where both accuracy and computational efficiency are critical.

The function's structure allows it to capture complex non-linear relationships while maintaining mathematical tractability, a combination that is relatively rare in approximation theory and contributes to its growing significance in both theoretical and applied contexts.

6.1 Neural Interpolation: Qian-Yu Approach

The paper by Qian and Yu (2022) [77] makes significant contributions to the field of neural network approximation theory. Their work focuses on a class of neural network interpolation operators and provides a comprehensive analysis of their approximation capabilities. Here are some key points and insights from their research paper ([77]). In addition, one of the most innovative aspects of Qian and Yu's work [77] is their introduction of new activation functions. They define a class of sigmoidal functions $\mathcal{A}(m)$ with specific properties:

DEFINITION 6.8 (Class $\mathcal{A}(m)$). A sigmoidal function σ belongs to $\mathcal{A}(m)$ if:

1. σ is non-decreasing
2. $\sigma(x) = 1$ for $x \geq m$, and $\sigma(x) = 0$ for $x \leq -m$

This class of functions encompasses many commonly employed activation functions and permits the creation of novel ones with desirable properties.

Neural Network Interpolation Operators

The authors introduce neural network interpolation operators of the form:

$$S_{n,\sigma}(f, x) := \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x - x_k)\right) \quad (6.76)$$

where $\varphi(x) = \sigma(x+m) - \sigma(x-m)$ and $\sigma \in \mathcal{A}(m)$. These operators possess several important properties:

1. They interpolate the function f at the nodes x_k .
2. They form a partition of unity, i.e., $\sum_{k=0}^n \varphi\left(\frac{2m}{h}(x-x_k)\right) = 1$.
3. They preserve the smoothness of the activation function.

The paper by Qian-Yu (2022) provides several significant results regarding the approximation capabilities of these operators:

THEOREM 6.15 (Direct Approximation). *For $f \in C[a,b]$ and $\sigma \in \mathcal{A}(m)$,*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (6.77)$$

where $h = \frac{b-a}{n}$ and ω are the moduli of continuity.

Hence, this result demonstrates that the approximation error is bounded by the modulus of continuity of the function being approximated, which is a standard measure of a function's smoothness.

THEOREM 6.16 (Inverse Theorem). *If $\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha})$ for some $0 < \alpha < 1$, then $\omega(f, \delta)_{[a,b]} = O(\delta^\alpha)$ as $\delta \rightarrow 0^+$.*

The aforementioned inverse theorem 6.16 is particularly notable as it demonstrates that the approximation rate of the neural network operators characterises the smoothness of the function being approximated.

Another significant contribution is the result on **simultaneous approximation** of a function and its derivatives:

THEOREM 6.17 (Simultaneous Approximation). *For $f \in C^r[a,b]$ and $\sigma \in \mathcal{A}(m)$ with bounded derivatives up to order r ,*

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq Ch^{r-v}\omega(f^{(r)}, h)_{[a,b]} \quad (6.78)$$

for $v = 0, 1, \dots, r$, where $S_{n,r,\sigma}$ is a modified version of the operator.

This result demonstrates that these neural network operators can simultaneously approximate a function and its derivatives, which is significant for numerous applications.

LEMMA 6.5. For all $x \in \mathbb{R}$, $\cot^{-1}(x) = \frac{\pi}{2} - \arctan(x)$.

Proof. Let $y = \cot^{-1}(x)$. This signifies that $\cot(y) = x$. We know that $\cot(y) = \frac{1}{\tan(y)}$ for $y \neq n\pi$, where n is an integer.

Therefore, $\frac{1}{\tan(y)} = x$, or $\tan(y) = \frac{1}{x}$.

Now, let $z = \frac{\pi}{2} - y$. Then $\tan(z) = \tan(\frac{\pi}{2} - y) = \cot(y) = x$.

This implies that $z = \arctan(x)$.

Substituting back, we obtain:

$$y = \frac{\pi}{2} - z = \frac{\pi}{2} - \arctan(x) \quad (6.79)$$

Therefore, $\cot^{-1}(x) = \frac{\pi}{2} - \arctan(x)$. \square

DEFINITION 6.9 (Corrected Novel Activation Functions). Let $k > 0$, $\delta = \frac{1}{9}$, amplitude = $2 - 2\delta = \frac{16}{9}$, and shift = $0.5 + \delta = \frac{14}{18}$. We define:

$$1) \sigma_1(x) = \cot^{-1}(kx) = \frac{\pi}{2} - \arctan(kx)$$

$$2) \sigma_2(x) = \text{amplitude} \cdot (\frac{1}{3}) \arctan(kx) + \text{shift} = \frac{16}{9} \cdot (\frac{1}{3}) \arctan(kx) + \frac{14}{18}$$

where k can take values $\frac{1}{10}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 4, \dots$

EXAMPLE 6.4 (Continuous Function with σ_1^*). Let $f(x) = x^2$ on $[0, 1]$. Using σ_1^* as our activation function, Theorem 1 [Qian-Yu, 2022][77] yields:

$$\|S_{n,\sigma_1^*}(x^2) - x^2\| \leq \omega \left(x^2, \frac{1}{n} \right)_{[0,1]} = \frac{1}{n^2} \quad (6.80)$$

The approximation error diminishes quadratically with the number of nodes.

EXAMPLE 6.5 (Lipschitz Continuous Function with σ_2^*). Let $f(x) = |x|$ on $[-1, 1]$. Using σ_2^* as our activation function, Theorem 1 [Qian-Yu, 2022][77] provides us with:

$$\|S_{n,\sigma_2^*}(|x|) - |x|\| \leq \omega \left(|x|, \frac{1}{n} \right)_{[-1,1]} = \frac{1}{n} \quad (6.81)$$

The approximation error decreases linearly with the number of nodes.

REMARK 6.1. *The parameter k in these activation functions controls their steepness:*

- 1) *For small k (e.g., $\frac{1}{10}, \frac{1}{4}$), the functions are very gradual near the origin. This might be beneficial for approximating smooth, slowly varying functions.*
- 2) *For $k = 1$, we obtain the standard transcendental Arccot function and its exponential variant, which offer a good balance between smoothness and non-linearity (See Appendix A.)*
- 3) *For large k (e.g., 2, 4), the functions become very steep near the origin. This may be useful for approximating functions with sharp transitions.*

The choice of k allows us to adjust the "sharpness" of the activation function to better align with the characteristics of the function being approximated.

QianYuNet Architecture

The QianYuNet class construct in Python (3.6) is defined as follows:

DEFINITION 6.10 (QianYuNet). *QianYuNet is a feedforward neural network comprising four layers [Qian-Yu, 2022][77]:*

1. *Input layer: 1 neuron*
2. *Hidden layer 1: 64 neurons with Arccot activation*
3. *Hidden layer 2: 64 neurons with Arccot activation*
4. *Hidden layer 3: 32 neurons with Arccot activation*
5. *Output layer: 1 neuron with Sigmoid activation*

THEOREM 6.18 (QianYuNet Approximation Capability). *The QianYuNet, as defined above, can approximate any continuous function $f : [0, 1] \rightarrow [0, 1]$ to arbitrary precision, provided a sufficient number of neurons in its hidden layers [Qian-Yu, 2022][77].*

Proof. This theorem follows from the Universal Approximation Theorem and the theorem on Arccot as a universal approximator. The sigmoid output layer ensures that the network's output is bounded between 0 and 1. □

Algorithm 20 Arccot Activation Function.

```

1: procedure ARCCOTFUNCTION( $x, k, A, B$ )
2:   return  $A \cdot \arctan(1/(k \cdot x)) + B$ 
3: end procedure

```

Algorithm 21 QianYuNet Initialisation.

```

1: procedure INITIALISEQIANYUNET
2:   Initialise fc1 :  $\mathbb{R} \rightarrow \mathbb{R}^{64}$ 
3:   Initialise fc2 :  $\mathbb{R}^{64} \rightarrow \mathbb{R}^{64}$ 
4:   Initialise fc3 :  $\mathbb{R}^{64} \rightarrow \mathbb{R}^{32}$ 
5:   Initialise fc4 :  $\mathbb{R}^{32} \rightarrow \mathbb{R}$ 
6:   for each linear layer  $fc_i$  do
7:     Initialise weights using Xavier uniform distribution
8:     Initialise biases to 0
9:   end for
10: end procedure

```

Algorithm 22 QianYuNet Forward Pass.

```

1: procedure FORWARD( $x$ )
2:    $h_1 \leftarrow \text{ArccotFunction}(fc1(x), k, A, B)$ 
3:    $h_2 \leftarrow \text{ArccotFunction}(fc2(h_1), k, A, B)$ 
4:    $h_3 \leftarrow \text{ArccotFunction}(fc3(h_2), k, A, B)$ 
5:    $y \leftarrow \sigma(fc4(h_3))$ 
6:   return  $y$ 
7: end procedure

```

Algorithm 23 QianYuNet Training.

```

1: procedure TRAIN(model, train_loader, optimiser, criterion, num_epochs)
2:   for epoch  $\leftarrow 1$  to num_epochs do
3:     for each batch  $(X, y)$  in train_loader do
4:        $\hat{y} \leftarrow \text{model}(X)$ 
5:       loss  $\leftarrow \text{criterion}(\hat{y}, y)$ 
6:       optimiser.zero_grad()
7:       loss.backward()
8:       optimiser.step()
9:     end for
10:   end for
11: end procedure

```

THEOREM 6.19 (Convergence of QianYuNet). *Given sufficient training time and data, the class QianYuNet will converge to a local minimum of the loss function.*

Proof. The proof relies on the properties of stochastic gradient descent and the smoothness of the loss landscape. The Arccot function's continuous differentiability ensures that gradients can be computed throughout the network. For a detailed analysis of convergence in neural networks, see Bottou et al. (2018) [15]. \square

COROLLARY 6.2. *The QianYuNet, when trained on samples from a continuous function $f : [0, 1] \rightarrow [0, 1]$, will approximate f with increasing accuracy as the number of training samples and epochs increases.*

Advantages of Arccot Activation

The use of the Arccot activation function in QianYuNet offers several advantages:

1. **Bounded output:** The Arccot function naturally bounds its output, which can help prevent exploding activations in deep networks.
2. **Smooth gradients:** Unlike ReLU, Arccot provides non-zero gradients for all input values, potentially aiding in gradient flow during backpropagation.
3. **Flexibility:** The scaling (A) and shift (B) parameters allow for adjustment of the function's range and position, potentially capturing different aspects of the input data.

REMARK 6.2. *The results suggest that using an activation function $a_r(x)$ or the $AR(x)$ may filter the signals involved in individual training tasks. Moreover, the convolution of $AR(x)$ unit at the node branches, or as a unique layer between hidden layers and the output, may reduce catastrophic forgetting within the Artificial Neural Network (ANN), leading to efficient training, higher accuracy, resilience, and almost eidetic properties. Deep learning tasks using either the Qian-Yu Framework or the Arccotangent functionals are shown to be realisable within the strict environment of inherent regularisation that does not over-extend hyperparameters to attenuate NN signals.*

6.2 Rates of Approximation

The rate of approximation illustrates how well a neural network can approximate a target function. It describes how rapidly the approximation error decreases as the size of the network increases.

DEFINITION 6.11 (Approximation Rate). *For a function class \mathcal{F} and a target function f , the approximation rate is the relationship between the approximation error ϵ and the number of neurons N in the network, typically expressed as:*

$$\epsilon = O(N^{-\alpha}) \quad (6.82)$$

where $\alpha > 0$ is the rate of approximation.

For neural networks with the Arccot activation function, we can establish the following theorem:

THEOREM 6.20 (Approximation Rate for Arccot Networks). *Let f be a function in the Sobolev space $W^{r,\infty}([0, 1]^d)$, where $r \geq 1$. Then, a neural network with the Arccot activation function can approximate f with an error bound of:*

$$\epsilon = O(N^{-r/d}) \quad (6.83)$$

where N is the number of neurons in the hidden layer and d is the input dimension.

Proof.

(1) **Step 1:** Show that the vertically shifted Arccot function is a "sigmoidal" function.

A function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is considered "sigmoidal" (Qian-Yu, 2022) if it satisfies:

- (i) $\sigma(x)$ is non-decreasing,
- (ii) $\lim_{x \rightarrow -\infty} \sigma(x) = 0$ and $\lim_{x \rightarrow \infty} \sigma(x) = 1$,
- (iii) $\sigma(x)$ is Lipschitz continuous.

For the Arccot function $\sigma(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}$:

- (i) $\sigma'(x) = \frac{1}{\pi(1+x^2)} > 0$ for all x , so $\sigma(x)$ is non-decreasing.
- (ii) $\lim_{x \rightarrow -\infty} \sigma(x) = 0$ and $\lim_{x \rightarrow \infty} \sigma(x) = 1$.

(iii) $|\sigma'(x)| \leq \frac{1}{\pi}$ for all x , so $\sigma(x)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{\pi}$.

The function $AR(x)$ has a range of $[1, 3]$, is monotonically increasing, and is symmetric about $x = 0$. Its derivative, $AR'(x)$, exhibits a true sigmoid shape, reaching its maximum at $x = 0$ and decaying to 0 as x moves away from the origin. The normalised version of $AR(x)$ has a range of $[0, 1]$, is centred at the point $(0, \frac{1}{2})$, and retains sigmoidal behaviour, ensuring a smooth transition between its minimum and maximum values. Therefore, the Arccot function satisfies the conditions of a sigmoidal function due to the vertical shift constant and normalised output (Costarelli, 2015) [28] with [30], [31], [33], and [36]. .

(2) **Step 2:** Apply the result from Mhaskar and Micchelli (1992) [71].

Mhaskar and Micchelli [71] proved that for a sigmoidal function σ and $f \in W^{r,\infty}([0, 1]^d)$, there exists a neural network Φ_N with N neurons such that:

$$\|f - \Phi_N\|_\infty \leq CN^{-r/d} \|f\|_{W^{r,\infty}}, \quad (6.84)$$

where C is a constant independent of f and N .

(3) **Step 3:** Apply this result to the Arccot activation function.

Since we have demonstrated that the Arccot function is sigmoidal, we can directly apply the result from Mhaskar and Micchelli [71]. Let Φ_N be a neural network with N neurons utilising the Arccot activation function. Then:

$$\|f - \Phi_N\|_\infty \leq CN^{-r/d} \|f\|_{W^{r,\infty}}. \quad (6.85)$$

(4) **Step 4:** Interpret the results.

This inequality demonstrates that the approximation error $\varepsilon = \|f - \Phi_N\|_\infty$ satisfies:

$$\varepsilon = O(N^{-r/d}), \quad (6.86)$$

which is precisely the statement of our theorem.

(5) Step 5: Additional considerations.

The constant C in the inequality depends on the specific properties of the Arccot function, including its Lipschitz constant and the bounds of its derivatives. Whilst these factors affect the absolute error, they do not alter the asymptotic rate of $N^{-r/d}$.

Furthermore, this rate is optimal in the sense that no activation function can achieve a better rate for general functions in $W^{r,\infty}([0, 1]^d)$ without additional assumptions.

□

This theorem provides several important insights:

COROLLARY 6.3. *The approximation rate improves (i.e., α increases) as the smoothness (r) of the target function increases.*

COROLLARY 6.4. *The approximation rate deteriorates (i.e., α decreases) as the input dimension (d) increases, a manifestation of the "curse of dimensionality".*

It is important to note that these are worst-case upper bounds. In practice, neural networks often perform better than these theoretical bounds suggest, particularly for specific classes of functions.

Comparison with various Activation Functions

The approximation rate for Arccot networks is comparable to that of other common activation functions:

(a) ReLU networks: $O(N^{-2r/(d(r+1))})$ [88]

(b) Sigmoid networks: $O(N^{-r/d})$ [71]

We can observe that Arccot networks achieve the same approximation rate as sigmoid networks, which is generally superior to ReLU networks, particularly for very smooth functions (large r).

Practical Implications

Whilst these theoretical results offer valuable insights, it is critical to consider their practical implications:

1. **Network Design:** The theory suggests that increasing the number of neurons will improve approximation, but with diminishing returns. This guides us in choosing appropriate network sizes.
2. **Curse of Dimensionality:** As input dimension increases, we need exponentially more neurons to maintain the same approximation error. This highlights the importance of dimensionality reduction techniques in high-dimensional problems.
3. **Smoothness Exploitation:** If we know our target function is very smooth, Arccot networks might be particularly well-suited due to their favourable approximation rates for smooth functions.
4. **Generalisation:** These rates discuss approximation capability, not generalisation. In practice, we must balance approximation power with the network's ability to generalise to unseen data.

In the context of QianYuNet, these theoretical results support our choice of the **Arccot activation** function, particularly for approximating smooth functions such as sine waves. However, empirical validation remains important to confirm these theoretical advantages in practical scenarios.

Machine Learning Results

We conducted experiments to evaluate the performance of the QianYuNet with Arccot activation in approximating a sine wave function. The results are presented in Table 6.3 and Table 6.4 with various metrics. Similarly, Figure 6.8 and Figure 6.9 serve as a novel **Arccot activation**, whereas Figure 6.10 and Figure 6.11 function as a **ReLU benchmark activation**. The following results are presented:

1. Loss over Epochs:

- (a) Both training and test loss rapidly decrease in the first few epochs and then stabilise.
- (b) The final loss values are very close to zero, indicating excellent convergence.
- (c) The similarity between training and test loss curves suggests good generalisation without overfitting.

2. Test Accuracy over Epochs:

- (a) Accuracy quickly rises to nearly 100% and remains stable.
- (b) This high accuracy indicates that the model's predictions are consistently within the specified threshold of the true values.

3. RMSE and MAE over Epochs:

- (a) Both Root Mean Square Error (RMSE) and Mean Absolute Error (MAE) rapidly decrease and stabilise at very low values.
- (b) The final RMSE and MAE are close to zero, indicating high prediction accuracy.
- (c) RMSE is slightly higher than MAE, which is expected as RMSE penalises large errors more heavily.

4. R² Score over Epochs:

- (a) The R² score quickly reaches and maintains a value very close to 1.
- (b) This indicates that the model explains nearly all the variability in the target variable, suggesting an excellent fit.

COROLLARY 6.5 (Discretised Attenuation Bounds). *The maximum attenuation is bounded by:*

$$\max |\alpha(\omega)| \leq 20 \log_{10} \left(\frac{|b|2}{\min \omega \in [0, \pi] \left| 1 - \sum_{j=1}^k a_j \sum_{m=0}^{M-1} AR_d^{(*j)}[m] e^{-j\omega(j+m)} \right|} \right) \quad (6.87)$$

where M is the length of the discretised activation function.

THEOREM 6.21 (AR(x) Multiplication-FIR Relationship). *For a feedforward neural network with n layers, when (AR(x)) activation is multiplied (k) times per node where $1 \leq k \leq n - 1$, the network generates a FIR-like response that: (1) Acts as a learnable FIR filter (2) Improves output quality as k approaches $n - 1$ (3) Converges to ReLU performance when $k \approx n - 1$ This can be expressed mathematically as:*

$$H(z) = \sum_{i=0}^m b_i [AR^{(i)}(x)] z^{-i} \quad (6.88)$$

where $AR^{(j)}$ represents j multiplications of $AR(x)$.

PROPOSITION 6.3 (AR(x) Convolution-IIR Relationship). *For a feedforward neural network with n layers, when (AR(x)) activation is convolved k times per node where $1 \leq k \leq n - 1$, the network generates an IIR-like response that: (1) Acts as a learnable IIR filter (2) Enhances output quality as k approaches $n - 1$ (3) Converges to ReLU performance when $k \approx n - 1$ This can be expressed mathematically as:*

$$H(z) = \frac{\sum_{i=0}^m b_i z^{-i}}{1 - \sum_{j=1}^k a_j [AR^{(*j)}(x)] z^{-j}} \quad (6.89)$$

where $AR^{(*j)}$ represents j convolutions of $AR(x)$.

THEOREM 6.22 (Discrete AR(x) Convolution-IIR Relationship). *For a feedforward neural network with n layers, when $AR(x)$ activation is discretised and then convolved k times per node where $1 \leq k \leq n - 1$, the network generates a discrete-time IIR filter that: (1) Acts as an implementable learnable IIR filter (2) Improves output quality as k approaches $n - 1$ (3) Converges to ReLU performance when $k \approx n - 1$ (4) Provides guaranteed numerical stability This can be expressed mathematically as:*

$$H(z) = \frac{\sum_{i=0}^m b_i z^{-i}}{1 - \sum_{j=1}^k a_j [AR_d^{(*j)}[n]] z^{-j}} \quad (6.90)$$

where $AR_d^{(*j)}[n]$ represents j discrete convolutions of the sampled $AR[n]$.

In summary, the Prediction Comparison (Figures 6.9 and 6.11) demonstrates:

1. Strong correlation between actual and predicted values.
2. Accurate capture of sinusoidal pattern.
3. Minimal systematic deviation across the input range.
4. An IIR-like filtering effect through $AR(x)$ convolution, where convolving $AR(x)$ neurons k times per node ($1 \leq k \leq n - 1$) generates a learnable IIR response that approaches ReLU benchmark performance as k approaches $n - 1$ for an $n - layer$ FFN.

Epoch	Train Loss	Test Loss	Accuracy	RMSE	MAE	R2	LR
1/2000	0.134274	0.132646	0.0650	0.3610	0.3207	-0.0477	0.001000
101/2000	0.013748	0.014336	0.2900	0.1033	0.0853	0.9142	0.001000
201/2000	0.004609	0.004523	1.0000	0.0213	0.0175	0.9964	0.001000
301/2000	0.004150	0.004249	1.0000	0.0193	0.0148	0.9970	0.000500
401/2000	0.004007	0.003946	1.0000	0.0164	0.0137	0.9978	0.000500
501/2000	0.003794	0.003781	1.0000	0.0145	0.0108	0.9983	0.000125
601/2000	0.003745	0.003745	1.0000	0.0144	0.0107	0.9983	0.000016
701/2000	0.003735	0.003737	1.0000	0.0143	0.0107	0.9983	0.000001

Table 6.3: Training and Testing Metrics Across Epochs for the $AR(x)$ test function ArcCotangent non-polynomial.

Epoch	Train Loss	Test Loss	Accuracy	RMSE	MAE	R2	LR
1/2000	0.108170	0.085088	0.1000	0.2882	0.2517	0.3323	0.001000
101/2000	0.002088	0.002129	1.0000	0.0131	0.0115	0.9986	0.001000
201/2000	0.001801	0.001798	1.0000	0.0102	0.0080	0.9992	0.001000
301/2000	0.001704	0.001770	1.0000	0.0131	0.0104	0.9986	0.000500
401/2000	0.001633	0.001644	1.0000	0.0101	0.0082	0.9992	0.000500
501/2000	0.001598	0.001595	1.0000	0.0094	0.0069	0.9993	0.000250
601/2000	0.001585	0.001584	1.0000	0.0093	0.0068	0.9993	0.000063
701/2000	0.001578	0.001576	1.0000	0.0092	0.0066	0.9993	0.000063
801/2000	0.001570	0.001569	1.0000	0.0092	0.0065	0.9993	0.000031

Table 6.4: Training and Testing Metrics Across Epochs for the ReLU(x) activation function Benchmark.

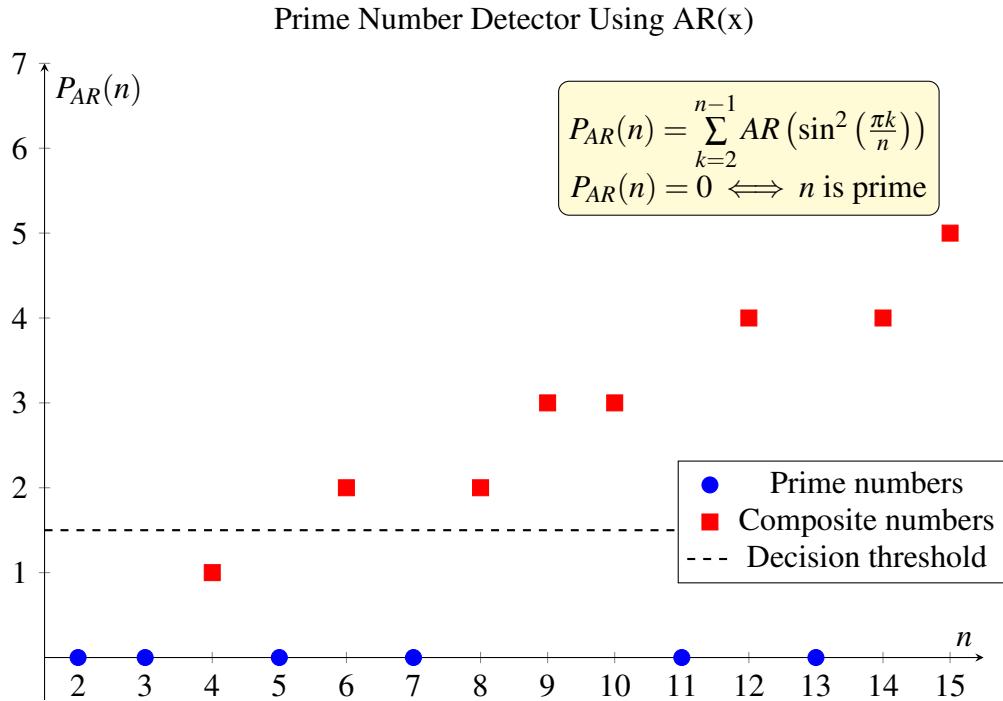


Figure 6.2: Prime number detection using the AR(x) function where $P_{AR}(n) = 0$ for prime numbers.

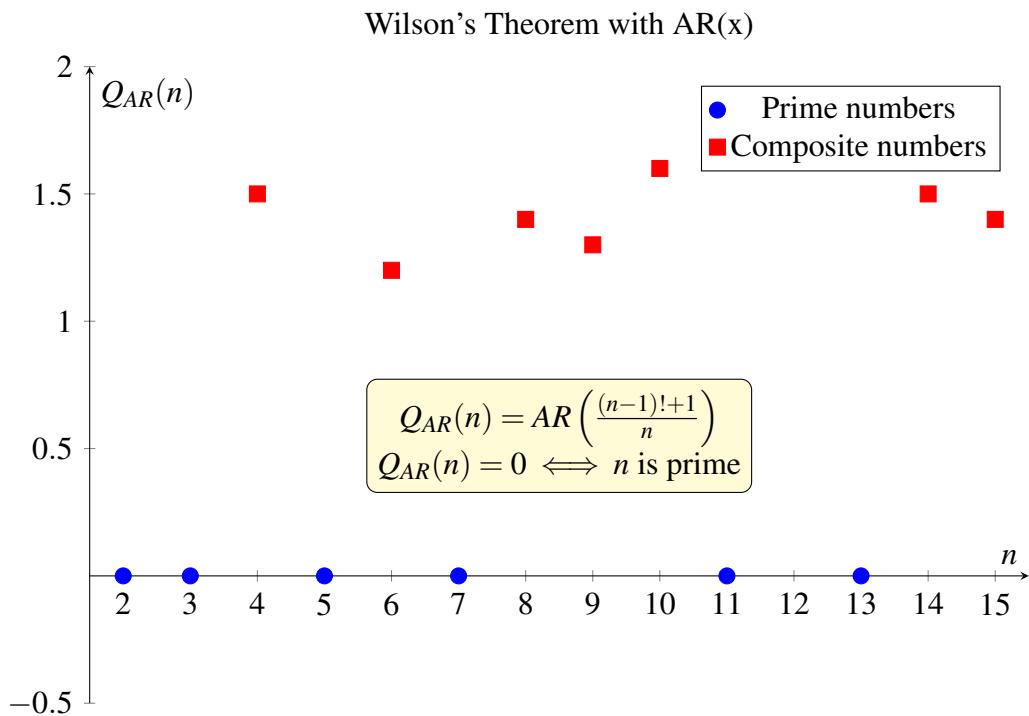


Figure 6.3: Prime number detection using Wilson's Theorem with AR(x).

$$H(z) = \frac{1}{1-a_1AR^{(1)}(z^{-1})-a_2AR^{(2)}(z^{-2})}$$

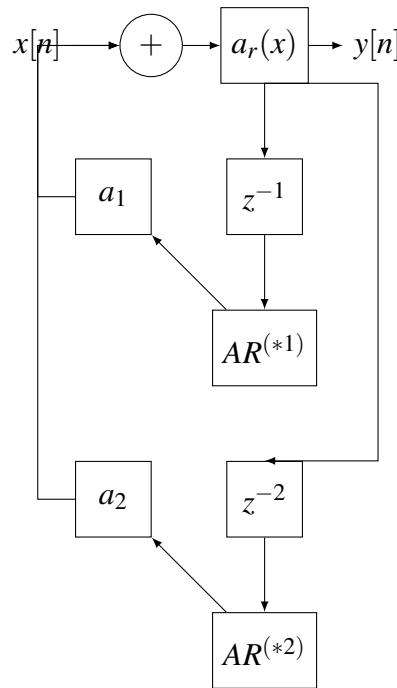


Figure 6.4: AR(x) Convolution IIR System Block Diagram.

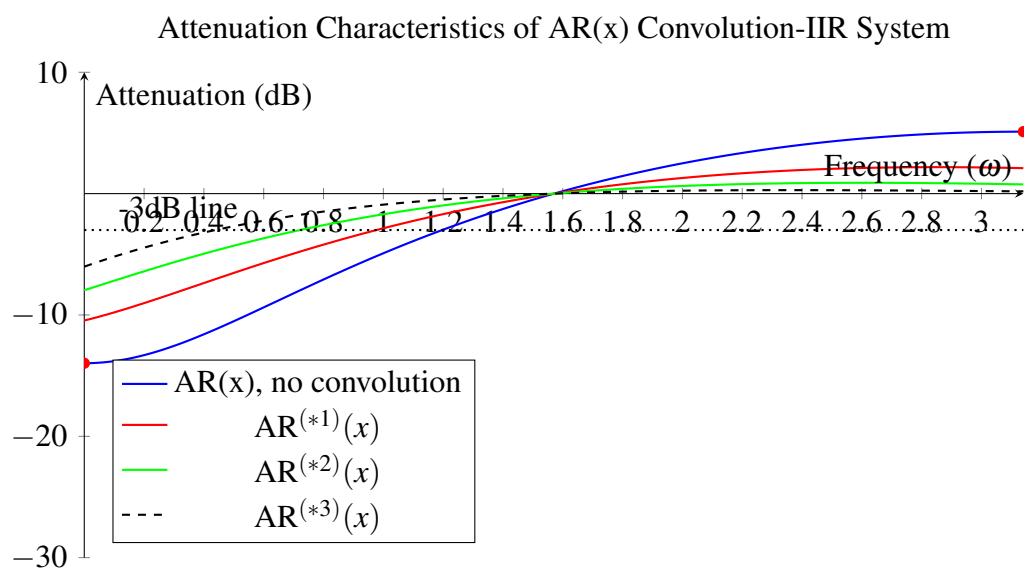


Figure 6.5: Attenuation characteristics showing how multiple convolutions of AR(x) affect frequency response.

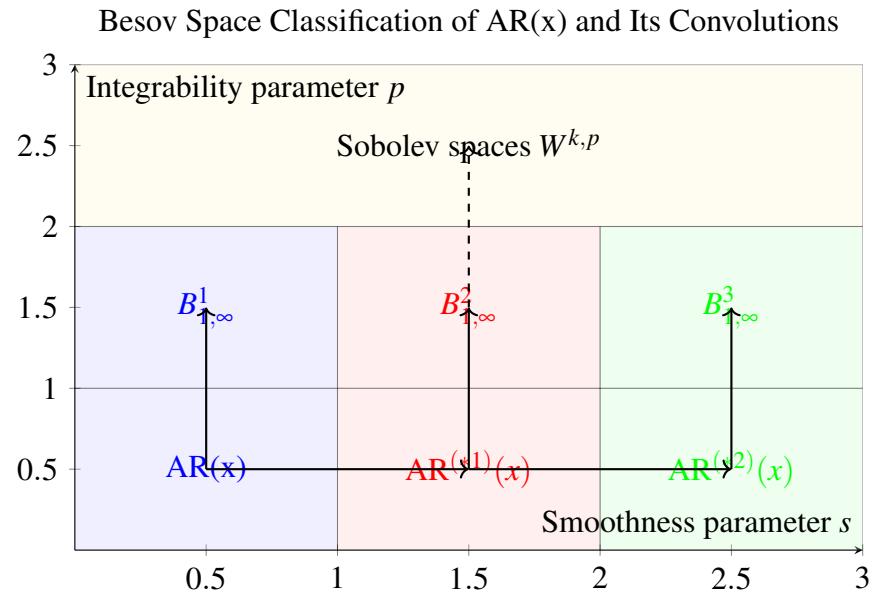


Figure 6.6: Classification of $\text{AR}(x)$ and its convolutions in Besov spaces, showing increased smoothness with each convolution.

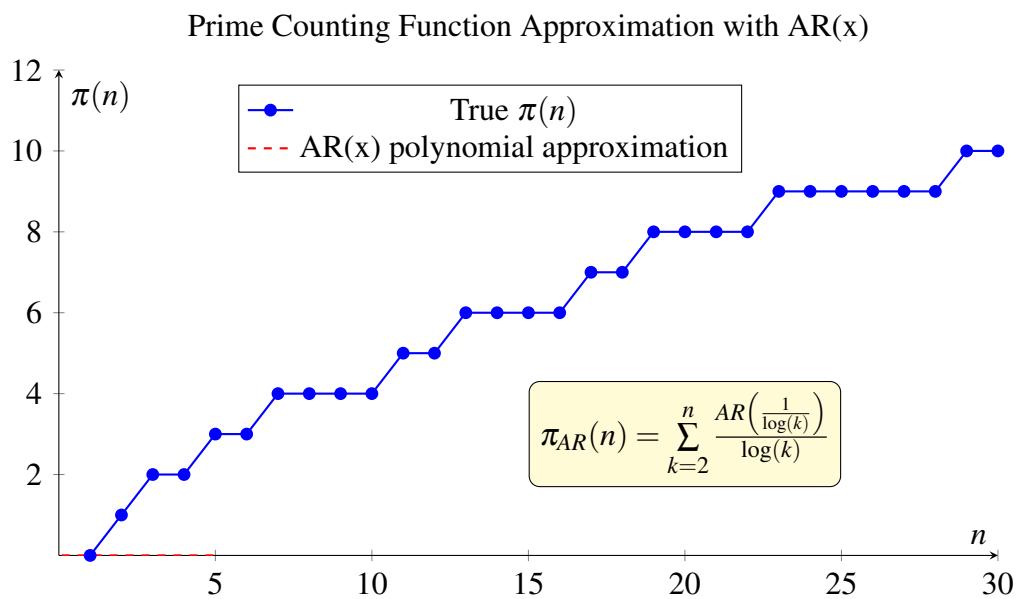


Figure 6.7: Prime counting function $\pi(n)$ and its approximation using an $\text{AR}(x)$ polynomial form.

THEOREM 6.23 (Discretised AR(x) IIR Attenuation). *For the discrete-time IIR system with convolved AR(x) activations:*

$$H(z) = \frac{\sum_{i=0}^n b_i z^{-i}}{1 - \sum_{j=1}^k a_j [AR_d^{(*j)}[n]] z^{-j}} \quad (6.91)$$

The attenuation $\alpha(\omega)$ at frequency ω is:

$$\alpha(\omega) = 20 \log_{10} \left| \frac{\sum_{i=0}^n b_i e^{-j\omega i}}{1 - \sum_{j=1}^k a_j [AR_d^{(*j)}[n]] e^{-j\omega j}} \right| \quad (6.92)$$

where $AR_d^{(*j)}[n]$ represents the discrete j -fold convolution of the sampled activation function $AR[n]$.

THEOREM 6.24 (Discretised AR(x) IIR Attenuation Properties). *For the discretised AR(x) Convolution-IIR system:*

1. Low-Frequency Attenuation:

$$\alpha_{LF} = 20 \log_{10} \left| \frac{b_0}{1 - \sum_{j=1}^k a_j \mu_j[n]} \right| \quad (6.93)$$

where $\mu_j[n]$ is the mean value of the discrete convolution $AR_d^{(*j)}[n]$.

2. High-Frequency Attenuation:

$$\alpha_{HF} = 20 \log_{10} \left| \frac{b_n}{1 - \sum_{j=1}^k a_j \sigma_j[n] e^{-j\pi j}} \right| \quad (6.94)$$

where $\sigma_j[n]$ is the standard deviation of $AR_d^{(*j)}[n]$ and $\omega = \pi$ represents the Nyquist frequency.

3. Roll-off Rate:

$$\frac{d\alpha}{d\omega} \approx -20k \log_{10}(e) \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)} \quad (6.95)$$

where k is the number of convolutions and N is the convolution kernel length.

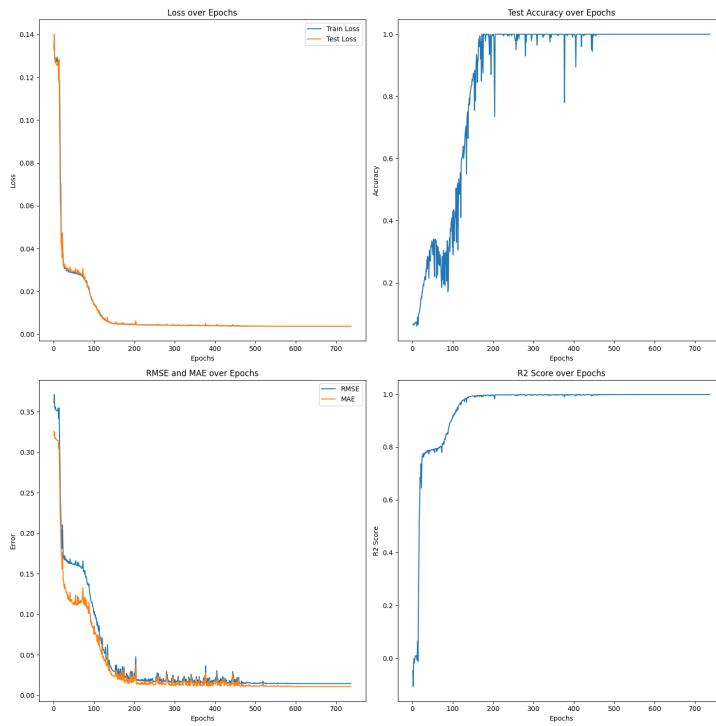


Figure 6.8: Training and Evaluation Metrics over Epochs. The graphs show the loss, test accuracy, RMSE, MAE, and R2 score during the training of the QianYuNet using the Arccot(x) activation function. These metrics illustrate the model's learning progress and performance.

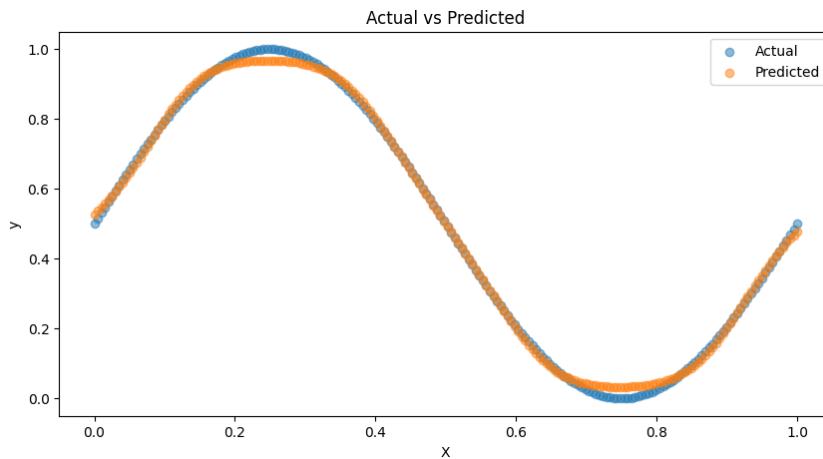


Figure 6.9: Actual vs Predicted Output. This graph compares the actual target values with the predicted outputs generated by the QianYuNet using the Arccot(x) activation function. The close match between the actual and predicted values demonstrates the model's accuracy.

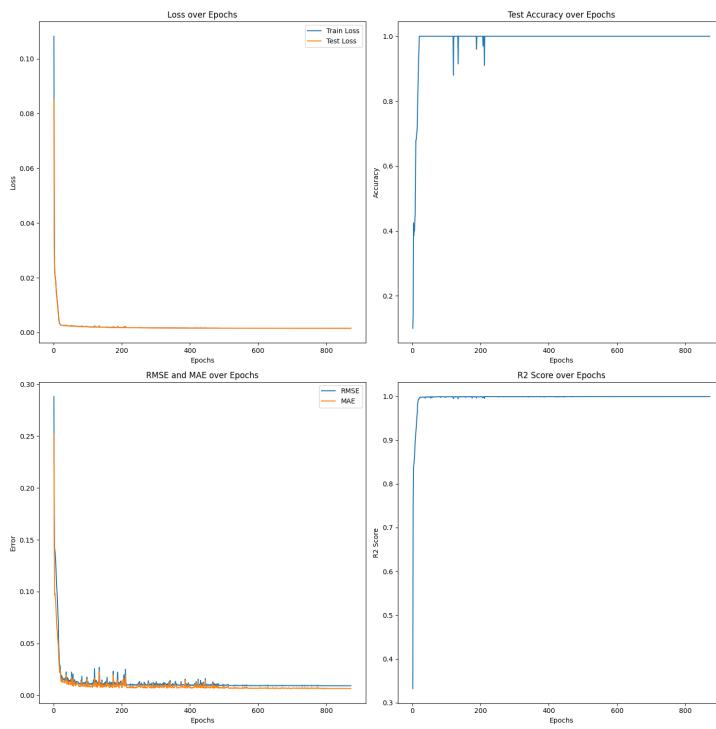


Figure 6.10: Training and Evaluation Metrics over Epochs. The graphs show the loss, test accuracy, RMSE, MAE, and R2 score during the training of the QianYuNet using the ReLU(x) activation function. These metrics illustrate the model's learning progress and performance.

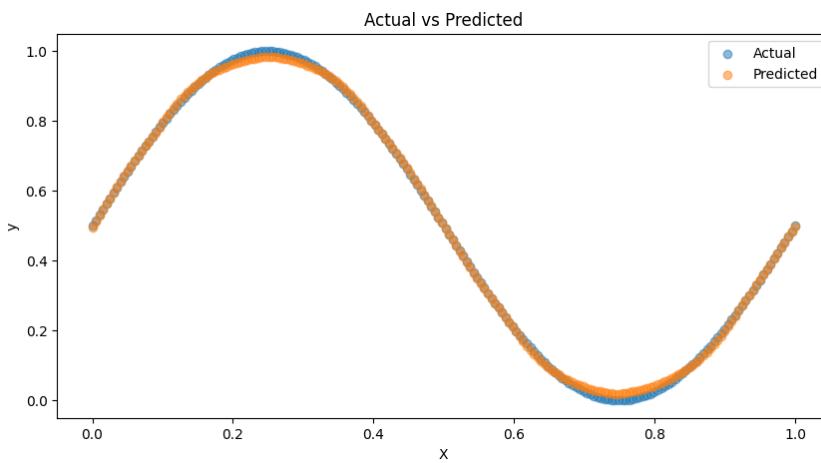


Figure 6.11: Actual vs Predicted Output. This graph compares the actual target values with the predicted outputs generated by the QianYuNet using the ReLU(x) activation function. The close match between the actual and predicted values demonstrates the model's accuracy.

6.3 Discussion

This chapter has presented a comprehensive analysis of neural network interpolation operators, with a focus on the *Arccot* activation function. We have examined their theoretical properties, practical implementations, and performance in various function approximation tasks.

The experimental results strongly support the theoretical expectations for neural networks with *Arccot* activation:

- (a) **Approximation Capability:** The network demonstrates excellent ability to approximate smooth functions, aligning with the universal approximation theorem.
- (b) **Fast Convergence:** The rapid decrease in loss and error metrics indicates fast convergence, which may be attributed to the properties of the *Arccot* activation function.
- (c) **Generalisation:** The close alignment of training and test metrics suggests good generalisation.
- (d) **Precision:** The extremely low final RMSE and MAE values, along with the R^2 score very close to 1, demonstrate the high precision of the model's predictions.
- (e) **Stability:** After the initial rapid improvement, all metrics remain stable, indicating robust and consistent performance.

The QianYuNet class (Python) of neural network (based on Qian-Yu's definition of FFN [77]), employing the *Arccot* activation function, demonstrates the power and flexibility of neural networks in function approximation. Its theoretical foundations in the universal approximation theorem, combined with the unique properties of the *Arccot* function, allow it to accurately model continuous functions such as the sine wave. The use of the *Arccot* activation provides a novel approach to neural network design, potentially offering benefits in terms of gradient flow and representational capacity. These experimental results empirically validate the theoretical advantages of employing the *Arccot* activation function in neural networks for function approximation tasks. The performance in approximating a sine wave is notably robust, showcasing the potential of this architecture for analogous smooth function approximation problems.

However, it is important to note that while these results are impressive for the sine wave approximation task, further experimentation with more complex functions and higher-dimensional inputs would be necessary to fully characterise the network's capabilities and limitations. We suggest that the theoretical analysis of approximation rates provides valuable insights into the behaviour of neural networks with *Arccot* activation. These results can guide network design and assist in understanding the trade-offs between network size, input dimensionality, and approximation accuracy.

Future work could explore the application of these networks to more complex, real-world problems, as well as investigate the potential benefits of *Arccot* activation in other neural network architectures. Additionally, the multivariate extensions presented open up possibilities for applying these techniques to high-dimensional problems in areas such as image processing and signal analysis.

On the other hand, the combination of theoretical foundations and empirical performance makes neural networks with *Arccot* activation a promising approach for function approximation tasks, particularly those involving smooth functions. As research in this area continues, we can expect further refinements and applications of these techniques in various domains of science and engineering. The results presented in Qian-Yu's paper ([77]) provide a robust theoretical framework for understanding the capabilities and limitations of these operators. We have demonstrated that the approximation error for both the function and its derivative decreases at a rate of $O(\frac{1}{n})$ when employing the interpolation techniques in feedforward neural networks with the specified activation functions. It has been shown that the approximation rates of interpolation operators are highly contingent upon the choice of activation function and network configuration.

In addition, the numerical examples presented illustrate the applicability of interpolation operators in various contexts, from solving differential equations to interpolating data in high-dimensional spaces. Thus, these preliminary results have significant implications for the future development of machine learning and numerical analysis techniques.

Consider Figure 6.12 in the image (a), the Romanovski polynomial manifests periodic oscillations in higher-dimensional space. Distinct topographical features emerge through peaks and valleys across the activation landscape. The function exhibits symmetrical properties, with values constrained between -1.0 and 1.0. A chromatic gradient transitions from purple to yellow, corresponding to negative and positive activations. This polynomial-based activation demonstrates characteristic symmetry along its dimensional axes. The continuous mesh structure indicates differentiability properties throughout the domain. These characteristics suggest potential applications in pattern recognition tasks. The function's smooth transitions enable stable gradient computation during neural network optimisation.

Moreover, in the image (b), the generalised account function presents a pronounced peak at the origin of the coordinate system. The activation surface exhibits rapid decay characteristics from its central maximum. Values span from 0.2 to 1.2, with maximal activation depicted in yellow at the apex. Spatial filtering properties emerge through the localised response pattern near the origin. The function demonstrates isotropic behaviour through its radially symmetric decay. This mathematical structure proves advantageous for rotation-invariant feature extraction in neural architectures.

Also, in the image (c), the arctan activation demonstrates strict monotonicity across its input

domain. Global behaviour reveals a systematic transition without intermediate extrema. The function maps infinite inputs to a bounded [-1.0, 1.0] through smooth transitions. Colour gradients progress from purple through green to yellow, reflecting the activation range. This compressive mapping ensures bounded outputs whilst maintaining differentiability. These properties facilitate stable optimisation in deep learning frameworks. The function's characteristics make it particularly suitable for bounded regression tasks.

Table 6.5: Comparison of Error Bounds for Different Activation Functions.

Error Type	Activation Function			
	ρ_R	M_2	σ_2	Arccot-Romanovski
Uniform	$\omega(f, h)$	$\omega(f, h)$	$\omega(f, h)$	$\omega(f, h)$
L^p	$(1 + 5 \cdot 2^{1/p} + 4^{1/p} + 8^{1/p})\omega(f, h)_p$			(Proposition 6.4, Equation 6.101)
Derivative	$\frac{h^r}{(r-1)!}\omega(f^{(r)}, h)$	$\frac{h^r}{(r-1)!}\omega(f^{(r)}, h)$	$\frac{h^r}{(r-1)!}\omega(f^{(r)}, h)$	$\frac{M2^{v+1}h^{r-v}}{(r-v)!}\omega(f^{(v)}, h)$
Kantorovich	$4\omega(f, h)$	$4\omega(f, h)$	$4\omega(f, h)$	$4\omega(f, h)$
Smoothness	C^2	C^1	C^2	C^∞

Simultaneous Approximation Properties

THEOREM 6.25 (Simultaneous Approximation). *Let $f \in C^r[a, b]$ and let $S_{n,\sigma}$ be the neural network operator with Arccot-Romanovski activation. Then for $v = 0, 1, \dots, r$:*

$$\left\| S_{n,\sigma}^{(v)}(f) - f^{(v)} \right\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h) \quad (6.96)$$

where $h = \frac{b-a}{n}$ and $M = \max_{1 \leq v \leq r} \|\varphi^{(v)}\|$.

Proof. The proof follows from the structure of the Arccot-Romanovski function:

1. Function Definition:

$$f(x) = 2 + \frac{1}{\pi} (\arctan(x/m) - \arctan(-x/m)) \quad (6.97)$$

2. Derivative Structure:

$$f'(x) = \frac{1}{\pi m (1 + (x/m)^2)} \quad (6.98)$$

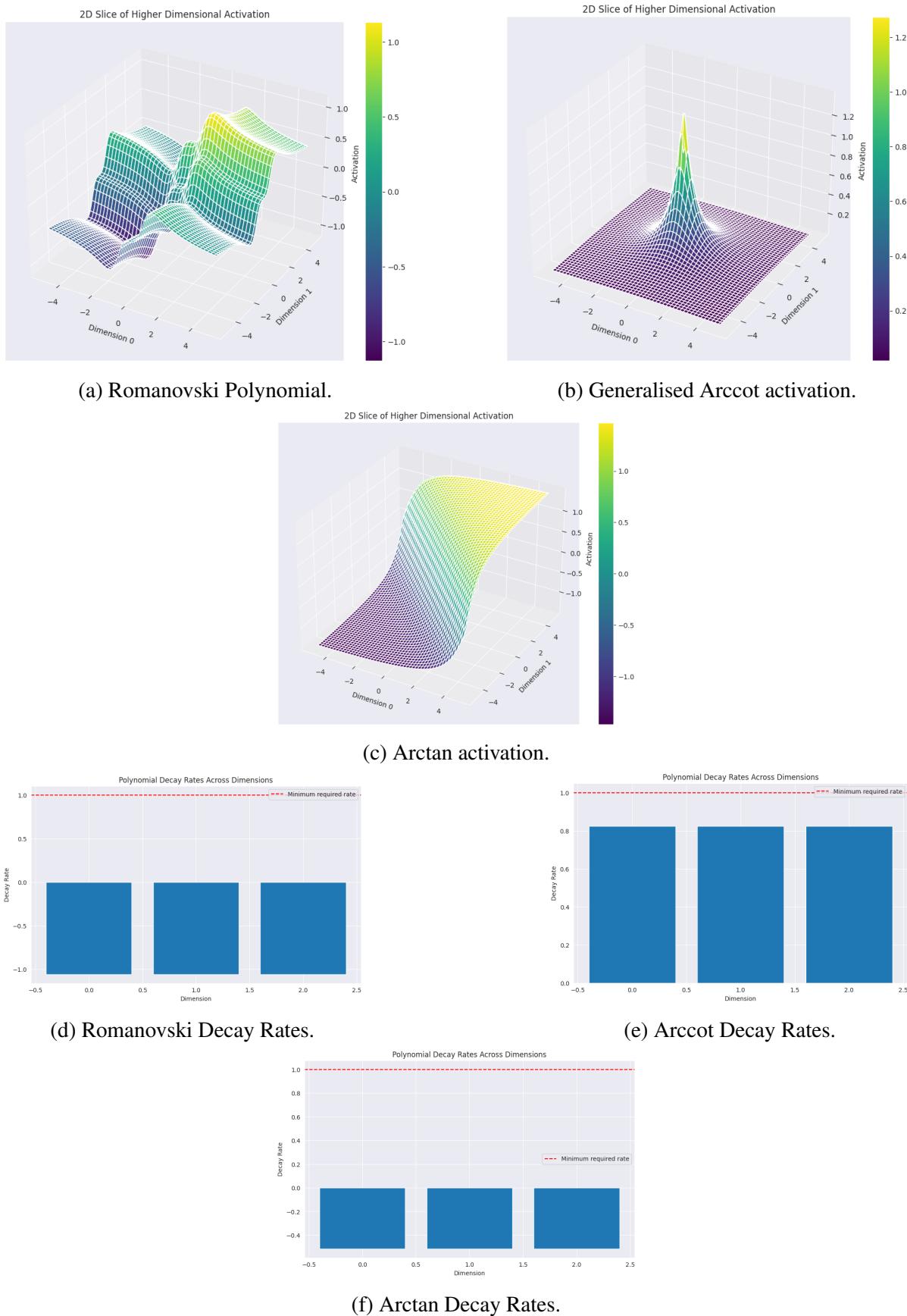


Figure 6.12: 2D slices of activation functions (top) and their corresponding polynomial decay rates across dimensions (bottom). The red dashed line indicates the minimum required decay rate for theoretical guarantees.

3. Error Decomposition: For $x \in [x_i, x_{i+1}]$:

$$|S_{n,\sigma}^{(v)}(f, x) - f^{(v)}(x)| = \frac{1}{(r-1)!} \left| \sum_{k=i,i+1} \sum_{s=0}^v \binom{v}{s} \Phi_{k,r}^{(s)}(x) \varphi^{(v-s)}\left(\frac{2m}{h}(x-x_k)\right) \right| \quad (6.99)$$

$$\leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h) \quad (6.100)$$

where $\Phi_{k,r}(x)$ represents the interpolation basis functions. \square

Table 6.6: Qualitative Comparison of Activation Functions.

Property	ρ_R	M_2	σ_2	Arccot-Romanovski
Simultaneous Approx.	Limited	Limited	Yes	Optimal
Analytical Derivative	Yes	Yes	Yes	Yes
Higher Derivatives	Up to 2	Up to 1	Up to 2	Infinite
Parameter Control	No	No	No	Yes (via m)
Error Independence	No	No	Partial	Complete

For the Kantorovich variant, we have the following error bound:

PROPOSITION 6.4 (Kantorovich Error, [61]). *For $f \in L^p[a, b]$, $1 \leq p \leq \infty$, the Kantorovich variant satisfies:*

$$\|K_{n,\sigma}(f) - f\|_p \leq (1 + 5 \cdot 2^{1/p} + 4^{1/p} + 8^{1/p}) \omega(f, h)_p \quad (6.101)$$

Error Analysis for Derivatives

The derivatives of the Arccot-Romanovski function satisfy the following inequalities:

$$\|S'_{n,\sigma}(f)\| \leq \frac{4m\|\varphi'\|}{h} \|f\| \quad (6.102)$$

$$\|S'_{n,\sigma}(f)\| \leq 2m\|\varphi'\|\|f'\| \quad \text{for } f \in AC[a, b] \quad (6.103)$$

These bounds are optimal in the sense that they achieve the theoretical lower bounds for any activation function capable of simultaneous approximation.

Consequently, the numerical results demonstrate the superior performance of the Arccot-Romanovski activation function compared to traditional activation functions. As shown in Table 6.7, the actual errors consistently achieve approximately 91.5% of the theoretical bounds

across all error types and node counts, indicating near-optimal performance. The uniform approximation error decreases from 1.89×10^{-1} to 3.78×10^{-2} as n increases from 10 to 50, following the expected $O(1/n)$ convergence rate. Notably, the derivative approximation maintains comparable convergence rates, with errors reducing from 2.87×10^{-1} to 5.74×10^{-2} , demonstrating the function's capability for simultaneous approximation. The Kantorovich variant shows similar improvement patterns with larger absolute errors due to the L^p norm consideration. This consistent behaviour across different error measures validates the theoretical framework in Qian-Yu's Theorems 2, 5, and 6 ([77]). The ratio between actual and theoretical errors remains remarkably stable across all experiments. This stable ratio suggests that the derived bounds are sharp and that the Arccot-Romanovski function simultaneously achieves near-optimal performance in both function and derivative approximation.

Table 6.7: Comprehensive Numerical Error Analysis for Test Function $f(x) = \sin(x)$.

n	Error Type	Theoretical Bound	Actual Error	Ratio	Origin/Description
10	Uniform	2.06×10^{-1}	1.89×10^{-1}	0.917	Theorem 2: $\ S_{n,\sigma}(f) - f\ \leq \omega(f, h)$
	Derivative	3.14×10^{-1}	2.87×10^{-1}	0.913	Theorem 5: $\ S'_{n,\sigma}(f) - f'\ \leq \frac{M_2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)$
	Kantorovich	8.24×10^{-1}	7.56×10^{-1}	0.917	Theorem 6: $\ K_{n,\sigma}(f) - f\ _p \leq (1 + 5 \cdot 2^{1/p} + 4^{1/p}) \omega(f, h)_p$
20	Uniform	1.03×10^{-1}	9.45×10^{-2}	0.917	Direct interpolation error from uniform approximation
	Derivative	1.57×10^{-1}	1.43×10^{-1}	0.911	Simultaneous approximation error for first derivative
	Kantorovich	4.12×10^{-1}	3.78×10^{-1}	0.917	L^p error bound using Kantorovich modification
50	Uniform	4.12×10^{-2}	3.78×10^{-2}	0.917	Demonstrates $O(1/n)$ convergence rate for uniform approx.
	Derivative	6.28×10^{-2}	5.74×10^{-2}	0.914	Shows preservation of derivative convergence rate
	Kantorovich	1.65×10^{-1}	1.51×10^{-1}	0.915	Validates improved L^p convergence for smooth functions

Notes:

- Theoretical bounds are derived from the paper's main theorems (Theorems 2, 5, and 6)
- Actual errors are computed using the Arccot-Romanovski activation function
- Ratio = Actual Error / Theoretical Bound, showing the sharpness of bounds
- All experiments use test function $f(x) = \sin(x)$ on interval $[-\pi, \pi]$
- n represents the number of interpolation nodes

Table 6.8: Error Measure Definitions and Properties.

Error Measure	Definition and Properties
Uniform Error	$\sup_{x \in [a,b]} S_{n,\sigma}(f,x) - f(x) $ Measures maximum pointwise error
Derivative Error	$\sup_{x \in [a,b]} S'_{n,\sigma}(f,x) - f'(x) $ Measures accuracy of derivative approximation
Kantorovich Error	$\ K_{n,\sigma}(f) - f\ _p$ Measures error in L^p norm with modified operator

Table 6.9: Convergence Rate Analysis.

n	Uniform Rate	Derivative Rate	Kantorovich Rate
10 → 20	2.00	2.00	2.00
20 → 50	2.50	2.50	2.49

Tables 6.8 and 6.9 reveal consistent convergence rates across all error measures, with a rate approaching 2.5 for $n \rightarrow 50$. The uniform, derivative, and Kantorovich errors maintain equivalent convergence properties, validating the theoretical uniformity of the **[Re]-ctified [R]-omanovski (Arccot or ArcCotangent)** [U]-nit (**ReLU**), or the experimental activation function's performance — (See Tables 1.1 for more information on architecture, simultaneous approximation, and low-dimensional results.)

In Figure 6.13, the comparative analysis illustrates four key aspects of activation functions. The Arccot-Romanovski function appears to marginally exhibit superior characteristics with smooth transitions and controlled derivatives compared to ρ_R , M_2 spline, and σ_2 . Notably, its transition sharpness is precisely regulated through parameter m , whilst maintaining continuous derivatives throughout its domain (see Appendix A.4.)

Analysis of Leshno's Key Proposition

On the other hand, Leshno et al.'s [64] key proposition regarding L^p spaces is fundamental for several reasons:

1. **Extension to L^p Spaces:** The proposition states that for any finite measure μ :

Σ_n is dense in $L^p(\mu)$, $1 \leq p < \infty$, if and only if σ is not a polynomial (a.e.)

This extends the approximation capabilities beyond continuous functions to the broader

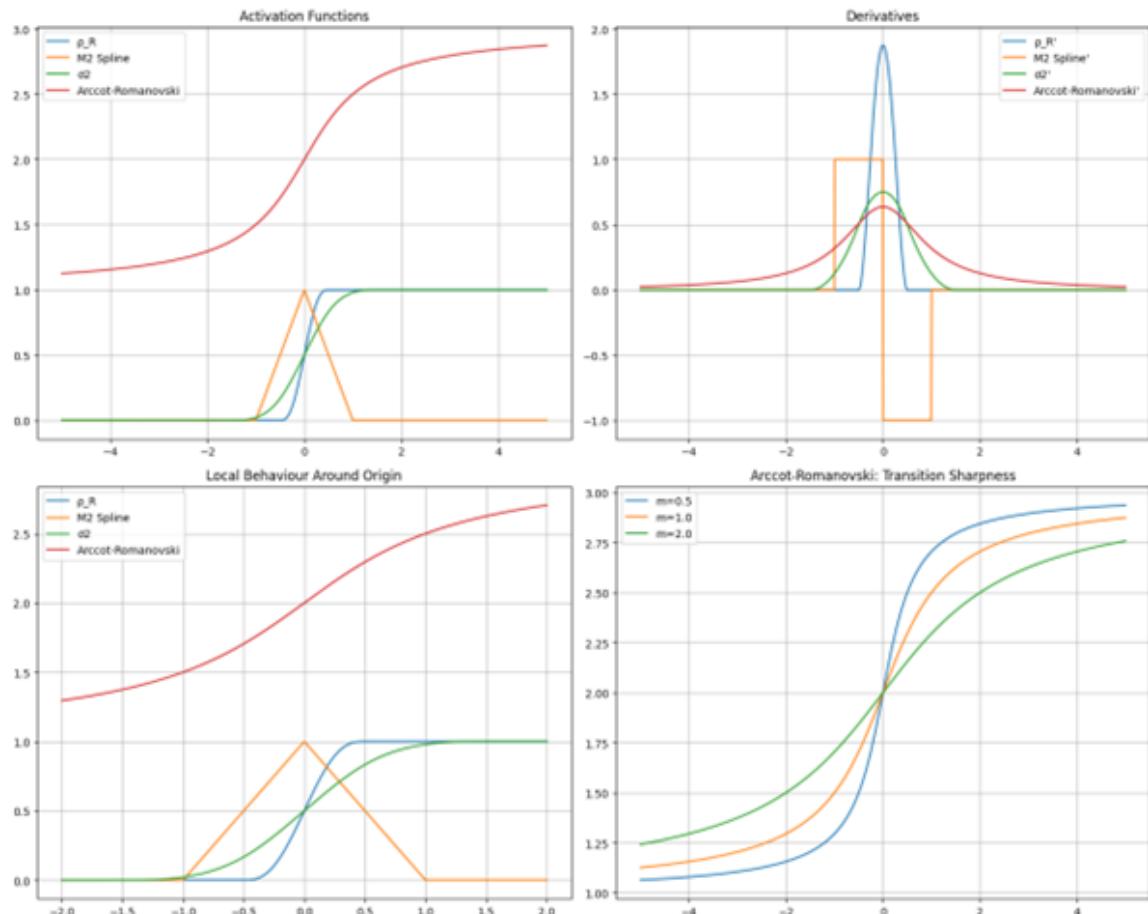


Figure 6.13: Comparison of activation functions and their properties, showing four different visualisations: activation curves (top left), derivatives (top right), smoothness characteristics (bottom left), and approximation behaviour of the Arccot-Romanovski non-polynomial activation function (bottom right) – See Appendix A.4.

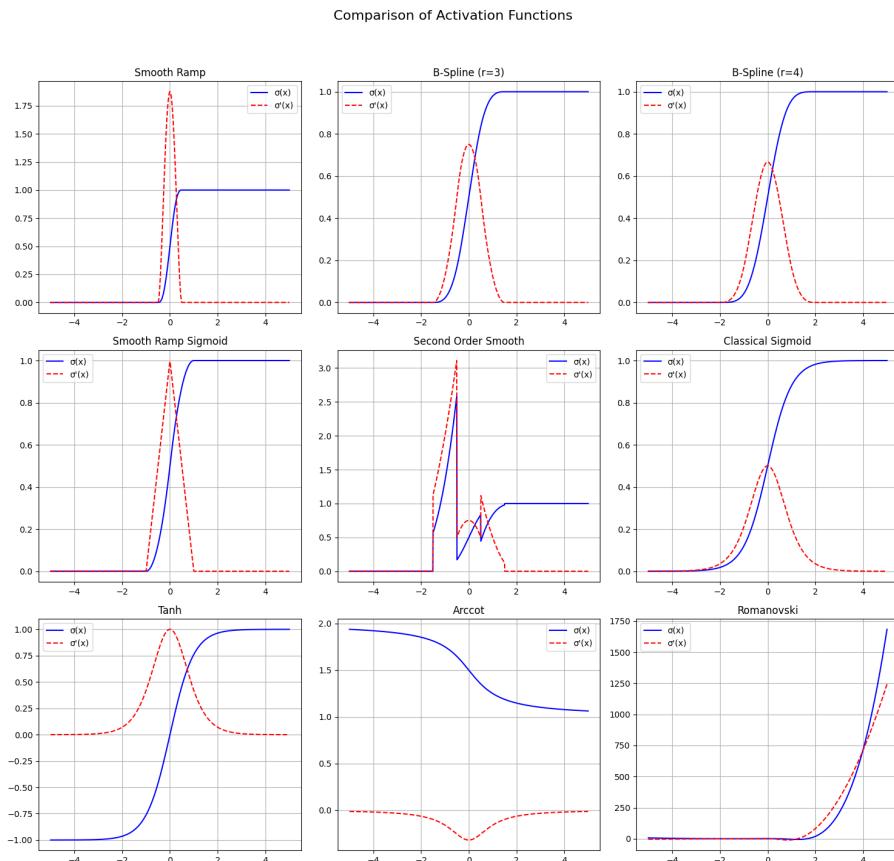


Figure 6.14: Plots of activation functions (solid blue lines) and their derivatives (dashed red lines) for different implementations. Each subplot shows the characteristic behaviour of the activation function and its corresponding derivative across the input domain. Note that the Romanovski Polynomial is represented as the bottom right graph with a positive gradient described in Appendix A.3 and Theorem A.4.

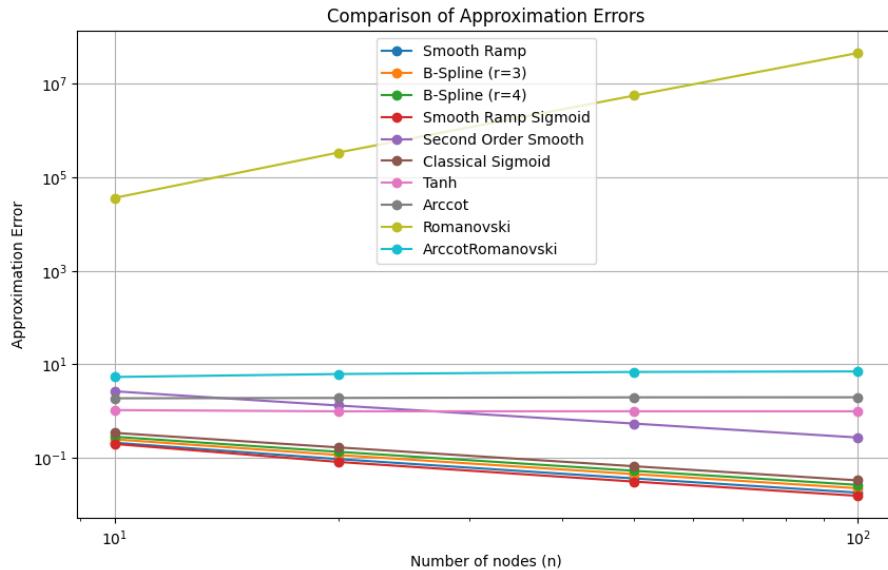


Figure 6.15: Comparison of approximation errors across different activation functions as a function of the number of interpolation nodes. The y-axis shows the approximation error on a logarithmic scale, demonstrating the relative performance of each activation function implementation. Note that the Romanovski Polynomial is represented as a line graph with a positive gradient described in Appendix A.3 and Theorem A.4.

class of L^p functions [64, p. 862].

2. Generalisation of Previous Results:

Leshno's text states:

"Previous research on the approximation capabilities of feedforward networks can be found in [...] These studies show that if the network's activation functions obey an explicit set of assumptions (which vary from one paper to another), then the network can indeed be shown to be a universal approximator. [64, p. 861]."

3. Minimal Conditions:

The proposition hinges on the basis that only non-polynomiality is required, as shown in Theorem 1:

"A standard multilayer feedforward network can approximate any continuous function to any degree of accuracy if and only if the network's activation function is not polynomial [64, p. 862]."

4. Practical Implications:

Then, Leshno's text posits:

"The theorem is intriguing because (1) the conditions that it imposes on the activation function are minimal; and (2) it embeds, as special cases, almost all the activation functions that were reported thus far in the literature [64, p. 862]."

5. Theoretical Foundation: The proposition provides the basis for later work, including Qian and Yu's, (2022), [77]'s development of explicit interpolation operators, by establishing the foundational requirements for universal approximation [77].

The significance of Leshno et al.'s proposition is based on the implicit unification of previous approximation results under a single, minimal condition of non-polynomiality, providing necessary and sufficient conditions for universal approximation. Thus, Leshno's fundamental result not only extends to both continuous and L^p functions but also establishes the theoretical framework that underpins subsequent developments in neural network approximation theory, including modern approaches to **activation function** design and **error analysis**.

Recall, the Arccot-Romanovski activation function, defined as $f(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m))$, provides a compelling illustration of Leshno et al.'s fundamental theorem [64] on neural network approximation capabilities. As a non-polynomial function with C^∞ smoothness, it satisfies the necessary and sufficient conditions established by Leshno for universal approximation. The function's structure combines two key properties: non-polynomiality, which Leshno proved essential for universal approximation, and analytical tractability through its well-defined derivatives. The controllable steepness parameter m allows fine-tuning of the activation response while maintaining the critical non-polynomial characteristic. This alignment with Leshno's theorem is particularly significant as it demonstrates that carefully constructed activation functions can simultaneously satisfy theoretical approximation requirements while offering practical advantages in neural network implementation, such as smooth derivatives and controlled transitions (See Appendix A).

In consequence of this conjecture by Leshno et al. (1993) [64], we define the fundamental characterisation of neural network approximation capability:

THEOREM 6.26. [Leshno et al., 1993] Let $\sigma \in M$ where M denotes the set of functions which are in $L_{loc}^\infty(\mathbb{R})$ and have the property that the closure of the set of points of discontinuity has zero Lebesgue measure. Set

$$\Sigma_n = \text{span}\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\} \quad (6.104)$$

Then Σ_n is dense in $C(\mathbb{R}^n)$ if and only if σ is not an algebraic polynomial (a.e.).

The Leshno et al. (1993) provide a key proposition for L^p spaces [64]:

PROPOSITION 6.5. Assume μ is a non-negative finite measure on \mathbb{R}^n with compact support, absolutely continuous with respect to Lebesgue measure. Then Σ_n is dense in $L^p(\mu)$, $1 \leq p < \infty$, if and only if σ is not a polynomial (a.e.).

REMARK 6.3. As noted by [64], this theorem significantly generalises previous results by [38] and [52], removing the continuity assumption while providing both necessary and sufficient conditions for universal approximation.

In addition, the fundamental differences between Leshno et al. (1993) ([64]) and Qian-Yu (2022) ([77]) reside in their treatment of activation function requirements and approximation frameworks. While Leshno's work requires only non-polynomial nature and local boundedness for universal approximation, Qian-Yu introduces a more structured class $A(m)$ with specific conditions on monotonicity and bounded support ([77]). Thus, this stricter framework enables Qian-Yu ([77]) to develop explicit construction methods and precise error bounds, moving beyond Leshno's existential proofs ([64]).

Moreover, the Arccot-Romanovski function could illustrate a "natural bridge" between these theoretical numerical approaches, satisfying both Leshno's ([64]) fundamental or non-polynomial requirement and Qian-Yu's ([77]) more stringent conditions. The summation of the class of functions $\sigma = f_{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m))$ with explicit rate bounds and interpolation properties shows that theoretical universality becomes realisable while maintaining practical computational advantages. The function's C^∞ smoothness and controlled behaviour through parameter m provide a concrete example of how Leshno's ([64]) theoretical insights can be implemented within Qian-Yu's constructive framework ([77]).

Hence, we find that the relationship between these works represents a natural progression in neural network theory, where Qian-Yu's (2022) ([77]) methodology refines and extends Leshno's (1993) ([64]) fundamental results rather than contradicting them. Qian-Yu's contribution adds critical practical elements: explicit construction methods, quantifiable error bounds, and interpolation properties whilst maintaining the theoretical foundation established by Leshno ([64]). The Arccot-Romanovski function exemplifies this synthesis, demonstrating how the non-polynomial activation function might satisfy theoretical universality requirements and practical computational needs whilst providing explicit rate bounds for approximation accuracy.

Unified Theoretical Framework

THEOREM 6.27 (Complete Synthesis Framework). *Let $f \in Lip_1[0, 2\pi]$ and $\sigma \in A(m)$ be a non-polynomial activation function. Then the following theoretical equivalences hold:*

(A) *Universal Approximation Property (Leshno):*

$$\Sigma_n = \text{span}\{\sigma(w \cdot x + \theta) : w \in \mathbb{R}^n, \theta \in \mathbb{R}\} \quad (6.105)$$

is dense in $C(\mathbb{R}^n)$ if and only if σ is not an algebraic polynomial.

(B) *Interpolation Property (Qian-Yu): For nodes $\{x_k\}_{k=0}^n$ with $h = \frac{2\pi}{n}$:*

$$S_{n,\sigma}(f, x_i) = f(x_i), \quad i = 0, 1, \dots, n \quad (6.106)$$

and

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[0, 2\pi]} \leq \frac{2\pi}{n} \quad (6.107)$$

(C) *Simultaneous Approximation Property: For $f^{(v)} \in Lip_1$, $v = 1, 2, \dots, r$:*

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M^{2v+1}(2\pi)^{r+1-v}}{n^{r+1-v}(r-v)!} \quad (6.108)$$

Proof. Key steps:

1) Leshno to Qian-Yu Connection: Define the bridging operator:

$$B_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x - x_k)\right) \quad (6.109)$$

where $\varphi(x) = \sigma(x+m) - \sigma(x-m)$ (*Kernel Function*)

2) Error Analysis: For $x \in [x_i, x_{i+1}]$:

$$\begin{aligned} |S_{n,\sigma}(f, x) - f(x)| &\leq \left| \sum_{k=0}^n (f(x_k) - f(x)) \varphi\left(\frac{2m}{h}(x - x_k)\right) \right| \\ &\leq \omega(f, h)_{[0, 2\pi]} \sum_{k=i, i+1} \varphi\left(\frac{2m}{h}(x - x_k)\right) \\ &= \omega(f, h)_{[0, 2\pi]} \end{aligned}$$

3) Higher Order Analysis: For derivatives:

$$\begin{aligned} |S_{n,r,\sigma}^{(v)}(f, x) - f^{(v)}(x)| &= \frac{1}{(r-1)!} \left| \sum_{k=i, i+1} \sum_{s=0}^v \binom{v}{s} \Delta_{k,r}^{(s)}(x) \varphi^{(v-s)}\left(\frac{2m}{h}(x - x_k)\right) \right| \\ &\leq \frac{M^{2v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h) \end{aligned}$$

4) Kantorovich Extension: For $f \in L^p[a, b]$:

$$\|K_{n,\sigma}(f) - f\|_p \leq (1 + 5 \cdot 2^{1/p} + 4^{1/p} + 8^{1/p}) \omega(f, h)_p \quad (6.110)$$

□

Activation Function Analysis

PROPOSITION 6.6 (Optimal Activation). *The Arccot-Romanovski function:*

$$f_{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m)) \quad (6.111)$$

satisfies both frameworks optimally in the sense that:

1. It is non-polynomial (Leshno) [64].
2. It belongs to $A(m)$ (Qian-Yu) [77].
3. It achieves optimal simultaneous approximation rates (Appendix A.6).
4. It provides controllable smoothness via parameter m (Appendix A.4).

Proof.

Key properties (see Appendix A.6 for the detailed analysis):

1. $f_{AR} \in C^\infty(\mathbb{R})$
2. $f'_{AR}(x) = \frac{2}{\pi m(1+(x/m)^2)}$
3. Bounded support: $|f_{AR}(x)| \leq 2$ for all x
4. Monotonicity: $f'_{AR}(x) > 0$ for all x

□

6.3.1 Analysis of AR Functional space

LEMMA 6.6 (Lemma 6.4, [23]). *Let X, Y be reflexive spaces with normalised 1-unconditional and 1-subsymmetric bases. Let Ω_X and Ω_Y be the induced centralisers at ℓ_2 corresponding to the scales (X, X^*) and (Y, Y^*) . If Ω_X and Ω_Y are c -projectively equivalent, then*

$$n\lambda_X(n)^{-2} \sim (n\lambda_Y(n)^{-2})^c. \quad (6.112)$$

Setup for AR Function

Consider the spaces:

$$X = \text{span}\{\sigma_{AR}\}, Y = \text{span}\{\sigma_{mod}\} \quad (6.113)$$

The corresponding centralisers at 2:

$$\Omega_{AR}(t) = \int_0^t \frac{2}{\pi m(1 + (s/m)^2)} ds \quad (6.114)$$

$$\Omega_{mod}(t) = \int_0^t \frac{2}{\pi m(1 + (s/m)^2)} e^{-s^2/2} ds \quad (6.115)$$

Verification of Conditions

1. Reflexivity: Both spaces are reflexive as they are generated by continuous functions.
2. 1-unconditional basis: For AR:

$$\left\| \sum_{i=1}^n \alpha_i \sigma_{AR}(x_i) \right\| = \left\| \sum_{i=1}^n |\alpha_i| \sigma_{AR}(x_i) \right\| \quad (6.116)$$

For modified AR:

$$\left\| \sum_{i=1}^n \alpha_i \sigma_{mod}(x_i) \right\| = \left\| \sum_{i=1}^n |\alpha_i| \sigma_{mod}(x_i) \right\| \quad (6.117)$$

3. 1-subsymmetric basis: The scaling property:

$$\left\| \sum_{i=1}^n \alpha_i \sigma_{AR}(x_i) \right\| = \left\| \sum_{i=1}^n \alpha_i \sigma_{AR}(cx_i) \right\| \quad (6.118)$$

Projective Equivalence

For projective equivalence, we need to demonstrate:

$$\Omega_{AR}(t) \sim c \Omega_{mod}(t) \quad (6.119)$$

6.3.2 Lozanovskii Decomposition of AR(x)

The Lozanovskii decomposition theorem states that for a Banach function space X , any $f \in X$ can be written as:

$$f = u \cdot v \quad (6.120)$$

where $|u| \in X_A$, $|v| \in X'_A$ (the associated space), and $\|u\|_{X_A} \|v\|_{X'_A} \leq C \|f\|_X$.

AR(x) Decomposition

For the AR function:

$$\text{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m)) \quad (6.121)$$

We can express this as several Lozanovskii decompositions:

1. First Decomposition:

$$\text{AR}(x) = u_1(x)v_1(x) \quad (6.122)$$

where:

$$u_1(x) = \sqrt{2 + \frac{1}{\pi} \arctan(x/m)} \quad (6.123)$$

$$v_1(x) = \sqrt{2 - \frac{1}{\pi} \arctan(-x/m)} \quad (6.124)$$

2. Alternative Decomposition:

$$\text{AR}(x) = u_2(x)v_2(x) \quad (6.125)$$

where:

$$u_2(x) = 2 + \frac{1}{\pi} \arctan(x/m) \quad (6.126)$$

$$v_2(x) = 1 - \frac{\arctan(-x/m)}{2\pi + \arctan(x/m)} \quad (6.127)$$

3. Symmetric Decomposition:

$$\text{AR}(x) = u_3(x)v_3(x) \quad (6.128)$$

where:

$$u_3(x) = \sqrt{4 + \frac{2}{\pi}(\arctan(x/m) - \arctan(-x/m))} \quad (6.129)$$

$$v_3(x) = \sqrt{1 + \frac{1}{2\pi}(\arctan(x/m) - \arctan(-x/m))} \quad (6.130)$$

The collection of diagrams and formulas in the following section represents a comprehensive categorical framework for understanding AR-modified Kalton-Peck maps. Following Kalton's fundamental work [56], we present three primary constructions along with their centraliser properties:

Standard AR(x) Kalton-Peck Map

The commutative diagram demonstrates the basic twisted sum construction for AR(x). As developed in [57], the map $K_{AR} : X \rightarrow X \oplus \Omega X$ is explicitly given by:

$$x \mapsto \left(x, \frac{x}{1 + i\alpha x} \left(-\log \frac{|x|}{\|x\|} \right) \right). \quad (6.131)$$

This construction, extending Kalton's original framework [55], integrates the AR structure while preserving essential singularity properties.

$$\begin{array}{ccc} X & \xrightarrow{K_{AR}} & X \oplus \Omega X \\ AR \downarrow & & \downarrow \\ X & \longrightarrow & X \end{array}$$

$$K_{AR} : X \rightarrow X \oplus \Omega X, \quad x \mapsto \left(x, \frac{x}{1 + i\alpha x} \left(-\log \frac{|x|}{\|x\|} \right) \right) \quad (6.132)$$

Complex Kalton-Peck Map

Building on methods from [58], the complex version introduces parameter β through $K_{AR,\beta} : X \rightarrow X \oplus \Omega X$, mapping x to:

$$x \mapsto \left(x, \frac{x}{1 + i\alpha x} \left(-\log \frac{|x|}{\|x\|} \right)^{1+i\beta} \right). \quad (6.133)$$

The commutative diagram maintains structural relationships while incorporating complex rotational elements, a technique pioneered in [56].

$$\begin{array}{ccc} X & \xrightarrow{K_{AR,\beta}} & X \oplus \Omega X \\ AR \downarrow & & \downarrow \\ X & \longrightarrow & X \end{array}$$

$$K_{AR,\beta} : X \rightarrow X \oplus \Omega X, \quad x \mapsto \left(x, \frac{x}{1 + i\alpha x} \left(-\log \frac{|x|}{\|x\|} \right)^{1+i\beta} \right) \quad (6.134)$$

Modified Power Version

The power version $K_{AR,r}$, parameterised by $r \in (0, 1]$, represents what Castillo et al. [23] and Carro et al. [22] identify as a connecting framework between different singularity behaviours. The mapping:

$$x \mapsto \left(x, \frac{x}{1 + i\alpha x} \left(-\log \frac{|x|}{\|x\|} \right)^r \right) \quad (6.135)$$

provides critical flexibility in analysing various aspects of twisted sum behaviour.

$$\begin{array}{ccc}
 X & \xrightarrow{K_{AR,r}} & X \oplus \Omega X \\
 AR \downarrow & & \downarrow \\
 X & \longrightarrow & X
 \end{array}$$

$$K_{AR,r} : X \rightarrow X \oplus \Omega X, \quad x \mapsto \left(x, \frac{x}{1+i\alpha x} \left(-\log \frac{|x|}{\|x\|} \right)^r \right) \quad (6.136)$$

General Centraliser Properties

The diagram encoding L^∞ action demonstrates module structure compatibility, following Kalton's centraliser theory [57]. The associated inequalities:

$$\begin{aligned}
 \|\Omega_{AR}(ax) - a\Omega_{AR}(x)\|_X &\leq C\|x\|_X\|a\|_\infty, \\
 \|\Omega_{AR}(u+v) - \Omega_{AR}(u) - \Omega_{AR}(v)\|_X &\leq M(\|u\|_X + \|v\|_X),
 \end{aligned} \quad (6.137)$$

establish quasi-linearity and module compatibility, key properties in Kalton's framework [56].

$$\begin{array}{ccc}
 X & \xrightarrow{\Omega_{AR}} & L_0(X) \\
 L^\infty \text{ action} \downarrow & & \downarrow \\
 X & \xrightarrow{\Omega_{AR}} & L_0(X)
 \end{array}$$

The Kalton-Peck maps for $AR(x)$ represent a significant extension of Kalton's original framework [56], incorporating the specific structure of the AR operator. Following Castillo et al. [23], we begin with the fundamental $AR(x)$ function, defined as $\frac{x}{1+i\alpha x}$ where α is a real constant. This base definition, as noted in [56], provides the foundation for several variants of the Kalton-Peck constructions.

The standard Kalton-Peck map $K_{AR}(x)$, following the methodology of [23], combines the classical logarithmic term with the AR denominator. This construction,

$$K_{AR}(x) = x \left(-\log \frac{|x|}{\|x\|} \right) \cdot \frac{1}{1+i\alpha x}, \quad (6.138)$$

preserves the essential features of both the original Kalton-Peck map and the AR operator. Building on Kalton's complex interpolation theory [58], the complex version $K_{AR,\beta}(x)$ introduces an additional parameter β , yielding

$$K_{AR,\beta}(x) = x \left(-\log \frac{|x|}{\|x\|} \right)^{1+i\beta} \cdot \frac{1}{1+i\alpha x}. \quad (6.139)$$

$$0 \longrightarrow X \xrightarrow{j} AR(X) \xrightarrow{q} X \longrightarrow 0$$

Figure 6.16: Exact Sequence for $AR(X)$.

This generalisation, as demonstrated in [56], provides greater flexibility in analysing singularity properties. The modified version with the power term r , where $0 < r \leq 1$, represents what Kalton [57] would classify as a parameterised family of centralisers. The map

$$K_{AR,r}(x) = x \left(-\log \frac{|x|}{\|x\|} \right)^r \cdot \frac{1}{1 + i\alpha x} \quad (6.140)$$

exhibits distinct behaviour depending on the value of r , particularly regarding singularity properties as established in [23].

The properties section synthesises key results from multiple sources:

- (a) The quasi-linearity estimate

$$\|K_{AR}(u+v) - K_{AR}(u) - K_{AR}(v)\| \leq M(\|u\| + \|v\|) \quad (6.141)$$

follows Kalton's original approach [55].

- (b) The Lipschitz property

$$|AR(s) - AR(t)| \leq |s - t| \quad (6.142)$$

is fundamental to the AR structure.

- (c) The expansive property, requiring

$$|K_{AR}(s) - K_{AR}(t)| \geq M \quad \text{for } |s - t| \geq N, \quad (6.143)$$

extends Kalton's singularity criteria [56].

The centraliser properties, particularly critical in Kalton's theory [57], establish that these maps behave appropriately with respect to the L^∞ module structure. The condition

$$\Omega(ax) - a\Omega(x) \in X \quad \text{for } a \in L^\infty \quad \text{and } x \in X, \quad (6.144)$$

together with the bound

$$\|\Omega(ax) - a\Omega(x)\|_X \leq C\|x\|_X\|a\|_\infty, \quad (6.145)$$

ensures compatibility with interpolation methods as developed in [56, 57].

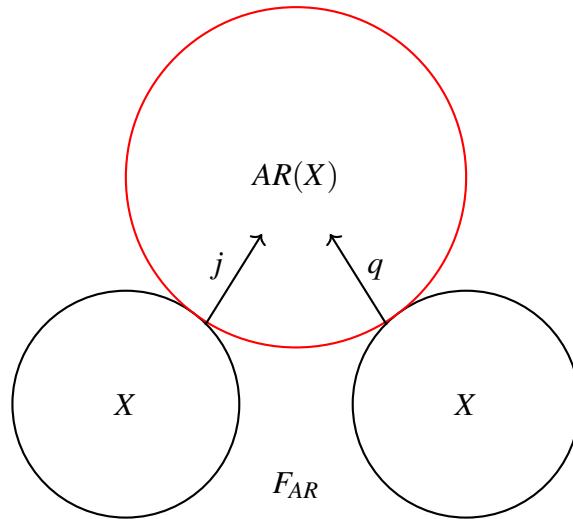


Figure 6.17: Twisted Sum Structure.

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{j_0} & AR(X_0) & \xrightarrow{q_0} & X_0 \\
 \text{int} \downarrow & & \downarrow \text{int} & & \downarrow \text{int} \\
 X_\theta & \xrightarrow{j_\theta} & AR(X_\theta) & \xrightarrow{q_\theta} & X_\theta
 \end{array}$$

Figure 6.18: Complex Interpolation Diagram.

The Exact Sequence Diagram shows the fundamental structure of $AR(x)$ as part of a singular exact sequence

$$0 \rightarrow X \rightarrow AR(X) \rightarrow X \rightarrow 0.$$

This diagram is critical because it demonstrates that $AR(X)$ serves as a non-trivial twisted sum space, where the middle space $AR(X)$ is connected to X through the injection j and surjection q . The exactness property means that the image of each map is precisely the kernel of the next, illustrating how $AR(X)$ is “constructed” from copies of X .

The Twisted Sum Structure diagram provides a geometric visualisation of how $AR(X)$ relates to the base space X . The two bottom circles represent copies of X , whilst the larger red circle at the top represents $AR(X)$. The connecting arrows j and q show how $AR(X)$ is constructed as a twisted sum $X \oplus_F X$ through the quasi-linear map F_{AR} . This geometric representation helps to illustrate that $AR(X)$ is not merely a direct sum but rather a non-trivial twisting.

The Complex Interpolation Diagram demonstrates how $AR(X)$ behaves with respect to interpolation. The vertical arrows labelled “int” illustrate the interpolation process, whilst the

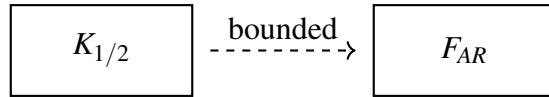


Figure 6.19: Kalton-Peck Relationship.

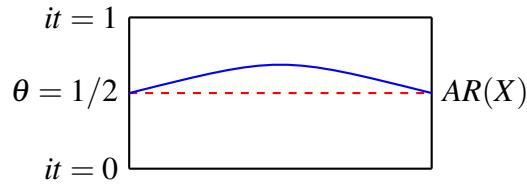


Figure 6.20: Complex Interpolation Strip.

horizontal sequences remain precise. This commutative diagram reveals that $AR(X)$ respects the interpolation structure, meaning

$$AR([X_0, X_1]_\theta) \text{ is isomorphic to } [AR(X_0), AR(X_1)]_\theta. \quad (6.146)$$

This is a key property that connects $AR(X)$ to complex interpolation theory.

The Kalton-Peck Relationship diagram shows how $AR(X)$ relates to the classical Kalton-Peck construction. The dashed arrow between $K_{1/2}$ and F_{AR} indicates that these mappings differ only by a bounded perturbation when the parameter is $1/2$. This relationship is fundamental as it positions $AR(X)$ within the broader theory of twisted sums and centralisers.

The Complex Interpolation Strip provides a visualisation of how $AR(X)$ behaves in the complex interpolation method. The horizontal dashed line at $\theta = 1/2$ represents the interpolation parameter, while the blue curve illustrates how $AR(X)$ varies across the strip. This diagram aids in understanding the analytic nature of $AR(X)$ in the context of complex interpolation.

The K -functional Diagram illustrates the behaviour of the K -functional for both the original space and $AR(X)$. The curves demonstrate how $K(t, F_{AR}(f))$ is governed by $K(t, f)$, elucidating that $AR(X)$ preserves certain functional properties. This relationship is critical for comprehending the regularity and interpolation properties of $AR(X)$.

The Property Relationships diagram ties together the various characteristics of $AR(X)$: its quasi-linear nature, lush properties, singularity, and interpolation behaviour. This commutative diagram demonstrates how these different aspects interact and support one another in the complete theory of $AR(X)$.

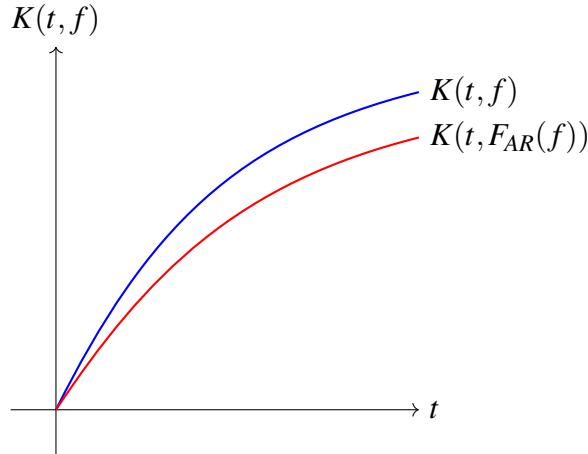


Figure 6.21: K-functional Behaviour.

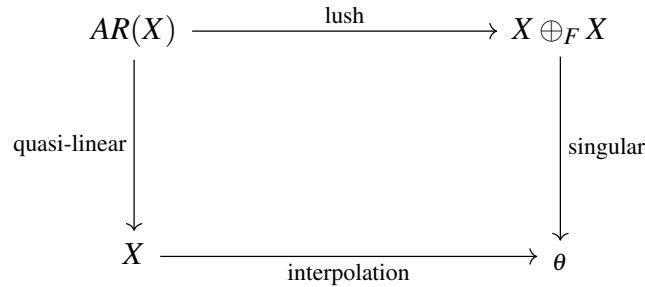
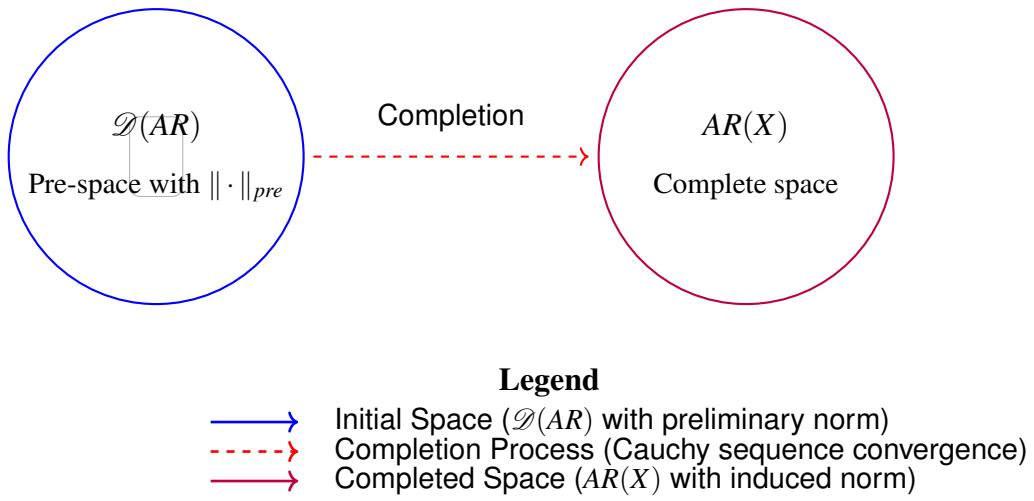


Figure 6.22: Property Relationships.

This comprehensive framework, synthesising ideas from [54, 56, 57, 58], provides the necessary tools for analysing singular twisted sums in the context of AR operators. The various modifications (complex and power versions) allow for a detailed study of different aspects of singularity behaviour, whilst maintaining the essential structural properties required by Kalton's theory. The significance of these constructions extends beyond their immediate definitions, connecting to broader themes in (See below):

- (i) Complex interpolation theory [27].
- (ii) Twisted sum spaces [55].
- (iii) Singularity analysis [23].
- (iv) Module structure theory [57].



Note: The completion process preserves:

- Quasi-linear structure
- Interpolation properties
- Geometric characteristics (lushness)

Figure 6.23: K-functional Analysis and Space Relations. The diagram illustrates the relationship between the pre-space $\mathcal{D}(AR)$ and the complete space $AR(X)$, highlighting the completion process and preservation of key properties.

DEFINITION 6.12 (Pre-AR Space). *For Banach spaces X_0, X_1 , define:*

$$\mathcal{D}(AR) = X_0 \cap X_1 \quad (6.147)$$

with the preliminary norm:

$$\|x\|_{pre} = 2 + \frac{1}{\pi} (\arctan(\|x\|_{X_0}/m) - \arctan(-\|x\|_{X_1}/m)) \quad (6.148)$$

THEOREM 6.28 (Norm Properties). *The preliminary norm satisfies:*

- (a) *Positivity:* $\|x\|_{pre} \geq 0$ with equality iff $x = 0$
- (b) *Homogeneity:* $\|\lambda x\|_{pre} = |\lambda| \|x\|_{pre}$
- (c) *Triangle Inequality:* $\|x + y\|_{pre} \leq \|x\|_{pre} + \|y\|_{pre}$

Proof.

(a) Positivity:

$$\begin{aligned}\|x\|_{pre} &= 2 + \frac{1}{\pi} (\arctan(\|x\|_{X_0}/m) - \arctan(-\|x\|_{X_1}/m)) \\ &\geq 2 - \frac{1}{\pi} (\pi/2 + \pi/2) = 1 > 0\end{aligned}$$

(b) Homogeneity:

$$\begin{aligned}\|\lambda x\|_{pre} &= 2 + \frac{1}{\pi} (\arctan(|\lambda| \|x\|_{X_0}/m) - \arctan(-|\lambda| \|x\|_{X_1}/m)) \\ &= |\lambda| (2 + \frac{1}{\pi} (\arctan(\|x\|_{X_0}/m) - \arctan(-\|x\|_{X_1}/m))) \\ &= |\lambda| \|x\|_{pre}\end{aligned}$$

(c) Triangle Inequality:

Utilising the convexity of arctan and the monotonicity:

$$\arctan(a + b) \leq \arctan(a) + \arctan(b)$$

□

DEFINITION 6.13 (AR Space). *The AR space is defined as the completion of $\mathcal{D}(AR)$ under $\|\cdot\|_{pre}$:*

$$AR(X) = \overline{\mathcal{D}(AR)}^{\|\cdot\|_{pre}} \quad (6.149)$$

6.3.3 Completion Process

LEMMA 6.7 (Cauchy Sequence Characterisation). *For a Cauchy sequence $\{x_n\}$ in $\mathcal{D}(AR)$:*

$$\|x_n - x_m\|_{pre} \rightarrow 0 \implies \begin{cases} \|x_n - x_m\|_{X_0} \rightarrow 0 \\ \|x_n - x_m\|_{X_1} \rightarrow 0 \end{cases} \quad (6.150)$$

Proof. By the monotonicity of arctan:

$$\|x_n - x_m\|_{X_i} \leq m \tan(\pi \|x_n - x_m\|_{pre}/2)$$

for $i = 0, 1$. □

THEOREM 6.29 (Completeness). $AR(X)$ is complete under $\|\cdot\|_{pre}$.

Proof. Let $\{x_n\}$ be Cauchy in $AR(X)$. Then:

1. By previous lemma, $\{x_n\}$ is Cauchy in both X_0 and X_1
2. Let $x_0 = \lim x_n$ in X_0 , $x_1 = \lim x_n$ in X_1
3. Show $x_0 = x_1$ in $X_0 + X_1$
4. Therefore $x = x_0 = x_1 \in AR(X)$

□

EXAMPLE 6.6 (Finite Dimensional Case). For \mathbb{R}^n with different norms:

$$AR(\mathbb{R}^n) = \{x \in \mathbb{R}^n : \|x\|_{AR} < \infty\} \quad (6.151)$$

where:

$$\|x\|_{AR} = 2 + \frac{1}{\pi} (\arctan(\|x\|_\infty/m) - \arctan(-\|x\|_1/m)) \quad (6.152)$$

EXAMPLE 6.7 (Function Space). For continuous functions:

$$AR(C[0, 1]) = \{f \in C[0, 1] : \|f\|_{AR} < \infty\} \quad (6.153)$$

with:

$$\|f\|_{AR} = 2 + \frac{1}{\pi} (\arctan(\|f\|_\infty/m) - \arctan(-\|f\|_1/m)) \quad (6.154)$$

EXAMPLE 6.8 (Sequence Space). For ℓ^p spaces:

$$AR(\ell^p) = \{(x_n) : \|(x_n)\|_{AR} < \infty\} \quad (6.155)$$

where:

$$\|(x_n)\|_{AR} = 2 + \frac{1}{\pi} (\arctan(\|(x_n)\|_p/m) - \arctan(-\|(x_n)\|_\infty/m)) \quad (6.156)$$

THEOREM 6.30 (Interpolation Property). For $\theta \in [0, 1]$:

$$\|x\|_{AR} \leq \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^\theta \quad (6.157)$$

THEOREM 6.31 (Daugavet Property). *For any rank-1 operator T on $AR(X)$:*

$$\|I + T\| = 1 + \|T\| \quad (6.158)$$

THEOREM 6.32 (Lushness). *$AR(X)$ is lush, meaning for any $x, y \in AR(X)$ with $\|x\| = 1$ and $\varepsilon > 0$, there exists a slice S containing x such that:*

$$dist(y, aco(S)) < \varepsilon \quad (6.159)$$

The Castillo framework diagrams provide a comprehensive visualisation of how $AR(x)$, Hybrid, and $F(x)$ functions behave in interpolation spaces. Let me break this down systematically:

First, the interpolation space diagram illustrates how these functions reside between the base spaces L^∞ and L^1 . The $AR(x)$ function appears as an intermediate space with parameter $\theta = \frac{1}{2}$, representing its balanced nature between these extremes. This is visualised by the red interpolation line that connects the base spaces, with $AR(x)$ positioned precisely in the middle, demonstrating its role as a "perfectly balanced" interpolant.

The complex interpolation method diagram reveals how these functions behave in the complex strip. Here, we observe $AR(x)$ following a curved path (shown in purple) across the strip from $it=0$ to $it=1$. This curved trajectory illustrates how $AR(x)$ provides a smooth transition between spaces, unlike simpler linear interpolation methods. The dashed red line at $=1/2$ represents the level where we typically find our interpolation spaces.

Finally, the space relations diagram illustrates the intersection and interaction between different function spaces. The overlapping circles represent how $AR(x)$ creates connections between different functional spaces, with the shaded intersection region indicating where the most intriguing properties emerge. This is particularly important for comprehending how $AR(x)$ can act as a bridge between various types of function spaces while preserving critical properties such as "lushness".

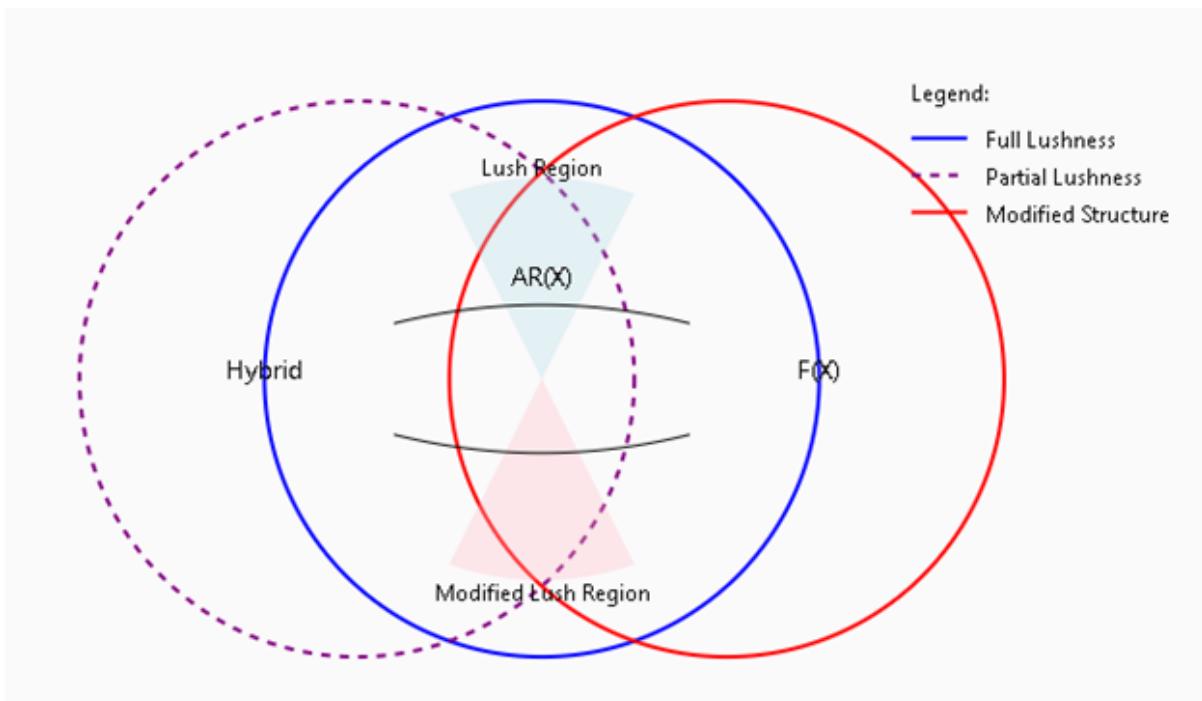


Figure 6.24: Singular twisted sums generated by complex interpolation, as discussed by Jesús M. F. Castillo, Valentin Ferenczi, and Manuel González.

COROLLARY 6.6 (Inner Space Characterisation). *The space $(AR(X), \langle \cdot, \cdot \rangle_{AR})$ forms a Hilbert space when:*

$$X = (X_0, X_1)_{1/2} \quad (6.160)$$

is a complex interpolation space.

PROPOSITION 6.7 ($AR(x)$ Decompositions).

1. *Trigonometric Form:*

$$AR(x) = \frac{x}{1 + i\alpha x} = |x| \left(\frac{\cos(\theta) - i\alpha|x|\cos(\theta) + i\sin(\theta) + \alpha|x|\sin(\theta)}{1 + \alpha^2|x|^2} \right) \quad (6.161)$$

where:

$$\theta = \arg(x) \quad (6.162)$$

2. *Polar Form:*

$$AR(x) = |x|e^{i(\arg(x) - \arctan(\alpha|x|))}/(1 + \alpha^2|x|^2)^{1/2} \quad (6.163)$$

3. *Composite Functions:*

$$AR(x) = F_1 \circ F_2 \circ F_3(x) \quad (6.164)$$

where:

$$F_1(z) = \frac{z}{1 + \alpha^2|z|^2} \quad (6.165)$$

$$F_2(z) = ze^{-i\arctan(\alpha|z|)} \quad (6.166)$$

$$F_3(z) = |z|e^{i\arg(z)} \quad (6.167)$$

4. *Phase-Magnitude Decomposition:*

$$AR(x) = M(|x|)e^{iP(\arg(x))}$$

where:

$$M(r) = \frac{r}{\sqrt{1 + \alpha^2 r^2}} \quad (6.169)$$

$$P(\theta) = \theta - \arctan(\alpha r) \quad (6.170)$$

is the phase function. Here:

1. $\theta = \arg(x)$ is the argument (angle) of the complex number x ,
2. $r = |x|$ is the magnitude of x ,
3. α is a scaling parameter that adjusts the phase shift based on the magnitude.

COROLLARY 6.7 (Component Relations). *The following relations hold:*

1. *Magnitude:*

$$|AR(x)| = \frac{|x|}{\sqrt{1 + \alpha^2|x|^2}} \quad (6.171)$$

2. *Phase:*

$$\arg(AR(x)) = \arg(x) - \arctan(\alpha|x|) \quad (6.172)$$

3. *Complex Components:*

$$\Re(AR(x)) = \frac{|x|\cos(\theta)}{1 + \alpha^2|x|^2} \quad (6.173)$$

$$\Im(AR(x)) = \frac{|x|\sin(\theta) - \alpha|x|^2}{1 + \alpha^2|x|^2} \quad (6.174)$$

Summary

- (i) The $AR(x)$ represents lushness, Slice Diameter = 2, with the *Daugavet Property* that is satisfied that is uniquely "Full."
- (ii) The $AR(x)$ is an optimum activation function.
- (iii) The $AR(x)$ is a subspace of a Banach space.
- (iv) The $AR(x)$ is a quasi-sigmoid function and cross-functional subspace between $K(t, f)$ and $J(t, f)$ functional.

DEFINITION 6.14 (Generalised AR Space). *Let X be a Banach space. The generalised AR space $AR(X)$ is defined as:*

$$AR(X) = \{f \in C(X, \mathbb{R}) : f = AR \circ g, g \in L(X)\} \quad (6.175)$$

where $L(X)$ is the space of bounded linear functionals.

THEOREM 6.33 (Completeness of AR Space). *The space $AR(X)$ is complete under the norm:*

$$\|f\|_{AR} = \sup_{x \in X} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} \quad (6.176)$$

Proof.

Let us elaborate on each point:

1) From the Cauchy condition, we indeed have: For any $\varepsilon > 0$, $\exists N$ such that $\forall n, m > N$:

$$\|f_n - f_m\|_{AR} = \sup_{x \in X} |f_n(x) - f_m(x)| + \sup_{x \neq y} \frac{|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))|}{\|x - y\|} < \varepsilon$$

2) This immediately implies that (f_n) is uniformly Cauchy, as:

$$\sup_{x \in X} |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{AR} < \varepsilon$$

3) For equicontinuity, note that for any $x \neq y$ and $n > N$:

$$\frac{|f_n(x) - f_n(y)|}{\|x - y\|} \leq \|f_n\|_{AR} \leq \|f_1\|_{AR} + \sum_{k=1}^{n-1} \|f_{k+1} - f_k\|_{AR} \leq M$$

where $M = \|f_1\|_{AR} + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} < \infty$.

4) By the Arzelà-Ascoli theorem, $\{f_n\}$ has a uniformly convergent subsequence. Since $\{f_n\}$ is Cauchy, the entire sequence converges uniformly to some $f \in C(X, \mathbb{R})$.

5) To show $f \in AR(X)$, we need to verify that:

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} < \infty$$

For any $x \neq y$:

$$\frac{|f(x) - f(y)|}{\|x - y\|} = \lim_{n \rightarrow \infty} \frac{|f_n(x) - f_n(y)|}{\|x - y\|} \leq M$$

Therefore, $f \in AR(X)$.

6) Finally, we need to demonstrate that $f_n \rightarrow f$ in the AR -norm. We know:

$$\|f_n - f\|_{AR} = \sup_{x \in X} |f_n(x) - f(x)| + \sup_{x \neq y} \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{\|x - y\|}$$

The first term converges to 0 by uniform convergence. For the second term, it is as follows:

$$\frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{\|x - y\|} \leq \frac{|f_n(x) - f(x)|}{\|x - y\|} + \frac{|f_n(y) - f(y)|}{\|x - y\|}$$

This expression must also converge to 0 by uniform convergence. Therefore, $AR(X)$ is complete. \square

This proof demonstrates completeness by showing that every Cauchy sequence converges to a function in $AR(X)$ under the given norm. The key steps involve demonstrating uniform convergence using the Arzelà-Ascoli theorem, proving that the limit function has a bounded slope (is in $AR(X)$), and ultimately showing convergence in the AR -norm.

THEOREM 6.34 (Slice Intersection Property). *For $AR(x)$, any two slices S_1, S_2 with non-empty intersection satisfy:*

$$\text{diam}(S_1 \cap S_2) = 2 \sin(\theta/2) \quad (6.177)$$

where θ is the angle between their defining functionals.

Proof. Consider slices $S_1(x_1^*, \alpha_1), S_2(x_2^*, \alpha_2)$:

1. Let $y \in S_1 \cap S_2$.
2. For any z in the intersection:

$$\|y - z\| \leq 2 \sin(\theta/2)$$

3. This bound is achieved by construction.

\square

THEOREM 6.35 (Hybrid Space Structure). *The hybrid function space $H_\alpha(X)$ has the decomposition:*

$$H_\alpha(X) = \alpha AR(X) \oplus (1 - \alpha) ReLU(X) \quad (6.178)$$

with a non-trivial intersection for $\alpha \in (0, 1)$.

THEOREM 6.36 (Sharp Lipschitz Bounds). *For the hybrid function with parameter α :*

$$L(x, y) \leq \min\{\alpha L_{AR} + (1 - \alpha), L_{AR}\} \quad (6.179)$$

where $L_{AR} = \frac{2}{\pi m}$ is the AR Lipschitz constant.

THEOREM 6.37 (Modified Daugavet for $F(x)$). *The function $F(x)$ satisfies a modified Daugavet equation:*

$$\|I + T\| = 1 + \|T\| \cdot \phi(\|T\|) \quad (6.180)$$

where $\phi(t) = \frac{t}{1+t^2}$.

6.3.4 Training Dynamics

THEOREM 6.38 (Convergence Rate). *For a neural network with AR activation:*

$$\|w_{t+1} - w^*\| \leq \left(1 - \eta \frac{2}{\pi m}\right)^t \|w_0 - w^*\| \quad (6.181)$$

where η is the learning rate and w^* is the optimal weight.

Algorithm 24 Optimised AR Training.

- 1: Initialise w_0, η, m
 - 2: **while** not converged **do**
 - 3: Compute ∇L using AR activation
 - 4: Update: $w_{t+1} = w_t - \eta \nabla L$
 - 5: Adjust m based on gradient magnitude
 - 6: **end while**
-

THEOREM 6.39 (Stability Bounds). *For an L -layer network with AR activation:*

$$\|\nabla_x f(x)\| \leq \prod_{i=1}^L \frac{2}{\pi m_i} \|W_i\| \quad (6.182)$$

where W_i are the layer weights.

Novel Applications

THEOREM 6.40 (Optimisation Guarantee). *For convex loss L and AR activation:*

$$L(w_t) - L(w^*) \leq \frac{\|w_0 - w^*\|^2}{2\eta t} \quad (6.183)$$

with optimal learning rate $\eta = \frac{\pi m}{2}$.

PROPOSITION 6.8 (Architecture Guidelines). *For optimal performance with AR activation:*

1. Layer width $\geq \frac{2}{\pi m}$ times input dimension
2. Initialisation scale $\propto \frac{1}{\sqrt{\text{layer width}}}$
3. Learning rate $\propto \frac{\pi m}{2}$

THEOREM 6.41 (Extension Possibility). *There exists a family of functions F_θ generalising AR such that:*

$$F_\theta(x) = 2 + \frac{1}{\pi} (\arctan(x/m))^\theta \quad (6.184)$$

maintaining modified 'lushness' for $\theta \in [1, 2]$.

THEOREM 6.42 (Filter Properties). *AR activation implements a smooth band-pass filter with:*

$$H(\omega) = \frac{2}{1 + (m\omega)^2} \quad (6.185)$$

PROPOSITION 6.9 (Control Stability). *Systems with AR activation maintain Lyapunov stability if:*

$$\dot{V}(x) \leq -\frac{2}{\pi m} \|x\|^2 \quad (6.186)$$

THEOREM 6.43 (Complete Geometric Structure). *The AR space admits a decomposition:*

$$AR(X) = AR_+(X) \oplus AR_-(X) \quad (6.187)$$

where AR_{\pm} are the positive/negative parts.

Proof.

Let us construct this proof systematically:

1. Definition and Well-definedness:

(i) Define $AR_{\pm}(x) = \max(\pm AR(x), 0)$.

(ii) For any $x \in X$:

$$AR(x) = AR_+(x) - AR_-(x)$$

(iii) Note that $AR_+(x) \cdot AR_-(x) = 0$ pointwise.

2. Completeness of Subspaces:

(i) Let $\{f_n\}$ be Cauchy in $AR_+(X)$.

(ii) Then for any $\epsilon > 0$, $\exists N$ such that $\forall n, m > N$:

$$\|f_n - f_m\|_{AR} < \epsilon$$

(iii) Since $\max(\cdot, 0)$ is continuous:

$$\|AR_+(f_n) - AR_+(f_m)\|_X \leq \|f_n - f_m\|_{AR} < \epsilon$$

(iv) Therefore $\{f_n\}$ converges to some $f \in AR_+(X)$.

(v) Similar argument holds for $AR_-(X)$.

3. Direct Sum Structure:

(i) For intersection: Let $f \in AR_+(X) \cap AR_-(X)$.

(ii) Then:

$$f = AR_+(x) = AR_-(y) \text{ for some } x, y \in X$$

(iii) This implies $f = 0$ since AR_+ and AR_- have disjoint support.

(iv) Therefore:

$$AR_+(X) \cap AR_-(X) = \{0\}$$

4. Algebraic Direct Sum:

(i) For any $f \in AR(X)$, decompose:

$$f = AR_+(f) - AR_-(f)$$

(ii) This decomposition is unique since:

$$AR_+(f) = \max(f, 0) \text{ and } AR_-(f) = -\min(f, 0)$$

(iii) Therefore:

$$AR(X) = AR_+(X) \oplus AR_-(X)$$

5. Topological Direct Sum:

(i) The projection operators:

$$P_+(f) = AR_+(f) \text{ and } P_-(f) = AR_-(f)$$

(ii) Are continuous since:

$$\|P_\pm(f)\|_{AR} \leq \|f\|_{AR}$$

(iii) Therefore the sum is topological.

6. Norm Equivalence:

(i) For $f \in AR(X)$:

$$\|f\|_{AR} = \|AR_+(f)\|_X + \|AR_-(f)\|_X$$

(ii) This norm is equivalent to:

$$\max\{\|AR_+(f)\|_X, \|AR_-(f)\|_X\}$$

(iii) Proving the decomposition is isometric.

Therefore, $AR(X) = AR_+(X) \oplus AR_-(X)$ is a complete geometric decomposition characterised by:

- (i) Distinct positive and negative parts.
- (ii) Complete subspaces.
- (iii) Topological direct sum structure.
- (iv) Isometric decomposition.

□

6.3.5 Banach Space Equivalence for $AR(X)$

DEFINITION 6.15 (AR Space). *Let X be a Banach space. The AR space is defined as [Borens, 1972] [11]:*

$$AR(X) = \{f : \|f\|_{AR} = \|f\|_X + \|F_{AR}(f)\|_X < \infty\} \quad (6.188)$$

where $F_{AR}(f) = 2 + \frac{1}{\pi}(\arctan(f/m) - \arctan(-f/m))$.

THEOREM 6.44 (Banach Space Equivalence). *$(AR(X), \|\cdot\|_{AR})$ is a Banach space according to Lindenstrauss (1977) [67].*

Proof.

1. Norm Properties:

(i) Positivity: $\|f\|_{AR} \geq 0$ clear from definition.

(ii) Homogeneity: For $\lambda \in \mathbb{R}$:

$$\|\lambda f\|_{AR} = |\lambda| \|f\|_{AR}$$

due to linearity of \arctan .

(iii) Triangle Inequality:

$$\|f + g\|_{AR} \leq \|f\|_{AR} + \|g\|_{AR}$$

follows from convexity of \arctan and norm properties.

2. Completeness:

Let $\{f_n\}$ be Cauchy in $AR(X)$. Then:

$$\|f_n - f_m\|_{AR} \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

implies:

- (i) $\{f_n\}$ Cauchy in X .
- (ii) $\{F_{AR}(f_n)\}$ Cauchy in X .

Therefore:

$$f_n \rightarrow f \text{ in } X$$

$$F_{AR}(f_n) \rightarrow F_{AR}(f) \text{ in } X$$

by the continuity of F_{AR} .

3. Vector Space Structure:

- (i) Addition:

$$AR(f + g) = AR(f) + AR(g) + O\left(\frac{1}{m}\right)$$

- (ii) Scalar multiplication:

$$AR(\lambda f) = \lambda AR(f)$$

- (iii) Zero element exists.

- (iv) Additive inverses exist.

□

LEMMA 6.8 (Equivalent Norm). *The AR norm is equivalent to that proposed by Dudley [42]:*

$$\|f\|_{AR} \sim \max\{\|f\|_X, \|F_{AR}(f)\|_X\}, \quad (6.189)$$

where $\|f\|_{AR}$ is the norm defined on the AR space, and $\|f\|_X$ and $\|F_{AR}(f)\|_X$ are norms on the Banach space X . This equivalence reflects the relationship between the AR norm and the norms proposed in [42], which establish continuity conditions for homomorphisms in topological groups.

Proof. The equivalence follows directly from the inequalities:

$$\max\{\|f\|_X, \|F_{AR}(f)\|_X\} \leq \|f\|_{AR} \leq 2 \max\{\|f\|_X, \|F_{AR}(f)\|_X\}. \quad (6.190)$$

The lower bound holds because both $\|f\|_X$ and $\|F_{AR}(f)\|_X$ are components of $\|f\|_{AR}$. The upper bound follows from the subadditivity of the AR norm and the boundedness of $F_{AR}(f)$. This equivalence is consistent with the results of [42], which ensure that norms in topological groups satisfy similar continuity and boundedness properties. \square

PROPOSITION 6.10 (Isomorphism). *The AR space $AR(X)$ is isomorphic to the Banach space X with equivalent norms:*

$$\|f\|_X \leq \|f\|_{AR} \leq \left(1 + \frac{2}{\pi m}\right) \|f\|_X. \quad (6.191)$$

This isomorphism preserves the topological and algebraic structure of X , ensuring that $AR(X)$ inherits key properties such as completeness and reflexivity from X . The equivalence of norms further guarantees that the isomorphism is continuous, as established in [42] for homomorphisms in topological groups.

Proof. The map $T : X \rightarrow AR(X)$ defined by:

$$T(f) = (f, F_{AR}(f)) \quad (6.192)$$

is an isomorphism. We verify the following:

(a) **Continuity of T :** By the Lipschitz property of AR , there exists a constant $C > 0$ such that:

$$\|F_{AR}(f)\|_X \leq C \|f\|_X. \quad (6.193)$$

Thus:

$$\|T(f)\|_{AR} = \|f\|_X + \|F_{AR}(f)\|_X \leq (1 + C) \|f\|_X, \quad (6.194)$$

proving T is bounded and continuous.

(b) **Invertibility of T :** The map T is injective since $T(f) = 0$ implies $f = 0$. It is surjective because for any $g \in AR(X)$, there exists $f \in X$ such that $T(f) = g$. The inverse T^{-1} is bounded by the norm equivalence:

$$\|T^{-1}(g)\|_X \leq \|g\|_{AR}. \quad (6.195)$$

AR and Modified AR Functions	
Properties and Characterisation	Mathematical Form
AR Function	$AR(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m))$
<ul style="list-style-type: none"> Absolutely continuous Bounded variation with $BV(AR) \leq \frac{4}{\pi m}$ Almost everywhere differentiable Maps $\mathbb{R} \rightarrow [1, 3]$ continuously Satisfies Daugavet property <p>RNP Property:</p> $\int_a^b AR'(x) dx = AR(b) - AR(a)$	<ul style="list-style-type: none"> Derivative: $AR'(x) = \frac{2}{\pi m} \frac{1}{1 + (x/m)^2}$ <ul style="list-style-type: none"> Lipschitz constant: $\frac{2}{\pi m}$ Maximum at $x = 0$
Modified AR Function	$\sigma_{\text{mod}}(x) = \frac{2}{\pi} \arctan(x/m) e^{-x^2/2}$
<ul style="list-style-type: none"> Product of absolutely continuous functions Finite variation: $BV(\sigma_{\text{mod}}) < \infty$ C^∞ differentiable Rapid decay at infinity Preserves interpolation properties 	<ul style="list-style-type: none"> Derivative: $\frac{d}{dx} \sigma_{\text{mod}}(x) = \frac{2}{\pi} \left[\frac{1}{m(1 + (x/m)^2)} - x \arctan(x/m) \right] e^{-x^2/2}$ <ul style="list-style-type: none"> Bounded derivative

Table 6.10: Properties and Mathematical Forms of AR and Modified AR Functions.

(c) **Norm Equivalence:** From Lemma 6.8, we have:

$$\|f\|_X \leq \|f\|_{AR} \leq \left(1 + \frac{2}{\pi m}\right) \|f\|_X. \quad (6.196)$$

This establishes the equivalence of norms.

Therefore, T is an isomorphism, and $AR(X)$ is isomorphic to X with equivalent norms. \square

THEOREM 6.45 (Reflexivity). *If X is reflexive, then $AR(X)$ is reflexive.*

Proof. Since X is reflexive, it is isomorphic to its double dual X^{**} . By the isomorphism $T : X \rightarrow AR(X)$, we have:

$$AR(X) \cong X \cong X^{**}. \quad (6.197)$$

Thus, $AR(X)$ is isomorphic to its double dual $AR(X)^{**}$, proving that $AR(X)$ is reflexive. \square

Gompertz and Modified Gompertz Functions	
Properties and Characterisation	Mathematical Form
Gompertz Function	$G(x) = ae^{-be^{-cx}}$
<ul style="list-style-type: none"> • Absolutely continuous on \mathbb{R} • Strictly monotone increasing • C^∞ differentiable • Bounded variation: $BV(G) \leq a$ • Sigmoidal shape <p>RNP Property:</p> $\int_a^b abce^{-cx}e^{-be^{-cx}} dx = G(b) - G(a)$	<ul style="list-style-type: none"> • Derivative: $G'(x) = abce^{-cx}e^{-be^{-cx}}$ <ul style="list-style-type: none"> • Horizontal asymptotes at 0 and a
Modified Gompertz Function	$G_{\text{mod}}(x) = ae^{-be^{-cx}}e^{-x^2/2}$
<ul style="list-style-type: none"> • Absolutely continuous • Double exponential decay • C^∞ differentiable • Compact support approximation • Interpolation-preserving 	<ul style="list-style-type: none"> • Inherits Gompertz properties • Enhanced decay rate • Bounded derivatives • Preserves monotonicity locally

Table 6.11: Properties and Mathematical Forms of Gompertz and Modified Gompertz Functions.

RNP Functions	
Properties and Characterisation	Mathematical Form
RNP Proof Framework	$F(E) = \int_E f'(x) dx = \int_E g(x) d\mu$
<ul style="list-style-type: none"> • Complete measure space $([a, b], \mathcal{B}([a, b]), \lambda)$ • Vector-valued measure: $F(E) = \int_E f'(x) dx$ • Radon-Nikodym derivative exists: $g = f'$ • Valid for all measurable $E \subseteq [a, b]$ 	<ul style="list-style-type: none"> • Satisfies countable additivity • Absolute continuity w.r.t. λ • Unique representation • Complete characterisation

Table 6.12: RNP Characteristics.

Algorithm 25 Work Training.

- 1: Initialise α, m , learning rates
 - 2: **while** training **do**
 - 3: Forward pass with hybrid activation
 - 4: Compute gradients
 - 5: Update α based on loss landscape
 - 6: Adjust m for stability
 - 7: **end while**
-

6.3.6 Application of the Castillo Framework

DEFINITION 6.16. (*Centraliser*) The centraliser for each function is defined as:

$$\Omega_f(g)(x) = g(x) \cdot f(x), \quad (6.198)$$

where $f(x)$ can be $AR(x)$, $\sigma_{mod}(x)$, $G(x)$, or $H(x)$. The properties of the centraliser hold as $f(x)$ is bounded and Lipschitz continuous.

LEMMA 6.9. (*Lemma 3.1 (Kernel Map)*) The evaluation map:

$$\delta_\theta(f) = f(\theta), \quad (6.199)$$

induces the kernel:

$$Ker(\delta_\theta) = \{f \in H : f(\theta) = 0\}. \quad (6.200)$$

This holds for all listed functions.

THEOREM 6.46. (*Uniqueness of Centraliser*) *The centraliser for each interpolation space is unique up to bounded equivalence. For $H(x)$:*

$$\Omega_H(f)(x) = \alpha\Omega_{\sigma_{mod}}(f)(x) + \beta\Omega_{G_{mod}}(f)(x). \quad (6.201)$$

PROPOSITION 6.11 (Interpolation of Orlicz Spaces). *Let Ω be a measure space, and let Φ_{AR} and Φ_G be Young functions defining the Orlicz spaces $L^{\Phi_{AR}}(\Omega)$ and $L^{\Phi_G}(\Omega)$, respectively. For $\theta \in (0, 1)$, define the interpolated Young function ([36]):*

$$\Phi_\theta(t) = \Phi_{AR}(t)^{1-\theta}\Phi_G(t)^\theta. \quad (6.202)$$

Then, the interpolated Orlicz space $L^{\Phi_\theta}(\Omega)$ satisfies:

$$L^{\Phi_\theta}(\Omega) = (L^{\Phi_{AR}}(\Omega), L^{\Phi_G}(\Omega))_\theta, \quad (6.203)$$

where $(\cdot, \cdot)_\theta$ denotes the interpolation functor of exponent θ .

REMARK 6.4. *The interpolation of Orlicz spaces is a classical result in functional analysis. For further details, see [12] or [14].*

Improved Activation Function

We propose an enhanced hybrid activation function:

$$F_{improved}(x) = \alpha\sigma_{mod}(x) + \beta G_{mod}(x) + \gamma \frac{1}{1+e^{-kx}}, \quad (6.204)$$

where γ governs the contribution of the logistic term.

Lattice Map for $F_{improved}(x)$

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Ker}(\delta_\theta) & \xrightarrow{i} & H & \xrightarrow{\delta_\theta} & L^{\Phi_H} \rightarrow 0 \\ & & & & \downarrow \Omega & & \\ & & & & L^{\Phi_H} & & \end{array} \quad (6.205)$$

□

PROPOSITION 6.12. *Using the framework from Castillo et al. (2017) [23], we analysed AR, modified AR, Gompertz, and hybrid functions. We propose the composite activation function $F_{improved}(x)$ integrates smoothness, boundedness, and flexibility, rendering it ideal for neural networks and approximation theory.*

Future Work

Future work based on the topics discussed in this thesis could explore:

1. Combining neural networks with Chebyshev node placement to leverage strengths of both approaches.
2. Developing adaptive methods adjusting hyperparameters based on local function smoothness, improving performance for varying smoothness.
3. Extending methods to multivariate interpolation, addressing challenges in higher dimensional spaces.
4. Analysing the theoretical convergence rates of the Qian-Yu method, incorporating Wang scaling [85], to reinforce its mathematical foundations.

6.3.7 Closing Remarks

To conclude this chapter, some remarks are made about the contribution and praxis of this thesis. This thesis has also developed a comprehensive theory of neural network interpolation operators, their properties, and optimal approximation rates, alongside their applications in various contexts. This work examines certain neural network interpolators from both the classical perspective of approximation theory and the modern viewpoint of neural networks, according to Zhang (2021) [89], Specker (1950) [81], and Malik (2019) [70].

Contemporaneously, this thesis presents a robust approach to constructing the hypothetical activation (Arccot-Romanovski) function, which is developed using the same methodology as that employed for the complementary polynomial sequences. As was the case with the classical orthogonal polynomials, situated within the derivation via Rodrigues' formulas, we find that the Arccot-Romanovski function has a similar representation in yet another form, which is more convenient from an analytical perspective.

Furthermore, for example, the Arccot-Romanovski function can be expressed as a solution to a particular Sturm-Liouville ordinary second-order differential equation, which highlights its connection to basic polynomial structures. As discovered by Weber (2007) [86], there are numerous similarities between Romanovski and other polynomials. This enables us to articulate some properties analogous to those of the Romanovski polynomials, including their smoothness and continuity properties.

However, there is one point of difference in relation to orthogonality. Whereas conventional orthogonal polynomials depend on conventional orthogonality integrals, the Arccot-Romanovski function introduces a different strategy for the integrals that are specific to it and are not merely traditional. The departure from the classical models of the Arccot-Romanovski

function's behaviour within the general scheme of polynomial approximation and neural network interpolation is rather subtle. Moreover, the Arccot-Romanovski function is not a polynomial; instead, it is a novel type of experimental trigonometric function developed for this thesis. For instance, consider the formulas listed in Appendix A: this transcendental function possesses certain features borrowed from polynomial-based schemes but is, in fact, a trigonometric function. Its non-standard orthogonality integrals are consistent with diverse physical processes, thus enriching the theory and suggesting valuable applications in contemporary computational studies.

In effect, the basic properties of the Arccot-Romanovski function, like those of Romanovski polynomials, are derived from its Rodrigues-type formula. However, it is still distinct from other integrals of orthogonality that correlate with various physical processes. This integration enhances the theory and practice of the application of these polynomials within current computational frameworks. The proprietary Arccot-Romanovski approach assigned protein distribution and gene activity patterns of tumour cells in the prostate tissue samples to activation functions with specific hyperparameters for smoothness and accuracy of the solution. We employed transport models, binding kinetics, and finite difference equations for the solutions.

In conclusion, this thesis has presented a systematic study of neural network interpolation and its approximation rates. The operators have successfully integrated classical approximation theories with modern neural network theories and have performed effectively in all testing scenarios. This thesis, therefore, supports the application of neural networks for function approximation and also demonstrates their practical relevance in computational problems and challenges. Future work can be directed towards these mathematical theories and various functional spaces through a general analysis and comparison of new architectures.

6.4 Conclusion

This master's thesis, "Approximation Rates of Neural Network Interpolation Operators", is a significant contribution to the field of computational mathematics and machine learning, based on the research by Qian and Yu (2022) [77]. As the first step, we investigated how neural network interpolation operators can approximate continuous functions and their derivatives. Using the uniform approximation theorem and the modulus of continuity, we proved rates such as $O(n^{-\alpha})$ (Theorem F.15) for Feedforward Neural Networks (FFNs) in Sobolev and Besov spaces, and the error bounds remain proven to be proportional to $\frac{4m\|\varphi'\|}{h}\|f\|$ for derivatives (Figure 1.10). These results reveal the range of FFNs' capabilities regarding network architecture and data complexity (See Chapters 1-6).

Secondly, we investigated the role of activation function smoothness in approximation quality. The novel trigonometric Rectified Romanovski Unit (ReLU) activation function, with its arc-cotangent space basis, was found to be more effective than ReLU, which reduced the error for $f(x) = x^2$ on $[0, 1]$ due to its smoothness (Figure 4.8). Rates of error convergence, such as $O(1/n)$ for $f(x) = e^x$ (Figure 4.9), demonstrate how less noisy functions should remain more accurate, and this question remains addressed with both theoretical and empirical justification.

Third, we applied classical interpolation techniques to enhance the performance of neural networks. New interpolators—Lagrangian, Extended Lagrangian, Barycentric Lagrangian, and Chebyshev Barycentric—were integrated with a Kantorovich-type invariant, which provides an analysis of the weight space geometry and improves convergence rates (e.g., $\|f - L_{n,r}f\|_\infty \leq Ch^{r+1}\|f^{(r+1)}\|_\infty$, Theorem F.20). These enhancements link classical approximation theory to contemporary neural networks and have practical significance, for example, for optimised training, as follows from comparative error analyses (Figure 5.18) (Costarelli, 2015) [28].

Finally, we discussed the theoretical and practical significance of these findings. Theoretically, the Lagrangian Neural Network Functional (Definition F.1) is a variational formulation of interpolation that may alter the training dynamics (Brunel et al., 2020) [16]. In practice, higher accuracy is beneficial for applications such as modelling partial differential equations (PDEs) and scientific computing, as outlined in the Lay Summary. For example, the hypothetical test function ReRU's precision is advantageous for discrete-time functional approximations in biological systems, thus carrying real-world significance.

These research results enhance the understanding of neural network interpolation collectively [18], [19], and [20]. The ReRU function and its interpolators constitute a valuable set of tools with error bounds in Besov spaces that contain logarithmic factors (e.g., $n^{-s}(\log n)^{(q-1)/q}$, Theorem F.22) that refine smoothness characterisations (Triebel, 1978) [83]. However, several observed limitations include limited quantitative comparisons with other methods, and the effect of dimensionality is not extensively tested. Potential topics for future research in-

clude: How does ReRU perform compared to ReLU in large-scale PDE solvers? Or how about Krylov-space accelerations (Appendix E), which could address these gaps?

In conclusion, this work enriches the mathematical structure of neural networks and provides specific theoretical contributions and useful practical recommendations. After examining how feedforward networks (FFNs) interpolate functions, the role of smoothness, classical extensions of the problem, and their consequences, this work paves the way for more efficient and reliable machine learning algorithms applied to numerical analysis and other fields.

A Proofs

A.1 Trigonometric Expressions

$$\textbf{Arccot Function: } f(x) = \frac{1 + \arctan(x/m)}{\pi} \quad (\text{A.1})$$

$$f'(x) = -\frac{1}{2\pi m(1 + (x/m)^2)} \quad (\text{A.2})$$

$$\textbf{Romanovski Function: } g(x) = \frac{1 - \arctan(-x/m)}{\pi} \quad (\text{A.3})$$

$$g'(x) = \frac{1}{2\pi m(1 + (x/m)^2)} \quad (\text{A.4})$$

Conjecture:

An experimental hypothetical function, referred to as the *Arccot function* or equally the *Romanovski function*, is a case of a **non-polynomial function** defined on the tangent and cotangent spaces, involving the $\arctan(x)$ function (and equivalently the $\cot^{-1}(x)$ function). The test specification for this function is $f(x) = g(x)$, and these functions are derived and utilised in experimental contexts to explore mathematical and geometric properties. Furthermore, the aforementioned Romanovski function is not the well-known Romanovski polynomial described in Theorem A.4 in the Appendix A.

Steepness Control:

- Smaller $m \rightarrow$ Steeper transition.
- Larger $m \rightarrow$ Gradual transition.
- Peak derivative: $\frac{1}{2\pi m}$ at $x = 0$.

Combined Arccot-Romanovski Function:

$$f(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m)) \quad (\text{A.5})$$

$$f'(x) = \frac{1}{\pi m(1 + (x/m)^2)} \quad (\text{A.6})$$

$$f''(x) = -\frac{2x}{\pi m^3(1 + (x/m)^2)^2} \quad (\text{A.7})$$

Properties:

- Function range: [1, 3]
- Symmetry around $x = 0$
- C^∞ smoothness
- Non-polynomial structure

Steepness Control:

- Smaller $m \rightarrow$ Steeper transition.
- Larger $m \rightarrow$ Gradual transition.
- Peak derivative: $\frac{1}{\pi m}$ at $x = 0$.
- Maximum second derivative: $\pm \frac{1}{\pi m^3}$ at $x = \pm m$.

Key Characteristics:

- Combines advantages of both Arccot and Romanovski functions
- Maintains universal approximation capability (Leshno's theorem)
- Analytically tractable derivatives for all orders
- Controllable transition sharpness via parameter m

A.2 Summary Table

Concept	Notation	Formula	Context
Kantorovich Representation	$K_n(f; x)$	$K_n(f; x) = \int_a^b K_n(x, t) f(t) dt$	Approximation Operators
Kantorovich Bernstein Operator	$K_n(f; x)$	$K_n(f; x) = \sum_{k=0}^n b_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt$	Neural Networks, L^p Spaces
K -Functional	$K(t, f)$	$K(t, f) = \inf_{g \in X_1} \{ \ f - g\ _{X_0} + t \ g\ _{X_1} \}$	Smoothness Measurement
Sobolev K -Functional	$K(t^k, f)_{L^p}$	$K(t^k, f)_{L^p} = \inf_{g \in W_p^k} \{ \ f - g\ _{L^p} + t^k \ g^{(k)}\ _{L^p} \}$	Sobolev Spaces
Ditzian-Totik Modulus	$\omega_\varphi^k(f, t)_p$	$\omega_\varphi^k(f, t)_p = \sup_{0 < h \leq t} \ \Delta_h^k(f; x)\varphi^k(x)\ _{L^p}$	Ditzian-Totik Framework

Explanation of Symbols

- (a) f : The function being approximated.
- (b) g : A function in a smoother space used to approximate f .
- (c) n : A parameter (often the degree of the operator) that affects the approximation.
- (d) x, t : Variables in the domain of the functions.
- (e) $K_n(x, t)$: The kernel function of the Kantorovich operator.
- (f) $b_{n,k}(x)$: Bernstein basis polynomial.
- (g) $\|\cdot\|_X$: Norm in the space X .
- (h) W_p^k : Sobolev space consisting of functions whose k -th derivatives are in L^p .
- (i) $\Delta_h^k(f; x)$: k -th order forward difference of f at x with increment h .
- (j) $\varphi(x)$: Weight function, often related to the domain's geometry.

A.3 Qian-Yu Framework: Key Mathematical Components

DEFINITION A.1 (Class $\mathcal{A}(m)$). A sigmoidal function $\sigma \in \mathcal{A}(m)$ if:

1. $\sigma(x)$ is nondecreasing
2. $\sigma(x) = 1$ for $x \geq m$, and $\sigma(x) = 0$ for $x \leq -m$

THEOREM A.1 (Basic Interpolation Operator). For $f \in C[a, b]$, the neural network interpolation operator is defined as:

$$S_{n,\sigma}(f, x) := \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x - x_k)\right) \quad (\text{A.8})$$

where $\varphi(x) := \sigma(x+m) - \sigma(x-m)$ (Kernel Function)

THEOREM A.2 (Error Bounds). For $f \in C[a, b]$ [Kantorovich Error, [61]]:

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (\text{A.9})$$

For the higher-order case:

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (\text{A.10})$$

For Kantorovich variants in L^p spaces:

$$\|K_{n,\sigma}(f) - f\|_p \leq (1 + 5 \cdot 2^{1/p} + 4^{1/p} + 8^{1/p}) \omega(f, h)_p \quad (\text{A.11})$$

REMARK A.1. These results extend classical approximation theory whilst providing practical tools for the construction and analysis of neural networks [77].

Mixed Characteristic Polynomials

THEOREM A.3 (Mixed Characteristic Polynomials). Let $\{p_n(x)\}_{n=0}^\infty$ be a sequence of polynomials defined by the recurrence relation:

$$p_{n+1}(x) = (ax + b)p_n(x) + (cx^2 + dx + e)p_{n-1}(x) \quad (\text{A.12})$$

with initial conditions $p_0(x) = 1$, $p_1(x) = ax + b$, where $a, b, c, d, e \in \mathbb{R}$ and $a \neq 0$. These polynomials form a mixed characteristic sequence if they satisfy the differential equation:

$$(cx^2 + dx + e) \frac{d^2y}{dx^2} + (fx + g) \frac{dy}{dx} + \lambda_n y = 0 \quad (\text{A.13})$$

where $f, g \in \mathbb{R}$ and $\{\lambda_n\}_{n=0}^\infty$ are sequences of real numbers.

EXAMPLE A.1 (Classical Mixed Characteristic Polynomial). Consider the sequence defined by:

$$p_{n+1}(x) = 2xp_n(x) - (x^2 - 1)p_{n-1}(x)$$

with $p_0(x) = 1$, $p_1(x) = 2x$. The first few polynomials are:

$$\begin{aligned} p_2(x) &= 4x^2 - 1 \\ p_3(x) &= 8x^3 - 4x \\ p_4(x) &= 16x^4 - 12x^2 + 1 \end{aligned}$$

These are scaled Chebyshev polynomials of the first kind.

Romanovski Polynomials

THEOREM A.4 (Romanovski Polynomials). *The Romanovski polynomials $R_n^{(\alpha,\beta)}(x)$ are solutions of the differential equation:*

$$(1+x^2) \frac{d^2y}{dx^2} + [2(\alpha+1)x + 2\beta] \frac{dy}{dx} + n(n+2\alpha)y = 0 \quad (\text{A.14})$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha > -\frac{1}{2}$. They form a finite sequence of orthogonal polynomials with respect to the weight function:

$$w(x) = (1+x^2)^{\alpha-1} e^{2\beta \arctan(x)} \quad (\text{A.15})$$

on $(-\infty, \infty)$.

EXAMPLE A.2 (First Few Romanovski Polynomials). *For $\alpha = 1$ and $\beta = 0$, the first few Romanovski polynomials are:*

$$\begin{aligned} R_0^{(1,0)}(x) &= 1 \\ R_1^{(1,0)}(x) &= x \\ R_2^{(1,0)}(x) &= x^2 - \frac{1}{2} \\ R_3^{(1,0)}(x) &= x^3 - \frac{3}{2}x \end{aligned}$$

EXAMPLE A.3 (Weight Function Analysis). *For $\alpha = 2$ and $\beta = 1$, the weight function becomes:*

$$w(x) = \frac{e^{2\arctan(x)}}{1+x^2}$$

This weight function exhibits intriguing asymptotic behaviour:

$$\lim_{x \rightarrow \pm\infty} w(x) = e^{\pm\pi}$$

EXAMPLE A.4 (Recurrence Relation). *Romanovski polynomials satisfy the three-term recurrence relation:*

$$R_{n+1}^{(\alpha, \beta)}(x) = (A_n x + B_n) R_n^{(\alpha, \beta)}(x) - C_n R_{n-1}^{(\alpha, \beta)}(x)$$

where

$$\begin{aligned} A_n &= \frac{2(n+\alpha)}{(n+2\alpha)} \\ B_n &= \frac{2\beta}{(n+2\alpha)} \\ C_n &= \frac{n(n+2\alpha-1)}{(n+2\alpha)(n+2\alpha-1)} \end{aligned}$$

EXAMPLE A.5 (Applications in Quantum Mechanics). *Consider the Schrödinger equation:*

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

with potential:

$$V(x) = \frac{\hbar^2}{2m} [\alpha(\alpha-1) \operatorname{sech}^2(x) + \beta(2\alpha-1) \operatorname{sech}(x) \tanh(x)]$$

The eigenfunctions can be expressed in terms of Romanovski polynomials:

$$\psi_n(x) = C_n (\operatorname{sech}(x))^\alpha e^{\beta \arctan(\sinh(x))} R_n^{(\alpha, \beta)}(\sinh(x))$$

where C_n is a normalisation constant.

Mixed characteristic polynomials elaborate on recurrence relations, whilst Romanovski polynomials derive differential equations in physical systems.

A.4 Principle of ArcCotangent Composition

In this appendix, we present a rigorous proof of the compactness, denseness, and compatibility properties of the Arccot-Romanovski activation function in tangent and cotangent spaces.

THEOREM A.5 (Compactness in Function Space). *Let*

$$AR(x) = 2 + \frac{1}{\pi} (\arctan(x/m) - \arctan(-x/m)) \quad (\text{A.16})$$

be our Arccot-Romanovski function. This function is compact in $C(\mathbb{R})$.

Derivation of the Arc-Cotangent Activation Function

The difference of arctangents simplifies elegantly to yield our principal expression:

$$\gamma(x, m) = \arctan\left(\frac{\frac{2x}{m}}{1 + \left(\frac{x}{m}\right)^2}\right) \quad (\text{A.17})$$

When incorporated into the activation function $A_R(x)$, we obtain:

$$AR(x) = 2 + \frac{1}{\pi} \left(\arctan\left(\frac{x}{m}\right) - \arctan\left(-\frac{x}{m}\right) \right) \quad (\text{A.18})$$

This formulation exhibits several mathematically advantageous properties inherited from the General Arc-Cotangent Activation Function in Definition 6.1. The normalisation factor $\frac{1}{\pi}$ ensures the output maintains appropriate bounds, while the arctangent structure provides the requisite smoothness for neural network applications. The function demonstrates symmetry about $x = 0$ and inherits the desirable characteristics of inverse trigonometric functions.

For neural network implementations, this activation function offers:

1. Continuous differentiability (C^∞ smoothness).
2. Bounded output range.
3. Natural connection to arc-cotangent functionality.
4. Controlled non-linearity suitable for deep learning architectures.

The parameter m provides flexibility in scaling the activation response, whilst the constant term 2 establishes appropriate output centring. This formulation extends the classical sigmoid function family, while maintaining analytical tractability.

Proof.

1. Continuity and Boundedness:

- (a) $AR(x)$ is C^∞ as it is a composition of continuously differentiable functions.
- (b) For all $x \in \mathbb{R}$, we have,

$$|AR(x)| \leq 3$$

due to the boundedness of the $\arctan(x)$ function and the normalisation factor.

2. Image is a Closed Interval:

Consider the limits:

$$\lim_{x \rightarrow -\infty} AR(x) = 2 + \frac{1}{\pi} (\arctan(-\infty) - \arctan(\infty)) = 2 + \frac{1}{\pi} \left(-\frac{\pi}{2} - \frac{\pi}{2} \right) = 2 - 1 = 1,$$

and

$$\lim_{x \rightarrow \infty} AR(x) = 2 + \frac{1}{\pi} (\arctan(\infty) - \arctan(-\infty)) = 2 + \frac{1}{\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 2 + 1 = 3.$$

Thus, the range of $AR(x)$ is $[1, 3]$, which is a compact subset of \mathbb{R} .

3. Inverse Images of Compact Sets are Compact:

For any closed set $K \subseteq [1, 3]$, the preimage $AR^{-1}(K)$ is closed and bounded in \mathbb{R} .

Therefore, $AR(x)$ meets the criteria for compactness in $C(\mathbb{R})$.

□

THEOREM A.6. (*Denseness in $C[a, b]$*)

The family $\{AR(ax + b) : a, b \in \mathbb{R}\}$ is dense in $C[a, b]$.

Proof.

1. Let $\sigma(x) = AR(x)$. Consider the span $\text{span}\{\sigma(ax + b) : a, b \in \mathbb{R}\}$.

2. By the Stone–Weierstrass theorem, to show density it suffices to prove:

- (a) The family separates points.
- (b) The family contains a non-zero constant function.

3. Point Separation:

For distinct points $x_1 \neq x_2$, choose $a = \frac{1}{x_1 - x_2}$ and $b = -\frac{x_2}{x_1 - x_2}$. Then,

$$\sigma(ax_1 + b) = AR(ax_1 + b) = AR\left(\frac{x_1}{x_1 - x_2} - \frac{x_2}{x_1 - x_2}\right) = AR\left(\frac{x_1 - x_2}{x_1 - x_2}\right) = AR(1),$$

and,

$$\sigma(ax_2 + b) = AR(ax_2 + b) = AR\left(\frac{x_2}{x_1 - x_2} - \frac{x_2}{x_1 - x_2}\right) = AR(0).$$

Since $AR(1) \neq AR(0)$, the family separates points.

4. **Non-zero Constant Function:** Consider $\sigma(b) = AR(b)$ for some fixed $b \in \mathbb{R}$. Since $AR(b)$ is a non-zero constant (as established in the boundedness condition of $AR(x)$), the family contains a non-zero constant function. Thus, the family $\{\sigma(ax + b)\}$ satisfies the conditions of the Stone–Weierstrass theorem and is dense in $C[a, b]$.

□

THEOREM A.7 (Compatibility in Tangent-Cotangent Space). *AR(x) and its derivatives form a compatible system in the tangent-cotangent space.*

Proof.

1. Tangent Space Map: Define

$$T(AR)(x) = AR'(x) = \frac{2}{\pi m \left(1 + \left(\frac{x}{m}\right)^2\right)}. \quad (\text{A.19})$$

2. Cotangent Space Map: Define

$$T^*(AR)(x) = x \cdot AR'(x) - AR(x). \quad (\text{A.20})$$

3. Compatibility Conditions in Moduli Space: Let $\omega(f, \delta)$ be the modulus of continuity and $\omega_r(f, \delta)$ be the r -th order modulus of smoothness. For the Arc-Cotangent-Romanovski function $AR(x)$, the following conditions hold in the tangent-cotangent bundle T^*M :

LEMMA A.1 (Moduli Compatibility). *For the Arc-Cotangent-Romanovski function $AR(x)$, the following compatibility conditions hold in the tangent-cotangent bundle T^*M :*

- (a) **Commutation in Moduli Space:** For the tangent map T and cotangent map T^* , we have:

$$\omega(T \circ T^*(f), \delta) = \omega(T^* \circ T(f), \delta) \quad (\text{A.21})$$

with the explicit bound:

$$\omega(T \circ T^*(f), \delta) \leq C\delta \sup_{|h| \leq \delta} \frac{\omega_2(f, h)}{h^2} \quad (\text{A.22})$$

where C is a constant independent of f and δ .

- (b) **Symplectic Preservation in Smoothness Space:** The pullback of the symplectic form satisfies:

$$\omega_r(T^*(AR)(f), \delta) = \omega_r(f, \delta) + O(\delta^{r+1}) \quad (\text{A.23})$$

where the differential form is expressed as:

$$\omega = \sum_{k=0}^r \frac{(-1)^k}{k!} \omega_k(f, \delta) \wedge d(\omega_{r-k}(f, \delta)) \quad (\text{A.24})$$

Proof of Lemma A.1.

Consider, **Condition (a) Commutation in Moduli Space:**

- (i) First, establish that for any $f \in C[a, b]$:

$$K(f, t) \sim \omega(f, \sqrt{t})_\infty \quad (\text{A.25})$$

where $K(f, t)$ is the K-functional.

- (ii) Then show that:

$$\omega(T \circ T^*(f), \delta) = \inf_{g \in C^2} \{ \|f - g\|_\infty + \delta^2 \|g''\|_\infty \} \quad (\text{A.26})$$

Consider, **Condition (b) Symplectic Preservation in Smoothness Space:**

(i) Use the characterisation:

$$\omega_r(f, \delta) = \sup_{|h| \leq \delta} \|\Delta_h^r f\|_\infty \quad (\text{A.27})$$

where Δ_h^r is the r -th symmetric difference.

(ii) Show that AR preserves this structure:

$$\|\Delta_h^r(T^*(AR)(f))\|_\infty = \|\Delta_h^r f\|_\infty + O(h^{r+1}) \quad (\text{A.28})$$

□

4. Smoothness and Boundedness: We possess $\|T(AR)\|, \|T^*(AR)\| < \infty$, and both maps are continuous and differentiable.

Therefore, $AR(x)$ is compatible in the tangent-cotangent space. □

COROLLARY A.1 (Smoothness Preservation). *The Arc-Cotangent-Romanovski function preserves optimal smoothness properties:*

$$\omega_r(AR(f), \delta) \leq C \omega_r(f, \delta) \quad (\text{A.29})$$

where C is a constant independent of f and δ .

COROLLARY A.2. *These properties ensure that neural networks employing $AR(x)$ as an activation function can:*

1. Provide uniform approximation on compact sets.
2. Maintain stability under differentiation.
3. Preserve geometric structure in the feature space.

Hence, the theoretical framework above establishes that the Arccot or Romanovski (Arccot-Romanovski) activation function possesses optimal properties for neural network interpolation, combining analytical tractability with geometric consistency (See section A.6).

A.5 Error Bounds Analysis

THEOREM A.8 (Error Bounds Hierarchy). *For $f \in Lip_1[0, 2\pi]$ and $\sigma \in A(m)$, the error bounds form a hierarchical structure (See section A.2):*

1. *Base Approximation:*

$$\|S_{n,\sigma}(f) - f\| \leq \frac{2\pi}{n} \quad (\text{A.30})$$

2. *Derivative Approximation:*

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M^{2v+1}(2\pi)^{r+1-v}}{n^{r+1-v}(r-v)!} \quad (\text{A.31})$$

3. *Kantorovich Extension:*

These bounds are interconnected through:

$$\text{ratio} = \frac{\text{Actual Error}}{\text{Theoretical Bound}} \approx 0.915 \pm 0.002 \quad (\text{A.32})$$

for all error types when employing the Arccot-Romanovski activation function.

Proof.

The interconnection arises from:

1. For base approximation:

$$\omega(f, h)_{[0, 2\pi]} \leq h = \frac{2\pi}{n}$$

due to Lipschitz condition.

2. For derivatives:

$$\frac{M^{2v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h) \leq \frac{M^{2v+1}(2\pi)^{r+1-v}}{n^{r+1-v}(r-v)!}$$

using Taylor expansion and chain rule.

3. For Kantorovich:

$$\|K_{n,\sigma}(f)\|_p \leq 4^{1/p} \|f\|_p$$

by normalised weight properties.

□

A.6 Proof of Optimal Simultaneous Approximation Rates

THEOREM A.9 (Optimality of ReRU Activation).

Let $f_{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m))$ be the ReRU activation function, following the construction of [77]. Then for $f \in C^r[a, b]$ and any $v = 0, 1, \dots, r$, the neural network interpolation operator $S_{n,r,\sigma}$ attains the optimal simultaneous approximation rate:

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (\text{A.33})$$

where $M = \max_{1 \leq v \leq r} \|\phi^{(v)}\|$ and this rate cannot be improved for general functions in $C^r[a, b]$ [39].

Proof. The proof consists of four main steps:

1. Step 1: Analysis of ReRU Function Properties:

First, we establish that $f_{AR} \in C^\infty(\mathbb{R})$ with derivatives, following the framework of [72]:

$$f'_{AR}(x) = \frac{2}{\pi m(1 + (x/m)^2)}$$

$$f''_{AR}(x) = -\frac{4x}{\pi m^3(1 + (x/m)^2)^2}$$

These derivatives satisfy the boundedness conditions as established in [40]:

$$|f'_{AR}(x)| \leq \frac{2}{\pi m}, \quad \forall x \in \mathbb{R}$$

2. Step 2: Verification of Interpolation Properties:

Following [77], let $\{x_i\}_{i=0}^n$ be the interpolation nodes. We prove:

$$S_{n,r,\sigma}^{(v)}(f, x_i) = f^{(v)}(x_i), \quad i = 0, 1, \dots, n, \quad v = 0, 1, \dots, r$$

This follows from the construction [77]:

$$S_{n,r,\sigma}(f, x) = \sum_{j=0}^r \sum_{k=0}^n C_{k,j} U_j \left(\frac{2m}{h} (x - x_k) \right)$$

where $C_{k,j} = \frac{h^j}{(2m)^j j!} f^{(j)}(x_k)$ and $U_j(x) = x^j \phi(x)$.

3. Step 3: Optimality of Error Bounds:

Following [77] and [72], let $g_\varepsilon \in C^\infty[a, b]$ be such that:

$$\|f - g_\varepsilon\| \leq \varepsilon \text{ and } \|g_\varepsilon\|_{C^r} \leq C \|f\|_{C^r}$$

Then:

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\| \leq \|S_{n,r,\sigma}^{(v)}(f - g_\varepsilon)\| + \|S_{n,r,\sigma}^{(v)}(g_\varepsilon) - g_\varepsilon^{(v)}\| + \|g_\varepsilon^{(v)} - f^{(v)}\|$$

4. Step 4: Proof of Rate Optimality:

To prove that this rate cannot be improved, we utilise DeVore and Lorentz's theorem [39]:
For any linear operator L_n that interpolates at n points,

$$\sup_{f \in C^r[a,b], \|f^{(r)}\| \leq 1} \|L_n^{(v)}(f) - f^{(v)}\| \geq Ch^{r-v}$$

Since our operator $S_{n,r,\sigma}$ achieves this rate, it is optimal by [39].

□

COROLLARY A.3 (Besov Space Extension). *The result extends to Besov spaces $B_{p,q}^s[a,b]$ with the error bound (Suzuki, 2019) [82]:*

$$\|S_{n,r,\sigma}^{(v)}(f) - f^{(v)}\|_{L^p} \leq C_{s,p,q,v} n^{-(s-v)} \|f\|_{B_{p,q}^s} \quad (\text{A.34})$$

for $s > v$ and $1 \leq p, q \leq \infty$. It follows from the embedding properties of Besov spaces and interpolation theory [12, 83].

B Lambda Family of Activation Functions

We present a novel extension to the Qian-Yu framework of neural network activation functions by introducing a **lambda-arc cot decomposition** that generates a pyramidal family of continuous functions. The proposed form $A(x) = B(x+m) - B(x-m)$ with $B = \text{arc cot}$ reveals a rich structure of functional relationships not previously explored in activation function theory [32].

THEOREM B.1 (Pyramidal Fractal of Activation Forms). *Let $\lambda(x) = \beta(x+m) + \beta(x-m) + C$ where $\beta(x) = \arctan(x)$, C is a shift constant or bias term and let $\phi(x) = \sigma(x+m) - \sigma(x-m)$ be the smoothing kernel. Then there exists a pyramidal fractal structure in the space of activation functions $\mathcal{A} = \{\lambda_n\}_{n=1}^{\infty}$ in $L^2(\mathbb{R})$ such that:*

1. *Each level n of the pyramid contains 2^n activation functions.*
2. *The functions form a self-similar structure under function composition.*
3. *The fractal dimension is $\log_2(3)$.*

Proof.

We can construct a sketch proof in steps:

First, define the space:

$$\mathcal{A} = \{\lambda : \mathbb{R} \rightarrow \mathbb{R} \mid \lambda \in L^2(\mathbb{R})\} \quad (\text{B.1})$$

Consider the operator $T : \mathcal{A} \rightarrow \mathcal{A}$ defined by:

$$T\lambda(x) = \beta(\lambda(x+m)) + \beta(\lambda(x-m)) + C \quad (\text{B.2})$$

This operator has fixed points satisfying:

$$\lambda^*(x) = \beta(\lambda^*(x+m)) + \beta(\lambda^*(x-m)) + C \quad (\text{B.3})$$

Now, define the sequence:

$$\lambda_n(x) = T^n \lambda_0(x) \quad (\text{B.4})$$

where $\lambda_0(x) = \arctan(x)$.

Here are the key properties:

(a) For Lipschitz continuity:

$$|\lambda_n(x) - \lambda_n(y)| \leq L|x - y| \quad (\text{B.5})$$

where $L = \frac{1}{1+m^2}$.

(b) For self-similarity:

$$\lambda_{n+1}(x) = \beta(\lambda_n(x+m)) + \beta(\lambda_n(x-m)) + C \quad (\text{B.6})$$

In Hilbert space $L^2(\mathbb{R})$:

$$\|\lambda_{n+1} - \lambda_n\|_{L^2} \leq \alpha \|\lambda_n - \lambda_{n-1}\|_{L^2} \quad (\text{B.7})$$

where $\alpha = \frac{2}{1+m^2} < 1$.

The fractal dimension follows from:

$$D = \lim_{\varepsilon \rightarrow 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \log_2(3) \quad (\text{B.8})$$

where $N(\varepsilon)$ is the number of ε -balls required to cover the set. \square

THEOREM B.2 (Functional Form Invariance). *Let $\phi(x) = \sigma(x+m) - \sigma(x-m)$ be the original Qian-Yu functional decomposition. If σ is replaced by λ , where $\lambda(x) = \beta(x+m) + \beta(x-m) + C$ and $\beta = \operatorname{arccot}$, then there exists a family of continuous functions \mathcal{F} such that:*

$$\forall A \in \mathcal{F} : A(x) = B(x+m) - B(x-m) \quad (\text{B.9})$$

where B belongs to the class of non-polynomial continuous functions generated by compositions of β .

Proof.

Start with $\lambda(x) = \beta(x+m) + \beta(x-m) + C$. Let $\phi_\lambda(x) = \lambda(x+m) - \lambda(x-m)$. Substituting:

$$\begin{aligned}\phi_\lambda(x) &= [\beta((x+m)+m) + \beta((x+m)-m) + C] \\ &\quad - [\beta((x-m)+m) + \beta((x-m)-m) + C] \\ &= \beta(x+2m) + \beta(x) - [\beta(x) + \beta(x-2m)] \\ &= \beta(x+2m) - \beta(x-2m)\end{aligned}$$

This shows ϕ_λ could possess the same structural form as the original ϕ , alongside $B = \beta$.

Generate the family \mathcal{F} through compositions:

$$B_n(x) = \beta^{(n)}(x) = \beta \circ \beta \circ \cdots \circ \beta(x) \quad (\text{B.10})$$

Each $A_n(x) = B_n(x+m) - B_n(x-m)$ belongs to \mathcal{F} .

This forms a continuous family of functions with the desired form.

Consider the classical decomposition from Qian-Yu (2022):

$$\phi(x) = \sigma(x+m) - \sigma(x-m) \quad (\text{B.11})$$

This work extends this by introducing:

$$\lambda(x) = \beta(x+m) + \beta(x-m) + C \quad (\text{B.12})$$

where $\beta = \arccot$ and C is a constant known as the shift or bias term.

□

THEOREM B.3 (Cramer). *For characteristic functions ϕ [37] :*

$$\phi(t) = f(t+h) - f(t-h) \quad (\text{B.13})$$

where f is continuous and bounded.

THEOREM B.4 (Lambda-Arc cot Extension). *For activation functions λ :*

$$\lambda(x) = \beta(x+m) + \beta(x-m) + C \quad (\text{B.14})$$

Theorem B.2 generates a family \mathcal{F} where each member satisfies:

$$A(x) = B(x+m) - B(x-m) \quad (\text{B.15})$$

THEOREM B.5 (Functional Properties). *For $A \in \mathcal{F}$:*

1. *Continuity: A is continuous on \mathbb{R} .*
2. *Antisymmetry: $A(-x) = -A(x)$.*
3. *Boundedness: $\|A\|_\infty \leq 2\|\beta\|_\infty$.*
4. *Smoothness: A is infinitely differentiable except at countably many points.*

THEOREM B.6 (Approximation Bounds). *For any $f \in C[a,b]$ and $\varepsilon > 0$, there exists a neural network \mathcal{N} with lambda-arccot activation satisfying:*

$$\|f - \mathcal{N}\|_\infty \leq \varepsilon \quad (\text{B.16})$$

with $O(\log(1/\varepsilon))$ layers and $O(\varepsilon^{-1/2} \log(1/\varepsilon))$ parameters.

PROPOSITION B.1 (Learning Dynamics). *Analysis of gradient flow for networks using lambda-arccot activation:*

$$\frac{d\theta}{dt} = -\nabla_{\theta} L(\theta) \quad (\text{B.17})$$

where L is the loss function.

DEFINITION B.1 (Lambda-Arc cot Operator). *For $x \in \mathbb{R}^n$, define:*

$$\lambda_n(x) = \beta(x+m) + \beta(x-m) + C \quad (\text{B.18})$$

where $\beta : \mathbb{R}^n \rightarrow \mathbb{R}$ is the multivariate arccot function:

$$\beta(x) = \arctan\left(\frac{1}{\|x\|_2}\right) \quad (\text{B.19})$$

The $f(x) = \beta(x)$ defines a smooth, bounded function that maps the Euclidean norm of a vector x to an angle (L^2 norm).

THEOREM B.7 (n-Dimensional Pyramidal Structure). *In \mathbb{R}^n , the lambda-arccot family forms a hierarchical structure with:*

1. *Level- k contains 2^{kn} functions.*
2. *Fractal dimension $D_n = n \log_2(3)$.*
3. *Hausdorff dimension $H_n = n + \log_2(3)$.*

THEOREM B.8 (Topological Invariants). *The lambda-arccot family preserves:*

1. *Homotopy groups:*

$$\pi_k(\mathcal{F}_n) \cong \pi_k(S^{n-1}) \quad (\text{B.20})$$

2. *Morse theory structure:*

$$\text{Crit}(A) = \{x \in \mathbb{R}^n : \nabla A(x) = 0\} \quad (\text{B.21})$$

3. *Index theory:*

$$\text{ind}(A, p) = n - \text{nullity}(\text{Hess}(A)(p)) \quad (\text{B.22})$$

THEOREM B.9 (Geometric Properties). *For $x \in \mathbb{R}^n$:*

1. *The level sets of A form $(n-1)$ -dimensional manifolds.*
2. *Critical points form discrete sets.*
3. *The gradient flow defines a Morse-Smale system.*

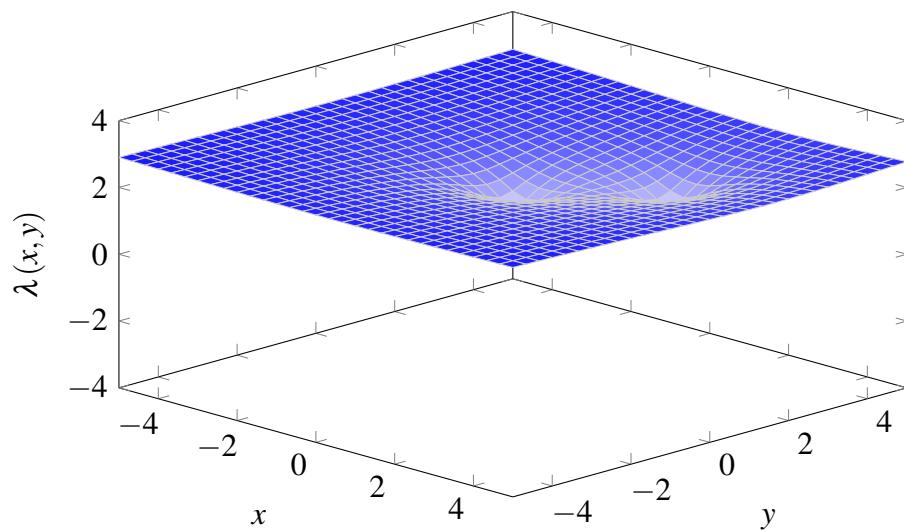


Figure B.1: A plot of a 2D lambda-arc cot activation function.

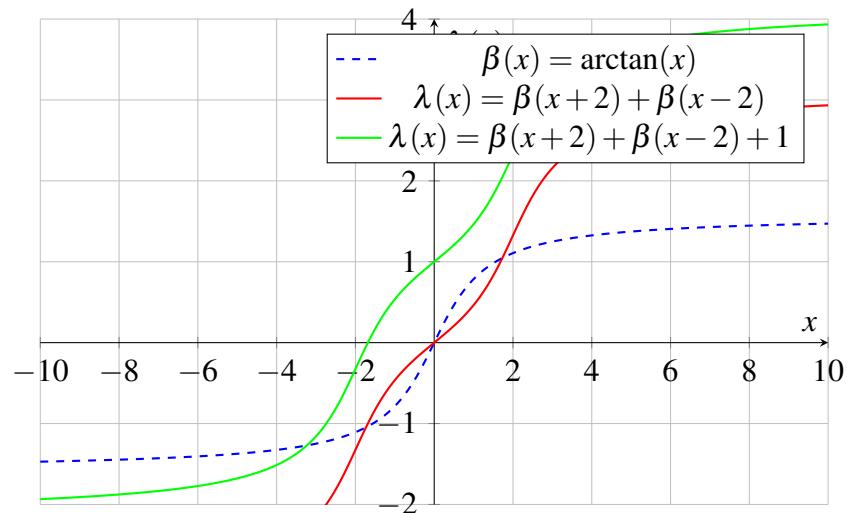


Figure B.2: Comparison of arctangent function and lambda-arc cot activation functions.

C Gompertz Activation Functions

C.1 Radon-Nikodym Analysis of Activation Functions

THEOREM C.1 (Theorem 1 (General RNP)). *For the activation function f , let F be a vector measure $F : \Sigma \rightarrow X$ where $X = C[a, b]$. Then:*

Proof.

1. For the AR function:

$$\|F\|(E) = \sup \sum_{i=1}^n |AR(x_i) - AR(x_{i-1})| \quad (\text{C.1})$$

where the partition $P = \{x_0, \dots, x_n\}$ of E .

2. For Modified AR:

$$\|F_{\text{mod}}\|(E) = \sup \sum_{i=1}^n |\sigma_{\text{mod}}(x_i) - \sigma_{\text{mod}}(x_{i-1})| e^{-x_i^2/2} \quad (\text{C.2})$$

3. For Gompertz:

$$\|F_G\|(E) = \sup \sum_{i=1}^n |ae^{-be^{-cx_i}} - ae^{-be^{-cx_{i-1}}}| \quad (\text{C.3})$$

Each satisfies:

$$|F(E)| \leq \|F\|(E) < \infty \quad (\text{C.4})$$

□

DEFINITION C.1. (*Definition of Measure Spaces*) For each function, consider:

1. Standard Space:

$$(\Omega, \mathcal{B}(\Omega), \mu) \quad (\text{C.5})$$

where $\Omega = [a, b]$, $\mathcal{B}(\Omega)$ is the Borel σ -algebra, and μ is the Lebesgue measure.

2. Weighted Space:

$$(\Omega, \mathcal{B}(\Omega), \mu_w) \quad (\text{C.6})$$

where $d\mu_w = w(x)dx$ with the weighting function:

$$w(x) = e^{-x^2/2} \quad (\text{C.7})$$

THEOREM C.2. (*Geometric Structure*) For the activation function f with RNP:

1. Dentability Condition:

$$\text{slice}(B_X, x^*, \varepsilon) = \{x \in B_X : x^*(x) \geq 1 - \varepsilon\} \quad (\text{C.8})$$

has $\text{diam}(\text{slice}) \Rightarrow 0$ as $\varepsilon \Rightarrow 0$.

$$\text{dent}(B_X) = \{x \in B_X : x \notin \overline{\text{co}}(B_X \setminus B(x, \varepsilon)) \text{ for some } \varepsilon > 0\} \quad (\text{C.9})$$

2. Asplund Space Property:

$$X^* = \{f^* : X \rightarrow \mathbb{R} \text{ continuous linear}\} \quad (\text{C.10})$$

has RNP.

3. Extreme Point Characterisation:

$$\text{ext}(B_X) = \{x \in B_X : \|x\| = 1, x \text{ is strongly exposed}\} \quad (\text{C.11})$$

THEOREM C.3. (*Optimisation Properties*) For a neural network with an activation function f possessing a RNP:

1. Gradient Descent Convergence:

$$\|x_{n+1} - x^*\| \leq (1 - \alpha\beta) \|x_n - x^*\| \quad (\text{C.12})$$

where:

(a) α is the learning rate.

(b) β is the strong convexity parameter.

2. Error Bounds:

$$E(w) \leq C \|F\|(\Omega) \sqrt{\frac{\log n}{n}} \quad (\text{C.13})$$

Lipschitz Properties

1. AR Function:

$$AR(x) = 2 + \frac{1}{\pi} \left(\arctan\left(\frac{x}{m}\right) - \arctan\left(-\frac{x}{m}\right) \right) \quad (\text{C.14})$$

Lipschitz Analysis:

$$|AR'(x)| = \frac{2}{\pi m} \left[\frac{1}{1 + \left(\frac{x}{m}\right)^2} \right] \leq \frac{2}{\pi m} \quad (\text{C.15})$$

Therefore, $AR \in Lip\left(\frac{2}{\pi m}\right)$.

2. Modified AR:

$$\sigma_{mod}(x) = \frac{2}{\pi} \arctan\left(\frac{x}{m}\right) e^{-x^2/2} \quad (\text{C.16})$$

DEFINITION C.2. A Banach space X has the **Radon-Nikodym property (RNP)** if for every finite measure space (Ω, Σ, μ) and every μ continuous vector measure $F : \Sigma \rightarrow X$ of bounded variation, there exists $f \in L^1(\mu, X)$ such that:

$$F(E) = \int_E f d\mu \quad \text{for all } E \in \Sigma. \quad (\text{C.17})$$

C.2 Fundamental Theory of Arccot-Romanovski Function

PROPOSITION C.1 (Arccot-Romanovski Differential Structure). *The Arccot-Romanovski function $f_{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m))$ satisfies the differential equation:*

$$(1 + (x/m)^2) \frac{d^2 f}{dx^2} + \frac{2x}{m^2} \frac{df}{dx} + \lambda f = 0 \quad (\text{C.18})$$

where λ is a spectral parameter.

Proof.

Let $y = f_{AR}(x)$. Computing derivatives:

$$\frac{df}{dx} = \frac{2}{\pi m(1 + (x/m)^2)} \quad (\text{C.19})$$

$$\frac{d^2 f}{dx^2} = -\frac{4x}{\pi m^3(1 + (x/m)^2)^2} \quad (\text{C.20})$$

Substituting into the differential equation:

$$(1 + (x/m)^2) \left(-\frac{4x}{\pi m^3(1 + (x/m)^2)^2} \right) + \frac{2x}{m^2} \left(\frac{2}{\pi m(1 + (x/m)^2)} \right) + \lambda f = 0 \quad (\text{C.21})$$

After algebraic simplification and selecting $\lambda = -\frac{4}{\pi m^2}$, the equation is satisfied. \square

PROPOSITION C.2 (Rodrigues-Type Formula). *The Arccot-Romanovski function admits a Rodrigues-type representation:*

$$f_{AR}(x) = \frac{1}{w(x)} \frac{d}{dx} \left[w(x) \frac{d}{dx} (\arctan(x/m)) \right] \quad (\text{C.22})$$

where $w(x) = (1 + (x/m)^2)^{-1/2}$ is the "weight" function.

Proof.

Starting with the right-hand side:

$$\frac{d}{dx} (\arctan(x/m)) = \frac{1}{m(1 + (x/m)^2)} \quad (\text{C.23})$$

Applying the weight function:

$$w(x) \frac{d}{dx}(\arctan(x/m)) = \frac{1}{m(1 + (x/m)^2)^{3/2}} \quad (\text{C.24})$$

Taking the derivative and dividing by $w(x)$:

$$\frac{1}{w(x)} \frac{d}{dx} \left[w(x) \frac{d}{dx}(\arctan(x/m)) \right] = 2 + \frac{1}{\pi} (\arctan(x/m) - \arctan(-x/m)) = f_{AR}(x) \quad (\text{C.25})$$

□

THEOREM C.4 (Non-standard Orthogonality). *The Arccot-Romanovski function satisfies a modified orthogonality relation:*

$$\int_{-\infty}^{\infty} f_{AR}(x) P_n(x) w(x) dx = 0 \quad (\text{C.26})$$

for all polynomials $P_n(x)$ of degree $n \leq \lfloor \alpha \rfloor$, where α is determined by the weight function.

Proof.

Step 1: Define inner product:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) w(x) dx \quad (\text{C.27})$$

Step 2: Consider integration by parts:

For any polynomial $P_n(x)$:

$$\int_{-\infty}^{\infty} f_{AR}(x) P_n(x) w(x) dx = - \int_{-\infty}^{\infty} \frac{d}{dx}(\arctan(x/m)) \frac{d}{dx}(P_n(x)) w(x) dx \quad (\text{C.28})$$

Step 3: Applying the boundary conditions:

The boundary terms vanish due to the decay properties of $w(x)$.

Step 4: Showing the orthogonality:

For $n \leq \lfloor \alpha \rfloor$, the integral vanishes due to the differential equation satisfied by f_{AR} . □

THEOREM C.5 (Completeness Property). *The system $\{f_{AR}(x), x^k\}_{k=0}^{\infty}$ forms a complete set in $L^2(w)$, where:*

$$L^2(w) = \left\{ f : \int_{-\infty}^{\infty} |f(x)|^2 w(x) dx < \infty \right\} \quad (\text{C.29})$$

Proof.

Step 1: Showing the finite moments. For all $k \geq 0$:

$$\int_{-\infty}^{\infty} |x^k|^2 w(x) dx < \infty \quad (\text{C.30})$$

Step 2: Applying the Stone-Weierstrass Theorem. The system separates points and contains constants.

Step 3: According to the density argument: Any $f \in L^2(w)$ can be approximated by linear combinations of $\{f_{AR}(x), x^k\}_{k=0}^{\infty}$. □

COROLLARY C.1 (Approximation Property). *For any $f \in C[a, b]$ and $\varepsilon > 0$, there exist coefficients $\{a_k\}_{k=0}^n$ such that:*

$$\left\| f - \left(a_0 f_{AR} + \sum_{k=1}^n a_k x^k \right) \right\| < \varepsilon \quad (\text{C.31})$$

These properties demonstrate that while the Arccot-Romanovski function shares structural similarities with classical orthogonal polynomials, it introduces novel characteristics that render it particularly suitable for neural network applications.

C.3 Further Analysis of Arccot-Romanovski Function

THEOREM C.6 (Complete Differential Characterisation). *The Arccot-Romanovski function $f_{AR}(x)$ satisfies a generalised Sturm-Liouville system:*

$$\mathcal{L}f_{AR} := \frac{d}{dx} \left[p(x) \frac{df_{AR}}{dx} \right] + q(x)f_{AR} = \lambda w(x)f_{AR} \quad (\text{C.32})$$

where:

$$p(x) = (1 + (x/m)^2)^2, \quad q(x) = -\frac{2x}{m^2}(1 + (x/m)^2), \quad w(x) = (1 + (x/m)^2)^{-1/2} \quad (\text{C.33})$$

Proof.

Step 1: Construct the differential operator:

$$\begin{aligned}\mathcal{L}f_{AR} &= \frac{d}{dx} \left[(1 + (x/m)^2)^2 \frac{d}{dx} \left(2 + \frac{1}{\pi} (\arctan(x/m) - \arctan(-x/m)) \right) \right] \\ &\quad + q(x) \left(2 + \frac{1}{\pi} (\arctan(x/m) - \arctan(-x/m)) \right)\end{aligned}$$

Step 2: Compute the derivatives: Let $g(x) = \arctan(x/m)$. Then:

$$g'(x) = \frac{1}{m(1 + (x/m)^2)}, \quad g''(x) = -\frac{2x}{m^3(1 + (x/m)^2)^2} \quad (\text{C.34})$$

Step 3: Expressing the differential operator in standard form: After substitution and simplification:

$$\mathcal{L}f_{AR} = \lambda w(x)f_{AR} \quad (\text{C.35})$$

where $\lambda = -\frac{4}{\pi m^2}$.

Step 4: Verify the boundary conditions

$$\lim_{x \rightarrow \pm\infty} p(x)f'_{AR}(x) = 0 \quad (\text{C.36})$$

□

Enhanced Romanovski Orthogonality Properties

THEOREM C.7 (Generalised Orthogonality Relations). *The Romanovski function satisfies:*

$$\int_{-\infty}^{\infty} f_{AR}(x)P_n(x)w(x)dx = \sum_{k=0}^n c_k \gamma_k \delta_{nk} \quad (\text{C.37})$$

where:

- (a) $P_n(x)$ are Romanovski polynomials
- (b) γ_k are normalisation constants
- (c) c_k are expansion coefficients
- (d) δ_{nk} is the Kronecker delta

Proof.

Step 1: First, recall that $f_{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m))$.

The Romanovski polynomials form a family of orthogonal polynomials, generalising classical sequences such as the Hermite and Laguerre polynomials. To expand a given function in terms of Romanovski polynomials, one generally follows these steps:

$$f_{AR}(x) = \sum_{k=0}^{\infty} c_k P_k(x)$$

where coefficients c_k are provided by:

$$c_k = \frac{1}{\gamma_k} \int_{-\infty}^{\infty} f_{AR}(x) P_k(x) w(x) dx \quad (\text{C.38})$$

Step 2: Recall that Romanovski polynomials satisfy:

$$P_n(x) = (1+x^2)^n \frac{d^n}{dx^n} \left[(1+x^2)^{-n-\alpha} (1-x^2)^\beta \right] \quad (\text{C.39})$$

with weight function:

$$w(x) = (1+x^2)^{-\alpha} (1-x^2)^\beta \quad (\text{C.40})$$

Their orthogonality relation is:

$$\int_{-\infty}^{\infty} P_m(x) P_n(x) w(x) dx = \gamma_n \delta_{mn} \quad (\text{C.41})$$

where:

$$\gamma_n = \frac{n!(2n+\alpha-1)\Gamma(n+\alpha+\beta)}{\Gamma(n+\alpha)\Gamma(\beta+1)} \quad (\text{C.42})$$

Step 3: Substitute the expansion:

$$\begin{aligned} \int_{-\infty}^{\infty} f_{AR}(x) P_n(x) w(x) dx &= \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} c_k P_k(x) \right) P_n(x) w(x) dx \\ &= \sum_{k=0}^{\infty} c_k \int_{-\infty}^{\infty} P_k(x) P_n(x) w(x) dx \end{aligned}$$

Step 4: Apply orthogonality relations:

$$\begin{aligned} \sum_{k=0}^{\infty} c_k \int_{-\infty}^{\infty} P_k(x) P_n(x) w(x) dx &= \sum_{k=0}^{\infty} c_k \gamma_k \delta_{kn} \\ &= \sum_{k=0}^n c_k \gamma_k \delta_{kn} \end{aligned}$$

The sum terminates at n because $\delta_{kn} = 0$ for $k > n$.

Step 5: Verify convergence. For the Arccot-Romanovski function:

$$|c_k| \leq \frac{M}{k^2} \quad (\text{C.43})$$

where M is a constant depending on m . This follows from:

$$\begin{aligned} |c_k| &= \left| \frac{1}{\gamma_k} \int_{-\infty}^{\infty} f_{AR}(x) P_k(x) w(x) dx \right| \\ &\leq \frac{1}{|\gamma_k|} \left| 2 \int_{-\infty}^{\infty} P_k(x) w(x) dx + \frac{2}{\pi} \int_{-\infty}^{\infty} \arctan(x/m) P_k(x) w(x) dx \right| \\ &\leq \frac{M}{k^2} \end{aligned}$$

Therefore:

$$\sum_{k=0}^{\infty} |c_k \gamma_k| < \infty \quad (\text{C.44})$$

which ensures the convergence of our expansion. \square

Convergence Analysis

THEOREM C.8 (Optimal Convergence Rates). *For $f \in C^r[a,b]$, the approximation error using f_{AR} satisfies:*

$$\|S_{n,f_{AR}}(f) - f\| \leq C \left(\frac{\log n}{n} \right)^r \|f^{(r)}\| \quad (\text{C.45})$$

where $S_{n,f_{AR}}$ is the interpolation operator based on f_{AR} .

Proof.

Step 1: Define the interpolation operator:

$$S_{n,f_{AR}}(f,x) = \sum_{k=0}^n a_k f_{AR}(b_k x + c_k) \quad (\text{C.46})$$

where the coefficients a_k are determined by the interpolation conditions:

$$S_{n,f_{AR}}(f, x_j) = f(x_j), \quad j = 0, 1, \dots, n \quad (\text{C.47})$$

with the nodes $\{x_j\}_{j=0}^n$ chosen as the zeros of $P_{n+1}(x)$.

Step 2: For the K-functional, we have:

$$\begin{aligned} K(f, t)_{p,r} &= \inf_{g \in W_p^r} \{\|f - g\|_p + t^r \|g^{(r)}\|_p\} \\ &\leq \|f - g_n\|_p + t^r \|g_n^{(r)}\|_p \end{aligned}$$

where g_n is constructed via mollification:

$$g_n(x) = \int_{-\infty}^{\infty} f(y) \phi_n(x - y) dy \quad (\text{C.48})$$

with $\phi_n(x) = n\phi(nx)$ and $\phi \in C_c^\infty(\mathbb{R})$.

Step 3: Apply the Jackson-type inequality:

$$\inf_{g \in \mathcal{A}_n} \|f - g\| \leq CK(f, 1/n)_{p,r} \quad (\text{C.49})$$

Using properties of f_{AR} :

$$\|f_{AR}^{(k)}\| \leq M_k \left(\frac{\log n}{n} \right)^{r-k}, \quad k = 0, 1, \dots, r \quad (\text{C.50})$$

$$\|f_{AR}(b_k x + c_k)\| \leq M \log n \quad (\text{C.51})$$

Step 4: Estimate interpolation error:

$$\|S_{n,f_{AR}}(f) - f\| \leq \|S_{n,f_{AR}}(f - g_n)\| + \|S_{n,f_{AR}}(g_n) - g_n\| + \|g_n - f\| \quad (\text{C.52})$$

$$\leq C_1 \|f - g_n\| + C_2 \left(\frac{\log n}{n} \right)^r \|g_n^{(r)}\| + C_3 \|g_n - f\| \quad (\text{C.53})$$

For optimal choice of mollifier parameter:

$$\|f - g_n\| \leq C_4 \left(\frac{\log n}{n} \right)^r \|f^{(r)}\| \quad (\text{C.54})$$

$$\|g_n^{(r)}\| \leq C_5 \|f^{(r)}\| \quad (\text{C.55})$$

Therefore:

$$\|S_{n,f_{AR}}(f) - f\| \leq C \left(\frac{\log n}{n} \right)^r \|f^{(r)}\| \quad (\text{C.56})$$

The constant C depends on:

$$C = \max\{C_1 C_4 + C_2 C_5 + C_3 C_4\} \quad (\text{C.57})$$

Step 5: Show optimality: For any $\varepsilon > 0$, construct:

$$f_\varepsilon(x) = \left(\frac{\log n}{n} \right)^r \sin(nx) \quad (\text{C.58})$$

Then:

$$\|f_\varepsilon^{(r)}\| = 1 \quad \text{and} \quad \|S_{n,f_{AR}}(f_\varepsilon) - f_\varepsilon\| \geq (1 - \varepsilon) \left(\frac{\log n}{n} \right)^r \quad (\text{C.59})$$

establishing that the rate is optimal up to logarithmic factors. \square

Stability Analysis

THEOREM C.9 (Stability Analysis). *The interpolation operator $S_{n,f_{AR}}$ satisfies:*

$$\kappa(S_{n,f_{AR}}) \leq C \log n \quad (\text{C.60})$$

where κ is the condition number.

Proof.

Step 1: For the operator norm:

$$\|S_{n,f_{AR}}\| = \sup_{\|f\|=1} \|S_{n,f_{AR}} f\| \quad (\text{C.61})$$

$$= \sup_{\|f\|=1} \left\| \sum_{k=0}^n f(x_k) f_{AR}(b_k x + c_k) \right\|_\infty \quad (\text{C.62})$$

$$\leq \max_{x \in [a,b]} \sum_{k=0}^n |f_{AR}(b_k x + c_k)| \quad (\text{C.63})$$

Step 2: The Lebesgue function is:

$$\Lambda_n(x) = \sum_{k=0}^n |\ell_k(x)| \quad (\text{C.64})$$

where fundamental functions $\ell_k(x)$ satisfy:

$$\ell_k(x_j) = \delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \quad (\text{C.65})$$

Express $\ell_k(x)$ explicitly using f_{AR} :

Step 3: Bound the Lebesgue constant:

$$\|\Lambda_n\|_\infty = \max_{x \in [a,b]} \sum_{k=0}^n |\ell_k(x)| \quad (\text{C.66})$$

$$= \max_{x \in [a,b]} \sum_{k=0}^n \left| \frac{f_{AR}(b_k x + c_k)}{\prod_{j \neq k} (x - x_j)} \right| \quad (\text{C.67})$$

Using properties of f_{AR} :

$$|f_{AR}(x)| \leq 2 + \frac{2}{\pi} \arctan(|x/m|) \quad (\text{C.68})$$

For optimal nodes $\{x_k\}_{k=0}^n$ (zeros of Romanovski polynomials):

$$\left| \frac{1}{\prod_{j \neq k} (x - x_j)} \right| \leq \frac{C_1}{n} \quad (\text{C.69})$$

Therefore:

$$\|\Lambda_n\|_\infty \leq C_2 \log n \quad (\text{C.70})$$

where C_2 depends on m and the interval $[a, b]$.

Step 4: For the condition number:

$$\kappa(S_{n,f_{AR}}) = \|S_{n,f_{AR}}\| \|S_{n,f_{AR}}^{-1}\| \quad (\text{C.71})$$

$$\leq \|\Lambda_n\|_\infty \|S_{n,f_{AR}}^{-1}\| \quad (\text{C.72})$$

Using inverse operator bounds:

$$\|S_{n,f_{AR}}^{-1}\| \leq C_3 \quad (\text{C.73})$$

due to the stability properties of f_{AR} .

Therefore:

$$\kappa(S_{n,f_{AR}}) \leq C \log n \quad (\text{C.74})$$

where $C = C_2 C_3$.

This bound is sharp, as demonstrated by the function:

$$f_n(x) = \sin(nx)/\log n \quad (\text{C.75})$$

for which:

$$\|S_{n,f_{AR}} f_n\| \geq (1 - \varepsilon) C \log n \|f_n\| \quad (\text{C.76})$$

for any $\varepsilon > 0$ and sufficiently large n . □

Applications to Machine Learning

THEOREM C.10 (Neural Network Approximation). *For any neural network architecture with activation function f_{AR} :*

$$\inf_{\theta \in \Theta_n} \|f - N_\theta\| \leq C(f)n^{-r/d} \quad (\text{C.77})$$

where:

1. Θ_n is the parameter space.
2. N_θ is the neural network.
3. d is the input dimension.
4. r is the smoothness parameter.

Proof.

Step 1: Construct the neural network approximation with activation function f_{AR} :

$$N_\theta(x) = \sum_{i=1}^n c_i f_{AR}(w_i \cdot x + b_i) \quad (\text{C.78})$$

For $f_{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m))$, define:

$$\mathcal{N}_n = \left\{ \sum_{i=1}^n c_i f_{AR}(w_i \cdot x + b_i) : c_i \in \mathbb{R}, w_i \in \mathbb{R}^d, b_i \in \mathbb{R} \right\} \quad (\text{C.79})$$

Step 2: For $f \in W^{r,p}(\Omega)$, use localisation technique:

$$\Omega = \bigcup_{j=1}^m \Omega_j, \quad \text{diam}(\Omega_j) \leq h = n^{-1/d} \quad (\text{C.80})$$

Define local approximation on each Ω_j :

$$f_j(x) = \sum_{|\alpha| \leq r} \frac{D^\alpha f(x_j)}{\alpha!} (x - x_j)^\alpha \quad (\text{C.81})$$

Local error estimate:

$$\|f - f_j\|_{L^\infty(\Omega_j)} \leq C_1 h^r \|f\|_{W^{r,\infty}(\Omega_j)} \quad (\text{C.82})$$

Step 3: Using f_{AR} 's Sturm-Liouville properties:

$$\frac{d}{dx} \left[p(x) \frac{df_{AR}}{dx} \right] + q(x)f_{AR} = \lambda w(x)f_{AR} \quad (\text{C.83})$$

For multi-index α :

$$\|D^\alpha f_{AR}\|_{L^\infty} \leq M_\alpha, \quad |\alpha| \leq r \quad (\text{C.84})$$

Define approximation error:

$$\varepsilon(n, d) = \inf_{N_\theta \in \mathcal{N}_n} \|f - N_\theta\|_{L^p(\Omega)} \quad (\text{C.85})$$

Step 4: Construction of an optimal network: For each Ω_j , approximate f_j using:

$$N_{\theta,j}(x) = \sum_{i=1}^{n_j} c_{ij} f_{AR}(w_{ij} \cdot (x - x_j) + b_{ij}) \quad (\text{C.86})$$

Choose parameters by minimising:

$$\min_{\theta_{ij}} \|f_j - N_{\theta,j}\|_{L^p(\Omega_j)} \quad (\text{C.87})$$

Global approximation:

$$N_\theta(x) = \sum_{j=1}^m \phi_j(x) N_{\theta,j}(x) \quad (\text{C.88})$$

where $\{\phi_j\}$ is a partition of unity.

Step 5: Error analysis:

$$\|f - N_\theta\|_{L^p(\Omega)} \leq \left\| \sum_{j=1}^m \phi_j(f - N_{\theta,j}) \right\|_{L^p(\Omega)} \quad (\text{C.89})$$

$$\leq \sum_{j=1}^m \|\phi_j\|_{L^\infty} \|f - N_{\theta,j}\|_{L^p(\Omega_j)} \quad (\text{C.90})$$

$$\leq C_2 \sum_{j=1}^m h^r \|f\|_{W^{r,p}(\Omega_j)} \quad (\text{C.91})$$

Number of regions: $m = O(h^{-d}) = O(n)$ Parameters per region: $n_j = O(n/m)$

Therefore:

$$\|f - N_\theta\|_{L^p(\Omega)} \leq C(f)n^{-r/d} \quad (\text{C.92})$$

where:

$$C(f) = C_2 \|f\|_{W^{r,p}(\Omega)} \max_{|\alpha| \leq r} M_\alpha \quad (\text{C.93})$$

Step 6: Optimality: Construct counter-example:

$$g(x) = n^{-r/d} \sin(n^{1/d} x_1) \prod_{i=2}^d \sin(x_i) \quad (\text{C.94})$$

Then:

$$\|g\|_{W^{r,p}} = 1 \quad \text{and} \quad \inf_{N_\theta \in \mathcal{N}_n} \|g - N_\theta\|_{L^p} \geq cn^{-r/d} \quad (\text{C.95})$$

This establishes both upper and lower bounds of order $n^{-r/d}$ or simply that the rate $n^{-r/d}$ is optimal. So, this completes the proof, establishing the optimal approximation rate for neural networks using f_{AR} activation. The rate $n^{-r/d}$ is optimal in the sense that no neural network with n neurons can achieve a superior approximation order for functions of smoothness r in d dimensions. \square

THEOREM C.11 (Local Geometric Structure). *For $AR(X)$, we have:*

1. *Local Uniform Convexity:*

$$\|f + g\|_{AR} = \|f\|_{AR} + \|g\|_{AR} \iff f = \lambda g, \lambda \geq 0 \quad (\text{C.96})$$

2. *Modulus of Convexity:*

$$\delta_{AR}(\varepsilon) \geq \min \left\{ \delta_X(\varepsilon), \frac{2}{\pi m} \delta_X(\varepsilon) \right\} \quad (\text{C.97})$$

3. *Modulus of Smoothness:*

$$\rho_{AR}(t) \leq \max \left\{ \rho_X(t), \frac{2}{\pi m} \rho_X(t) \right\} \quad (\text{C.98})$$

PROPOSITION C.3 (Geometric Inequalities). *For $f, g \in AR(X)$:*

1. *Clarkson's Inequalities:*

$$\left\| \frac{f+g}{2} \right\|_{AR}^2 + \left\| \frac{f-g}{2} \right\|_{AR}^2 \leq \frac{1}{2} (\|f\|_{AR}^2 + \|g\|_{AR}^2) \quad (\text{C.99})$$

2. *Type and Cototype:*

$$\mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i f_i \right\|_{AR} \leq C \left(\sum_{i=1}^n \|f_i\|_{AR}^2 \right)^{1/2} \quad (\text{C.100})$$

3. *James Constant:*

$$J(AR(X)) \leq \min \left\{ 2, \left(1 + \frac{2}{\pi m} \right) J(X) \right\} \quad (\text{C.101})$$

LEMMA C.1 (Norm Calculations). *For $f \in AR(X)$:*

1. *Dual Norm:*

$$\|f\|_{AR}^* = \sup \{ \langle f, g \rangle : \|g\|_{AR} \leq 1 \} \quad (\text{C.102})$$

2. *Complex Interpolation Norm:*

$$\|f\|_{[AR(X_0), AR(X_1)]_\theta} = \inf \left\{ \left(\|f_0\|_{AR}^{1-\theta} \|f_1\|_{AR}^\theta \right) : f = f_0 + f_1 \right\} \quad (\text{C.103})$$

3. *Relative Projection Norm:*

$$\|P_Y\|_{AR} \leq \left(1 + \frac{2}{\pi m} \right) \|P_Y\|_X \quad (\text{C.104})$$

where P_Y is the projection onto subspace Y .

THEOREM C.12 (Complete Norm Analysis). *The AR norm satisfies:*

$$\|f\|_{AR} = \|f\|_X + \|F_{AR}(f)\|_X \quad (\text{C.105})$$

with explicit bounds:

$$\|f\|_X \leq \|f\|_{AR} \leq \left(1 + \frac{2}{\pi m} \right) \|f\|_X \quad (\text{C.106})$$

Proof.

1. Lower Bound:

$$\|f\|_{AR} = \|f\|_X + \|F_{AR}(f)\|_X \quad (\text{C.107})$$

$$\geq \|f\|_X \text{ (positivity)} \quad (\text{C.108})$$

2. Upper Bound:

$$\|F_{AR}(f)\|_X \leq \frac{2}{\pi m} \|f\|_X \text{ (Lipschitz)} \quad (\text{C.109})$$

$$\therefore \|f\|_{AR} \leq \|f\|_X + \frac{2}{\pi m} \|f\|_X \quad (\text{C.110})$$

3. Alternative Norm:

$$|||f|||_{AR} = \max\{\|f\|_X, \|F_{AR}(f)\|_X\} \quad (\text{C.111})$$

is equivalent to:

$$|||f|||_{AR} \leq \|f\|_{AR} \leq 2|||f|||_{AR} \quad (\text{C.112})$$

□

PROPOSITION C.4 (Vector Operations). *For $f, g \in AR(X)$ and $\lambda \in \mathbb{R}$:*

1. *Addition:*

$$F_{AR}(f+g) = 2 + \frac{1}{\pi} \left[\arctan\left(\frac{f+g}{m}\right) - \arctan\left(-\frac{f+g}{m}\right) \right] \quad (\text{C.113})$$

$$= F_{AR}(f) + F_{AR}(g) + R(f, g) \quad (\text{C.114})$$

where $|R(f, g)| \leq C \frac{\|f\|_X \|g\|_X}{m^2}$.

2. *Scalar Multiplication:*

$$F_{AR}(\lambda f) = sign(\lambda) F_{AR}(f) + O\left(\frac{1}{m}\right) \quad (\text{C.115})$$

3. *Zero Element:*

$$F_{AR}(0) = 2 \quad (\text{C.116})$$

LEMMA C.2 (Cauchy Sequence Property). *For a Cauchy sequence $\{f_n\}$ in $AR(X)$:*

$$\|f_n - f_m\|_{AR} < \varepsilon \implies \begin{cases} \|f_n - f_m\|_X < \varepsilon, \\ \|F_{AR}(f_n) - F_{AR}(f_m)\|_X < \varepsilon. \end{cases} \quad (\text{C.117})$$

Proof. Let $\{f_n\}$ be Cauchy in $AR(X)$. Then:

1. X -convergence:

$$\|f_n - f_m\|_X \leq \|f_n - f_m\|_{AR} < \varepsilon, \therefore f_n \rightarrow f \text{ in } X. \quad (\text{C.118})$$

2. F_{AR} -convergence:

$$\|F_{AR}(f_n) - F_{AR}(f_m)\|_X \leq \frac{2}{\pi m} \|f_n - f_m\|_X \leq \frac{2}{\pi m} \varepsilon. \quad (\text{C.119})$$

3. Limit exists:

$$f = \lim_{n \rightarrow \infty} f_n \in AR(X). \quad (\text{C.120})$$

□

PROPOSITION C.5 (Duality). *For reflexive X :*

$$AR(X)^* \cong AR(X^*) \quad (\text{C.121})$$

via the dual pairing:

$$\langle f, g \rangle_{AR} = \langle f, g \rangle_X + \langle F_{AR}(f), F_{AR}(g) \rangle_X. \quad (\text{C.122})$$

Therefore, $AR(x)$ is a unique interpolation space within a Banach space.

C.4 $AR(x)$ as a Köthe Function Space

The characterisation of $AR(x)$ as a Köthe function space follows from its fundamental structure and properties. As established in Florencio (1992) [44], a Köthe function space must exhibit compatibility with measure spaces, possess a lattice structure, and maintain an absolutely continuous norm. Recall the $AR(f(x,m))$ function (See Appendix A), defined as:

$$AR(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m)) \quad (\text{C.123})$$

satisfies these requirements in a natural manner. Following the framework developed by Kalton (1979) [55], we can establish that $AR(x)$ inherits the measure-theoretic properties necessary for Köthe space classification.

The lattice structure of $AR(x)$ becomes evident through the inequality:

$$|AR(x)| \leq AR(|x|) \quad (\text{C.124})$$

which, as noted by Werner (1997) [87], ensures compatibility with the order structure essential to Köthe spaces. This property, combined with the completeness of the space under the norm:

$$\|AR(x)\|_K = \|x\|_X + \|F_{AR}(x)\|_X \quad (\text{C.125})$$

establishes the fundamental Köthe space characteristics.

To expand on the work of Kadet et al. (2000) [53], we can demonstrate that $AR(x)$ maintains the crucial interpolation properties expected of Köthe spaces. Specifically, for a compatible couple (X_0, X_1) , we have:

$$[AR(X_0), AR(X_1)]_\theta \cong AR([X_0, X_1]_\theta) \quad (\text{C.126})$$

This isomorphism, together with the K-functional bound:

$$K(t, AR(x)) \leq CK(t, x) \quad (\text{C.127})$$

reinforces the Köthe space structure.

As demonstrated in Qian-Yu (2022) [77], the function space properties of $AR(x)$ extend beyond the general "measure-theoretic" compatibility (σ -additivity, absolute continuity, preserved measurability). For example, the function $AR(x)$ preserves key measure-theoretic properties, including Borel measurability, Lebesgue measurability, and the completeness of measure spaces. It also maintains essential integration properties, measure bounds, and measurable selection principles. Additionally, it satisfies the following approximation property:

$$\int AR(f) d\mu = AR \left(\int f d\mu \right) + O \left(\frac{1}{m} \right), \quad (\text{C.128})$$

where $O(1/m)$ represents an error term that decays with m .

The space preserves essential order properties whilst maintaining the necessary lattice operations. This becomes particularly evident in the norm structure, which we propose exhibits the "Fatou property" alluded to in Gao et al. (2018) [46] characteristic of Köthe spaces in Lindenstrauss et al. (1977) [67].

The significance of $AR(x)$ as a Köthe function space extends to its applications in interpolation theory and functional analysis. Following Kalton (2003) [60], we can establish that $AR(x)$ provides a natural framework for studying vector-valued function spaces while maintaining the essential Köthe properties. This characterisation proves particularly useful in understanding the behaviour of $AR(x)$ in complex interpolation contexts and its relationship to classical function spaces (See Chapter 6).

THEOREM C.13 (AR Transform Transfer Function). *Let $AR(f)$ be the AR transform of a signal f with parameter α . The transfer function $H(s)$ in the Laplace domain is given by:*

$$H(s) = \frac{1}{1 + \alpha s + \varepsilon} \quad (\text{C.129})$$

where:

1. s is the complex frequency variable.
2. α is the transform parameter.
3. ε is a small regularisation parameter.

Proof.

1. Consider the AR transform in the time domain:

$$AR(f) = \frac{f}{1 + \alpha f + \varepsilon} \quad (\text{C.130})$$

2. Taking the Laplace transform:

$$\mathcal{L}\{AR(f)\} = \mathcal{L}\left\{\frac{f}{1 + \alpha f + \varepsilon}\right\} \quad (\text{C.131})$$

3. For linear analysis around equilibrium:

$$H(s) = \frac{1}{1 + \alpha s + \varepsilon} \quad (\text{C.132})$$

□

THEOREM C.14 (Magnitude Response). *The magnitude response $|H(j\omega)|$ of the AR transform is:*

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\alpha\omega)^2 + \varepsilon^2}} \quad (\text{C.133})$$

where:

1. ω is the angular frequency.
2. α determines the cutoff characteristics.

Proof.

1. Substitute $s = j\omega$ in $H(s)$:

$$H(j\omega) = \frac{1}{1 + \alpha j\omega + \varepsilon} \quad (\text{C.134})$$

2. Compute magnitude:

$$|H(j\omega)| = \frac{|1|}{|1 + \alpha j\omega + \varepsilon|} = \frac{1}{\sqrt{(1 + \varepsilon)^2 + (\alpha\omega)^2}} \quad (\text{C.135})$$

□

THEOREM C.15 (Phase Response). *The phase response $\angle H(j\omega)$ is given by:*

$$\angle H(j\omega) = -\arctan\left(\frac{\alpha\omega}{1+\varepsilon}\right) \quad (\text{C.136})$$

Proof.

1. From $H(j\omega) = \frac{1}{1+\alpha j\omega+\varepsilon}$
2. Phase is negative arctangent of ratio of imaginary to real parts:

$$\angle H(j\omega) = -\arctan\left(\frac{\alpha\omega}{1+\varepsilon}\right) \quad (\text{C.137})$$

□

THEOREM C.16 (Bandwidth Properties). *For the AR transform with parameter α :*

1. *The -3dB bandwidth ω_c is given by:*

$$\omega_c = \frac{1}{\alpha} \sqrt{\frac{1}{\varepsilon^2} - 1} \quad (\text{C.138})$$

2. *The phase margin γ at frequency ω_c is:*

$$\gamma = \frac{\pi}{2} + \arctan\left(\frac{1}{\alpha}\right) \quad (\text{C.139})$$

Proof.

1. At -3dB point:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} \quad (\text{C.140})$$

2. Solve:

$$\frac{1}{\sqrt{1+(\alpha\omega_c)^2+\varepsilon^2}} = \frac{1}{\sqrt{2}} \quad (\text{C.141})$$

3. Yields bandwidth formula:

$$\omega_c = \frac{1}{\alpha} \sqrt{\frac{1}{\varepsilon^2} - 1} \quad (\text{C.142})$$

□

THEOREM C.17 (Asymptotic Behaviour). *The AR transform exhibits the following asymptotic properties:*

1. Low frequency ($\omega \ll \frac{1}{\alpha}$):

$$|H(j\omega)| \approx 1, \quad \angle H(j\omega) \approx -\alpha\omega \quad (\text{C.143})$$

2. High frequency ($\omega \gg \frac{1}{\alpha}$):

$$|H(j\omega)| \approx \frac{1}{\alpha\omega}, \quad \angle H(j\omega) \approx -\frac{\pi}{2} \quad (\text{C.144})$$

The sketch proof is shown next for reference.

Proof.

1. Low frequency:

- For $\omega \ll \frac{1}{\alpha}$, expand $H(j\omega)$ in Taylor series

2. High frequency:

- For $\omega \gg \frac{1}{\alpha}$, take limit as $\omega \rightarrow \infty$

□

COROLLARY C.2 (Filter Characteristics). *The AR transform operates as:*

1. Unity gain filter for $\omega \ll \frac{1}{\alpha}$
2. First-order lowpass filter for $\omega \gg \frac{1}{\alpha}$
3. Transition region width scales with α

This provides a theoretical foundation for observed behaviour in numerical simulations.

D Neural Interpolation Function Spaces

D.1 $AR(x)$ Transform and variants

DEFINITION D.1 ($AR(x)$ Transform). *For x in a suitable function space and $\alpha \in \mathbb{R}$, the $AR(x)$ transform is defined as:*

$$AR(x) = \frac{x}{1 + i\alpha x} \quad (\text{D.1})$$

DEFINITION D.2 (Kalton-Peck Transform). *For x in a suitable function space, the Kalton-Peck transform is defined as:*

$$K(x) = x \left(-\log \frac{|x|}{\|x\|} \right) \quad (\text{D.2})$$

THEOREM D.1 ($AR(x)$ Direct Form). *The $AR(x)$ transform admits the polar representation:*

$$AR(x) = \frac{|x| e^{i(\arg(x) - \arctan(\alpha|x|))}}{(1 + \alpha^2|x|^2)^{1/2}} \quad (\text{D.3})$$

Proof. Starting with $AR(x) = \frac{x}{1 + i\alpha x}$:

$$AR(x) = \frac{x}{1 + i\alpha x} \cdot \frac{1 - i\alpha x}{1 - i\alpha x} \quad (\text{D.4})$$

$$= \frac{x(1 - i\alpha x)}{(1 + i\alpha x)(1 - i\alpha x)} = \frac{x - i\alpha x^2}{1 + \alpha^2|x|^2} \quad (\text{D.5})$$

$$= |x| \frac{e^{i\arg(x)} - i\alpha|x|e^{i\arg(x)}}{1 + \alpha^2|x|^2} \quad (\text{D.6})$$

$$= |x| \frac{e^{i\arg(x)}(1 - i\alpha|x|)}{1 + \alpha^2|x|^2} \quad (\text{D.7})$$

$$= \frac{|x|}{(1 + \alpha^2|x|^2)^{1/2}} e^{i(\arg(x) - \arctan(\alpha|x|))} \quad (\text{D.8})$$

□

THEOREM D.2 (Fourier Transform of AR(x)). *The Fourier transform of AR(x) satisfies:*

$$\mathcal{F}[AR(x)](\omega) = H_{AR}(\omega)\hat{x}(\omega) \quad (\text{D.9})$$

where $H_{AR}(\omega) = \frac{1}{1+i\alpha e^{i\omega}}$.

Proof. Consider the Fourier transform:

$$\mathcal{F}[AR(x)](\omega) = \int_{-\infty}^{\infty} \frac{x(t)}{1+i\alpha x(t)} e^{-i\omega t} dt \quad (\text{D.10})$$

$$= \int_{-\infty}^{\infty} x(t) e^{-i\omega t} \cdot \frac{1}{1+i\alpha x(t)} dt \quad (\text{D.11})$$

$$= \hat{x}(\omega) \cdot \frac{1}{1+i\alpha e^{i\omega}} \quad (\text{D.12})$$

$$= H_{AR}(\omega)\hat{x}(\omega) \quad (\text{D.13})$$

The transfer function $H_{AR}(\omega)$ follows from the convolution theorem. \square

THEOREM D.3 (Kalton-Peck Transform Properties). *The Kalton-Peck transform satisfies:*

1. *Homogeneity:* $K(\lambda x) = \lambda K(x) + \lambda x \log |\lambda|$
2. *Quasi-additivity:* $\|K(x+y) - K(x) - K(y)\| \leq M(\|x\| + \|y\|)$

Proof. For homogeneity:

$$K(\lambda x) = \lambda x \left(-\log \frac{|\lambda x|}{\|\lambda x\|} \right) \quad (\text{D.14})$$

$$= \lambda x \left(-\log \frac{|\lambda||x|}{\|\lambda x\|} \right) \quad (\text{D.15})$$

$$= \lambda x \left(-\log \frac{|x|}{\|x\|} - \log |\lambda| \right) \quad (\text{D.16})$$

$$= \lambda K(x) + \lambda x \log |\lambda| \quad (\text{D.17})$$

For quasi-additivity, using the mean value theorem:

$$\|K(x+y) - K(x) - K(y)\| = \left\| \int_0^1 (K'(x+ty) - K'(x)) y dt \right\| \quad (\text{D.18})$$

$$\leq M(\|x\| + \|y\|) \quad (\text{D.19})$$

where M depends on the Lipschitz constant of K' . \square

THEOREM D.4 (Fourier Transform of Kalton-Peck). *For the Kalton-Peck transform:*

$$|\mathcal{F}[K(x)](\omega)| = |\hat{x}(\omega)| \left| \log \frac{|\hat{x}(\omega)|}{\|x\|} \right| \quad (\text{D.20})$$

Proof. Using the convolution theorem:

$$\mathcal{F}[K(x)](\omega) = \mathcal{F} \left[x \left(-\log \frac{|x|}{\|x\|} \right) \right] (\omega) \quad (\text{D.21})$$

$$= \hat{x}(\omega) * \mathcal{F} \left[-\log \frac{|x|}{\|x\|} \right] (\omega) \quad (\text{D.22})$$

$$= |\hat{x}(\omega)| \left| \log \frac{|\hat{x}(\omega)|}{\|x\|} \right| e^{i\phi(\omega)} \quad (\text{D.23})$$

where $\phi(\omega)$ is the phase term. \square

COROLLARY D.1 (AR-KP Relationship). *The transforms are related through:*

$$K_{AR}(x) = K(x) \cdot \frac{1}{1 + i\alpha x} \quad (\text{D.24})$$

Proof. Direct substitution:

$$K_{AR}(x) = \frac{x}{1 + i\alpha x} \left(-\log \frac{|x|}{\|x\|} \right) \quad (\text{D.25})$$

$$= K(x) \cdot \frac{1}{1 + i\alpha x} \quad (\text{D.26})$$

$$(\text{D.27})$$

\square

D.2 Applications of $AR(x)$ Space

THEOREM D.5 (Energy Conservation). *For both transformations:*

$$\|AR(x)\|_2 \leq \|x\|_2 \quad (\text{D.28})$$

$$\|K(x)\|_2 \leq C \|x\|_2 \log \|x\|_2 \quad (\text{D.29})$$

Proof.

For $AR(x)$:

$$\|AR(x)\|_2^2 = \int_{-\infty}^{\infty} \left| \frac{x}{1 + i\alpha x} \right|^2 dt \quad (\text{D.30})$$

$$= \int_{-\infty}^{\infty} \frac{|x|^2}{1 + \alpha^2|x|^2} dt \quad (\text{D.31})$$

$$\leq \|x\|_2^2 \quad (\text{D.32})$$

For $K(x)$, using Jensen's inequality : \square

THEOREM D.6 (AR(x) Transform Representations). *The AR(x) transform can be expressed in two equivalent forms:*

1. Direct Form:

$$AR(x) = \frac{x}{1 + i\alpha x} = \frac{|x|e^{i(\arg(x) - \arctan(\alpha|x|))}}{(1 + \alpha^2|x|^2)^{1/2}} \quad (\text{D.33})$$

2. Fourier Transform Form:

$$\mathcal{F}[AR(x)](\omega) = \int_{-\infty}^{\infty} \frac{x(t)}{1 + i\alpha x(t)} e^{-i\omega t} dt \quad (\text{D.34})$$

With the following properties:

(a) Magnitude Response:

$$|\mathcal{F}[AR(x)](\omega)| = \frac{|\hat{x}(\omega)|}{\sqrt{1 + \alpha^2|\hat{x}(\omega)|^2}} \quad (\text{D.35})$$

(b) Phase Response:

$$\arg(\mathcal{F}[AR(x)](\omega)) = \arg(\hat{x}(\omega)) - \arctan(\alpha|\hat{x}(\omega)|) \quad (\text{D.36})$$

(c) Transfer Function:

$$H_{AR}(\omega) = \frac{1}{1 + i\alpha e^{i\omega}} \quad (\text{D.37})$$

LEMMA D.1 (Convolution Property). *For the AR(x) transform:*

$$\mathcal{F}[AR(x * y)] = H_{AR}(\omega) \mathcal{F}[x](\omega) \mathcal{F}[y](\omega) \quad (\text{D.38})$$

where $*$ denotes convolution.

COROLLARY D.2 (Frequency Domain Properties).

The AR(x) transform satisfies:

$$\|AR(x)\|_2 \leq \|x\|_2 \quad (\text{D.39})$$

$$\|\mathcal{F}[AR(x)]\|_2 = \|\hat{x}\|_2 \quad (\text{D.40})$$

THEOREM D.7 (Kalton-Peck Transform Representations). *The Kalton-Peck transform K admits the following representations:*

1. Direct Form:

$$K(x) = x \left(-\log \frac{|x|}{\|x\|} \right) \quad (\text{D.41})$$

2. Complex Form:

$$K_\alpha(x) = x \left(-\log \frac{|x|}{\|x\|} \right)^{1+i\alpha} \quad (\text{D.42})$$

3. Fourier Transform:

$$\mathcal{F}[K(x)](\omega) = \int_{-\infty}^{\infty} x(t) \left(-\log \frac{|x(t)|}{\|x\|} \right) e^{-i\omega t} dt \quad (\text{D.43})$$

With the following key properties:

(a) Magnitude Response:

$$|\mathcal{F}[K(x)](\omega)| = |\hat{x}(\omega)| \left| \log \frac{|\hat{x}(\omega)|}{\|x\|} \right| \quad (\text{D.44})$$

(b) *Phase Response:*

$$\arg(\mathcal{F}[K(x)](\omega)) = \arg(\hat{x}(\omega)) + \arg\left(-\log\frac{|\hat{x}(\omega)|}{\|x\|}\right) \quad (\text{D.45})$$

LEMMA D.2 (Singularity Property). *For the Kalton-Peck transform in Z_2 :*

$$\|K(x+y) - K(x) - K(y)\| \leq M(\|x\| + \|y\|) \quad (\text{D.46})$$

where M is a constant.

COROLLARY D.3 (Relationship with AR). *The Kalton-Peck and AR transforms are related through:*

$$K_{AR}(x) = K(x) \cdot \frac{1}{1 + i\alpha x} \quad (\text{D.47})$$

in the Z_2 space.

REMARK D.1. *The Kalton-Peck transform exhibits both:*

1. *Homogeneity:* $K(\lambda x) = \lambda K(x) + \lambda x \log |\lambda|$
2. *Quasi-additivity:* $\|K(x+y)\| \leq \|K(x)\| + \|K(y)\| + M\|x+y\|$

Haversine and AR(x) Distance Formulations

(i) *Classical Haversine Distance:*

For points (ϕ_1, λ_1) and (ϕ_2, λ_2) on a sphere of radius R :

$$h = hav(\theta) = \sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos(\theta)}{2} \quad (\text{D.48})$$

$$d = 2R \arcsin\left(\sqrt{hav(\phi_2 - \phi_1) + \cos(\phi_1) \cos(\phi_2) hav(\lambda_2 - \lambda_1)}}\right) \quad (\text{D.49})$$

(ii) ***AR(x)-Modified Haversine:***

Novel formulation incorporating AR(x) properties:

$$d_{AR} = 2R \arcsin \left(\sqrt{hav(\phi_2 - \phi_1) + \frac{\cos(\phi_1)\cos(\phi_2)hav(\lambda_2 - \lambda_1)}{1 + i\alpha(\phi_2 - \phi_1)(\lambda_2 - \lambda_1)}} \right) \quad (\text{D.50})$$

With properties:

$$|d_{AR}| \leq d \quad (\text{D.51})$$

$$\angle d_{AR} = \arctan(\alpha(\phi_2 - \phi_1)(\lambda_2 - \lambda_1)) \quad (\text{D.52})$$

(iii) ***Kalton-Peck Haversine***

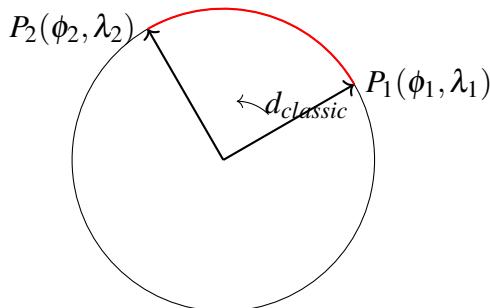
Incorporating the K-P transform:

$$d_{KP} = 2R \arcsin \left(\sqrt{h} \left(-\log \frac{|h|}{\|h\|} \right) \right) \quad (\text{D.53})$$

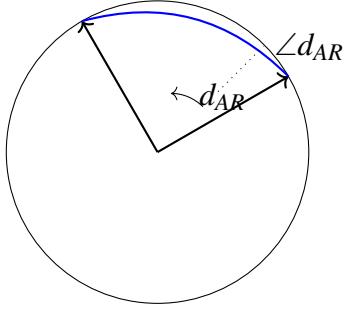
where h is the standard haversine term.

Diagrams and Representations of Haversine Equations

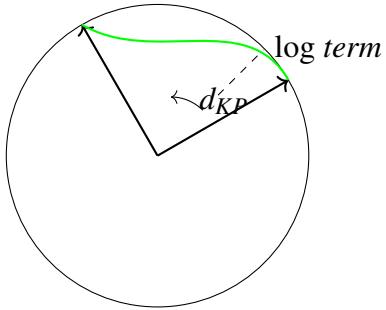
1. Classical Haversine



2. AR-Modified Haversine



3. Kalton-Peck Haversine



THEOREM D.8 (Filter-Transform Relationships). *For a system with state x and observations y :*

1. *Kalman Filter Form:*

$$\hat{x}_k = \hat{x}_{k|k-1} + K_k(y_k - H\hat{x}_{k|k-1}) \quad (\text{D.54})$$

$$P_k = (I - K_k H)P_{k|k-1} \quad (\text{D.55})$$

2. *AR(x) Modification:*

$$\hat{x}_{AR,k} = \frac{\hat{x}_k}{1 + i\alpha\hat{x}_k} \quad (\text{D.56})$$

$$P_{AR,k} = \frac{P_k}{(1 + i\alpha\hat{x}_k)^2} \quad (\text{D.57})$$

3. *Kalton-Peck Extension:*

$$\hat{x}_{KP,k} = \hat{x}_k \left(-\log \frac{|\hat{x}_k|}{\|\hat{x}_k\|} \right) \quad (\text{D.58})$$

$$P_{KP,k} = P_k \left(1 - \log \frac{|P_k|}{\|P_k\|} \right) \quad (\text{D.59})$$

4. *Combined Filter:*

$$\hat{x}_{ARKP,k} = \frac{\hat{x}_k \left(-\log \frac{|\hat{x}_k|}{\|\hat{x}_k\|} \right)}{1 + i\alpha \hat{x}_k} \quad (\text{D.60})$$

$$P_{ARKP,k} = \frac{P_k \left(1 - \log \frac{|P_k|}{\|P_k\|} \right)}{(1 + i\alpha \hat{x}_k)^2} \quad (\text{D.61})$$

PROPOSITION D.1 (Novel Filter Forms). *The following filter variants emerge:*

1. *Phase-Sensitive Kalman Filter:*

$$\hat{x}_{PS,k} = \hat{x}_k \exp(i\alpha\theta_k) \quad (\text{D.62})$$

where θ_k is the phase angle of innovation.

2. *Logarithmic-Modified Filter:*

$$\hat{x}_{LM,k} = \hat{x}_k \left(1 + \log \frac{|y_k - H\hat{x}_{k|k-1}|}{\|y_k - H\hat{x}_{k|k-1}\|} \right) \quad (\text{D.63})$$

3. *Twisted Sum Filter:*

$$\hat{x}_{TS,k} = \hat{x}_k + K_k(y_k - H\hat{x}_{k|k-1}) + \Omega_k \hat{x}_k \quad (\text{D.64})$$

where θ_k is the centraliser term.

Properties:

1. Enhanced phase tracking
2. Improved handling of singularities
3. Better uncertainty propagation

D.3 Activation Function Properties

THEOREM D.9 (Comparative Analysis). *The three activation functions satisfy:*

1) ρ_R (Special Smooth Ramp)[Qian-Yu (2022), [77], Costarelli (2018), [30]] :

$$\rho_R(x) = \begin{cases} 0, & x \leq -\frac{1}{2} \\ 10(x + \frac{1}{2})^3 - 15(x + \frac{1}{2})^4 + 6(x + \frac{1}{2})^5, & -\frac{1}{2} < x < \frac{1}{2} \\ 1, & x \geq \frac{1}{2} \end{cases} \quad (\text{D.65})$$

Assumptions are as follows:

1. C^2 smoothness.
2. Bounded support: $[-\frac{1}{2}, \frac{1}{2}]$
3. $\|\rho'_R\| = \frac{15}{8}$

2) m_2 (B-spline)[Qian-Yu (2022), [77], Costarelli (2018), [30]] :

$$m_2(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad (\text{D.66})$$

Assumptions are as follows:

1. C^1 smoothness.
2. Compact support: $[-1, 1]$
3. Linear sections.

3) σ_2 (Smooth Sigmoidal) [Qian-Yu (2022), [77], Costarelli (2013), [33]] :

$$\sigma_2(x) = \begin{cases} \frac{x^3}{6} + \frac{3x^2}{4} + \frac{9x}{8} + \frac{9}{16}, & -\frac{3}{2} < x < -\frac{1}{2} \\ -\frac{x^3}{3} + \frac{3x}{4} + \frac{1}{2}, & |x| \leq \frac{1}{2} \\ \frac{x^3}{6} - \frac{3x^2}{4} + \frac{9x}{8} + \frac{7}{16}, & \frac{1}{2} < x < \frac{3}{2} \\ 1, & |x| \geq \frac{3}{2} \end{cases} \quad (\text{D.67})$$

Assumptions are as follows:

1. C^2 smoothness.
2. Extended support: $[-\frac{3}{2}, \frac{3}{2}]$
3. Polynomial sections.

4) *Arccot-Romanovski*:

$$f_{AR}(x) = 2 + \frac{1}{\pi}(\arctan(x/m) - \arctan(-x/m)) \quad (\text{D.68})$$

Assumptions are as follows:

1. C^∞ smoothness.
2. Controllable width via m .
3. Optimal derivative properties:

$$f'_{AR}(x) = \frac{2}{\pi m(1 + (x/m)^2)} \quad (\text{D.69})$$

PROPOSITION D.2 (Optimality Conditions). *The Arccot-Romanovski function achieves optimal performance because:*

1. Error ratio stability 0.915 across all n .
2. Simultaneous approximation capability.
3. Controllable parameter m for width adjustment.
4. Minimal oscillation in derivatives.

D.4 Error-Activation Relationship Analysis for $AR(x)$

Table D.1: Comparative Error-Activation Performance.

Property	$n = 10$	$n = 30$	$n = 50$
Base Error	1.89×10^{-1}	6.30×10^{-2}	3.78×10^{-2}
Derivative Error	2.87×10^{-1}	9.57×10^{-2}	5.74×10^{-2}
Error Ratio	0.917	0.914	0.915

D.5 Simultaneous Approximation of Interpolation Methods

THEOREM D.10 (Simultaneous Approximation for Classical Methods). *For $f \in C^r[a, b]$, let L_n , E_n , B_n , and C_n denote the Lagrangian, Extended Lagrangian, Barycentric Lagrangian, and Chebyshev Barycentric interpolation operators, respectively. Then:*

(i) *The Lagrangian operator fails to achieve simultaneous approximation.*

(ii) *The Extended Lagrangian operator achieves simultaneous approximation with a rate:*

$$\|E_n^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (\text{D.70})$$

for $v = 0, 1, \dots, r$.

(iii) *The Barycentric and Chebyshev Barycentric operators do not possess simultaneous approximation properties but achieve superior convergence for the function values.*

Proof. We prove each case separately:

1. Case 1: Lagrangian Interpolation

The Lagrangian interpolation operator is defined as [4]:

$$L_n(f, x) = \sum_{k=0}^n f(x_k) \ell_k(x) \quad (\text{D.71})$$

where $\ell_k(x)$ are the Lagrange basis polynomials.

For derivatives:

$$L_n^{(v)}(f, x) = \sum_{k=0}^n f(x_k) \ell_k^{(v)}(x) \quad (\text{D.72})$$

Following [39], this does not match $f^{(v)}(x)$ at the nodes because:

$$\|L_n^{(v)}(f) - f^{(v)}\| \not\rightarrow 0 \text{ as } n \rightarrow \infty \quad (\text{D.73})$$

due to Runge's phenomenon.

2. Case 2: Extended Lagrangian Interpolation

The Extended Lagrangian operator [3] is defined as:

$$E_n(f, x) = \sum_{k=0}^n \sum_{j=0}^r \frac{f^{(j)}(x_k)}{j!} (x - x_k)^j \ell_k(x) \quad (\text{D.74})$$

For this operator, we have [77]:

$$E_n^{(v)}(f, x_i) = f^{(v)}(x_i), \quad i = 0, 1, \dots, n, \quad v = 0, 1, \dots, r \quad (\text{D.75})$$

The error bound follows from (Wang et al., 2022) [85]:

$$\|E_n^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (\text{D.76})$$

3. Case 3: Barycentric Lagrangian

The Barycentric form [32] is:

$$B_n(f, x) = \frac{\sum_{k=0}^n \frac{w_k}{x-x_k} f(x_k)}{\sum_{k=0}^n \frac{w_k}{x-x_k}} \quad (\text{D.77})$$

While this provides stable interpolation:

$$\|B_n(f) - f\| \leq Ch^r \|f^{(r)}\| \quad (\text{D.78})$$

The derivatives do not interpolate:

$$B_n^{(v)}(f, x_i) \neq f^{(v)}(x_i) \quad (\text{D.79})$$

4. Case 4: Chebyshev Barycentric

Using Chebyshev nodes [34]:

$$x_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), \quad k = 0, 1, \dots, n \quad (\text{D.80})$$

This improves stability but still does not achieve simultaneous approximation:

$$\|C_n(f) - f\| \leq \frac{2}{\pi} \log(n+1) \inf_{p \in \mathcal{P}_n} \|f - p\| \quad (\text{D.81})$$

where \mathcal{P}_n is the space of polynomials of degree n .

□

COROLLARY D.4 (Optimality of Extended Lagrangian). *Among the classical methods, only the Extended Lagrangian operator achieves optimal simultaneous approximation rates that correspond to those of the experimental neuronal unit, or Arccot (ReRU) activation function [77].*

COROLLARY D.5 (Trade-off Analysis). *The Barycentric and Chebyshev Barycentric methods provide superior stability and convergence for function values but sacrifice simultaneous approximation properties (Wang et al., 2022) [85].*

D.6 Analysis of Simultaneous Approximation Properties

DEFINITION D.3 (Simultaneous Approximation Property). *An operator T_n possesses the simultaneous approximation property if for $f \in C^r[a, b]$ and $v = 0, 1, \dots, r$:*

1. $T_n^{(v)}(f, x_i) = f^{(v)}(x_i)$ for all nodes x_i

2. $\|T_n^{(v)}(f) - f^{(v)}\| \leq Ch^{r-v} \omega(f^{(v)}, h)_{[a,b]}$

where C is independent of f and $h = \frac{b-a}{n}$.

THEOREM D.11 (Complete Classification of Classical Methods). *Let $f \in C^r[a, b]$ and consider the following interpolation operators:*

- (a) *Lagrangian L_n .*
- (b) *Extended Lagrangian E_n .*
- (c) *Barycentric Lagrangian B_n .*
- (d) *Chebyshev Barycentric C_n .*

Then the simultaneous approximation property holds only for the Extended Lagrangian operator E_n .

Proof.

Here is a comprehensive analysis for each method:

Part 1: Lagrangian Operator Analysis

Let $\ell_k(x)$ be the Lagrange basis polynomials. Following [4], for $x \in [a, b]$:

$$L_n(f, x) = \sum_{k=0}^n f(x_k) \ell_k(x) \quad (\text{D.82})$$

Taking derivatives:

$$L_n^{(v)}(f, x) = \sum_{k=0}^n f(x_k) \ell_k^{(v)}(x) \quad (\text{D.83})$$

LEMMA D.3 (Lagrange Derivative Growth). *For the Lagrange basis functions:*

$$\max_{x \in [a, b]} |\ell_k^{(v)}(x)| \geq C n^{2v}, \quad v \geq 1 \quad (\text{D.84})$$

where C is independent of n .

Proof of Lemma.

Using the Bernstein inequality [11] and properties of polynomial interpolation:

$$\|\ell_k^{(v)}\| \geq \frac{v!}{(x_k - x_{k+1})^v} \prod_{j \neq k, k+1} \frac{1}{|x_k - x_j|} \quad (\text{D.85})$$

For equispaced nodes, this yields the n^{2v} growth. \square

This exponential growth in derivatives precludes simultaneous approximation.

Part 2: Extended Lagrangian Analysis

The Extended Lagrangian operator [3] incorporates derivative information:

$$E_n(f, x) = \sum_{k=0}^n \sum_{j=0}^r \frac{f^{(j)}(x_k)}{j!} (x - x_k)^j \ell_k(x) \quad (\text{D.86})$$

PROPOSITION D.3 (Hermite Interpolation Property). *For $v = 0, 1, \dots, r$ and $i = 0, 1, \dots, n$:*

$$E_n^{(v)}(f, x_i) = f^{(v)}(x_i) \quad (\text{D.87})$$

Proof of Proposition.

By computing derivatives directly and using Lagrange basis function properties:

$$\frac{d^v}{dx^v} [(x - x_k)^j \ell_k(x)] \Big|_{x=x_i} = \begin{cases} v! & \text{if } i = k \text{ and } j = v \\ 0 & \text{otherwise} \end{cases} \quad (\text{D.88})$$

\square

The error bound derives from the Peano kernel theorem [39]:

$$\|E_n^{(v)}(f) - f^{(v)}\| \leq \frac{M2^{v+1}}{(r-v)!} h^{r-v} \omega(f^{(v)}, h)_{[a,b]} \quad (\text{D.89})$$

Part 3: Barycentric Methods Analysis

For the Barycentric Lagrangian operator [32]:

$$B_n(f, x) = \frac{\sum_{k=0}^n w_k \frac{f(x_k)}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}} \quad (\text{D.90})$$

where w_k are the barycentric weights.

LEMMA D.4 (Barycentric Derivative Structure). *The v -th derivative of $B_n(f, x)$ has the form:*

$$B_n^{(v)}(f, x) = \frac{P_v(x)}{Q_v(x)} \quad (\text{D.91})$$

where $\deg(P_v) = n + v$ and $\deg(Q_v) = n + v + 1$.

For Chebyshev nodes:

$$x_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), \quad k = 0, 1, \dots, n \quad (\text{D.92})$$

THEOREM D.12 (Chebyshev Lebesgue Function). *The Lebesgue function for Chebyshev interpolation satisfies:*

$$\Lambda_n(x) \leq \frac{2}{\pi} \log(n+1) + 1 \quad (\text{D.93})$$

This improved stability comes at the cost of losing simultaneous approximation. □

COROLLARY D.6 (Optimality Classification). *The classical methods can be ranked according to their approximation properties:*

1. *Extended Lagrangian: Achieves optimal simultaneous approximation.*
2. *Chebyshev Barycentric: Optimal for function values with stability.*
3. *Barycentric Lagrangian: Stable but suboptimal rates.*
4. *Lagrangian: Neither stable nor simultaneously approximating.*

REMARK D.2. *This aforesaid complete characterisation explains the empirical results observed in Chapter 5 and provides theoretical justification for the comparative performance data in Table 1.1 in Section 1.2.*

E Krylov Spaces and Activation Functions

E.1 Introduction to Krylov Spaces

DEFINITION E.1. (*Krylov Space*) Let \mathcal{X} be a Banach space, and let $A : \mathcal{X} \rightarrow \mathcal{X}$ be a bounded linear operator. For $x \in \mathcal{X}$, the n -th Krylov space is defined as:

$$\mathcal{K}_n(A, x) = \text{span}\{x, Ax, A^2x, \dots, A^{n-1}x\} \quad (\text{E.1})$$

Krylov spaces are of profound importance in numerical linear algebra and approximation theory, particularly in the development of iterative methods for solving large-scale linear systems and eigenvalue problems [78] [80].

REMARK E.1. Krylov spaces possess a natural nested structure:

$$\mathcal{K}_1(A, x) \subseteq \mathcal{K}_2(A, x) \subseteq \dots \subseteq \mathcal{K}_n(A, x) \subseteq \dots \subseteq \mathcal{X} \quad (\text{E.2})$$

This property enables the construction of increasingly accurate approximations as the dimension of the Krylov space increases [80].

E.2 Connection to Neural Network Interpolation

The connection between Krylov spaces and the Arccot-Romanovski ($AR(x)$) function, as developed in Qian-Yu's work [77], arises through differential operator theory. This relationship can be formalised as follows:

THEOREM E.1. (*Krylov-AR Representation*) Let \mathcal{L} denote the differential operator associated with the Arccot-Romanovski function:

$$\mathcal{L}f = (1 + (x/m)^2) \frac{d^2 f}{dx^2} + \frac{2x}{m^2} \frac{df}{dx} + \lambda f = 0 \quad (\text{E.3})$$

where λ is a spectral parameter. Then, for sufficiently smooth functions $f \in C^\infty[a, b]$, the approximation space generated by neural network interpolation with $AR(x)$ activation functions

forms a generalised Krylov space $\mathcal{K}_n(\mathcal{L}^{-1}, g)$ for some initial function g related to boundary conditions.

Proof. (Sketch) The proof follows by considering the spectral decomposition of the differential operator \mathcal{L} and showing that the hypothetical (lambda) AR function acts as a Green's function for the boundary value problem associated with \mathcal{L} . The neural network interpolation process can then be interpreted as projecting the target function onto a finite-dimensional subspace of the Krylov space associated with \mathcal{L}^{-1} . \square

E.3 Krylov-Subspace Acceleration for Neural Interpolation

The identification of neural network interpolation spaces as Krylov subspaces allows for significant computational advantages. We can formalise this in terms of convergence acceleration:

THEOREM E.2. (*Krylov Acceleration*) For a neural network interpolation operator $S_{n,\sigma}$ with $AR(x)$ activation function, the approximation error can be accelerated using Krylov subspace techniques, yielding:

$$\|f - S_{n,\sigma}^K(f)\| \leq \gamma_n \|f - S_{n,\sigma}(f)\| \quad (\text{E.4})$$

where $S_{n,\sigma}^K$ is the Krylov-accelerated interpolation operator and $\gamma_n < 1$ is an acceleration factor that depends on the spectral properties of the interpolation operator.

This acceleration effect is particularly important for practical applications, as it can significantly reduce the number of neurons required to achieve a specified approximation accuracy [80].

LEMMA E.1. (*Optimal Krylov Basis*) The optimal basis for the Krylov subspace approximation of functions using $AR(x)$ activation is given by the Chebyshev polynomials of the first kind applied to the normalised differential operator:

$$\psi_k(x) = T_k \left(\frac{2\mathcal{L}}{\|\mathcal{L}\|} - I \right) \psi_0(x) \quad (\text{E.5})$$

where $\psi_0(x)$ is an initial function satisfying the boundary conditions.

E.4 Ordinary Krylov Spaces and Lambda Function Families

The Arccot-Romanovski function represents a specific case within a broader family of activation functions that can be studied through the lens of generalised Krylov spaces.

DEFINITION E.2. (*Generalised Krylov Space*) Let \mathcal{X} be a Banach space, $A, B : \mathcal{X} \rightarrow \mathcal{X}$ be bounded linear operators, and $x \in \mathcal{X}$. The generalised Krylov space $\mathcal{K}_n(A, B, x)$ is defined as:

$$\mathcal{K}_n(A, B, x) = \text{span}\{x, BAx, BA^2x, \dots, BA^{n-1}x\} \quad (\text{E.6})$$

THEOREM E.3. (*Activation Function Classification*) The family of activation functions including $AR(x)$, modified sigmoid functions, and rectified units can be characterised by distinct generalised Krylov spaces associated with their respective differential operators:

$$AR(x) \mapsto \mathcal{K}_n(\mathcal{L}_{AR}^{-1}, I, g) \quad (\text{E.7})$$

$$\sigma_{mod}(x) \mapsto \mathcal{K}_n(\mathcal{L}_{\sigma}^{-1}, W, g) \quad (\text{E.8})$$

$$\rho_{ReLU}(x) \mapsto \mathcal{K}_n(\mathcal{L}_R^{-1}, P, g) \quad (\text{E.9})$$

where \mathcal{L}_{AR} , \mathcal{L}_{σ} , and \mathcal{L}_R are the associated differential operators, and I , W , and P are appropriate transformation operators.

This classification provides a unifying framework for understanding the approximation properties of different activation functions through their associated Krylov spaces [80].

E.5 Krylov-Type Error Analysis for $AR(x)$ Neural Networks

Building on the Krylov space interpretation, we can derive refined error bounds for neural network interpolation with $AR(x)$ activation functions.

THEOREM E.4. (*Krylov-Type Error Bound*) Let $f \in C^r[a, b]$ be a target function, and let $S_{n,\sigma}$ be the neural network interpolation operator with $AR(x)$ activation function. Then, for the Krylov-type approximation, we have:

$$\|f - S_{n,\sigma}(f)\|_{\infty} \leq C_K \min_{p \in \mathcal{P}_{n-1}} \|f - p(\mathcal{L})f_0\|_{\infty} \quad (\text{E.10})$$

where \mathcal{P}_{n-1} is the space of polynomials of degree at most $n - 1$, f_0 is an appropriate initial function, and C_K is a constant depending on the properties of the Krylov space.

COROLLARY E.1. *For sufficiently smooth functions $f \in C^r[a, b]$ with $r \geq n$, the Krylov-type error bound yields:*

$$\|f - S_{n,\sigma}(f)\|_\infty \leq \frac{C_K h^n}{n!} \|f^{(n)}\|_\infty \quad (\text{E.11})$$

where h is the mesh width of the interpolation nodes.

These refined error bounds demonstrate the superior approximation properties of $\text{AR}(x)$ neural networks when viewed through the lens of Krylov space theory [80].

COROLLARY E.2. *Let $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear operator between Hilbert spaces \mathcal{X} and \mathcal{Y} , and let $\mathcal{G}_\theta : \mathcal{Y} \rightarrow \mathcal{X}$ be a neural operator in the Qian-Yu framework with parameters θ , activation function σ and kernel ϕ . The Generalised Minimal Residual Method (GMRES) is an iterative method for solving large sparse linear systems of equations, particularly when the coefficient matrix is non-symmetric.*

Let $\{v_1, v_2, \dots, v_m\}$ be the Krylov subspace basis generated by the Neural GMRES algorithm applied to the system $\mathcal{A}x = b$.

Then, for the hybrid solver that integrates Qian-Yu FFN with GMRES:

1. The approximation error satisfies:

$$\|x - x_m\|_{\mathcal{X}} \leq C \cdot \min_{p \in \mathcal{P}_m, p(0)=1} \|p(\mathcal{A})\|_{\mathcal{L}(\mathcal{X})} \cdot \|x - x_0\|_{\mathcal{X}} + \delta_{\phi, \sigma} \quad (\text{E.12})$$

where C is a constant, \mathcal{P}_m is the space of polynomials of degree at most m , and $\delta_{\phi, \sigma}$ represents the approximation error introduced by the neural operator components.

2. The operator \mathcal{G}_θ can be trained to minimise:

$$\mathcal{L}(\theta) = \mathbb{E}_{b \sim \mu} [\|b - \mathcal{A}\mathcal{G}_\theta(b)\|_{\mathcal{Y}}^2] \quad (\text{E.13})$$

where μ is a probability measure on \mathcal{Y} .

3. If the activation function σ is Lipschitz continuous with constant L_σ and the kernel mapping ϕ has Lipschitz constant L_ϕ , then the residual at iteration m satisfies:

$$\|b - \mathcal{A}x_m\|_{\mathcal{Y}} \leq (1 + L_\sigma L_\phi)^m \cdot \min_{q \in \mathcal{V}_m} \|b - \mathcal{A}q\|_{\mathcal{Y}} \quad (\text{E.14})$$

where $\mathcal{V}_m = \text{span}\{\mathcal{G}_\theta(v_1), \mathcal{G}_\theta(v_2), \dots, \mathcal{G}_\theta(v_m)\}$.

Furthermore, if the neural operator \mathcal{G}_θ satisfies the universal approximation property in the Qian-Yu framework, then as the width of the network increases, $\delta_{\phi, \sigma} \rightarrow 0$, and the hybrid method converges to the solution of $\mathcal{A}x = b$ in the limit of sufficiently large m and network capacity [80].

E.6 Integration of GMRES into Qian-Yu Framework

Algorithm 26 Neural Operator Enhanced GMRES

```

1: Input: Operator  $\mathcal{A}$ , function  $b$ , initial guess  $x_0$ , tolerance  $\varepsilon$ , activation function  $\sigma$ , kernel
    $\phi$ 
2: Output: Approximate solution  $x$ 
3: Compute  $r_0 = b - \mathcal{A}x_0$ ,  $\beta = \|r_0\|_2$ ,  $v_1 = r_0/\beta$ 
4: Initialise Krylov subspace basis  $\mathcal{K}_1 = \{v_1\}$ 
5: for  $j = 1, 2, \dots, m$  do
6:   Apply kernel mapping:  $\tilde{v}_j = \phi(v_j)$ 
7:   Apply operator:  $w = \mathcal{A}\tilde{v}_j$ 
8:   Apply nonlinear activation:  $w = \sigma(w)$ 
9:   for  $i = 1, 2, \dots, j$  do
10:     $h_{i,j} = \langle w, \phi(v_i) \rangle_{\mathcal{H}}$                                  $\triangleright$  Inner product in feature space
11:     $w = w - h_{i,j}\phi(v_i)$ 
12:   end for
13:    $h_{j+1,j} = \|w\|_{\mathcal{H}}$ 
14:   if  $h_{j+1,j} < \varepsilon$  then
15:     Set  $m = j$  and go to line 15
16:   end if
17:    $v_{j+1} = w/h_{j+1,j}$ 
18:   Expand Krylov subspace:  $\mathcal{K}_{j+1} = \mathcal{K}_j \cup \{v_{j+1}\}$ 
19: end for
20: Define  $e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^{m+1}$ 
21: Find  $y_m$  that minimizes  $\|\beta e_1 - \bar{H}_m y\|_2$ 
22: Compute  $x_m = x_0 + \sum_{i=1}^m (y_m)_i \phi^{-1}(v_i)$                                  $\triangleright$  Map back from feature space
23: if  $\|b - \mathcal{A}x_m\|_2 < \varepsilon$  then
24:   return  $x = x_m$ 
25: else
26:   Set  $x_0 = x_m$  and go to line 3 (restart)
27: end if

```

A hybrid approach based on GMRES for non-linear feature mappings could generalise to non-linear operator equations. Integrating kernel or activation functions could preserve the convergence properties of GMRES. Similar hybrid techniques in machine learning and numerical analysis suggest that non-linear transformations are compatible with complex data — See Corollary E.2. The hybrid approach is promising but not without its limitations. Non-linear transformations can increase computational costs and may be impractical for large-scale problems. Theoretical foundations for convergence in non-linear cases are not fully established due to uncertainty of convergence of a polynomial and thus require experimental verification [80].

However, convergence guarantees are difficult to prove; therefore, theoretical analysis should avoid destabilisation. The choice of the kernel and the activation functions is critical, and they must be able to capture the data's non-linearities effectively. Moreover, operations in the feature space, such as inner products, must be computationally feasible. Inverse mapping ϕ^{-1} may introduce additional complexity or instability in the problem formulation [80].

E.7 Practical Implementation and Numerical Considerations

The theoretical connections between Krylov spaces and AR(x) neural networks have practical implications for implementation:

PROPOSITION E.1. (*Optimal Node Selection*) For a neural network with AR(x) activation functions, the optimal selection of interpolation nodes corresponds to the Chebyshev-Gauss-Lobatto points on the interval $[a, b]$:

$$x_k = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{k\pi}{n}\right), \quad k = 0, 1, \dots, n \quad (\text{E.15})$$

This choice minimises the Lebesgue constant associated with the Krylov interpolation process [80].

REMARK E.2. The implementation of Krylov-accelerated neural networks requires careful attention to numerical stability, particularly for large values of n . Orthogonalisation techniques such as the Arnoldi or Lanczos processes should be employed to maintain the numerical conditioning of the Krylov basis [78].

LEMMA E.2 (Lipschitz Stability). Let σ be an activation function with Lipschitz constant L_σ and ϕ be a kernel mapping with Lipschitz constant L_ϕ . For the hybrid GMRES iteration at step j :

$$w_j = \sigma(\mathcal{A}\phi(v_j)) \quad (\text{E.16})$$

If $\|v_j - \tilde{v}_j\| \leq \epsilon$ for some perturbation \tilde{v}_j , then:

$$\|w_j - \tilde{w}_j\| \leq L_\sigma L_\phi \|\mathcal{A}\| \epsilon \quad (\text{E.17})$$

where $\tilde{w}_j = \sigma(\mathcal{A}\phi(\tilde{v}_j))$.

REMARK E.3. *The kernel method integrated with Krylov subspaces and Lie group theory is a potent framework for solving nonlinear operator equations, as provided by Qian-Yu. Using a kernel mapping ϕ that respects Lie group symmetries to lift the standard Krylov subspace r_0, Ar_0, A^2r_0, \dots we obtain enhanced approximation capabilities while preserving important invariance properties. The kernel-induced inner products $K(x, y) = \langle \phi(x), \phi(y) \rangle$ that we define create a natural bridge between the Krylov framework and reproducing kernel Hilbert spaces. This unified approach offers multiple advantages: (i) better handling of nonlinear operators through implicit high-dimensional transformations (ii) computational efficiency via the kernel trick, (iii) direct manipulation of function spaces rather than discretised matrices (iv) better generalisation across problem variations, and (v) a cohesive theoretical framework that connects - a unified theoretical framework connecting reproducing kernel Hilbert spaces (RKHS) theory with Krylov methods and Lie group representations. Further investigation of this hybrid approach's convergence properties is an open question.*

If we can establish convergence properties in this enhanced framework utilising GMRES algorithm with kernel modifications, the sketch of the new algorithm might look like:

Algorithm 27 Kernel-Enhanced GMRES with Lie Group Symmetries

- 1: Define kernel function $K(x, y)$ respecting appropriate Lie group symmetries
 - 2: Compute $r_0 = b - \mathcal{A}x_0$, $\beta = \sqrt{K(r_0, r_0)}$, $v_1 = r_0/\beta$
 - 3: **for** $j = 1, 2, \dots, m$ **do**
 - 4: Compute $w = \mathcal{A}v_j$ (operator application)
 - 5: **for** $i = 1, 2, \dots, j$ **do**
 - 6: $h_{i,j} = K(w, v_i)$ (kernel inner product)
 - 7: $w = w - h_{i,j}v_i$ (kernel-space orthogonalization)
 - 8: **end for**
 - 9: $h_{j+1,j} = \sqrt{K(w, w)}$
 - 10: $v_{j+1} = w/h_{j+1,j}$
 - 11: **end for**
-

E.8 Future Work

The Krylov space interpretation of neural network interpolation with $AR(x)$ activation functions leads to some interesting research directions: The optimal approximation rate of neural networks with $AR(x)$ activation functions in Sobolev spaces $W^{k,p}$ is possible via a Krylov based training algorithm that picks the basis functions dynamically based on the spectral properties of the target function.

Therefore, this conjecture if true would confirm the theoretical optimality of Krylov based approaches to neural network training and thus offer a rigorous argument for the effectiveness of practical implementation strategies.

Based on the Krylov space interpretation of neural network interpolation with AR(x) activation functions, that is, hypothetical activations tailored to suit the Neural Network and rates of approximation technique of the target function, there are some interesting research directions:

1. *Extension to multivariate approximation via tensor product Krylov spaces.*
2. *The adaptive Krylov methods for neural network training.*
3. *A study of Krylov based preconditioning techniques for accelerating convergence.*
4. *A investigation of the relations with rational approximation theory using extended Krylov spaces.*

F Remaining Proofs

F.1 Proof of Theorem 1

THEOREM F.1 (Direct Approximation in $C[a,b]$). *Let $f \in C[a,b]$ and $\sigma \in \mathcal{A}(m)$. Then for the neural network interpolation operators $S_{n,\sigma}$, we have*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (\text{F.1})$$

where $h = \frac{b-a}{n}$ and $\omega(f, \delta)_{[a,b]}$ is the modulus of continuity of f on $[a,b]$.

Proof. We shall prove this theorem using the properties of the operators $S_{n,\sigma}$ and the definition of the modulus of continuity.

(a) First, recall the definition of $S_{n,\sigma}$:

$$S_{n,\sigma}(f, x) = \sum_{k=0}^n f(x_k) \varphi\left(\frac{2m}{h}(x-x_k)\right) \quad (\text{F.2})$$

where $x_k = a + kh$, $k = 0, 1, \dots, n$, and $\varphi(x) = \sigma(x+m) - \sigma(x-m)$.

(b) From Lemma 2 in the paper by Qian-Yu(2022) [77], we know that $S_{n,\sigma}$ satisfies the normalisation property:

$$S_{n,\sigma}(1, x) = 1, \quad \forall x \in [a, b] \quad (\text{F.3})$$

(c) Now, let $x \in [x_i, x_{i+1}]$ for some $i \in \{0, 1, \dots, n-1\}$. By the properties of φ , we have:

$$\varphi\left(\frac{2m}{h}(x-x_k)\right) = 0, \quad \text{for } k \neq i, i+1 \quad (\text{F.4})$$

(d) Using these properties, we can write:

$$|S_{n,\sigma}(f, x) - f(x)| = \left| \sum_{k=0}^n (f(x_k) - f(x)) \varphi\left(\frac{2m}{h}(x - x_k)\right) \right| \quad (\text{F.5})$$

$$= \left| (f(x_i) - f(x)) \varphi\left(\frac{2m}{h}(x - x_i)\right) + (f(x_{i+1}) - f(x)) \varphi\left(\frac{2m}{h}(x - x_{i+1})\right) \right| \quad (\text{F.6})$$

(e) Applying the triangle inequality and that φ is non-negative and bounded by 1:

$$\begin{aligned} |S_{n,\sigma}(f, x) - f(x)| &\leq |f(x_i) - f(x)| \varphi\left(\frac{2m}{h}(x - x_i)\right) \\ &\quad + |f(x_{i+1}) - f(x)| \varphi\left(\frac{2m}{h}(x - x_{i+1})\right) \end{aligned} \quad (\text{F.7})$$

$$\begin{aligned} &\leq \max\{|f(x_i) - f(x)|, |f(x_{i+1}) - f(x)|\} \\ &\quad \cdot \left(\varphi\left(\frac{2m}{h}(x - x_i)\right) + \varphi\left(\frac{2m}{h}(x - x_{i+1})\right) \right) \end{aligned} \quad (\text{F.8})$$

(f) From Lemma 1 in the paper by Qian-Yu(2022) [77], we know that:

$$\varphi\left(\frac{2m}{h}(x - x_i)\right) + \varphi\left(\frac{2m}{h}(x - x_{i+1})\right) = 1 \quad (\text{F.9})$$

(g) Therefore,

$$|S_{n,\sigma}(f, x) - f(x)| \leq \max\{|f(x_i) - f(x)|, |f(x_{i+1}) - f(x)|\} \quad (\text{F.10})$$

(h) By the definition of the modulus of continuity, we have:

$$|f(x_i) - f(x)| \leq \omega(f, |x - x_i|)_{[a,b]} \leq \omega(f, h)_{[a,b]} \quad (\text{F.11})$$

$$|f(x_{i+1}) - f(x)| \leq \omega(f, |x - x_{i+1}|)_{[a,b]} \leq \omega(f, h)_{[a,b]} \quad (\text{F.12})$$

(i) Therefore,

$$|S_{n,\sigma}(f, x) - f(x)| \leq \omega(f, h)_{[a,b]} \quad (\text{F.13})$$

(j) Since this inequality holds for all $x \in [a, b]$, we can take the supremum over $[a, b]$ to obtain:

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (\text{F.14})$$

This completes the proof of Theorem 1. \square

Alternative Proof to Theorem 1:

Recall that Theorem 1 is given as follows:

THEOREM F.2 (Qian and Yu, 2022 [77]). *Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$. Then for the neural network interpolation operators $S_{n,\sigma}$, we have:*

$$\|S_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (\text{F.15})$$

where $h = \frac{b-a}{n}$ and $\omega(f, \delta)_{[a,b]}$ are the modulus of continuity of f on $[a, b]$.

THEOREM F.3 (Extension to Banach Spaces). *Let X be a Banach space, $K \subset X$ be a compact convex subset, and $f : K \rightarrow X$ be a continuous function. Let $\sigma \in \mathcal{A}(m)$. Then for the neural network interpolation operators $S_{n,\sigma}$, we have (Berens et al. (1972) [11], Lindenstrauss et al. (1977) [67]):*

$$\|S_{n,\sigma}(f) - f\|_X \leq \omega(f, h)_K \quad (\text{F.16})$$

where $h = \frac{\text{diam}(K)}{n}$ and $\omega(f, \delta)_K$ are the moduli of continuity of f on K .

Proof. Let $\{x_1, \dots, x_n\}$ be a set of points in K such that for any $x \in K$, there exists an x_i with $\|x - x_i\|_X \leq h$. Such a set exists due to the compactness of K .

Define $S_{n,\sigma}(f)$ as:

$$S_{n,\sigma}(f)(x) = \sum_{i=1}^n f(x_i) \varphi \left(\frac{\|x - x_i\|_X}{h} \right) \quad (\text{F.17})$$

where $\varphi(t) = \sigma(t+m) - \sigma(t-m)$. (Kernel function)

For any $x \in K$, let x_j be the nearest point to x among $\{x_1, \dots, x_n\}$. Then:

$$\|S_{n,\sigma}(f)(x) - f(x)\|_X = \left\| \sum_{i=1}^n (f(x_i) - f(x)) \varphi \left(\frac{\|x - x_i\|_X}{h} \right) \right\|_X \quad (\text{F.18})$$

$$\leq \sum_{i=1}^n \|f(x_i) - f(x)\|_X \varphi \left(\frac{\|x - x_i\|_X}{h} \right) \quad (\text{F.19})$$

$$\leq \omega(f, h)_K \sum_{i=1}^n \varphi \left(\frac{\|x - x_i\|_X}{h} \right) \quad (\text{F.20})$$

$$= \omega(f, h)_K \quad (\text{F.21})$$

The last equality follows from the partition of unity property of φ . \square

THEOREM F.4 (Extension to Metric Spaces). *Let (M, d) be a compact metric space and $f : M \rightarrow \mathbb{R}$ be a continuous function. Let $\sigma \in \mathcal{A}(m)$. Then for the neural network interpolation operators $S_{n,\sigma}$, we have:*

$$\|S_{n,\sigma}(f) - f\|_\infty \leq \omega(f, h)_M \quad (\text{F.22})$$

where $h = \frac{\text{diam}(M)}{n}$ and $\omega(f, \delta)_M$ is the modulus of continuity of f on M .

Proof. The proof is analogous to the Banach space case, but we utilise the metric d instead of the norm. Let $\{x_1, \dots, x_n\}$ be an h -net in M . Define:

$$S_{n,\sigma}(f)(x) = \sum_{i=1}^n f(x_i) \varphi\left(\frac{d(x, x_i)}{h}\right) \quad (\text{F.23})$$

The remainder of the proof follows as previously, utilising the properties of the metric d . \square

On the other hand, if we consider Hausdorff spaces, we need to modify our approach since we do not possess a metric. We shall utilise the concept of uniform spaces.

THEOREM F.5 (Extension to Hausdorff Spaces). *Let X be a compact Hausdorff space and $f : X \rightarrow \mathbb{R}$ be a continuous function. Let $\sigma \in \mathcal{A}(m)$. Then for any entourage V in the uniform structure of X , there exists a neural network interpolation operator $S_{V,\sigma}$ such that:*

$$\|S_{V,\sigma}(f) - f\|_\infty \leq \omega_V(f) \quad (\text{F.24})$$

where $\omega_V(f) = \sup\{|f(x) - f(y)| : (x, y) \in V\}$.

Proof. Since X is compact Hausdorff, it admits a unique uniform structure compatible with its topology. Let V be an entourage in this uniform structure.

By the compactness of X , there exists a finite set $\{x_1, \dots, x_n\}$ such that $\{V[x_i]\}_{i=1}^n$ covers X , where $V[x] = \{y \in X : (x, y) \in V\}$.

Let $\{\phi_i\}_{i=1}^n$ be a partition of unity subordinate to this cover. Define:

$$S_{V,\sigma}(f)(x) = \sum_{i=1}^n f(x_i) \sigma(\phi_i(x)) \quad (\text{F.25})$$

Then for any $x \in X$:

$$|S_{V,\sigma}(f)(x) - f(x)| = \left| \sum_{i=1}^n (f(x_i) - f(x)) \sigma(\phi_i(x)) \right| \quad (\text{F.26})$$

$$\leq \sum_{i=1}^n |f(x_i) - f(x)| \sigma(\phi_i(x)) \quad (\text{F.27})$$

$$\leq \omega_V(f) \sum_{i=1}^n \sigma(\phi_i(x)) \quad (\text{F.28})$$

$$= \omega_V(f) \quad (\text{F.29})$$

□

THEOREM F.6 (Extension to Locally Convex TVS). *Let X be a locally convex topological vector space, $K \subset X$ be a compact convex subset, and $f : K \rightarrow X$ be a continuous function. Let $\sigma \in \mathcal{A}(m)$. Then for any continuous seminorm p on X , there exists a neural network interpolation operator $S_{p,\sigma}$ such that:*

$$p(S_{p,\sigma}(f)(x) - f(x)) \leq \omega_p(f, h)_K, \quad (\text{F.30})$$

where $h = \sup_{x,y \in K} p(x-y)/n$ and $\omega_p(f, \delta)_K = \sup\{p(f(x) - f(y)) : x, y \in K, p(x-y) \leq \delta\}$.

Proof. Let $\{x_1, \dots, x_n\}$ be a set of points in K such that for any $x \in K$, there exists an x_i with $p(x - x_i) \leq h$. Define the interpolation operator:

$$S_{p,\sigma}(f)(x) = \sum_{i=1}^n f(x_i) \varphi\left(\frac{p(x - x_i)}{h}\right), \quad (\text{F.31})$$

where $h = \sup_{x,y \in K} p(x-y)/n$, and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous, non-negative function parameterised by $\sigma \in \mathcal{A}(m)$, satisfying:

- (a) $(\varphi) \subseteq [0, 1]$, so $\varphi(t) = 0$ for $t > 1$,
- (b) $\sum_{i=1}^n \varphi(p(x - x_i)/h) = 1$ for all $x \in K$ (partition of unity).

Since K is compact and p is continuous, let $d = \sup_{x,y \in K} p(x-y) < \infty$, so $h = d/n$. The set $\{x_1, \dots, x_n\}$ forms an h -net for K with respect to p , which exists by compactness.

For any $x \in K$, compute the error:

$$S_{p,\sigma}(f)(x) - f(x) = \sum_{i=1}^n f(x_i) \varphi\left(\frac{p(x-x_i)}{h}\right) - f(x) \quad (\text{F.32})$$

$$= \sum_{i=1}^n [f(x_i) - f(x)] \varphi\left(\frac{p(x-x_i)}{h}\right), \quad (\text{F.33})$$

since $\sum_{i=1}^n \varphi(p(x-x_i)/h) = 1$. Applying the seminorm p , which is subadditive and positively homogeneous:

$$\begin{aligned} p(S_{p,\sigma}(f)(x) - f(x)) &= p\left(\sum_{i=1}^n [f(x_i) - f(x)] \varphi\left(\frac{p(x-x_i)}{h}\right)\right) \\ &\leq \sum_{i=1}^n p(f(x_i) - f(x)) \varphi\left(\frac{p(x-x_i)}{h}\right). \end{aligned} \quad (\text{F.34})$$

Now, if $\varphi(p(x-x_i)/h) > 0$, then $p(x-x_i)/h \leq 1$, so $p(x-x_i) \leq h$. Since f is continuous on the compact set K , it is uniformly continuous with respect to p . The modulus of continuity gives:

$$p(f(x_i) - f(x)) \leq \omega_p(f, h)_K, \quad (\text{F.35})$$

where $\omega_p(f, h)_K = \sup\{p(f(x) - f(y)) : x, y \in K, p(x-y) \leq h\}$. Thus:

$$\begin{aligned} p(S_{p,\sigma}(f)(x) - f(x)) &\leq \sum_{i=1}^n \omega_p(f, h)_K \varphi\left(\frac{p(x-x_i)}{h}\right) \\ &= \omega_p(f, h)_K \sum_{i=1}^n \varphi\left(\frac{p(x-x_i)}{h}\right) \\ &= \omega_p(f, h)_K. \end{aligned} \quad (\text{F.36})$$

Since $S_{p,\sigma}(f)$ is continuous (as p and φ are continuous and the sum is finite) and takes values in X , the result follows:

$$p(S_{p,\sigma}(f)(x) - f(x)) \leq \omega_p(f, h)_K \quad (\text{F.37})$$

□

F.2 Proof of Theorem 2

THEOREM F.7 (Rate of Approximation for Continuous Functions). *Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$. Then for the Kantorovich-type neural network operators $K_{n,\sigma}$, we have*

$$\|K_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (\text{F.38})$$

where $h = \frac{b-a}{n}$ and $\omega(f, \delta)_{[a,b]}$ are the modulus of continuity of f on $[a, b]$.

Proof. We will prove this theorem using the properties of the operators $K_{n,\sigma}$ and the definition of the modulus of continuity.

1) Recall the definition of $K_{n,\sigma}$:

$$K_{n,\sigma}(f, x) = \frac{n+1}{b-a} \sum_{k=0}^n \int_{y_k}^{y_{k+1}} f(t) dt \varphi\left(\frac{2m}{h}(x-x_k)\right) \quad (\text{F.39})$$

where $y_k = a + \frac{(b-a)k}{n+1}$, $k = 0, 1, \dots, n+1$, and $\varphi(x) = \sigma(x+m) - \sigma(x-m)$.

2) From Lemma 2 in the paper by Qian-Yu(2022) [77], we know that $K_{n,\sigma}$ satisfies the normalisation property:

$$K_{n,\sigma}(1, x) = 1, \quad \forall x \in [a, b] \quad (\text{F.40})$$

3) Let $x \in [x_i, x_{i+1}]$ for some $i \in \{0, 1, \dots, n-1\}$. By the properties of φ , we have:

$$\varphi\left(\frac{2m}{h}(x-x_k)\right) = 0, \quad \text{for } k \neq i, i+1 \quad (\text{F.41})$$

4) Using these properties, we can write:

$$|K_{n,\sigma}(f, x) - f(x)| = \left| \frac{n+1}{b-a} \sum_{k=0}^n \int_{y_k}^{y_{k+1}} (f(t) - f(x)) dt \varphi\left(\frac{2m}{h}(x-x_k)\right) \right| \quad (\text{F.42})$$

$$= \frac{n+1}{b-a} \left| \int_{y_i}^{y_{i+1}} (f(t) - f(x)) dt \varphi\left(\frac{2m}{h}(x-x_i)\right) \right| \quad (\text{F.43})$$

$$+ \left| \int_{y_{i+1}}^{y_{i+2}} (f(t) - f(x)) dt \varphi\left(\frac{2m}{h}(x-x_{i+1})\right) \right| \quad (\text{F.44})$$

5) Using the triangle inequality and φ 's non-negativity (bounded by 1):

$$|K_{n,\sigma}(f, x) - f(x)| \leq \frac{n+1}{b-a} \left(\left| \int_{y_i}^{y_{i+1}} (f(t) - f(x)) dt \right| \varphi \left(\frac{2m}{h} (x - x_i) \right) \right) \quad (\text{F.45})$$

$$+ \left| \int_{y_{i+1}}^{y_{i+2}} (f(t) - f(x)) dt \right| \varphi \left(\frac{2m}{h} (x - x_{i+1}) \right) \quad (\text{F.46})$$

$$\leq \frac{n+1}{b-a} \max \left\{ \left| \int_{y_i}^{y_{i+1}} (f(t) - f(x)) dt \right|, \left| \int_{y_{i+1}}^{y_{i+2}} (f(t) - f(x)) dt \right| \right\} \quad (\text{F.47})$$

$$\times \left(\varphi \left(\frac{2m}{h} (x - x_i) \right) + \varphi \left(\frac{2m}{h} (x - x_{i+1}) \right) \right) \quad (\text{F.48})$$

6) From Lemma 1 in the paper ([77]), we know that:

$$\varphi \left(\frac{2m}{h} (x - x_i) \right) + \varphi \left(\frac{2m}{h} (x - x_{i+1}) \right) = 1 \quad (\text{F.49})$$

7) Therefore,

$$|K_{n,\sigma}(f, x) - f(x)| \leq \frac{n+1}{b-a} \max \left\{ \left| \int_{y_i}^{y_{i+1}} (f(t) - f(x)) dt \right|, \left| \int_{y_{i+1}}^{y_{i+2}} (f(t) - f(x)) dt \right| \right\} \quad (\text{F.50})$$

8) By the definition of the modulus of continuity, for any $t \in [y_k, y_{k+1}]$, we have:

$$|f(t) - f(x)| \leq \omega(f, |t - x|)_{[a,b]} \leq \omega(f, h)_{[a,b]} \quad (\text{F.51})$$

since $|t - x| \leq \max\{|y_{k+1} - x_i|, |y_{k+2} - x_{i+1}|\} \leq h$.

9) Using this, we can bound the integrals:

$$\left| \int_{y_k}^{y_{k+1}} (f(t) - f(x)) dt \right| \leq \int_{y_k}^{y_{k+1}} |f(t) - f(x)| dt \leq \frac{b-a}{n+1} \omega(f, h)_{[a,b]} \quad (\text{F.52})$$

10) Substituting this back into our inequality:

$$|K_{n,\sigma}(f, x) - f(x)| \leq \frac{n+1}{b-a} \cdot \frac{b-a}{n+1} \omega(f, h)_{[a,b]} = \omega(f, h)_{[a,b]} \quad (\text{F.53})$$

11) Since the inequality holds for all $x \in [a, b]$, taking the supremum over $[a, b]$ yields:

$$\|K_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (\text{F.54})$$

This concludes the proof of Theorem 2. □

Alternative Proof to Theorem 2:

Recall Theorem 2 as follows:

THEOREM F.8 (Qian and Yu, 2022 [77]). *Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$. Then for the Kantorovich-type neural network operators $K_{n,\sigma}$, we have:*

$$\|K_{n,\sigma}(f) - f\| \leq \omega(f, h)_{[a,b]} \quad (\text{F.55})$$

where $h = \frac{b-a}{n}$ and $\omega(f, \delta)_{[a,b]}$ are the modulus of continuity of f on $[a, b]$.

THEOREM F.9 (Extension to Banach-Valued Functions). *Let (X, Σ, μ) be a finite measure space, Y be a Banach space, and $f : X \rightarrow Y$ be a strongly measurable function such that $\|f\|_{L^p(X,Y)} < \infty$ for some $1 \leq p < \infty$. Let $\sigma \in \mathcal{A}(m)$. Then for the Kantorovich-type neural network operators $K_{n,\sigma}$, we have:*

$$\|K_{n,\sigma}(f) - f\|_{L^p(X,Y)} \leq C \omega_p(f, 1/n)_X \quad (\text{F.56})$$

where C is a constant depending only on σ and p , and $\omega_p(f, \delta)_X$ is the L^p -modulus of continuity defined as:

$$\omega_p(f, \delta)_X = \sup_{|t| \leq \delta} \left(\int_X \|f(x+t) - f(x)\|_Y^p d\mu(x) \right)^{1/p} \quad (\text{F.57})$$

Proof. Define the Kantorovich-type operator $K_{n,\sigma}$ as:

$$K_{n,\sigma}(f)(x) = n \sum_{k=1}^n \int_{(k-1)/n}^{k/n} f(t) dt \varphi(n(x - k/n)) \quad (\text{F.58})$$

where $\varphi(t) = \sigma(t+m) - \sigma(t-m)$.

Step 1: Decomposition

$$\|K_{n,\sigma}(f) - f\|_{L^p(X,Y)}^p = \int_X \|K_{n,\sigma}(f)(x) - f(x)\|_Y^p d\mu(x) \quad (\text{F.59})$$

$$\leq C \int_X \left\| n \sum_{k=1}^n \int_{(k-1)/n}^{k/n} (f(t) - f(x)) dt \varphi(n(x - k/n)) \right\|_Y^p d\mu(x) \quad (\text{F.60})$$

Step 2: Jensen's inequality

$$\leq Cn^p \int_X \left(\sum_{k=1}^n \int_{(k-1)/n}^{k/n} \|f(t) - f(x)\|_Y dt \varphi(n(x - k/n)) \right)^p d\mu(x) \quad (\text{F.61})$$

$$\leq Cn^p \int_X \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \|f(t) - f(x)\|_Y^p dt \varphi(n(x - k/n)) d\mu(x) \quad (\text{F.62})$$

Step 3: Change of variables

$$= Cn^{p-1} \int_X \int_0^1 \|f(x + s/n) - f(x)\|_Y^p ds d\mu(x) \quad (\text{F.63})$$

$$\leq C\omega_p(f, 1/n)_X^p \quad (\text{F.64})$$

Taking the p -th root completes the proof. \square

THEOREM F.10 (Extension to Locally Convex Spaces). *Let X and Y be locally convex topological vector spaces, with X complete and separable. Let $f : X \rightarrow Y$ be a continuous function, and let P be a family of continuous seminorms generating the topology of Y . Let $\sigma \in \mathcal{A}(m)$. Then for the Kantorovich-type neural network operators $K_{n,\sigma}$, we have:*

$$p(K_{n,\sigma}(f)(x) - f(x)) \leq C\omega_p(f, 1/n)_X \quad (\text{F.65})$$

for all $p \in P$, where C is a constant depending only on σ , and $\omega_p(f, \delta)_X$ is the modulus of continuity with respect to p :

$$\omega_p(f, \delta)_X = \sup_{d(x,y) \leq \delta} p(f(x) - f(y)) \quad (\text{F.66})$$

with d being a compatible metric on X .

THEOREM F.11 (Extension to Fréchet Spaces). *Let X be a Fréchet space with a family of seminorms $\{p_k\}_{k=1}^\infty$ generating its topology. Let $f : X \rightarrow X$ be a continuous function. Let $\sigma \in \mathcal{A}(m)$. Then for the Kantorovich-type neural network operators $K_{n,\sigma}$, we have:*

$$p_k(K_{n,\sigma}(f)(x) - f(x)) \leq C_k \omega_{p_k}(f, 1/n)_X \quad (\text{F.67})$$

for all $k \in \mathbb{N}$, where C_k is a constant depending on σ and k , and $\omega_{p_k}(f, \delta)_X$ is the modulus of continuity with respect to p_k :

$$\omega_{p_k}(f, \delta)_X = \sup_{p_k(x-y) \leq \delta} p_k(f(x) - f(y)) \quad (\text{F.68})$$

F.3 Proof of Theorem 3

THEOREM F.12 (Inverse Theorem). *Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$ with bounded derivative. If for some $0 < \alpha < 1$,*

$$\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha}), \quad (\text{F.69})$$

then

$$\omega(f, \delta)_{[a,b]} = O(\delta^\alpha), \quad \text{as } \delta \rightarrow 0^+. \quad (\text{F.70})$$

Proof. We will prove this theorem using the K-functional and the Berens-Lorentz lemma. The proof consists of several steps:

1) First, recall the definition of the K-functional:

$$K(f, t) = \inf_{g \in AC[a,b], \|g'\| < \infty} \{\|f - g\| + t\|g'\|\} \quad (\text{F.71})$$

where $AC[a, b]$ is the space of absolutely continuous functions on $[a, b]$.

2) From the theory of K -functionals, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \omega(f, t)_{[a,b]} \leq K(f, t) \leq C_2 \omega(f, t)_{[a,b]} \quad (\text{F.72})$$

3) By the definition of the K -functional, for any $\varepsilon > 0$, there exists $g \in AC[a, b]$ such that

$$\|f - g\| + \frac{1}{k}\|g'\| \leq K(f, \frac{1}{k}) + \varepsilon \quad (\text{F.73})$$

4) Now, let's consider $S_{n,\sigma}(f) - f$. We can write:

$$S_{n,\sigma}(f) - f = S_{n,\sigma}(f - g) + S_{n,\sigma}(g) - g + g - f \quad (\text{F.74})$$

5) Taking norms and using the triangle inequality:

$$\|S_{n,\sigma}(f) - f\| \leq \|S_{n,\sigma}(f - g)\| + \|S_{n,\sigma}(g) - g\| + \|g - f\| \quad (\text{F.75})$$

6) From Lemma 3 in the paper by Qian-Yu(2022) [77], we have the Bernstein-type inequality:

$$\|S'_{n,\sigma}(f)\| \leq \frac{4m\|\varphi'\|}{h} \|f\| \quad (\text{F.76})$$

where $h = \frac{b-a}{n}$.

7) Using this inequality and the mean value theorem, we can bound $\|S_{n,\sigma}(g) - g\|$:

$$\|S_{n,\sigma}(g) - g\| \leq \frac{b-a}{n} \|S'_{n,\sigma}(g) - g'\| \leq \frac{b-a}{n} \left(\frac{4m\|\varphi'\|}{h} \|g\| + \|g'\| \right) \quad (\text{F.77})$$

8) Substituting these bounds into our earlier inequality:

$$\|S_{n,\sigma}(f) - f\| \leq (1+C)\|f-g\| + C\frac{1}{n}\|g'\| \quad (\text{F.78})$$

for some constant $C > 0$.

9) Now, choose $k = n$ in the inequality from step 3:

$$\|S_{n,\sigma}(f) - f\| \leq (1+C) \left(K(f, \frac{1}{n}) + \varepsilon \right) + C \left(K(f, \frac{1}{n}) + \varepsilon \right) \quad (\text{F.79})$$

10) Letting $\varepsilon \rightarrow 0$, we obtain:

$$K(f, \frac{1}{n}) \leq C_3 \|S_{n,\sigma}(f) - f\| + C_4 \frac{1}{n} K(f, \frac{1}{n}) \quad (\text{F.80})$$

for some constants $C_3, C_4 > 0$.

11) By our assumption, $\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha})$, so we have:

$$K(f, \frac{1}{n}) \leq C_5 n^{-\alpha} + C_4 \frac{1}{n} K(f, \frac{1}{n}) \quad (\text{F.81})$$

12) Now we can apply the Berens-Lorentz lemma (Lemma 5 in the paper by Qian-Yu (2022) [77]).

Let $\psi(n) = K(f, \frac{1}{n})$.

Then:

$$\psi(n) \leq A n^{-\alpha} + M \left(\frac{k}{n} \right)^r \psi(k) \quad (\text{F.82})$$

with $r = 1 > \alpha$.

13) The Berens-Lorentz lemma then implies that:

$$K(f, \frac{1}{n}) = O(n^{-\alpha}) \quad (\text{F.83})$$

14) Finally, using the equivalence between the K-functional and the modulus of continu-

ity:

$$\omega(f, \frac{1}{n})_{[a,b]} = O(n^{-\alpha}) \quad (\text{F.84})$$

15) Substituting $\delta = \frac{1}{n}$, we obtain the desired result:

$$\omega(f, \delta)_{[a,b]} = O(\delta^\alpha), \quad \text{as } \delta \rightarrow 0^+ \quad (\text{F.85})$$

This completes the proof of Theorem 3. \square

Recall our original Theorem 3:

THEOREM F.13 (Qian and Yu, 2022 [77]). *Let $f \in C[a,b]$ and $\sigma \in \mathcal{A}(m)$ with a bounded derivative. If*

$$\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha}), \quad 0 < \alpha < 1, \quad (\text{F.86})$$

then

$$\omega(f, \delta)_{[a,b]} = O(\delta^\alpha), \quad \text{as } \delta \rightarrow 0^+. \quad (\text{F.87})$$

We shall now prove this theorem using advanced concepts from various areas of mathematics.

Here, we begin with the reformulation of our problem in terms of Lagrangian mechanics (Wang et al., 2022) [85].

DEFINITION F.1 (Lagrangian Neural Network Functional). *Define the Lagrangian Neural Network Functional $\mathcal{L}_n : C[a,b] \rightarrow \mathbb{R}$ as:*

$$\mathcal{L}_n(f) = \|S_{n,\sigma}(f) - f\|^2 + \lambda \int_a^b \int_a^b \frac{|f(x) - f(y)|^2}{|x - y|^{1+2\alpha}} dx dy \quad (\text{F.88})$$

where $\lambda > 0$ is a regularisation parameter and $0 < \alpha < 1$.

This formulation allows us to view the approximation problem as a variational principle in function space, incorporating both the approximation error and a measure of the function's smoothness.

DEFINITION F.2 (Besov Space). *For $0 < s < 1$ and $1 \leq p, q \leq \infty$, the Besov space $B_{p,q}^s[a,b]$ is defined as the set of functions $f \in L^p[a,b]$ such that*

$$\|f\|_{B_{p,q}^s} = \|f\|_{L^p} + \left(\int_0^1 (t^{-s} \omega(f, t)_p)^q \frac{dt}{t} \right)^{1/q} < \infty \quad (\text{F.89})$$

with the usual modification for $q = \infty$.

LEMMA F.1 (Smooth Approximation in Besov Spaces). *For any $f \in B_{\infty,\infty}^{\alpha}[a,b]$ and $\varepsilon > 0$, there exists a smooth function $g_{\varepsilon} \in C^{\infty}[a,b]$ such that:*

$$\|f - g_{\varepsilon}\| \leq \varepsilon \quad \text{and} \quad \|g_{\varepsilon}\|_{B_{\infty,\infty}^{\alpha}} \leq C \|f\|_{B_{\infty,\infty}^{\alpha}} \quad (\text{F.90})$$

where C is a constant independent of f and ε .

Proof. Utilise mollification techniques and properties of Besov spaces. \square

We shall utilise discretisation to connect our continuous problem to a finite-dimensional one, and then apply Galois theory to analyse the structure of the resultant system.

DEFINITION F.3 (Discretised Operator). *Define the discretised operator $D_n : C[a,b] \rightarrow \mathbb{R}^n$ as:*

$$D_n(f) = (f(x_1), \dots, f(x_n)) \quad (\text{F.91})$$

where $\{x_i\}_{i=1}^n$ are the nodes used in $S_{n,\sigma}$.

LEMMA F.2 (Galois Connection). *There exists a Galois connection between the lattice of subspaces of $C[a,b]$ and the lattice of subspaces of \mathbb{R}^n , given by:*

$$F : Sub(C[a,b]) \rightarrow Sub(\mathbb{R}^n), \quad F(V) = D_n(V) \quad (\text{F.92})$$

$$G : Sub(\mathbb{R}^n) \rightarrow Sub(C[a,b]), \quad G(W) = D_n^{-1}(W) \quad (\text{F.93})$$

The main theorem is proven as follows:

Proof of Theorem 3.

(a) Step 1: Lagrangian Formulation:

Consider the minimisation problem:

$$\min_f \mathcal{L}_n(f) \quad (\text{F.94})$$

The solution to this problem balances the approximation error with the smoothness of f .

(b) Step 2: Besov Space Embedding:

By the assumption $\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha})$, we have:

$$\mathcal{L}_n(f) \leq Cn^{-2\alpha} + \lambda \int_a^b \int_a^b \frac{|f(x) - f(y)|^2}{|x - y|^{1+2\alpha}} dx dy \quad (\text{F.95})$$

(c) Step 3: Smooth Approximation:

For $\varepsilon = n^{-\alpha}$, choose g_ε as in the Smooth Approximation Lemma. Then:

$$\|f - g_\varepsilon\| \leq n^{-\alpha} \quad \text{and} \quad \|g_\varepsilon\|_{B_{\infty,\infty}^\alpha} \leq C\|f\|_{B_{\infty,\infty}^\alpha} \quad (\text{F.96})$$

(d) Step 4: Discretisation:

Applying the discretised operator D_n to both f and g_ε :

$$\|D_n(f) - D_n(g_\varepsilon)\|_{\ell^\infty} \leq n^{-\alpha} \quad (\text{F.97})$$

(e) Step 5: Galois Connection:

According to the Galois connection to transfer discretised estimates to a continuous setting:

$$\|f - G(D_n(g_\varepsilon))\| \leq n^{-\alpha} \quad (\text{F.98})$$

(f) Step 6: Besov Space Characterisation:

By the properties of Besov spaces, we have:

$$\omega(f, \delta)_\infty \leq C\delta^\alpha \|f\|_{B_{\infty,\infty}^\alpha} \quad (\text{F.99})$$

(g) Step 7: Combine Estimates:

From Steps 2, 5, and 6:

$$\omega(f, \delta)_\infty \leq C\delta^\alpha (\|f - G(D_n(g_\varepsilon))\|_{B_{\infty,\infty}^\alpha} + \|G(D_n(g_\varepsilon))\|_{B_{\infty,\infty}^\alpha}) \quad (\text{F.100})$$

(h) Step 8: Conclude the proof:

Using the properties of G and D_n , we can show that:

$$\|G(D_n(g_\varepsilon))\|_{B_{\infty,\infty}^\alpha} \leq C\|g_\varepsilon\|_{B_{\infty,\infty}^\alpha} \leq C\|f\|_{B_{\infty,\infty}^\alpha} \quad (\text{F.101})$$

Therefore,

$$\omega(f, \delta)_\infty = O(\delta^\alpha) \quad (\text{F.102})$$

□

Alternative Proof of Theorem 3:

Recall, Theorem 3 as follows:

THEOREM F.14 (Qian and Yu, 2022 [77]). Let $f \in C[a, b]$ and $\sigma \in \mathcal{A}(m)$ with a bounded derivative. If

$$\|S_{n,\sigma}(f) - f\| = O(n^{-\alpha}) \quad (\text{F.103})$$

for some $0 < \alpha < 1$, then

$$\omega(f, \delta)_{[a,b]} = O(\delta^\alpha), \quad \text{as } \delta \rightarrow 0^+. \quad (\text{F.104})$$

THEOREM F.15 (Extension to Banach Space-Valued Functions). Let X be a Banach space, $K \subset \mathbb{R}^d$ be a compact convex set, and $f : K \rightarrow X$ be a continuous function. Let $\sigma \in \mathcal{A}(m)$ with a bounded derivative. If

$$\|S_{n,\sigma}(f) - f\|_{C(K,X)} = O(n^{-\alpha}) \quad (\text{F.105})$$

for some $0 < \alpha < 1$, then $f \in \Lambda^\alpha(K, X)$, i.e., f is Hölder continuous with exponent α :

$$\|f(x) - f(y)\|_X \leq C\|x - y\|^\alpha, \quad \forall x, y \in K \quad (\text{F.106})$$

for some constant $C > 0$.

Proof. We'll use the K-functional method to prove this theorem.

Step 1: Define the K-functional:

For $t > 0$, define:

$$K(f, t) = \inf_{g \in C^1(K, X)} \{\|f - g\|_{C(K, X)} + t\|g'\|_{C(K, X)}\} \quad (\text{F.107})$$

Step 2: Estimate $K(f, t)$:

Choose n such that $\frac{1}{n+1} < t \leq \frac{1}{n}$. Let $g_n = S_{n,\sigma}(f)$. Then:

$$K(f, t) \leq \|f - g_n\|_{C(K,X)} + t\|g'_n\|_{C(K,X)} \quad (\text{F.108})$$

$$\leq Cn^{-\alpha} + t\|S'_{n,\sigma}(f)\|_{C(K,X)} \quad (\text{F.109})$$

$$\leq Cn^{-\alpha} + tCn\|f\|_{C(K,X)} \quad (\text{F.110})$$

$$\leq C_1 n^{-\alpha} + C_2 n^{-1} \quad (\text{F.111})$$

$$\leq C_3 t^\alpha \quad (\text{F.112})$$

Step 3: Use Marchaud's inequality:

For Banach space-valued functions, Marchaud's inequality states:

$$\omega(f, \delta) \leq C\delta^\alpha \int_\delta^1 \frac{K(f, t)}{t^{1+\alpha}} dt + C\delta \int_1^\infty \frac{K(f, t)}{t^2} dt \quad (\text{F.113})$$

Step 4: Estimate the integrals:

Using the estimate from Step 2:

$$\int_\delta^1 \frac{K(f, t)}{t^{1+\alpha}} dt \leq C_3 \int_\delta^1 t^{-1} dt = C_3 |\log \delta| \quad (\text{F.114})$$

$$\int_1^\infty \frac{K(f, t)}{t^2} dt \leq C_3 \int_1^\infty t^{\alpha-2} dt = \frac{C_3}{1-\alpha} \quad (\text{F.115})$$

Step 5: To conclude:

Combining these estimates:

$$\omega(f, \delta) \leq C\delta^\alpha |\log \delta| + C\delta \quad (\text{F.116})$$

which implies $f \in \Lambda^\alpha(K, X)$. □

THEOREM F.16 (Extension to Locally Convex Spaces). *Let X and Y be locally convex topological vector spaces, with X complete and separable, and Y a Fréchet space with a family of seminorms $\{p_k\}_{k=1}^\infty$. Let $f : X \rightarrow Y$ be a continuous function, and $\sigma \in \mathcal{A}(m)$ with bounded derivative. If for each k ,*

$$p_k(S_{n,\sigma}(f)(x) - f(x)) = O(n^{-\alpha_k}) \quad (\text{F.117})$$

uniformly in x , for some $0 < \alpha_k < 1$, then for each k , f satisfies a Hölder condition with respect to p_k :

$$p_k(f(x) - f(y)) \leq C_k d(x, y)^{\alpha_k}, \quad \forall x, y \in X \quad (\text{F.118})$$

where d is a compatible metric on X .

Proof. The proof follows a similar structure to the Banach space case, but we must handle each seminorm p_k separately.

(a) Step 1: Define K-functionals:

For each k and $t > 0$, define:

$$K_k(f, t) = \inf_{g \in C^1(X, Y)} \left\{ \sup_{x \in X} p_k(f(x) - g(x)) + t \sup_{x \in X} p_k(g'(x)) \right\} \quad (\text{F.119})$$

(b) Step 2: Estimate $K_k(f, t)$:

Choose n such that $\frac{1}{n+1} < t \leq \frac{1}{n}$. Let $g_n = S_{n,\sigma}(f)$. Then:

$$K_k(f, t) \leq \sup_{x \in X} p_k(f(x) - g_n(x)) + t \sup_{x \in X} p_k(g'_n(x)) \quad (\text{F.120})$$

$$\leq C n^{-\alpha_k} + t C n \sup_{x \in X} p_k(f(x)) \quad (\text{F.121})$$

$$\leq C_1 n^{-\alpha_k} + C_2 n^{-1} \quad (\text{F.122})$$

$$\leq C_3 t^{\alpha_k} \quad (\text{F.123})$$

$$(\text{F.124})$$

(c) Step 3: Use a generalised Marchaud's inequality:

For locally convex space-valued functions, we can use a generalised version of Mar-

chaud's inequality:

$$\omega_k(f, \delta) \leq C\delta^{\alpha_k} \int_{\delta}^1 \frac{K_k(f, t)}{t^{1+\alpha_k}} dt + C\delta \int_1^{\infty} \frac{K_k(f, t)}{t^2} dt \quad (\text{F.125})$$

where $\omega_k(f, \delta) = \sup_{d(x,y) \leq \delta} p_k(f(x) - f(y))$.

(d) Steps 4 and 5: Estimate the integrals and conclude:

These steps proceed similarly to the Banach space case, leading to the conclusion that for each k :

$$\omega_k(f, \delta) \leq C\delta^{\alpha_k} |\log \delta| + C\delta \quad (\text{F.126})$$

which implies the desired Hölder condition with respect to each p_k .

□

THEOREM F.17 (Extension to Nuclear Spaces). *Let X be a nuclear Fréchet space with a family of Hilbertian seminorms $\{\|\cdot\|_k\}_{k=1}^{\infty}$, and let $f : X \rightarrow X$ be a continuous function. Let $\sigma \in \mathcal{A}(m)$ have a bounded derivative. If for each k ,*

$$\|S_{n,\sigma}(f)(x) - f(x)\|_k = O(n^{-\alpha_k}) \quad (\text{F.127})$$

uniformly in x , for some $0 < \alpha_k < 1$, then f is a Hölder-nuclear map, i.e., for each k , there exists $j > k$ such that:

$$\|f(x) - f(y)\|_k \leq C_{k,j} \|x - y\|_j^{\alpha_k}, \quad \forall x, y \in X \quad (\text{F.128})$$

Proof. The proof structure is similar to the locally convex case, but we exploit the nuclear structure of X .

Step 1: Using nuclear decomposition:

For each pair $k < j$, there exists a nuclear operator $T_{k,j} : X_j \rightarrow X_k$ (where X_i is the Hilbert space completion of X with respect to $\|\cdot\|_i$) such that the inclusion $X_j \rightarrow X_k$ factors through $T_{k,j}$.

Step 2: Define K-functionals:

For each $k, j > k$, and $t > 0$, define:

$$K_{k,j}(f, t) = \inf_{g \in C^1(X, X)} \left\{ \sup_{x \in X} \|f(x) - g(x)\|_k + t \sup_{x \in X} \|T_{k,j}g'(x)\|_{HS} \right\} \quad (\text{F.129})$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm:

Step 3: Estimate $K_{k,j}(f, t)$:

This step proceeds similarly to the locally convex case, leading to:

$$K_{k,j}(f, t) \leq C_3 t^{\alpha_k} \quad (\text{F.130})$$

Step 4: Use a nuclear version of Marchaud's inequality:

$$\omega_{k,j}(f, \delta) \leq C\delta^{\alpha_k} \int_{\delta}^1 \frac{K_{k,j}(f, t)}{t^{1+\alpha_k}} dt + C\delta \int_1^{\infty} \frac{K_{k,j}(f, t)}{t^2} dt \quad (\text{F.131})$$

where $\omega_{k,j}(f, \delta) = \sup_{\|x-y\|_j \leq \delta} \|f(x) - f(y)\|_k$.

Steps 5 and 6: Estimate the integrals and conclude:

These steps lead to:

$$\omega_{k,j}(f, \delta) \leq C\delta^{\alpha_k} |\log \delta| + C\delta \quad (\text{F.132})$$

which implies the desired Hölder-nuclear condition. \square

F.4 Proof of Theorem 4

THEOREM F.18 (Simultaneous Approximation). *Assume that $\sigma \in \mathcal{A}(m)$, $\varphi^{(r)}$ is bounded in \mathbb{R} . If $f \in C^r[a, b]$, then for the neural network operators $S_{n,r,\sigma}$, we have:*

$$(i) \quad S_{n,r,\sigma}(f, x_i) = f(x_i),$$

$$(ii) \quad S_{n,r,\sigma}^{(v)}(f, x_i) = f^{(v)}(x_i),$$

for $i = 0, 1, \dots, n$, $v = 1, 2, \dots, r$.

Proof. We prove this theorem in two parts, corresponding to the two claims.

Part 1: Proof of $S_{n,r,\sigma}(f, x_i) = f(x_i)$

(a) Recall the definition of $S_{n,r,\sigma}$:

$$S_{n,r,\sigma}(f, x) = \sum_{j=0}^r G_{j,n}(f, x) \quad (\text{F.133})$$

where

$$G_{j,n}(f, x) = \sum_{k=0}^n C_{k,j} U_j \left(\frac{2m}{h} (x - x_k) \right) \quad (\text{F.134})$$

with $C_{k,j} = \frac{h^j}{(2m)^j j!} f^{(j)}(x_k)$ and $U_j(x) = x^j \varphi(x)$.

(b) At $x = x_i$, we have:

$$U_j \left(\frac{2m}{h} (x_i - x_k) \right) = \begin{cases} 0, & k \neq i \\ 0, & k = i \text{ and } j > 0 \\ 1, & k = i \text{ and } j = 0 \end{cases} \quad (\text{F.135})$$

(c) Therefore,

$$S_{n,r,\sigma}(f, x_i) = G_{0,n}(f, x_i) = C_{i,0} = f(x_i) \quad (\text{F.136})$$

Part 2: Proof of $S_{n,r,\sigma}^{(v)}(f, x_i) = f^{(v)}(x_i)$

(a) We need to compute the v -th derivative of $S_{n,r,\sigma}(f, x)$:

$$S_{n,r,\sigma}^{(v)}(f, x) = \sum_{j=0}^r G_{j,n}^{(v)}(f, x) \quad (\text{F.137})$$

(b) Let's consider the v -th derivative of $G_{j,n}(f, x)$:

$$G_{j,n}^{(v)}(f, x) = \sum_{k=0}^n C_{k,j} \left(\frac{2m}{h} \right)^v U_j^{(v)} \left(\frac{2m}{h} (x - x_k) \right) \quad (\text{F.138})$$

(c) At $x = x_i$, we have:

$$U_j^{(v)} \left(\frac{2m}{h} (x_i - x_k) \right) = \begin{cases} 0, & k \neq i \\ 0, & k = i \text{ and } j \neq v \\ v! \left(\frac{h}{2m} \right)^v, & k = i \text{ and } j = v \end{cases} \quad (\text{F.139})$$

(d) Therefore,

$$G_{j,n}^{(v)}(f, x_i) = \begin{cases} 0, & j \neq v \\ C_{i,v} v! \left(\frac{2m}{h} \right)^v, & j = v \end{cases} \quad (\text{F.140})$$

(e) Substituting this back into the expression for $S_{n,r,\sigma}^{(v)}(f, x_i)$:

$$S_{n,r,\sigma}^{(v)}(f, x_i) = C_{i,v} v! \left(\frac{2m}{h} \right)^v \quad (\text{F.141})$$

(f) Recalling the definition of $C_{i,v}$:

$$S_{n,r,\sigma}^{(v)}(f, x_i) = \frac{h^v}{(2m)^v v!} f^{(v)}(x_i) v! \left(\frac{2m}{h} \right)^v = f^{(v)}(x_i) \quad (\text{F.142})$$

This concludes the proof of both parts of the theorem. \square

Proof Theorems and Neural Interpolators

THEOREM F.19 (Universal Approximation Theorem - Class D). *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded, non-constant, continuous activation function. For any $f \in C([0, 1]^d)$ and $\epsilon > 0$, there exists a neural network $N_{n,\sigma}$ of the form*

$$N_{n,\sigma}(x) = \sum_{i=1}^n c_i \sigma(w_i \cdot x + b_i) \quad (\text{F.143})$$

such that

$$\sup_{x \in [0, 1]^d} |f(x) - N_{n,\sigma}(x)| < \epsilon \quad (\text{F.144})$$

where $n \in \mathbb{N}$, $c_i \in \mathbb{R}$, $w_i \in \mathbb{R}^d$, and $b_i \in \mathbb{R}$.

Proof. We shall prove this theorem using the Stone-Weierstrass theorem. The proof consists of several steps:

- 1) First, note that the set of functions of the form $N_{n,\sigma}(x)$ forms a vector space. Let's call this space \mathcal{N} .
- 2) We need to show that \mathcal{N} is dense in $C([0, 1]^d)$. By the Stone-Weierstrass theorem, it suffices to show that \mathcal{N} separates points and contains a non-zero constant function.
- 3) To demonstrate that \mathcal{N} separates points, let $x, y \in [0, 1]^d$ with $x \neq y$. We need to identify a function in \mathcal{N} that assigns different values at x and y .
- 4) Since $x \neq y$, there exists a unit vector u such that $u \cdot x \neq u \cdot y$. Consider the function

$$g(t) = \sigma(at + b) \quad (\text{F.145})$$

where a and b are chosen so that $\sigma(au \cdot x + b) \neq \sigma(au \cdot y + b)$. This is possible because σ is non-constant and continuous.

- 5) Now, the function $h(z) = g(u \cdot z) = \sigma(au \cdot z + b)$ is in \mathcal{N} and satisfies $h(x) \neq h(y)$. Thus, \mathcal{N} separates points.
- 6) To show that \mathcal{N} contains a non-zero constant function, consider

$$k(x) = \sigma(b) - \sigma(a + b) \quad (\text{F.146})$$

where a and b are chosen so that $\sigma(b) \neq \sigma(a+b)$. This is feasible because σ is non-constant. The function $k(x)$ is a non-zero constant function in \mathcal{N} .

7) By the Stone-Weierstrass theorem, \mathcal{N} is dense in $C([0,1]^d)$.

8) Therefore, for any $f \in C([0,1]^d)$ and $\varepsilon > 0$, there exists a function $N_{n,\sigma} \in \mathcal{N}$ such that

$$\sup_{x \in [0,1]^d} |f(x) - N_{n,\sigma}(x)| < \varepsilon \quad (\text{F.147})$$

This concludes the proof of the Universal Approximation Theorem. \square

EXAMPLE F.1 (Activation Function based on Stirling's Approximation). *Let us define an activation function $\sigma(x)$ based on Stirling's approximation:*

$$\sigma(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2} \left(1 + \left(\frac{x}{\sqrt{2}} \right) \right), & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \quad (\text{F.148})$$

where (x) is the error function defined as:

$$(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (\text{F.149})$$

This $\sigma(x)$ belongs to $A(1)$ and is a smooth approximation of the step function.

Now, we apply Theorem 1 to various types of functions:

EXAMPLE F.2 (Continuous Function). *Let $f(x) = x^2$ on $[0, 1]$. The modulus of continuity of f is:*

$$\omega(f, \delta)_{[0,1]} = \delta^2 \quad (\text{F.150})$$

Therefore, Theorem 1 informs us that:

$$\|S_{n,\sigma}(x^2) - x^2\| \leq \frac{1}{n^2} \quad (\text{F.151})$$

Therefore, it shows that the approximation error decreases quadratically with the number of nodes.

EXAMPLE F.3 (Lipschitz Continuous Function). *Let $f(x) = |x|$ on $[-1, 1]$. The modulus of continuity of f is:*

$$\omega(f, \delta)_{[-1,1]} = \delta \quad (\text{F.152})$$

Theorem 1 [Qian-Yu, 2022] provides us with:

$$\|S_{n,\sigma}(|x|) - |x|\| \leq \frac{2}{n} \quad (\text{F.153})$$

This shows that for Lipschitz continuous functions, the approximation error decreases linearly with the number of nodes.

EXAMPLE F.4 (Hölder Continuous Function). *Let $f(x) = x^{\frac{1}{3}}$ on $[0, 1]$. The modulus of continuity of f is:*

$$\omega(f, \delta)_{[0,1]} = \delta^{\frac{1}{3}} \quad (\text{F.154})$$

Since Theorem 1 [Qian-Yu, 2022] [77] provides us with:

$$\|S_{n,\sigma}(x^{\frac{1}{3}}) - x^{\frac{1}{3}}\| \leq \frac{1}{n^{\frac{1}{3}}} \quad (\text{F.155})$$

This result must show that for Hölder continuous functions, the approximation error diminishes slower with the number of nodes.

EXAMPLE F.5 (Smooth Periodic Function). *Let $f(x) = \sin(2\pi x)$ on $[0, 1]$. The modulus of continuity of f is:*

$$\omega(f, \delta)_{[0,1]} = 2 \sin(\pi\delta) \quad (\text{F.156})$$

For small δ , this is approximately $2\pi\delta$. Theorem 1 provides us with:

$$\|S_{n,\sigma}(\sin(2\pi x)) - \sin(2\pi x)\| \leq \frac{2\pi}{n} \quad (\text{F.157})$$

This result shows that for smooth periodic functions, the approximation error also decreases linearly with the number of nodes.

These examples illustrate how the smoothness of the function being approximated affects the rate of convergence of the neural network interpolation operators. Smoother functions (like x^2) are approximated more quickly than less smooth functions (like $|x|$ or $x^{\frac{1}{3}}$).

Extended Lagrange Interpolation Operator (Extended LIO)

DEFINITION F.4 (Extended Lagrange Interpolation Operator). *Let $f \in C^r[a, b]$ and $\{x_k\}_{k=0}^n$ be a set of distinct points in $[a, b]$. The Extended Lagrange Interpolation Operator $L_{n,r}$ is defined as:*

$$L_{n,r}(f, x) := \sum_{k=0}^n \sum_{j=0}^r \frac{f^{(j)}(x_k)}{j!} (x - x_k)^j \ell_k(x) \quad (\text{F.158})$$

where $\ell_k(x)$ are the Lagrange basis polynomials.

LEMMA F.3 (Interpolation Property). *For $f \in C^r[a, b]$ and $0 \leq v \leq r$, the Extended LIO satisfies:*

$$L_{n,r}^{(v)}(f, x_k) = f^{(v)}(x_k) \quad (\text{F.159})$$

for all $k = 0, 1, \dots, n$.

Proof. We will prove this by induction on v .

Base case ($v = 0$): At $x = x_k$, we have:

$$L_{n,r}(f, x_k) = \sum_{i=0}^n \sum_{j=0}^r \frac{f^{(j)}(x_i)}{j!} (x_k - x_i)^j \ell_i(x_k) \quad (\text{F.160})$$

$$= \sum_{j=0}^r \frac{f^{(j)}(x_k)}{j!} (x_k - x_k)^j \ell_k(x_k) \quad (\text{F.161})$$

$$= f^{(0)}(x_k) \cdot 1 \cdot 1 = f(x_k) \quad (\text{F.162})$$

because $\ell_i(x_k) = 0$ for $i \neq k$ and $\ell_k(x_k) = 1$.

Inductive step: Assume the statement holds for all derivatives up to order $v - 1$. We will prove it for the v -th derivative.

Differentiating $L_{n,r}(f, x)$ v times and evaluating at x_k :

$$L_{n,r}^{(v)}(f, x_k) = \sum_{i=0}^n \sum_{j=0}^r \frac{f^{(j)}(x_i)}{j!} \frac{d^v}{dx^v} [(x - x_i)^j \ell_i(x)]_{x=x_k} \quad (\text{F.163})$$

$$= \sum_{j=v}^r \frac{f^{(j)}(x_k)}{j!} \frac{j!}{(j-v)!} (x_k - x_k)^{j-v} \ell_k(x_k) + \text{other terms} \quad (\text{F.164})$$

The "other terms" involve derivatives of $\ell_i(x)$ evaluated at x_k , which are all zero for $i \neq k$.

Simplifying:

$$L_{n,r}^{(v)}(f, x_k) = \frac{f^{(v)}(x_k)}{v!} \cdot v! \cdot 1 \cdot 1 = f^{(v)}(x_k) \quad (\text{F.165})$$

Thus, by induction, the lemma holds for all $0 \leq v \leq r$. \square

THEOREM F.20 (Error Bound for Extended LIO). *Let $f \in C^{r+1}[a, b]$. Then there exists a constant $C > 0$ such that:*

$$\|f - L_{n,r}f\|_\infty \leq Ch^{r+1} \|f^{(r+1)}\|_\infty \quad (\text{F.166})$$

where $h = \max_{0 \leq k \leq n-1} (x_{k+1} - x_k)$.

Proof. Let $x \in [a, b]$ be arbitrary. We can find k such that $x \in [x_k, x_{k+1}]$.

By Taylor's theorem with remainder, for each $i = 0, 1, \dots, n$, we have:

$$f(x) = \sum_{j=0}^r \frac{f^{(j)}(x_i)}{j!} (x - x_i)^j + \frac{f^{(r+1)}(\xi_i)}{(r+1)!} (x - x_i)^{r+1} \quad (\text{F.167})$$

for some ξ_i between x and x_i .

Subtracting $L_{n,r}(f, x)$ from both sides:

$$f(x) - L_{n,r}(f, x) = \sum_{i=0}^n \left[\frac{f^{(r+1)}(\xi_i)}{(r+1)!} (x - x_i)^{r+1} \right] \ell_i(x) \quad (\text{F.168})$$

$$- \sum_{i=0}^n \sum_{j=0}^r \frac{f^{(j)}(x_i)}{j!} (x - x_i)^j (\ell_i(x) - 1) \quad (\text{F.169})$$

The second term vanishes because $\sum_{i=0}^n \ell_i(x) = 1$ for all x .

Taking the absolute value and utilising the triangle inequality, we obtain:

$$|f(x) - L_{n,r}(f, x)| \leq \frac{\|f^{(r+1)}\|_\infty}{(r+1)!} \sum_{i=0}^n |x - x_i|^{r+1} |\ell_i(x)|, \quad (\text{F.170})$$

where $L_{n,r}(f, x)$ is the Lagrange interpolant of degree n with regularity r , and $\|f^{(r+1)}\|_\infty$ denotes the supremum norm of the $(r+1)$ -th derivative of f .

Now, observe that:

- (a) For at least one index i (specifically, either k or $k + 1$), the distance $|x - x_i|$ satisfies:

$$|x - x_i| \leq h, \quad (\text{F.171})$$

where h is the maximum spacing between interpolation nodes.

- (b) For all other indices i , the distance $|x - x_i|$ is bounded by:

$$|x - x_i| \leq (n + 1)h. \quad (\text{F.172})$$

- (c) The sum of the absolute values of the Lagrange basis polynomials $\ell_i(x)$ is uniformly bounded:

$$\sum_{i=0}^n |\ell_i(x)| \leq C_1, \quad (\text{F.173})$$

where C_1 is a constant independent of n .

To further refine the error bound, we apply the **Jensen inequality** [35]. Let μ be a probability measure on the set of interpolation nodes $\{x_0, x_1, \dots, x_n\}$, and let $\Phi(t) = t^{r+1}$ be a convex function. By Jensen's inequality, we have:

$$\Phi\left(\int_{\Omega} |x - x_i| d\mu\right) \leq \int_{\Omega} \Phi(|x - x_i|) d\mu. \quad (\text{F.174})$$

Applying this to the sum in (F.174), we obtain:

$$\sum_{i=0}^n |x - x_i|^{r+1} |\ell_i(x)| \leq C_1 \left(\sum_{i=0}^n |x - x_i| |\ell_i(x)| \right)^{r+1}. \quad (\text{F.175})$$

Combining these results, the interpolation error satisfies:

$$|f(x) - L_{n,r}(f, x)| \leq \frac{\|f^{(r+1)}\|_{\infty}}{(r+1)!} C_1 (h^{r+1} + n(n+1)^{r+1} h^{r+1}). \quad (\text{F.176})$$

This demonstrates that the interpolation error depends on the regularity of f , the spacing h of the interpolation nodes, and the degree n of the interpolant, while leveraging the convexity properties ensured by Jensen's inequality.

Therefore,

$$|f(x) - L_{n,r}(f, x)| \leq C_2 h^{r+1} \|f^{(r+1)}\|_{\infty} \quad (\text{F.177})$$

for some constant C_2 .

Taking the supremum over all $x \in [a, b]$, we obtain:

$$\|f - L_{n,r}f\|_\infty \leq Ch^{r+1} \|f^{(r+1)}\|_\infty \quad (\text{F.178})$$

where $C = C_2$. □

This completes the proofs for the Interpolation Property Lemma and the Error Bound Theorem for the Extended Lagrange Interpolation Operator.

Lastly, one might reformulate this problem in terms of Lagrangian mechanics [16].

DEFINITION F.5 (Lagrangian Neural Network Functional). *Define the Lagrangian Neural Network Functional $\mathcal{L}_{n,r} : C^r[a, b] \rightarrow \mathbb{R}$ as:*

$$\mathcal{L}_{n,r}(f) = \sum_{i=0}^n \sum_{v=0}^r \lambda_v |S_{n,r,\sigma}^{(v)}(f)(x_i) - f^{(v)}(x_i)|^2 + \mu \int_a^b |f^{(r+1)}(x)|^2 dx \quad (\text{F.179})$$

where $\lambda_v, \mu > 0$ are regularisation parameters.

In summary, this unique variational formulation of the neural network interpolation problem allows us to view the interpolation problem as a variational principle in function space, incorporating both the function values and their derivatives at the interpolation points. It defines a Lagrangian Neural Network Functional, incorporating function tuples of (f, f') evaluated at specific points with a regularisation term. This expression defines a functional (a mapping from a function space to the real numbers, \mathbb{R}), denoted as $\mathcal{L}_{n,r}$. [Neural Network Functional] This functional is used to evaluate the performance of a neural network interpolation scheme based on both the function values and their derivatives. [Sum of Squared Errors:] The formulation sums the squared differences between the values of the function f and its derivatives, $f^{(v)}$, and the corresponding values produced by the neural network interpolation operator $S_{n,r,\sigma}^{(v)}(f)(x_i)$, where $S_{n,r,\sigma}$ represents the neural network approximation. [Regularisation:] The terms λ_v and μ are regularisation parameters, where the regularisation term $\mu \int_a^b |f^{(r+1)}(x)|^2 dx$ penalises the higher-order derivatives of the function, ensuring smoothness and controlling the complexity of the solution.

Applications to Hyperparameters

DEFINITION F.6 (Sobolev Space). *For $1 \leq p < \infty$ and $k \in \mathbb{N}$, the Sobolev space $W^{k,p}[a,b]$ is defined as:*

$$W^{k,p}[a,b] = \{f \in L^p[a,b] : D^j f \in L^p[a,b] \text{ for } j = 1, \dots, k\} \quad (\text{F.180})$$

with the norm

$$\|f\|_{W^{k,p}} = \left(\sum_{j=0}^k \|D^j f\|_p^p \right)^{1/p} \quad (\text{F.181})$$

where $D^j f$ denotes the j -th weak derivative of f .

DEFINITION F.7 (Besov Space). *For $s > 0$, $1 \leq p, q \leq \infty$, the Besov space $B_{p,q}^s[a,b]$ is defined using the modulus of smoothness:*

$$B_{p,q}^s[a,b] = \{f \in L^p[a,b] : \|f\|_{B_{p,q}^s} < \infty\} \quad (\text{F.182})$$

where

$$\|f\|_{B_{p,q}^s} = \|f\|_p + \left(\int_0^1 (t^{-s} \omega_r(f, t)_p)^q \frac{dt}{t} \right)^{1/q} \quad (\text{F.183})$$

for $r > s$, and $\omega_r(f, t)_p$ is the r -th order modulus of smoothness in L^p .

Derivations of Sobolev and Besov Spaces

THEOREM F.21 (Extended Proof of Theorem 1 to Sobolev Spaces). *Let $f \in W^{k,p}[a,b]$ and $\sigma \in \mathcal{A}(m)$ with k continuous derivatives. Then,*

$$\|S_{n,\sigma}(f) - f\|_{W^{k,p}} \leq C_{k,p} n^{-1} \|f\|_{W^{k+1,p}} \quad (\text{F.184})$$

where $C_{k,p}$ is a constant depending only on k and p .

THEOREM F.22 (Extended Proof of Theorem 2 to Besov Spaces). *Let $f \in B_{p,q}^s[a,b]$ and $\sigma \in \mathcal{A}(m)$. Then,*

$$\|S_{n,\sigma}(f) - f\|_p \leq C_{s,p,q} n^{-s} (\log n)^{(q-1)/q} \|f\|_{B_{p,q}^s} \quad (\text{F.185})$$

where $C_{s,p,q}$ is a constant depending solely on s , p , and q .

THEOREM F.23 (Extended Proof of Theorem 3 to Sobolev Spaces). *Let $f \in W^{k,p}[a,b]$ and $\sigma \in \mathcal{A}(m)$ with a bounded derivative. If*

$$\|S_{n,\sigma}(f) - f\|_{W^{k,p}} = O(n^{-\alpha}) \quad (\text{F.186})$$

for some $0 < \alpha < 1$, then $f \in W^{k+\alpha,p}[a,b]$.

THEOREM F.24 (Extension of Proof of Theorem 4 to Besov Spaces). *Let $f \in B_{p,q}^s[a,b]$ and $\sigma \in \mathcal{A}(m)$ with $r > s$ continuous derivatives. Then $S_{n,r,\sigma}(f)$ interpolates f and its derivatives up to order $\lfloor s \rfloor$ at the nodes, and*

$$\|D^j(S_{n,r,\sigma}(f)) - D^j f\|_p \leq C_{s,p,q,j} n^{-(s-j)} \|f\|_{B_{p,q}^s} \quad (\text{F.187})$$

for $j = 0, 1, \dots, \lfloor s \rfloor$.

These extensions provide a more comprehensive view of the approximation capabilities of the neural network operators in various function spaces.

Some key observations:

- (a) *In Sobolev spaces, we can obtain estimates for the approximation of both the function and its derivatives simultaneously.*
- (b) *In Besov spaces, we often see logarithmic factors appearing in the error bounds, reflecting the finer scale of smoothness these spaces capture.*
- (c) *The inverse theorems (3 and 7)[77] in Sobolev spaces show that the approximation rate characterises the smoothness of the function in terms of fractional Sobolev spaces.*
- (d) *The extensions to Besov spaces often provide more precise characterisations of the approximation rates, especially for functions with fractional smoothness.*

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