Theorem 1. Let (C, η) be a relaxed-CNF clause where C has m literals and η is the threshold on literals. An equivalent compact encoding of (C, η) into a CNF formula $F = \bigwedge_i C_i$ requires $\binom{m}{m-\eta+1}$ clauses where each clause is distinct and has $m - \eta + 1$ literals of (C, η). Therefore, the total number of literals in F is $(m - \eta + 1)\binom{m}{m-\eta+1}$.

Proof. (Theorem 1) Let $F = \bigwedge_i C_i$ be a CNF formula of $\binom{m}{m-\eta+1}$ clauses where each clause C_i is distinct and has $m - \eta + 1$ literals of C. F is an equivalent CNF formula of clause (C, η) if $\sigma \models (C, \eta) \Leftrightarrow \sigma \models F$.

We will first prove $\sigma \models (C, \eta) \Rightarrow \sigma \models F$. Since $\sigma \models (C, \eta)$, at least η literals in C are $true$ and let T be the set of those η literals. Therefore, according to pigeon hole theorem we cannot construct any C_i with $m - \eta + 1$ literals such that $T \cap C_i = \emptyset$. Hence, at least 1 literal in C_i is also in T. Therefore, $\sigma \models (C_i, 1) \Rightarrow \sigma \models F$.

We prove $\sigma \models F \Rightarrow \sigma \models (C, \eta)$ by contradiction. Let assume $\sigma \models F \Rightarrow \sigma \not\models (C, \eta)$. Hence at least $m - \eta + 1$ literals in C must be $false$. Then there exists a clause $C_{i'}$ that consists of these $m - \eta + 1$ literals. But $F = \bigwedge_i C_i$ must include $C_{i'}$ and $\sigma \models C_{i'}$, which is a contradiction. Therefore $\sigma \models F \Rightarrow \sigma \models (C, \eta)$. Hence F is an equivalent CNF formula of (C, η).

Now we prove that F is a compact encoding. Suppose F excludes clause $C_{i'}$ but includes all other clauses as well as clauses with greater than $m - \eta + 1$ variables. Now we set the variables in $C_{i'}$ as $false$ and every other variables $true$. Then F is satisfiable. But $(C_{i'}, \eta)$ is not satisfiable because each literal in $C_{i'}$ is $false$. Therefore, F must include all $\binom{m}{m-\eta+1}$ clauses and the total literal count in F is $(m - \eta + 1)\binom{m}{m-\eta+1}$. $\square$