A Tutorial on Hybrid Finite Element Formulation

Assumed Stress-Displacement Field

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Outline

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Conventional Formulation for FEM

$$\int_{\Omega} \mathbf{v}.(\nabla.\tau + \mathbf{b})\, d\Omega + \int_{\Gamma_t} \mathbf{v}.(\mathbf{t} - \bar{\mathbf{t}})\, d\Gamma = \mathbf{0}\ \forall \mathbf{v} \in V_u$$

- Ω - domain
- Γ - boundary.
- Γ_t - traction specified boundary.
- $\mathbf{v}$ - variation of the displacement.
- $\mathbf{t}$ - tractions on the boundary
- $\bar{\mathbf{t}}$ - specified tractions
- τ - stress tensor
  - $\tau = \mathbf{C} : \varepsilon(\mathbf{u})$
  - $\mathbf{C}$ fourth order constitutive tensor.
  - $\varepsilon(\mathbf{u})$ linear strain tensor
    - $\frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$
  - $\mathbf{u}$ Displacements

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- Find $\mathbf{u}$ for all possible $\mathbf{v}$
- Use the identity
  - $\mathbf{v}.(\nabla.\tau) = \nabla.(\tau\mathbf{v}) - \tau : \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$
    - $\frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$ denoted by $\varepsilon(\mathbf{v})$
  - Used symmetry of τ
- Use Divergence theorem
  - cancel out the term $\int_{\Gamma_t} \mathbf{v}.\mathbf{t}\, d\Gamma$
- This results in

$$\int_{\Omega} \mathbf{v}.\mathbf{b}\, d\Omega + \int_{\Gamma_t} \mathbf{v}.\bar{\mathbf{t}}\, d\Gamma = \int_{\Omega} \varepsilon(\mathbf{v}) : \tau\, d\Omega\ \forall \mathbf{v} \in V_u$$

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$$\int_{\Omega} \mathbf{v}.\left(\nabla.\tau+\mathbf{b}\right) d\Omega+\int_{\Gamma_t} \mathbf{v}.\left(\mathbf{t}-\bar{\mathbf{t}}\right) d\Gamma=\mathbf{0}\ \forall \mathbf{v}\in V_u$$

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- τ is replaced by $\mathbf{C} : \varepsilon(\mathbf{u})$
- $\mathbf{v}$ and $\mathbf{u}$ are interpolated

Outline

Hybrid Formulation for FEM

$$\int_{\Omega} \mathbf{v}.\left(\nabla.\tau+\mathbf{b}\right) d\Omega + \int_{\Gamma_t} \mathbf{v}.\left(\mathbf{t}-\bar{\mathbf{t}}\right) d\Gamma + \int_{\Omega} \sigma : \left(\varepsilon\left(\mathbf{u}\right) - \mathbf{C}^{-1} : \tau\right) d\Omega = \mathbf{0}$$

$$\forall\left(\mathbf{v},\sigma\right) \in \left(V_u \times V_\tau\right)$$

- σ Variation of the stress field.
- Results in two sets of equations
  - set variation in each to zero

$$\int_{\Omega} \varepsilon(\mathbf{v}) : \tau d\Omega = \int_{\Omega} \mathbf{v}.\mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{v}.\bar{\mathbf{t}} d\Gamma$$

$$\int_{\Omega} \sigma : \left(\varepsilon\left(\mathbf{u}\right) - \mathbf{C}^{-1} : \tau\right) d\Omega = 0$$

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- $\mathbf{v}$ and $\mathbf{u}$ interpolated
- σ and τ interpolated

Outline

Getting the matrix equations

- $\mathbf{u} = N\hat{\mathbf{u}}$ and $\mathbf{v} = N\hat{\mathbf{v}}$
  - $\hat{\mathbf{u}}$ *and* $\hat{\mathbf{v}}$ are nodal displacements and its variations.
  - viewed as constants helping to describe the displacement field within.
- $\tau = P\hat{\beta}$ and $\sigma = P\hat{\gamma}$
  - $\hat{\beta}$ *and* $\hat{\gamma}$ are constants in an element that help in description of the stress field within.
- This results in $\varepsilon(\mathbf{u}) = B\hat{\mathbf{u}}$ and $\varepsilon(\mathbf{v}) = B\hat{\mathbf{v}}$
  - B is the strain-displacement matrix

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$$\int_{\Omega} \sigma : \left(\varepsilon\left(\mathbf{u}\right) - \mathbf{C}^{-1} : \tau\right) d\Omega = 0$$

Getting the matrix equations

$$\hat{\mathbf{v}}^T \left(\int_\Omega B^T P d\Omega \right) \hat{\beta} = \hat{\mathbf{v}}^T \left(\int_\Omega N^T \mathbf{b} d\Omega \right) + \hat{\mathbf{v}}^T \left(\int_{\Gamma_t} N^t \bar{\mathbf{t}} d\Gamma \right)$$

$$\hat{\gamma}^T \int_\Omega P^T \left(B\hat{\mathbf{u}} - \mathbf{C}^{-1} P \hat{\beta} \right) d\Omega = 0$$

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$$\hat{\gamma}^T \left(\int_\Omega P^T B d\Omega \right) \hat{\mathbf{u}} - \hat{\gamma}^T \left(\int_\Omega P^T \mathbf{C}^{-1} P d\Omega \right) \hat{\beta} = 0$$

Getting the matrix equations

- Eliminating the Variations we get

$$\overbrace{\left(\int_\Omega B^T P d\Omega\right)}^{G^T}\hat{\beta} = \overbrace{\left(\int_\Omega N^T \mathbf{b} d\Omega\right) + \left(\int_{\Gamma_t} N^t \bar{\mathbf{t}} d\Gamma\right)}^{\hat{f}}$$

$$\overbrace{\left(\int_\Omega P^T B d\Omega\right)}^{G}\hat{\mathbf{u}} - \overbrace{\left(\int_\Omega P^T \mathbf{C}^{-1} P d\Omega\right)}^{H}\hat{\beta} = 0$$

Getting the matrix equations

- The Matrix equations

$$G^T \hat{\beta} = \hat{f}$$

$$G\hat{\mathbf{u}} - H\hat{\beta} = 0$$

Getting the matrix equations

- Condensing out the $\hat{\beta}$

$$\left(G^T H^{-1} G\right) \hat{\mathbf{u}} = \hat{f}$$

- Can be split across elements

$$\sum_e \left(G_e^T H_e^{-1} G_e\right) \hat{\mathbf{u_e}} = \hat{f_e}$$

- So every element stiffness matrix formulation needs a matrix inversion.

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Obtaining the 'P' Matrix

- Governed by the stress shape functions.
- Defined in local co-ordinate system of the element.
  - Transformed to global coordinate system
  - At every point inside the element
  - $\tau = J^T \tau_{loc} J$
    - J Jacobian matrix
    - Linear in τ_{loc}
    - For vectorized τ one can get a Transformation matrix T size 6×6 in 3D
- P_{loc} These are of size $6 \times num\ \beta s$ in 3D
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'P' Matrix an example

- In local coords for a 4 node quad
  - $$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \beta_1 + \beta_2 y \\ \beta_3 + \beta_4 x \\ \beta_5 \end{bmatrix}$$
  - $$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{P_{loc}} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix}$$
- Transformation matrix
  - $$\begin{bmatrix} J_{11}^2 & J_{21}^2 & 2J_{11}J_{21} \\ J_{12}^2 & J_{22}^2 & 2J_{12}J_{22} \\ J_{11}J_{12} & J_{21}J_{22} & J_{11}J_{22} + J_{12}J_{21} \end{bmatrix}$$

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Stress Shape Functions the rule of the thumb

- τ_{xx}- differentiation of the displacement field u_x
- τ_{yy}- differentiation of the displacement field u_y
- τ_{zz}- differentiation of the displacement field u_z
- τ_{xy}- intersection of $\tau_{xx}$and τ_{yy}
- τ_{xz}- intersection of $\tau_{xx}$and τ_{zz}
- τ_{yz}- intersection of $\tau_{yy}$and τ_{zz}

Outline

Pseudo - Code

For every gauss point:

> displacement shape functions (disp)
> displacement derivatives (derdisp)
> P_{local}
> Jacobian J
> J^{-1}
> $Det(J)$
> T - using J
> B - using J^{-1}derdisp
> P - $T \times P_{local}$
> $G = G + P^T \times B \times Det(J)$
> $H = H + P^T \times C \times P \times Det(J)$

end for

$KE = G^T H^{-1} G$

Thank You

For Further Reading I

C.S. Jog. A 27-node hybrid brick and a 21-node hybrid wedge element for structural analysis, 2007