3.1 Pigeonhole Principle

Given $n+1$ pigeons that we place into n pigeonholes, then there 15 at least 1 pigeonhole containing 2 or more pigeons (Theorem 3.1.1).

Example

1. Given 367 people, at least 2 share a birthday, suppose
   - pigeon: people
   - pigeonholes: 366 dates on the calendar
2. n married couples ($2n$ people total). How many people must be selected to guarantee selection of a married couple? $n+1$

   Why? Use pigeonhole principle.
   - Pigeons: people ($2n$)
   - Pigeonholes: each marriage as a category (n).
     By P.P., need $n+1$ for a repeat (2 people in same marriage).
3. Given m integers $a_1, a_2, ..., a_n$, there exist integers k and l with $0 \leq k < l \leq m$ such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.
   - pigeonholes: remainders upon division by m (m possibilities are 0,1,...,m-1).
   - pigeons: we will start by considering the sums

$$\begin{aligned}
&a_1 \\
&a_1 + a_2 + a_3 \\
&\vdots \\
&a_1 + a_2 + a_3 + \ldots + a_m
\end{aligned}$$

Case 1: Each of the m pigeons fits into a different pigeonhole has 1 pigeon. So "remainder 0 pigeonhole" applies to one of these sums. So, there is a sum $a_1+a_2+a_3+\ldots+a_l$ divisible by m and we are done.

In fact, this happens any time one of these sums is divisible by m.

Case 2: No sum amongst our in m sums is divisible by m. We now have $m-1$ pigeonholes and m pigeons. By P.P., there exist k and $l(l > k)$ such that $a_1+a_2+a_3+\ldots+a_k$ and $a_1 + a_2 + a_3 + \ldots + a_{k+1} + \ldots + a_l$. Both have remainder r when divided by m. So, $(a_1 + a_2 + a_3 + \ldots + a_l) - (a_1 + a_2 + a_3 + \ldots + a_k) = a_{k+1} + a_{k+2} + ... + a_l$ is divisible by m.

4. Chess master has 11 weeks to prepare. The master plays at least one game a day. But no more than 12 games during any calendar week.

   <u>Goal:</u> Show there is some succession of consecutive days in which exactly 21 gives are played.

   Let a_i be the total number of games played from day 1 through day 2.

$$1 \leq a_1 < a_2 < a_3 < \ldots < a_{77} \leq 132$$

$$22_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \leq 153$$

   - pigeons: integers $a_1, a_2, ..., a_{77}, a_1 + 21, a_2 + 21, ..., a_{77} + 21$
   - Pigeonholes: integers 1,2,...,153

   By P.P., there are two of those integers that are equal.

   So, there exist i and j (where $i \neq j$) such that

$$\underbrace{\cancel{a_i = a_j} \text{ or } \cancel{a_i + 21 = a_j + 21}}_{\substack{\text{Impossible because at least} \\ \text{one game played per day}}} \text{ or } a_i = a_j + 21$$

   So,

$$a_i = a_j + 21 \Leftrightarrow \underbrace{a_i - a_j}_{\substack{\text{\# of games played from} \\ \text{day } j+1 \text{ through day } i}} = 21$$

5. We choose 101 integers from 1,2,...,200. Show there are 2 such that 1 divides another.

   - pigeons: 101 integers
   - pigeonholes: Greatest odd factors of our integers.

     i.e., given $1 \leq n \leq 200$, write $n = 2^k \cdot a$ where a is odd.

     100 pigeonholes: a=1,3,5,7,9,...,199

3.2 Strong Form of Pigeonhole Principal

Theorem 3.2.1 If $q_1\ q_2,\ \ldots, q_n$ are positive integers and $q_1 + q_2 + \ldots + q_n - n + 1$ objects are placed into n boxes then

- Box 1 has at least q_1 objects or
- Box 2 has at least q_2 objects or

  $\vdots$
- Box n has at least q_n objects

Previously, we considered $q_1 = q_2 = \ldots = q_n = 2$.

Corollary 3.2.2 Let n and r be positive integers if $n(r-1)+1$ objects are distributed into n boxes then there is at least one box containing at least r objects.

This is *Theorem 3.2.1* with $q_{=}q_2 = \ldots = q_n = r$.

Example Need a basket with at

- at least 8 apples or
- at least 6 bananas or
- at least 9 oranges.

How many individual pieces of fruit do we need to achieve this?

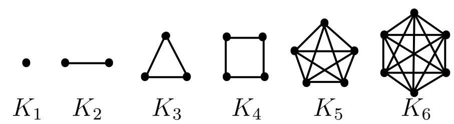
$\Rightarrow$ By Theorem 3.2.1, we need 7+5+8+1=21 pieces of fruit.

3.3 Ramsey Numbers

Goal: Show that amongst any group of 6 people, there 3 people that either all know each other or all don't know each other.

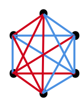
Idea: We will model this by considering colorings of the edges of a complete graph.

A complete graph on n vertices (called K_n) is a collection of n vertices with one edge drawn between any given pair of vertices.



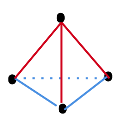
We represent 6 people as vertices of a K_6 and we draw a red edge between them if they know each other.

So, our goal boils down to showing that, if we color the edges of K_6 either red or blue, we obtain either a red K_3 or a blue K_3 within our K_6.



Why does this happen?

Fix a vertex of our K_6. There are 5 edges connected to this vertex. By Pigeonhole Principal, at least 3 of these edges are red or blue.



If any of the vertices connecting these edges are connected by the same color edge, there is a K_3 of that color. If not, these edges for a K_3 of the other color.

Feb 19, 2019(Tue)
Last time, we showed $K_6 \to K_3,\ K_3$.

Theorem 3.3.1 Ramsey's Theorem
If $m, n \geq 2$ are integers then there is a positive integer p such that

$$K_p \to K_m, K_n$$

i.e., given m and n, then there is a p such that 2-coloring the edges of K_p gives a K_m of one color or a K_n of another color.

Given m and n (integers ≥ 2), there is a smallest value p such that $K_p \to K_m, K_n$. We call this p the Ramsey number of m and n, notated $r(m, n)$.

Facts

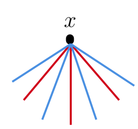
$$r(3,3) = 6 \qquad r(2,n) = n \qquad r(m,2) = m$$

Proof outline We double induct on both m and n.

Base Cases: $m = 2$ and $n = 2$
Since $r(m, 2)$ and $r(2, n)$ exist, the theorem is true for $m = 2$ and $n = 2$.

Induction Hypothesis
Assume $m \geq 3$ and $n \geq 3$. Assume $r(m, n-1)$ and $r(m-1, n)$ both exist. Let $p = r(m, n-1) + r(m-1, n)$. Suppose each edge of K_p is colored either red or blue.

Let x be a vertex of K_p.



Let R_x be the vertices connected to x with a red edge and B_x be the vertices connected to x with a blue edge.

$$|R_x| + |B_x| = p - 1 = r(m,\ n-1) + r(m-1,\ n) - 1$$

So,

$$\left.\begin{array}{r} |R_x| \geq r(m-1,\ n) \\ \text{or } |B_x| \geq r(m,\ n-1) \end{array}\right\} \begin{array}{c} \text{If both false then} \\ |R_x| + |B_x| \leq p-2 \end{array}$$

Either way, we have a red K_m or a blue K_n.

This proof shows

$$r(m,n) \leq r(m-1,n) + r(m,n-1) \qquad \text{(m,n} \geq 3)$$

Let

$$f(m,\ n) = \binom{m+n-2}{m-1} \qquad \text{(m,n} \geq 2)$$

$$\binom{m+n-2}{m-1} = \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} \qquad \text{(Pascal's Formula)}$$

So,

$$f(m,n) = f(m,n-1) + f(m-1,n)$$

$$r(m,n) \leq \binom{m+n-2}{m-1}$$