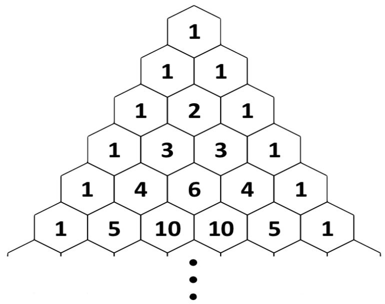
# 5 Binomial Theorem

*Goal*: Expand $(x+y)^n$ quickly.

## 5.1 Pascal's Triangle




Each number is the sum of the two above it.

$$
\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^2 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 &= (x+y)(x+y)^3 \\
&= (x+y)(x^3+3x^2y+3xy^2+y^3) \\
&= x^4+3x^3y+3x^2y^2+xy^3+x^3y+3x^2y^2+3xy^3+y^4 \\
&= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\end{aligned}
$$

Recall: Pascal's Formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

What this tells us is that Pascal's triangle is also

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

Properties

- Pascal's Triangle has vertical reflective symmetry because $\binom{n}{k} = \binom{n}{n-k}$
- Rows add to $2^n$ because $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

## 5.2 Binomial Theorem

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \left[ = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k} \right]$

*Example*

$$\begin{aligned}(x+y)^6 &= \sum_{k=0}^{6} \binom{6}{k} x^{6-k} y^k \\ &= \binom{6}{0} x^6 y^0 + \binom{6}{1} x^5 y^1 + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x^1 y^5 + \binom{6}{6} x^0 y^6\end{aligned}$$

Proof Idea 1:

$$(x+y)^n = \underbrace{(x+y)(x+y)\ldots(x+y)}_{n \text{ times}}$$

We know this will be a sum of monomials $x^{n-k}y^k$. (The number of x and y equals $n$)

The coefficient is the number of ways to choose $x$ (or $y$) at each stage. i.e.

$$\binom{n}{n-k} = \binom{n}{k}$$

Special Case (Theorem. 5.2.2): Let $y = 1$ in binomial theorem.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{n-k} x^k$$

## 5.3 Unimodality of Binomial Coefficients

Combinatorial Identities

$$k\binom{n}{k} = n\binom{n-1}{k-1} \tag{5.2}$$

set $x = y = 1$ in binomial theorem to get:

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} \tag{5.3}$$

set $x = 1$ and $y = -1$ in binomial theorem to get:

$$(1-1)^n = 2^n = \binom{n}{0}(-1)^1 + \binom{n}{1}(-1)^2 + \ldots + \binom{n}{n}(-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n\binom{n}{n} \tag{5.4}$$

Add (5.3) and (5.4)

$$2^n + 0 = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} + \binom{n}{0} - \binom{n}{1} + \binom{n}{2} = \binom{n}{3} + \ldots + (-1)^n\binom{n}{n}$$

$$2^n = 2\binom{n}{0} + 2\binom{n}{2} + 2\binom{n}{4} + \ldots + 2\begin{cases}\binom{n}{n-1} & \text{if } n \text{ odd}\\ \binom{n}{n} & \text{if } n \text{ even}\end{cases}$$

$$(5.6) \qquad 2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots$$

(all terms are eventually 0)

Subtracting 5.4 from 5.3, we get

$$(5.7) \qquad 2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

Differentiate both sides with respect to $x$

$$n(x+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + n\binom{n}{n}x^{n-1}$$

Letting $x = 1$, we obtain

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} \tag{5.8}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n} \tag{5.16}$$

are equivalent to

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$$

These count $\binom{2n}{n}$ in 2 different ways, directly and by looking at how many elements are taken from each of two subcollections of $n$ things.

$\binom{r}{k}$ makes sense if $r$ is a real number and $k$ is any integer. In this case,

$$\binom{r}{k} = \begin{cases}\frac{r(r-1)(r-2)\ldots(r-k+1)}{k!} & \text{if } k \geq 1\\ 1 & \text{if } k = 0\\ 0 & \text{if } k =\leq -1\end{cases}$$

Example $r = \frac{1}{2},\ k = 3$

$$\binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!} = \frac{3}{48} = \frac{1}{16}$$
$$\binom{\sqrt{2}}{-1} = 0$$
$$\binom{4 \cdot 8}{0} = 1$$
$$\binom{-4}{5} = \frac{(-4)\cdot(-5)\cdot(-6)\cdot(-7)\cdot(-8)}{5!} - 56$$

It is still true that

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1} \qquad \text{(Pascal's Formula)}$$
$$k\binom{r}{k} = r\binom{r-1}{k-1}$$

If $k$ is a positive integer,

$$\begin{aligned}\binom{r}{k} &= \binom{r-1}{k} + \binom{r-1}{k-1} \\ &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-2}{k-2} \\ &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \binom{r-3}{k-3} \\ &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \ldots + \binom{r-k}{1} + \binom{r-k}{0} \\ &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \ldots + \binom{r-k}{1} + \binom{r-k-1}{0} + \\ &\quad \cancel{\binom{r-k-1}{-1}}\end{aligned}$$

Replace $r$ with $r+k+1$

$$\binom{r}{0} + \binom{r+1}{1} + \binom{r+2}{2} + \ldots + \binom{r+k}{k} = \binom{r+k+1}{k} \qquad (5.18)$$
$$(1)$$

For $n,\ k$ positive integers,

$$\begin{aligned}\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ &= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-1}{nk-1} \\ &= \binom{n-3}{k} + \binom{n-3}{k-1} + \binom{n-2}{k-1} + \binom{n-1}{k-1} \\ &\vdots \\ &= \cancel{\binom{0}{k}} + \cancel{\binom{0}{k-1}} + \binom{1}{k-1} + \ldots + \binom{n-1}{k-1}\end{aligned}$$

## 5.4 Multinomial Theorem

Suppose we need to compute $(x+y+z)^3$

1. We could multiply it all out. If we are clever, we will automate it with some combinatorial reasoning.

2. We could use the binomial theorem *twice*.

$$\begin{aligned}(x+(y+z))^3 &= \binom{3}{0}x^3 + \binom{3}{1}x^2(y+z)^1 + \binom{3}{2}x(y+z)^2 + \binom{3}{3}(y+z)^3 \\ &= \binom{3}{0}x^3 + \binom{3}{1}\binom{1}{0}x^2y + \binom{3}{1}\binom{1}{1}x^2z + \binom{3}{2\binom{2}{0}}xy^2 + \binom{3}{2}\binom{2}{1}xyz + \\ &\quad \binom{3}{2}\binom{2}{1}xz^2 + \binom{3}{3}\binom{3}{0}y^3 + \binom{3}{3}\binom{3}{1}y^2z + \binom{3}{3}\binom{3}{2}yz^2 + \binom{3}{3}\binom{3}{3}z^3\end{aligned}$$

Notation:

$$\underbrace{\binom{n}{n_1\ n_2\ n_3 \ldots n_t}}_{n_i \text{ is multinomial coefficient}} = \frac{n!}{n_1!\ n_2!\ n_3! \ldots n_t!}$$

This refers to the number of ways to arrange $n$ objects that include

$n_1$ identical objects

$n_2$ identical objects

$\vdots$

$n_t$ identical objects

Note that $\sum_{n=1}^{t} = n$.

*Example* Permutation of WISCONSIN

$$\binom{9}{1\ 2\ 2\ 1\ 1\ 2} = \frac{9!}{1!2!2!1!1!2!}$$

Note that

$$\binom{n}{k} = \binom{n}{k(n-k)}$$

Also

$$\binom{n}{n_1 n_2 \dots n_t} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t}$$

**Multinomial Theorem**

*Example*

$$\begin{aligned}(x+y+z)^3 &= (x+y+z)(x+y+z)(x+y+z)\\ &= \binom{3}{3}x^3+\binom{3}{2\ 1}x^2y+\binom{3}{1\ 2}xy^2+\binom{3}{3}y^3+\binom{3}{2\ 1}x^2z+\binom{3}{1\ 2}xz^2+\\ &\binom{3}{1\ 1\ 1}xyz+\binom{3}{2\ 1}y^2z+\binom{3}{1\ 2}yz^2+\binom{3}{3}z^3\end{aligned}$$

**Theorem 5.4.1 Multinomial Theorem**

$$(x_1+x_2+x_3+...+x_t)^n = \sum\binom{n}{n_1 n_2 \dots n_t}x_1^{n_1}x_2^{n_2}x_3^{n_3}\dots x_t^{n^t}$$

where the sum is over all non-negative integers $n_1,\ n_2,\ ...,\ n_t$ such that $n_1+n_2+...+n_t = n$.

*Example* Find the coefficient of $x^4y^3z^2w$ in $(x+y+z+w)^{10}$.

$$\binom{10}{4\ 3\ 2\ 1} = \frac{10!}{4!3!2!1!}$$

*Example* Find the coefficient of $x^4y^3z^2w$ in $(x+y+z+w)^{10}$

It is 0 because 4+3+2+2=11, so there is no such term.

*Pascal's Formula*

$$\binom{n}{n_1\ n_2\ \dots\ n_t} = \binom{n-1}{n_1-1\ n_2 \dots n_t} + \binom{n-1}{n_1\ n_2-1 \dots n_t} + \dots + \binom{n-1}{n_1\ n_2 \dots n_t-1}$$

## 5.5 Newton's Extended Binomial Theorem

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-y} \qquad \text{(for any real number } \alpha\text{)}$$

*Special Case* Let $z$ be a complex number with $|z| < 1$. Then

$$(1+z)^\alpha = \sum k = 0^\infty \binom{\alpha}{k} z^k$$

*Example*: $\alpha = 3$

$$\begin{aligned}(1+z)^3 = \sum k = 0^\infty \binom{3}{k} z^k &= \binom{3}{0} + \binom{3}{1} z + \binom{3}{2} z^2 + \binom{3}{3} z^3 + \binom{3}{4} z^4 + ...\\ &= 1 + 3z + 3z^2 + z^3 + 0 + ...\\ &= 1 + 3z + 3z^2 + z^3\end{aligned}$$

*Example*: $\alpha = -1$

$$\begin{aligned}(1+z)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} z^k &= \binom{-1}{0} + \binom{-1}{1} z + \binom{-1}{2} z^2 + \binom{-1}{3} z^3 + \dots\\ &= 1 - z + z^2 - z^3 + z^4 - ...\end{aligned}$$

This is just the sum of an infinite geometric series in disguise.

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + ...$$

$$\begin{aligned}\binom{-1}{k} &= \frac{(-1)(-1-1)(-1-2)\dots(-1-(k-1))}{k!}\\ &= \frac{(-1)(-2)(-3)\dots(-k)}{k!} = (-1)^k \cdot \frac{k!}{k!} = (-1)^k\end{aligned}$$

<u>Recall</u>: $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

By Newton's Extended Binomial Theorem,

$$
\begin{aligned}
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k \\
&= \binom{-n}{0}(-x)^0 + \binom{-n}{1}(-x)^1 + \binom{-n}{2}(-x)^2 + \binom{-n}{3}(-x)^3 + \cdots \\
&= 1 + \frac{(-n)}{1!}(-x) + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}(-x^3) + \cdots \\
&= 1 + \binom{n}{1}x + \frac{(n+1)!}{2!(n-1)!}x^2 + \frac{(n+2)!}{3!(n-1)!}x^3 + \cdots + \frac{(n+r-1)!}{r!(n-1)!}x^r + \cdots \\
&= 1 + \binom{n}{1}x + \frac{(n+1)!}{2!(n-1)!}x^2 + \frac{(n+2)!}{3!(n-1)!}x^3 + + \frac{(n+r-1)!}{r!(n-1)!}x^r + \cdots \\
&= 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \cdots + \binom{n+r-1}{r}x^r + \cdots \\
(1-x)^{-n} &= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r \\
&= (1 + x + x^2 + \cdots)^n \\
&= \underbrace{(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)\cdots(1 + x + x^2 + \cdots)}_{n\ times}
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{1+x} = (1_x)^{1/2} &= \sum_{k=0}^{\infty} \binom{1/2}{k} x^k \\
&= \binom{1/2}{0} + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \cdots + \binom{1/3}{r}x^r + \cdots \\
&= 1 + \frac{\left(\frac{1}{2}\right)}{1!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)}{3!}x^3 + \cdots + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\cdots\left(\frac{1}{2}-r+1\right)}{r!}x^r + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots + \frac{(-1)^{r-1}}{r \cdot 2^{2r-1}}\binom{2r-2}{r-1}x^r + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k \cdot 2^{2k-1}} \binom{2k-2}{k-1} x^k
\end{aligned}
$$

*Example:* Estimate $\sqrt{20}$
We could treat this as $\sqrt{1+19}$, but we need to have $|x| < 1$ for a good estimate.

$$\begin{aligned}
\sqrt{20} = \sqrt{16+4} \qquad &= \sqrt{16}\left(\sqrt{1+\frac{1}{4}}\right) = 4\sqrt{1+\frac{1}{4}} \\
&\approx 4\left(1+\frac{1}{2}-\frac{1}{8}\left(\frac{1}{4}\right)^2+\frac{1}{16}\left(\frac{1}{4}\right)^3\right) \\
&= \left(1+\frac{1}{8}-\frac{1}{128}+\frac{1}{1024}\right) \\
&= 4.472
\end{aligned}$$