2.1 Four Basic counting Principles

2.1.1 Addition Principle

Ex: Suppose you are eating lunch at a place with 5 different sandwiches and 3 different soups. How many ways are there to order 1 meal (soup or sandwich)? **5+3 = 8**

Let S be a set. A partition of S is a collection of subsets $S_1, S_2, ..., S_m$ such that each element of S appears in *exactly* one of $S_1, S_2, ..., S_m$.

$$S = S_1 \cup S_2 \cup ... \cup S_m$$
$$\phi = S_i \cap S_j \qquad (i \neq j \quad 1 \leq i,\ j \leq m)$$

Addition Principle
Let $S_1, S_2, ..., S_m$ be a partition of finite S. Then

$$\underbrace{|S|}_{\text{\# of elements in S}} = |S_1| + |S_2| + \ldots + |S_m|$$

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2.1.2 Multiplication Principle

Let S be a set of ordered pairs (a,b) of objects where the 1st object a comes from a set of p objects and for each choice of a, the 2nd object b has q choices. Then

$$|S| = p \times q$$

This comes from repeated application of the addition principle

$$|S| = \overbrace{\underbrace{q}_{\substack{q \text{ choices for } b \\ \text{given an } a}} + q + \ldots + q}^{p} = p \times q$$

Equivalently, if a first task has p outcomes and, no matter what the outcome of the first task is, the second task has q outcomes, then the number of ways of accomplishing both tasks is $p \times q$.

Example Ed is picking an outfit. He has 6 distinct shirts and 4 distinct pairs of shorts. How many possible outfits (shirt and shorts) does he have? $6 \times 4 = 24$

Example Suppose we order a pizza with

- 3 possible sizes
- 2 different crust styles

- One of 11 different toppings

How many choices of pizza do we have? $3 \times 2 \times 11 = 66$

How many 2-digit numbers have distinct and non-zero digits? $\underline{9} \times \underline{8} = 72$

How many 2-digit numbers have distinct digits? $\underline{9} \times \underline{9} = 81$

2.1.3 Subtraction Principle

Suppose set A is a subset of a larger set U.

$$\text{Let } \overline{A} = U \setminus A = \{x \in U : x \notin A\}$$

$$\text{Then } |A| = |U| - |\overline{A}| \text{ or } |\overline{A}| = |U| - |A|$$

Example We are generating 6-digit passwords from the symbols 0,1,2,...,9 and a,b,c,...,z. How many passwords have a repeated digit?

- $|U|$:= The number of set of all passwords = 36^6
- $|\overline{A}|$:= The number of set of passwords with no repeated digit = $36\times35\times34\times33\times32\times31$
- $|A|$:= The number of set of passwords with a repeated digit = $|U| - |\overline{A}|$

2.1.4 Division Principle

Let S be a finite set partitioned into k parts such that each part contains the same number of objects. Then the number of parts of the partition is given by

$$k = \frac{|S|}{\text{Common Size of Each Partition}}$$

So, the size of each partition is $\frac{|S|}{k}$.

Example Suppose there are 740 pigeons residing in a collection of pigeonholes such that each pigeonhole contains 5 pigeons. How many pigeonholes are there?

$$k = \frac{740}{5} = 148$$

Example You want to make a nonempty fruit basket from 6 oranges and 9 apples. How many ways to do this?

Questions

- Are all apples identical? **Yes**
- Are all oranges identical? **Yes**

$$\underbrace{\underline{7} \times \underline{10}}_{\text{possible \# of oranges/apples}} -1 = 69$$

4 Types of Counting Questions

- Arrangement and Selection
- Repetition and No Repetition

2.2 Permutations

Permutations are arrangements without repetition. Given a set S of a distinct objects, the r-permutations of S are the arrangements of r elements from S in a distinct order.

Example Let $S = \{a, b, c\}$. The 2-permutations of S are

$$\begin{matrix} ab & ba & ca \\ ac & bc & cb \end{matrix}$$

By the multiplication principle, we can readily compute $P(n, r)$, the number of r-permutations of a set S with n distinct elements.

$$P(n, r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$$

Notation

$$\begin{aligned} P(n, r) &= \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \times (n-r) \times (n-r-1) \times \ldots \times 2 \times 1}{(n-r) \times (n-r-1) \times \ldots \times 2 \times 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Thus

$$0! = 1$$

2.3 Combination of Sets

Given a set S with n distinct elements the $r-$combinations of s are the subsets of s with r elements. We compute the number of $r-$combinations using the permuation formula and the division principle. The number of $r-$combinations of a set with n elements is donoted $\binom{n}{r}$.

$$\binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!}{r!(n-r)!}$$

Circular Permutations Suppose we are arranging objects around a circular table (evenly space) that is free to rotate.

Example 3 people (A, B, C) at a circular table. (Only 2 distinct arrangement.???) In general, for n people seated at a circular table with n seats. Then, there are

$$\frac{P(n,n)}{n} = \frac{n!}{n} = (n-1)!$$

distinct ways of seating them.

Theorem 2.2.2 The number of $r-$circular permutations or n object is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$$

Example How many ways are there to seat to people at a circular table if 2 people will not sit next to each other?

(i) Total seating arrangement: $\frac{10!}{10} = 9!$

(ii) The number of cases 2 people are sitting together: $2 \times \frac{9!}{9} = 2 \times 8!$ (treat 2 as 1)

Then subtract (i)-(ii).

Another approach

- Seat person 1: 10 ways
- Seat person 2: 7 ways
- Seat anyone else: 8! ways.

Hence

$$\frac{10 \times 7 \times 8!}{10}$$

Example How many positive integers are factors of $3^4 \times 5^2 \times 11^7 \times 13^8$?

Factors are $3^i \times 5^j \times 11^k \times 13^l$.

$$0 \leq i \leq 4, \quad 0 \leq j \leq 2, \quad 0 \leq k \leq 7, \quad 0 \leq l \leq 8$$

$\Rightarrow 5 \times 3 \times 8 \times 9$

Example How many integers from 0 to 10,000(inclusive) have exactly 1 digit equal to 5?

Assume we are looking at 4-digit numbers but we allow leading zeros?

Example How many ways to order the 26 letters of the alphabet so that none of a e i o u occur consecutive?

There are 21! ways of arranging ——. By multiplication principle, $21! \times p(22, 5) = 21! \times \frac{22!}{17!}$.

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Recall

$$\binom{n}{0}+\binom{n}{1}+...+\binom{n}{n}=2^n$$

Theorem 2.3.3(Pascal's Formula): For integers n and k with $n \geq 1$ and $1 \leq k \leq n$,

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{k-1}{k}$$

proof. Suppose we have a set of n distinct elements, and we choose k of them. We know there are $\binom{n}{k}$ was to do this. On the other hand, we can do so in a fashion that pays close attention to one element of our set (the "Golden Egg").

$\binom{n-1}{k}$ ways to not choose the golden egg (and $k-1$ other items). So, $\binom{n-1}{k}+\binom{n-1}{k-1}$ ways total.

$\Rightarrow$ These count the same thing, so $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$

2.4 Permutations of Multisets

$\underline{Multisets} are sets that may contain multiple instances of the same thing.$

Example Suppose 5 people each buy one of 3 available computer models and there are <u>infinitely</u> many computers available in each model.How many ways can these people purchase their computers? $3 \times 3 \times 3 \times 3 \times 3 = 3^5$

In this example, if we let A, B, and C denote our computer models, we are counting permutations of the multiset

$$\{\infty \cdot A,\ \infty \cdot B,\ \infty \cdot C\}$$

In general, the multiset $\{n_1 k_1,\ n_2 k_2, ..., n_r k_r\}$ indicates

$$\text{Axxx } n_i \text{ could be } \infty \begin{cases} n_1 \text{ elements } k_1 \\ n_2 \text{ elements } k_2 \\ \vdots \end{cases}$$

Theorem 2.4.1 Let S be a multiset with objects of r different types, where each element has infinite repetition number. Then the number of m-permutations is r^m. (Book uses different notation.)

proof. Use multiplication principle.

Example How many ways are there to arrange all the leeters in MISSISSIPPI?

Solution 1. Treat repeated letters as distinct, to obtain 11! permutation. Now, we use the division principle and divide by $\underbrace{4!}_{4I_s} \times \underbrace{4!}_{4S_s} \times \underbrace{2!}_{2P_s}$ to compensate for repetition. This gives us $\frac{11!}{4!4!2!}$ total permutations.

Solution 2. Use combinations! Choose positions for each letter.

- $\binom{11}{4}$ choices for I_s
- $\binom{7}{4}$ choices for S_s
- $\binom{3}{2}$ choices for P_s
- $\binom{1}{1}$ choices for I_s

Generalizing this example, we have...

Theorem 2.4.2 Let S be a multiset with objects of k different types with finite repetition numbers $n_1,\ n_2, ..., n_k$, respectively. Let the size of S be $n = n_1 + n_2 + ... + n_k$. Then the number of permutations of S equals

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

Theorem 2.4.3 Let n be a positive integer and $n_1,\ n_2, ..., n_k$ positive integers such that $n_1 + n_2 + ... + n_k = n$. The number of ways to partition n objects into k labelled boxes such that

$$\begin{cases} \text{Box 1 contains } n_1 \text{ items} \\ \text{Box 2 contains } n_2 \text{ items} \\ \vdots \\ \text{Box k contains } n_k \text{ items is} \end{cases}$$

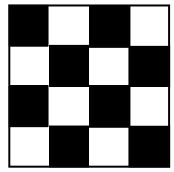
$$\frac{n!}{n_1!n_2!\dots n_k!}$$

If the boxes are not labelled and $n_1 = n_2 = ... = n_k$ then the number of partitions equals

$$\frac{n!}{k!n_1!n_2!...n_k!}$$

Suppose we are given an $n \times n$ chessboard and we wish to place n identical rooks so that no two rooks are in attacking position. (i.e. they do not share a row or column)

Example $n = 4$



There are $n!$ ways to do this:

Go column by column

- First column has 1 rook, n possible rows.
- Second column has 1 rook, $n-1$ possible rows.
- Third column has 1 rook, $n-2$ possible rows.
  $\vdots$
- *nth* column has 1 rook, 1 possible rows.

Use multiplication principle.

Now suppose the n rooks are distinct. This gives us $(n!)^2$ ways of placing the rooks in non-attacking position:

- n! configurations
- n! rearrangements of the rooks in each configuration.

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Theorem 2.4.4 The number of ways to place n rooks in non-attacking position on an $n \times n$ chess board with

$$\sum_{i=1}^{k} n_i = n \begin{cases} n_1 \text{ rooks of color 1} \\ n_2 \text{ rooks of color 2} \\ \vdots \\ n_k \text{ rooks of color k} \end{cases}$$

$$= \frac{(n!)^2}{n_1!n_2!\dots n_k!}$$

2.5 Combinations of Multisets

Example Suppose we order 12 doughnuts of 8 different varieties ("infinitely" many of each variety available).

To solve this, we think of there being 8-1=7 dividers between types of doughnuts. (We assume all doughnuts of each type are grouped together. It's ok to have zero doughnuts of a given type.)

So we have 19 spaces for doughuts and dividers. We choose 7 of these for the dividers.

$$\underline{G}\ \underline{G}\ \underline{G}\ \underbrace{\underline{/}}_{\text{Divider}}\ \underline{J}\ \underline{J}\ \underline{/}\ \underline{C}\ \underline{C}\ \underline{C}\ \underline{C}\ \underline{/}\ \underline{/}\ \underline{L}\ \underline{/}\ \underline{F}\ \underline{/}\ \underline{/}\ \underline{K}$$

So the number of doughnut orders is $\binom{12+8-1}{8-1} = \binom{19}{7}$.

Recall: $\binom{19}{7} = \binom{19}{12} \leftarrow$ ways to position the doughnuts themselves.

Theorem 2.5.1 Let S be a multiset with objects of k different types, each with infinite repetition number. Then the number of r-combinations of S equals

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

Proof Idea This is just like the doughnut example with r doughnuts and k varieties.

Note that our doughnut example gave us solutions to

$$\sum_{i=1}^{8} n_i = 12$$

What if $n_2 \geq 2$? Then let $n_2' = n_2 - 2$. So $n_2' \geq 0$. Look at $n_1 + n_2' + n_3 + \ldots + n_8 = 10$

Example What is the number of non-decreasing sequences of length r taken from the numbers $1, 2, ..., k$?

We are choosing r things from the multiset $\{\infty \cdot 1,\ \infty \cdot 2, \ldots,\ \infty\}$. So we have

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1} \text{ choices.}$$

Example the number of ways to distribute 9 identical apples to 3 kids $\binom{9+3-1}{3-1}$ (9 apples, 3 "varieties" based on which kid gets each.)

$$\underline{\text{Kid}}\ \underline{1\text{'s}}\ \underline{\ }\ \underline{/}\ \underline{\text{Kid}}\ \underline{2\text{'s}}\ \underline{/}\ \underline{\ }\ \underline{\text{Kid}}\ \underline{3\text{'s}}\ \underline{\ }$$

2.6 Probability

Idea There is an experiment ζ with a set of outcomes sample space S. An event E is a subset of S. We will assume all outcomes are equally likely.

Example Suppose our experiment is rolling two [6-sided] dice (one blue, one red).

$$\begin{aligned} S = \{&(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\\ &(1,2),(2,2),(3,2),(4,2),(5,2),(\mathbf{6},\mathbf{2})\\ &(1,3),(2,3),(3,3),(4,3),(\mathbf{5},\mathbf{3}),(6,3)\\ &(1,4),(2,4),(3,4),(\mathbf{4},\mathbf{4}),(5,4),(6,4)\\ &(1,5),(2,5),(\mathbf{3},\mathbf{5}),(4,5),(5,5),(6,5)\\ &(1,6),(\mathbf{2},\mathbf{6}),(3,6),(4,6),(5,6),(6,6)\} \end{aligned}$$

Let E be the event that the dice roll adds to 8.

$$E = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

- S has 36 elements
- E has 5 elements

In this case, we say that the probability of event E occurring is $\frac{5}{36}$. In generally, the probability of event of event E is

$$P(E) = \frac{|E|}{|S|}.$$

Note that $0 \le P(E) \le 1$.

Cool Trick: Sometimes figuring out the probability that something will not happen and subtracting this from 1 is easier.

In other words,

$$P(\overline{E}) = 1 - P(E).$$

Example To compute the probability that 2 people in a group share a birthday, compute the probability that they do not share a birthday. Given n people, there are 365^n birthday possibilities (Ignoring Feb 29).

There are $P(365, n)$ ways they can have distinct birthdays (Assuming $n \le 365$). So, the probability of an unshared birthday is

$$\frac{\frac{365!}{(365-n)!}}{365^n}$$

. The probability of a shared birthday is

$$1-\frac{\frac{365!}{(365-n!)}}{365^n}$$