4 Generating Permutations and Combinations

4.1 Generating Permutation

Potentially useful formula Sterling's Formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

<u>Idea</u> We can write down all the permutations of n objects by "weaving" n across the permutations of $n-1$ objects.

Example(p.90)

	1		2		3	**4**
	1		2	**4**	3	
	1	**4**	2		3	
4	1		2		3	
4	1		3		2	
	1	**4**	3		2	
	1		3	**4**	2	
	1		3		2	**4**
	3		1		2	**4**
	3		1	**4**	2	
	3	**4**	1		2	
4	3		1		2	
4	3		2		1	
	3	**4**	2		1	
	3		2	**4**	1	
	3		2		1	**4**
	2		3		1	**4**
	2		3	**4**	1	
	2	**4**	3		1	
4	2		3		1	
4	2		1		3	
	2	**4**	1		3	
	2		1	**4**	3	
	2		1		3	**4**

Algorithm for generating permutations of {1, 2, ..., n}

- Start with $\overleftarrow{1}\,\overleftarrow{2}\,\ldots\,\overleftarrow{n}$
  (A mobile integer is one whose arrow points to an adjacent smaller integer.)

  1. Find the largest mobile integer m.
  2. Switch m with the adjacent integer that its arrow points to.
  3. Switch the direction of all arrows above integers p with $p > m$.

Example n=4

$$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}$$
$$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}\,\overleftarrow{3}$$
$$\overleftarrow{1}\,\overleftarrow{4}\,\overleftarrow{2}\,\overleftarrow{3}$$
$$\overleftarrow{4}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}$$
$$\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}$$
$$\overleftarrow{1}\,4\,\overleftarrow{3}\,\overleftarrow{2}$$
$$\overleftarrow{1}\,\overleftarrow{3}\,\overrightarrow{4}\,\overleftarrow{2}$$
$$\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}$$
$$\overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{4}$$
$$\vdots$$

until

$$\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{3}\,\overrightarrow{4}$$

Nothing mobile, so we stop!

4.2 Inversions in Permutations

Let $i_1 i_2 i_3 \ldots i_n$ be a permutation of $\{1,2,\ldots,n\}$. An <u>inversion</u> is a pair $(i_k,\ i_l)$ such that $k < l$ but $i_k > i_l$.

Example 6 3 5 1 2 4 has inversions

$$\begin{array}{ll} (6,3) & (3,1)(5,1) \\ (6,5) & (3,2)(5,2) \\ (6,1) & \quad\;\;\, (5,4) \\ (6,1) & \\ (6,2) & \\ (6,4) & \end{array}$$

There are 10 total inversions.

For the permutation $i_1 i_2 i_3 \ldots i_n$, let a_j be the number of inversions whose second component is j. The sequence $a_1 a_2 a_3 \ldots a_n$ is called the <u>inversion sequence</u>. (a_j equals the number of integers that precede j in the permutation that are greater than j).

Example $a_1 = 3 \quad a_2 = 3 \quad a_3 = 1 \quad a_4 = 2 \quad a_5 = 1 \quad a_6 = 0$ So, the permutation 6 3 5 1 2 4 has inversion sequence 3 3 1 2 1 0.

Theorem 4.2.1 Let $b_1,\ b_2,\ ..., b_n$ be integers satisfying

$$0 \leq b_1 \leq n-1,\ 0 \leq b_2 \leq n-2,\ 0 \leq b_{n-1} \leq 1,\ b_n = 0$$

. Then there is a unique permutation whose inversion sequence is $b_1 b_2 \dots b_n$.

Proof Use an algorithm (see pp.94-96 of Brualdi for details)
Let 5 1 4 3 0 1 1 0 be an inversion sequence.

<u>Algorithm I</u> (right to left):
Each b_k tells us how many entries in permutation bigger than k precede k.

8

87

867

5867

58647

586437

5286437

52864137

<u>Algorithm II</u> (left to right):
Place each k in open spot $b_k + 1$

$\underline{5}\ \underline{2}\ \underline{8}\ \underline{6}\ \underline{4}\ \underline{1}\ \underline{3}\ \underline{7}$

4.3 Generating Combinations

Goal Given a set of n distinct elements, write down all 2^n subsets of our set.

Approach count n-digit binary integers.
<u>Ex</u> n=4 S={A, B, C, D} $0 \leq k \leq 2^n - 1$

k	n-bit binary representation of k	subset of S
0	0000	ϕ
1	0001	{A}
2	0010	{B}
3	0011	{A,B}
4	0100	{C}
5	0101	{A,C}
6	0110	{B,C}
7	0111	{A,B,C}
8	1000	{D}
9	1001	{A,D}
10	1010	{B,D}
11	1011	{A,B,D}
12	1100	{C,D}
13	1101	{A,C,D}
14	1110	{B,C,D}
15	1111	{A,B,C,D}

This ordering is called lexicographic ordering of the sequence of 0s and 1s.

The order of the third column 'subset of S' is called the squashed order.

4.4 Generating r-subsets

Recall the squashed order of the subsets of {A,B,C,D}.

0	0	0	ϕ
0	0	1	{A}
0	1	0	{B}
0	1	1	{A,B}
1	0	0	{C}
1	0	1	{A,C}
1	1	0	{B,C}
1	1	1	{A,B,C}

2-subsets: {A,B}, {A,C}, {B,C} $\Rightarrow$ squashed ordering of 2-subsets inherited from the squashed ordering of all subsets.

Example Squashed order of 2-subsets of {A,B,C,D} is

$$\{A,B\} \quad \{A,C\} \quad \{A,D\}$$

$$\{B,C\} \quad \{B,D\} \quad \{C,D\}$$

Big Idea Lexicographic order is essentially alphabetic order.

Example Lexicographic order of 3-subsets of $\{1,2,3,4,5\}$ (should be $\binom{5}{3} = 10$ of these)

$$\begin{array}{ccccccc} 1 & 2 & 2 & \quad & 1 & 4 & 5 \\ 1 & 2 & 4 & & 2 & 3 & 4 \\ 1 & 2 & 5 & & 2 & 3 & 5 \\ 1 & 3 & 4 & & 2 & 4 & 5 \\ 1 & 3 & 5 & & 3 & 4 & 5 \end{array}$$