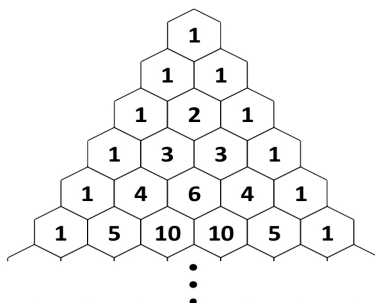# 5   Binomial Theorem

Goal: Expand $(x + y)^n$ quickly.

5.1   Pascal's Triangle

Each number is the sum of the two above it.

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = (x + y)(x + y)^3$$
$$= (x + y)(x^3 + 3x^2y + 3xy^2 + y^3)$$
$$= x^4 + 3x^3y + 3x^2y^2 + xy^3 + x^3y + 3x^2y^2 + 3xy^3 + y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

<u>Recall</u>: Pascal's Formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

What this tells us is that Pascal's triangle is also

$$\binom{0}{0}$$
$$\binom{1}{0} \qquad \binom{1}{1}$$
$$\binom{2}{0} \qquad \binom{2}{1} \qquad \binom{2}{2}$$
$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$
$$\binom{4}{0} \qquad \binom{4}{1} \qquad \binom{4}{2} \qquad \binom{4}{3} \qquad \binom{4}{4}$$

Properties

- Pascal's Triangle has vertical reflective symmetry because $\binom{n}{k} = \binom{n}{n-k}$

- Rows add to 2^n because $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

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5.2 Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \left[= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k} \right]$$

Example

$$
\begin{aligned}
(x+y)^6 &= \sum_{k=0}^{6} \binom{6}{k} x^{6-k} y^k \\
&= \binom{6}{0} x^6 y^0 + \binom{6}{1} x^5 y^1 + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x^1 y^5 + \binom{6}{6} x^0 y^6
\end{aligned}
$$

Proof Idea 1:

$$(x+y)^n = \underbrace{(x+y)(x+y)\dots(x+y)}_{n \text{ times}}$$

We know this will be a sum of monomials $x^{n-k} y^k$. (The number of x and y equals n)

The coefficient is the number of ways to choose x (or y) at each stage. i.e.

$$\binom{n}{n-k} = \binom{n}{k}$$

Special Case (Theorem. 5.2.2): Let $y = 1$ in binomial theorem.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{n-k} x^k$$

5.3 Unimodality of Binomial Coefficients

Combinatorial Identities

$$k\binom{n}{k} = n\binom{n-1}{k-1} \tag{5.2}$$

set $x = y = 1$ in binomial theorem to get:

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} \tag{5.3}$$

set $x = 1$ and $y = -1$ in binomial theorem to get:

$$(1-1)^n = 2^n = \binom{n}{0}(-1)^1 + \binom{n}{1}(-1)^2 + \ldots + \binom{n}{n}(-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} \tag{5.4}$$

Add (5.3) and (5.4)

$$2^n + 0 = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} + \binom{n}{0} - \binom{n}{1} + \binom{n}{2} = \binom{n}{3} + \ldots + (-1)^n \binom{n}{n}$$

$$2^n = 2\binom{n}{0} + 2\binom{n}{2} + 2\binom{n}{4} + \ldots + 2 \begin{cases} \binom{n}{n-1} & \text{if } n \text{ odd} \\ \binom{n}{n} & \text{if } n \text{ even} \end{cases}$$

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots \tag{5.6}$$

(all terms are eventually 0)

Subtracting 5.4 from 5.3, we get

$$2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots \tag{5.7}$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

Differentiate both sides with respect to x

$$n(x+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + n\binom{n}{n}x^{n-1}$$

Letting $x = 1$, we obtain

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} \tag{5.8}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{5.16}$$

are equivalent to

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

These count $\binom{2n}{n}$ in 2 different ways, directly and by looking at how many elements are taken from each of two subcollections of n things.

$\binom{r}{k}$ makes sense if r is a real number and k is any integer. In this case,

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)(r-2)\ldots(r-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k =\leq -1 \end{cases}$$

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<u>Example</u> $r = \frac{1}{2}$, $k = 3$

$$\binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!} = \frac{3}{48} = \frac{1}{16}$$

$$\binom{\sqrt{2}}{-1} = 0$$

$$\binom{4 \cdot 8}{0} = 1$$

$$\binom{-4}{5} = \frac{(-4) \cdot (-5) \cdot (-6) \cdot (-7) \cdot (-8)}{5!} - 56$$

It is still true that

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1} \qquad \text{(Pascal's Formula)}$$

$$k\binom{r}{k} = r\binom{r-1}{k-1}$$

If k is a positive integer,

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

$$= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-2}{k-2}$$

$$= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \binom{r-3}{k-3}$$

$$= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \ldots + \binom{r-k}{1} + \binom{r-k}{0}$$

$$= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \ldots + \binom{r-k}{1} + \binom{r-k-1}{0} +$$

$$\cancel{\binom{r-k-1}{-1}}$$

Replace r with $r + k + 1$

$$\binom{r}{0} + \binom{r+1}{1} + \binom{r+2}{2} + \ldots + \binom{r+k}{k} = \binom{r+k+1}{k} \qquad (5.18)$$

$$(1)$$

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For $n,\ k$ positive integers,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-1}{nk-1}$$

$$= \binom{n-3}{k} + \binom{n-3}{k-1} + \binom{n-2}{k-1} + \binom{n-1}{k-1}$$

$$\vdots$$

$$= \cancel{\binom{0}{k}} + \cancel{\binom{0}{k-1}} + \binom{1}{k-1} + \ldots + \binom{n-1}{k-1}$$

5.4 Multinomial Theorem

Suppose we need to compute $(x+y+z)^3$

1. We could multiply it all out. If we are clever, we will automate it with some combinatorial reasoning.

2. We could use the binomial theorem *twice*.

$$(x+(y+z))^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2(y+z)^1 + \binom{3}{2}x(y+z)^2 + \binom{3}{3}(y+z)^3$$

$$= \binom{3}{0}x^3 + \binom{3}{1}\binom{1}{0}x^2y + \binom{3}{1}\binom{1}{1}x^2z + \binom{3}{2\binom{2}{0}}xy^2 + \binom{3}{2}\binom{2}{1}xyz +$$

$$\binom{3}{2}\binom{2}{1}xz^2 + \binom{3}{3}\binom{3}{0}y^3 + \binom{3}{3}\binom{3}{1}y^2z + \binom{3}{3}\binom{3}{2}yz^2 + \binom{3}{3}\binom{3}{3}z^3$$

<u>Notation</u>:

$$\underbrace{\binom{n}{n_1\ n_2\ n_3 \ldots n_t}}_{n_i \text{ is multinomial coefficient}} = \frac{n!}{n_1!\ n_2!\ n_3! \ldots n_t!}$$

This refers to the number of ways to arrange n objects that include

$\quad n_1$ identical objects

$\quad n_2$ identical objects

$\quad\quad \vdots$

$\quad n_t$ identical objects

Note that $\sum\limits_{n=1}^{t} = n.$

Example Permutation of WISCONSIN

$$\binom{9}{1 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 2} = \frac{9!}{1!2!2!1!1!2!}$$

Note that

$$\binom{n}{k} = \binom{n}{k(n-k)}$$

Also

$$\binom{n}{n_1 n_2 \dots n_t} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\dots\binom{n-n_1-n_2-\dots-n_{t-1}}{n_t}$$

Multinomial Theorem
Example

$$(x+y+z)^3 = (x+y+z)(x+y+z)(x+y+z)$$
$$= \binom{3}{3}x^3 + \binom{3}{2\,1}x^2y + \binom{3}{1\,2}xy^2 + \binom{3}{3}y^3 + \binom{3}{2\,1}x^2z + \binom{3}{1\,2}xz^2 +$$
$$\binom{3}{1\,1\,1}xyz + \binom{3}{2\,1}y^2z + \binom{3}{1\,2}yz^2 + \binom{3}{3}z^3$$

Theorem 5.4.1 Multinomial Theorem

$$(x_1 + x_2 + x_3 + \dots + x_t)^n = \sum \binom{n}{n_1 n_2 \dots n_t} x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n^t}$$

where the sum is over all non-negative integers n_1, n_2, ..., n_t such that $n_1 + n_2 + \dots + n_t = n$.

Example Find the coefficient of $x^4 y^3 z^2 w$ in $(x+y+z+w)^{10}$.

$$\binom{10}{4\,3\,2\,1} = \frac{10!}{4!3!2!1!}$$

Example Find the coefficient of $x^4 y^3 z^2 w$ in $(x+y+z+w)^{10}$

It is 0 because 4+3+2+2=11, so there is no such term.

Pascal's Formula

$$\binom{n}{n_1 \; n_2 \; \ldots \; n_t} = \binom{n-1}{n_1 - 1 \; n_2 \ldots n_t} + \binom{n-1}{n_1 \; n_2 - 1 \ldots n_t} + \ldots + \binom{n-1}{n_1 - 1 \; n_2 \ldots n_t - 1}$$

5.5　Newton's Extended Binomial Theorem

$$(x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-y} \qquad \text{(for any real number } \alpha)$$

Special Case Let z be a complex number with $|z| < 1$. Then

$$(1 + z)^\alpha = \sum k = 0^\infty \binom{\alpha}{k} z^k$$

Example: $\alpha = 3$

$$(1 + z)^3 = \sum k = 0^\infty \binom{3}{k} z^k = \binom{3}{0} + \binom{3}{1} z + \binom{3}{2} z^2 + \binom{3}{3} z^3 + \binom{3}{4} z^4 + \ldots$$
$$= 1 + 3z + 3z^2 + z^3 + 0 + \ldots$$
$$= 1 + 3z + 3z^2 + z^3$$

Example: $\alpha = -1$

$$(1 + z)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} z^k = \binom{-1}{0} + \binom{-1}{1} z + \binom{-1}{2} z^2 + \binom{-1}{3} z^3 + \ldots$$
$$= 1 - z + z^2 - z^3 + z^4 - \ldots$$

This is just the sum of an infinite geometric series in disguise.

$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + \ldots$$

$$\binom{-1}{k} = \frac{(-1)(-1-1)(-1-2)\ldots(-1-(k-1))}{k!}$$
$$= \frac{(-1)(-2)(-3)\ldots(-k)}{k!} = (-1)^k \cdot \frac{k!}{k!} = (-1)^k$$

Recall: $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

By Newton's Extended Binomial Theorem,

$$
\begin{aligned}
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{-n}{k}(-x)^k \\
&= \binom{-n}{0}(-x)^0 + \binom{-n}{1}(-x)^1 + \binom{-n}{2}(-x)^2 + \binom{-n}{3}(-x)^3 + \cdots \\
&= 1 + \frac{(-n)}{1!}(-x) + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}(-x^3) + \cdots \\
&= 1 + \binom{n}{1}x + \frac{(n+1)!}{2!(n-1)!}x^2 + \frac{(n+2)!}{3!(n-1)!}x^3 + \cdots + \frac{(n+r-1)!}{r!(n-1)!}x^r + \cdots \\
&= 1 + \binom{n}{1}x + \frac{(n+1)!}{2!(n-1)!}x^2 + \frac{(n+2)!}{3!(n-1)!}x^3 + + \frac{(n+r-1)!}{r!(n-1)!}x^r + \cdots \\
&= 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \cdots + \binom{n+r-1}{r}x^r + \cdots \\
(1-x)^{-n} &= \sum_{r=0}^{\infty} \binom{n+r-1}{r}x^r \\
&= (1 + x + x^2 + \cdots)^n \\
&= \underbrace{(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{n\ times}
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{1+x} = (1_x)^{1/2} &= \sum_{k=0}^{\infty} \binom{1/2}{k}x^k \\
&= \binom{1/2}{0} + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \cdots + \binom{1/3}{r}x^r + \cdots \\
&= 1 + \frac{\left(\frac{1}{2}\right)}{1!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)}{3!}x^3 + \cdots + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\cdots\left(\frac{1}{2}-r+1\right)}{r!}x^r + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots + \frac{(-1)^{r-1}}{r \cdot 2^{2r-1}}\binom{2r-2}{r-1}x^r + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k \cdot 2^{2k-1}}\binom{2k-2}{k-1}x^k
\end{aligned}
$$

Example: Estimate $\sqrt{20}$

We could treat this as $\sqrt{1 + 19}$, but we need to have $|x| < 1$ for a good estimate.

$$\sqrt{20} = \sqrt{16 + 4} \qquad = \sqrt{16}\left(\sqrt{1 + \frac{1}{4}}\right) = 4\sqrt{1 + \frac{1}{4}}$$

$$\approx 4\left(1 + \frac{1}{2} - \frac{1}{8}\left(\frac{1}{4}\right)^2 + \frac{1}{16}\left(\frac{1}{4}\right)^3\right)$$

$$= \left(1 + \frac{1}{8} - \frac{1}{128} + \frac{1}{1024}\right)$$

$$= 4.472$$