5.16 By integrating the binomial expansion, prove that, for a positive integer n,

$$1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}.$$

Consider binomial expansion of $(1+x)^n$.

$$\begin{aligned}(1+x)^n &= 1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n\\ \int(1+x)^n dx &= \int\left[1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n\right]dx\\ \frac{1}{n+1}(1+x)^{n+1} &= x+\frac{x^2}{2}\binom{n}{1}+\frac{x^3}{3}\binom{n}{2}+\cdots+\frac{x^{n+1}}{n+1}\binom{n}{n}+C\end{aligned}$$

when plugging in $x=0$, C turns out to be $\frac{1}{n+1}$.

$$\frac{1}{n+1}(1+x)^{n+1}=x+\frac{x^2}{2}\binom{n}{1}+\frac{x^3}{3}\binom{n}{2}+\cdots+\frac{x^{n+1}}{n+1}\binom{n}{n}+\frac{1}{n+1}$$

Consider $x=1$.

$$\begin{aligned}\frac{2^{n+1}}{n+1} &= 1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}+\frac{1}{n+1}\\ \frac{2^{n+1}}{n+1}-\frac{1}{n+1} &= 1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}\\ \frac{2^{n+1}-1}{n+1} &= 1+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}\end{aligned}$$

5.19 Sum the series $1^2 + 2^2 + 3^2 + \cdots + n^2$ by observing that

$$m^2 = 2\binom{m}{2} + \binom{m}{1}$$

and using the identity (5.19).

$$\sum_{m=1}^{n} m^2 = 2\sum_{m=1}^{n}\binom{m}{2} + \sum_{m=1}^{n}\binom{m}{1}$$

By suing the identity

$$\binom{n+1}{k+1} = \sum_{i=0}^{n}\binom{i}{k}$$

$$\begin{aligned}
\sum_{m=1}^{n} m^2 &= 2\cdot\binom{n+1}{2+1} + \binom{n+1}{1+1} \\
&= 2\cdot\frac{(n+1)!}{3!(n-2)!} + \frac{(n+1)!}{2!(n-1)!} \\
&= 2\cdot\frac{(n+1)\cdot n\cdot(n-1)(n-2)}{3!} + \frac{(n+1)n}{2!} \\
&= \frac{n(n+1)\cdot[2(n-1)+3]}{6} \\
&= \frac{n(n+1)(2n-1)}{6} = 1^2 + 2^2 + 3^2 + \cdots + n^2
\end{aligned}$$

5.37 Use the multinomial theorem to show that, for positive integers n and t,

$$t^n = \sum \binom{n}{n_1 n_2 \cdots n_t},$$

where the summation extends over all nonnegative integral solutions $n_1, n_2, ..., n_t$ of $n_1 + n_2 + ... + n_t = n$.

Recall Multinomial Theorem

$$(x_1 + x_2 + x_3 + ... + x_t)^n = \sum \binom{n}{n_1 n_2 \dots n_t} x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n^t}$$

To show the given equation on the question, consider all x_i to be 1. Then

$$\begin{aligned} t^n = (\underbrace{1 + 1 + \cdots + 1}_{t})^n &= \sum \binom{n}{n_1 n_2 \dots n_t} 1^{n_1} 1^{n_2} 1^{n_3} \dots 1^{n^t} \\ &= \sum \binom{n}{n_1 n_2 \dots n_t} \end{aligned}$$

5.38 Use the multinomial theorem to expand $(x_1 + x_2 + x_3)^4$.

By the theorem

$$\begin{aligned}
(x_1 + x_2 + x_3)^4 &= \sum_{n_1+n_2+n_3=4} \binom{4}{n_1\ n_2\ n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3} \\
&= \frac{4!}{4!}(x_1^4 + x_2^4 + x_3^4) + \\
&\quad \frac{4!}{1!\ 3!}(x_1^1x_2^3 + x_1^3x_2^1 + x_1^1x_3^3 + x_1^3x_3^1 + x_2^1x_3^3 + x_2^3x_3^1) + \\
&\quad \frac{4!}{2!\ 2!}(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + \\
&\quad \frac{4!}{1!\ 1!\ 2!}(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2) \\
&= x_1^4 + x_2^4 + x_3^4 + \\
&\quad 4(x_1^1x_2^3 + x_1^3x_2^1 + x_1^1x_3^3 + x_1^3x_3^1 + x_2^1x_3^3 + x_2^3x_3^1) + \\
&\quad 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + \\
&\quad 12(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2)
\end{aligned}$$

5.39 Determine the coefficient of $x_1^3x_2x_3^4x_5^2$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$.

$$\begin{aligned}
\binom{10}{3\ 1\ 4\ 2} &= \frac{10!}{3!\ 1!\ 4!\ 2!} \\
&= 10 \cdot 9 \cdot 4 \cdot 7 \cdot 5 = 12600
\end{aligned}$$

5.40 What is the coefficient of $x_1^3x_2^3x_3x_4^2$ in the expansion of

$$(x_1 - x_2 + 2x_3 - 2x_4)^9?$$

$$\binom{9}{3\ 3\ 1\ 2} \cdot 1^3 \cdot (-1)^3 \cdot 2^1 \cdot (-2)^2 = -40320$$

5.46 Use Newton's binomial theorem to approximate $\sqrt{30}$.

$$\sqrt{30} = \sqrt{25+5} = 5\sqrt{\left(1+\frac{1}{5}\right)} = 5\left(1+\frac{1}{5}\right)^{1/2}$$

By Newton's Binomial Theorem

$$\begin{aligned}
&= 5\left[1 + \frac{\frac{1}{2}\cdot\frac{1}{5}}{1!} + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(\frac{1}{5}\right)^2}{2!} + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{5}\right)^3}{3!} + \cdots\right] \\
&= 5\left[1 + \frac{1}{10} - \frac{1}{8}\cdot\frac{1}{5^2} + \frac{3}{48}\frac{1}{5^3} + \cdots\right] \\
&= 5\left[1 + \frac{1}{10} - \frac{1}{200} + \frac{1}{2000}\right] \\
&\approx 5.4775
\end{aligned}$$

5.47 Use Newton's binomial theorem to approximate $10^{1/3}$.

$$\begin{aligned}
10^{1/3} &= (8+2)^{1/3} = \left[2^3\left(1+\frac{1}{4}\right)\right]^{1/3} \\
&= 2\left[1 + \frac{\frac{1}{3}\cdot\frac{1}{4}}{1!} + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\left(\frac{1}{4}\right)^2}{2!} + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(\frac{1}{4}\right)^3}{3!} + \cdots\right] \\
&= 2\left[1 + \frac{1}{12} - \frac{1}{9}\cdot\frac{1}{16} + \frac{5}{3^4}\frac{1}{4^3} + \cdots\right] \\
&\approx 2.15407
\end{aligned}$$