Question 3.1
Determine which of the following functions $d_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ determines a metric.

Note that the definition of metric space.

(a) d(p,q) > 0 if $p \neq q$; d(p,p) = 0

(b) d(p,q) = d(q,p)

(c) d(p,q) $\leq$ d(p,r) + d(r,p) $\quad \forall\, r \in \mathbb{R}$.

(i) $d_1(x, y) = (x - y)^2$

  (c) Consider d_1(1,3)

  - d_1(1,3) = 4
  - d_1(1,2) = 1; d_1(2,3) = 1
  - d_1(1,3) $\nleq$ d_1(1,2) + d_1(2,3)
  - Thus this is not a metric.

(ii) $d_2(x, y) = \sqrt{|x - y|}$

  (a) $d_2(x, y) > 0$ if $p \neq q$; d(x,x) = 0

  - If x = y, $d_2(x, y) = 0$;
    Otherwise $d_2(x, y) > 0$ since the absolute value and the square root is always positive by definition.

  (b) $d_2(x, y) = d_2(y, x)$

  - $d_2(x, y) = \sqrt{|x - y|} = \sqrt{|-1| \cdot |y - x|} = \sqrt{|y - x|} = d_2(y, x)$

  (c) $d_2(x, y) \leq d_2(x, z) + d_2(z, y) \quad \forall\, z \in \mathbb{R}$.

  - $d_2(x, y) = \sqrt{|x - z|}$
  - $d_2(x, z) = \sqrt{|x - z|}; \quad d_2(z, y) = \sqrt{|z - y|} \quad \forall\, z \in \mathbb{R}$
  - $d_2(x, y) \leq d_2(x, z) + d_2(z, y)$

(iii) $d_3(x, y) = |x^2 - y^2|$

  (a) Consider d_3(1,-1) where $x \neq y$

  - d_3(1,-1) = 0 $\ngtr$ 0
  - Thus this is not a metric.

(iv) $d_4(x, y) = |x - 2y|$

(a) Consider d_4(1,1)

- $d_4(1,1) = 1 \neq 0$
- Thus this is not a metric.

(v) $d_5(x, y) = \frac{|x-y|}{1+|x-y|}$

(a) If $x = y$, $|x - y| = 0 \Leftrightarrow d_5(x, y) = 0$;
Otherwise $d_5(x, y) = \frac{|x-y|}{1+|x-y|} > 0$

(b) $d_5(x, y) = \frac{|x-y|}{1+|x-y|} = \frac{|-1|\cdot|y-x|}{1+|-1|\cdot|y-x|} = \frac{|y-x|}{1+|y-x|} = d_5(y, x)$

(c) $d_5(x, y) = \frac{|x-y|}{1+|x-y|}$
$d_5(x, z) = \frac{|x-z|}{1+|x-z|}$; $d_5(z, y) = \frac{|z-x|}{1+|z-x|}$ $\forall\, z \in \mathbb{R}$.
$\Rightarrow$ $\mathrm{d}_5(x, y) \leq d_5(x, z) + d_5(z, y)$

Question 3.2

Let $A_1, A_2, A_3, ...$ be subsets of a metric space.

(i) If $B_n = \cup_{i=1}^n A_i$, prove that $\overline{B}_n = \cup_{i=1}^n \overline{A}_i$.

- $n = 2$
  - $\overline{A_1} \cup \overline{A_2} = (A_1 \cup A_1') \cup (A_2 \cup A_2')$ by the definition of closed set where A_i' is a set of all limit points in A.
  - $(A_1 \cup A_1') \cup (A_2 \cup A_2') = (A_1 \cup A_2) \cup (A_1' \cup A_2')$ $= (A_1 \cup A_2) \cup (A_1 \cup A_2)' = \overline{A_1 \cup A_2}$
- $n = 3$
  - $\overline{A_1} \cup \overline{A_2} \cup \overline{A_3} = \overline{A_1 \cup A_2} \cup \overline{A_3} = ((A_1 + A_2) + (A_1 + A_2)') + (A_3 + A_3')$ $= (A_1 + A_2 + A_3) + (A_1 + A_2 + A_3)' = \overline{A_1 + A_2 + A_3}$
- $B_n = \cup_{i=1}^n A_i$
  - Therefore $\overline{\cup_{i=1}^n A_i} = \cup_{i=1}^n \overline{A_i}$

(ii) If $B = \cup_{n=1}^\infty A_n$, prove that $\overline{B} \supset \cup_{n=1}^\infty \overline{A}_i$.

- Recall that a closed set $\overline{B}_n = B_n \cup B_n'$ (where B_n' is a set of all its limit points).
- Suppose $A_n = \frac{1}{n}$
- Since A_n is open, $0 \notin A_n'$; However, $0 \in B_n'$ as n goes to ∞ due to its closeness.
- Hence $\overline{B} \supset \cup_{n=1}^\infty \overline{A_i}$ is said to be proper.

Question 3.3

Let E be a subset of a metric space and let $E = E^\circ$

(i) E° is open

- Let $x \in E^\circ$, and suppose $\exists\, y \in E$ and $r > 0$ such that $d(x, y) < r$.
- Let $h = r - d(x, y)$, and $\exists\, \alpha$ such that $d(y, \alpha) < h < r$.
- Then, by the Triangle inequality, $d(x, \alpha) \leq d(x, y) + d(y, \alpha)$
- Since $d(y, \alpha) < h,\;\; d(y, x) + d(y, \alpha) < d(x, y) + h = r$
- Hence $\alpha \in E$. i.e. $y \in E^\circ$.
- This follows all such points are in E°.
- Thus E° is open.

(ii) E is open iff $E = E^\circ$

- $\Rightarrow$ Obviously E° is included in E ($E^\circ \subset E$).
  By the definition of openness, every point of E is an interior point of E.
  i.e. $E^\circ \subset E$. Thus $E^\circ = E$.
- $\Leftarrow$ Since every point of E is an interior point of E, E is open.

(iii) If $G \subset E$ and G is open, the $G \subset E^\circ$ (so E° is the largest open subset of E).

- Take $\alpha \in G$. Then $\alpha \in E^\circ$ since $G \subset E$ and E° is the largest open subset of E.
- Hence $G \subset E$.

(iv) $(E^\circ)^c = \overline{E^c}$, where the overline denotes closure.

- $(E^\circ)^c \subseteq \overline{E^c}$
  - Suppose $\exists\, x \in \overline{A^c}$.
  - Then for every $\epsilon > 0,\; B(x, \epsilon) \cup A^c \neq \phi$.
  - i.e. there are some overlapped area between any ball around x and A^c.
  - This strictly implies $x \notin A$ or $x \notin A^\circ$, but $x \in (A^\circ)^c$.
- $(E^\circ)^c \supseteq \overline{E^c}$
  - Suppose $\exists\, x \in (A^\circ)^c$.
  - Then there is no such ball $B(x, \epsilon) \subseteq A$ for every ϵ.
  - i.e. there are some overlapped area between any ball around x and A^c.
  - This also shows $x \in \overline{A^c}$.

- Hence we proved $(E^\circ)^c = \overline{E^c}$

Question 3.4
Let E be a subset of a metric space. Either prove or find a counterexample to:

(i) $E^\circ = (\bar{E})^\circ$

- Obviously $E = E^\circ$ and $\overline{E} =(\overline{E})^\circ$
- Suppose $E = (1,2) \cup (2,3)$. Then $\overline{E} = \{1\} \cup \{2\} \cup \{3\} \cup (1,3) = [1,3]$.
- Hence $E^\circ \neq (\bar{E})^\circ$, but $E^\circ \subset (\bar{E})^\circ$

(ii) $\overline{E} = \overline{(E^\circ)}$

- Suppose $E = \mathbb{Q}$ (The whole space of rational number).
- Then $\overline{E} = \mathbb{R} \setminus \mathbb{Q}$.
- $(\overline{E^\circ}) = (\overline{\mathbb{Q}^\circ}) = \phi$ since irrational is dense in $\mathbb{R}$.
- Thus $\overline{E} \neq \overline{\mathbb{Q}^\circ}$

Question 3.5
A metric space is called separable if it contains a countable dense subset. Prove that $\mathbb{R}^n$ is separable.
Hint: Consider the set of points with rational coordinates.

- A handy example for a countable dense subset would be $\mathbb{Q} \in \mathbb{R}$.
- Take $p = (p_1, ..., p_n) \in \mathbb{R}^k$.
- Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $p \in \mathbb{Q} \cup \mathbb{Q}'$ (where $\mathbb{Q}'$ is a set of limit points of $\mathbb{Q}$).

- In order p to be limit points of $\mathbb{Q}$, $\exists\ B_r(p)$ such that a ball around p with radius r.
- Take $q = (q_1, ..., q_n) \in \mathbb{Q}$ such that $q_i \neq p_i$ for $i = 1, ..., n$.
- Let $\alpha = \frac{r}{n}$, and we obtain an inequality by the triangle theorem,

$$d(p, q) = \sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2} < \sqrt{\frac{r^2}{n} + ... + \frac{r^2}{n}} = \sqrt{\frac{nr^2}{n}} = r$$

- Hence we conclude that $q \in B_r(p)$, and p is limit points of $\mathbb{Q}$.
- Thus $\mathbb{Q}^k$ is a countable dense subset in $\mathbb{R}^k$

Question 3.6
Consider $\mathbb{Q}$ with the usual distance $d(p,q) = |p-q|$ as a metric space and consider the subset

$$E = \{p \in \mathbb{Q} \mid 2 < p^2 < 3\}.$$

Prove that E is closed and bounded in $\mathbb{Q}$ but that E is not compact in Q. Is E open in $\mathbb{Q}$?

- E is closed in $\mathbb{Q}$.
  - In order to prove E is closed, it should be proved that E is closed in $\mathbb{Q}$.
  - Recall the definition of closed set: A set is closed if every lmit point of E is in E.
  - Suppose $x \in \mathbb{Q}$ is a limit point of E.
  - Obviously $p^2 \neq 2$ or $p^2 \neq 3$.
  - Proof by contradiction
    * Suppose $p^2 < 2 \Leftrightarrow (\sqrt{2}-p)(\sqrt{2}+p) > 0 \quad \Leftrightarrow \quad \sqrt{2} < |p|$
    * For some r, $r + |x| = \sqrt{2}$
    * If $q \in N_r(x)$, then
    $$|q| \leq |x-q| + |x| < |x| + r = \sqrt{2}$$
    * i.e. $q^2 < 2 \Leftrightarrow x^2 < 2$ which contradicts that x is a limit point of E.
    * Hence $2 < x^2$.
    * If $x^2 > 3 \Leftrightarrow (|x| + \sqrt{3})(|x| - \sqrt{3}) > 0 \Leftrightarrow |x| - \sqrt{3} > 0$
    * Suppose $s = |x| - \sqrt{3}$.
    * Similarly,
    $$|q| \geq \ -|x-q| \geq |x| - s = \sqrt{3}$$
    * Hence for any q is in $N_s(x)$.
    * i.e. $q \notin E \Leftrightarrow$ x is not a limit point of E.
  - Thus we conclude that E is closed in $\mathbb{Q}$.
- E is bounded in $\mathbb{Q}$
  - $\sqrt{2} < p < \sqrt{3}\ ; -\sqrt{3} < p < -\sqrt{2} \quad \Rightarrow \quad -\sqrt{3} < p < \sqrt{3} \quad \Rightarrow \quad |E| < 2$
    $\Rightarrow E$ is bounded by 2.
- Is E open in $\mathbb{Q}$?
  - Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists\ \alpha \in \mathbb{Q}, \epsilon > 0$ such that $|p - \alpha| < \epsilon$ and a ball around p $B_\epsilon(p) \subset E$.
  - i.e. For all x, $x \in E \in E^\circ \Rightarrow E$ is open.