Question 7.1
Let $f\colon X \to \mathbb{R}$ be a continuous function. Prove that the zero set of f, that is the set

$$\{x \mid f(x) = 0\}$$

is closed.

- Note that the Corollary of Theorem 4.8 on Rudin states *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.*
- Suppose $g(f) = f^{-1}(\{0\})$. By the Corollary above, $g(f)$ is closed if f is continuous.

- Let x be a limit point of $g(f)$. Then $\exists$ a sequence $\{x_n\}$ in $g(f)$ such that $d(x_n, x) \to 0$.
- f is continuous at x. So $\exists\, \delta > 0 \;\; \forall \epsilon > 0$ such that $d(x_n, x) < \delta \Leftrightarrow |f(x_n) - f(x)| < \epsilon$.
- As $\{x_n\}$ converges to x, $\exists\, N$ such that $n \geq N$; hence $|f(x_n) - f(x)| < \epsilon$.
- Since $f(x_0) = 0,\; |f(x_n) - f(x)| = d(f(x)) < \epsilon \;\; \forall\, \epsilon > 0$.
- This leads to $f(x) = 0$, or $x \in g(f)$, and $g(f)$ is, therefore, closed since it contains all its limit points.

Question 7.2
We say that a function $f\colon X \to Y$ is Lipschitz if there exists a real number $L > 0$ such that

$$d(f(x_1), f(x_2)) \leq L d(x_1, x_2)$$

for all $x_1, x_2 \in X$. Show that every Lipshitz function is continuous.

- Fix $c \in X$, and let $x_1 \in X$ converges to c and $\epsilon > 0$ be given.
- Suppose $\exists\, N$ such that $n > N$ and $d(x_1 - c) \leq \epsilon / L$.
- Now we hold,
$$d(f(x_1) - f(c)) \leq L d(x_1, c) \leq \epsilon$$
- i.e. $f(x_1)$ converges to $f(c)$.
- The continuity of f on X is proved.

Question 7.3
Let $p \in X$ be a fixed point. Prove that the function $f(x) = d(x,p)$ is Lipschitz, and hence continuous.

- Suppose $x, y \in X$. So $f(x) = d(x,p) \leq d(x,y) + d(y,p) = d(x,y) + f(y)$
- $\Leftrightarrow f(x) - f(y) \leq d(x,y) \Leftrightarrow f(y) - f(x) \leq d(y,x) = d(x,y) \Leftrightarrow |f(x), f(y)| \leq d(x,y)$
- As there exists Lipshitz constant which is 1,
$$|f(x) - f(y)| \leq 1 \cdot d(x,y),$$
$f(x)$ is Lipschitz and, hence continuous.

Question 7.4
Let $E \subset X$ and define a function $f\colon X \to R$ by
$$f(x) = \inf\{d(x,p) \mid p \in E\}.$$
Show that f is Lipschitz, and hence continuous.

- Suppose $x, y \in X$.
- Let $p \in E$. So $f(x) = \inf\{d(x,p) \mid p \in E\} \leq d(x,p) \leq d(x,y) + d(y,p)$.
- $f(x) - d(x,y) \leq \inf\{d(y,p) \mid p \in E\} = f(y) \Leftrightarrow f(x) - f(y) \leq d(x,y)$
- Since $f(y) - f(x) \leq d(y,x) = d(x,y)$ also, we obtain $|f(x) - f(y)| \leq 1 \cdot d(x,y)$.
- Thus f is Lipschitz.

Question 7.5
Let $K \subset X$ be compact and suppose that $x \in X \setminus K$. Prove that there exists a nearest point of K to x, that is, that there exists $p \in K$ such that $d(x,p) \leq d(x,q)$ for all $q \in K$.

- The set $\{d(x,q) \mid q \in K\}$ is a non-empty subset of $(0, \infty)$.
- So there exists the infumum and a sequence $q_n \in K$ such that
$$d(x, q_n) \to D = \inf\{d(x,q) \mid q \in K\}$$
- $\{q_n\}$ has a convergent subsequence $\{q_{n_k}\}$ since it is in a compact set.
(Note that a topological space is sequentially compact if every infinite sequence has a convergent subsequent.)
- Since $d(x, q_{n_k})$ also converges to D, we may just assume that $x_n \to q$ in X.
- Since compact sets are closed, $p \in K$ and we just need to check that $d(x,p) = D$.

- Put $\epsilon > 0$. By the definition of infimum and the convergence of the distnace, $\exists\ n$ such that

$$|d(q, q_n) - D| < \frac{\epsilon}{2} \ \ \& \ \ d(q_n, p) < \frac{\epsilon}{2}$$

  Hence

$$|d(x, p) - D| < |d(x, p) - d(x, q_n)| + |d(x, q_n) - D| < \epsilon \ \text{ for any } \epsilon > 0$$

- Thus $d(x, p) = D$ as desired and the infimum of the distance is atained.

Question 7.6

Define the function $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{|x|}$. Is f Lipschitz?

- Suppose that $x, y \in \mathbb{R}$. To prove whether f is Lipschitz, we need to find a Lipschitz constant C such that $|\sqrt{x} - \sqrt{y}| \leq Cd(x, y)$ which holds on any interval.
- Let $y = 2x$. Then we obtain

$$\frac{\sqrt{x} - \sqrt{2x}}{|x - 2x|} = \frac{1}{\sqrt{x}} - \frac{\sqrt{2}}{\sqrt{x}}$$

- However, in case where $x < 0$, C may go to negative. Thus f is not Lipschitz.

Question 7.7

Let $f, g : X \to Y$ be continuous functions and suppose that $E \subset X$ is dense.

(i) Prove that $f(E)$ is dense in $f(X)$.

- Let $y \in f(X)$. Hence $\exists\ x \in X$ such that $f(x) = y$.
- Since $y \in f(X)$, x is unable to be an element of E. i.e. x is strictly in X.
- Since E is dense in X, $\exists$ a sequence $\{x_n\} \in E$ such that $x_n \to x$.
- Further, due to continuity of f, $f(x_n) \to f(x)$.
- However, this does not imply that $y = f(x) = f(x_n)$ because $y \notin f(E)$.
- Hence we can conclude that $y = f(x)$ is a limit point of $f(E)$.

(ii) Suppose now that $f(p) = g(p)$ for all $p \in E$. Prove that $f(x) = g(x)$ for all $x \in X$.

- Remind that $X = E \cup E^c$. What we need to show is that $f(p) = g(p)$ for every $p \in E^c$.
- Since E is dense in X, $\exists$ a sequence $\{p_n\} \in E$ such that $p_n \to p \in E^c$ as $n \to \infty$.
- Since $f(p_n) = g(p_n)$, where $p_n \in E$, f and g are continuous.
- Thus $f(p) = g(p)$ holds.