Question 2.1
A complex number $\in \mathbb{C}$ is said to be algebraic if there exists integers $a_0, ..., a_n$ such that

$$a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_0 = 0$$

Prove that the set of all algebraic numbers is countable. Hint: For every positive integer N, there are only finitely many equations with

$$n + |a_0| + |a_1| + ... + |a_n| = N.$$

- Suppose S_N be the set of equations that are equal to a positive integer N.
- Since there are only finitely many equations for a single N, as stated above, the collection of these equations, S_N, is at most countable.
- For each algebraic number $a_i z^{n-i}$, where $i_{\mathbb{Z}\geq 0}$, an equation for S_N can be formed for some N.
- Thus the set of all algebraic number is countable.

Question 2.2
Prove that there exist real numbers that are not algebraic.

- Suppose not. Assume there exists real numbers that are algebraic.
- A set of algebraic number is countable, as we proved on the Question 2.1.
- Hence the set of those real numbers is countable.
- This contradicts the statement we prove.

Question 2.3
Is the set of all irrational numbers countable?

- No.

  Because the set of all irrational numbers cannot be 1-1 paired with any countable set such as $\mathbb{Z}$.

  Or if the set of all irrational numbers is countable, then the set of all real number consists of irrational real and rational real numbers is countable, which is impossible.

Question 2.4
Prove that each of the following distance functions $d : \mathbb{R}^2$ x $\mathbb{R}^2 \to \mathbb{R}$ on $\mathbb{R}^2$ define a metric.

Recall the definition of metric space.

(i) d(p,q) $>$ 0 if $p \neq q$; d(p,p) $=$ 0

(ii) d(p,q) $=$ d(q,p)

(iii) d(p,q) $\leq$ d(p,r) + d(r,p) $\forall\, r \in \mathbb{R}$.

$$d(x, y) = \begin{cases} |x - y| & \text{if } x = \lambda y, \text{ some } \lambda \in \mathbb{R} \\ |x| + |y| & \text{if } x \neq \lambda y, \text{ for any } \lambda \in \mathbb{R}. \end{cases}$$

- if $x = \lambda y$
  (i) By the definition of absolute value, $|x - y| \geq 0$
  $d(x, y) = |x - y| > 0$ only and only if $x \neq y$.
  Otherwise, in case of $x = y, d(x, y) = |x - y| = 0$.

  (ii) $d(x, y) = |x - y| = |y - x| = d(y, x)$

  (iii) $d(x, y) \leq d(x, z) + d(z, y) \Leftrightarrow |x - y| \leq |x - z| + |z - y| \;\; \forall\, z \in X$.

- if $x \neq \lambda y$
  (i) Then d(x,y) $= |x| + |y| > 0$

  (ii) d(x,y) $= |x| + |y| = |y| + |x| =$ d(y,x)

  (iii) d(x,y) $= |x| + |y| \leq |x| + |z| + |z| + |y| \;\; \forall z \in \mathbb{R}$

Thus this is a metric space.

Writing x = (x_1, x_2), y = (y_1, y_2),

$$d(x,y) = \begin{cases} |x_2 - y_2| & \text{if } \mathrm{x}_1 = y_1 \\ |x_2| + |y_1 - x_1| + |y_2| & \text{if } x_1 \neq y_1 \end{cases}$$

- if $x_1 = y_1$
  - (i) d(x,y) = $|x_2 - y_2| > 0$ if and only if $x_2 \neq y_2$ d(x,y) = 0 if and only if $x_2 = y_2$
  - (ii) d(x,y) = $|x_2 - y_2| = |-1| \cdot |y_2 - x_2| = |y_2 - x_2|$ = d(y,x)
  - (iii) d(x,y) = $|x_2 - y_2| = |x_2 - z_2 + z_2 - y_2| \leq |x_2 - z_2| + |z_2 + y_2| \;\; \forall z_2 \in \mathbb{R}$
- if $x_1 \neq y_1$
  - (i) d(x,y) = $|x_2| + |y_1 - x_1| + |y_2| > |y_1 - x_1| > 0$
    d(x,y) = 0 is not the case for this condition since $x_1 \neq y_1$
  - (ii) if d(x,y) = $|x_2| + |y_1 - x_1| + |y_2| = |y_2| + |x_1 - y_1| + |x_2| = |y_2| + |-1| \cdot |-x_1 + y_1| + |x_2|$ $= |y_2| + |y_1 - x_1| + |x_2|$ = d(y,x)
  - (iii)
    - if $z_1 = x_1$ (or $z_1 = y_1$)
      d(x,z) + d(z,y) = $|x_2| + |y_1 - z_1| + |x_1 - z_1| + |y_2|$
      $= |x_2| + |y_1 - z_1| + |z_1 - x_2| + |y_2|$
      $\geq |x_2| + |y_1 - z_1| + |x_2 - z_2 + y_2|$
      $= |x_2| + |y_1 - z_1| + |y_2|$
    - if $x_1 \neq y_1 \neq z_1$
      d(x,z) + d(z,y) = $|x_2| + |z_1 - x_1| + |z_2| + |z_2| + |y_1 - z_1| + |y_2|$
      $\geq |x_2| + |y_1 - z_1| + |z_1 - x_1| + |y_2|$
      $\geq |x_2| + |y_1 - z_1 + z_1 - x_1| + |y_2| = |x_2| + |y_1 - x_1| + |y_2|$
      =d(x,y)

Question 2.5
For each of the two metrics in Q2.4, find

(i) $B_{1/2}((1,1))$, that is the ball of radius 1/2 centered at (1,1).
$\Rightarrow B_{\frac{1}{2}} = \{(x,y) \in \mathbb{R}^2 \mid d\{(1,1),(x,y)\} < \frac{1}{2}\}$

- $d(x,y) = \begin{cases} |x-y| & \text{if } x = \lambda y \text{, some } \lambda \in \mathbb{R} \\ |x|+|y| & \text{if } x \neq \lambda y \text{, for any } \lambda \in \mathbb{R}. \end{cases}$

  - **Case 1**: (x,y) $= \lambda(1,1)$

    d((1,1),(x,y)) $= |(1,1)-(x,y)|$
    $= \sqrt{(x-1)^2+(y-1)^2} < \frac{1}{2}$
    $\Leftrightarrow (x-1)^2+(y-1)^2 < \frac{1}{4}$

  - **Case 2**: (x,y) $\neq \lambda(1,1)$

    d((0,1),(x,y)) $= |(1,1)|+|(x,y)|$
    $= \sqrt{2}+\sqrt{x^2+y^2} < \frac{1}{2} \Leftrightarrow \sqrt{x^2+y^2} < \frac{1}{2}-\sqrt{2}$
    $\Leftrightarrow x^2+y^2 < (\frac{1}{2}-\sqrt{2})^2$

- $d(x,y) = \begin{cases} |x_2-y_2| & \text{if } \mathrm{x}_1 = y_1 \\ |x_2|+|y_1-x_1|+|y_2| & \text{if } x_1 \neq y_1 \end{cases}$

  - **Case 1**: $x_1 = 1$

    d((1,1),(x,y)) $= |y-1| < \frac{1}{2}$
    $\Leftrightarrow -\frac{1}{2} < y-1 < \frac{1}{2} \Leftrightarrow \frac{1}{2} < y < \frac{3}{2}$

  - **Case 2**: $x_1 \neq 1$

    d((1,1), (x,y)) $= |y|+|x-1|+|1| < \frac{1}{2}$
    $|y|+|x-1| < -\frac{1}{2}$, which is impossible.

(ii) B_2 $((1,1))$, that is the ball of radius 2 centered at (1,1).
$\Rightarrow B_2(1,1) = \{(x,y) \in \mathbb{R}^2 \mid d\{(1,1),(x,y)\} < 2\}$

- $d(x,y) = \begin{cases} |x-y| & \text{if } x = \lambda y, \text{ some } \lambda \in \mathbb{R} \\ |x|+|y| & \text{if } x \neq \lambda y, \text{ for any } \lambda \in \mathbb{R}. \end{cases}$
  - **Case 1**: (x,y) = $\lambda(1,1)$

    d((1,1),(x,y)) = $|(1,1)-(x,y)|$
    $= \sqrt{(x-1)^2+(y-1)^2} < 2$
    $\Leftrightarrow (x-1)^2+(y-1)^2 < 4$

  - **Case 2**: (x,y) $\neq \lambda(1,1)$

    d((0,1),(x,y)) = $|(1,1)| + |(x,y)|$
    $= \sqrt{2} + \sqrt{x^2+y^2} < 2 \Leftrightarrow \sqrt{x^2+y^2} < 2-\sqrt{2}$
    $\Leftrightarrow x^2+y^2 < (2-\sqrt{2})^2$

- $d(x,y) = \begin{cases} |x_2-y_2| & \text{if } \mathrm{x}_1 = y_1 \\ |x_2|+|y_1-x_1|+|y_2| & \text{if } x_1 \neq y_1 \end{cases}$
  - **Case 1**: $x_1 = 1$

    d((1,1),(x,y)) = $|y-1| < 2$
    $\Leftrightarrow\ -2 < y-1 < 2\ \Leftrightarrow\ -1 < y < 3$

  - **Case 2**: $x_1 \neq 1$

    d((1,1), (x,y)) = $|y| + |x-1| + |1| < 2$
    $|y| + |x-1| < 1$

Question 2.6
Prove that the set $\{(x, y) \in \mathbb{R}^2 \mid y < |x|\}$ is open with respect to the Euclidean metric.

- **Def**: Open Set
  A subset A of R^n is said to be **open** (in R^n) if and only if for every $x \in A \;\; \exists \, \epsilon > 0$ such that $B_\epsilon(x) \subseteq A$.