Question 1
Suppose A is a non-empty, bounded subset of $\mathbb{R}$. Is it always the case that $infA \leq supA$? Give either a proof or a counterexample.

- Suppose $\alpha = supA, \beta = infA$. Then $x \leq \alpha,\ \beta \leq x\ \ \forall\, x \in A$.
- Proof by contradiction: Suppose $supA < infA$
- Then $\alpha < \beta \leq x \in A$, which contradicts to $\beta \leq x \leq \alpha$
- Thus $infA \leq supA$.
- Consider a set $A = \{999\}$. In case of this, $infA = supA = \{999\}$.
- Hence $infA$ and $supA$ does not always have to be $infA < supA$.

Question 2
Suppose A and B are non-empty and bounded subsets of $\mathbb{R}$ such that $A \subset B$.

(i) Prove $supA \leq supB$

- Since $A \subset B$, $supB$ is an upper bound for A such that $a \leq supB\ \ \forall a \in A$.
- As a subset of B, the greatest element of A, $supA$, cannot be larger than any element of B.
- Hence $supA \leq supB$.

(ii) Is it always true that $infA \leq supB$? Give a proof or counterexample.

- By definition, $infA \leq supA$.
- We proved $supA \leq supB$ in (i).
- Hence $infA \leq supB$.

(iii) Is it always true that $infB \leq supA$? Prove or give a contradiction.

- As we proved previously, $infB \leq infA$.
- By definition, $infA \leq supA$.
- Hence $infB \leq supA$.

Question 3

Let $A \subset \mathbb{R}$ be a non-empty bounded set of integers. Prove that $supA$ is an integer.

- Suppose not. Consider $\alpha = supA$ such that $x \leq \alpha \;\; \forall\, x \in A$.
- Then $\exists\, x$ such that $\alpha - 1 < x \leq \alpha$.
- However, there must be only one integer in the interval $(\alpha - 1, \alpha)$.
- Clearly x is an integer, and it lies in the interval.
- i.e. x is the only integer in the interval.
- Hence, there is no such α as a supremum of A.
- Thus the least upper bound of A should be an integer.

Question 4
Let $A = \{\frac{n}{2n+1} | n \in \mathbb{N}\}$. Prove that $supA = \frac{1}{2}$.

- $\frac{n}{2n+1} < \frac{n}{2n} = \frac{1}{2}$ leads to that $\frac{1}{2}$ is an upper bound for A.
- Suppose $x = supA < \frac{1}{2}$.
- Then $\frac{n}{2n+1} \leq x \Leftrightarrow n \leq 2nx + x \Leftrightarrow n(1-2x) \leq x \Leftrightarrow n \leq \frac{x}{1-2x}$
- However, by Archimedean Property, $n > \frac{x}{1-2x} > 0$ for some n.
- Therefore, our initial assumption, $x < \frac{1}{2}$, is not valid.
- Since $\frac{1}{2}$ is an upper bound and $x \nless \frac{1}{2}$, $supA = \frac{1}{2}$.

Question 5
Let $A = \{\frac{2^n}{2^n+1} | n \in \mathbb{N}\}$. Prove that $supA = 1$.

- $\frac{2^n}{2^n+1} < \frac{2^n}{2^n} = 1$ meaning 1 is an upper bound for A.
- Consider $x - \epsilon < \frac{2^n}{2^n+1} \leq x$, and let $x = supA$.
- Suppose $\frac{2^{n+1}}{2^{n+1}+1}$ such that $x < \frac{2^{n+1}}{2^{n+1}+1}$
- In order $\frac{2^{n+1}}{2^{n+1}+1}$ on the right hand side to make $\frac{2^n}{2^n+1}$ in the middle, multiply by $\frac{2^{n+1}+1}{2(2^n+1)}$.
- Put $x - \epsilon = x \cdot \frac{2^{n+1}+1}{2(2^n+1)}$.
- Then $x \cdot \frac{2^{n+1}+1}{2(2^n+1)} < \frac{2^n}{2^n+1} \leq x$, and multiply by $\frac{2(2^n+1)}{2^{n+1}+1}$ this inequality.
- Then we obtain $x < \frac{2^{n+1}}{2^{n+1}+1} \in A$.
- Since 1 is an upper bound for A and $x \nless 1$, $supA = 1$.

Question 6

Archimedean property and variations

(i) Prove the Archimedean property of the real numbers directly from the least upper bound axiom.

- Suppose $x, y \in \mathbb{R}$ such that $x > 0$. Then $\exists\, n \in \mathbb{N}$ such that $y < nx$.
- Let A be the set of all nx, and $\alpha = supA$.
- Consider $x > 0 \Leftrightarrow -x < 0$. Then $\alpha - x < \alpha$.
- Consider, further, an integer m such that $\alpha - x < mx < \alpha$ where $mx \in A$ and $\alpha - x$ is lower bound of A.
- Then $\alpha - x < mx \Leftrightarrow \alpha < (m+1)x$.
- Since $(m+1)x \in A$, α cannot be a supremum of A.
- Therefore there exists such n satisfies $y < nx$.

(ii) Show that for any pair of real numbers $x < y$ there is a rational number $r \in \mathbb{Q}$ with $x < r < y$.

- $x < y \Leftrightarrow 0 < y - x$. By Archimedean Property, $\exists\, n$ such that
  $1 < n(y - x) \Leftrightarrow 1 + nx < ny$ where $n \in \mathbb{N}$.
- Consider integers m_1, m_2 such that $-m_1 < nx, nx < m_2$.
- Consider, further, m such that $-m_2 < m < m_1$, $m - 1 < nx \leq m$
- Then we obtain an inequality $nx \leq m < nx + 1 < ny \Leftrightarrow nx \leq m < ny$
- Further, $x \leq \frac{m}{n} < y$. Obviously $\frac{m}{n} \in \mathbb{Q}$ since both $m, n \in \mathbb{Z} \subset \mathbb{Q}$.
- We proved such r between x and y.

(iii) Show that the set $A = \{3n | n \in \mathbb{N}\}$ is unbounded, i.e. show that for every real number x, there is an integer n with $3n > x$.

- Suppose not. Suppose A is bounded.
- Then $\exists\, \alpha = supA$ such that $x \leq \alpha\ \ \forall\, x \in A$.
- However, by Archimedean Property, $n \cdot 1 > x \Leftrightarrow 3n > n > x$.
- Hence the set A is unbounded.

(iv) Show that the set $\{\frac{n^2}{n+1} | n \in \mathbb{N}\}$ is unbounded.

- Suppose not. Suppose A is bounded.
- Then $\exists\, \alpha = supA$ such that $x \leq \alpha\ \ \forall\, x \in A$.
- However, by Archimedean Property, $n \cdot 1 > x \Leftrightarrow n \cdot \frac{n}{n+1} > x$.
- Hence the set A is unbounded.

(v) Show that the set $\{n!|n \in \mathbb{N}\}$ is unbounded.

- Suppose not. Suppose A is bounded.
- Then $\exists\ \alpha = supA$ such that $x \leq \alpha\ \ \forall\ x \in A$.
- However, by Archimedean Property, $n \cdot 1 > x \Leftrightarrow n! > n > x$.
- Hence the set A is unbounded.

(vi) Show that the set $\{\sqrt{n}|n \in \mathbb{N}\}$ is unbounded.

- Suppose not. Then $\exists\ \alpha = supA$ such that $\sqrt{n} \leq \alpha$.
- Consider $\sqrt{n+1}$ such that $\alpha < \sqrt{n+1}$.
- In order $\sqrt{n+1}$ make to $\sqrt{n}$, multiply by $\frac{\sqrt{n}}{\sqrt{n+1}}$.
- Put $\alpha - \epsilon = \alpha \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$ where $\alpha - \epsilon < \sqrt{n} \leq \alpha$.
- Then $\alpha \cdot \frac{\sqrt{n}}{\sqrt{n+1}} < \sqrt{n} < \alpha$, and multiply by $\frac{\sqrt{n+1}}{\sqrt{n}}$.
- Then we obtain $\alpha < \sqrt{n+1} \in A$.
- Hence we showed that α is not $supA$.
- Thus A is unbounded.

(vii) Let $A = \{a_1, a_2, a_3, ...\}$ be a set of real numbers where $a_{n+1} \geq a_n + 1$ holds for all $n \in \mathbb{N}$. Show that A is unbounded.

- Proof by contradiction: Suppose $\exists\ \alpha = supA$.
- Then we can obtain $\alpha - \epsilon < a_n \leq \alpha$.
- Fix $\epsilon = 1$. $\alpha - 1 < a_n \Leftrightarrow \alpha < a_n + 1 \leq a_{n+1} \in A$.
- Hence $\alpha \neq supA$.
- We proved that A is unbounded.

Question 7

Let $E = \{\frac{1}{n} | n \in \mathbb{N}\}$, and let $F = E \cup 0$. Find all the limit points of E and of F. Are either of E or F closed?

- Recall the definition limit point. A limit point is a point such that $\{N_\epsilon(x) \cap E\} \setminus \{x\}$.
- Let $x = 0$. Then choose an arbitrary ϵ; the interval is becoming $(0 - \epsilon,\ 0 + \epsilon)$.
- No matter how it is small, there must be $\frac{1}{n}$ such that $\frac{1}{n} < 0 + \epsilon$
- We proved $\exists$ a point within the interval other than $x = 0$.
- Hence 0 is a limit point of E.

- Meanwhile, consider any other point. Suppose $x = 1$.
- Then the interval is $(1 - \epsilon,\ 1 + \epsilon)$.
- However, in case where $\epsilon < \frac{1}{2}$, $\{N_\epsilon(x) \cap E\} \setminus \{x\} = \phi$.
- Hence any other $\frac{1}{n}$ is unable to be a limit point.
- Thus E is an open set.

- Since 0 is the only and all limit point of E, $F = E \cup \{0\}$ is a closed set by definition.

Question 8

Let $A = \{\frac{m}{m+2} | m \in \mathbb{N}\}$ and $B = \{\frac{m}{m-2} | m \in \mathbb{N}, m \geq 3\}$. Find all limit points of the set A and of the set B.

- Since $\frac{m}{m+2} < \frac{m+2}{m+2} = 1$, 1 is an upper bound for A.
- Since $1 = \frac{m-2}{m-2} < \frac{m}{m-2}$, 1 is an lower bound for B.

- Each $\frac{m}{m+2}$ or $\frac{m}{m-2}$ is an isolated point because $\{N_\epsilon(x) \cap \{\frac{m}{m+2}$ or $\frac{m}{m-2}\}\} \setminus \{x\} = \phi$ for some ϵ.
- However, in case where $x = 1$, $\{N_\epsilon(x) \cap \{\frac{m}{m+2}$ or $\frac{m}{m-2}\}\} \setminus \{x\} \neq \phi \ \ \forall\, \epsilon$.
- Thus all limit points of the set A and B is $\{1\}$.

Question 9

Find a subset $E \subset \mathbb{R}$ with exactly three limit points. (justify your answer.)

$$\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{1 + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$$

Proof was given in Question 7.

Question 10

Let $E \subset \mathbb{R}$ be a bounded subset, and let m be a limit point of E. Show that $m \leq supE$.

- Proof by contradiction: Suppose $m > supE$.
- If m is a limit point, then $m \in \{\{B_\epsilon(x) \cap E\} \setminus \{x\} \mid x \in \mathbb{R}, \epsilon > 0\} \neq \phi$.
- Consider $\epsilon = \frac{d(m,supE)}{2}$. Then m is strictly detached from E.
- i.e. $m \in \{\{B_\epsilon(x) \cap E\} \setminus \{x\} \mid x \in \mathbb{R}, \epsilon > 0\} = \phi$ meaning m is not a limit point.
- Thus $m \leq supE$.

Question 11

Let (X, d) be some metric space, $a \in X$ some point in X, and $r > 0$.

(i) Show that the set $E = \{x \in X \mid d(x, a) > r\}$ is open.

- Consider $h > 0$ such that $d(x, a) = r + h$ for some h, and let $\epsilon = \frac{h}{2}$.
- Then $\{\{B_\epsilon(x) \cap E\} \setminus \{x\} \mid x \in \mathbb{R}, \epsilon > 0\} = \phi$
- i.e. $E^c \cap B_\epsilon(x) = \phi$, so x is an interior point of E.
- Thus E is open.

(ii) Let F be some subset of $B_r(a)$, and let p be a limit point of F. Show that $d(p, a) \leq r$.

- Suppose not: $d(p, a) > r$.
- Then $\exists\ h$ such that $d(p, a) = r + h$.
- Consider $h > 0$ such that $d(p, a) = r + h$ for some h, and let $\epsilon = \frac{h}{2}$.
- Then $\{\{B_\epsilon(p) \cap E^c\} \setminus \{p\} \mid p \in \mathbb{R}, \epsilon > 0\} = \phi$, meaning p is not a limit point.
- Hence $d(p, a) \leq r$ so that p is to be a limit point.

Question 12

Find all limit point points of the set $\mathbb{N} \subset \mathbb{R}$.

- Since $n \in \mathbb{N}$ is discrete, $\exists\ \epsilon > 0$ such that $\{\{N_\epsilon(x) \cap \mathbb{N}\} \setminus \{x\} \mid x \in \mathbb{R}\} = \phi$.
- Thus the set of all limit points of $\mathbb{N}$ is ϕ.

Question 13

(About open and closed sets) Notation: $A^c = X \setminus A = \{x \in X \mid x \notin A\}$.

(i) True or false? if $A \subset \mathbb{R}$ is open then A contains none of its limit points (i.e. prove the statement, or give a counterexample.)

- By the definition of open set, if $\{E^c \cap N_\epsilon(x)\} = \phi$, then the set is open.
- Hence $\exists$ a limit point of A $p \in \{\{B_\epsilon(p) \cap A\} \setminus \{p\}\} \neq \phi$.
- Therefore this statement is false.

(ii) True or false? If $A \subset \mathbb{R}$ is open then there is a limit point p of A that lies outside A.

- Consider a point p which is exactly on the boundary.
- $p \notin A$, but $\{\{B_\epsilon(p) \cap A\} \setminus \{p\}\} \neq \phi$.
- Hence this statement is true.

(iii) True or false? If $E \subset X$ is closed, then E contains no limit points of its complement(i.e.E contains no limit points of E^c.)

- E^c is an open set for this case.
- As we considered in (i), one of limit points of an open set E^c which is exactly on the boundary is not an element of E^c, but it is still a limit point of E^c.
- This statement is false.

(iv) True or false? If $E \subset X$ is open, then E contains no limit points of its complement(i.e.E contains no limit points of E^c.)

- Suppose any point $p \in E^c$
- Consider h such that $r + h = d(x, p)$ for some $h > 0$, $x \in E$, and let $\epsilon = \frac{h}{2}$.
- Then $\{\{B_\epsilon(p) \cap E\} \setminus \{p\}\} = \phi$. i.e. there is no such p that is a limit point of E.
- Therefore this statement is true.