Questions 4.1
Let (X, d) be a metric space. Prove that a sequence (x_n) in X converges $x_n \to x \in X$ if and only if every subsequence $x_{n_k} \to x$.

- $\Rightarrow$ If every subsequence $x_{n_k} \to x$, a sequence (x_n) in X converges $x_n \to x \in X$
  - If every subsequence converges $x_{n_k} \to x$, $\exists\ K$ such that $k \geq K$ and $n_k \geq N$.
  - So $|x_{n_k} - x| < \epsilon$.
  - Therefore $\exists\ N$ such that $n \geq N$ and further x_n such that $|x_n - x| < \epsilon$.
  - Thus $lim_{k\to\infty}\, x_n = x$.
- $\Leftarrow$ If a sequence (x_n) in X converges $x_n \to x \in X$, every subsequence $x_{n_k} \to x$.
  - If a sequence (x_n) in X converges $x_n \to x \in X$,
    $\exists\ N$ such that $n \geq N$ implying $|x_n - x| < \epsilon$.
  - Then $\exists\ K$ such that $k \geq K$ and $n_k \geq N$.
  - Hence $\exists$ a subsequence x_{n_k} such that $|x_{n_k} - x| < \epsilon$ also.
  - Thus $lim_{k\to\infty}\, x_{n_k} = x$.

Question 4.2
Let (X, d) be a metric space and let (x_n) be a sequence in X. Prove $x_n \to x \in X$ if and only if $d(x_n, x) \to 0$ in $\mathbb{R}$.

- $\Leftarrow$ If $x_n \to x \in X$, $d(x_n, x) \to 0$ in $\mathbb{R}$.
  - If $x_n \to x \in X, \exists\ N$ such that $n \geq N$ and $|x_n - x| < \epsilon$ where $\epsilon > 0$.
  - So we obtain $-\epsilon < |x_n - x| < \epsilon$.
  - Since we chose an arbitrary ϵ, it can be as small as possible as long as it is greater than 0.
  - Take $lim_{n\to\infty} - \frac{\epsilon}{n} < |x_n - x| < \frac{\epsilon}{n}$.
  - Thus $d(x_n, x)$ converges to 0 eventually.
- $\Rightarrow$ If $d(x_n, x) \to 0$ in $\mathbb{R}$, $x_n \to x \in X$.
  - Since $d(x_n, x)$ is converging to 0, we cannot say it is equal to 0.
    i.e. it is greater than 0.
  - Then consider $\epsilon > 0$ such that $0 < d(x_n, x) = |x_n - x| < \epsilon$.
  - Then $\exists\ N$ such that $n \geq N$.
  - Thus $x_n \to x \in X$.

Question 4.3
Let (X, d) be a metric space, $C \subset X$.

(i) Let C be closed, (x_n) a sequence in C. Prove that if $x_n \to x \in X$, then $x \in C$.

- Suppose by contradiction that $x_n \in C$, but $x \in X \setminus C$ where is open.
- By definition of open, $\exists\ B_\epsilon(x) \subset X \setminus C$.
  i.e. the ball $B_\epsilon(x)$ is completely contained in $X \setminus C$.
- Hence $\exists\ N$ such that $n \geq N, x_n \in B_\epsilon(x)$.
- This leads to $x_n \subset X \setminus C$. However the sequence x_n is in C.
- Thus this contradicts to the assumption we made and we proved that
  if $x_n \to x \in X$, then $x \in C$ as required.

(ii) Suppose that for every sequence (x_n) in C such that $x_n \to x \in X$ that $x \in C$. Prove that C is closed.

- $C \subset X$ and $\exists$ a limit point x of C.
- This implies that $\exists$ a sequence x_n that converges to x in C also.
- Since the given limit point x and x_n are proven to be in the set C, C is a closed set.

Question 4.4
Let $A \subset \mathbb{R}$ and suppose (x_N) is a sequence of upper bounds for A, that is, suppose that for every $n \in \mathbb{N}$, x_n is an upper bound for A. Suppose $x_n \to x$. Prove that x is an upper bound for A.

- x_n is an upper bound of A. Then $\exists\ a \in A$ such that $a \leq min\{x_n\} \leq x_n$.
- Since $x_n \to x$, $\exists\ N \in \mathbb{N}$ such that $n \geq N$ and $|x_n - x| < \epsilon \ \ \forall\ \epsilon > 0$.
- i.e. $-\epsilon < x_n - x < \epsilon \Leftrightarrow x - \epsilon < x_n < x + \epsilon$

- Suppose by contradiction that $x + \epsilon < supA \ \ \forall\ \epsilon > 0$. Then $x_n < x + \epsilon < supA$.
- Since x_n is an upper bound of A, $supA \leq x_n$.
- And by the inequality we obtained above, $supA \leq x_n < x + \epsilon'$ which contradicts with our assumption.
- Hence x is also an upper bound of A.

Question 4.5
Let $(x_n), (y_n), (z_n)$ be sequences of real numbers such that $x_n \to y, z_n \to y$ and, for all $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. Prove that $y_n \to y$ also.

Take $\epsilon > 0$ and suppose $\exists\ N', N'' \in \mathbb{N}$ such that

(i) $|x_n - y| < \epsilon$ for $n \geq N'$
$\Rightarrow y - \epsilon < x_n < y + \epsilon$

(ii) $|z_n - y| < \epsilon$ for $n \geq N''$
$\Rightarrow y - \epsilon < z_n < y + \epsilon$

- Since $x_n \leq y_n \leq z_n,\ \ y - \epsilon < x_n \leq y_n \leq z_n < y + \epsilon$.
- Hence $y - \epsilon < y_n < y + \epsilon$. i.e. $|y_n - y| < \epsilon$.
- Thus y_n converges to y also.

Question 4.6
For a sequence (x_n) of real numbers, define the arithmetic mean σ_n by

$$\sigma_n = \frac{x_1 + \ldots + x_n}{n}.$$

(i) Prove that if $lim_{n\to\infty} x_n = x$, then $lim_{n\to\infty} \sigma_n = x$.

(ii) Prove by a counterexample that the converse statement is not true. More precisely, construct a sequence (x_n) that does not converge such that $\sigma_n \to 0$.