1 Compact Sets

Def Let (X, d) be a metric space, $K \subset X$. An **open cover** of K is a collection $\{G_\alpha\}$ of open sets such that $K \subset_\alpha G_\alpha$

Def Let $K \subset X$. K is **compact** if every open cover$\{G_\alpha\}$ of K has a finite subcover.
i.e. $\exists\ \alpha_1, ..., \alpha_n$ such that $K \subset \bigcup_{\alpha=1}^{n} G_\alpha$

- In order to show a given set is compact, there exists a set of *finite open covers* has to have finite sub-covers.
- If it is not compact, there is a single open cover that has no finite sub-cover.

Example: Compact

- Let $K = [0, 1], X = \mathbb{R}$
- Consider $\{G_\alpha\} \cup \{G_0\} \cup \{G_1\}$ where
  (i) $G_\alpha = (\frac{\alpha}{2}, 1)\ \forall\ \alpha \in (0, 1)$
  (ii) $G_0 = (-\epsilon, \epsilon)$
  (iii) $G_1 = (1 - \epsilon, 1 + \epsilon)$ for some $\epsilon > 0$
- Then $\{G_\alpha\} \cup \{G_0\} \cup \{G_1\}$ is an open cover of [0,1].
- It has finite subcovers $\{G_\alpha, G_0, G_\epsilon\}$
- Thus $K = [0.1]$ is compact.

Example: Not Compact
Let $E = (0, 1),\ X = \mathbb{R}$

- E is an open cover itself, but that does not mean it is compact.
- Let $G_\alpha = (\frac{\alpha}{2}, 1)\ \forall\ \alpha \in (0, 1)$. Then $\{G_\alpha\}$ is open covers of E.
- But we are unable to take a finite collection of G_α.
- Hence E is not a compact set.

Theorem Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.

Remark: We know that any metric (X, d) is both open and closed relative to itself. However, this is not true for compactness.

Proof

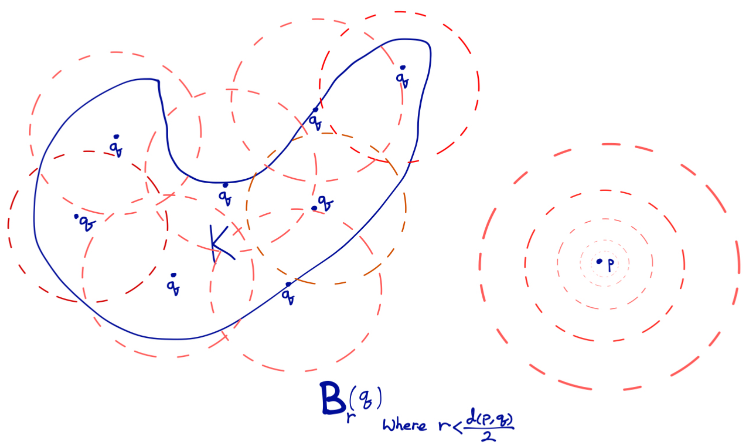
- $\Rightarrow$ Suppose K is compact relative to Y.
- Let $\{G_\alpha\}$ be a collection of sets open relative to X. i.e. $K \subset \bigcup\limits_\alpha G_\alpha$
- Let $V_\alpha = G_\alpha \cap Y \ \ \forall\, \alpha$, so V_α is open relative to Y.
- For $K \subset Y$, $K = K \cap Y \subset \bigcup\limits_\alpha G_\alpha \cap Y = \bigcup\limits_\alpha V_\alpha$.
- So V_α is an open cover of K relative to Y.
- Hence, $\exists\ \alpha_1, ..., \alpha_n$ such that $K \subset \cap_{i=1}^n V_{\alpha_i}$
- Now $K \subset \bigcup\limits_{i=1}^n V_{\alpha_i} = \bigcup\limits_{i=1}^n (G_{\alpha_i} \cap Y) \subset \bigcup\limits_{i=1}^n G_{\alpha_i}$
- So K is compact relative to X.

- $\Leftarrow$ Suppose K is compact relative to X.
- Assume V_α is an open cover of K relative to Y and G_α is open in X i.e. $V_\alpha = G_\alpha \cap Y$.
- So $K \subset \bigcup\limits_\alpha V_\alpha \subset \bigcup\limits_\alpha G_\alpha$ i.e. G_α is an open cover of K in X.
- As K is compact relative to X, $\exists\ \alpha_1, ..., \alpha_n$ such that $K \subset \bigcup\limits_{i=1}^n G_{\alpha_i}$.
- Hence $K = K \cap Y \subset \bigcup\limits_{i=1}^n G_{\alpha_i} \cap Y = \bigcup\limits_{i=1}^n V_{\alpha_i}$
- Since $\exists\ V_1, ..., V_\alpha$ is a finite subcover, K is compact relative to Y.

Theorem Compact subsets K of metric spaces X, $K \subset X$, are closed.
Note. Closed does not have to imply compactness, but compactness implies closed.
e.g. $\mathbb{R}$ *is closed, but not compact.*

Proof.

- We shall prove that the complement of K, K^c, is an open subset of X.
- Suppose $p \in X,\ p \notin K,$ but $q \in K$.
- Let V_q is neighborhoods of p and W_q is those of q whose radius is less than $\frac{d(p,q)}{2}$.
- Since K is compact, there are **finitely many** points $q_1, ..., q_n$ in K such that
$$K \subset W_{q_1} \cup \ldots \cup W_{q_n} = W$$
- If $V = V_{q_1} \cap \ldots \cap V_{q_1}$, then V is a neighborhood of p which does not intersect W.
- Hence $V \subset K^c$, so that p is an interior point of K^c.



Theorem Closed subsets of compact sets are compact.

Proof.

- Suppose $F \subset K \subset X$, F is closed relative to X, and K is compact.
- Let $\{V_\alpha\}$ be an open cover of F.
- If F^c is adjoined to $\{V_\alpha\}$, we obtain an open cover Ω of K.
- Since K is compact, $\exists$ a finite subcollection Φ of Ω which overs K, and F also.
- If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F.
- We have thus shown that a finite subcollection of $\{V_\alpha\}$ covers F.

Corollary If F is closed, K is compact, then $F \cap K$ is compact.

Theorem If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Proof.

(i) Suppose by contradiction that $\bigcap\limits_{\alpha} K_\alpha = \phi$.

(ii) Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$. (G_α is open and K_α is closed.)

(iii) Assume that no point of K_1 belongs to every K_α.

(iv) Then the sets G_α form an open cover of K_1

(v) Suppose $\exists\, x \in K_1$, and then $x \in G_\alpha = K_\alpha^c$ due to (iii).

(vi) Hence $\{G_\alpha\}$ is an open cover of K_1. i.e. $\exists\, \alpha_1, ..., \alpha_n$ such that $K_1 \subset \bigcup\limits_{i=1}^{n} G_{\alpha_i}$.

(vii) Due to (ii), we can obtain that

$$K_1 \cap \left(\bigcup_{i=1}^{n} G_{\alpha_i} \right)^c = K_1 \cap \bigcup_{i=1}^{n} K_{\alpha_i} = \phi$$

which contradicts to the initial assumption.

Theorem If E is an infinite subset of a compact set K, then E has a limit point in K.

- Suppose by contradiction that E does not have a limit point in K.
- Then every $x \in K$ has a neighborhood V_x such that $V_x \cap E$ has at most one point.
- It is clear that no finite subcollection of $\{V_q\}$ can cover E and K also.
- This contradicts to the initial assumption.

Theorem If a set E in R^k has one of the following three properties, then it has other two:

(i) E is compact.

(ii) E is closed and bounded.

(iii) Every infinite subset of E has a limit point in E.

Remark. $(i) \Longleftrightarrow (ii)$ are Heine-Borel Theorem.

Proof.

(i) Every finite subset of E has a limit point in E. This is proved in the previous theorem.

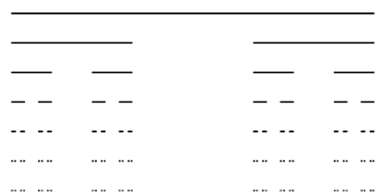
(ii) As E is bounded, $\exists$ an n-cell I such that $E \subset I$.
Since n-cells are compact, E is closed subset of a compact set. Hence E is compact.

(iii) Every infinite subset of E has a limit point in E implies that E is closed and bounded.

- $\Rightarrow$ Suppose by contradiction that E is not bounded.
- Then $\exists\ x_k \in E\ \ \forall\ k$ such that $|x_k| > k$. (Note that $|x_k|$ is norm.)
- Now $\{x_k\}$ is an infinite subset of E.
- But x_k has no limit point in $\mathbb{R}^n$, so not in E.

- $\Leftarrow$ Suppose by contradiction that E is not closed.
- Then $\exists$ a limit point x_0 of E such that $x_0 \in \mathbb{R}^n \setminus E$.
- So $\forall\ n \in \mathbb{N}$, $\exists\ x)n \in B_{1/n}(x_0) \cap E$ and $x_n \neq x_0$.
- $\{x_n\}$ is infinite in E, so it has a limit point $y \in E$.
- Consider $|x_n - y| \geq |x_0 - y| - |x_n - x_0| \geq |x_0 - y| - \frac{1}{n} \geq \frac{1}{2}|x_0 - y|$
for sufficiently large n (as $|x_0 - y| > 0$).
- Then $B_{\frac{|x_0-y|}{2}}(y)$ contains only finitely many x.
- Hence y is not a limit point.

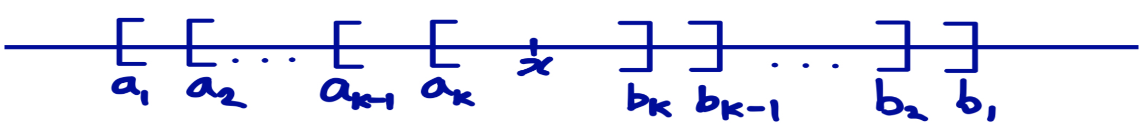
Example Cantor Set



- Let $E_0 = [0,1]$
- Remove the middle third: $E_1 = [0, \frac{1}{2}] \cup [\frac{2}{3}, 1]$
- Repeat: $E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ (still closed, bounded, and compact)

  $\vdots$
- Eventually, we get a sequence $E_0 \supset E_1 \supset E_2 \supset \ldots$ such that E_n is compact $\forall\ n$.
- Hence $\bigcap\limits_{n=1}^{\infty} E_n \neq \phi$ is compact having infinitely many points.

Theorem Let n be a positive integer. If $\{I_k\}$ is a sequence of cube or n-cells such that $I_k \supset I_{k+1}$ (n=1,2,3,...), then $\bigcap\limits_{k=1}^{\infty} I_k$ is not empty.

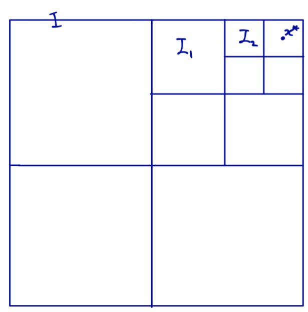


Proof.

- First, we consider the case $n = 1$. So $n-$cells lie in the interval.
- So we have $[a_1, b_1] \supset [a_2, b_2] \supset \ldots$ In particular, $a_k < b_k \leq b_1\ \forall\ k$.
- So b_1 is an upper bound for E. Hence $\exists\ x = sup\ E$
- For any $k, m \in \mathbb{N},\ a_k \leq a_{k+m} \leq b_k$.
- So b_k is an upper bound of $E\ \ \forall\ k \in \mathbb{N}$.
- Hence $x \leq b_k\ \ \forall\ k$ as $x = sup\ E$.
- So $a_k \leq x \leq b_k\ \ \forall\ k \in \mathbb{N}$, so $x \in [a_k, b_k]$.
- Hence $x \in \bigcap\limits_{k=1}^{\infty} [a_k, b_k]$

- Now consider n-cells, $I_1 \supset I_2 \supset \ldots$ and write $I_k = [a_{k_j}, b_{k_j}] \times \ldots \times [a_{k_n}, b_{k_n}]$.
- Then for each $j = 1, \ldots, n$, the intervals $[a_{k_n}, b_{k_n}]$ are nested:
  $[a_{1j}, b_{1j}] \supset [a_{2j}, b_{2j}] \supset \ldots \supset [a_{kj}, b_{kj}] \supset \ldots$
- By first part of the proof, $\exists\, x_j \in \bigcap\limits_{k=1}^{\infty} [a_{k_j}, b_{k_j}]$
- Then $x^* = (x_1^*, ..., x_n^*) \in \bigcap\limits_{k=1}^{\infty} ([a_{k_1}, b_{k_1}] \times \ldots \times [a_{k_n}, b_{k_n}]) = \bigcap\limits_{k=1}^{\infty} I_k$

Theorem n-cells are compact.



- Let I be an n-cell, $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$
- Define $\delta = \left(\sum\limits_{i=1}^{n} (b_i - a_i)^2 \right)^{1/2}$ (max distance between two points.)
- Let $\{g_\alpha\}$ be an open cover of I.
- Suppose by contradiction that there is no finite subcover.
- For $j = 1, \ldots, n$, let $c_j = \frac{a_j + b_j}{2}$ be the midpoint of $[a_j, b_j]$ can divide to 2^n cell.
- Divide each interval into $[a_j, c_j]$ and $[c_j, b_j]$ and hence divide I into 2^n n-cells.
- At least one of these n-cells does not have finite subcover from $\{G_\alpha\}$, call it I_1.
- Repeating, we get a sequence $I \supset I_1 \supset I_2 \supset \ldots$ of n-cells such that I_k does not have a finite subcover from collection $\{G_\alpha\}$
- And if $x, y \in I_k$, then $|x - y| \leq \delta 2^{-k}$
- By the previous theorem, $\exists\, x^* \in \bigcap\limits_{k=1}^{\infty} I_k$.
- As $\{G_\alpha\}$ is an open cover, $\exists\, \alpha_0$ such that $X^* \in G_{\alpha_0}$.
- As G_{α_0} is open, $\exists\, r > 0$ such that $B_r(x^*) \subset G_\alpha$.

- Now as I_k is such that $|x-y| \leq 2^{-k}d \ \ \forall \ x, y \in I_k$ taking k sufficiently large so that $2^{-k}\delta < r$, we have $I_k \subset B_r(x_0) \subset G_{x_0}$
- As I_k has no finite subcover from $\{G_\alpha\}$.

Corollary Every bounded infinite subset $E \subset \mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

Proof. Since E is bounded, $\exists$ an n-cell I such that $I \supset E$. As I is compact, the infinite subset E has a limit point in $I \subset \mathbb{R}^n$