Question 1.1

Let $F = 0, 1$ be the set called the paddock, defined in class, and define addition and multiplication on F by

$$0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0,$$

and

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1$$

- Is F a field?
  - Check if addition is closed.
    - (A1) 0 + 1 = 0 $\in F$
    - (A2) 0 + 1 = 1 + 0 = 1
    - (A3)
      - (0 + 1) + 1 = 1 + 1 = 0
        0 + (1 + 1) = 0 + 0 = 0
        Thus (0 + 1) + 1 = 0 + (1 + 1)
      - (0 + 1) + 0 = 1 + 0 = 1
        0 + (1 + 0) = 0 + 1 = 1
        Thus (0 + 1) + 0 = 0 + (1 + 0)
    - (A4)
      - 0 + 1 = 1 $\in F$
      - 0 + 0 = 0 $\in F$
    - (A5)
      - 0 + (-0) = 0
      - 1 + (-1) = 0
    - Therefore F is closed under addition.
  - Check if multiplication is closed.
    - (M1) $0 \cdot 1 = 0 \in F$
    - (M2) $0 \cdot 1 = 1 \cdot 0 = 0$
    - (M3)
      - $(1 \cdot 0) \cdot 0 = 1 \cdot (0 \cdot 0) = 0$
      - $(1 \cdot 0) \cdot 1 = 1 \cdot (0 \cdot 1) = 0$
    - (M4)
      - $1 \cdot 0 = 0$
      - $1 \cdot 1 = 1$
    - (M5)
      - $\frac{1}{1} \cdot 1 = 1$
    - Therefore F is closed under multiplication.

Thus F is a field.

- Is F an ordered field?
  - In case where $1 > 0$
  - $1 + 1 > 1 + 0$
  - $0 > 1$, which is a contradiction

Thus F is not an ordered field

Question 1.2
Prove that there is no rational number $p \in \mathbb{Q}$ such that $2^p = 6$.

- Argue by contradiction. Suppose $\exists$ a rational number p such that $2^p = 6$.
- If there were such a p, we could write $\frac{a}{b}$ where a and b are integers that are not both even.
- Then $2^p = 2^{\frac{a}{b}} = 6$. Hence $2^a = 6^b = (2 \cdot 3)^b = 2^b \cdot 3^b \Leftrightarrow \mathbf{2^{a-b} = 3^b}$.
- Since $6 > 2^1$, $p = \frac{a}{b} > 1$, which means $a > b$.
- Thus $2^{a-b} > 2$, and 2^{a-b} must be even.
- On the other hand, 3^b must be odd.
- i.e. $2^{a-b} = 3^b$ is impossible.
- Thus there is no such p that satisfies the condition $2^p = 6$.

Question 1.3
Use the Archimedean property to prove that for any pair of rational numbers $p < q$, there exists an irrational number x such that $p < x < q$.
Hint: You may use without proof the fact that if $r \neq 0$ is rational and x is irrational, then $r + x$ and rx are both irrational.

- We are given $p < q$, or equivalently $q - p > 0$.
- According to the Archimedean property, there is a positive integer n, such that $n(q - p) > 2$ (instead of 1).
- Let $x = \frac{q-p}{2}$, and clearly we obtain $p < x < q$, and $\frac{q-p}{2} > \frac{1}{n} \Leftrightarrow \frac{1}{n} < x$.
- Hence $p < x + \frac{1}{n} < 2x = q - p < q$.
- Consider irrational $\frac{1}{n\sqrt{2}}$ in lieu of $\frac{1}{n}$.
- Since $\frac{1}{n\sqrt{2}}$ is still positive, it preserves the inequalities $p < x + \frac{1}{n\sqrt{2}} < q$.
- Since the addition of rational x and irrational $\frac{1}{n\sqrt{2}}$ is irrational number, there exists an irrational such that $p < x + \frac{1}{n\sqrt{2}} < q$.

Question 1.4
Let $A \subset \mathbb{R}$ be non-empty and bounded below. Define $-A = \{-a \mid a \in A\}$. Prove that

$$\text{inf A = - sup(-A)}$$

- inf A $\leq$ -sup(-A)
  - Since inf A is an infimum for A, inf A $\leq a \ \ \forall a \in A$,
  - Then -inf A $\geq -a \ \ \forall -a \in -A$.
  - In other words, -inf A is an upper bound for -A.
  - Hence sup(-A) $\leq$ -inf A $\Leftrightarrow$ inf A $\leq$ -sup(-A)
- inf A $\geq$ -sup(-A)
  - Since A is bounded below, -A is bounded above.
  - Then $-a \leq$ sup(-A) $\Leftrightarrow a \geq$ -sup(-A)
  - This means that -sup(-A) is a lower bound for A implying inf(A) $\geq$ -sup(-A).

inf A $\leq$ -sup(-A) and inf A $\geq$ -sup(-A) at the same time. Thus inf A = -sup(-A)

Question 1.5
Let $A, B \subset \mathbb{R}$ be non-empty and bounded subsets of the real numbers, and let $\lambda \in \mathbb{R}$ be a constant.

(a) Let $\lambda A = \{\lambda A \mid a \in A\}$. Prove that if $\lambda \geq 0$, then sup$\lambda A = \lambda$sup A, and if $\lambda < 0$, then sup $\lambda A = \lambda$inf A.

   i. $\lambda \geq 0$
   - In case where $\lambda = 0$, its supremum and infimum are both 0.
   - Fix $\epsilon > 0$, and suppose $\exists\ \alpha \in A$ such that sup A - $\epsilon < \alpha \leq$ sup A by definition of supremum.
   - Multiply by λ each side of the inequality above

   $$\lambda \text{sup A} - \lambda\epsilon < \lambda\alpha \leq \lambda \text{sup A}.$$

   - Hence λsup A is a supremum of $\lambda\alpha$. At the same time, sup λA is also a supremum of $\lambda\alpha$ since $\lambda\alpha \in \lambda$A.
   - Thus sup λA = λsup A

   ii. $\lambda < 0$
   - Consider the inequalities we stated above sup A - $\epsilon < \alpha \leq$ sup A.
   - Multiply this inequalities by λ

   $$\lambda \text{sup A} \leq \lambda\alpha < \lambda \text{sup A} - \lambda\epsilon$$

- Since we proved that sup λA = λsup A, we obtain

$$\text{sup } \lambda \text{A} \leq \lambda\alpha \;\Leftrightarrow\; \alpha \geq \frac{1}{\lambda} \cdot \text{sup } \lambda\text{A}$$

- This means that inf $\alpha = \frac{1}{\lambda} \cdot$ sup λA.
- Hence λinf α = sup λA

(b) Let $A + B = \{a + b \mid a \in A,\ b \in B\}$. Prove sup(A + B) = sup A + sup B.

- By definition, sup A $\geq$ a, and sup B $\geq$ b.
- Thus **sup A + sup B $\geq$ a + b**.
- Suppose $\epsilon > 0$, and consider a and b such that a $>$ sup A $- \frac{\epsilon}{2}$, b $>$ sup B $- \frac{\epsilon}{2}$.
- Then **a + b $>$ sup A + sup B -** ϵ.
- Thus sup(A + B) $\geq$ a + b = sup A + sup B - ϵ.

- Suppose x such that $x \in A + B$
- $x \leq$ sup A + sup B
- sup x $\leq$ sup A + sup B $\Leftrightarrow$ sup (A + B) $\leq$ sup A + sup B

We obtain both sup(A + B) $\geq$ = sup A + sup B and sup (A + B) $\leq$ sup A + sup B. Thus sup(A + B) = sup A + sup B.

(c) Let AB = $\{a \cdot b \mid a \in A,\ b \in B\}$. Either prove or provide a counterexample to sup(AB) = (sup A)(sup B).

- Assume $\exists$ A $= \{1, 2, 3\}$, B $= \{-1, -2, -3\}$. Then AB $= \{-9, -6, -4, -3, -2, -1\}$.
- sup AB = -1, sup A = 3, sup B = -1 respectively.
- sup A $\cdot$ sup B = -3 which is not equal to sup AB = -1.

Question 1.6

Let $A \subset \mathbb{R}$ be non-empty and bounded above. Prove that α = sup A if and only if $x \leq \alpha$ for every $x \in A$ and for every $\epsilon > 0$, there exists $x \in A$ with $\alpha - \epsilon < x \leq \alpha$.

- α = sup A **if** α = sup A if and only if $x \leq \alpha$ for every $x \in A$ and for every $\epsilon > 0$, there exists $x \in A$ with $\alpha - \epsilon < x \leq \alpha$.
  - It is obvious that α is an upper bound for A $\forall x \leq \alpha$.
  - Argue by contradiction. Suppose there exists β such that sup A = β.
  - i.e. $\beta < \alpha$ and $x \leq \beta\ \ \forall x \in A \Leftrightarrow 0 < \alpha - \beta$.
  - Suppose $\exists\, \epsilon$ such that $\epsilon = \frac{\alpha-\beta}{2} > 0$
  - i.e. $\alpha - \epsilon < \alpha$
  - Hence $\beta < \alpha \Leftrightarrow 2\beta < \alpha + \beta \Leftrightarrow \beta < \frac{\alpha+\beta}{2} = \epsilon \Leftrightarrow \beta < \alpha - \epsilon$

  - Hence further we obtain $\beta < \alpha - \epsilon < \alpha$.
  - i.e. $x < \alpha - \epsilon < \alpha \ \ \forall x \in A$, which is contradicted by the given condition that there exists $x \in A$ with $\alpha - \epsilon < x \leq \alpha$.
  - Thus α must be the least upper bound for A.

- $x \leq \alpha$ for every $x \in A$ and for every $\epsilon > 0$, there exists $x \in A$ with $\alpha - \epsilon < x \leq \alpha$ **if** $\alpha =$ sup A
  - $x \leq \alpha$ for every $x \in A$
    * Argue by contradiction. Suppose $\exists\, x > \alpha \ \ \forall x \in A$.
    * By definition, α is not an upper bound of A if $x > \alpha$.
    * Therefore, sup A must be equal to α such that $x \leq \alpha$ for every $x \in A$.
  - For every $\epsilon > 0$, there exists $x \in A$ with $\alpha - \epsilon < x \leq \alpha$
    * Argue by contradiction. Suppose $\exists \ \ x \leq \alpha - \epsilon$ for some x.
    * i.e. sup A $\neq \alpha$, but sup A $= \alpha - \epsilon$, which is contradiction.
    * Hence $\alpha - \epsilon < x \leq \alpha \ \ \forall x \in A$