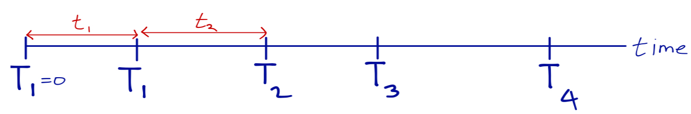
1 Renewal Process

1.1 Law of Large Number



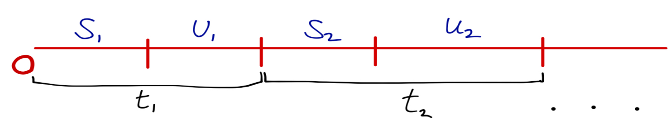
- Assumption: $\{t_i\}_{i\geq 1}$ are IID non-negative random variables.
- Common CDF: $F(t) = P(t_i \leq t)$ where F is a distribution with $F(0) = P(t_i \leq 0) = 0$.
  - At time 0, t_i ends and the following t_{i+1} starts.
  - i.e. $T_n = t_1 + \ldots + t_n$ gives the time t_n ends. (for $n \geq 1$)
- Common Mean: $\mu = \mu_F = E(t_i)$
- $N(t) = \max\{n : T_N \leq t\}$ =the number of events in [0,t]
- Special case: rate λ Poisson process whose $\{t_i\} \sim$ IID Exp(λ).

1.1.1 Example

(i) Let x is a recurrent state for a Markov chain. Start the Markov chain at x. $T_0 = 0$ and denote T_n as time of n^{th} return to x. So $t_i = T_i - T_{i-1}$. Then $\{t_i\} \geq 1$ are IID, so this is a renewal process.

(ii) Imagine a machine that alternates between a functional state and being under repair. Assume that after repair, the machine is again *like new*.

s_i = length of the i^{th} functioning cycle. u_i = length of the i^{th} repair cycle.
$t_i = s_i + u_i$



1.1.2 Theorem 3.1

Law of Large Number(LLN) for the counting process:

$$\frac{N(t)}{t} \to \frac{1}{\mu} \qquad \text{(as } t \to \infty\text{, with probability 1)}$$

Proof
Recall the Strong Law of Large Number(SLLN): if $\{X_i\}$ and IID,

$$X_i \geq 0,\ S_n = \sum_{i=1}^{n} x_i$$

then

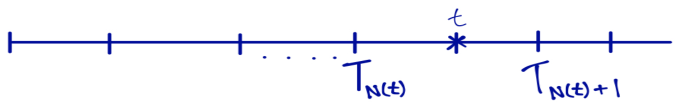
$$\frac{S_n}{n} \to EX_1 \quad \text{with probability 1}$$

Applying SLLN to $\{t_i\}$:

$$\frac{T_n}{n} \xrightarrow[n\to\infty]{w.p.1} \mu$$

Let $t \to \infty$.

$$\underbrace{\frac{T_{N(t)}}{N(t)}}_{\mu} \leq \underbrace{\frac{t}{N(t)}}_{\mu \text{ as } t\to\infty} \leq \underbrace{\frac{T_{N(t)+1}}{N(t)+1}}_{\substack{\mu \\ \text{This works} \\ \text{even in the case} \\ \mu = \infty}} \cdot \underbrace{\frac{N(t)+1}{N(t)}}_{1}$$



$$\text{as } t \to \infty, \ N(t) \to \infty$$

1.1.3 Renewal-reward Process

To the i^{th} cycle is associated a random *renewed* r_i.

Assumption:

$$\{(t_i, r_i)\}_{i\geq 1} = \{(t_i, r_i), (t_2, r_2), ...\}$$

are IID.

$$R(t) = \sum_{i=1}^{N(t)} r_i = \text{total } \textit{renewal} \text{ up to time } t$$

1.1.4 Theorem 3.3

Assume $E(r_i)$ is finite. Then

$$\frac{R(t)}{t} \xrightarrow[t\to\infty]{w.p.1} \frac{E(r_1)}{E(t_1)}$$

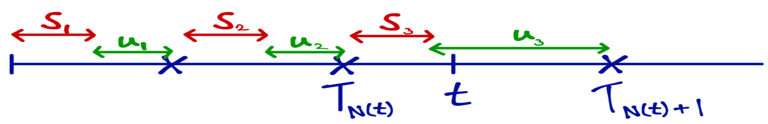
Check if there's something more

1.1.5 Example

Machine with cycles of functioning $(s_1, s_2, s_3, ...)$ and repair $(u_1, u_2, u_3, ...)$ Over the long term, what fraction of time is the machine functional?

Let $t_i = s_i + u_i$ (cycle length) and $r_i = s_i$ ("reward" = length of functional cycle).

$$\frac{R(t)}{t} = \frac{1}{t}\sum_{i=1}^{N(t)} s_i = \frac{\text{functional time up to time } t}{t} + \underbrace{\frac{\text{small discrepancy}}{t}}_{0}$$



1.1.6 Example: Car replacement policy

Let A be the price at a car after the trade-in. She keeps the car until either the car breaks down or car is T years old. Let $h(t)$ =PDF of the lifetime of this car. If the car breaks down, there is an additional cost B when the car is traded in. What is the long-term cost per time unit at this policy?

Assumption:

$$\frac{E(r_1)}{E(t_1)}$$

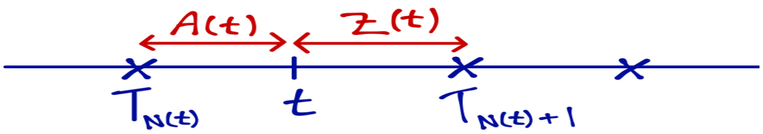
Recall: If Y has PDF h then

$$E[g(Y)] = \int_0^{\infty} yh(y)dy$$

$$E(t_i) = \int_0^{T} yh(y)dy + T\int_T^{\infty} h(y)dy \qquad \text{(since T is maximum time for a period)}$$

$$E[r_i] = A + B\underbrace{\int_0^{T} h(y)dy}_{\substack{\text{probability that}\\ \text{can break down}\\ \text{before time T}}}$$

1.2 Age and Residual Lifetime



$A(t) =$ age at the current cycle $= t - T_{N(T)}$ $\qquad Z(t) =$ residual lifetime $= T_{N(t)+1} - t$

(also called forward recurrent time)

1.2.1 Example

Consider a Poisson process with rate λ. For every t, what are the distribution's of $A(t)$, $Z(t)$, and $Z(t) \sim \exp(t)$?

$$P(A(t) > u) = P(\underbrace{N(t-u, t] = 0}_{\text{Poisson}}) = e^{-\lambda t}$$

Except for the cut-off due to the time origin, $A(t)$ is also exponential.
Claim: Let $t_1' = Z(t) = T_{N(t)+1} - t \quad t_i' = t_{N(t)+i},\ i \geq 2$
Then $\{t_i'\}_{i\geq 1}$ are independent and $\{t_i'\}_{i\geq 2}$ are IID, with the same distribution F.

1.2.2 Definition

Let S be an $\mathbb{Z}_{\geq 0}$ valued random variable. Then S is a stopping time for the sequence $\{t_i\}_{i\geq 1}$ if $\{S = n\}$ depends only on $t_1, ..., t_n$.

1.2.3 Lemma

Let $\{t_i\}_{i\geq 1}$ be IID and S a stopping time. Then $\{t_{s+i}\}_{i\geq 1}$ is independent of $(S, \{t_i\}_{i\leq s})$ and $\{t_{s+i}\}_{i\geq 1} \overset{d}{=} \{t_i\}_{i\geq 1}$

Proof

$$P\{\ \underbrace{S = n,\ (t_1, ..., t_n) \in B,}_{\text{An event determined by } (t_1, ..., t_n)}\ (t_{n+1}, t_{n+2}, ...) \in U\}$$
$$= P\{S = n,\ (t_1, ..., t_n) \in B\} \cdot \underbrace{P\{(t_{n+1}, t_{n+2} ...) \in U\}}_{P\{(t_1, t_2, ...) \in U\}}$$

Analogy from recurrent, aperiodic with invariant π:

$$P_x(X_n = y) \xrightarrow{n\to\infty} \pi(y)$$

Guided by the Markov chain example, we look for a limit distribution for $(A(t), Z(t))$. The ideal result would be

$$P(A(t) > x,\ Z(t) > y) \xrightarrow{t\to\infty} \text{(something)}$$

We don't have the technology for proving this. But we can establish a slightly weaken result.

$$\frac{1}{t}\int_0^t P(A(s) > x,\ Z(s) > y)ds \xrightarrow{t\to\infty} \text{(something)}$$

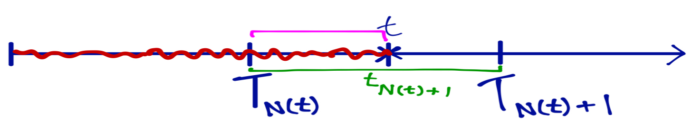
$$\frac{1}{t}\int_0^t P(A(s) > x,\ Z(s) > y)ds = \frac{1}{t}\int_0^t E[\ \underbrace{1_{A(s)>x,\ Z(s)>y}}_{\substack{\text{indicator r.v.}\\ \text{of the event}\\ \{A(s)>x, Z(s)>y\}}}\ ds]$$

$$= \underbrace{E[\frac{1}{t}\int_0^t 1_{A(s)>x,\ Z(s)>y}ds]}_{\substack{\text{This average inside the E[..]}\\ \text{we can handle with the}\\ \text{renewal-reward LLN}}}$$

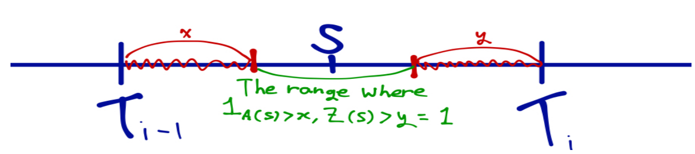
1.2.4 Theorem 3.9

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t 1_{A(s)>x,\ Z(s)>y}ds = \frac{1}{E(t_1)}\int_{x+y}^{\infty} P(t_1 > z)dz \text{ w.p.1, for any } x, y \geq 0$$

$$\frac{1}{t}\int_0^t 1_{A(s)>x, Z(s)>y}ds = \frac{1}{t}\sum_{i=1}^{N(t)}\int_{T_{i-1}}^{T_i} 1_{A(s)>x,\ Z(s)>y}ds + \underbrace{\frac{1}{t}}_{\substack{0\\ \text{as } t\to\infty}}\int_{T_{N(t)}}^{t} 1_{A(s)>x,\ Z(s)>y}\ ds\ \leq\ t_{N(t)+1}$$



Let's calculate $r_i =$



<u>2 cases</u>:

$$\begin{cases} x+y \leq t_i : r_i = t_i - (x+y) \\ x+y > t : r_i = 0 \end{cases}$$

Notation: for red x, $x^+ = \max(x, 0)$. e.g. $7^+ = 7, (-3)^+ = 0$

$\mathrm{r}_i = (t_i - (x+y))^+$.

By the renewal-reward LLN,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N(t)} r_i = \frac{E(r_1)}{E(t_1)}$$

Useful formula: For $X \geq 0$, then

$$E(X) = \int_0^\infty P(X > s) ds.$$

proof.

$$\begin{aligned}
E(X) &= E\left(\int_0^X ds\right) = E\left(\int_0^\infty 1_{X>s} ds\right) \\
&= \int_0^\infty E(1_{X>s}) ds = \int_0^\infty P(X > s) ds \\
E(r_1) &= E\left[(t_1 - (x+y))^+\right] = \int_0^\infty P\{\underbrace{(t_1 - (x+y))^+ > s}_{\text{for } s>0}\} ds
\end{aligned}$$

$$(t_1 - (x+y))^+ > s \Leftrightarrow t_1 - (x+y) > s$$
$$\Leftrightarrow t_1 > s + x + y$$

$$= \int_0^\infty p(t_1 > s + x + y) ds = \int_{x+y}^\infty P(t_1 > z)$$

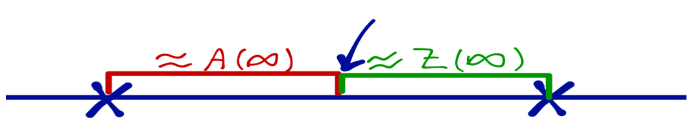
Fact: if random variables $X_n \to X$ w.p.1 and $|X_n| \leq c$ (constant) $\forall n$ then $E(X_n) \to E(X)$

Take E[...] over Theorem 3.9:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P(A(s) > x,\ Z(s) > y) ds = \frac{1}{E(t_1)} \int_{x+y}^\infty P(t_1 > z) dz$$

Let $A(\infty),\ Z(\infty)$ represent *limiting* or long-term age and residual lifetime. Our result gives

$$P(A(\infty) > x,\ Z(\infty) > y) = \frac{1}{E(t_1)} \int_{x+y}^\infty P(t_1 > z) dz$$



1.2.5 Example: $t_i \sim \mathbf{Exp}(\lambda),\ E(t_1) = \frac{1}{\lambda}$

$$
\begin{aligned}
P(A(\infty) > x,\ Z(\infty) > y) &= \frac{1}{E(t_1)} \int_{x+y}^{\infty} P(t_1 > z)dz \\
&= \lambda \int_{x+y}^{\infty} e^{-\lambda z} dz = e^{-\lambda x} \cdot e^{-\lambda y} \\
&\Rightarrow \text{so } A(\infty) \text{ and } Z(\infty) \text{ are independent and both are Exp}(\lambda)
\end{aligned}
$$

1.2.6 Example: $t_i \sim$ Unif(0,1), $Et_1 = \frac{1}{2}$

$$
\begin{aligned}
P(A(\infty) > x,\ Z(\infty) > y) &= \frac{1}{E(t_1)} \int_{x+y}^{\infty} \underbrace{P(t_1 > z)}_{=\begin{cases} 1-z & 0<z<1 \\ 0 & z \geq 1 \end{cases}} dz \\
&= 2\int_{x+y}^{1} (1-z)dz = 2\int_{0}^{1-(x+y)} udu = (1-x-y)^2
\end{aligned}
$$

$$
P(A(\infty) > x),\ Z(\infty) > y) = \begin{cases} 0 & x+y \geq 1 \\ (1-x-y)^2 & x+y < 1 \end{cases}
$$

Marginals:

$$
\begin{aligned}
&P(Z(\infty) > y) = P(A(\infty) > 0,\ Z(\infty) > y) = (1-y)^2 \\
&P(A(\infty) > x) = (1-x)^2
\end{aligned}
$$

$(1-x)^2(1-y)^2 = (1-x-y)^2$ is not true except for some special x,y. Hence $A(\infty)$ and $Z(\infty)$ are not independent.

Back to the general results:

$$
P(A(\infty > x,\ Z(\infty) > y) = \frac{1}{E(t_1)} \int_{x+y}^{\infty} P(t_1 > z)dz
$$

Let's find g= PDF of $Z(\infty)$, and also $E(Z(\infty))$. Assume that t_1 has PDF f_{t_1}.

$$
P(Z(\infty) > y) = P(A(\infty) > 0,\ Z(\infty) > y) = \frac{1}{E(t_1)} \int_{y}^{\infty} P(t_1 > z)dz
$$

$$
g(y) = -\frac{d}{dy} P(z(\infty) > y) = \frac{P(t_1 > y)}{E(t_1)}
$$

1.2.7 Example: $t_1 \sim \mathbf{Exp}(\lambda)$

Another useful formula: Assume $X \geq 0,\ h(0) = 0$

$$E[h(X)] = E\left[\int_0^X h'(s)ds\right] = E\left[\int_0^\infty h'(s)1_{X>s}ds\right]$$
$$= \int_0^\infty h'(s)E[1_{X>s}]ds = \int_0^\infty h'(s)P(X > s)ds$$

$$g(y) = \frac{P(t_1 > y)}{E(t_1)} = \frac{e^{-\lambda y}}{1/\lambda} = \lambda e^{-\lambda y}$$

$$E[Z(\infty)] = \int_0^\infty y g(y)dy = \frac{1}{E(t_1)}\int_0^\infty yP(t_1 > y)dy = \frac{\frac{1}{2}E[t_1^2]}{E(t_1)}$$

1.2.8 Theorem

Let now t_1' have the distribution of $Z(\infty)$, and t_2', t_3', ... have the IID distribution of t_1. Then we get a *stationary renewal process* whose probability are constant in time: in particular, P(the number of arrivals in (a,b] = m) = P(the number of arrivals in (s+a,s+b)=m)