1. Consider an $M/M/\infty$ queue (Durrett Example 4.16) where customers arrive at rate λ, and the service time for each server is a rate μ exponential random variable. Let X(t) denote the number of customers in the system at time t. Assume that $X(0) = 0$.

   (a) State the Kolmogorov forward equations for this process.

   Rates for CTMC

$$\begin{cases} q(n, n+1) = \lambda & \text{for n=0,1,2,...} \\ q(n, n-1) = n\mu & \text{for n=1,2,3,...} \end{cases} \qquad \begin{cases} \lambda_0 = \lambda \\ \lambda_{n\geq 1} = \lambda + n\mu \end{cases}$$

   The forward equation

$$\frac{d}{dt}[p_t(i,j)] = \sum_{k \neq jr} q(i,k)p_t(k,j) - \lambda_i p_t(i,j)$$

$$\begin{cases} j \geq 1 : p_t'(i,j) = p_t(i,\ j-1)\lambda + p_t(i,\ j+1)\underbrace{(j+1)\mu}_{n\mu} - (\lambda + j\mu)p_t(i,\ j) \\ j = 0 : p_t'(i,0) = p_t(i,1)\mu - \lambda_0 p_t(i,0) \end{cases}$$

   (b) Set $M(t) = E[X(t)]$. Prove that

$$\frac{dM}{dt} = \lambda - \mu M(t)$$

   Hint. Use the equations from (a). Do not hesitate to differentiate series term by term.

$$\begin{aligned} \frac{d}{dt}M(t) &= \frac{d}{dt}E[\underbrace{X(t)}_{p_t(0,j)}] = \frac{d}{dt}\sum_{j=0}^{\infty} jp_t(0,j) = \sum_{j=1}^{\infty} jp_t'(0,j) \\ &= \sum_{j=1}^{\infty} j[p_t(0,\ j-1)\lambda + p_t(0,\ j+1)(j+1)\mu - (\lambda + j\mu)p_t(0,\ j)] \\ &= \lambda \underbrace{\sum_{j=1}^{\infty}(j-1)p_t(0,\ j-1)}_{M(t)} + \underbrace{\lambda\sum_{j=1}^{\infty} p_t(0,\ j-1)]}_{1} \\ &+ \sum_{j=1}^{\infty} \mu \underbrace{j(j+1)}_{(j+1)^2-(j+1)} p_t(0,\ j+1) - \lambda \underbrace{\sum_{j=1}^{\infty} jp_t(0,j)}_{M(t)} - \sum_{j=1}^{\infty} \mu j^2 p_t(0,j) \end{aligned}$$

$$= \cancel{\lambda \cdot M(t)} + \lambda + \mu \underbrace{\cancel{\sum_{j=1}^{\infty}(j+1)^2 p_t(0,\ j+1)}}_{p_t(0,1)} - \mu \underbrace{\sum_{j=1}^{\infty}(j+1)p_t(0,\ j+1)}_{M(t)}$$

$$-\cancel{\lambda \cdot M(t)} - \mu \underbrace{\cancel{\sum_{j=1}^{\infty} j^2 p_t(0,\ j)}}_{p_t(0,1)}$$

$$= \lambda - \mu M(t)a$$

(c) Solve the differential equation for M(t). (A reminder about linear ODEs is appended to this HW sheet.)

$$\begin{aligned}
\frac{dM(t)}{dt} &= \lambda - \mu M(t) \\
e^{\mu t}\frac{dM(t)}{dt} &= e^{\mu t}\lambda - \mu e^{\mu t} M(t) \\
e^{\mu t}\lambda &= \underbrace{e^{\mu t}\frac{dM(t)}{dt} + \mu e^{\mu t} M(t)}_{[e^{\mu t}\cdot M(t)]'} \\
e^{\mu t}\cdot M(t) &= \int [e^{\mu t}\cdot M(t)]' dt = \int e^{\mu t}\lambda dt = \frac{\lambda}{\mu}e^{\mu t} + C \\
M(t) &= \frac{\lambda}{\mu} + \frac{C}{e^{\mu t}} \quad \Leftrightarrow \quad M(0) = \frac{\lambda}{\mu} + C = 0 \qquad \text{since X(0)=0} \\
M(t) &= \frac{\lambda}{\mu}\left(1 - \frac{1}{e^{\mu t}}\right)
\end{aligned}$$

(d) Evaluate $\lim_{t\to\infty} M(t)$.The stationary distribution for $X(t)$ is given in Example 4.16 of Durrett's book. Compare the limit you found to the expected value of the stationary distribution.

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} \frac{\lambda}{\mu}\left(1 - \frac{1}{e^{\mu t}}\right) = \frac{\lambda}{\mu}$$

2. Consider an M/M/2 queue where customers arrive at rate λ and the rate for each server is μ. However, arriving customers who see N customers already in the system leave and never return. Assume $N > 2$. Let $X(t)$ denote the number of customers in the system at time t. Find the stationary distribution for $X(t)$. (The M/M/s queue appears in Durrett's examples 4.3 and 4.17.)

$$\begin{cases} q(n, n+1) = \lambda & \text{for n=0,1,2,...,N-1} \\ q(n, n-1) = \mu & \text{for n=1,2,3,...,N} \end{cases}$$

Since $\pi \cdot Q = 0$ should be achieved,

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \dots & \pi_N \end{bmatrix} \begin{bmatrix} -\lambda & \lambda & 0 & \dots & & & \\ \mu & -(\mu+\lambda) & \lambda & 0 & \dots & & \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \dots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & \mu & -(\mu+\lambda) & \lambda \\ & & & & 0 & \mu & -\mu \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}$$

$$\begin{aligned} \lambda\pi_0 = \mu\pi_1 \quad &\Leftrightarrow \quad \pi_1 = \frac{\lambda}{\mu}\pi_0 \\ \lambda\pi_1 = \mu\pi_2 \quad &\Leftrightarrow \quad \pi_2 = \frac{\lambda}{\mu}\pi_1 \\ &\vdots \\ \lambda\pi_{n-1} = \mu\pi_n \quad &\Leftrightarrow \quad \pi_n = \frac{\lambda}{\mu}\pi_{n-1} = \left(\frac{\lambda}{\mu}\right)^n \pi_0 \end{aligned}$$

$$\begin{aligned} \sum_{j=0}^{N} \pi_j = 1 \quad &\Leftrightarrow \quad \pi_0 \sum_{j=0}^{N} \left(\frac{\lambda}{\mu}\right)^j = \pi_0 \cdot \frac{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}{1 - \frac{\lambda}{\mu}} = 1 \\ &\Leftrightarrow \pi_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} \end{aligned}$$

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}$$

For the $M/M/2$ queue, the rates are

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & \cdots \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & \text{N-1} \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & \text{N} \arrow[l, bend left, "2\mu"]
\end{tikzcd}$$

$$\begin{cases} q(j,\ j+1) = \lambda & \text{for j=0,1,...,N-1} \\ q(j,\ j-1) = 2\mu & \text{for j=2,...,N-1} \\ q(1,0) = \mu \end{cases}$$

$$\begin{aligned}
\pi(j) &= \frac{q(0,1)q(1,2)\ldots q(j-1,j)}{q(j,j-1)q(j-1,j-2)q(j-2,j-3)\ldots q(1,0)} \cdot \pi(0) \\
&= \frac{\lambda^j}{(2\mu)^{j-1}\mu}\pi(0) = \frac{1}{2^{j-1}}\left(\frac{\lambda}{\mu}\right)^j \pi(0) = 2\cdot\left(\frac{\lambda}{2\mu}\right)^j \pi(0)
\end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{N} \pi(j) = 1 \quad &\Leftrightarrow \quad \pi_0 + \sum_{j=0}^{N} 2\cdot\left(\frac{\lambda}{2\mu}\right)^j \pi(0) = 1 \\
&\Leftrightarrow \pi_0 \left(1 + 2\cdot\frac{1-\left(\frac{\lambda}{2\mu}\right)^N}{1-\frac{\lambda}{2\mu}}\cdot\frac{\lambda}{2\mu}\right) = 1 \\
\pi_0 &= \frac{1}{\left(1 + 2\cdot\frac{1-\left(\frac{\lambda}{2\mu}\right)^N}{1-\frac{\lambda}{2\mu}}\cdot\frac{\lambda}{2\mu}\right)} \qquad \pi_j = 2\left(\frac{\lambda}{2\mu}\right)^j \pi_0
\end{aligned}$$

3. (a) Consider the special case of the previous problem in which $\lambda_1 = \lambda_2 = 1$, and $\mu_1 = \mu_2 = 3$, and find the stationary probabilities.

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 3 & -4 & 0 & 1 \\ 3 & 0 & -4 & 1 \\ 0 & 3 & 3 & -6 \end{pmatrix} \qquad \pi = \begin{pmatrix} \frac{9}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \end{pmatrix}$$

such that $\pi Q = 0$

(b) Suppose they upgrade their telephone system so that a call to one line that is busy is forwarded to the other phone and lost if that phone is busy. Find the new stationary probabilities.

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 3 & -5 & 0 & 2 \\ 3 & 0 & -5 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix} \qquad \pi = \begin{pmatrix} \frac{9}{17} & \frac{3}{17} & \frac{3}{17} & \frac{2}{17} \end{pmatrix}$$

such that $\pi Q = 0$

4. A hemoglobin molecule can carry one oxygen or one carbon monoxide molecule. Suppose that the two types of gases arrive at rates 1 and 2 and attach for an exponential amount of time with rates 3 and 4, respectively. Formulate a Markov chain model with state space $\{+,\ 0,\ -\}$ where $+$ denotes an attached oxygen molecule, $-$ an attached carbon monoxide molecule, and 0 a free hemoglobin molecule and find the long-run fraction of time the hemoglobin molecule is in each of its three states.

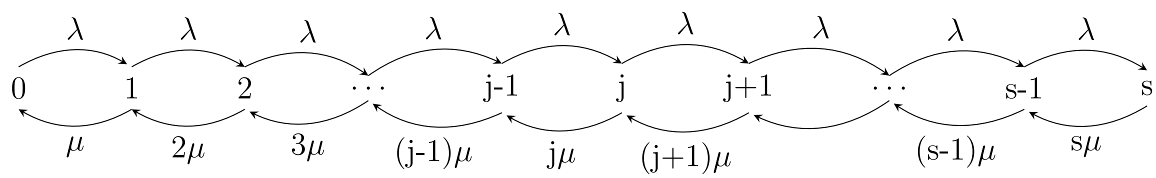
$$Q = \begin{array}{cccc} & + & 0 & - \\ + & -3 & 3 & 0 \\ 0 & 1 & -3 & 2 \\ - & 0 & 4 & -4 \end{array}$$

$$\pi = \begin{bmatrix} \pi_+ & 3\pi_+ & \frac{3}{2}\pi_+ \end{bmatrix} \qquad \text{(such that } \pi Q = 0\text{)}$$

$$\pi_+ + 3\pi_+ + \frac{3}{2}\pi_= \frac{11}{2}\pi_+ = 1 \quad \pi_+ = \frac{2}{11}$$

$$\Rightarrow \pi = \begin{bmatrix} \frac{2}{11} & \frac{6}{11} & \frac{3}{11} \end{bmatrix}$$

5. Consider an $M/M/s$ queue with no waiting room. In words, requests for a phone line occur at a rate λ. If one of the s lines is free, the customer takes it and talks for an exponential amount of time with rate . If no lines are free, the customer goes away never to come back. Find the stationary distribution. You do not have to evaluate the normalizing constant.



$$\begin{cases} q(j,\ j+1) = \lambda & \text{for j=0,1,...,s-1} \\ q(j,\ j-1) = j\mu & \text{for j=1,2,...,s} \end{cases}$$

Consider the birth and death chain.

$$\begin{aligned} \pi(j) &= \frac{\lambda_{j-1}\ldots\lambda_0}{\mu_j\ldots\mu_1}\pi(0) & \qquad \text{(where } q(j,\ j+1) = \lambda_j \text{ and } q(j,\ j-1) = \mu_j\text{)} \\ &= \frac{(\lambda_\mu)^j}{j!}\pi(0) \end{aligned}$$