Question 1
Copy machine 1 is in use now. Machine 2 will be turned on at time t. Suppose that the machines fail at rate λ_i. What is the probability that machine 2 is the first to fail?

- Let A be the event that Machine 1 does not fail before time t and B be the event that Machine 2 fails first after time t.
- The probability we try to get is

$$P(A, B) = \frac{P(B \mid A)}{P(A)}$$

- Let us think
  - $\tau_1 \sim \exp(\lambda_1)$ as the case A
  - $t + \tau_2 \sim \exp(\lambda_2)$ as the case B
- Then we can construct this conditional probability as below.

$$P(t + \tau_2 < \tau_1) = P(t + \tau_2 < \tau_1 \mid t < \tau_1) \cdot P(t < \tau_1) = P(t + \tau_2 < \tau_1 \mid t < \tau_1) \cdot e^{-\lambda_1 t}$$

Recall that $P(T > t) = 1 - P(T \leq t) = 1 - \left(1 - e^{-\lambda t}\right) = e^{-\lambda t}$

- By the Memoryless Property, $P(t + \tau_2 \mid t < \tau_1) = P(\tau_2 < \tau_1)$.
- So

$$P(t + \tau_2 < \tau_1 \mid t < \tau_1) \cdot e^{-\lambda_1 t} = P(\tau_2 < \tau_1) \cdot e^{-\lambda_1 t}$$

$$= \iint_{\tau_2 < \tau_1} f_{\tau_2(x)} f_{\tau_1} \, dydx \cdot e^{-\lambda_1 t} = e^{-\lambda_1 t} \left(\int_0^\infty \lambda_2 e^{-\lambda_2 x} \left(\int_x^\infty \lambda_1 e^{-\lambda_1 y} dy \right) dx \right)$$

$$= e^{-\lambda_1 t} \left(\int_0^\infty \lambda_2 e^{-\lambda_2 x} \left(e^{-\lambda_1 x} \right) dx \right) = e^{-\lambda_1 t} \left(\int_0^\infty \lambda_2 e^{-(\lambda_1 + \lambda_2) x} dx \right)$$

$$= e^{-\lambda_1 t} \left(\int_0^\infty \lambda_2 e^{-(\lambda_1 + \lambda_2) x} dx \ \cdot \ \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} \right) = e^{-\lambda_1 t} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \int_0^\infty \lambda_2 e^{-(\lambda_1 + \lambda_2) x} \ \cdot \ (\lambda_1 + \lambda_2) dx \right)$$

- Since

$$\int_0^\infty \lambda_2 e^{-(\lambda_1 + \lambda_2) x} \ \cdot \ (\lambda_1 + \lambda_2) dx = 1,$$

the remainder is the probability that machine 2 is the first to fail

$$e^{-\lambda_1 t} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

Question 2
Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S,T\}$ and $V = \max\{S,T\}$. Find

(a) EU

- Since $U = \min\{S,T\}$ is independent,

$$P(U > t) = P(S > t, T > t) = P(S > t) \cdot P(T > t) = e^{-\lambda t} \cdot e^{-\mu t} = e^{-(\lambda+\mu)t}$$

- Hence the expectation EU is

$$\frac{1}{\lambda+\mu}$$

(b) $E(V-U)$. Compute first $P(V-U > s)$ for $s > 0$ either by integrating densities of S and T or by conditioning on the events $S < T$ and $T < S$. From $P(V-U > s)$ deduce the density function $f(v-u)$ of $V-U$, and then the mean $E(V-U)$ by integrating the density.

- $P(V-U > s)$

$$= \int_0^\infty \int_0^\infty (\max\{S,T\} - \min\{S,T\}) \cdot \lambda e^{-\lambda s} \mu e^{-\mu t} \, dsdt$$

$$= \int_0^\infty \int_0^\infty (s-t)\lambda\mu e^{-\lambda s - \mu t} \, dsdt\Big|_{s>t} + \int_0^\infty \int_0^\infty (t-s)\lambda\mu e^{-\lambda s - \mu t} \, dsdt\Big|_{s<t}$$

Let $x = s - t$.

$$\begin{aligned}
\int_0^\infty \int_0^\infty (s-t)\ \lambda\mu\ e^{-\lambda s - \mu t} \ dsdt|_{s>t} &= \lambda\mu \cdot \int_0^\infty \int_t^\infty (s-t)\ e^{-\lambda s-\mu t} \ dsdt \qquad (1) \\
&= \lambda\mu \cdot \int_0^\infty \int_0^\infty x\ e^{-\lambda(x+t)-\mu t} \ dxdt \\
&= \lambda\mu \cdot \int_0^\infty x\ e^{-\lambda x} \left(\int_0^\infty e^{-t(\lambda+\mu)} dt \right) dx \\
&= \lambda\mu \cdot \frac{1}{\lambda+\mu} \cdot \int_0^\infty x\ e^{-\lambda x} dx \\
&= \lambda\mu \cdot \frac{1}{\lambda+\mu} \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda} = \frac{\mu}{\lambda(\lambda+\mu)}
\end{aligned}$$

Through the same procedure for the other double integral ($s < t$), we are able to get

$$\frac{\lambda}{\mu(\lambda+\mu)}$$

(c) $EV = E(\max\{S, T\})$

$$P(S < t)P(T < t) = (1 - e^{-\lambda t})(1 - e^{-\mu t}) = (1 - e^{-\mu t} - e^{\lambda t} + e^{-t(\mu+\lambda)})$$

$$\frac{d}{dt}(1 - e^{-\mu t} - e^{\lambda t} + e^{-t(\mu+\lambda)}) = \left(\frac{1}{\mu}e^{-\mu t} + \frac{1}{\lambda}e^{-\lambda t} - \frac{1}{\mu+\lambda}e^{-t(\mu+\lambda)}\right)$$

$$EV = \frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\mu+\lambda} = \frac{\lambda\mu + \lambda^2 + \mu^2 + \lambda\mu}{\lambda\mu(\mu+\lambda)} - \frac{\lambda\mu}{\lambda\mu(\mu+\lambda)} = \frac{\lambda^2 + \lambda\mu + \mu^2}{\lambda\mu(\lambda+\mu)}$$

Finally, check that your answers to (a),(b),(c) satisfy $E(V - U) = E(V) - E(U)$.

$$E(V - U) = \frac{\mu^2 + \lambda^2}{\lambda\mu(\lambda+\mu)}$$

$$E(V) - E(U) = \frac{\lambda^2 + \lambda\mu + \mu^2}{\lambda\mu(\lambda+\mu)} - \frac{1}{\lambda+\mu} = \frac{\lambda^2 + \mu^2}{\lambda\mu(\lambda+\mu)}$$

Question 3

In a hardware store you must first go to server 1 to get your goods and then go to a server 2 to pay for them. Suppose that the times for the two activities are exponentially distributed with means 6 minutes and 3 minutes. Find the answer when times for the two activities are exponentially distributed with rates λ and μ.

For symmetric notation, let A_i denote the amount of time Al spends at server i, and B_i the amount of time Bob spends at server i, for $i = 1, 2$.

Hint. Draw a picture of the time line to understand how Al and Bob move through the servers. Use your answer from 2.7. This problem requires no integration.

- Variables
  - A : The total amount of time Al spent in the store.
  - A_1: The amount of time Al to get his goods at server 1
  - A_2: The amount of time Al pay his goods at server 2
  - B : The total amount of time Bob spent in the store.
  - B_1: The amount of time Bob to get his goods at server 1
  - B_2: The amount of time Bob pay his goods at server 2
- Things to consider.
  - Since Al is already with server 1, we must consider the amount of time Al spend there which is A_1 at the beginning.
  - So we break into only two cases:
    (i) The server 2 is free, but Bob has not finished yet at server 1. ($A_2 < B_1$)
    $A_1 + B_1 + B_2$

(ii) The server 2 is still with Al, but the Bob has done at server 1. $(B_1 < A_2)$
$A_1 + B_1 + (A_2 - B_1)_{>0} + B_2 = A_1 + B_2 + A_2$

- So $B = A_1 + B_2 + \max\{A_2, B_1\}$
- $E(B) = E(A_1) + E(B_2) + E(\max\{A_2, B_1\})$

$$= \frac{1}{\mu} + \frac{1}{\lambda} + \left(\frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\lambda + \mu}\right)$$

(Consider Question 2 (c))

Question 4

A machine has two critically important parts and is subject to three different types of shocks. Shocks of type i occur at times of a Poisson process with rate λ_i. Shocks of types 1 break part 1, those of type 2 break part 2, while those of type 3 break both parts. Let U and V be the failure times of the two parts.
Hint. Let N_i denote a Poisson process of rate λ_i. For part (a), express the event in terms of $N_1, N-2$ and N_3. Part (b) follows quickly from (a). For part (c), check whether $P(U > s, V > t)$ equals $P(U > s) \cdot P(V > t)$.

(a) Find $P(U > s, V > t)$

Let $T_{i\ (i=1,2,3)}$ is the times of shock types 1,2,3 occur.

$$P(U > s,\ V > t) = P(T_1 > s_2,\ T_2 > t_2,\ t_3 > \max\{s, t\})$$
$$= P(T_1 > s)P(T_2 > t)P(T_3 > \max\{s, t\}) = e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max\{s,t\}}$$

(b) Find the distribution of U and the distribution of V.

$$\begin{aligned} P(U \leq s) &= 1 - P(V > s) \\ &= 1 - (P(T_1 > s, T_3 > s) \\ &= 1 - (P(T_1 > s)P(T_3 > s)) \\ &= 1 - \left(e^{-\lambda_1 s - \lambda_3 s}\right) = 1 - \left(e^{-s(\lambda_1 + \lambda_3)}\right) \end{aligned}$$

$$\begin{aligned} P(V \leq t) &= 1 - P(V > s) \\ &= 1 - (P(T_2 > s, T_3 > s) \\ &= 1 - (P(T_2 > s)P(T_3 > s)) \\ &= 1 - \left(e^{-\lambda_2 s - \lambda_3 s}\right) = 1 - \left(e^{-s(\lambda_2 + \lambda_3)}\right) \end{aligned}$$

(c) Are U and V independent? **No**

$$P(U > s,\ V > t) = e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max\{s,t\}} \neq \left(e^{-s(\lambda_1+\lambda_3)}\right)\cdot\left(e^{-s(\lambda_2+\lambda_3)}\right) = P(U > s)\cdot P(V > t)$$

Question 5

Consider a Poisson process of rate λ and let s be a fixed positive number. Let σ be the random amount of time from s till the next arrival. In symbols,

$$\sigma = \left(\min_{k:T_k>s} T_k\right) - s.$$

Calculate rigorously the probability $P(\sigma > a)$ for real $a > 0$ by using the density functions of the arrival times T_k and the interarrival times τ_k.

Hint. Draw a picture of the time line. Since $s + \sigma$ is one of the arrival times, you can decompose the probability $P(\sigma > a)$ into different cases according to which T_k is equal to $s + \sigma$.

$$\begin{aligned}
P(\sigma > a) &= \sum_{k=0}^{\infty} P(\sigma > a,\ T_k \le s < T_{k+1}) \\
&= \sum_{k=0}^{\infty} P(T_{k+1} - s > a,\ T_{k+1} > s,\ T_k \le s) \\
&= \sum_{k=0}^{\infty} P(T_{k+1}a + s,\ T_{k+1} > s,\ T_k \le s) \\
&= \sum_{k=0}^{\infty} P(T_{k+1} > a + s,\ T_k \le s) \\
&= \sum_{k=0}^{\infty} P(T_k \le s,\ T_{k+1} > a + s) \\
&= \sum_{k=0}^{\infty} P(T_k \le s,\ T_{k+1} > a + s - T_k) \\
&= \sum_{k=0}^{\infty} \int_0^s \int_{s+a-t}^{\infty} f_{T_k}(t) f_{T_{k+1}}(u)\, dudt \\
&= \sum_{k=0}^{\infty} \int_0^s \lambda e^{-\lambda t} \frac{\lambda t^{k-1}}{(k-1)!} \int_{s+a-t}^{\infty} \lambda e^{-\lambda u}\, dudt \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda(s+a)} \int_0^s t^{k-1}\, dt \\
&= \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} e^{-\lambda(s+a)} = e^{-\lambda(s+a)} \cdot e^{\lambda s} = e^{-\lambda a}
\end{aligned}$$