

Philosophically exactly like in discrete time: given the present state, the past and the future are independent. But the math becomes more technical. Throughout, the state space S is finite or countably infinite.

Definition Let $\{X_t : 0 \leq t < \infty\}$ be a stochastic process. Then X is a Markov chain if for all $0 \leq s_0 < \dots < s_n < s < t$ and all $x_0, \dots, x_n, x, y \in S$:

$$P(X_t = y \mid X_s = x, X_{s_n} = x_n, \dots, X_{s_0} = x_0) = P(X_t = y \mid X_s = x)$$

wherever the conditioning event has positive probability.

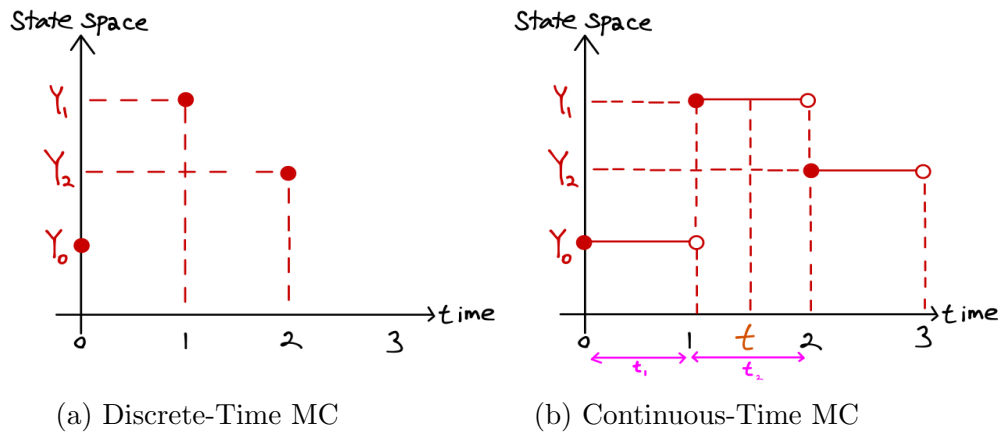
We will only deal with the time-homogeneous case where

$$P(X_t = y \mid X_s = x) = P(X_{t-s} = y \mid X_0 = x)$$

The transition probability is

$$P_t^{(x,y)} = P(X_t = y \mid X_0 = x)$$

Since there are uncountably many transition matrices $\{P_t(x, y)\}_{x,y \in S}$, the transition probability is not as useful as in discrete time. It even turns out to be extremely hard to calculate in some cases.



The Markov property force the holding times t_1, t_2, t_3, \dots to be exponential random variables. By the same token, the sequence of states visited has to be a discrete-time Markov chain.

Recall the following:

Let $\eta_1, \eta_2, \dots, \eta_n$ be independent. $\eta \sim (\lambda)$. $\zeta = \min\{\eta_1, \dots, \eta_n\}$. $I =$ the index i s.t. $\eta_i = \zeta$.

$$\zeta \perp I \quad \zeta \sim \exp(\lambda_1 + \dots + \lambda_n)$$

$$P(\zeta > t) = \prod_{i=1}^n P(\eta_i > t) = \prod_{i=1}^n e^{-\lambda_i t} = e^{-(\lambda_1 + \dots + \lambda_n)t} \quad P(I = 1) = \frac{\lambda_1}{\sum_{j=1}^n \lambda_j}$$

Special case Let (Y_n) be a discrete-time Markov chain with transition matrix $u = (u(x, y))_{x, y \in S}$. Let $N(t)$ be a rate λ Poisson process. Then $X_t = Y_{N(t)}$ is a continuous time Markov chain. In this **SPECIAL CASE** we can calculate the transition probability:

$$\begin{aligned}
 P_t(x, y) &= P(x_t = y \mid X_0 = x) = P(X_t = y \mid Y_0 = x) \\
 &= \sum_{n=0}^{\infty} P(X_t = y, N(t) = n \mid Y_0 = x) \\
 &\quad (N(t): \text{the number of jumps by time } t) \\
 &= \sum_{n=0}^{\infty} P(Y_n = y, N(t) = n \mid Y_0 = x) \\
 &= \sum_{n=0}^{\infty} P(N(t) = n) P(Y_n = y \mid Y_0 = x) \\
 &= \sum_{n=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^n}{n!} \cdot u^n(x, y)
 \end{aligned}$$

Note that $Z \sim \text{Poisson}(\mu)$ then

$$P(Z = k) = \frac{e^{-\mu} \mu^k}{k!}$$

What takes the place of the transition probabilities as the fundamental object? **Rates**.

Definition For $x \neq y$ in S , the Rate of jumping from x to y is

$$q(x, y) = \lim_{h \rightarrow 0} \frac{P_h(x, y)}{h}$$

This limit will exist in all our examples.

Note that

$$u^{(0)}(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

$$u^{(0)} = I = \text{the identity matrix}$$

Back to the previous **SPECIAL CASE**:

$$\begin{aligned}
 \frac{1}{h} p_h(x, y) &= \frac{1}{h} \sum_{n=0}^{\infty} \frac{e^{\lambda h} (\lambda h)^n}{n!} u^n(x, y) \\
 &\vdots \\
 n=0 \text{ term} &= 0! \\
 &\vdots \\
 &= \underbrace{0}_{n=0} \underbrace{e^{-\lambda h} \lambda u(x, y)}_{\substack{n=1 \\ \lambda u(x, y) \text{ as } h \rightarrow 0}} + \underbrace{\frac{1}{h} \sum_{n=2}^{\infty} \frac{e^{-\lambda h} (\lambda h)^n}{n!} u^{(n)}(x, y)}_{\substack{0 \leq \cdot \leq \frac{1}{h} h^2 \sum_{n=2}^{\infty} \frac{\lambda^n h^{n-2}}{n!} \\ \leq h \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \leq h e^{\lambda}}}
 \end{aligned}$$

This term doesn't matter later on

Hence

$$q(x, y) = \lambda u(x, y)$$

where λ is the rate of Poisson clock and $u(x, y)$ is the probability of jumping to y .

"little oh of h" $o(h)$ means a quantity that when divided by h goes to 0 as $h \rightarrow 0$, $\frac{o(h)}{h} \xrightarrow{h \rightarrow 0} 0$

→ Hence forth we will give our examples in terms of rates.

Ex. Let $N(\cdot)$ be a rate λ Poisson process. Then $N(\cdot)$ is a Markov chain with state space $S = \mathbb{Z}_{\geq 0}$ and rates

Continuous Time Markov chains are determined by the rates

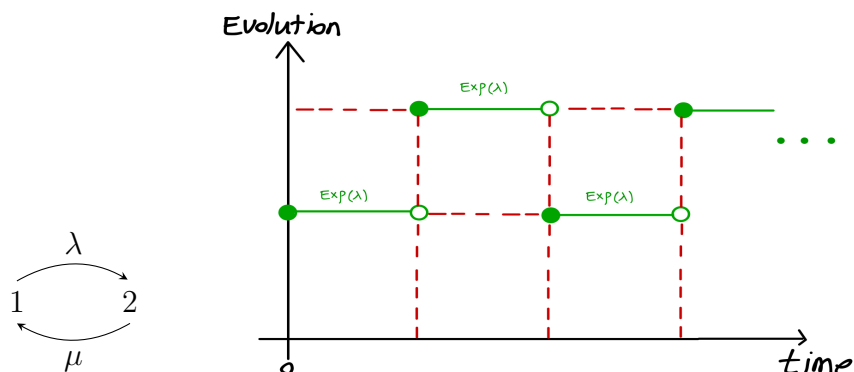
$q(x, y) = \lim_{h \rightarrow 0} \frac{p_h(x, y)}{h}$ = exponential rate of jumping to y , when the present state is x

Given rates $\{q(x, y)\}_{x \neq y}$

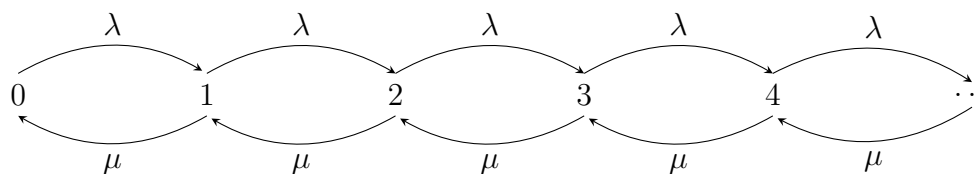
1. Construct the Markov chain X_t
2. Compute transition probability $P_t = (p_t(x, y))_{x, y \in S}$
3. Determine invariant distributions

1. Construct the Markov chain X_t . Suppose $X_0 = 1$.

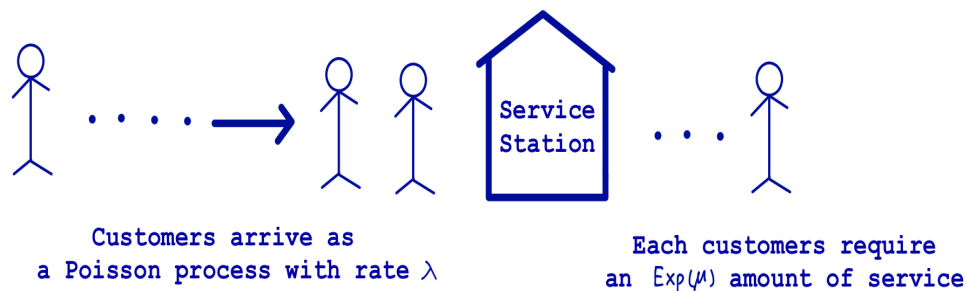
- $S = \{1, 2\}$.



- Compute transition probability $P_t = (p_t(x, y))_{x, y \in S}$



- Birth-death chain (Population Model)
- M/M/1 queue



Example

Suppose the current state is 3. Then the rate of jumping out of the state 3 is $\lambda + \mu$. And

$$\begin{cases} P(\text{move to 4}) = \frac{\lambda}{\lambda + \mu} \\ P(\text{move to 2}) = \frac{\mu}{\lambda + \mu} \end{cases}$$

Construction of X_t , given the rates $\{q(x, y)\}_{x \neq y}$

$$\lambda_x = \sum_{y : y \neq x} q(x, y) = \text{total rate to jump out of state } x$$

If $\lambda_x = \infty$, the Markov chain leaves state x immediately, so we can ignore such states & assume each $\lambda_x < \infty$. $\lambda_x = 0$ means that x is absorbing.

If $0 < \lambda_x < \infty$, we define a transition probability

$$r(x, y) = \frac{q(x, y)}{\lambda_x}$$

Aside: Suppose $W \sim \exp(1)$

Let $\tilde{W} = W\lambda$

$$P(\tilde{W} > t) = P(W > \lambda t) = e^{-\lambda t} \Rightarrow \tilde{W} \sim \exp(\lambda)$$

Ingredients of the construction:

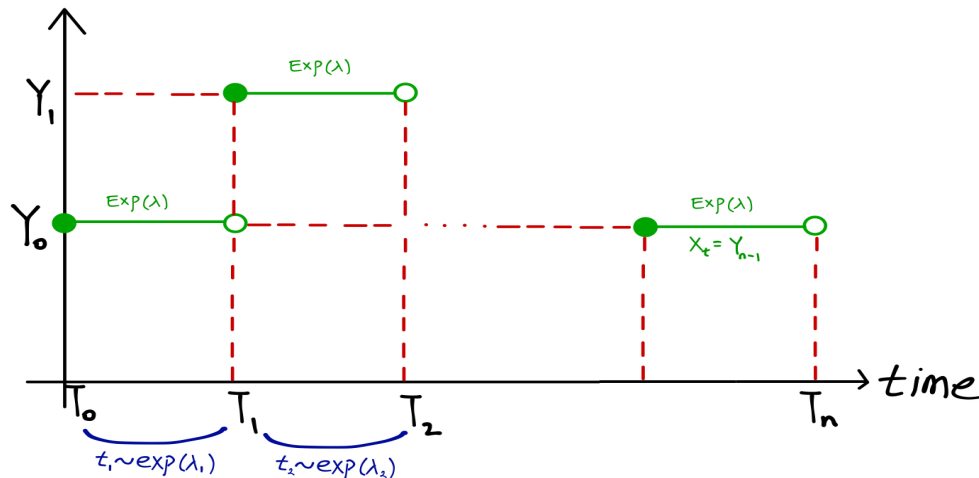
- (i) A discrete-time Markov chain $(Y_n)_{n \geq 0}$ with $Y_0 \sim \mu$.
- (ii) IID sequence $(\tau_k)_{k \geq 0}$ with $\tau_k \sim \exp(1)$.

- $T_0 = 0$
- $T_1 = t_1 = \frac{\tau_0}{\lambda_{Y_0}}$
- $X_t = Y_0$ for $0 \leq t < T_1$
- $t_2 = \frac{\tau_1}{\lambda_{Y_1}} \quad T_2 = T_1 + t_2$
- $X_t = Y_1$ for $T_1 \leq t < T_2$

Suppose we have constructed up to $T_n = t_1 + \dots + t_n$ such that $X_t = y_k$ for $T_k \leq t < T_{k+1}$ for $k = 0, \dots, n-1$.

Next step:

$$t_{n+1} = \frac{\tau_n}{\lambda_{Y_n}} \quad T_{n+1} = T_n + t_{n+1} \quad X_t = Y_n \text{ for } T_n \leq t < T_{n+1}$$



Compute transition probability $P_t = (p_t(x, y))_{x, y \in S}$ (4.2 on Durrett)

This construction does satisfy

$$\lim_{h \rightarrow 0} \frac{P_h(x, y)}{h} = q(x, y)$$

Differential equation: for transition probabilities.

Chapman-Kolmogorov equations:

$$p_{t+s}(x, y) = \sum_z p_t(x, z) p_s(z, y)$$

Think of $h > 0$ small small

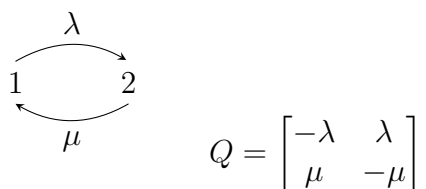
$$\begin{aligned} p_{t+h} - p_t(x, y) &= \sum_z P_h(x, z) p_t(z, y) - p_t(x, y) \\ &= \sum_{z \neq x} p_h(x, z) p_t(z, y) + (p_h(x, x) - 1) p_t(x, y) \\ \frac{p_{t+h} - p_t(x, y)}{h} &= \underbrace{\sum_{z \neq x} \frac{p_h(x, z)}{h} p_t(z, y)}_{\xrightarrow{h \rightarrow 0} q(x, z)} + \underbrace{\frac{p_h(x, x) - 1}{h} p_t(x, y)}_{\xrightarrow{h \rightarrow 0} -\sum_{z \neq x} \frac{p_h(x, z)}{h} \xrightarrow{h \rightarrow 0} -\sum_{z \neq x} q(x, z) = -\lambda_x} \\ \frac{d}{dt} p_t(x, y) &= \sum_{z \neq x} q(x, z) p_t(z, y) + (-\lambda_x) p_t(x, y) = \sum_z Q_{x, z} p_t(z, y) \end{aligned}$$

$$\frac{d}{dt} P_t = Q P_t \quad P_0 = I \quad \text{Kolmogorov's Backward Equation}$$

$$\frac{d}{dt} P_t = \left[\frac{d}{dt} p_t(x, y) \right]_{x, y \in S}$$

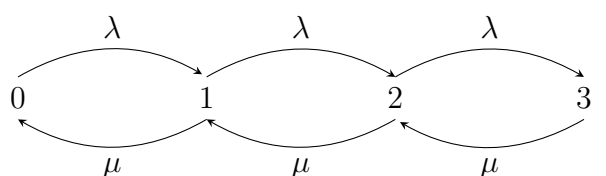
Similar argument leads to Kolmogorov's forward equation.

$$P'_t = P_t Q$$

Examples

Generator Matrix Q

$$Q_{x,z} = \begin{cases} q(x,z) & z \neq x \\ -\lambda_x & z = x \end{cases}$$

2. $\mu/\mu/1$ queue

$$Q = \begin{matrix} & \begin{matrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{matrix} \\ \begin{matrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix} & \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -(\lambda + \mu) & \lambda & 0 & 0 & \dots \\ 0 & \mu & -(\lambda + \mu) & \lambda & 0 & \dots \\ & & & & & \ddots \end{bmatrix} \end{matrix}$$

A probability measure π on S is an invariant distribution if $\pi = \pi P_t \quad \forall t \geq 0$.**Lemma 4.3** π is invariant $\Leftrightarrow \pi Q = 0$ *Proof* Assume π is invariant

$$\Rightarrow \pi = \pi P_t \quad \forall t \geq 0$$

$$0 = \frac{d}{dt}(\pi P_t) = \pi \frac{d}{dt}(P_t) = \pi P_t Q = \pi Q$$

$$\begin{aligned} \text{Assume } \pi Q = 0 \quad & \frac{d}{dt}(\pi P_t) = \pi \frac{d}{dt}(P_t) = \pi Q P_t = 0 \cdot P_t = 0 \\ & \pi P_t = \pi P_0 = \pi \end{aligned}$$

So π is invariant.

Today

1. Detailed Balance
2. Queing Models
3. Matrix Exponentials & Solving Kolmogorov's Equations
4. Blow-up

1. Detailed Balance

Last time: π is invariant if and only if $\pi Q = 0$.

Definition π is a reversible distribution (satisfies detailed balance) if

$$\pi(x)q(x, y) = \pi(y)q(y, x) \quad \forall x \neq y$$

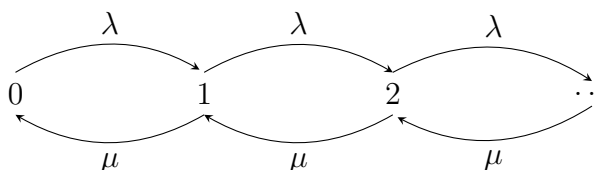
Fact: If we start the Markov chain with a reversible distribution π , the time reversed process has the same distribution as the original process.

Theorem 4.5 Detailed balance $\Rightarrow \pi$ is invariant.

Proof.

$$\begin{aligned} (\pi Q)_y &= [\dots] \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \\ &= \sum_x \pi(x)q(x, y) = \pi(y)q(y, y) + \sum_{x \neq y} \pi(x)q(x, y) \\ &= \pi(y)(-\lambda_y) + \sum_{x \neq y} \pi(y)q(y, x) = \pi(y) \left(0 - \lambda_y + \underbrace{\sum_x q(y, x)}_{\text{total rate out of } y} \right) \end{aligned}$$

Example: M/M/1 queue



$$\begin{cases} q(n, n+1) = \lambda & \text{for } n \geq 0 \\ q(n, n-1) = \mu & \text{for } n \geq 1 \end{cases}$$

Try to find a reversible distribution:

$$\begin{aligned} \pi_n q(n, n+1) &= \pi_{n+1} q(n+1, n) \\ \Leftrightarrow \lambda \pi_n &= \mu \pi_{n+1} \\ \Leftrightarrow \pi_{n+1} &= \frac{\lambda}{\mu} \pi_n \end{aligned}$$

So then

$$\pi_n = \frac{\lambda}{\mu} \pi_{n-1} = \left(\frac{\lambda}{\mu}\right)^2 \pi_{n-2} = \dots = \left(\frac{\lambda}{\mu}\right)^n \pi_0$$

continue: To have a probability distribution, we want

$$1 = \sum_{n=0}^{\infty} \pi_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \pi_0$$

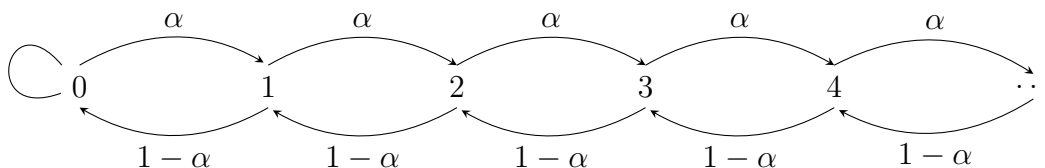
If $\frac{\lambda}{\mu} < 1$, we can solve for π_0 :

$$\begin{aligned} 1 &= \pi_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \\ &= \pi_0 \frac{1}{1 - \frac{\lambda}{\mu}} \Rightarrow \pi_0 = 1 - \frac{\lambda}{\mu} \end{aligned}$$

Answer: provided $\lambda < \mu$, where λ is arrival rate and μ is service rate, we have a reversible distribution

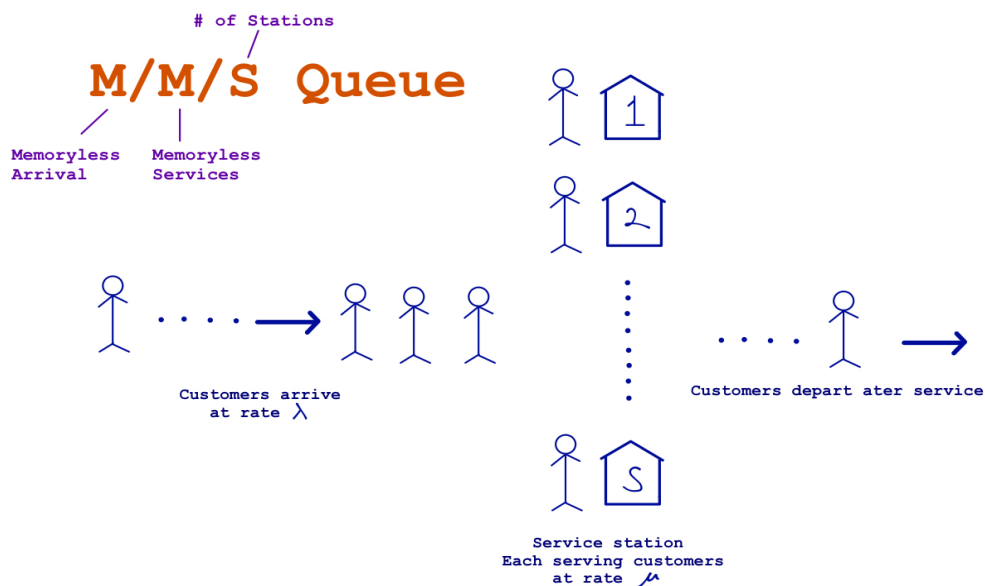
$$\pi_n = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n \quad \text{for } n \geq 0 \quad (\text{shifted geometric})$$

Recall from HomeWork 3:



This Markov chain has an invariant distribution if and only if $\alpha < \frac{1}{2}$.

2. Queuing Models.



Suppose X_t = the number of customers in the system at time t , and let state space $S = \{0, 1, 2, \dots\}$. Then what is the rates?

$$q(n, n+1) = \lambda, \quad n \geq 0$$

$$q(n, n-1) = \begin{cases} n\mu & 1 \leq n < s \\ s\mu & n \geq s \end{cases}$$

Fundamental fact: If W_1, \dots, W_n are independent random variables $W_i \sim \exp(\lambda_i)$, and $W = \min_{i \leq n} W_i$ then $W \sim \exp(\lambda_1 + \dots + \lambda_n)$.

M/M/S queue with balking

Suppose an arriving customer who sees n customers in the system, leaves with probability b_n .

Rates:

$$q(n, n+1) = a_n \lambda \text{ where } a_n = 1 - b_n$$

$$q(n, n-1) \text{ same as before}$$

M/M/ ∞ queue

Same rules as before except now the number of servers is unlimited so each arriving customers goes directly into service.

$$\begin{aligned}q(n, n+1) &= \lambda \\ q(n, n-1) &= n\mu\end{aligned}$$

Detailed Balance

$$\begin{aligned}\pi(n)q(n, n+1) &= \pi(n+1)q(n+1, n) \\ \pi(n)\lambda &= \pi(n+1) \cdot (n+1)\mu \\ \Rightarrow \pi(n+1) &= \frac{\lambda/\mu}{n+1}\pi(n) = \frac{(\lambda/\mu)^2}{(n+1)n}\pi(n-1) \\ &= \dots = \frac{(\lambda/\mu)^{n+1}}{(n+1)!}\pi_0^{e/-\lambda/\mu}\end{aligned}$$

So

$$\pi(n) = e^{-\lambda/\mu} \frac{(\lambda/\mu)^n}{n!} \sim \text{Poisson}\left(\frac{\lambda}{\mu}\right)$$

Scalar Version

$$\begin{aligned}x'(t) &= qx(t) \\ x(0) &= 1\end{aligned}$$

Solution:

$$\begin{aligned}x(t) &= e^{tq} \\ e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!}\end{aligned}$$

$$\begin{aligned}P'_t &= QP_t = P_tQ \\ P_0 &= I\end{aligned}$$

Define a matrix exponential

$$e^{tQ} = \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!}$$

Fact: this series converges for any finite matrix.

Let's see if $P_t = e^{tQ}$ solves.

$$\begin{aligned}
 \frac{d}{dt}e^{tQ} &= \frac{d}{dt} \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{d}{dt} \left(\frac{t^k Q^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{k t^{k-1} Q^k}{k!} = \sum_{k=0}^{\infty} \frac{t^{k-1} Q^k}{(k-1)!} \\
 &= Q \sum_{k=0}^{\infty} \frac{t^{k-1} Q^{k-1}}{(k-1)!} = Q e^{tQ} \quad (\text{Or, similarly, } = e^{tQ} Q)
 \end{aligned}$$

We have shown here $P_t = e^{tQ}$ satisfies

$$\begin{aligned}
 P'_t &= Q P_t = P_t Q \\
 e^{tQ} &= I + \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!} \quad (Q^0 = I) \\
 t = 0 : e^{0Q} &= I
 \end{aligned}$$

In some literature, e^{tQ} is used as an alternative to P_t .

Examples

$$\begin{array}{ccc}
 & \lambda & \\
 1 & \xrightarrow{\quad} & 2 \\
 & \xleftarrow{\quad} & \\
 & \mu &
 \end{array} \quad Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Diagonalize Q :

Idea: write $Q = B \Lambda B^{-1}$ where B is is eigen-vectors and Λ is diagonal matrix of eigenvalues.

$$Q^n = B \underbrace{\Lambda B^{-1} \cdot B \Lambda B^{-1} \cdots B \Lambda B^{-1}}_I = B \Lambda B^{-1}$$

$$\begin{array}{ll}
 \text{Eigenvalues} & 0 \quad -(\lambda + \mu) \\
 \text{Eigenvectors} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} \lambda \\ -\mu \end{pmatrix}
 \end{array} \quad B = \begin{bmatrix} 1 & \lambda \\ 1 & -\mu \end{bmatrix}$$

$$\begin{aligned}
B^{-1} &= \frac{1}{\lambda + \mu} \begin{bmatrix} \mu & \lambda \\ 1 & -1 \end{bmatrix} = B \underbrace{\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} 0 & 0 \\ 0 & -\mu - \lambda \end{bmatrix}^n \right)}_{\begin{bmatrix} 1 & 0 \\ 0 & e^{-t(\lambda+\mu)} \end{bmatrix}} B^{-1} \\
&= \begin{bmatrix} 1 & \lambda \\ 1 & -\mu \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t(\lambda+\mu)} \end{bmatrix} \frac{1}{\lambda + \mu} \begin{bmatrix} \mu & \lambda \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{\lambda + \mu} \begin{bmatrix} 1 & \lambda e^{-t(\lambda+\mu)} \\ 1 & -\mu e^{-t(\lambda+\mu)} \end{bmatrix} \begin{bmatrix} \mu & \lambda \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{\lambda + \mu} \begin{bmatrix} \mu + \lambda e^{-t(\lambda+\mu)} & \lambda - \lambda e^{-t(\lambda+\mu)} \\ \mu - \mu e^{-t(\lambda+\mu)} & \lambda + \mu e^{-t(\lambda+\mu)} \end{bmatrix} \\
\lim_{t \rightarrow \infty} P_t &= \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}
\end{aligned}$$