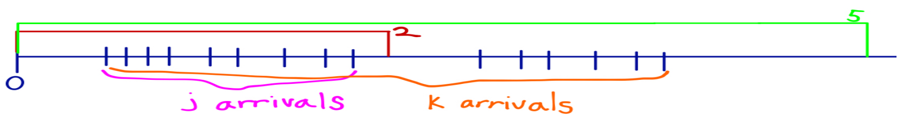
1. Let $\{N(t) : t \geq 0\}$ be a rate λ Poisson process, $\{T_k\}_{k\geq 1}$ the arrival times of the process, and $\tau_k = T_k - T_{k-1}$ for $k \geq 1$ the interarrival times. Calculate the probabilities below. When unspecified non-negative integers j and/or k appear in the question, your answer should cover all possible cases. When units are used, suppose the time unit is second.

(a) $P(N(2) = j,\ N(5) = k)$



(i) $0 < j \leq k$

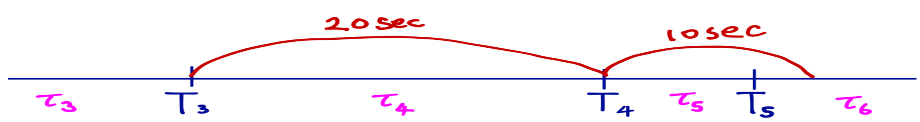
$$P(N(2) = j) \cap P(N(5) = k) = P(N(2) = j) \cap P(N(5) - N(2) = k - j)$$

$$\left(e^{-2\lambda} \cdot \frac{(2\lambda)^j}{j!}\right) \cdot \left(e^{-3\lambda} \cdot \frac{(3\lambda)^{k-j}}{(k-j)!}\right) = e^{-5\lambda} \cdot \frac{2^j 3^{k-j} \lambda^k}{k!}$$

(ii) There is no probability in cases where $j > k$ or $j \leq 0$.

$$P(N(2) = j,\ N(5) = k) = \begin{cases} e^{-5\lambda} \cdot \frac{2^j 3^{k-j} \lambda^k}{k!} & \text{if } 0 < j \leq k \\ 0 & \text{if } j \leq 0 \text{ or } j > k \end{cases}$$

(b) P(after the 3rd arrival there are no arrivals for 20 seconds, but after that the next two arrivals come within 10 seconds)

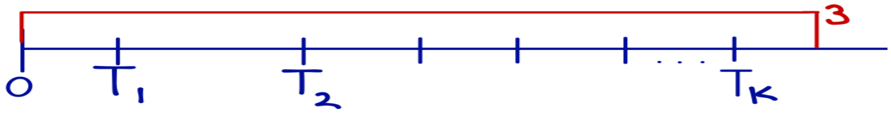


$$\begin{aligned} P(20 < t_4 < 30,\ 0 < t_5 < 30 - t_4) &= \iint_{\{(t_4,t_5)|20<t_4<30,0<t_5<30-t_4\}} f_4(t_4) f_5(t_5) dt_5 dt_4 \\ &= \int_{20}^{30} \int_0^{30-t_4} \lambda^2 e^{-\lambda t_4} e^{-\lambda t_5} dt_5 dt_4 \\ &= \int_{20}^{30} \lambda e^{-4} \{1 - e^{-\lambda(30-t_4)}\} dt_4 \\ &= e^{-20\lambda} - e^{-30\lambda} - 10\lambda e^{-30\lambda} \end{aligned}$$

Or

$$\begin{aligned} P(M(20) = 0,\ M(30) \geq 2) &= P(M(30 - M(20) \geq 2 \mid M(20) = 0) \cdot P(M(20) = 0) \\ &= \{1 - P(M(30) - M(20) \geq 2 \mid M(20) = 0)\} \cdot e^{-20\lambda} \\ &= \{1 - e^{-10\lambda} - 10\lambda e^{-10\lambda}\} \cdot e^{-20\lambda} \\ &= e^{-20\lambda} - e^{-30\lambda} - 10\lambda e^{-30\lambda} \end{aligned}$$

(c) $P(N(3) = k \mid T_2 \leq 3)$



$$= \frac{P(N(3) = k) \cap P(N(3) \geq 2)}{P(T_2 \leq 3)} = \frac{P(N(3) = k)}{P(T_2 \leq 3)} = \frac{e^{-3\lambda}\frac{(3\lambda)^k}{k!}}{1 - e^{-3\lambda} - 3\lambda e^{-3\lambda}}$$

$$= \frac{1}{k!}\frac{(3\lambda)^k}{e^{3\lambda} - 1 - 3\lambda}$$

(d) $P(N(2) = k \mid T_3 > 4)$

$$= \frac{P(N(2) = k \mid N(4) < 3)}{P(N(4) < 3)} = \frac{P(N(2) = k)(N(2) < 3 - k)}{P(N(4) < 3)}$$

$$= \frac{\left(e^{-2\lambda}\frac{(2\lambda)^k}{k!}\right)\left(\sum_{k=0}^{3-k-1} P(N(2) = 3 - k)\right)}{\sum_{k=0}^{2} P(N(4) = k)} = \frac{\left(e^{-2\lambda}\frac{(2\lambda)^k}{k!}\right)\left(\sum_{k=0}^{3-k-1} e^{-2\lambda}\frac{(2\lambda)^k}{k!}\right)}{\sum_{k=0}^{2} e^{-4\lambda}\frac{(4\lambda)^k}{k!}}$$

(e) $P(T_2 \leq 3 \mid N(4) = 5)$. Explain how your answer can be expressed in terms of a certain binomial probability mass function.

$$P(T_2 \leq 3 \mid N(4) = 5) = \frac{\sum_{n=2}^{5}\left(\cancel{e^{-3\lambda}}\cdot\frac{(3\cancel{\lambda})^n}{n!}\right)\left(\cancel{e^{-\lambda}}\cdot\frac{\cancel{\lambda}^{5-n}}{(5-n)!}\right)}{\cancel{e^{-4\lambda}}\cdot\frac{(4\cancel{\lambda})^5}{(5-n)!}}$$

$$= \frac{5!}{4^5}\left[\left(\frac{3^2}{2!\cdot 3!}\right) + \left(\frac{3^3}{3!\cdot 2!}\right) + \left(\frac{3^4}{4!\cdot 1!}\right) + \left(\frac{3^5}{5!\cdot 0!}\right)\right]$$

$$= \sum_{n=2}^{5}\binom{5}{n}\left(\frac{3}{4}\right)^n\cdot\left(\frac{1}{4}\right)^{5-n}$$

2.36 Customers arrive at an automated teller machine at the times of a Poisson process with rate of 10 per hour. Suppose that the amount of money withdrawn on each transaction has a mean of \$30 and a standard deviation of \$20. Find the mean and standard deviation of the total withdrawals in 8 hours.

Given the time $[0,t]$, let

- Y_n be the amount of n^{th} withdrawal
- $W(t)$ be total amount withdrawn: $W(t) = Y_1 + Y_2 + ... + Y_n$
- $N(t)$ be the number of customers
- μ be the mean and σ be the standard deviation

Then we can formulate

$$W(t) = \sum_{i=1}^{N(t)} Y_i$$

For the mean,

$$E[W(t)] = \sum_{n=0}^{N(t)} \underbrace{E\left[W(t) \mid N(t) = n\right]}_{nE[Y_i]=nE[Y]} \cdot \underbrace{P(N(t) = n)}_{e^{-t\lambda}\frac{(t\lambda)^n}{n!}}$$

$$nE[Y_i] \cdot e^{-t\lambda}\frac{(t\lambda)}{n} \cdot \underbrace{\sum_{n=1}^{N(t)} \frac{(t\lambda)^{n-1}}{(n-1)!}}_{\text{Taylor Expansion: } e^{t\lambda}} = \cancel{n}E[Y_i] \cdot \cancel{e^{-t\lambda}}\frac{(t\lambda)}{\cancel{n}} \cdot \cancel{e^{t\lambda}} = t\lambda\mu$$

$$= 8 \cdot 10 \cdot \$30 = \$2,400$$

For the standard deviation, $\sigma = \sqrt{Var[W(t)]} = \sqrt{E[W^2(t)] - E^2[W(t)]}$

$$E[W^2(t)] = \sum_{n=0}^{N(t)} E\left[\underbrace{W^2(t) \mid N(t) = n}_{(Y_1+...+Y_n)^2}\right] \cdot \underbrace{P(N(t) = n)}_{e^{-t\lambda}\frac{(t\lambda)^n}{n!}}$$

$$E[W^2(t) \mid N(t) = n] = \underbrace{E[Y_1^2] + E[Y_2^2] + \ldots + E[Y_n^2]}_{n \text{ terms}} + 2\underbrace{(E[Y_1]E[Y_2] + E[Y_1]E[Y_3] + E[Y_2]E[Y_3] + \ldots)}_{\frac{n(n-1)}{2} \text{ terms}}$$

Since this is $i.i.d$, $E[Y_1] = E[Y_2] = ... = E[Y_n]$

$$E[W^2(t) \mid N(t) = n] = n \cdot E[Y^2] + n(n-1)E^2[Y]$$

Remind that $E[Y^2] = \sigma^2 + \mu^2$. Then

$$E[W^2(t) \mid N(t) = n] = n(\sigma^2 + \mu^2) + n(n-1) \cdot \mu^2 = n\sigma^2 + n^2\mu^2$$

$$E[W^2(t)] = \sum_{n=0}^{N(t)} (n\sigma^2 + n^2\mu^2) \cdot e^{-t\lambda} \frac{(t\lambda)^n}{n!} = \sigma^2 \underbrace{\sum_{n=0}^{N(t)} n e^{-t\lambda} \frac{(t\lambda)^n}{n!}}_{Expectation} + \mu^2 \underbrace{\sum_{n=0}^{N(t)} n^2 e^{-t\lambda} \frac{(t\lambda)^n}{n!}}_{Expectation\ of\ square}$$
$$= \sigma^2 E[N(t)] + \mu^2 E[N^2(t)]$$

Hence

$$Var[W(t)] = E[W^2(t)] - E^2[W(t)] = \sigma^2 E[N(t)] + \mu^2 E[N^2(t)] - (\mu \underbrace{\lambda t)^2}_{E^2[N(t)]}$$

Note that by Lemma 2.2 on Durrett, $N(t)$ has a Poisson distribution with means λt,

$$\sigma^2 E[N(t)] + \mu^2 \underbrace{(E[N^2(t)] - E^2[N(t)])}_{Var(N(t)}$$

We also know that mean is equal to variance for Poisson distribution. Hence

$$Var[W(t)] = \sigma^2 E[N(t)] + \mu^2 E[N(t)] = (\sigma^2 + \mu^2)\lambda t = (\$20^2 + \$30^2)10 \cdot 8 = 104,000$$

Thus the standard deviation is $\sqrt{104,000}$.

2.38 Let S_t be the price of stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable $X_i > 0$ with mean μ and variance σ^2. That is

$$S_t = S_0 \prod_{i=1}^{N(t)} X_i$$

where the product is 1 if $N(t) = 0$. Find $ES(t)$ and Var$S(t)$.

$$\begin{aligned}
E[S_t] &= E\left[S_0 \prod_{i=1}^{N(t)} X_i\right] = S_0 \prod_{i=1}^{N(t)} E[X_i] = S_0 \mu^{N(t)} = S_0 \sum_{k=0}^{\infty} E[X_i \mid N(t) = k] \cdot P[N(t) = k] \\
&= S_0 \sum_{k=0}^{\infty} \mu^k e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!} = S_0 e^{-\lambda t} \underbrace{\sum_{k=0}^{\infty} \frac{(\mu \lambda t)^k}{k!}}_{e^{\mu\lambda t}} = S_0 e^{\lambda t(\mu - 1)}
\end{aligned}$$

$$\begin{aligned}
Var[S_t] &= E[S_t^2] - E^2[S_t] = E\left[S_0^2 \prod_{i=1}^{N(t)} X_i^2\right] - (S_0 e^{\lambda t(\mu-1)})^2 \\
&= S_0^2 \prod_{i=1}^{N(t)} E[X_i^2] - S_0^2 e^{2(\mu-1)\lambda t} = S_0^2 \prod_{i=1}^{N(t)} (\sigma^2 + \mu^2) - S_0^2 e^{2(\mu-1)\lambda t} \\
&= S_0^2 (\sigma^2 + \mu^2)^{N(t)} - S_0^2 e^{2(\mu-1)\lambda t} = S_0^2 \sum_{k=0}^{\infty} (\sigma^2 + \mu^2)^k \cdot \left(e^{-t\lambda} \cdot \frac{(t\lambda)^k}{k!}\right) - S_0^2 e^{2(\mu-1)t\lambda} \\
&= S_0^2 e^{-t\lambda} \sum_{k=0}^{\infty} \frac{((\mu^2 + \sigma^2)t\lambda)^k}{k!} - S_0^2 e^{2(\mu-1)} = S_0^2 e^{-t\lambda} e^{(\mu^2+\sigma^2)t\lambda} - S_0^2 e^{2(\mu-1)t\lambda} \\
&= S_0^2 (e^{t\lambda((\mu^2+\sigma^2)-1)} - e^{2(\mu-1)t\lambda})
\end{aligned}$$

2.44 Ellen catches fish at times of a Poisson process with rate 2 per hour. 40% of the fish are salmon, while 60% of the fish are trout. What is the probability she will catch exactly 1 salmon and 2 trout if she fishes for 2.5 hours?

Let $N_s(t)$ be the number of salmon Ellen catches for time t with rate $2 \times 40\% = 0.8$ and N_t be that of trout with rate $2 \times 60\% = 1.2$. Then we can formulate the probability Ellen will catch exactly 1 salmon and 2 trout for 2.5 hours as below.

$$\begin{gathered}
P(N_s(2.5) = 1,\ N_t(2.5) = 2) = P(N_s(2.5) = 1) \cdot P(N_t(2.5) = 2) \\
\left(e^{-2.5\times0.8} \frac{(-2.5 \times 0.8)^2}{2!}\right) \cdot \left(e^{-2.5\times1.2} \frac{(-2.5 \times 1.2)^1}{1!}\right) = 9e^{-5}
\end{gathered}$$

2.48 When a power surge occurs on an electrical line, it can damage a computer without a surge protector. There are three types of surges: small surges occur at rate 8 per day and damage a computer with probability 0.001; medium surges occur at rate 1 per day and will damage a computer with probability 0.01; large surges occur at rate 1 per month and damage a computer with probability 0.1. Assume that months are 30 days.

(a) What is the expected number of power surges per month?

$$\underbrace{8 \times 30}_{\text{Small Surges}} + \underbrace{1 \times 30}_{\text{Medium Surges}} + \underbrace{1}_{\text{Large Surges}} = 271$$

(b) What is the expected number of computer damaging power surges per month?

$$8 \times 30 \times 0.001 + 1 \times 30 \times 0.01 + 1 \times 0.1 = 0.64$$

(c) What is the probability a computer will not be damaged in one month? $t = 1$ since 30-days is the unit time for this question.

- Small Surge: $\lambda = 0.001 \times (8 \cdot 30)$=0.24

$$P(\text{No damage due to small surge}) = P(N(t) = 0) = e^{-0.24} \cdot \frac{0.24^0}{0!} = e^{-0.24}$$

- Medium Surge: $\lambda = 0.01 \times (1 \cdot 30) = 0.3$

$$P(\text{No damage due to medium surge}) = P(N(t) = 0) = e^{-0.3} \cdot \frac{0.3^0}{0!} = e^{-0.3}$$

- Large Surge: $\lambda = 0.1 \times 1 = 0.1$

$$P(\text{No damage due to large surge}) = P(N(t) = 0) = e^{-0.1} \cdot \frac{0.1^0}{0!} = e^{-0.1}$$

Hence the probability a computer will not be damaged in one month is

$$e^{-0.24} \times e^{-0.3} \times e^{-0.1} = e^{-0.64}$$

(d) What is the probability that the first computer damaging surge is a small one?

$$\frac{e^{-0.24}}{e^{-0.64}}$$

2.49 Wayne Gretsky scored a Poisson mean 6 number of points per game. 60% of these were goals and 40% were assists (each is worth one point). Suppose he is paid a bonus of 3K for a goal and 1K for an assist.

(a) Find the mean and standard deviation for the total revenue he earns per game.

Let G_i be the number of score he made and A_i be that of assist per game.

Average Goals per game: $0.6 \cdot 6 = 3.6$ Average Assists per game: $0.4 \cdot 6 = 2.4$

$$E[Goals] = \underbrace{E[N]}_{3000} \cdot \underbrace{E[G_i]}_{3.6} = 10800 \qquad E[Assists] = \underbrace{E[N]}_{1000} \cdot \underbrace{E[A_i]}_{2.4} = 2400$$

Now let X_i be the points he gain on i^{th} game and $P = X_1 + X_2 + ... + X_n$. Then

$$E[P] = 10800 + 2400 = 13200$$

$$Var[P] = 3000^2 \cdot 3.6 + 1000^2 \cdot 2.4$$

(b) What is the probability that he has 4 goals and 2 assists in one game?

$$P(\text{Poisson}(1) = 4) \cdot P(\text{Poisson}(1) = 2) = \left(e^{-3.6} \cdot \frac{3.6^4}{4!}\right) \cdot \left(e^{-2.4} \cdot \frac{2.4^2}{2!}\right)$$

(c) Conditional on the fact that he had 6 points in a game, what is the probability he had 4 in the first half?

$$P(N(1/2) = 4 \mid N(1) = 6) = \binom{6}{4} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^4 = 0.234375$$

2.51 Two copy editors read a 300-page manuscript. The first found 100 typos, the second found 120, and their lists contain 80 errors in common. Suppose that the authors typos follow a Poisson process with some unknown rate per page, while the two copy editors catch errors with unknown probabilities of success p_1 and p_2. Let X_0 be the number of typos that neither found. Let X_1 and X_2 be the number of typos found only by 1 or only by 2, and let X_3 be the number of typos found by both.

(a) Find the joint distribution of (X_0, X_1, X_2, X_3).

Rates for those independent Poisson process are

- $X_0 :\ (1-p_1)(1-p_2)\lambda$
- $X_1 :\ p_1(1-p_2)\lambda$
- $X_2 :\ p_2(1-p_1)\lambda$
- $X_3 :\ p_1p_2$

$$P(X_0 = n_0,\ X_1 = n_1,\ X_2 = n_2,\ X_3 = n_3) = e^{-300(1-p_1)(1-p_2)\lambda}\frac{(300(1-p_1)(1-p_2)\lambda)^{n_0}}{n_0!}\times$$
$$e^{-300p_1(1-p_2)\lambda}\frac{(300p_1(1-p_2)\lambda)^{n_1}}{n_1!}\times e^{-300p_2(1-p_1)\lambda}\frac{(300p_2(1-p_1)\lambda)^{n_2}}{n_2!}\times e^{-300p_1p_2\lambda}\frac{(300p_1p_2)^{n_3}}{n_3!}$$

Since $((1-p_1)((1-p_2)+p_2)+p_1((1-p_2)+p_2) = 1$,

$$P(X_0 = n_0,\ X_1 = n_1,\ X_2 = n_2,\ X_3 = n_3)$$
$$= e^{-300\lambda}300\lambda^{n_0+n_1+n_2+n_3}\times\frac{(1-p_1)(1-p_2)^{n_0}+(p_1(1-p_2))^{n_1}+p_2(1-p_1)^{n_2}+(p_1p_2)^{n_3}}{n_0!n_1!n_2!n_3!}$$

(b) Use the answer to (a) to find an estimates of p_1, p_2 and then of the number of undiscovered typos.

$$E[X_1] = 300\cdot\lambda p_1(1-p_2) = 20$$
$$E[X_2] = 300\cdot\lambda p_2(1-p_1) = 40$$
$$E[X_3] = 300\cdot\lambda p_1p_2 = 80$$

$$\frac{E[X_2]}{E[X_3]} = \frac{1-p_1}{p_1} = \frac{1}{p_1}-1 = \frac{1}{2} \quad\Leftrightarrow\quad p_1 = \frac{2}{3}$$
$$2\cdot\frac{2}{3}(1-p_2) = \frac{1}{3}\cdot p_2 \quad\Leftrightarrow\quad \frac{1}{p_2}-1 = \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4} \quad\Leftrightarrow\quad p_2 = \frac{4}{5}$$
$$300\cdot\lambda\frac{2}{3}\cdot\frac{4}{5} = 160\lambda = 80 \quad\Leftrightarrow\quad \lambda = \frac{1}{2}$$

Hence

$$E[X_0] = 300\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{5} = 10$$