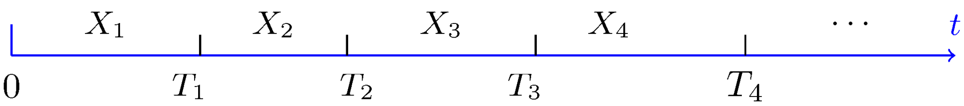
1 Poisson Process



- $N(t)$: The number of arrivals in $[0, t]$.
- $N(I)$: The number of arrivals in $I \subset [0, \infty)$.

Characterization

- iid
- $N(b) - N(a) \sim$Poisson($(b-a)$)
  $N(s_1),\ N(s_2) - N(s_1), \dots, N(s_n) - N(s_{n-1})$ are independent

1.1 Thinning Property

$N(t)$ is a Poisson process with rate . We assign a type $Y_j \in \{1, 2, , \dots, l\}$ for the j^{th} arrival. We assume that $Y_1, Y_2, \dots,$ are *iid*.

Let $N_j(t)$ be the number of arrivals with type J in $[0, t]$. Then N_j is a Poisson process of rate $P_j \cdot d$. $N_1, N_{2,j}$ are independent.

1.1.1 Example

Vehicles arrive at a roll both with a rate of $2/min$. The probability that a given vehicles is a truck is 2/3.

(a) P(exactly 2 cars and 3 trucks arriving in the next 5 minutes)

- Arriving vehicles form a Poisson process with rate $\lambda = 2$.
- Let N_1 be the number of arrivals of trucks and N_2 be that of arrivals of cars.
- Then, for trucks, the rate of Poisson process is $\frac{2}{3}\lambda$ and the rate of Poisson process for cars is $\frac{1}{3}\lambda$.
- So $P(N_1(5) = 3,\ N_2(5) = 2) = P(N_1(5) = 3)P(N_2(5) = 2) = \frac{(\frac{20}{3})^3}{3!}e^{-\frac{20}{3}} \cdot \frac{(\frac{10}{3})^2}{2!}e^{-\frac{10}{3}}$

(b) P(First arrival is a truck and the second one is a car) $= \frac{2}{3} \cdot \frac{1}{3}$

(c) Given that 20 trucks arrived in an hour what is expected the number of cars within the same hour? $\frac{1}{3} \cdot 60 = 20$

The given 20 trucks actually does not affect the expected number of cars within the same time period.

1.2 Superposition Property

Suppose that $N_1,\ N_2, \ldots, N_j$ are independent Poisson processes with the rate of N_j is λ_j. We look at the union of all arrivals. This process is also a Poisson process, the rate $\lambda_1 + \lambda_2 + \ldots + \lambda_j$. The types of the arrivals will form an iid sequence where the probability of type $j = \frac{\lambda_j}{\lambda_1+\ldots+\lambda_j}$.

Proof. Let N_j be a counting functino of the j^{th} Poisson process. Denote by N the counting function of arrivals $N(t) = N_1(t) + N_2(t) + \ldots + N_j(t)$

$$N(b) - N(a) = \sum_{j=1}^{j} \underbrace{(N_j(b) - N_j(a))}_{Poisson(\lambda_j(b-a))} \sim \text{ Poisson}\left((b-a)\sum_{j=1}^{k} \lambda_j\right)$$

1.2.1 Example

Customers arrive at a ticket counter. 30 girls arrive per an hour and 20 boys arrive per an hour.

(a) What is the expected waiting time between the first and third customer?

- The arrival process of the customers is a Poisson process with rate 50/hour: Girls:$\frac{3}{5}$ Boys:$\frac{2}{5}$
- $\tau_1,\ \tau_2, \ldots \sim \exp(50)$
  $E[\tau_2 + \tau_3] = E[\tau_2] + E[\tau_3] = \frac{1}{50} + \frac{1}{50} = \frac{1}{25}$

(b) P(The first 3 customers are all)=$\left(\frac{3}{5}\right)^3$

1.3 Conditioning the Poisson Process

We want to consider on $\{N(t) = k\}$. What is the (conditional) distribution of the k points in $[0, t)$.

- $k = 1$

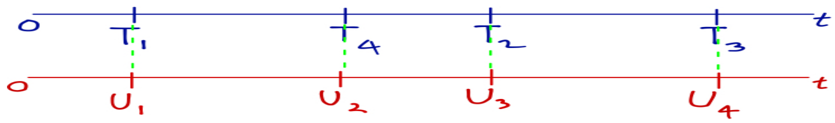
$$P(T_1 \leq s \mid N(t) = 1) = \frac{P(T_1 \leq s)(N(t) = 1)}{P(n(t) = 1)} = \frac{(P(N(s) = 1)d,\ N(t) - N(s) = 0)}{P(n(t) = 1)}$$

$$= \frac{P(N(s) = 1)P(N(t) - N(s) = 0)}{P(N(t) = 1)} = \frac{(\lambda s)e^{-\lambda s} \cdot e^{-\lambda(t-s)}}{(\lambda t)e^{-\lambda t}} = \frac{s}{t}$$

$$P(T_1 \leq s \mid N(t) = 1) = \begin{cases} 1 & \text{if } s \geq t \\ \frac{s}{t} & \text{if } 0 < s < t \\ 0 & \text{if } s \leq 0 \end{cases}$$

Theorem Let ≥ 1. Then the conditional distribution of $(T_1,\ T_2, \ldots, T_k)$ given that $N(t) = k$

Theorem 2.14 If we condition on $N(t) = n$, then the vector $(T_1,\ T_2, \ldots,\ T_n)$ has the same distribution as $(V_1,\ V_2, \ldots,\ V_n)$ and hence the set of arrival times $(T_1,\ T_2, \ldots, T_n)$ has the same distribution as $\{U_1,\ U_2, \ldots, U_n\}$



To construct the distribution $(T_1,\ T_2, \ldots, T_n)$ given $N(t) = n$

(i) Put on $[0, t]$ n $i.i.d$ uniform random points $(U_1, U_2, \ldots, U_n)$

(ii) Let $V_1 < V_2 < \ldots < V_n$ be the ordered set of values $(U_1, \ldots, U_n)$

(iii) Then $(V_1, \ldots, V_n)$ has the same joint distribution as $(T_1,\ T_2, \ldots, T_n)$ conditioned on $N(t) = n$

Now $(T_1,\ T_2, \ldots, T_n)$ is the new distribution. Consequently, the joint PDF of $(T_1,\ T_2, \ldots, T_n)$ conditional on $N(t) = n$ is $f(x_1, \ldots, x_n) = \frac{n!}{t^n}$ on the set $\underbrace{\{0 < x_1 < x_2 < \ldots < x_n < t\}}_{\text{These random variables are ordered}}$

Aside

- If X is uniform $X \sim$Unif[a,b], then X has PDF

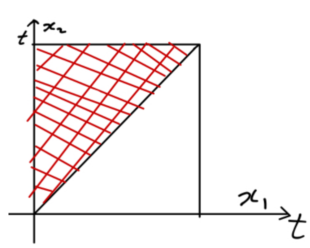
$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

.

- $\vec{x} = (x_1, \ldots, x_n)$ is uniform on a set $H \subset \mathbb{R}^n$ if $\vec{x}$ has joint PDF

$$f(x_1, \ldots, x_n) = \begin{cases} \frac{1}{VOL(H)} & (x_1, \ldots, x_n) \in H \\ 0 & (x_1, \ldots, x_n) \notin H \end{cases}$$

The volume of uniform density on the set $\{0 < x_1 < x_2 < \ldots < x_n < 1\} = \frac{t^n}{n!}$



Closer look at the case n=2
If X has CDF of F and PDF of f, then $f(x) = F'(x)$.

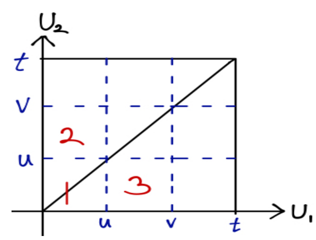
Case $n = 2$: The joint CDF of (X, Y) is $F(x, y) = p(X \leq x, Y \leq y)$.
If (X, Y) has joint PDF f, then connection between

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)dvdu$$

If f is continuous at (x, y),

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)$$

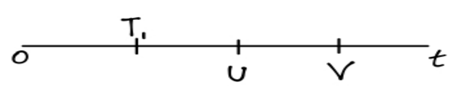
Let U_1, U_2 be independent uniform random variables on $[0, t]$. Let V_1, V_2 be $V_1 = \min(U_1, U_2),\ \max(U_1, U_2)$. Find joint distribution function CDF $F_{V_1 V_2}$ of (V_1, V_2). Let $u < v$.



$$\begin{aligned}
F(u,v) &= P(V_1 \leq u,\ V_2 \leq v) \\
&= P(U_1 \leq u,\ U_2 \leq U) + P(U_1 \leq u,\ u < U_2 < v) + P(u < U_1 \leq v,\ U_2 \leq u) \\
&= \left(\frac{u}{t}\right)^2 + \left(\frac{u}{t}\right)\left(\frac{v-u}{t}\right) + \left(\frac{v-u}{t}\right)\left(\frac{u}{t}\right) = \frac{u^2 + 2uv - 2u^2}{t^2}
\end{aligned}$$

$$F_{V_1,V_2}(u,v) = \left(\frac{u}{v}\right)^2 + \frac{2u(v-u)}{t^2} \qquad \text{for } 0 < u < v < t$$

Let $N(\cdot)$ be a rate λ Poisson Process condition on $N(t) = 2$. Find the conditional joint PDF. $(0 \leq u < v \leq t)$



$$\begin{aligned}
F_{T_1,T_2}(u, v \mid N(t) = 2) &= P(T_1 \leq u,\ T_2 \leq v \mid N(t) = 2) \\
&= P(N(u) \geq 1\ N(v) = 2 \mid N(t) = 2) \\
&= P(N(u) = 1,\ N(v) = 2 \mid N(t) = 2) \\
&\quad + P(N(u) = 2,\ N(v) = 2 \mid N(t) = 2)
\end{aligned}$$

$$= \frac{P(N(u)=1,\ N(v)=2,\ N(t)=2}{P(N(t)=2} + \frac{P(N(u)=2,\ N(v)=2,\ N(t)=2}{P(N(t)=2}$$

$$= \frac{P(N(u)=1,\ N(v)-N(u)=1,\ N(t)-N(v)=0)}{P(N(t)=2} + \frac{P(N(u)=2,\ N(v)-N(t)=2}{P(N(t)=2}$$

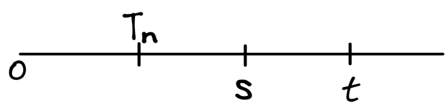
$$= \frac{e^{-u\lambda}u\lambda e^{-\lambda(v-u)}\lambda(v-u)e^{-\lambda(t-v)} + e^{-u\lambda}\frac{(u\lambda)^2}{2}e^{-\lambda(t-u)}}{e^{-t\lambda\cdot\frac{(-t\lambda)^2}{2}}} = \frac{2u(v-u)}{t^2} + \frac{u^2}{t^2} = \frac{2uv-u^2}{t^2}$$

Conditional Joint Density Function PDF

$$f_{T_1,T_2}(u,v \mid N(t)=2)\frac{\partial^2}{\partial u \partial v}F_{T_1,T_2}(u,v \mid N(t)=2) = \frac{2}{t^2}$$

Find the conditional expectation: $E[T_n \mid N(t)=n]$

The conditional joint density function PDF $f_{T_n}(s \mid N(t)=n) = \frac{d}{ds}\left(\frac{s}{t}\right)^n = \frac{ns^{n-1}}{t^n}$ on the set $\{(x_1,\ldots,x_n) \mid 0 < x_1 < \ldots < x_n < t\}$



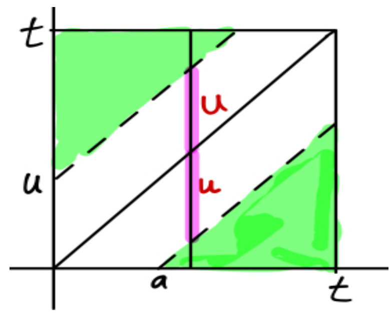
To find PDF of T_n, let's first find $P(T_n \le s \mid N(t)=n) = \left(\frac{s}{t}\right)^n$.

$$f_{T_n}(s \mid N(t)=n) = \frac{d}{ds}\left(\frac{s}{t}\right)^n = \frac{ns^{n-1}}{t^n} \qquad (0 \le s \le t)$$

$$E[T_n \mid N(t)=n] = \int_0^t \frac{s \cdot ns^{n-1}}{t^n}ds = \frac{n}{n+1}t$$

P(arrivals at least u time units apart | N(t)=2) $= \int_0^{t-a}\int_{x+a}^{t}\frac{2}{t^2}dydx = \frac{(t-a)^2}{t^2}$

$$\text{Probability} : \frac{\text{shaded area}}{t^2} = \frac{(t-u)^2 \cdot \frac{1}{2} \cdot 2}{t^2}$$



1.4 Practice Exam

1.d Let N be the proccess of ice cream cones sold with rate 3.

λ_c : Rate for chocolate$= 0.7 \cdot 3$ λ_s : Rate for strawberry $= 0.2 \cdot 3$ λ_v : Rate for vanilla$= 0.1 \cdot 3$

$$\begin{aligned}
P(N_s(6,9] = 2 \mid N(7,9] = 5) &= \frac{P(N_5(6,9] = 2,\ N(7,9] = 5)}{P(N(7,9] = 5)} \\
&= \sum_{j=0}^{2} \frac{P(N_s(6,7] = j,\ N_s(7,9) = 2-j,\ N(7,9] = 5)}{P(N(7,9] = 5)} \\
&= \underbrace{\sum_{j=0}^{2} \frac{P(N_s(6,7] = j)\ \overbrace{P(N_s(7,9] = 2-j,\ N(7,0] = 5)}^{\substack{P(N_s(7,9]=2-j,\ N_{cv}(7,9]=3+j): \\ \text{Independece by the thinning property}}}}{P(N(7,9] = 5)}}_{\text{Independence of arrivals in (6,7] \& (7,9]}} \\
&= \sum_{j=0}^{2} \frac{e^{\frac{3}{5}}\left(\frac{3}{5}\right)^j \cdot \frac{1}{j!} \cdot e^{-\frac{6}{5}}\left(\frac{6}{5}\right)^{2-j}\frac{1}{(2-j)!} \cdot e^{-\frac{24}{5}}\left(\frac{24}{5}\right)^{3+j}\frac{1}{(3+j)!}}{e^{-6}\frac{6^5}{5!}} \\
&= \sum_{j=0}^{2} \underbrace{e^{-\frac{3}{5}}\left(\frac{3}{5}\right)^j \frac{1}{j!}}_{\substack{\text{Poisson probability for} \\ \text{strawberries in (6,7]}}} \underbrace{\binom{5}{2-j}\left(\frac{1}{5}\right)^{2-j}\left(\frac{4}{5}\right)^{3+j}}_{\substack{\text{Binomial probability of} \\ \text{strawberries in (7,9],} \\ \text{comes from conditioning on} \\ \text{the total number of sales}}}
\end{aligned}$$

Example: $P(T_3 \leq s \mid N(t) = 4)$

$$\begin{aligned}
P(T_3 \leq s \mid N(t) = 4) = P(N(s) \geq 3 \mid N(t) = 4) &= P(N(s) = 3 \mid N(t) = 4) + P(N(s) = 4 \mid N(t) = 4) \\
&= P(\text{out of 4 independent uniforms, 3 land in [0,5]}) \\
&\quad + P(\text{out of 4 independent uniforms, all 4 land in [0,5]}) \\
&= 4\left(\frac{s}{t}\right)^3\left(1 - \frac{s}{t}\right) + \left(\frac{s}{t}\right)^4
\end{aligned}$$

Longer way: $P(T_3 \leq s \mid N(t) = 4)$

$$\begin{aligned}
&= \frac{P(N(s) = 3,\ N(t) = 4)}{P(N(t) = 4)} + \frac{P(N(s) = 4,\ N(t) = 4)}{P(N(t) = 4)} \\
&= \frac{P(N(s) = 3,\ N(s,t] = 1)}{P(N(t) = 4)} + \frac{P(N(s) = 4,\ N(s,t] = 0)}{P(N(t) = 4)} \\
&= \frac{e^{-s\lambda}(s\lambda)^3\frac{1}{3!} \cdot e^{-(t-s)\lambda}(t-s)\lambda}{e^{-t\lambda}(t\lambda)^4\frac{1}{4!}} + \frac{e^{-s\lambda}(s\lambda)^4\frac{1}{4!} \cdot e^{-(t-s)\lambda}(t-s)\lambda}{e^{-t\lambda}(t\lambda)^4\frac{1}{4!}} \\
&= \frac{4!}{3!}\frac{s^3(t-s)}{t^4} + \frac{s^4}{t^4}
\end{aligned}$$

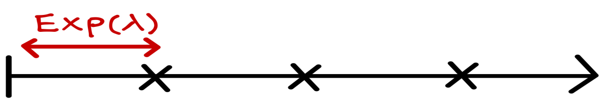
Fact

Conditional on $N(t)=4$, $(T_1,\ T_2,\ T_3\ ,T_4)$ has conditional PDF $f(x_1,\ x_2,\ x_3,\ x_4)=\frac{4!}{t^4}$ on the set $\underbrace{\{(x_1,\ x_2,\ x_3,\ x_4):\ 0<x_1<x_2<x_3<x_4<t\}}_{\text{whose volume is } \frac{t^4}{4!}}$

$$P(T_3\le s\mid N(t)=4)=\int\dots\int\frac{4!}{t^4}dx_4dx_3dx_2dx_1=\int_0^s dx_1\int_{x_1}^s dx_2\int_{x_2}^s dx_3\int_{x_3}^t dx_4\frac{4!}{t^4}$$

2.27 Person waits for a bus. Time till arrival of bus is Unif(0,1). Cars go by at rate 6. Each car gives this person a ride with probability 0.5. What is the probability that this person rides the bus.

Remark: Arrivals come as a rate λ Poisson process then interarrival times are Exp(λ).



$$f_S(s)=\begin{cases}3e^{-3s} & s>0\\ 0 & s\le 0\end{cases}\qquad f_U(s)=\begin{cases}1 & s\in(0,1)\\ 0 & s\notin(0,1)\end{cases}$$

$$\begin{aligned}P(U<S)&=\iint_{u<s}\underbrace{f_U(u)f_S(s)duds}_{\text{Joint PDF of (U,S)}}\\&=\int_0^1 1\ du\int_u^\infty 3e^{-3s}ds\\&=\int_0^1 e^{-3u}du=\frac{1}{3}\int_0^1 3e^{-3u}du\\&=\frac{1-e^{-3}}{3}\end{aligned}$$