1. Let A and B be two events on the same sample space.

(a) Suppose $P(A) = p, P(B) = q$, and $P(A|B) = r$. Deduce $P(A \cap B)$ and $P(B|A)$.

- $P(A \cap B)$
  - Conditional probability $P(A|B)$ is defined as $\frac{P(A \cap B)}{P(B)}$. i.e. $P(A|B) = \frac{P(A \cap B)}{P(B)}$
  - $P(A|B)$ and $P(B)$ are given above; r and q respectively.
  - Thus $P(A \cap B) = P(A|B) \cdot P(B) = rq$.
- $P(B|A)$
  - Conditional probability $P(B|A) = \frac{P(B \cap A)}{P(A)}$
  - $P(B \cap A) = P(A \cap B) = rq$ as we proved above.
  - Thus $P(B|A) = \frac{rq}{p}$

(b) Give an example where $P(A|B) > P(A)$ and neither A nor B is equal to the whole space or the empty set. (This means that you give a sample space Ω, a probability measure P on Ω, and define two events A and B on Ω that satisfy the question.)

- Let's roll a die. Suppose A is a probability of getting one of $\{1, 2, 3\}$, and B = A.
- Then A$\cap B = \{1, 2, 3\}$, so P(A$\cap B) = \frac{1}{2}$.
- Meanwhile, $P(A) = P(B) = \frac{1}{2}$; hence $P(A)P(B) = \frac{1}{4}$.
- Thus $P(A|B) = \frac{P(A \cap B)}{P(B)} > P(A) \Leftrightarrow P(A \cap B) > P(A)P(B) \Leftrightarrow \frac{1}{2} > \frac{1}{4}$

(c) Give an example where $P(A|B) < P(A)$ and neither A nor B is equal to the whole space or the empty set.

- Let's roll a die again. Suppose A is a probability of getting one of 1,2,3, and B is a that of getting even number.
- Then $P(A) = \frac{1}{2}, P(B) = \frac{1}{2}$, and $P(A \cap B) = P(\{2\}) = \frac{1}{6}$
- Hence $P(A)P(B) = \frac{1}{4} > P(A \cap B) = \frac{1}{6}$.
- Thus it satisfies $P(A|B) < P(A) \Leftrightarrow P(AB) < P(A)P(B)$.

2. Let p, q, r be positive reals such that $p+q+r=1$ and let n be a positive integer. Let (X,Y,Z) have multinomial distribution with parameters $(n, 3, p, q, r)$. This means that the joint probability mass function is

$$P(X=k, Y=l, Z=m)=\frac{n!}{k!\ l!\ m!}p^k q^l r^m$$

for integers $k, l, m \geq 0$ such that $k+l+m=n$. This distribution arises from the following experiment: perform n independent repetitions of a trial with three possible outcomes with probabilities p, q, r and let X, Y, Z count how many times these outcomes appear.

Calculate the probability mass function of $W=X+Y$ and identify its distribution by name. After the calculation, give an intuitive justification for the answer you obtained. *Hints*: Figure out what the possible values of W are. To calculate $P(W=a)$ for each possible value a, decompose $P(W=a)$ according to the different values of (X, Y, Z) that make up the event $W=a$. Use the PMF of (X, Y, Z).

Since W is given in terms of X and Y, Z should be removed for convenience.

- $k+l+m=n \Leftrightarrow m=n-k-l$
- $p+q+r=1 \ \Leftrightarrow \ r=1-p-q$

$$P(X=k, Y=l, Z=m)=\frac{n!}{k!\ l!\ m!}p^k q^l r^m \Rightarrow P(X=k, Y=l)=\frac{n!}{k!\ l!\ (n-k-l)!}p^k q^l (1-p-q)^{n-k-l}$$

$$P(X+Y=w \mid X+Y=W,\ W=w)=\sum_{k=0}^{w} P(X=k, Y=w-k)$$

$$=\sum_{k=0}^{w}\frac{n!}{k!(w-k)!(n-w)!}p^k q^l (1-p-q)^{n-k-l}=\frac{n!}{\mathbf{w}!(n-w)!}(1-p-q)^{n-w}\sum_{k=0}^{w}\frac{\mathbf{w}!}{k!(w-k)!}p^k q^{w-k}$$

$$=\frac{n!}{\mathbf{w}!(n-w)!}(1-p-q)^{n-w}\ \sum_{k=0}^{w}\binom{w}{k}\ p^k\ q^{w-k}=\frac{n!}{w!(n-w)!}(1-p-q)^{n-w}\ (p+q)^w$$

$$=\frac{n!}{w!(n-w)!}(p+q)^w\ (1-(p+q))^{n-w}$$

Thus

$$P(W=X+Y)=\frac{n!}{w!(n-w)!}(p+q)^w\ (1-(p+q))^{n-w}$$

whose distribution is binomial:

$$P(W=X+Y)\sim Bin(n, p+q)$$

3. Let $n \geq 2$ and $0 < p < 1$. Let $X_1, X_2, ..., X_n$ be *i.i.d.* Bernoulli random variables with success probability p and $S_n = X1 + + Xn$. The Bernoulli assumption means that each X_i has PMF $P(X_i = 1) = p = 1 - P(X_i = 0)$.

(a) Compute the conditional joint probability mass function of the random vector $(X_1, ..., X_n)$, given that $S_n = k$. That is, find

$$P(X_1 = a_1, X_2 = a_2, ..., X_n = a_n \mid S_n = k)$$

for all vectors $(a_1, ..., a_n)$ of zeros and ones, and all $k \in \{0, ..., n\}$.
Hints. Use the definition of conditional probability. Which vectors $(a_1, ..., a_n)$ are compatible with $S_n = k$? Check that your conditional PMF sums to 1 over all vectors $(a_1, ..., a_n)$.

The meaning of the given condition $S_n = k$ is the sum of $a_i \ \forall\ i \in \mathbb{Z}_{\geq 1}$. i.e. $\sum_{i=1}^{n} a_i = k$

Since a_i is either 0 or 1 and $S_n = k$ is defined as above, simply

$$P(X_1 = a_1, X_2 = a_2, ..., X_n = a_n \mid S_n = k) = \frac{1}{\binom{n}{k}}$$

(b) Deduce whether $X_1, ..., X_n$ are conditionally independent, given $S_n = k$. (The general definition is that random variables $X_1, ..., X_n$ are conditionally independent, given an event B such that $P(B) > 0$, if

$$P(X_1 = a_1, X_2 = a_2, ..., X_n = a_n \mid B) = \prod_{i=1}^{n} P(X_i = a_i \mid B)$$

for all possible values $a_1, ..., a_n$).
Hints. Consider cases $k = 0,\ 0 < k < n$ and $k = n$.

(i) k = 0

- $P(X_1 = a_1, X_2 = a_2, ..., X_n = a_n \mid S_n = 0) =$
  $P(X_1 = 0, X_2 = 0, ..., X_n = 0 \mid S_n = 0) =$
  $\prod_{i=1}^{n} P(X_i = 0 \mid S_n = 0) = \frac{1}{\binom{n}{0}} = 1$

(ii) k = n

- $P(X_1 = a_1, X_2 = a_2, ..., X_n = a_n \mid S_n = n) =$
  $P(X_1 = 1, X_2 = 1, ..., X_n = 1 \mid S_n = n) =$
  $\prod_{i=1}^{n} P(X_i = 1 \mid S_n = n) = \frac{1}{\binom{n}{n}} = 1$

(iii) $0 < k < n$

- $P(X_1 = a_1, X_2 = a_2, ..., X_n = a_n \mid S_n = k) = \frac{1}{\binom{n}{k}}$
- $\prod_{i=1}^{n} P(X_i = a_i \mid S_n = k) =$
  $\prod_{i=1}^{k} P(X_i = 1 \mid S_n = k) \cdot \prod_{i=1}^{n-k} P(X_i = 0 \mid S_n = 0)$

  where
  - $\prod_{i=1}^{k} P(X_i = 1 \mid S_n = k) = \frac{k}{n}$
  - $\prod_{i=1}^{n-k} P(X_i = 0 \mid S_n = 0) = 1 - \prod_{i=1}^{k} P(X_i = 1 \mid S_n = k) = 1 - \frac{k}{n} = \frac{n-k}{n}$

- Hence $\prod_{i=1}^{n} P(X_i = a_i \mid S_n = k) = (\frac{k}{n})^k \cdot (\frac{n-k}{n})^{n-k}$
- Obviously, $\frac{1}{\binom{n}{k}} \neq (\frac{k}{n})^k \cdot (\frac{n-k}{n})^{n-k}$
  i.e. $P(X_1 = a_1, X_2 = a_2, ..., X_n = a_n \mid S_n = k) \neq \prod_{i=1}^{n} P(X_i = a_i \mid S_n = k)$

- Thus, in case of either k = 0 or k = n, $X_1, ..., X_n$ are independent;
  Otherwise, they are not for $n \geq 2$.

1.1 A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, ...$ that are 0 or 1 with probability 1/2 each. For $n \geq 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)th$ and nth tosses. Is X_n a Markov chain?

- If it is to be a Markov Chain,
  $P(X_n = i \mid X_{n-1} = j,\ X_{n-2} = k) = P(X_n = i \mid X_{n-1} = j)$ should be satisfied.
  - Consider $P(X_3 = 2 \mid X_2 = 1)$
    $P(X_3 = 2 \mid X_2 = 1) = \frac{P(X_3=2,\ X_2=1)}{P(X_2=1)} = \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{1}{4}$
  - Let's consider one more restriction: In case of $X_1 = 0$
    $P(X_3 = 2 \mid X_2 = 1,\ X_1 = 0) = P(Y_3 = 1) = \frac{1}{2}$
  - Thus $P(X_3 = 2 \mid X_2 = 1) \neq P(X_3 = 2 \mid X_2 = 1,\ X_1 = 0)$, so this is not a Markov Chain.

1.2 Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

Consider the transitional probability of X_n: $p(i, j)$ where

- The Left Urn
  - i: the number of white ball
  - $5 - i$: the number of black balls
- The Right Urn
  - i: the number of black balls
  - $5 - i$: the number of white balls

We can break this into three different cases.

(a) $i = j$
This is the case when we choose white balls from both urns: No changes.

$$p(i, j) = \frac{i}{5} \cdot \frac{5-i}{5} \cdot 2$$

(b) $i = j + 1$
This is the case when we choose a black ball from the left urn, and a white ball from the right urn.

$$p(i, j) = \frac{5-i}{5} \cdot \frac{5-i}{5}$$

(c) $i = j - 1$
This is the case when we choose a white ball from the left urn, and a black ball from the right urn.

$$p(i, j) = \frac{i}{5} \cdot \frac{i}{5}$$

Thus the transitional probability for X_n is

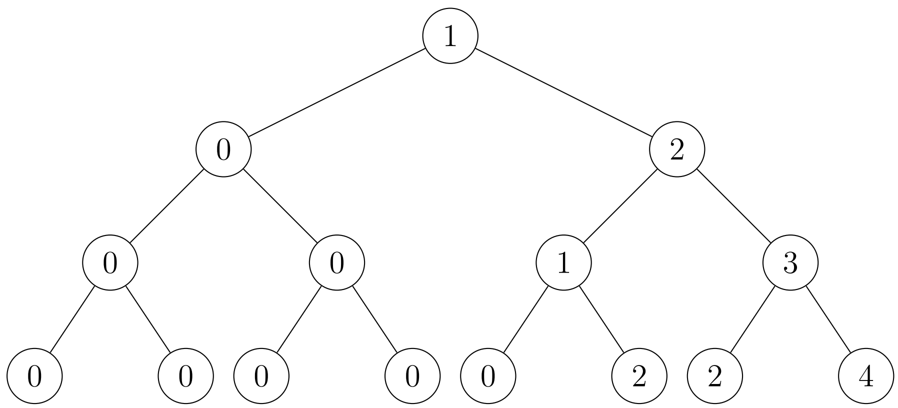
$$p(i, j) = \begin{cases} 1 & \text{for } X_n = 0 \text{ or } 5 \\ \frac{i}{5} \cdot \frac{5-i}{5} \cdot 2 & \text{for i = j} \\ \frac{5-i}{5} \cdot \frac{5-i}{5} & \text{for i = j + 1} \\ \frac{i}{5} \cdot \frac{i}{5} & \text{for i = j - 1} \\ 0 & \text{otherwise} \end{cases}$$

1.5 Consider a gamblers ruin chain with N = 4. That is, if $1 \leq i \leq 3$, $p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0,0) = 1$ and $p(4,4) = 1$. Compute $p^3(1,4)$ and $p^3(1,0)$.

(i) $p^3(1,4)$

- $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$: 0.4 x 0.4 x 0.4 = 0.064

(ii) $p^3(1,0)$



- $1 \Rightarrow 0 \Rightarrow 0 \Rightarrow 0$: 0.6 x 1 x 1 = 0.6
- $1 \Rightarrow 2 \Rightarrow 1 \Rightarrow 0$: 0.4 x 0.6 x 0.6 = 0.144

$\Rightarrow p^3(1,0) = 0.6 + 0.144 = 0.744$

1.6 A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4.

(a) Find the transition matrix for the chain.

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

where 1: Airport, 2: Hotel B, 3: Hotel C

(b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

- The probability of being in Airport at time 2
  - Airport $\to$ Hotel B $\to$ Airport: $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$
  - Airport $\to$ Hotel C $\to$ Airport: $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$
    $\Rightarrow \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$

  The probability of being in the Airport for next time is $\frac{1}{2}$.

- The probability of being in Hotel B at time 2
  - Airport $\to$ Hotel C $\to$ Hotel B: $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

  The probability of being in the Airport for next time is 0.

- The probability of being in Hotel C at time 2
  - Airport $\to$ Hotel B $\to$ Hotel C: $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

  The probability of being in the Hotel B for next time is $\frac{1}{4}$.

Thus

$$P(X_3 = \text{Hotel B} \mid X_0 = \text{Airport}) = \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{4} = \frac{13}{32}$$

1.7 Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$.

(a) Compute its transition probability.

	RR	RS	SR	SS
RR	0.6	0.4	0	0
SR	0.6	0.4	0	0
RS	0.6	0	0	0.4
SS	0	0	0.3	0.7

(b) Compute the two-step transition probability.

$$p^2 = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0.6 & 0 & 0 & 0.4 \\ 0 & 0 & 0.3 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0.6 & 0 & 0 & 0.4 \\ 0 & 0 & 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0.36 & 0.24 & 0.12 & 0.28 \\ 0.18 & 0 & 0.21 & 0.61 \end{bmatrix}$$

	RR	RS	SR	SS
RR	0.6	0.4	0	0
SR	0.6	0.4	0	0
RS	0.36	0.24	0.12	0.28
SS	0.18	0	0.21	0.61

(c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

$$P\{X_2 = SR \mid X_0 = SS\} + P\{X_2 = RR \mid X_0 = SS\} = 0.21 + 0.18 = 0.39$$