Philosophically exactly like in discrete time: given the present state, the past and the future are independent. But the math becomes more technical. Throughout, the state space S is finite or countably infinite.

Definition Let $\{X_t : 0 \le t < \infty\}$ be a stochastic process. Then X is a Markov chain if for all $0 \le s_0 < \ldots < s_n < s < t$ and all $x_0, \ldots, x_n, x, y \in S$:

$$P(X_t = y \mid X_s = x,\ X_{s_n} = x_n, \ldots, X_{s_0} = x_0) = P(X_t = y \mid X_s = x)$$

wherever the conditioning event has positive probability.

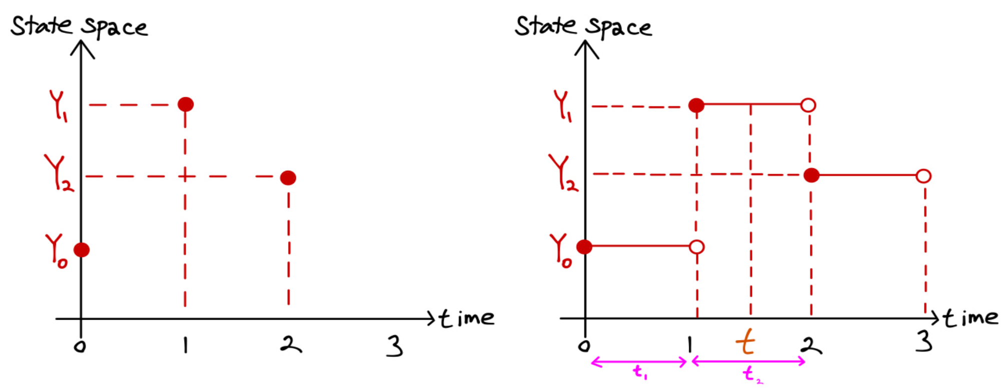
We will only deal with the time-homogeneous case where

$$P(X_t = y \mid X_s = x) = P(X_{t-s} = y \mid X_0 = x)$$

The transition probability is

$$p_t(x, y) = P(X_t = y \mid X_0 = x)$$

Since there are uncountably many transition matrices $\{p_t(x, y)\}_{x,y \in S}$, the transition probability is not as useful as in discrete time. It even turns out to be extremely hard to calculate in some cases.



(a) Discrete-Time MC (b) Continuous-Time MC

The Markov property forces the holding times $t_1, t_2, t_3, \ldots$ to be exponential random variables. By the same token, the sequence of states visited has to be a discrete-time Markov chain.

Recall the following:
Let $\eta_1, \eta_2, \ldots, \eta_n$ be independent. $\eta \sim (\lambda)$. $\zeta = \min\{\eta_1, \ldots, \eta_n\}$. I = the index i s.t. $\eta_1 = \zeta$.

$$\zeta \perp\!\!\!\perp I \qquad \zeta \sim \exp(\lambda_1 + \ldots + \lambda_n)$$

$$P(\zeta > t) = \prod_{i=1}^{n} P(\eta_i > t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\lambda_1 + \ldots + \lambda_n)t} \qquad P(I = i) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

Special case Let (Y_n) be a discrete-time Markov chain with transition matrix $u = (u(x,y))_{x,y\in S}$. Let $N(t)$ be a rate λ Poisson process. Then $X_t = Y_{N(t)}$ is a continuous time Markov chain. In this **SPECIAL CASE** we can calculate the transition probability:

$$\begin{aligned}
P_t(x,y) &= P(x_t = y \mid X_0 = x) = P(X_t = y \mid Y_0 = x) \\
&= \sum_{n=0}^{\infty} P(X_t = y,\ N(t) = n \mid Y_0 = x) \\
&\quad \text{(N(t): the number of jumps by time t)} \\
&= \sum_{n=0}^{\infty} P(Y_n = y,\ N(t) = n \mid Y_0 = x) \\
&= \sum_{n=0}^{\infty} P(N(t) = n) P(Y_n = y \mid Y_0 = x) \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \cdot u^n(x,y)
\end{aligned}$$

Note that $Z \sim \text{Poisson}(\mu)$ then

$$P(Z = k) = \frac{e^{-\mu}\mu^k}{k!}$$

What takes the place of the transition probabilities as the fundamental object? **Rates**.

Definition For $x \neq y$ in S, the <u>Rate</u> of jumping from x to y is

$$q(x,y) = \lim_{h\to 0} \frac{p_h(x,y)}{h}$$

This limit will exist in all our examples.

Note that

$$u^{(0)}(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

$$u^{(0)} = I = \text{the identity matrix}$$

Back to the previous **SPECIAL CASE:**

$$
\begin{aligned}
\frac{1}{h}p_h(x,y) &= \frac{1}{h}\sum_{n=0}^{\infty}\frac{e^{\lambda h}(\lambda h)^n}{n!}u^n(x,y)\\
&\vdots\\
&\text{n=0 term} = 0!\\
&\vdots\\
&= \underbrace{0}_{n=0}\,\underbrace{e^{-\lambda h}\lambda u(x,y)}_{\substack{n=1\\ \lambda u(x,y)\text{ as } h\to 0}} + \underbrace{\frac{1}{h}\sum_{n=2}^{\infty}\frac{\overbrace{e^{-\lambda h}}^{\le 1}(\lambda h)^n}{n!}\overbrace{u^{(n)}(x,y)}^{\le 1}}_{\substack{0\le\cdot\le\frac{1}{h}h^2\sum_{n=2}^{\infty}\frac{\lambda^n h^{n-2}}{h!}\\ \le h\sum_{n=2}^{\infty}\frac{\lambda^n}{n!}\le he^{\lambda}\\ \text{This term doesn't matter later on}}}
\end{aligned}
$$

Hence

$$q(x,y) = \lambda u(x,y)$$

where λ is the rate of Poisson clock and $u(x,y)$ is the probability of jumping to y.

'little o(h) of h' o means a quantity that when divided by h goes to 0 as $h \to 0$, $\frac{o(h)}{h} \xrightarrow{h\to 0}$

$\Rightarrow$ Hence forth we will give our examples in terms of rates.

Ex. Let $N(\cdot)$ be a rate λ Poisson process. Then $N(\cdot)$ is a Markov chain with stable space $S = Z_{\geq 0}$ and rates

Continuous Time Markov chains are determined by the rates

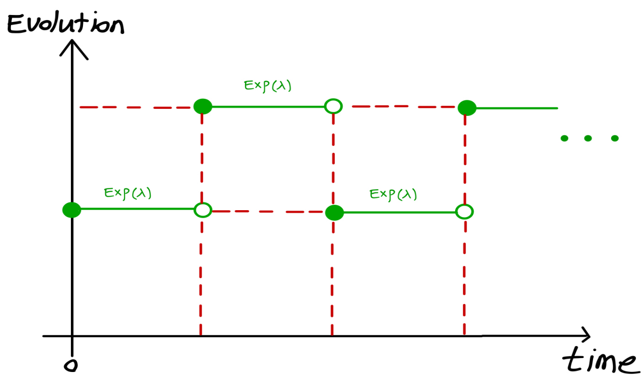
$$q(x,y) = \lim_{h\to 0} \frac{p_h(x,y)}{h} = \text{exponential rate of jumping to } y \text{, when the present state is } x$$

Given rates $\{q(x,y)\}_{x\neq y}$

1. Construct the Markov chain X_t
2. Compute transition probability $P_t = (p_t(x,y))_{x,y\in S}$
3. Determine invariant distributions

1. Construct the Markov chain X_t. Suppose $X_0 = 1$.
   - $S = \{1, 2\}$.

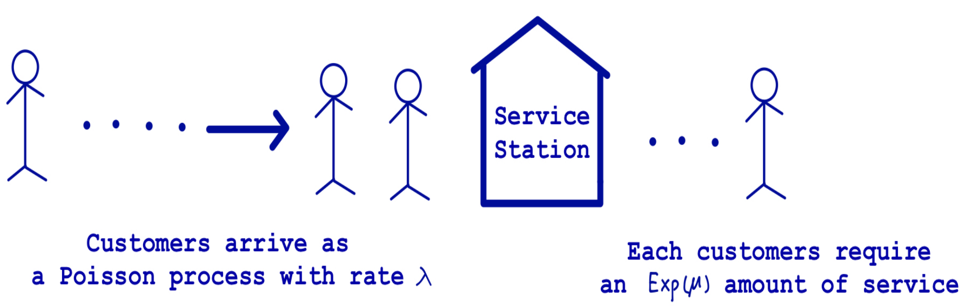
$$\begin{tikzcd} 1 \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \end{tikzcd}$$



   - Compute transition probability $P_t = (p_t(x,y))_{x,y\in S}$

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 4 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & \cdots \arrow[l, bend left, "\mu"] \end{tikzcd}$$

     - Birth-death chain (Population Model)
     - M/M/1 queue



Example

Suppose the current state is 3. Then the rate of jumping out of the state 3 is $\lambda + \mu$. And

$$\begin{cases} \text{P(move to 4)} = \frac{\lambda}{\lambda+\mu} \\ \text{P(move to 2)} = \frac{\mu}{\lambda+\mu} \end{cases}$$

Construction of X_t, given the rates $\{q(x,y)\}_{x\neq y}$

$$\lambda_x = \sum y : y \neq xq(x,y) = \text{total rate to jump out of state } x$$

If $\lambda_x = \infty$, the Markov chain leaves state x immediately, so we can ignore such states & assume each $\lambda_x < \infty$. $\lambda_x = 0$ means that x is absorbing.

If $0 < \lambda_x < \infty$, we define a transition probability

$$r(x,y) = \frac{q(x,y)}{\lambda_x}$$

Aside: Suppose $W \sim \exp(1)$
Let $W = \tilde{W}\lambda$

$$P(\tilde{W} > t) = P(W > \lambda t) = e^{-\lambda t} \Rightarrow \tilde{W} \sim \exp(\lambda)$$

Ingredients of the construction:

(i) A discrete-time Markov chain $(Y_n)_{n\geq 0}$ with $Y_0 \sim \mu$.

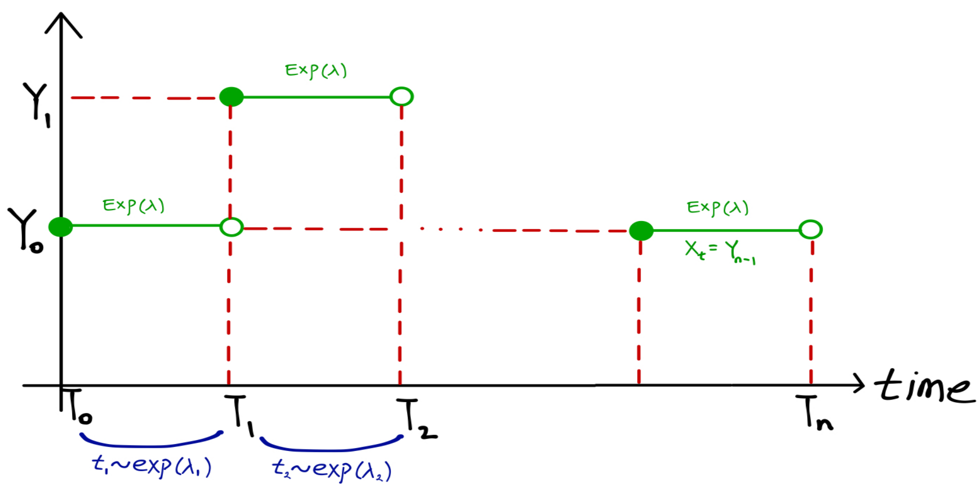
(ii) IID sequence $(\tau_k)_{k\geq 0}$ with $\tau_k \sim \exp(1)$.

- $T_0 = 0$
- $T_1 = t_1 = \frac{\tau_0}{\lambda_{Y_0}}$
- $X_t = Y_0$ for $0 \leq t < T_1$
- $t_2 = \frac{Y_1}{\lambda_{Y_1}}$ $\qquad T_2 = T_1 + t_2$
- $X_t = Y_1$ for $T_1 \leq t < T_2$

Suppose we have constructed up to $T_n = t_1 + ... + t_n$ such that $X_t = y_k$ for $T_k \leq t < T_{k+1}$ for $k = 0, ..., n-1$.

Next step:

$$t_{n+1} = \frac{\tau_n}{\lambda_{Y_n}} \qquad T_{n+1} = T_n + t_{n+1} \qquad X_t = Y_n \text{ for } T_n \leq t < T_{n+1}$$



Compute transition probability $P_t = (p_t(x,y))_{x,y\in S}$ (4.2 on Durrett)

This construction does satisfy

$$\lim_{h\to 0}\frac{p_h(x,y)}{h} = q(x,y)$$

Differential equation: for transition probabilities.
Chapmnan-Kolmogorov equations:

$$p_{t+s}(x,y) = \sum_z p_t(x,z)p_s(z,y)$$

Think of $h > 0$ small small

$$\begin{aligned}
p_{t+h} - p_t(x,y) &= \sum_z P_h(x,z)p_t(z,y) - p_t(x,y) \\
&= \sum_{z\neq x} p_h(x,z)p_t(z,y) + (p_h(x,x)-1)p_t(x,y) \\
\frac{p_{t+h} - p_t(x,y)}{h} &= \underbrace{\sum_{z\neq x}\frac{p_h(x,z)}{h}p_t(z,y)}_{\xrightarrow{h\to 0} q(x,z)} + \underbrace{\frac{p_n(x,x)-1}{h}p_t(x,y)}_{-\sum_{z\neq x}\frac{p_h(x,z)}{h}\xrightarrow{h\to 0} -\sum_{z\neq x} q(x,z) == \lambda_x} \\
\frac{d}{dt}p_t(x,y) &= \sum_{z\neq x} q(x,z)p_t(z,y) + (-\lambda_x)p_t(x,y) = \sum_z Q_{x,z}p_t(z,y) \\
\frac{d}{dt}P_t &= QP_t \qquad P_0 = I
\end{aligned}$$

Kolmogorov's Backward Equation

$$\frac{d}{dt}P_t = \left[\frac{d}{dt}p_t(x,y)\right]_{x,y\in S}$$

Similar argument leads to Kolmogorov's forward equation.

$$P_t' = P_tQ$$

Examples

$$\begin{tikzcd} 1 \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \end{tikzcd} \qquad Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Generator Matrix Q

$$Q_{x,z} = \begin{cases} q(x,z) & z \neq x \\ -\lambda_x & z = x \end{cases}$$

2. $\mu/\mu/1$ queue

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "\mu"] \end{tikzcd}$$

$$Q = \begin{array}{c|cccccc} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \\ \mathbf{0} & -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mathbf{1} & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \dots \\ \mathbf{2} & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & \dots \\ \mathbf{3} & & & & & & \end{array}$$

A probability mean μ on S is an invariant distribution if $\pi =_t \quad \forall t \geq 0$.

Lemma 4.3 π is invariant $\Leftrightarrow \ \pi Q = 0$
Proof Assume π is invariant

$\Rightarrow \ \pi = \pi P_t \ \ \forall t \geq 0$

$$\begin{aligned} & 0 = \frac{d}{dt}(\pi P_t) = \pi \frac{d}{dt}(P_t) = \pi P_t Q = \pi Q \\ \textit{Assume } \pi Q = 0 \qquad & \frac{d}{dt}(\pi P_t) = \pi \frac{d}{dt}(P_t) = \pi Q P_t = 0 \cdot P_t = 0 \\ & \pi P_t = \pi P_0 = \pi \end{aligned}$$

So π is invariant.

Today

1. Detailed Balance
2. Queing Models
3. Matrix Exponentials & Solving Kolmogorov's Equations
4. Blow-up

1. Detailed Balance

Last time: π is invariant if and only if $\pi Q = 0$.

Definition π is a <u>reversible distribution</u> (satisfies detailed balance) if

$$\pi(x)q(x,y) = \pi(y)q(y,x) \ \ \forall x \neq y$$

<u>Fact:</u> If we start the Markov chain with a reversible distribution π, the time reversed process has the same distribution as the original process.

Theorem 4.5 Detailed balance $\Rightarrow$ π is invariant.
Proof.

$$
\begin{aligned}
(\pi Q)_y &= [\ldots\ldots] \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \\
&= \sum_x \pi(x)q(x,y) = \pi(y)q(y,y) + \sum_{x\neq y} \pi(x)q(x,y) \\
&= \pi(y)(-\lambda y) + \sum_{x \neq y} \pi(y)q(y,x) = \pi(y) \left(= 0 - \lambda_y + \underbrace{\sum_x q(y,x)}_{\text{total rate out of y}} \right)
\end{aligned}
$$

Example: M/M/1 queue

$$
\begin{tikzcd}
0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & \cdots \arrow[l, bend left, "\mu"]
\end{tikzcd}
$$

$$\begin{cases} q(n, n+1) = \lambda & \text{for } n \geq 0 \\ q(n, n-1) = \mu & \text{for } n \geq 1 \end{cases}$$

Try to find a reversible distribution:

$$\begin{aligned} \pi_n q(n, n+1) &= \pi_{n+1} q(n+1, n) \\ &\Leftrightarrow \lambda \pi_n = \mu \pi_{n+1} \\ &\Leftrightarrow \pi_{n+1} = \frac{\lambda}{\mu} \pi_n \end{aligned}$$

So then

$$\pi_n = \frac{\lambda}{\mu} \pi_{n-1} = \left(\frac{\lambda}{\mu}\right)^2 \pi_{n-2} = \ldots = \left(\frac{\lambda}{\mu}\right)^n \pi_0$$

<u>continue:</u> To have a probability distribution, we want

$$1 = \sum_{n=0}^{\infty} = \pi_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \pi_9$$

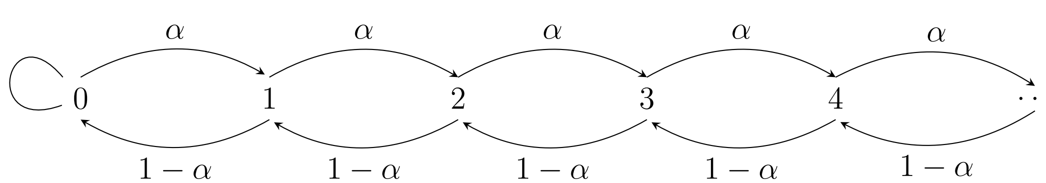
If $\frac{\lambda}{\mu} < 1$, we can solve for π_0:

$$\begin{aligned} 1 &= \pi \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \\ &= \pi_0 \frac{1}{1 - \frac{\lambda}{\mu}} \Rightarrow \pi_0 = 1 - \frac{\lambda}{\mu} \end{aligned}$$

Answer: provided $\lambda < \mu$, where λ is arrival rate and μ is service rate, we have a reversible distribution

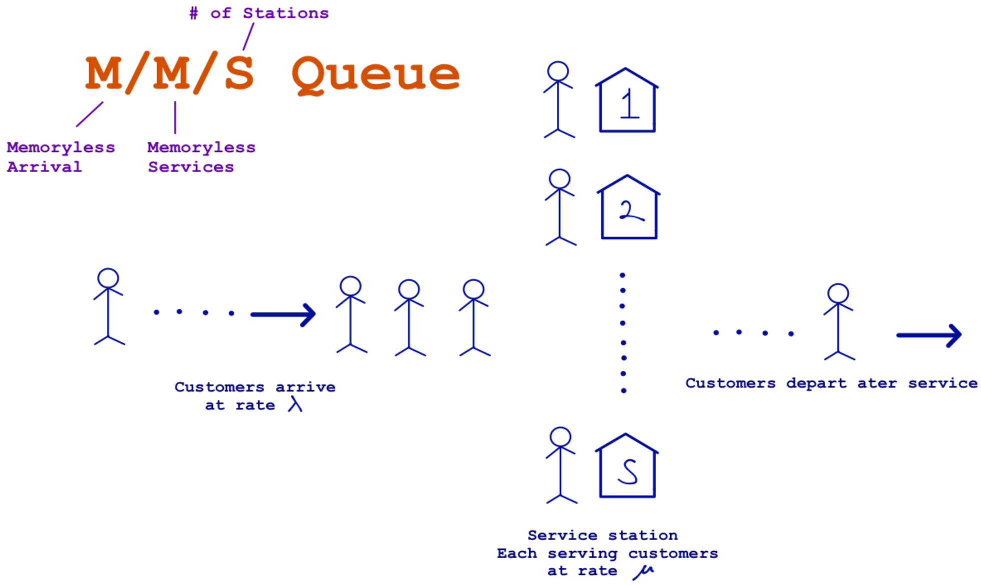
$$\pi_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \quad \text{for } n \geq 0 \qquad \text{(shifted geometric)}$$

Recall from HomeWork 3:



This Markov chain has an invariant distribution if and only if $\alpha < \frac{1}{2}$.

2. Queuing Models.



Suppose X_t = the number of customers in the system at time t, and let state space $S = \{0, 1, 2, ...\}$. Then what is the rates?

$$q(n, n+1) = \lambda, \ n \geq 0$$
$$q(n, n-1) = \begin{cases} n\mu & 1 \leq n < s \\ s\mu & n \geq s \end{cases}$$

Fundamental fact: If $W_1, ..., W_n$ are independent random varialbes $W_i \sim \exp(\lambda_i)$, and $W = \min\limits_{i \leq 1 \leq n} W_i$ then $W \sim \exp(\lambda_1 + ... + \lambda_n)$.

M/M/S queue with balking

Suppose an arriving customer who sees n customers in the system, leaves with probability b_n.

Rates:

$$q(n, n+1) = a_n\lambda \text{ where } a_n = 1 - b_n$$
$$q(n, n-1) \text{ same as before}$$

M/M/∞ queue
Same rules as before except now the number of servers is unlimited so each arriving customers goes directly into service.

$$\begin{aligned} q(n, n+1) &= \lambda \\ q(n, n-1) &= n\mu \end{aligned}$$

Detailed Balance

$$\begin{aligned} \pi(n)q(n, n+1) &= \pi(n+1)q(n+1, n) \\ \pi(n)\lambda &= \pi(n+1) \cdot (n+1)\mu \\ \Rightarrow \pi(n+1) &= \frac{\lambda/\mu}{n+1}\pi(n) = \frac{(\lambda/\mu)^2}{(n+1)n}\pi(n-1) \\ &= \ldots = \frac{(\lambda/\mu)^{n+1}}{(n+1)!}\pi_0^{e/-\lambda/\mu} \end{aligned}$$

So

$$\pi(n) = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!} \sim \text{Poisson}\left(\frac{\lambda}{\mu}\right)$$

Scalar Version

$$\begin{aligned} x'(t) &= qx(t) \\ x(0) &= 1 \end{aligned}$$

Solution:

$$\begin{aligned} x(t) &= e^{tq} \\ e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned}$$

$$\begin{aligned} P_t' &= QP_t = P_tQ \\ P_0 &= I \end{aligned}$$

Define a matrix exponential

$$e^{tQ} = \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!}$$

<u>Fact:</u> this series converges for any finite matrix.

Let's see if $P_t = e^{tQ}$ solves.

$$\begin{aligned}
\frac{d}{dt}e^{tQ} &= \frac{d}{dt}\sum k = 0^{\infty}\frac{t^kQ^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{d}{dt}\left(\frac{t^kQ^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{kt^{k-1}Q^k}{k!} = \sum_{k=0}^{\infty}\frac{t^{k-1}Q^k}{(k-1)!} \\
&= Q\sum_{k=0}^{\infty}\frac{t^{k-1}Q^{k-1}}{(k-1)!} = Qe^{tQ} \qquad\qquad (\text{Or, similarly, } =e^{tQ}Q)
\end{aligned}$$

We have shown here $P_t = e^{tQ}$ satisfies

$$\begin{gathered}
P_t' = QP_t = P_tQ \\
e^{tQ} = I + \sum_{k=0}^{\infty}\frac{t^kQ^k}{k!} \qquad (Q^0 = I) \\
t = 0 : e^{0\cdot Q} = I
\end{gathered}$$

In some literature, e^{tQ} is used as an alternative to P_t.

Examples

$$\begin{tikzcd}
1 \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"]
\end{tikzcd} \qquad Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Diagonalize Q:
Idea: write $Q = B \wedge B^{-1}$ where B is is eigen-vectors and $\wedge$ is diagonal matrix of eigenvalues.

$$Q^n = B \wedge \underbrace{B^{-1}\cdot B}_{I} \wedge B^{-1}\cdot \ldots B \wedge B^{-1} = B \wedge B^{-1}$$

$$\begin{array}{lcc} \text{Eigenvalues} & 0 & -(\lambda+\mu) \\ \text{Eigenvectors} & \binom{1}{1} & \binom{\lambda}{-\mu} \end{array} \qquad B = \begin{bmatrix} 1 & \lambda \\ 1 & -\mu \end{bmatrix}$$

$$B^{-1} = \frac{1}{\lambda+\mu}\begin{bmatrix}\mu & \lambda \\ 1 & -1\end{bmatrix} = B \underbrace{\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix}0 & 0 \\ 0 & -\mu-\lambda\end{bmatrix}^n\right)}_{\begin{bmatrix}1 & 0 \\ 0 & e^{-t(\lambda+\mu)}\end{bmatrix}} B^{-1}$$

$$\begin{aligned}
&= \begin{bmatrix}1 & \lambda \\ 1 & -\mu\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & e^{-t(\lambda+\mu)}\end{bmatrix}\frac{1}{\lambda+\mu}\begin{bmatrix}\mu & \lambda \\ 1 & -1\end{bmatrix} \\
&= \frac{1}{\lambda+\mu}\begin{bmatrix}1 & \lambda e^{-t(\lambda+\mu)} \\ 1 & -\mu e^{-t(\lambda+\mu)}\end{bmatrix}\begin{bmatrix}\mu & \lambda \\ 1 & -1\end{bmatrix} \\
&= \frac{1}{\lambda+\mu}\begin{bmatrix}\mu+\lambda e^{-t(\lambda+\mu)} & \lambda-\lambda e^{-t(\lambda+\mu)} \\ \mu-\mu e^{-t(\lambda+\mu)} & \lambda+\mu e^{-t(\lambda+\mu)}\end{bmatrix}
\end{aligned}$$

$$\lim_{t\to\infty} P_t = \begin{bmatrix}\frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}\end{bmatrix} = \begin{bmatrix}\pi \\ \pi\end{bmatrix}$$