Question 1

Consider a discrete time Markov chain with transition probabilities $p(i,j)$, with state space $\{1,2,...,10\}$, and assume $X_0 = 3$. Express

$$P(X_6 = 7, X_5 = 3 \mid X_4 = 1, X_9 = 3)$$

in terms of the (if necessary multi-step) transition probabilities.

$$P(X_6 = 7, X_5 = 3 \mid X_4 = 1, X_9 = 3) = \frac{P(X_6 = 7, X_5 = 3, X_4 = 1, X_9 = 3)}{P(X_4 = 1, X_9 = 3)}$$

$$= \frac{P(X_9 = 3 \mid X_6 = 7, X_5 = 3, X_4 = 1) \cdot P(X_6 = 7 \mid X_5 = 3, X_4 = 1) \cdot P(X_5 = 3 \mid X_4 = 1) \cdot P(X_4 = 1)}{P(X_9 = 3 \mid X_4 = 1)P(X_4 = 1)}$$

$$= \frac{P(X_9 = 3 \mid X_6 = 7) \cdot P(X_6 = 7 \mid X_5 = 3) \cdot P(X_5 = 3 \mid X_4 = 1) \cdot \cancel{P(X_4 = 1)}}{P(X_9 = 3 \mid X_4 = 1) \cdot \cancel{P(X_4 = 1)}}$$

$$= \frac{P(X_9 = 3 \mid X_6 = 7) \cdot P(X_6 = 7 \mid X_5 = 3) \cdot P(X_5 = 3 \mid X_4 = 1)}{P(X_9 = 3 \mid X_4 = 1)}$$

$$= \frac{p^3(7,3) \cdot p^1(3,7) \cdot p^1(1,3)}{p^5(1,3)}$$

Question 2

Consider a discrete time Markov chain with transition probabilities $p(i,j)$, with state space $\{1, 2, ..., 10\}$. Assume, moreover, that T is a stopping time with the properties $P_1(T < \infty) = 1$ and $P_1(X_T = 3) = 1$. Express

$$P_1(X_{T+6} = 7, X_{T+5} = 3 \mid X_{T+4} = 1, X_{T+9} = 3)$$

in terms of the (if necessary multi-step) transition probabilities.

We can expand the given conditional probability as below.

$$P_1(X_{T+6} = 7, X_{T+5} = 3 \mid X_{T+4} = 1, X_{T+9} = 3) = \frac{P_1(X_{T+6} = 7, X_{T+5} = 3, X_{T+4} = 1, X_{T+9} = 3)}{P_1(X_{T+4} = 1, X_{T+9} = 3)}$$

Since the stopping time T for P_1 is finite and $P_1(X_T = 3) = 1$, it can be transformed as follows preserving the existing distribution by the strong Markov property.

$$\frac{P_3(X_4 = 1, X_5 = 3, X_6 = 7, X_9 = 3)}{P_3(X_9 = 3, X_4 = 1)}$$

$$= \frac{P_3(X_9 = 3 \mid X_6 = 7, X_5 = 3, X_4 = 1) \cdot P_3(X_6 = 7 \mid X_5 = 3, X_4 = 1) \cdot P_3(X_5 = 3 \mid X_4 = 1) \cdot P_3(X_4 = 1)}{P_3(X_9 = 3 \mid X_4 = 1) \cdot P_3(X_4 = 1)}$$

$$= \frac{P(X_9 = 3 \mid X_6 = 7) \cdot P(X_6 = 7 \mid X_5 = 3) \cdot P(X_5 = 3 \mid X_4 = 1) \cdot \cancel{P(X_4 = 1)}}{P(X_9 = 3 \mid X_4 = 1) \cdot \cancel{P(X_4 = 1)}}$$

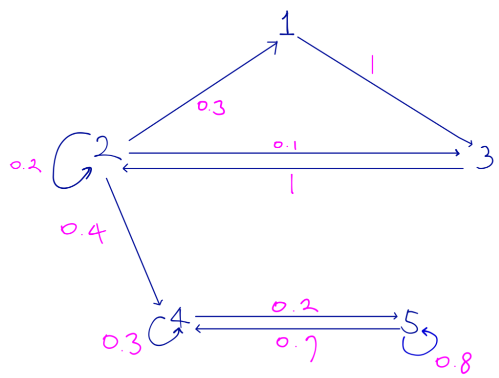
$$= \frac{p^3(7,3) \cdot p^1(3,7) \cdot p^1(1,3)}{p^5(1,3)}$$

We obtain the same transition probability as Question 1.

Question 3
Consider the discrete time Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0.2 & 0.1 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$

(a) Draw the transition graph. Identify the transient and the recurrent states.



- Transient: $\{1, 2, 3\}$
- Recurrent: $\{4, 5\}$

(b) Let $R_2 = max\{n \geq 0 : X_n = 2\}$. Prove that $P_2(R_2 < \infty) = 1$.

- Since $\{4, 5\}$ is recurrent, it has to encounter 4 at some point.
- Then there is no probability to go back to 2.
- Hence it is not possible to stop at 2 infinitely many times i.e. R_2, the last stop at the state 2, is finite.
- Thus $P_2(R_2 < \infty) = 1$

(c) Is R_2 a stopping time? Why?

- By the definition of stopping time, stopping at time n depends only on the past and present states, or $X_0, ..., X_n$.
- However, it is impossible to expect when R_2 is stopped without consideration of its future status.
  e.g.) In case where it goes to 4 and never come back to 2.
- Thus R_2 is not a stopping time.

(d) Calculate $P_3(X_{R_2+1} = 4 \mid X_{R_2} = 2, R_2 = 8)$

- Since $8th$ state is the last stop at the state 2, the next state must be 4 where is the recurrent state.
- Thus $P_3(X_{R_2+1} = 4 \mid X_{R_2} = 2, R_2 = 8) = 1$

Question 4

Consider a Markov chain with state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}$$

Decide which of the following is a stopping time:

(a) $T_1 = min\{n \geq 6 : X_n = 2\}$: Yes

- This is a stopping time since T_1 can only be determined by looking at the current state.

(b) $T_2 = min\{n \geq 1 : X_{n+1} = 2\}$: No

- Since we need to know the future state X_{n+1} to expect T_2, it is not a stopping time.

(c) $T_3 = min\{n \geq 2 : X_{n-1} = 2\}$: Yes

- This is a stopping time since it requires to know just the past state for T_3.

(d) $T_4 = min\{n \geq T_1 : X_{n-1} = 2\}$: Yes

- Since the given T_1 considers the current state and the given condition of T_4 considers the previous state, not future, it is said to be a stopping time.

(e) $T_5 = min\{n \geq 10 : X_n = X_{n-1}\}$: Yes

- This is a stopping time since it requires to know just the previous state for T_5.

(f) $T_6 = min\{n \geq 1 : X_n = X_5\}$: No

- Similar to (b), we need to know the future state X_5, when the current state is $1 \leq n \leq 5$. Hence it is not a stopping time.

(g) $T_7 = 10$: Yes

- T_7 is given as 10. We know when and what occurs. Hence T_7 is a stopping time.

Question 5

Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. (No need to give reasons here, but draw the transition graphs.)

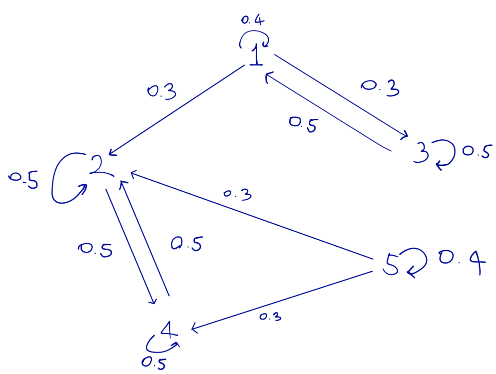
(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

(c)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.3	.4	0
4	0	.6	0	.4	0
5	.3	0	0	0	.7

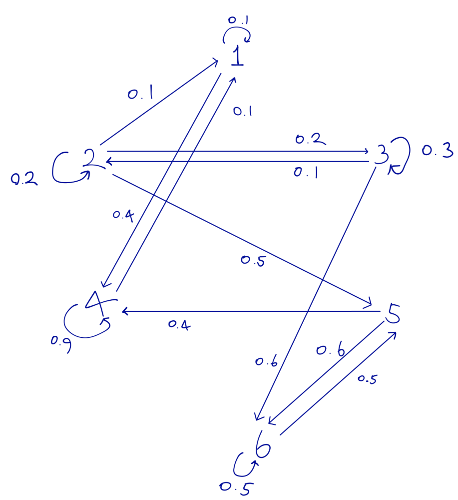
(d)	**1**	**2**	**3**	**4**	**5**	**6**
1	.8	0	0	.2	0	0
2	0	.5	0	0	.5	0
3	0	0	.3	.4	.3	0
4	.1	0	0	.9	0	0
5	0	.2	0	0	.8	0
6	.7	0	0	.3	0	0

(a)



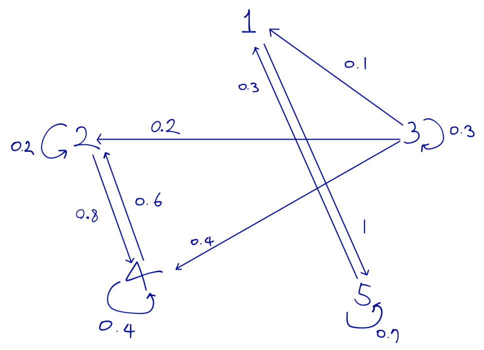
- Transient Set: $\{1\}, \{3\}, \{5\}$
- Irreducible Closed Set: $\{2, 4\}$

(b)



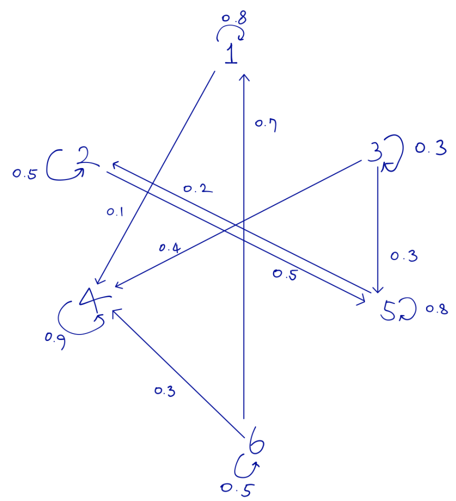
- Transient Sets: $\{2\}, \{3\}$
- Irreducible Closed Sets: $\{1, 4, 5, 6\}$

(c)



- Transient Sets: $\{3\}$
- Irreducible Closed Sets: $\{1, 5\}, \{2, 4\}$

(d)



- Transient Sets: $\{3\}, \{6\}$
- Irreducible Closed Sets: $\{2, 5\}, \{1, 4\}$

Question 6

Consider a discrete time Markov chain with state space $\mathbb{Z} = \{..., 2, 1, 0, 1, 2, ...\}$ and one step transition probabilities given by

$$p(i, i+1) = p \qquad \text{for all } i \in \mathbb{Z}$$

$$p(i, i-1) = q \qquad \text{for all } i \in \mathbb{Z}$$

and zero otherwise, with $p, q \geq 0$ and $p + q = 1$. (This is simple random walk.)

(a) Assume that $p = 0$. Find all the closed sets.

Since $p = 0 \Leftrightarrow q = 1$, it goes to only backwards. Hence the closed set is

$$A_j = \{i \leq j \mid \forall i \in \mathbb{Z}\} \text{ for all integer } j.$$

(b) Assume that $p = 0$. Find all the irreducible sets.

We know that $i \to j$ if and only if $j \leq i$. Hence, no irreducible set can contain two states $i < j$, because we have $i \to j$. It follows that irreducible sets have at most one state. Besides, all sets of the form$\{i\}$ for some $i \in \mathbb{Z}$ are irreducible, because we always have $i \to i$. These sets are therefore all the irreducible sets of the chain for $p = 0$.

(c) Now assume that $p, q > 0$. Prove this statement: either all the states are recurrent or all the states are transient.

- Suppose $\exists\, i \in \mathbb{Z}$. Then obviously i communicates with j such that $j = i+1$ when $p(i, j) = p$.
- In case of $p(j, i) = q$, j communicates with i backwards.
- In other words, the state i is recurrent.
- By Lemma 1.9 on Durrett, there states that if x is recurrent, and $x \to y$, the y is recurrent.
- Thus $\forall$ state $i \in \mathbb{Z}$ are recurrent.

Question 7
Let $\{X_n\}_{n\geq 0}$ be a Markov Chain with transition probability $\{p(x,y)\}_{x,y\in\mathbb{S}}$ with some countable state space $\mathbb{S}$. That is , the process satisfies

$$P(X_{n+1} = y \mid X_n = x_n, ..., X_0 = x_0) = p(x_n, y) \tag{1}$$

for all states $x_0, ..., x_n, y$ such that the conditioning event has positive probability.

(a) Using (1) (and general properties of probability and conditional probability), show that for any $0 < k \leq n$,

$$P(X_{n+1} = y \mid X_n = x_n, ..., X_k = x_k) = p(x_n, y) \tag{2}$$

whenever the conditioning event has positive probability.

$$P(X_{n+1} = y \mid X_n = x_n, ..., X_k = x_k) = \frac{P(X_{n+1} = y, X_n = x_n, ..., X_k = x_k)}{P(X_n = x_n, ..., X_k = x_k)}$$

$$= \frac{P(X_{n+1} = y|X_n = x_n) \cdot P(X_n = x_n|X_{n-1} = x_{n-1}) \dots P(X_{k+1} = x_{k+1}|X_k = x_k) \cdot P(X_k = x_k)}{P(X_n = x_n|X_{n-1} = x_{n-1}) \dots P(X_{k+1} = x_{k+1}|X_k = x_k) \cdot P(X_k = x_k)}$$

$$= \frac{P(X_{n+1} = y|X_n = x_n) \cancel{P(X_n = x_n|X_{n-1} = x_{n-1})} \dots \cancel{P(X_{k+1} = x_{k+1}|X_k = x_k)}\, \cancel{P(X_k = x_k)}}{\cancel{P(X_n = x_n|X_{n-1} = x_{n-1})} \dots \cancel{P(X_{k+1} = x_{k+1}|X_k = x_k)} \cdot \cancel{P(X_k = x_k)}}$$

$$= P(X_{n+1} = y|X_n = x_n)$$

(b) Using (2), show that

$$P(X_{n-1} = x, X_{n+1} = z \mid X_n = y) = P(X_{n-1} = x \mid X_n = y) \cdot P(X_{n+1} = z \mid X_n = y)$$

for all states x, y, z such that $P(X_n = y) > 0$. This is a special case of the statement that for a Markov chain, *given the present, the past and the future are independent.*

$$P(X_{n-1} = x, X_{n+1} = z \mid X_n = y) = \frac{P(X_{n-1} = x, X_n = y, X_{n+1} = z)}{P(X_n = y)}$$

$$= \frac{P(X_{n+1} = z|X_n = y) \cdot P(X_n = y|X_{n-1} = x) \cdot P(X_{n-1} = x)}{P(X_n = y)}$$

$$= \frac{P(X_{n+1} = z, X_n = y)}{P(X_n = y)} \cdot \frac{P(X_n = y, X_{n-1} = x)}{P(X_{n-1} = x)} \cdot P(X_{n-1} = x) \cdot \frac{1}{P(X_n = y)}$$

$$= \frac{P(X_{n+1} = z, X_n = y)}{P(X_n = y)} \cdot \frac{P(X_n = y, X_{n-1} = x)}{\cancel{P(X_{n-1} = x)}} \cdot \cancel{P(X_{n-1} = x)} \cdot \frac{1}{P(X_n = y)}$$

$$= \frac{P(X_{n+1} = z, X_n = y)}{P(X_n = y)} \cdot \frac{P(X_n = y, X_{n-1} = x)}{P(X_n = y)} = P(X_{n+1=x}|X_n = y) \cdot P(X_{n-1} = x|X_n = y)$$