Question 1

Consider the discrete time Markov chain with the state space $S = \{1, 2, 3, 4, 5\}$ and the transition matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0 & 0.2 & 0.4 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Let $N(y)$ be the total number of visits to the state y for all time, excluding a possible visit at time *zero*. In symbols,

$$N(y) = \sum_{n=1}^{\infty} 1_{(X_n = y)}.$$

$E_1[N(3)] = 1 + E_3[N(3)]$

$E_2[N(3)]$

- $= 0.2 \cdot E_1[N(3)] + 0.4 \cdot (E_3[N(3)] + 1)$
- $= 0.2 \cdot (1 + E_3[N(3)]) + 0.4 \cdot (E_3[N(3)] + 1)$
- $= 0.6 \cdot E_3[N(3)] + 0.6$
- $= 0.6 \, (E_3[N(3)] + 1)$

$E_3[N(3)]$

- $= 0.3 \cdot E_1[N(3)] + 0.1 \cdot E_2[N(3)]$
- $= 0.3 \cdot (1 + E_3[N(3)]) + 0.1 \cdot 0.6 \, (1 + E_3[N(3)])$
- $= 0.3 + 0.3 \cdot E_3[N(3)] + 0.06 \cdot E_3[N(3)] + 0.06$
- $= 0.36 \, (1 + E_3[N(3)])$
- $\Leftrightarrow 64 E_3[N(3)] = 36 \Leftrightarrow E_3[N(3)] = 0.5625$

Hence
$$E_2[N(3)] = 0.6(0.5625 + 1) = 0.9375$$

$$E_1[N(3)] = 1.5625$$

In case of $E_4[N(4)]$, it goes to ∞ since $\{4, 5\}$ is an irreducible closed set and also null recurrent having the probability of $1/2$.

Question 2

Success run chain. Imagine a never ending succession of free throws with a basketball. Each throw is independently either a basket with probability α or a miss with probability $1 - \alpha$. Assume $0 < \alpha < 1$. After the nth throw, let X_n be the number of consecutive baskets since the last miss, in other words the length of the current success run. X_n is called the *success run* chain. For example, if throw 7 is a miss, throws 8, 9, and 10 are baskets, and throw 11 again a miss, then $X_7 = 0, X_8 = 1, X_9 = 2, X_{10} = 3$, and $X_{11} = 0$. Thus every basket increases X_n by 1, and every miss sends X_n back to 0. X_n is a Markov chain with state space $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, ...\}$.

(a) Draw the transition graph of the success run chain and give its transition probability. Classify states according to recurrence/transience. (Note that the state space is infinite so recurrence/transience is not a trivial matter.)
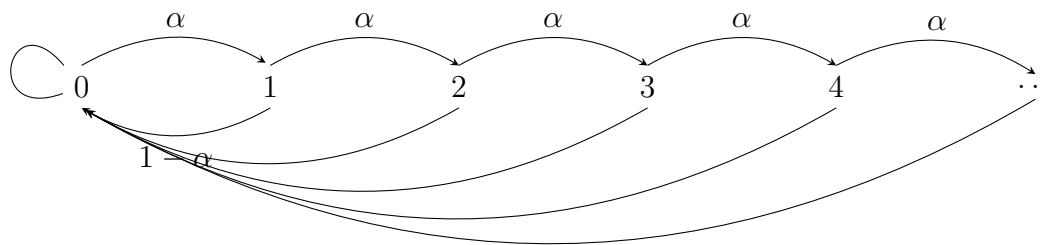
Since the chain is irreducible, all states are recurrent.

(b) Find the invariant distribution or determine that one does not exist. If you propose an invariant distribution, check that the probabilities add up to 1. Is it a familiar probability distribution with a name?

$$
\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \dots \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \dots \end{pmatrix} \begin{pmatrix} 1-\alpha & \alpha & 0 & 0 & \dots \\ 1-\alpha & 0 & \alpha & 0 & \dots \\ 1-\alpha & 0 & 0 & \alpha & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

- $\pi_0 = \sum_{k=0}^{n}(1-\alpha)\pi_k = (1-\alpha)\pi_0 + \sum_{k=1}^{n}(1-\alpha)\pi_k$
- If there exists an invariant distribution, $\sum_{k=0}^{n}(1-\alpha)\pi_k = 1$.
- i.e. $\sum_{k=1}^{n}(1-\alpha)\pi_k = 1 - \pi_0$
- So $\pi_0 = \sum_{k=0}^{n}(1-\alpha)\pi_k = (1-\alpha)\pi_0 + (1-\pi_0) = 1 - \alpha$

Further

(i) $\pi_1 = \alpha\pi_0 = \alpha(1-\alpha)$ (ii) $\pi_2 = \alpha(\alpha(1-\alpha)) = \alpha^2(1-\alpha)$

Hence we can deduce $\pi_k = \alpha^k(1-\alpha)$ by induction.

Thus we can see that the sum of all π_k goes to 1.

$$
\sum_{k=0}^{n} \alpha^k(1-\alpha) = \frac{1}{1-\alpha} \cdot (1-\alpha) = 1
$$

(c) Change the model slightly. Let $0 < \beta < 1$ be another parameter and suppose that immediately after a miss, the next throw succeeds with probability β. But as before, after a successful free throw, the next shot succeeds with probability α. Find again the invariant distribution π. Find also all the probabilities $P_0(T_0 = k)$ for positive integers k. Using these probabilities compute the expectation $E_0[T_0]$. Check that your answers satisfy the identity $1/\pi(0) = E_0[T_0]$ as the theory states.

Hint. You may need to evaluate a series of the type $\sum_{k=1}^{\infty} kx^{k-1}$ for $|x| < 1$. There are several ways to do this. One goes by $\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{k} x^{k-1}$ and switches the order of summation. Another observes that $\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} \frac{d}{dx} x^k = \frac{d}{dx} (\sum_{k=0}^{\infty} x^k)$, evaluates the series and then differentiates.

$$
\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \ldots \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \ldots \end{pmatrix} \begin{pmatrix} 1-\beta & \beta & 0 & 0 & 0 & \ldots \\ 1-\alpha & 0 & \alpha & 0 & 0 & \ldots \\ 1-\alpha & 0 & 0 & \alpha & 0 & \ldots \\ 1-\alpha & 0 & 0 & 0 & \alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

- $\pi_0 = (1-\beta)\pi_0 + (1-\alpha)\pi_1 + (1-\alpha)\pi_2 + \ldots = (1-\beta)\pi_0 + \sum_{k=1}^{n}(1-\alpha)\pi_k$
  $= (1-\beta)\pi_0 + (1-\alpha)(1-\pi_0)$

- $\cancel{\pi_0} = \cancel{\pi_0} - \beta\pi_0 + 1 - \pi_0 - \alpha + \alpha\pi_0 \Leftrightarrow \pi_0 = \frac{1-\alpha}{1-\alpha+\beta}$

- $\pi_1 = \beta\pi_0 = \frac{\beta(1-\alpha)}{1-\alpha+\beta}$

- $\pi_2 = \beta\pi_1 = \frac{\alpha\beta(1-\alpha)}{1-\alpha+\beta}$

  $\vdots$

- $\pi_k = \alpha\pi_{k=1} = \frac{\alpha^{k-1}\beta(1-\alpha)}{1-\alpha+\beta}$

Hence

$$
\sum_{k=0}^{n} \pi_k = \frac{1-\alpha}{1-\alpha+\beta} + \frac{\beta(1-\alpha)}{1-\alpha+\beta} + \frac{\alpha\beta(1-\alpha)}{1-\alpha+\beta} + \frac{\alpha^2\beta(1-\alpha)}{1-\alpha+\beta} + \ldots
$$

$$
= \frac{1-\alpha}{1-\alpha+\beta} + \sum_{k=1}^{n} \frac{\alpha^{k-1}\beta(1-\alpha)}{1-\alpha+\beta} = \frac{1-\alpha}{1-\alpha+\beta} + \frac{\beta}{1-\alpha+\beta} \cdot \sum_{k=1}^{n} \alpha^{k-1}(1-\alpha)
$$

Since we proved $\sum_{k=1}^{n} \alpha^{k-1}(1-\alpha) = 1$ in (b),

$$
\frac{1-\alpha}{1-\alpha+\beta} + \frac{\beta}{1-\alpha+\beta} = \frac{1-\alpha+\beta}{1-\alpha+\beta} = 1
$$

(d) Imagine that the chain of part (c) has been running for a very long time. What is the probability that the next two shots succeed?

Hint. For a sanity check of your answer, see that if $\beta = \alpha$ your answer is consistent with the initial description of the free throws.

$$
\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \ldots \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \ldots \end{pmatrix} \begin{pmatrix} 0 & \beta & 0 & 0 & 0 & \ldots \\ 0 & 0 & \alpha & 0 & 0 & \ldots \\ 0 & 0 & 0 & \alpha & 0 & \ldots \\ 0 & 0 & 0 & 0 & \alpha & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

- $\pi_0 \cdot \beta \cdot \alpha$
- $\pi_1 \cdot \alpha \cdot \alpha$
- $\pi_2 \cdot \alpha \cdot \alpha$
- $\pi_3 \cdot \alpha \cdot \alpha$
- $\vdots$
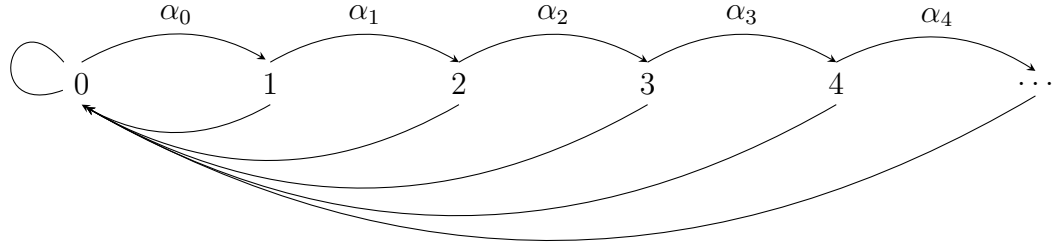- $\pi_k \cdot \alpha \cdot \alpha$

$$
P = \pi_0 \cdot \beta \cdot \alpha + \pi_1 \cdot \alpha \cdot \alpha + \pi_2 \cdot \alpha \cdot \alpha + \ldots + \pi_k \cdot \alpha \cdot \alpha
$$

$$
= \alpha\beta\pi_0 + \alpha^2 \sum_{k=1}^{n} \pi_k = \alpha\beta \frac{1-\alpha}{1-\alpha+\beta} + \alpha^2 \sum_{k=1}^{n} \frac{\alpha^{k-1}\beta(1-\alpha)}{1-\alpha+\beta}
$$

$$
= \alpha\beta \frac{1-\alpha}{1-\alpha+\beta} + \frac{\alpha^2\beta}{1-\alpha+\beta} \cdot \sum_{k=1}^{n} \alpha^{k-1}(1-\alpha)
$$

$$
= \alpha\beta \frac{1-\alpha}{1-\alpha+\beta} + \frac{\alpha^2\beta}{1-\alpha+\beta} = \frac{\alpha\beta - \alpha^2\beta}{1-\alpha+\beta} + \frac{\alpha^2\beta}{1-\alpha+\beta} = \frac{\alpha\beta}{1-\alpha+\beta}
$$

(e) Now suppose that we have parameters $\alpha_0, \alpha_1, \alpha_2, \ldots$ all strictly between 0 and 1, and we specify that, after k consecutive successes, the next shot succeeds with probability α_k. That is, now the transition probabilities are $p(k, k+1) = \alpha_k$, $p(k, 0) = 1 - \alpha_k$. Is it possible to choose the numbers $\alpha_k \in (0, 1)$ so that the MC is transient?

- Due to its transiency, $p_{00} < 1$, so $P_0(T_0 = \infty) = \prod_{k=0}^{\infty} \alpha_i > 0$.
- Choose α_k such that Markov Chain is transient: $\alpha_k = 1 - \frac{1}{2^k}$.
- Then

$$\lim_{n \to \infty} \prod_{k=0}^{\infty} \left(1 - \frac{1}{2^k}\right) = \lim_{n \to \infty} \prod_{k=0}^{\infty} \left(1 - \frac{1}{2^k}\right)^{\log e = 1} = \lim_{n \to \infty} e^{\log \prod_{k=0}^{\infty} \left(1 - \frac{1}{2^k}\right)}$$

$$= \lim_{n \to \infty} e^{\log \sum_{k=0}^{\infty} \left(1 - \frac{1}{2^k}\right)} \geq \lim_{n \to \infty} e^{\log \sum_{k=0}^{\infty} -\frac{1}{2^k}} > 0$$
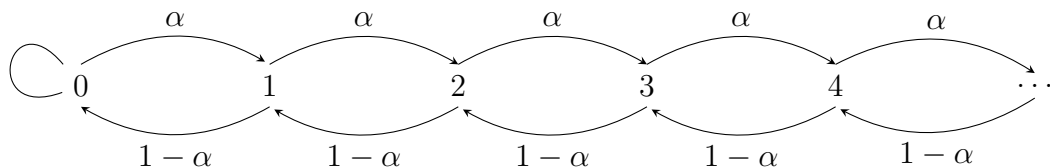
Thus we proved that there exists such $\alpha_k \in (0, 1)$ that the Markov Chain is transient.

Question 3
Reflected Random Walk. Fix a parameter $0 < \alpha < 1$. Consider the Markov chain with state space $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, ...\}$ and transition probability.

$$p(0,0) = 1 - \alpha, p(x, x+1) = \alpha \text{ for } x \geq 0, \text{ and } p(x, x-1) = 1 - \alpha \text{ for } x \geq 1.$$

(a) Specify for which values of α there exists an invariant distribution.
Find the invariant distribution for those values of α for which it exists.

$$\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \dots \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \dots \end{pmatrix} \begin{pmatrix} 1-\alpha & \alpha & 0 & 0 & \dots \\ 1-\alpha & 0 & \alpha & 0 & \dots \\ 0 & 1-\alpha & 0 & \alpha & \dots \\ 0 & 0 & 1-\alpha & 0 & \dots \\ 0 & 0 & 0 & 1-\alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(i) $\pi_0 = (1-\alpha)\pi_0 + (1-\alpha)\pi_1$
$= \pi_0 - \alpha\pi_0 + \pi_1 - \alpha\pi_1$
$\Leftrightarrow \pi_0 = \frac{(1-\alpha)\pi_1}{\alpha}$

(ii) $\pi_1 = \alpha\pi_0 + (1-\alpha)\pi_2$
$= \alpha \cdot \frac{(1-\alpha)\pi_1}{\alpha} + (1-\alpha)\pi_2$
$\Leftrightarrow \pi_1(1 - (1-\alpha)) = (1-\alpha)\pi_2$
$\Leftrightarrow \pi_1 = \frac{1-\alpha}{\alpha}\pi_2$

By induction, we obtain $\pi_n = \frac{1-\alpha}{\alpha}\pi_{n+1} = (\frac{\alpha}{1-\alpha})^n \pi_0$.

Hence

$$\sum_{n>0}^{\infty} \left(\frac{\alpha}{1-\alpha}\right)^n \pi_0 = \frac{\pi_0}{1 - \frac{\alpha}{1-\alpha}} = \frac{\pi_0}{\frac{1-2\alpha}{1-\alpha}} = \frac{(1-\alpha)\pi_0}{1-2\alpha} = 1$$

$$\Leftrightarrow \pi_0 = \frac{1-2\alpha}{1-\alpha} \Rightarrow \pi_n = \left(\frac{\alpha}{1-\alpha}\right)^n \left(\frac{1-2\alpha}{1-\alpha}\right)$$

(b) For which values of α is the MC recurrent and for which values transient? Here you probably cannot give a rigorous proof, but you should be able to give an intuitively plausible argument. Note that when you are away from the origin, the reflected random walk behaves just like ordinary random walk.

   (i) $\alpha < 1/2$
     As α gets smaller, it is more probable to come back to 0. However this recurrence is finite, so we can consider this to be a positive recurrent case.

  (ii) $\alpha = 1/2$
     In case of $\alpha = \frac{1}{2}$, recurrence will be occurred infinitely many times (null-recurrence). It is always coming back to 0.
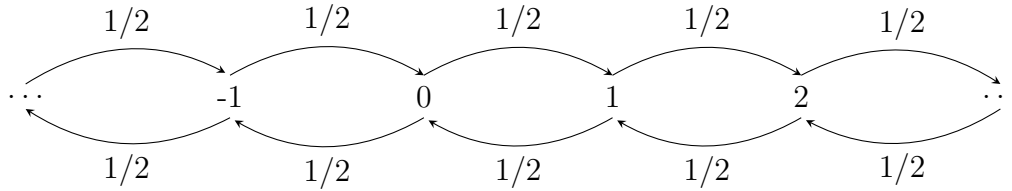
 (iii) $\alpha > 1/2$
     As α gets larger, it is more probable to go forward, which is transient.

Question 4
Symmetric Simple Random Walk. Consider the Markov chain with state space
$S = \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. The transition probabilities for this chain are

$$p(j, j - 1) = \frac{1}{2}, p(j, j + 1) = \frac{1}{2}$$

for all $j \in S$. Find all the stationary measures and stationary distributions for this
Markov chain.

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & ... \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & ... \end{pmatrix} \begin{pmatrix} ... & ... & ... & ... & ... & ... & \vdots \\ \vdots & \ddots & ... & ... & ... & ... & \vdots \\ \vdots & ... & 1/2 & 0 & 1/2 & ... & ... & \vdots \\ \vdots & ... & ... & 1/2 & 0 & 1/2 & ... & \vdots \\ \vdots & ... & ... & ... & 1/2 & 0 & 1/2 & \vdots \\ \vdots & ... & ... & ... & ... & \ddots & 0 & \vdots \\ \vdots & ... & ... & ... & ... & ... & \ddots & \vdots \end{pmatrix}$$

$$\Rightarrow \pi_k = \frac{1}{2}\pi_{k-1} + \frac{1}{2}\pi_{k+1} \Leftrightarrow 2\pi_k = \pi_{k-1} + \pi_{k+1} \Leftrightarrow \pi_{k+1} - \pi_k = \pi_k - \pi_{k-1}$$

Since this is an arithmetic sequence which has same difference between adjacent terms,
π_k can be expressed as $\pi_k = a + bk$.

Since π_k should be non-negative, $b = 0$. However, then, $\sum_{k=0}^{n} \pi_k$ does not have to be 0.
Thus it diverges and there does not exist a stationary distribution for the symmetric
simple random walk.

Exercise 1.13 Consider the Markov chain with transition matrix:

$$
\begin{array}{ccccc}
 & 1 & 2 & 3 & 4 \\
1 & 0 & 0 & 0.1 & 0.9 \\
2 & 0 & 0 & 0.6 & 0.4 \\
3 & 0.8 & 0.2 & 0 & 0 \\
4 & 0.4 & 0.6 & 0 & 0
\end{array}
$$

(a) Compute p^2.

$$
p^2 = \begin{bmatrix} 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.6 & 0.4 \\ 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.6 & 0.4 \\ 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}
$$

(b) Find the stationary distributions of p and all of the stationary distributions of p^2.

- The stationary distribution of p.

$$
\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.6 & 0.4 \\ 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}
$$

(i) $0.8\pi_3 + 0.4\pi_4 = \pi_1$ (ii) $0.2\pi_3 + 0.6\pi_4 = \pi_2$

(iii) $0.1\pi_1 + 0.6\pi_2 = \pi_3$ (iv) $0.9\pi_1 + 0.4\pi_2 = \pi_4$

Hence, we obtain following equations.

$$
\pi_1 + \pi_2 = \pi_3 + \pi_4 = \frac{1}{2}
$$

Then (ii) + (iii)

$$
\pi_2 + \pi_3 = 0.1\pi_1 + 0.6\pi_2 + 0.2\pi_3 + 0.6\pi_4
$$

$$
0.4\pi_2 + 0.8\pi_3 = 0.1\pi_1 + 0.6\pi_4
$$

$$
0.4(0.2\pi_3 + 0.6\pi_4) + 0.8\pi_3 = 0.1(0.8\pi_3 + 0.4\pi_4) + 0.6\pi_4
$$

$$
\cancel{0.08\pi_3} + 0.24\pi_4 + 0.8\pi_3 = \cancel{0.08\pi_3} + 0.04\pi_4 + 0.6\pi_4
$$

$$
0.24\pi_4 + 0.8\pi_3 = 0.64\pi_4 \Leftrightarrow 0.8\pi_3 = 0.4\pi_4 \Leftrightarrow 2\pi_3 = \pi_4
$$

Plugging $2\pi_3 = \pi_4$ into (i) and (ii),

$$
0.8\pi_3 + 0.8\pi_3 = 1.6\pi_3 = \pi_1 \qquad 0.2\pi_3 + 1.2\pi_3 = 1.4\pi_3 = \pi_2
$$

Thus the stationary distribution of p is

$$
\pi = \begin{bmatrix} \dfrac{4}{15} & \dfrac{7}{30} & \dfrac{1}{6} & \dfrac{1}{3} \end{bmatrix}
$$

- The stationary distribution of p^2.

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix} \cdot \begin{bmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}$$

(i)　$0.44\pi_1 + 0.64\pi_2 = \pi_1$　　　　　　　(ii)　$0.56\pi_1 + 0.36\pi_2 = \pi_2$

(iii)　$0.2\pi_3 + 0.4\pi_4 = \pi_3$　　　　　　　(iv)　$0.8\pi_3 + 0.6\pi_4 = \pi_4$

Hence, we obtain following equations.

$$0.56\pi_1 = 0.64\pi_2 \iff \pi_2 = \frac{7}{8}\pi_1 \qquad 0.8\pi_3 = 0.4\pi_4 \iff \pi_3 = \frac{1}{2}\pi_4$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = \frac{15}{8}\pi_1 + 3\pi_3 = 1$$

Thus the stationary distribution is

$$\pi = \begin{bmatrix} \pi_1 & \frac{7}{8}\pi_1 & \dfrac{1 - \frac{15}{8}\pi_1}{3} & \dfrac{2(1 - \frac{15}{8}\pi_1)}{3} \end{bmatrix}$$

(c)　Find the limit of $p^{2n}(x, x)$ as $n \to \infty$.

To find the limit, we have to use the irreducible recurrent classes.
For states $\{1, 2\}$ in the upper-left part of the matrix,

- $\lim\limits_{n \to \infty} p^{2n}(1, 1) = \pi^1(1) = 8/15$

- $\lim\limits_{n \to \infty} p^{2n}(2, 2) = \pi^1(2) = 7/15$

For states $\{1, 2\}$ in the lower-right part of the matrix,

- $\lim\limits_{n \to \infty} p^{2n}(3, 3) = \pi^2(3) = 1/3$

- $\lim\limits_{n \to \infty} p^{2n}(4, 4) = \pi^2(4) = 2/3$

```
>>> p2                              >>> p4                                >>> p6
array([[0.44, 0.56, 0.  , 0.  ],    array([[0.552, 0.448, 0.  , 0.  ],    array([[0.5296, 0.4704, 0.    , 0.    ],
       [0.64, 0.36, 0.  , 0.  ],           [0.512, 0.488, 0.  , 0.  ],           [0.5376, 0.4624, 0.    , 0.    ],
       [0.  , 0.  , 0.2 , 0.8 ],           [0.  , 0.  , 0.36 , 0.64 ],           [0.    , 0.    , 0.328 , 0.672 ],
       [0.  , 0.  , 0.4 , 0.6 ]])          [0.  , 0.  , 0.32 , 0.68 ]])          [0.    , 0.    , 0.336 , 0.664 ]])


>>> p8                                    >>> p12
array([[0.53408, 0.46592, 0.    , 0.    ],  array([[0.5333632, 0.4666368, 0.        , 0.        ],
       [0.53248, 0.46752, 0.    , 0.    ],         [0.5332992, 0.4667008, 0.        , 0.        ],
       [0.    , 0.    , 0.3344 , 0.6656 ],         [0.        , 0.        , 0.333376 , 0.666624 ],
       [0.    , 0.    , 0.3328 , 0.6672 ]])        [0.        , 0.        , 0.333312 , 0.666688 ]])
```

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Calculating the matrix multiplication through Python ($\text{pi} = p^i$ where $i = \mathbb{Z}_{2,4,6,...}$) also found that, as $n \to \infty$, $p^{2n}(x,x)$ converges to

$$
\begin{bmatrix}
8/15 & 7/15 & 0 & 0 \\
8/15 & 7/15 & 0 & 0 \\
0 & 0 & 1/3 & 2/3 \\
0 & 0 & 1/3 & 2/3
\end{bmatrix}
$$

Exercise 1.32 The weather in a certain town is classified as rainy, cloudy, or sunny and changes according to the following transition probability is

$$
\begin{array}{c|ccc}
 & \mathbf{R} & \mathbf{C} & \mathbf{S} \\
\mathbf{R} & 1/2 & 1/4 & 1/4 \\
\mathbf{C} & 1/4 & 1/2 & 1/4 \\
\mathbf{S} & 1/2 & 1/2 & 0
\end{array}
$$

In the long run what proportion of days in this town are rainy? cloudy? sunny?

This chain is aperiodic, closed irreducible. i.e. $p^n(x,y) \xrightarrow[n \to \infty]{} \pi(y) \ \forall \ x, y \in S$

Hence the convergence theorem is needed to find its stationary distribution.

$$
\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}
\begin{pmatrix}
1/2 & 1/4 & 1/4 \\
1/4 & 1/2 & 1/4 \\
1/2 & 1/2 & 0
\end{pmatrix}
= \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}
$$

$$\frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 = \pi_1 \tag{1}$$

$$\frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 = \pi_2 \tag{2}$$

$$\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 \qquad\quad = \pi_3 \tag{3}$$

Plugging (3) into (2)

$$
\frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\left(\frac{1}{4}(\pi_1 + \pi_2)\right) = \left(\frac{1}{2} + \frac{1}{8}\right)\pi_1 + \left(\frac{1}{4} + \frac{1}{8}\right)\pi_2
$$

$$
= \frac{5}{8}\pi_1 + \frac{3}{8}\pi_2 = \pi_2 \Leftrightarrow \pi_1 = \pi_2
$$

Hence in (3),

$$
\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 = \frac{1}{2}\pi_1 = \pi_3 \Leftrightarrow \pi_1 = 2\pi_3
$$

Finally

$$
\pi_1 + \pi_2 + \pi_3 = 1 \Leftrightarrow 2\pi_3 + 2\pi_3 + \pi_3 = 5\pi_3 = 1
$$

$$
\Leftrightarrow \pi_3 = \frac{1}{5}, \text{ so } \pi_1 = \pi_2 = \frac{2}{5}
$$

$$
\pi = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}
$$