Modelling Complex Systems: Project Sheet 2

April 17, 2018

The deadline for this first exercise sheet is midnight Monday 7th of May.

Please submit hand-ins on Studentportalen. All code should be submitted as an appendix and not as part of the answer to the hand-ins. Please feel free to submit videos illustrating your results where appropriate, via Studentportalen or uploaded elsewhere.

There will be no extensions given, apart from for medical or genuine personal reasons.

1 Population dynamics

This exercise will be covered in the lab session on Thursday the 12th of April. Please start to answer the questions before you come to the lab session.

In this project we implement a stochastic model of population dynamics. Imagine an environment consisting of n resource sites and a population of X_t individuals. We assign individuals to sites uniformly at random, so each site has an equal probability of being chosen. We then apply the following rule. If exactly **two** individuals land on the same site they produce b offspring. These new individuals pass to the next generation. If three or more individuals occupy the same site then they all die and produce no offspring. Empty sites or sites containing just one individual also produce no offspring (i.e. the individual dies without reproducing). The new population A_{t+1} now consists of all the surviving offspring and which now choose sites at random and repeat the procedure assigning individuals completely at random between sites.

1. Implement the above model and show how the population changes through time for $b = 5$, $b = 10$, $b = 20$, $b = 30$ and $b = 40$, with $n = 1000$ and an initial population $A_0 = n$. What happens for the same parameter values when you set A_0 to be small? By running the simulation for different values of b within a range [1:1:50] and $n = 1000$ draw a phase transition diagram for your model. **(3 points)**

2. Derive a mean-field equation for your model. Show that the steady state $x_* = 0$ is always stable. Find conditions in terms of b for the existence of two further non-zero steady states. **(3 points)**

3. Calculate (numerically) and plot the Lyapunov exponent for your mean-field equation as a function of b. Identify the value of b at which the populations become chaotic. Explain why in your simulations from question 1 the populations become extinct for large b. **(4 points)**

2 Groups of friends

This exercise will be covered in the lab session on Friday the 20th of April. Please start to answer the questions before you come to the lab session.

Consider the following model of how students make friends. The world initially consists of N friends, and each group contains only one lonely individual. Then on each time step of the model we pick one group at random

- If we pick a group containing only one student ($i = 1$), then he joins another group. The probability of choosing a particular new group increases with the number of students in the group, so that if the new group has k students then the probability he joins that group is proportional to k/N (the student can also again pick the group consisting of himself!). This can be thought of a tendency of lone students to want to join larger groups.

- If we pick a group containing more than one student ($i > 1$), then with probability ri the group splits and all students are placed alone in groups of size one. This can be thought of as the probability of falling out with each other increasing with group size. With probability $1 - ri$ the group does not split.

We are interested in the distribution of friend group sizes arising from this model.

1. Implement a simulation model of the above description. Simulate the model with $N = 100$ individuals. First investigate $r = 0.01$. Make a log-log plot of group size vs frequency. To do this properly you should run the simulation at least 20 times for a sufficiently large number of time steps and make a histogram over a combination of all of the final distributions. Investigate and plot how the group size distribution changes with r. **(4 points)**

2. Write down a Master equation for the model. Let $\pi(k, t)$ be the proportion of groups of size k and time t and write equations for transitions between proportions. You should write one equation down for the general case, $k > 1$ and a separate equation for $k = 1$. Solve the Master

equation to show that for $k > 1$

$$\pi(k) = \frac{Kc^k}{k}$$

gives a solution for the distribution when N is large. Find c and K expressed in terms of r, N and $\pi(1)$. For $r = 0.01$ plot a comparison of the distribution from your simulation and your solution to the Master equation. You can use your simulation to approximate $\pi(1)$ **(4 points)**

3 Network Epidemics

This exercise will be covered in the lab session on Friday the 20th of April. Please start to answer the questions before you come to the lab session.

1. Create a random undirected social network with 5000 individuals and a link density of 0.0016. That is, every pair of individuals has a 0.16% chance of being linked to each other. Plot a histogram of the degree distribution. What kind of distribution is this? What is the average degree of this network? **(2 points)**

2. Simulate the following process on the network. Start with 100 random infected individuals. Every day, an individual connected to n infected individuals becomes infected with probability

$$P_{infected}(n) = 1 - e^{(-pn)}$$

per day, where p is a constant. An infected individual recovers with constant probability $r = 0.03$ per day. Recovered individuals can become infected again. Plot the number of infected individuals over the first 1000 days of the epidemic, using $p = 0.01$. Then run the simulation for $p = 0.001, 0.002, \ldots, 0.01$. Plot the number of infected individuals against time for each of these values on the same graph. Finally, plot the number of infected individuals on day 1000 against r/p for as you vary p. **(4 points)**

3. Now create a social network using preferential attachment. Start with $n = 2$ individuals linked together. Now add individuals, $n > 2$, to the network one after another. Every new person initially links to just one other randomly chosen individual, such that, the probability of the newly added individual linking to individual i is proportional to the degree (number of existing links) of individual i (denoted as k_i). That is the probability of linking is equal to

$$\frac{k_i}{2(n-2)}$$

Try 100 people first to make sure your algorithm works, and then build a network of 5000 individuals. Plot a histogram of the degree distribution on a log-log scale. What kind of distribution is this? What is the average degree of the network? **(2 points)**

4. Write down a Master equation for the probability $\pi(k, t)$ that a randomly chosen individual has k friends after t time. Show that as $t \to \infty$, the proportion of individuals with degree k satisfies

$$\pi(k, t) = \frac{c}{k(k+1)(k+2)}$$

where you should find the value of the constant c. Compare this theoretical distribution to the one you got from the simulation. **(2 points)**

5. Repeat question 2 for this preferential attachment network. Make sure the initially infected individuals are chosen randomly. Describe how the equilibrium number of infected individuals depends on the infection probability p. How is this different to the random network? Discuss this result in the context of the spread of memes on the Internet. **(4 points)**

4 Flocks and Predators

This exercise will be covered in the lab session on Friday the 27th of April. Please start to answer the questions before you come to the lab session.

First run the `Align2D.m` model provided on studentportalen. This implements a Vicsek alignment model with a fixed radius of alignment. We now develop a version of the model where interactions are only with the n nearest neighbours. Note that in this exercise you do not need to assume periodic boundary conditions (which are difficult to implement).

1. Implement a new version of the model, where instead of a radius of interaction, the number of neighbours is fixed. Specifically, on each time step, each particle calculates the average direction of its n nearest neighbors irrespective of how far away they are. Use a total number of particles $N = 40$, domain size $L = 10$, and angular noise $e = 0.5$. Run this for $J = 200$ time steps, at each time calculating the global alignment as defined in the lecture. Plot the alignment as a function of time for $n = 4$. **(4 points)**

2. Add a very weak force that pulls the particles toward the centre of mass of *all* particles. The parameter value should be set so that the particles stay together but still move like a flock. With this force active, run this simulation for various values of noise e and number of neighbours n. Make a two dimensional heat map of how the steady-state alignment depends on these two parameters. **(4 points)**

3. Give 20 of the particles a large n and give another 20 particles a small n in the same simulation. Make up your own rules of motion for an extra particle that acts as a "predator". This predator should try to get close to your other particles, eating them if they come within a small distance (Tip: making a particle's co-ordinates and direction equal NaN [Not a Number] will make it disappear). It may be useful to simulate with different colours for the two types of prey and for the predator. Find out whether the particles with large n or or the ones with small n are consumed faster and explain why. **(6 points)**