Odd and Even Functions with their Fourier Series, Half range Expansion of Fourier Series.

Differential Equations Presentation

Group Members

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- Introduction to Odd and Even Functions

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- Introduction to Fourier Series and Fourier Transform

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- Fourier Series for Odd and Even Functions

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- Half-range expansion of Fourier Series

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- Comparison and Applications

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- Practical Examples

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- Conclusion

Introduction to Odd and Even Functions

Sayeed Tauheed Shah (14143)

Definitions

Even Functions

- A function f(x) is called **even** if, for all values of x in the domain of the function, the following condition holds

$$f(-x)=f(x)$$

Example: f(x) = x^2

Odd Functions

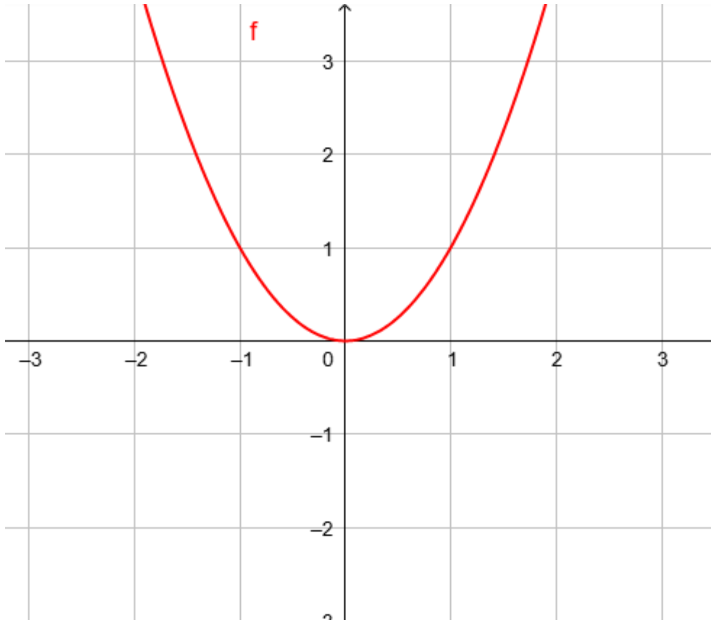
- A function f(x) is called **odd** if, for all values of x in the domain of the function, the following condition holds

$$f(-x)= -f(x)$$

Example: f(x) = x^3

Graphical Representations

Even Functions



Odd Functions

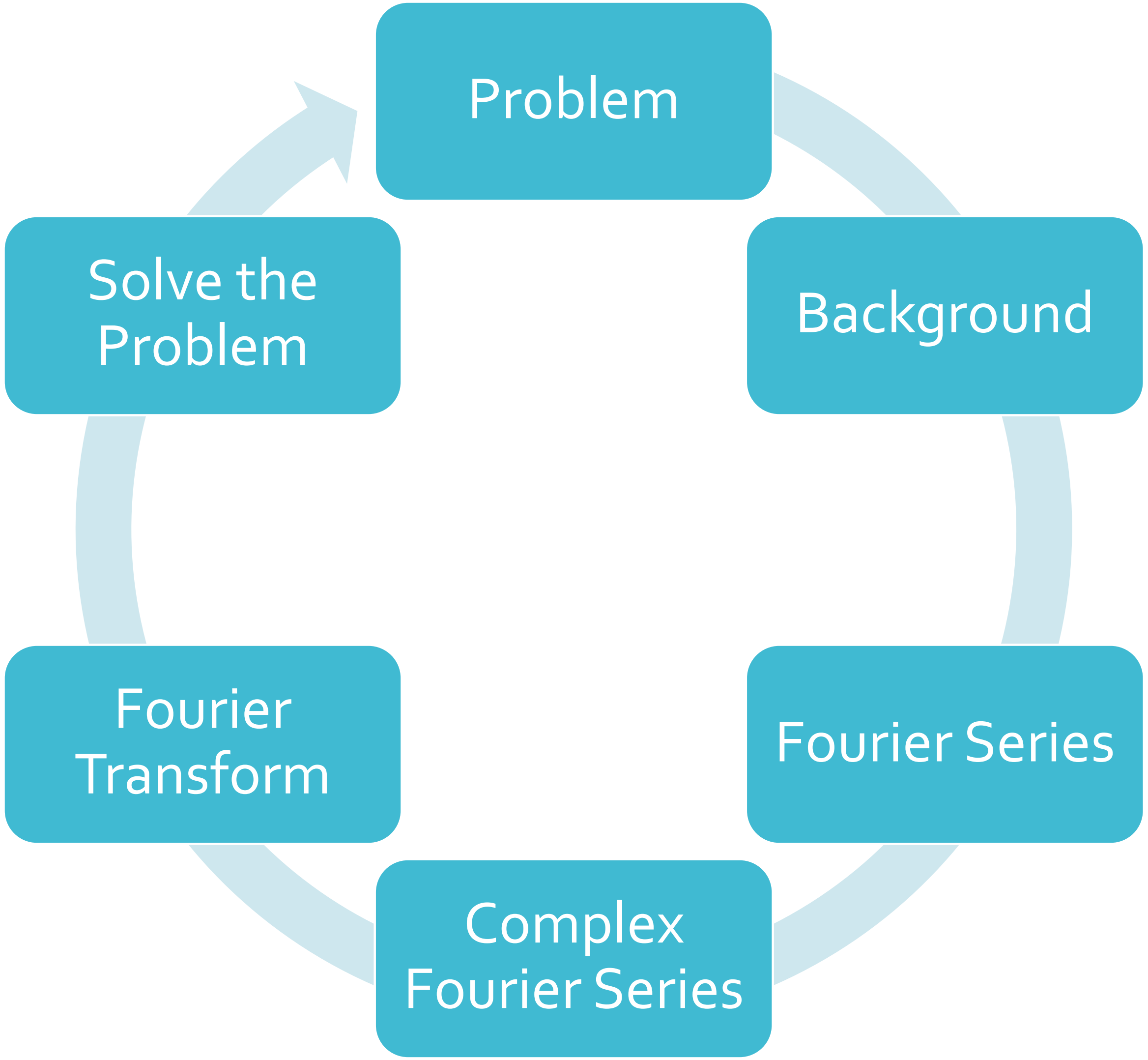
Key Properties

- Sum of Even Functions: The sum of two even functions is even.
- Sum of Odd Functions: The sum of two odd functions is odd.
- Sum of Even and Odd Functions: The sum of an even and an odd function is neither even nor odd in general.

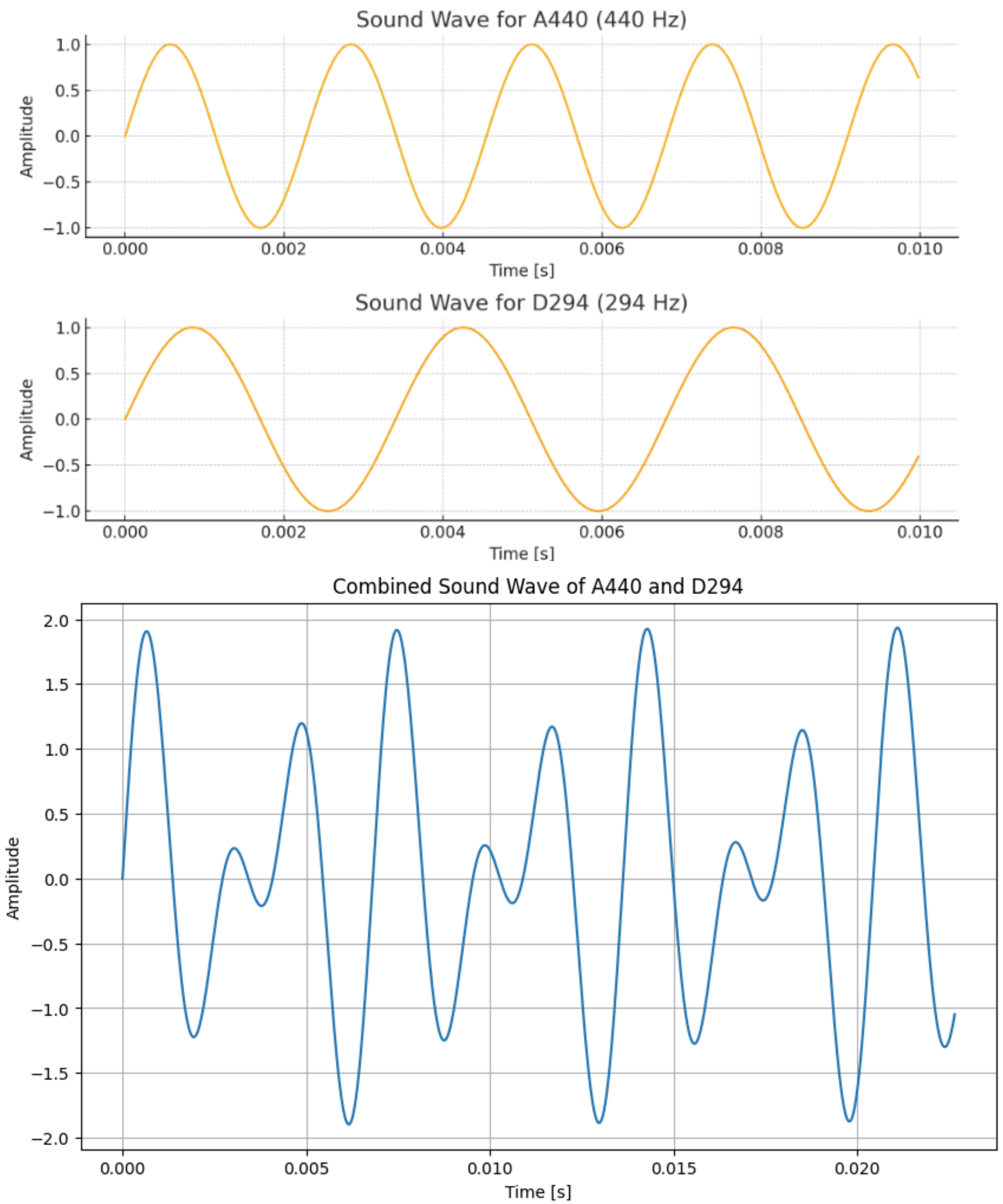
Introduction to Fourier Series and Fourier Transform

Taimoor Ul Islam (14031)

Flow


Problem
Background
Fourier Series
Complex Fourier Series
Fourier Transform
Solve the Problem

The Problem



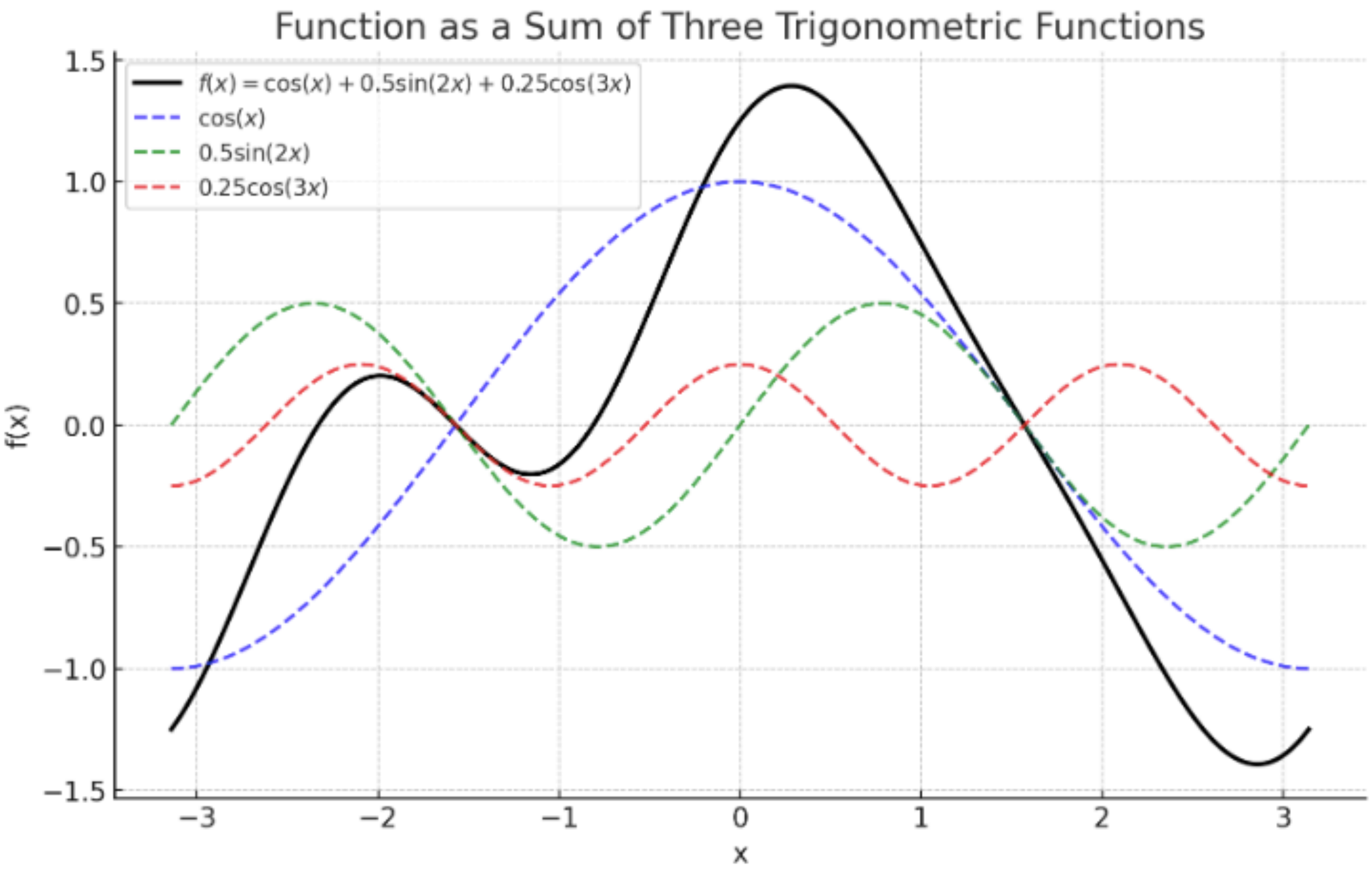
Sound Wave for A440 (440 Hz)
Amplitude
Time [s]
Sound Wave for D294 (294 Hz)
Amplitude
Time [s]
Combined Sound Wave of A440 and D294
Amplitude
Time [s]

Background

- The Fourier series was introduced by Joseph Fourier in his work on the theory of heat
- He published his seminal work, *"Théorie analytique de la chaleur"* (*The Analytical Theory of Heat*), in 1822.
- How can the distribution of heat in a rod or a body be mathematically represented and predicted over time?

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

Fourier Series



Mathematical Formulation

Fourier series when x is period on domain [−π,π]

$$f(x) = \sum_{n=0}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

Fourier series when x is periodon domain [o,L]

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right)\right]$$

$$b_n = \frac{2}{T} \int_{0}^{T} f(x) \sin\left(\frac{2\pi nx}{T}\right) dx \qquad a_n = \frac{2}{T} \int_{0}^{T} f(x) \cos\left(\frac{2\pi nx}{T}\right) dx$$

Complex Fouries Series

Fourier Series for real valued function

$$f(t) = \sum_{k=0}^{\infty} \left(a_k \cos\left(k\frac{\pi}{L}t\right) + b_k \sin\left(k\frac{\pi}{L}t\right)\right)$$

Euler's Formula

$$e^{ix} = \cos(x) + i\sin(x)$$

Fourier Series for complex valued function

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{L}t}$$

$$c_k = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-ik\frac{\pi}{L}t}\,dt$$

Fourier Transform

Fourier Series for real valued function

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{L}t} \quad c_k = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-ik\frac{\pi}{L}t}\, dt$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

Inverse Fourier Transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

Fourier Series for Odd and Even Functions

Hamad Khan (13654)

Formula Simplification

- Fourier Series for Odd and Even Functions Simplification of Fourier series for odd functions (sine series).
- Simplification of Fourier series for even functions (cosine series).

Formula Derivation

Even Function

$f(x) = f(-x)$

$\cos x$, x^2, $x \sin x$

$\ominus \ \ominus$

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$$

ODD Function

$f(x) = -f(-x)$

$\sin x$, x, x^3, $x \cos x$

$\ominus \ \oplus$

$$\int_{-l}^{l} f(x)\,dx = 0$$

General formular

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n(x)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n(x)\,dx$$

If $f(x)$ is even

$$a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos n(x)\,dx$$

$$b_n = 0$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n(x)$$

Half-range expansion of Fourier Series

Shayan Amir (13664)

What is Half-Range Expansion?

- **The Half-Range Expansion is used to represent a non-periodic function over a finite interval using Fourier series.**
- **We apply it when the function is defined only over a limited range.**
- **Normally, Fourier series assumes that the function repeats indefinitely. In half-range expansion, we modify the Fourier series so that it only represents the function over a half interval.**

Significance:

- Representation of Non-Periodic Functions: Half-range expansion allows us to represent functions that are not periodic but defined on a specific interval (like [o,L][o,L]).
- Simplified Calculations: Instead of working with full-period functions, we can break down complex problems into simpler parts.

Applications of Half-Range Expansion

- Signal Processing: When signals are non-periodic and confined to a limited range, half-range expansion is useful.
- Heat Transfer Problems: In cases like a heated bar or wire, the temperature distribution is non-periodic but occurs within a limited range.
- Mechanical Vibrations: Analyzing vibrations within a finite interval rather than periodic motion.
- Electrical Engineering: Representing currents or voltages that are non-repetitive but occur within a certain time frame.

Formulas for Half-Range Expansion

Half-Range Sine Series (when the function is odd):

For a function f(x) defined on [0,L], the Fourier sine series is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Formulas for Half-Range Expansion:

If $f(x)$f(x) is an even function, the Fourier cosine series is used:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where,

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Derivation: Half-Range Sine Series

We want to express a given function $f(x)$, defined on the interval $[0, L]$, as a sum of sine terms. The series is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where b_n are the coefficients to be determined.

Step 2: Orthogonality of Sine Functions

The sine functions $\sin\left(\frac{n\pi x}{L}\right)$ are orthogonal over $[0, L]$. That is, for integers m and n:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{L}{2}, & \text{if } m = n. \end{cases}$$

This property is the foundation for isolating the coefficients b_n.

Derivation: Half-Range Sine Series

Step 3: Multiply by a Sine Function

Multiply both sides of the assumed series by $\sin\left(\frac{m\pi x}{L}\right)$, where m is a positive integer:

$$f(x)\sin\left(\frac{m\pi x}{L}\right) = \left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi x}{L}\right).$$

Step 4: Integrate Both Sides

Integrate both sides over the interval $[0, L]$:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx.$$

Derivation: Half-Range Sine Series

Step 5: Apply Orthogonality

Using the orthogonality property of sine functions, all terms in the summation on the right-hand side vanish except when $n = m$. This simplifies the integral:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = b_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx.$$

Step 6: Simplify the Integral

From the orthogonality property, we know:

$$\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}.$$

Step 7: Solve for b_m

Rearrange to solve for b_m:

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

Substitute the expression for b_m back into the series:

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)\sin\left(\frac{n\pi x}{L}\right).$$

Derivation: Half-Range cosine Series

Half-Range Cosine Series for $f(x) = L - x$

For the **Half-Range Cosine Series:**

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where the Fourier cosine coefficients a_n are given by:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Substitute $f(x) = L - x$:

$$a_n = \frac{2}{L}\int_0^L (L - x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Derivation: Half-Range cosine Series

Now, split the integral:

$$a_n = \frac{2}{L}\left(\int_0^L L\cos\left(\frac{n\pi x}{L}\right)dx - \int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

First Integral:

$$\int_0^L L\cos\left(\frac{n\pi x}{L}\right)dx = L\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

The integral of cosine is:

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0$$

Derivation: Half-Range cosine Series

Apply the integration by parts formula:

$$\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \left[\frac{L}{n\pi}x\sin\left(\frac{n\pi x}{L}\right)\right]_0^L - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$

Evaluating the first term:

$$\left[\frac{L}{n\pi}x\sin\left(\frac{n\pi x}{L}\right)\right]_0^L = \frac{L}{n\pi}\cdot L\sin(n\pi) = 0$$

Now, solve the second term:

$$\frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{n\pi}\cdot\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

Derivation: Half-Range cosine Series

This gives:

$$\frac{L^2}{n^2\pi^2}(1-(-1)^n)$$

Thus, the second integral becomes:

$$\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{L^2}{n^2\pi^2}(1-(-1)^n)$$

Final Expression for a_n:

Substitute back into the formula for a_n:

$$a_n = \frac{2}{L}\left(0-\frac{L^2}{n^2\pi^2}(1-(-1)^n)\right)$$

Derivation: Half-Range cosine Series

Thus, we get:

$$a_n = \frac{2L}{n^2\pi^2}\left(1-(-1)^n\right)$$

This shows that a_n is non-zero only for **odd values of** n. Therefore, the **Half-Range Cosine Series** for $f(x) = L - x$ is:

$$f(x) = \sum_{n\ \text{odd}} \frac{4L}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$$

Comparison and Applications

Hamad Ali (14109)

1. Comparison

Definition:

- Full Fourier Series: Represents periodic functions using both sine and cosine terms (or exponential form).
- Half Fourier Series: Represents periodic functions only in a half-range (e.g., sine-only or cosine-only expansion).

Range of Application:

- **Full Fourier Series:** Defined over a complete period $[-L, L]$.
- **Half Fourier Series:** Defined over a half-period $[0, L]$, focusing on specific symmetry properties.

1.2 Comparison

Symmetry:

Full Fourier Series: Useful for functions with no specific symmetry or mixed symmetry.

Half Fourier Series: Best for functions with even or odd symmetry.

Terms Included:

Full Fourier Series: Both sine (sin) and cosine (cos) terms.

Half Fourier Series:

Even Symmetry: Only cosine terms.

Odd Symmetry: Only sine terms.

1.1 Full-Range Fourier Series Formula

- Formula:
- $f(x) = a_0 + \Sigma\, [a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]$

- Coefficients:
- $a_0 = (1/2L) \int[-L, L]\, f(x)\, dx$
- $a_n = (1/L) \int[-L, L]\, f(x) \cos(n\pi x/L)\, dx$
- $b_n = (1/L) \int[-L, L]\, f(x) \sin(n\pi x/L)\, dx$

1.2 Half-Range Fourier Series Formulas

- Half-Range Cosine Series:
- $f(x) = (a_0/2) + \Sigma [a_n \cos(n\pi x/L)]$
- $a_n = (2/L) \int[0, L] f(x) \cos(n\pi x/L)\, dx$

- Half-Range Sine Series:
- $f(x) = \Sigma [b_n \sin(n\pi x/L)]$
- $b_n = (2/L) \int[0, L] f(x) \sin(n\pi x/L)\, dx$

2. Key Differences: Full-Range vs Half-Range

- \- Full-Range:
- * Defined over symmetric intervals (e.g., [−L, L]).
- * Requires sine and cosine terms.

- \- Half-Range:
- * Defined over one-sided intervals (e.g., [0, L]).
- * Exploits function symmetry (even or odd).

3.1 Applications full Fourier Series

Full Fourier Series Applications:

- General-purpose modeling of periodic signals, such as electrical signals and sound waves.
- Analysis of vibrations in mechanical systems.
- Engineering applications: signal processing and communications.

3.2 Applications of Half Fourier Series

Half Fourier Series Applications:

- Heat conduction problems over a semi-infinite domain (e.g., metal rod heated from one end).
- Structural analysis of beams with specific boundary conditions (e.g., clamped/free).
- Wave equation solutions with boundary constraints.

Summary of Applications

Type	Key Application	Example
Full range fourier Series	General Periodic Functions	Electrical Circuits, Sound Wave Analysis
Half range Sine Series	Problems with Odd symmetry	Heat Transfer, String Vibrations, Fluid Dynamics
Half range Cosine Series	Problems with even symmetry	Signal Processing, Heat Distribution, Optics

Practical Examples

Amar (11637)

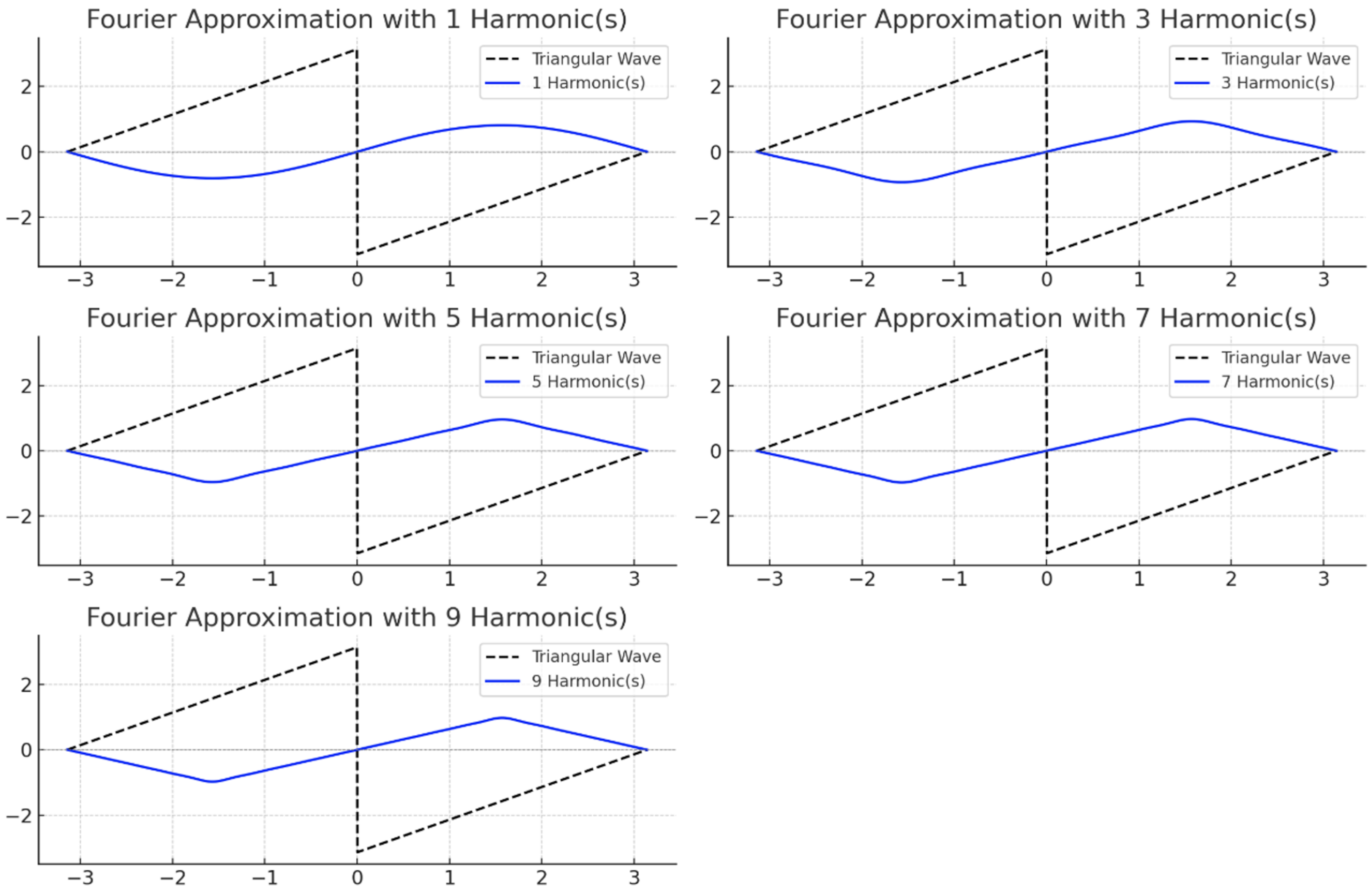
Odd and Even Functions with their Fourier Series, Half range Expansion of Fourier Series

(11637) Ammar UD Din

Triangular / Sawtooth Wave

These waves are commonly used in:

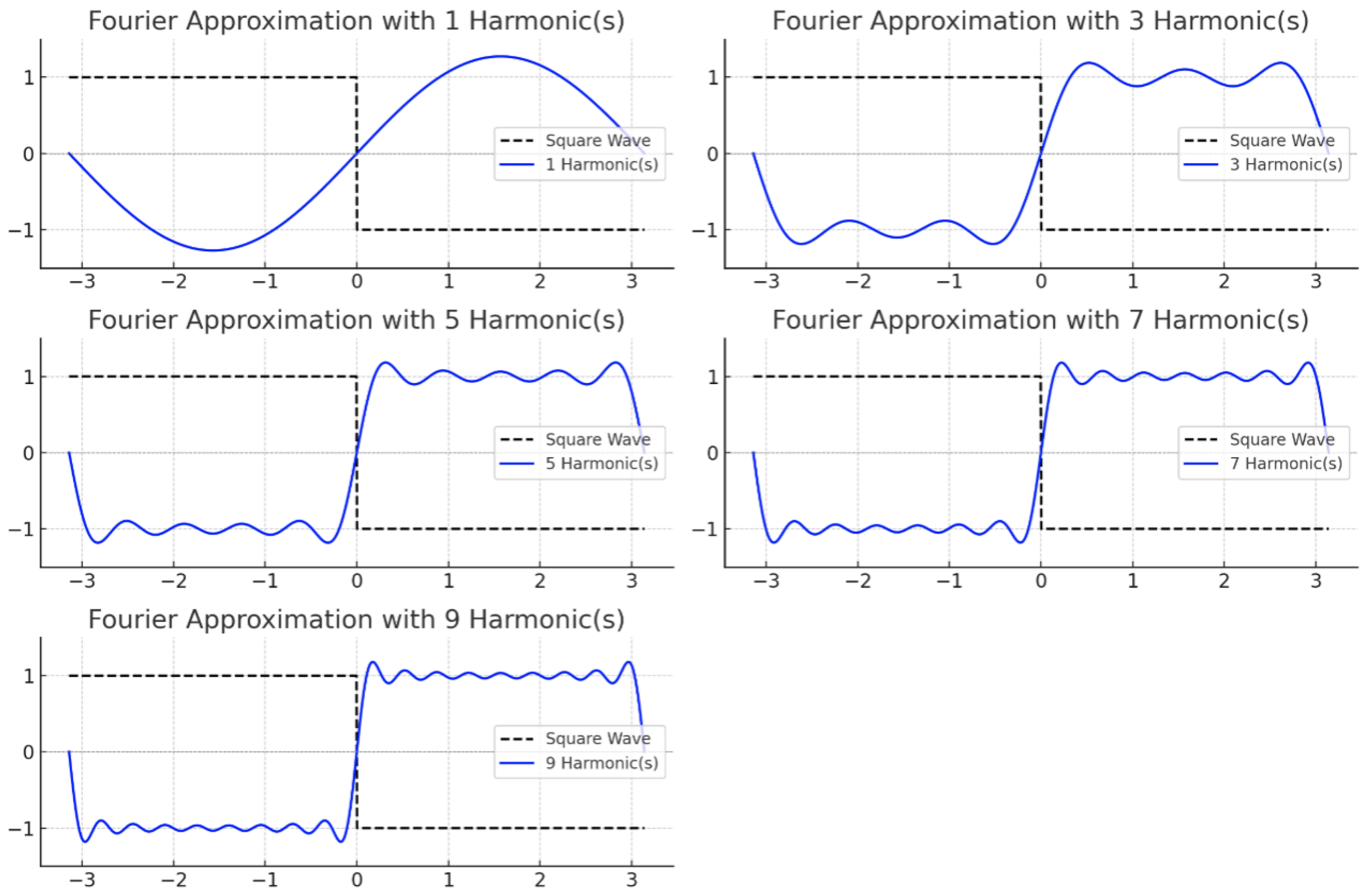
- Electronics
- Audio synthesis
- Signal processing



Square wave

These types of functions appear in:

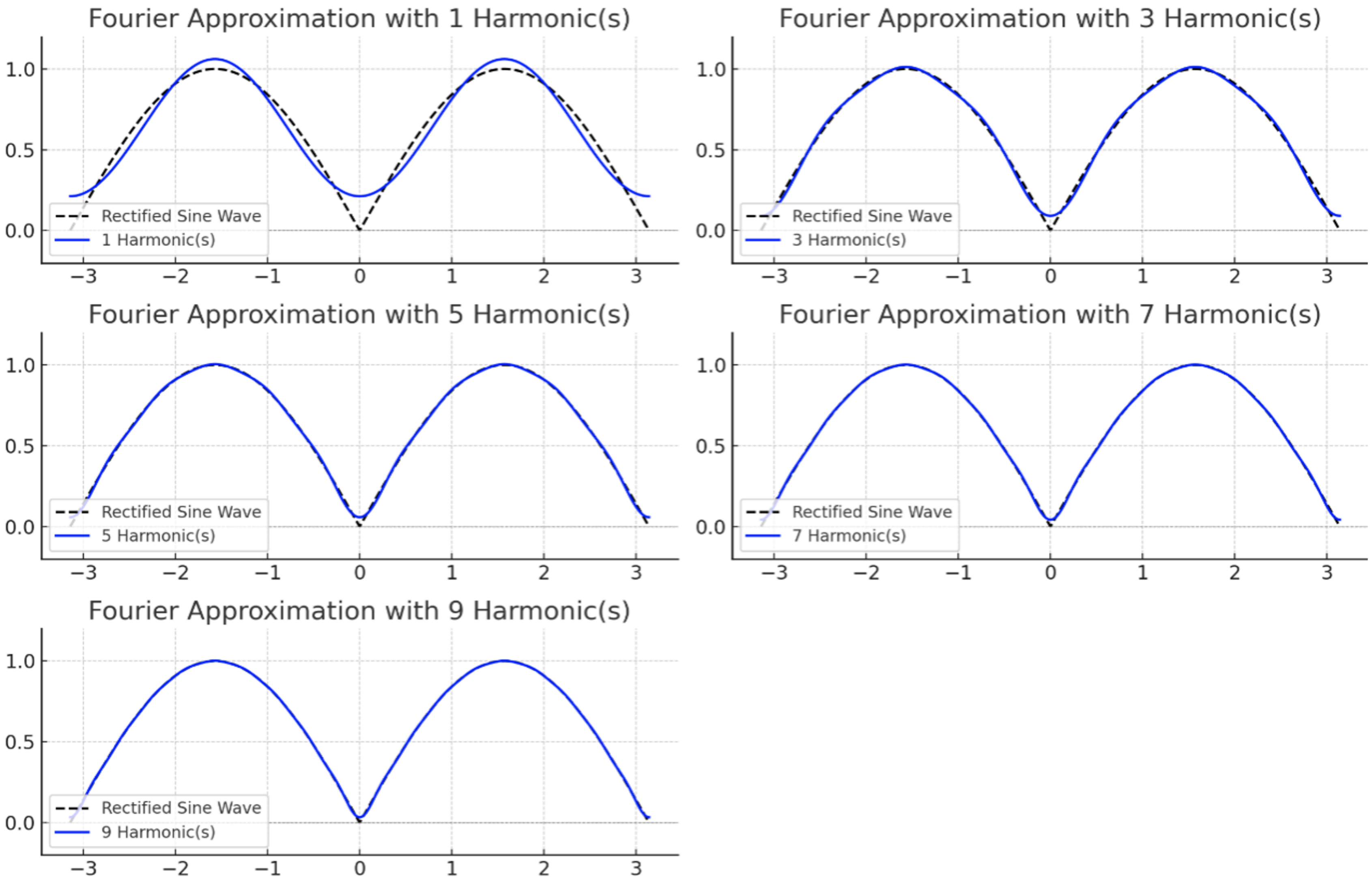
- Digital Electronics
- Clock Signals
- Music Synthesis
- Control Systems



Rectified sine wave

These types of functions appear in:

- AC power systems
- Signal Processing



Numerical Example:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right) \right]$$

$$a_0 = \frac{1}{T} \int_0^T f(x)\, dx$$

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n x}{T}\right) dx$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n x}{T}\right) dx$$

Find the **half-range sine series** for $f(x) = x$ on $[0, 1]$.

Solution:

1. **Given Function:** $f(x) = x$ on $[0, 1]$.
2. **Fourier Coefficients** (b_n):

$$b_n = \frac{2}{1} \int_0^1 x \sin(n\pi x)\, dx$$

Step-by-Step Calculation:

1. **Integral**: Using integration by parts:

$$I = \int_0^1 x \sin(n\pi x)\, dx$$

Let $u = x$, $dv = \sin(n\pi x)dx$. Then:

$$u = x, \quad du = dx, \quad v = -\frac{\cos(n\pi x)}{n\pi}$$

Integration by parts gives:

$$I = \left[-\frac{x\cos(n\pi x)}{n\pi}\right]_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}dx$$

2. **Simplify:**

- Evaluate the boundary term:
$$\left[-\frac{x\cos(n\pi x)}{n\pi}\right]_0^1 = -\frac{\cos(n\pi)}{n\pi} + 0$$
Since $\cos(n\pi) = (-1)^n$:
$$= -\frac{(-1)^n}{n\pi}$$
- Solve the remaining integral:
$$\int_0^1 \frac{\cos(n\pi x)}{n\pi}dx = \frac{\sin(n\pi x)}{(n\pi)^2}\Big|_0^1 = \frac{\sin(n\pi)}{(n\pi)^2} - \frac{\sin(0)}{(n\pi)^2} = 0$$

3. **Final Result:** The coefficient becomes:

$$b_n = \frac{2}{1} \cdot \left(-\frac{(-1)^n}{n\pi} \right) = \frac{2(-1)^{n+1}}{n\pi}$$

Half-Range Sine Series:

The half-range sine series for $f(x) = x$ is:

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Conclusion

Abdul Muqeet Paracha ()

Key Points

1) Even functions are symmetric about the y-axis
2) Odd functions are symmetric about the origin
3) Fourier series is a mathematical tool used to represent a periodic function as a sum of sine and cosine functions
4) Complex periodic signals into simpler sinusoidal components.
5) Simplifying computations through symmetry.
6) Enable efficient and compact representation of functions.
7) Separating signals into meaningful components for analysis and applications.

Thank you!

Fee Free to ask any question.