Dimensionality Reduction

Question 1: Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7, 3/7, 6/7] and another is [6/7, 2/7, -3/7]. Let the third column be [x, y, z]. Since the length of the vector [x, y, z] must be 1, there is a constraint that $x^2+y^2+z^2 = 1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

Given Matrix M = $2/7\ 6/7\ x\ 3/7\ 2/7\ y\ 6/7\ -3/7\ z$

If columns in a matrix form an orthonormal base, then:

- □ The dot product between any two columns is Zero.

Then, C1.C3 □ 0

- ⇨ $\left(\frac{2}{7}\right).x + \left(\frac{3}{7}\right).y + \left(\frac{6}{7}\right).z = 0$ □ equation 1
- ⇨ $C2.C3 = 0$
- ⇨ $\left(\frac{6}{7}\right).x + \left(\frac{2}{7}\right).y - \left(\frac{3}{7}\right).z = 0$ □ equation 2

Rewriting the above equations, we get

1 □ $2x + 3y + 6z = 0$

2 □ $6x + 2y - 3z = 0$

Eq.1/6 □ $\frac{x}{3} + \frac{y}{2} + z = 0$ □ equation 3

Eq.2/3 □ $2x + \left(\frac{2}{3}\right)y - z = 0$ □ equation 4

Eq.3 + Eq.4 □ $\left(\frac{7}{3}\right)x + \left(\frac{7}{6}\right)y = 0$

- ⇨ $42x + 24y = 0$
- ⇨ $14x + 8y = 0$
- ⇨ $x = \left(-\frac{1}{2}\right)y$

Eq.1 x 3 □ $6x + 9y + 18z = 0$ □ equation 5

Eq.5 – Eq.2 □ $7y + 21z = 0$

- ⇨ $y = -3z$

Therefore $x:y:z = -3:6:-2$

Question 2: Find the eigenvalues and eigenvectors of the following matrix:

2	3
3	10

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.

Given Vector A = $[2\ 3\ 3\ 10\ ]$

We know that $\det det\ (A - \lambda I)\ = 0$ equation 1

A - λi = $[2\ 3\ 3\ 10\ ]$ - $[\lambda\ 0\ 0\ \lambda\ ]$ = $[2 - \lambda\ 3\ 3\ 10 - \lambda\ ]$

$det(A - \lambda I)$ = (2 - λ)(10 - λ) – 9

- 20 - 12λ + λ^2 – 9
- $\lambda^2 - 12\lambda + 11$

Hence the Eigen values $\lambda = 1,\ 11$

<u>**For**</u>$\lambda = 1$**:**

- $(A - \lambda I)(V) = 0$
- $[2 - \lambda\ 3\ 3\ 10 - \lambda\ ][V1\ V2\ ]$ = 0
- $V1 = - 3V2$
- $Eigen\ Vector\ = \left(1\ - \frac{1}{3}\right) * K_1$

<u>**For**</u>$\lambda = 1$**:**

- $[2 - 11\ 3\ 3\ 10 - 11\ ][V1\ V2\ ] = 0$
- $- 9V2 + 3V2 = 0$
- $Eigen\ Vector\ = (1\ 3\ ) * K$

Question 3: Suppose [1,3,4,5,7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

$V = [1,\ 3,\ 4,\ 5,\ 7]$

Unit Eigen Vector in the same direction

- $\left(\frac{1}{\sqrt{1+9+25+16+49}}\right)[1,\ 3,\ 4,\ 5,\ 7]$
- $\left(\frac{1}{\sqrt{99}}\right)[1,\ 3,\ 4,\ 5,\ 7]$
- $\left(\frac{1}{10}\right)[1,\ 3,\ 4,\ 5,\ 7]$

Question 4: Suppose we have three points in a two dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

Given elements in two dimensional space are (1, 1), (2, 2) and (3, 4)

➪ M = $[1\ 1\ 2\ 2\ 3\ 4]$

Transpose of $M(M^T)$ = $[1\ 2\ 3\ 1\ 2\ 4]$

Therefore $N = M^T * M$

➪ $[1\ 2\ 3\ 1\ 2\ 4] * \ [1\ 1\ 2\ 2\ 3\ 4]$
➪ $[1 + 4 + 9\ 1 + 4 + 12\ 1 + 4 + 12\ 1 + 4 + 16]$
➪ $[14\ 17\ 17\ 21]$

Question 5: Consider the diagonal matrix M =

1	0	0
0	2	0
0	0	0

Compute its Moore-Penrose pseudoinverse.

Moore-Penrose pseudoinverse $M^t = \ [1\ 0\ 0\ 0\ y2\ 0\ 0\ 0\ 0]$

Question 6: When we perform a CUR decomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

1	2	3
4	5	6
7	8	9
10	11	12

Calculate the probability distribution for the rows.

<u>Probability distribution of the rows</u>:

The sum of squares of elements of A is 1+4+9+16+25+35+49+64+81+100+121+144 = 650

Squared frobenius norm for all rows and their probabilities are:

<u>Row 1:</u>

$$1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

Probability = $\frac{14}{659} = 0.021$

<u>Row 2:</u>

$$4^2 + 5^2 + 6^2 = 16 + 25 + 36 = 77$$

Probability = $\frac{77}{650} = 0.118$

Row 3:

$$7^2 + 8^2 + 9^2 = 49 + 64 + 81 = 194$$

Probability = $\frac{194}{650} = 0.298$

Row 4:

$$10^2 + 11^2 + 12^2 = 100 + 121 + 144$$

Probability = $\frac{365}{650} = 0.561$

Probability distribution of rows is □ 0.021, 0.118, 0.298, 0.561