THIS PAPER IS NOT TO BE REMOVED FROM THE EXAMINATION HALL

UNIVERSITY OF LONDON

CM1020

BSc EXAMINATION

COMPUTER SCIENCE

Discrete Mathematics

Tuesday 3 March 2020 : 10.00 – 12.00

Time allowed: 2 hours

DO NOT TURN OVER UNTIL TOLD TO BEGIN

INSTRUCTIONS TO CANDIDATES:

This examination paper is in two parts: Part A and Part B. You should answer **ALL** of question 1 in Part A and **TWO** questions from Part B. Part A carries 40 marks, and each question from Part B carries 30 marks. If you answer more than **TWO** questions from **Part B** only your first **TWO** answers will be marked.

All answers must be written in the answer books; answers written on the question paper will not be marked. You may write notes in your answer book. Any notes or additional answers in the answer book(s) should be crossed out.

The marks for each part of a question are indicated at the end of the part in [.] brackets. There are 100 marks available on this paper.

Calculators are not permitted in this examination.

PART A

Candidates should answer **ALL** of Question 1 in Part A.

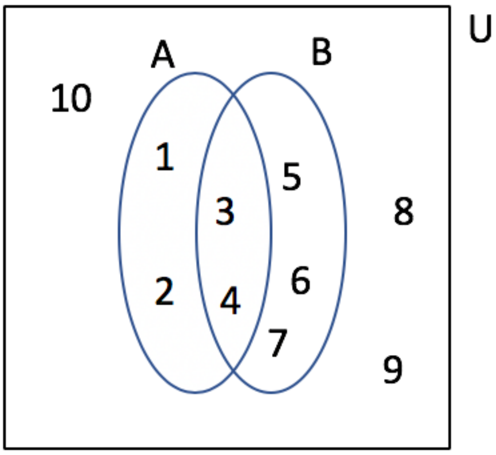
Question 1

(a) Let A be a set with n elements. What is the number of subsets which can be formed from A?

Choose ONE option. [4]

i. 2^n
ii. $2n$
iii. n^2
iv. $\frac{n}{2}$

(b) Given the following Venn diagram representing two sets A and B, subsets of the universal set U:

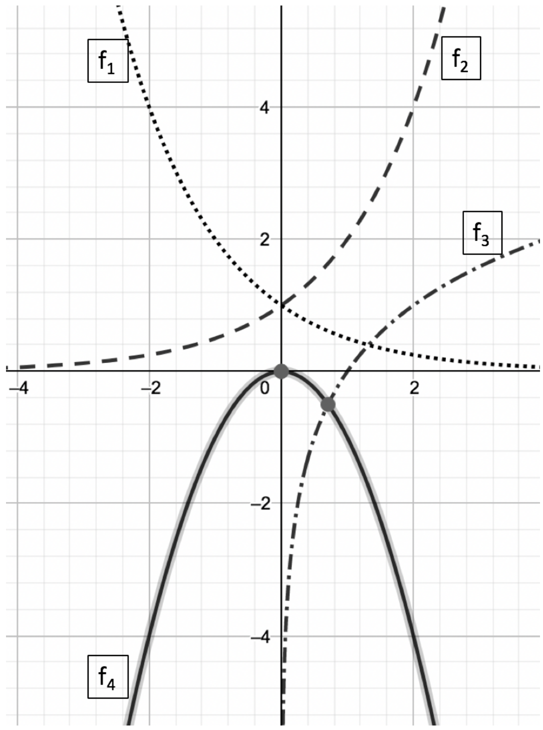


Which one of the following sets represents $(\overline{A \oplus B})$?

Choose ONE option. [4]

i. $\{1, 2, 5, 6, 7\}$
ii. $\{3, 4, 8, 9, 10\}$
iii. $\{8, 9, 10\}$
iv. none of the other options is correct

(c) The following graph shows the curves of four functions, f_1, f_2, f_3 and f_4:



Which one of these functions is not invertible?
Choose ONE option. [4]

i. f_1
ii. f_2
iii. f_3
iv. f_4

(d) Let p and q be the following propositions where p means 'James has a cell phone' and q means 'James has a laptop computer'. Which one of the following logical expressions is equivalent to a correct formalisation of the sentence below?

'James does not have a cell phone or he does not have a laptop computer.'

Choose ONE option. [4]

i. $\neg(p \vee q)$

ii. $\neg p \wedge \neg q$

iii. $\neg(p \wedge q)$

iv. $\neg p \vee q$

(e) Given the statement $S(x)$: '2 divides $x^2 + 1$', select the right statement from the following.

Choose ONE option. [4]

i. S can be expressed using propositional logic

ii. S is not a proposition, as its truth value is a function depending on x

iii. The truth value of $S(3)$ is false

iv. The truth value of $S(2)$ is true

(f) The number of ways k objects that can be selected from n categories of objects where repetition is not allowed and the ordering of the outputs is not important can be calculated using which one of the following formulations?

Choose ONE option. [4]

i. $\frac{n!}{k!(n-k)!}$

ii. $\frac{n!}{(n-k)!}$

iii. $\frac{n!}{k!}$

iv. $n(n-k)$

(g) Which one of the following degree sequences cannot represent a simple graph?

Choose ONE option. [4]

i. $3, 3, 2, 2$
ii. $2, 2, 2, 2$
iii. $1, 1, 1, 1$
iv. $4, 3, 3, 2$

(h) Which one the following correctly defines a Hamiltonian path?

Choose ONE option. [4]

i. A Hamiltonian path in a graph G is a path that uses each edge in G once
ii. A Hamiltonian path in a graph G is a path that visits each vertex in G once
iii. A Hamiltonian path is a trail in which neither vertices nor edges are repeated
iv. A Hamiltonian path is a walk in which no edge is repeated

(i) Which one the following correctly defines an Eulerian path?

Choose ONE option. [4]

i. An Eulerian path in a graph G is a path that uses each edge in G once
ii. An Eulerian path in a graph G is a path that visits each vertex once.
iii. An Eulerian path is a trail in which neither vertices nor edges are repeated
iv. An Eulerian path is a walk in which no edge is repeated

(j) Let $S = \{1, 2, 3\}$ and $\mathcal{R}$ be a relation on elements in S with

$$\mathcal{R} = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,3)\}$$

Which one of the following statements is correct about the relation $\mathcal{R}$?
Choose ONE option. [4]

i. $\mathcal{R}$ is reflexive
ii. $\mathcal{R}$ is symmetric
iii. $\mathcal{R}$ is transitive
iv. $\mathcal{R}$ is anti-symmetric.

PART B

Candidates should answer any **TWO** questions from Part B.

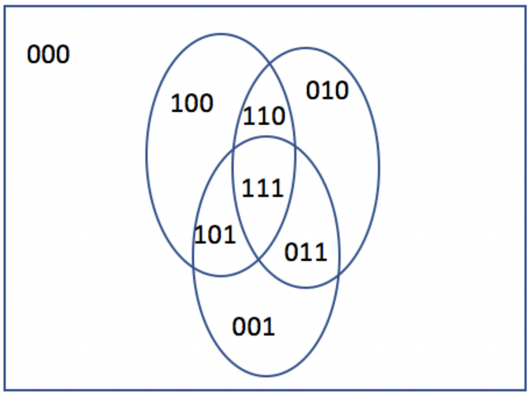
Question 2 Set theory, Propositional and Predicate logic & Combinatorics

(a) i. Rewrite the following three sets using the listing method:

- $A = \{n^3 + n^2 + -(-1)^n : n \in \mathbb{Z}$ and $0 \leq n < 4\}$
- $B = \{n + \frac{1}{n} : n \in \mathbb{Z}^+$ and $n < 6\}$
- $C = \{1 + (-1)^n : n \in \mathbb{Z}^+\}$

[3]

ii. The following Venn diagram represents three sets A, B and C intersecting in the most general way. Three binary digits are used to refer to each one of the eight regions in this diagram. In terms of A, B and C, write the set representing the area comprising the regions 100, 010 and 001. The answer must be written in its simplest form using the $\oplus$ operator only.



[3]

iii. Given three sets A, B and C, prove that the expression $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$ is equivalent to $B \cap C$ by re-writing the expression using algebraic laws. State the name of each law used. [4]

(b) Let p and q be two propositions for which $p \to q$ is false. Determine the truth value for each of the following: $p \wedge q$; $\neg p \vee q$; $q \to p$ and $\neg q \to \neg p$.

[4]

(c) Let p, q and r denote the following statements:

p : 'I finish writing my computer program before lunch'

q : 'I play football in the afternoon'

r : 'The sun is shining'

i. Write the following statements into their corresponding symbolic forms:

- If the sun is shining and I finish writing my computer program, I shall play football this afternoon.
- I will play football this afternoon only if the sun is shining and I finish writing my computer program.

[4]

ii. Write in words the contrapositive of the following statement:

'If the sun is shining and I finish writing my computer program, I shall play football this afternoon.' [2]

(d) Let A be the set of all students, $p(x)$ be the proposition: 'x plays football' and $S(x)$ is the set of all students in the same class as x' where x is an element of A.

Use rules of inference with quantifiers to formalise the three following statements:

i. None of the students plays football. [2]

ii. Not every student plays football. [2]

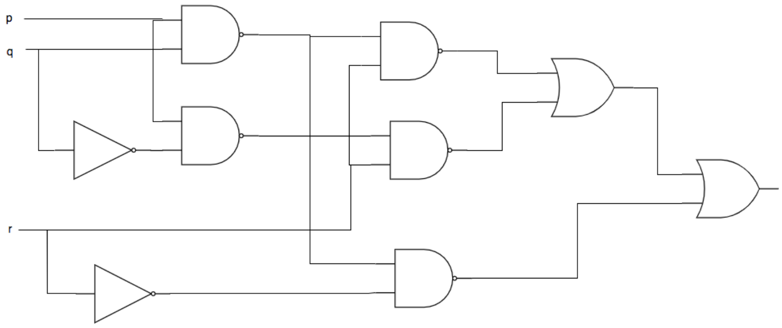
iii. There exists a student, none of whose classmates play football. [2]

(e) Suppose that there are eight runners in a race. The winner receives a gold medal, the person in second place receives a silver medal, and the person in third place receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties? Show your working.

[4]

Question 3 Boolean Algebra, Functions & Recursion

(a) Given the following logical circuit with three inputs p, q and r:



i. Identify the logical gates used in this circuit. [2]

ii. What is the logical expression of the output of this circuit? [4]

iii. Simplify the logical expression in (ii). Explain your answer. [4]

iv. Draw in your answer book the resulting simplified circuit. [2]

(b) Name two properties a function has to satisfy to be a bijective function. [2]

(c) Consider the following function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 2x + 3$.

i. Show f is a bijective function. [4]

ii. Find the inverse function f^{-1}. [4]

(d) Consider the Fibonacci recurrence relation:

$f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$ and $f_1 = 1$

What are the values of f_2 and f_3? [2]

(e) Let $S_n = \sum\limits_{i=1}^{i=n}(2i - 1) = n^2$ for all $n \in \mathbb{Z}^+$.

i. Find S_1 and S_2. [1]

ii. Prove by induction that $S_n = n^2$ for all $n \in \mathbb{Z}^+$. [5]

Question 4 Graph Theory, Trees & Relations

(a) i. Give a definition of a bipartite graph. [2]

ii. Name an algorithm that is used to find the shortest path between vertices in a weighted graph. [2]

iii. Is it possible to draw a simple graph with a degree sequence, 5, 4, 3, 2, 2? If it is possible, draw the graph. If it is not possible, explain why you cannot draw the graph. [3]

(b) A graph G with 5 vertices: $a,\ b,\ c,\ d,\ e$ has the following adjacency list:

$a : b,\ e$

$b : a,\ c,\ d$

$c : b,\ d$

$d : b,\ c,\ e$

$e : d,\ a.$

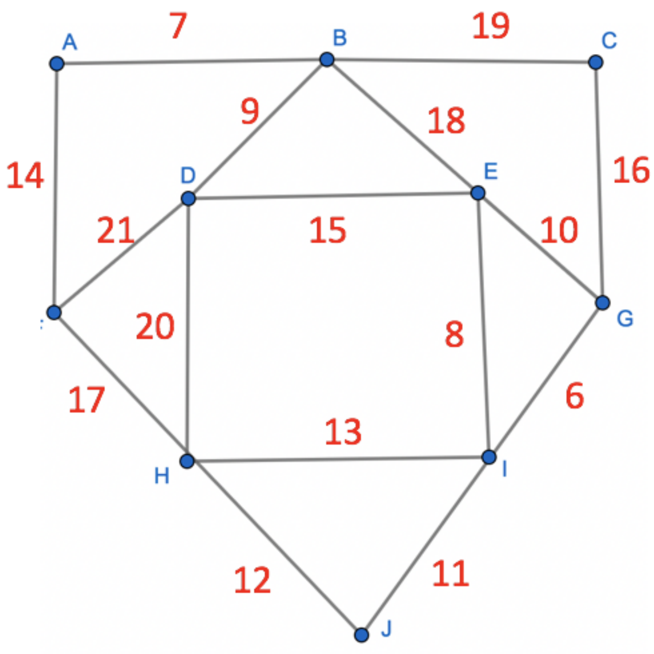
i. Draw this graph, $G.$ [2]

ii. Find the adjacency matrix of the graph G ? [2]

(c) What is the number of edges in a tree with n vertices? [2]

(d) Explain how to find the minimum spanning tree in a weighted graph using Kruskal's algorithm. [2]

(e) Consider the following undirected weighted graph with 10 vertices, $A, B, \cdots J$. The number (weight) on each edge represent the distance, in kilometres between the pairs of vertices.



i) Using Kruskal's algorithm find the minimum spanning tree of this graph. [3]

ii) What is the length of the resulting spanning tree? [2]

(f) Let S be the set of integers $\{5, 6, 7, 8, 9, 10\}$. $\mathcal{R}$ is a relation defined on S by the following condition such that,
for all $x, y \in S$, xRy if $(x + y) \mod 2 = 0$.

i. Draw the digraph of $\mathcal{R}$. [2]

ii. Explain why $\mathcal{R}$ is or is not each of the following:

- reflexive;
- symmetric;
- anti-symmetric;
- transitive.

In the cases where the given property does not hold, provide a counter example to justify this. [4]

iii. Is $\mathcal{R}$ a partial order? Explain your answer. [1]

iv. Is $\mathcal{R}$ an equivalence relation? If the answer is yes, write down the equivalence classes for this relation and if the answer is no, explain why. [3]

END OF PAPER

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