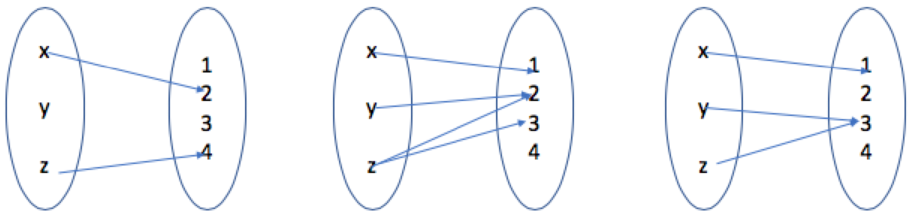
Discrete Mathematics

Tutorial sheet | Funtions

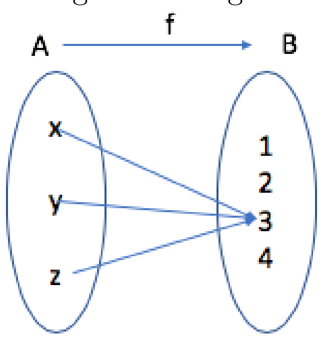
Question 1.
Let A and B be two sets with $A = \{x, y, z\}$ and $B = \{1, 2, 3, 4\}$. Which of the following arrow diagrams define functions from A to B?



(i) (ii) (iii)

Solution:
In (i) there is an element x of A that is not mapped to any element of B as there is no arrow coming from x. Hence, it is not a function as not every element A has an image in B. In (ii) there is an element z of A that is mapped to two elements, 2 and 3, of B. Hence, it is not a function as in a function every element of the domain needs to be mapped to unique element in the co-domain. However, the arrow diagram in (iii) defines a function as every element of A is mapped to unique image in B.

Question 2.
Let A and B be two sets with $A = \{x, y, z\}$ and $B = \{1, 2, 3, 4\}$. Let f from A to B defined by the following arrow diagram:



1. Write the domain, the co-domain and the range of f.
2. Find $f(x)$ and $f(y)$.
3. Write down the set of pre-images of 3 and the set of pre-images of 1 .
4. represent f as a set of ordered pairs.

Solution:

1. $D_f = A$, $co - D_f = B$ and $R_f = \{3\}$.
2. $f(x) = 3$ and $f(y) = 3$.
3. pre-images of $3 = \{x, y, z\}$ and pre-images of $1 = \emptyset$.
4. f as as set of ordered pairs is $\{(x, 3), (y, 3), (z, 3)\}$.

Question 3.

The Hamming distance function is very important in coding theory. It gives a measure of the difference between two strings of 0's and 1's that have the same length. Let S_n be the set of all strings of 0's and 1's of length n. The Humming function H is defined as follows:

$H : \quad S_n \times S_n \to \mathbb{N} \cup \{0\}$

$\quad (s, t) \to H(s, t) =$ The number of positions in which s and t have different values.

For $n = 5$, Find $H(11111, 00000)$, $H(11000, 00000)$, $H(00101, 01110)$ and $H(10001, 01111)$.

Solution:

$H(11111, 00000) = 5$, $H(11000, 00000) = 2$, $H(00101, 01110) = 3$ and $H(10001, 01111) = 4$.

Question 4.

Digital messages consist of a finite sequence of 0's and 1's. When they are communicated across a transmission channel, they are frequently coded in special ways to reduce the chance that they will be garbled by interfering noise in the transmission lines. A simple way to encode a message of 0's and 1's is to write each bit three times, for example: the message 0010111 would be encoded as 000 000 111 000 111 111 111.

Let A be the set of all strings of 0's and 1's and let E and D be the encoding and the decoding function on the set A defined for each string, s, in A as follows:

$E(s) =$ The string obtained from s by replacing each bit of s with the same bit written three times.

$D(s) =$ The string obtained from s by replacing each consecutive triple of three identical bits of s by a single copy of that bit.

Find $E(0110), E(0101), D(000111000111000111111)$ and $D(111111000111000111000000)$

Solution:

$E(0110) = 000111111000, E(0101) = 000111000111, D(000111000111000111111) = 0101011$ and $D(111111000111000111000000) = 11010100$

Question 5.

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{a, b, c, d\}$ and $C = \{w, x, y, z\}$ be three sets.
Let f and g be two functions defined as follows:
$f : A \to B$ is defined by the following table.

x	1	2	3	4	5	6
$f(x)$	a	b	a	c	d	d

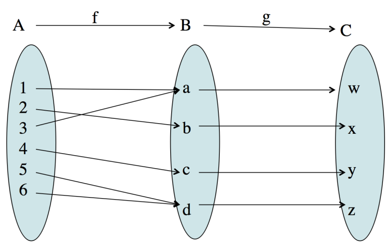
$g : B \to C$ is defined by the following table.

x	a	b	c	d
$g(x)$	w	x	y	z

1. Draw arrow diagrams to represent the function f and g.
2. List the domain; the co-domain and the range of f and g.
3. Find $f(1)$, the ancestor (pre-image) of d. and (g o f)(3)
4. Show that f is not a one to one function.
5. Show that f is an onto function.
6. Show that g is both one to one and onto.

Solution:

1. Arrow diagram



2. $D_f = A = \{1, 2, 3, 4, 5, 6\}$ $Co - D_f = B = \{a, b, c, d\}$ $R_f = \{a, b, c, d\}$
   $D_g = B = \{a, b, c, d\}$ $Co - D_g = C = \{w, x, y, z\}$ $R_g = \{w, x, y, z\}$
3. $f(1) = a,$ (pre-image) of $d = \{5, 6\}$. $(gof)(3) = g(f(3)) = g(a) = w$
4. f is not a one to one function as $f(5) = f(6) = d$.

5. The arrow diagram shows that every element in the co-domain has at least one pre-image, hence, the function f is an onto function.
6. It is clear from the arrow diagram that every element of the range of g has a unique pre-image, hence, g is a one to one function. $R_g = Co - D_g$, hence, g is an onto function.

Question 6.
Suppose you read that a function $f : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ is defined by the formula $f(m, n) = \frac{m}{n}$ for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}^+$.

1. Is f a one to one function?
2. Is f an onto function?

Solution:

1. $f(1, 1) = f(2, 2) = 1$ hence, more than one input can lead to the same output. Hence, f is not a one to one function.
2. Every rational number can be writing with a positive denominator, hence, f is an onto function.

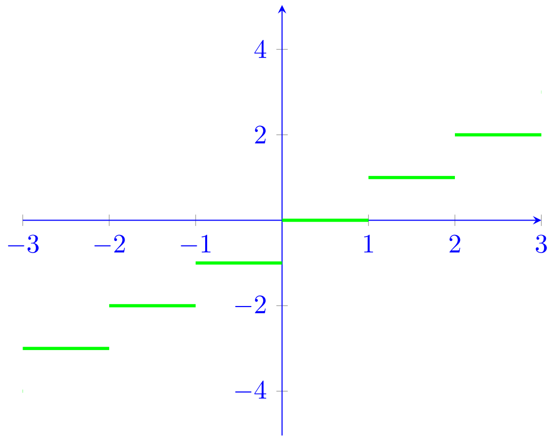
Question 7.

Given a function f definted by $f(x) = \lfloor x \rfloor$. where $f : \mathbb{R} \to \mathbb{Z}$,

1. Plot the graph of the function f(x) for $x \in [-3, 3]$.
2. Use this graph to find $\lfloor \pi \rfloor$, $\lfloor -2.5 \rfloor$, $\lfloor -1 \rfloor$.
3. Use the graph in (1) to show that f is not a one to one (not injective) function.
4. Is f onto (surjective)? Justify your answer.

Solution:

1. Graph:



2. $3 < \pi = 3.14 < 4 < 4 \implies \lfloor \pi \rfloor = 3$, $\lfloor -2.5 \rfloor = -3$ and $\lfloor -1 \rfloor = -1$
3. The graph shows that each element of the range has more than one pre-image, i.e. $\lfloor 2.5 \rfloor = \lfloor 2. \rfloor = 2$. Therefore, the floor function is not a one to one function.
4. for all n in $\mathbb{Z}$ there exists at least one pre-images x=n in $\mathbb{R}$ such that $\lfloor x \rfloor = n$. Therefore every element of co-domain has a pre-image, hence, the the floor function is an onto function.

Question 8.
Let S denote the set of all 3 bit binary strings and $B = (0, 1, 2, 3)$. The function $f : S \to B$ is defined by the rule

f(x)= the number of zeros in x for each $x \in S$.

Find the following.

1. The domain of f.
2. $f(001)$ and $f(101)$.
3. The set of ancestors of 2.
4. The range of f.
5. Say whether or not f is one to one, giving a reason for your answer.
6. Say whether or not f is onto, giving a reason for your answer.

Solution:

1. $D_f = \{000, 001, 010, 011, 100, 101, 110, 111\}$
2. $f(001) = 2$ and $f(101) = 1$.
3. The set of ancestors of $2 = \{001, 010, 100\}$.

4. The range of $f = \{0, 1, 2, 3\}$.
5. f is not one to one as 2 has more than one ancestor.
6. f is onto as the Range of f $R_f = Co - D_f = \{0, 1, 2, 3\}$.

Question 9.

Let $f(x) = x \bmod 3$, where $f(x)$ is the remainder when x is divided by 3, and $f : \mathbb{Z}^+ \to \{0, 1, 2\}$.

1. Find $f(7)$ and $f(12)$.
2. Find the ancestors of 2.
3. Say whether or not $f(x)$ is one to one, justifying your answer.
4. Say whether or not $f(x)$ is onto, justifying your answer.

Solution:

1. $f(7) = 7\ mod\ 3 = 1$ and $f(12) = 12\ mod\ 3 = 0$.
2. ancestors of $2 = \{2, 5, 8, 11, \cdots\}$.
3. $f(x)$ is not one to one as $f(2) = f(5)$ and $2 \neq 5$
4. $f(x)$ is onto as each element in $\{0, 1, 2\}$ has at least one pre-image.

Question 10.

Given the following function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 4x - 1$, for any real number x.

1. Is f a one to one function? Prove or give a counter-example.
2. Is f an onto function? Prove or give a counter-example.
3. Is f invertible? and why? if the answer yes define f^{-1}.

Solution:

1. One-to-one: To prove that f is a one to one function, we need to show that given two real number a and b if $f(a) = f(b)$ then $a = b$. $f(a) = f(b) \implies 4a - 1 = 4b - 1 \implies 4a = 4b \implies a = b$. Thus, f is one to one function.
2. Onto: To prove that f is onto, you must prove that for all $y \in \mathbb{R}$ there exist $x \in \mathbb{R}$ such that $f(x) = y$. Given a real number y, we need to show that there exists a real number x such that $y = 4x - 1$.

   if such real number x exists, then $4x - 1 = y \implies 4x = y + 1 \implies x = \frac{y+1}{4} \in \mathbb{R}$

   Hence, $\forall y \in \mathbb{R}, \exists x = \frac{y+1}{4} \in \mathbb{R}$ such that $f(x) = y$. Therefore, f is an onto function.

3. f is both a one to one and an onto function. Hence , f is invertible and the inverse function is defined as follow:

$$f^{-1} : \mathbb{R} \to \mathbb{R} \text{ with } f^{-1}(x) = \frac{x+1}{4}, \forall x \in \mathbb{R}$$

Question 11.
Given the following function $f : \mathbb{Z} \to \mathbb{Z}$ with $g(x) = 4x - 1$, for any real number x.

1. Is g a one to one function? Prove or give a counterexample.
2. Is g an onto function? Prove or give a counterexample.
3. Is g invertible? and why? if the answer yes define g^{-1}.

Solution:

1. One-to-one: To prove that f is a one to one function, we need to show that given two integers a and b if $g(a) = g(b)$ then $a = b$. $g(a) = g(b) \implies 4a-1 = 4b-1 \implies 4a = 4b \implies a = b$. Thus, g is one to one function.
2. Onto: To prove that g is onto, you must prove that for all $m \in \mathbb{Z}$ there exist $n \in \mathbb{Z}$ such that $g(x) = y$. Given an integer m, we need to show that there exists an integer n such that $m = 4n - 1$.
   If such integer n exists, then $4n - 1 = m \implies 4n = m + 1 \implies n = \frac{m+1}{4}$
   For$m = 0, n = \frac{0+1}{4} = \frac{1}{4}$ which is not an integer. hence, 0 has no pre-image, thus g is not an onto function.
3. g is a one to one but not an onto function. Thus, , g is not invertible and hence, g^{-1} doesn't exist.

Question 12.

Given the following function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x^2 - 1$, for any real number x.

1. What is co-domain and the range of h
2. Is h a one to one function? Prove or give a counterexample.
3. Is h an onto function? Prove or give a counterexample.
4. Is h invertible? and why? if the answer yes define h^{-1}.

Solution:

1. $co - D_h = \mathbb{R}$ and $R_h = [-1, +\infty[$

2. One-to-one: $h(2) = 2^2 - 1 = 3$ and $h(-2) = (-2)^2 - 1 = 3$, hence,$h(-2) = h(2)$ but $-2 \neq 2$. Thus h is not a one to one function.

3. Onto: To prove that h is onto, you must prove that for all $x \in \mathbb{R}$ there exist $x \in \mathbb{R}$ such that $h(x) = y$. However, all negative real numbers less than -1 have no pre-images. for example -2 has no pre-images. Thus, h is not an onto function.

4. h is neither a one to one nor an onto function. Thus, , h is not invertible and hence, h^{-1} doesn't exist.

Question 13.

Given the following function $h : [0, +\infty[\rightarrow [-1, +\infty[$with $h(x) = x^2 - 1$, for any real number x.

1. What is co-domain and the range of h

2. Is h a one to one function? Prove or give a counterexample.

3. Is h an onto function? Prove or give a counterexample.

4. Is h invertible? and why? if the answer yes define h^{-1}.

5. On the same graph, plot the curve of h and that of h^{-1} if it exists.

Solution:

1. $co - D_h = [-1, +\infty[$ and $R_h = [-1, +\infty[$

2. One-to-one: To prove that h is a one to one function, we need to show that given two real non negative number a and b if $h(a) = h(b)$ then $a = b$. $h(a) = h(b) \implies a^2 - 1 = b^2 - 1 \implies a^2 = b^2 \implies a = b$, as a, b are non-negative real numbers. Thus, h is one to one function.

3. Onto: To prove that h is onto, you must prove that for all $y \in [-1, +\infty[$ there exist $x \in [0, +\infty[$ such that $h(x) = y$. Given a real number $y \geq -1$, we need to show that there exists a real number $x \geq 0$ such that $y = x^2 - 1$.

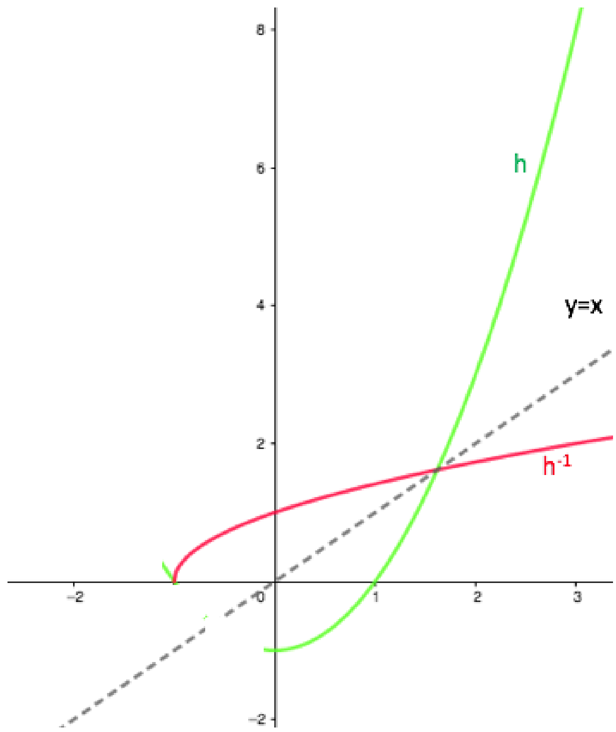
   if such real number x exists, then $x^2 - 1 = y \implies x^2 = y + 1 \implies x = \sqrt{y+1}$ which is in $D_h = [0, +\infty[$ as $y \geq -1$.

   Hence, $\forall y \in [-1, +\infty[, \exists x = \sqrt{y+1} \in [0, +\infty[$ such that $h(x) = y$. Therefore, h is an onto function.

4. h is both a one to one and an onto function. Hence , f is invertible and the inverse function is defined as follow:

$$h^{-1} : [-1, +\infty[\rightarrow [0, +\infty[\text{ with } h^{-1}(x) = \sqrt{x+1}, \forall x \in [-1, +\infty[$$

5. The diagram below shows the curves of h in green and h^{-1} in red. it also shows these curves symmetric with respect to the line $y = x$



Question 14.
Consider the following function $f : \mathbb{R} \to \mathbb{R}^+$ with $f(x) = 2^{x+3}$.

1. Show that f is a bijective function.
2. Find the inverse function f^{-1}.
3. Plot the both curves of f and of f^{-1} on the same graph.

Solution:

1. To show that f is a bijective function, we need to show that f is both a one-to-one and an onto function.
   One-to-one: Given two real number a and b, we need to show that if $f(a) = f(b)$ then $a = b$.

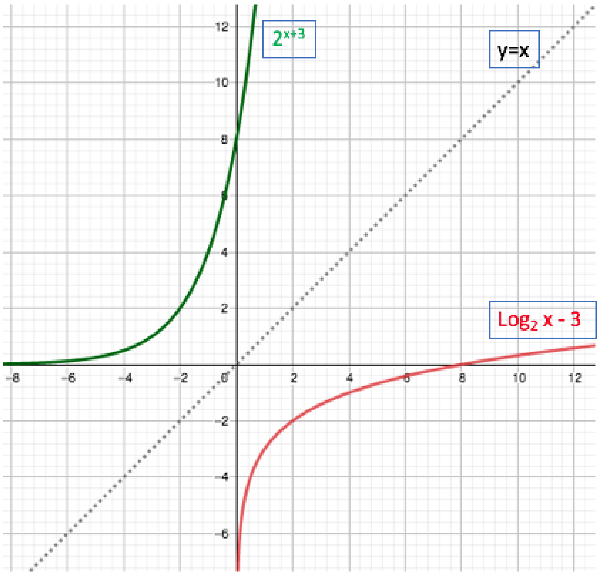
   $$f(a) = f(b) \implies 2^{a+3} = 2^{b+2} \implies \log_2(2^{a+3}) = \log_2(2^{b+2}) \implies a+3 = b+2 \implies a = b$$

   hence, f is a one-to-one function.
   Onto: To prove that f is onto, you must prove that for all $y \in \mathbb{R}^+$ there exist $x \in \mathbb{R}$ such that $f(x) = y$. Given a real number positive y, we need to show that there exists a real number x such that $y = 2^{x+3}$.
   if such real number x exists, then $2^{x+3} = y \implies \log_2(2^{x+3}) = \log_2 y \implies x + 3 = \log_2 y \implies x = \log_2 y - 3$ which is in $\mathbb{R}$

Hence, $\forall y \in \mathbb{R}^+, \exists x = \log_2 y - 3 \in \mathbb{R}$ such that $f(x) = y$. Therefore, f is an onto function.

Thus, f is a bijection.

2. $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$. with $f^{-1}(x) = \log_2 x - 3$
3. The diagram below shows the curves of f in green and h^{-1} in red. it also shows these curves symmetric with respect to the line $y = x$



Question 15.

Consider the following function $f : \mathbb{R} - \{-1\} \to \mathbb{R}$ with $f(x) = \frac{2x}{x+1}$.

1. Show that f is a one to one function.
2. Show that f is not an onto funcion.

Solution:

1. To prove that f is a one to one function, we need to show that given $a, b \in \mathbb{R} - \{-1\}$, if $f(a) = f(b)$ then $a = b$. $f(a) = f(b) \implies \frac{2a}{a+1} = \frac{2b}{b+1} \implies 2a(b+1) = 2b(a+1) \implies 2ab + 2a = 2ba + 2b \implies 2a = 2b \implies a = b$. Thus, f is a one to one function.
2. To prove that f is onto, you must prove that for all $y \in \mathbb{R}$ there exist $x \in \mathbb{R} - \{-1\}$ such that $f(x) = y$.

   if such real number x exists, then $\frac{2x}{x+1} = y \implies 2x = yx + y \implies 2x - y \implies x = \frac{y}{2-y}$, this doesn't exist if $y = 2$. Hence, 2 has no pre-image. Thus f is not an onto function

Question 16.

Find the inverse of the following functions:

1. $f(x) = e^{x^2-5}$
2. $g(x) = e^x + 5$

Solution:

1. To find the inverse we write $y = e^{x^2-5}$ and try to find x in terms of y.

$$y = e^{x^2-5} \implies \ln y = x^2 - 5 \implies x^2 = \ln y + 5 \implies x = \sqrt{\ln y + 5}$$

Thus, $f^{-1} = \sqrt{\ln x + 5}$

2. To find the inverse we write $y = e^x + 5$ and try to find x in terms of y.

$$y = e^x + 5 \implies y = 5 = 2^x \implies x = \ln(y - 5)$$

Thus, $g^{-1}(x) = \ln(x - 5)$

Question 17.
Find the inverse of the following functions:

1. $f(x) = \ln(x + 2) + 2$
2. $g(x) = \log_2(x - 5) + 3$

Solution:

1. To find the inverse we write $y = \ln(x + 2) + 2$ and try to find x in terms of y.

$$y = \ln(x + 2) + 2 \implies y - 2 = \ln(x + 2) \implies e^{y-2} = x + 2 \implies x = e^(y - 2) - 2$$

Thus, $f^{-1} = e^{x-2} - 2$

2. To find the inverse we write $y = \log_2(x - 5) + 3$ and try to find x in terms of y.

$$y = \log_2(x - 5) + 3 \implies y - 3 = \log_2(x - 5) \implies 2^{y-3} = x - 5 \implies x = 2^{y-3} + 5$$

Thus, $g^{-1}(x) = 2^{x-3} + 5$

Question 18.
Let A, B and C be three sets/and $f : A \to B$ and $g : B \to C$ be two functions. Prove that if gof is an onto function then g must be onto.
Solution:
Proof: $f : A \to B$ and $f : B \to C$. if $g\o f$ is onto hence for all z in C there exists $x \in A$ such that $z = (gof)(x) = g(f(x))$. hence, there exists $y = f(x) \in B$ such that $z = g(y)$. Thus g is an onto function.,

Question 19.
Let A, B and C be three sets and $f : A \to B$ and $g : B \to C$ be two functions. Prove that if $g\o f$ is a one to one function then f must be one to one.

Solution:
Proof: $f : A \to B$ and $g : B \to C$. Given that gof is a one to one function, we must show that for all $a, b \in A$ if$a \neq b$ then $f(a) \neq f(b)$.
gof is a one to one function, hence, Given $a, b \in A$ with $a \neq b$ then $(gof)(a) \neq (gof)(b)$, this implies that $g(f(a)) \neq g(f(b))$. Thus $f(a) \neq f(b)$ and this implies that f is a one to one function.
hence, there exists $y = f(a) \in B$ such that $z = g(y)$. Thus g is an onto function.,

Question 20.

Let $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ with $f(x, y) = x + \sqrt{2}y$ for all $x, y \in \mathbb{Q}$
Is f a one to one function? Prove or give a counter-example.

Solution:
Given $(a, b)$and (c, d) in $\mathbb{Q} \times \mathbb{Q}$ with $f(a, b) = f(c, d)$ we need to show that $a = c$ and $b = d$.
$f(a, b) = f(c, d) \implies a + \sqrt{2}b = c + \sqrt{2}d,$
Case 1: if $a = c$ then $b = d$
Case 2: if $b = d$ then $a = c$
Case 3: if $a \neq c$ and $b \neq d$, then $\sqrt{2} = \frac{a-c}{d-b}$ which is a rational. This is a contradiction as $\sqrt{2}$ is irrational and can't be written as fraction. Hence, $a = c$ and $b = d$. Thus f is a one to one function.

End of questions