

Question 1.

Let \mathcal{S} be a set and \mathcal{R} be a relation on \mathcal{S} . Explain what it means (you are expected to give mathematical definitions.) to say that \mathcal{R} is:

1. reflexive;
2. symmetric;
3. anti-symmetric;
4. transitive;
5. an equivalence relation;
6. a partial order.

In each case give an example of a relation which has the given property and another relation which does not have it.

Solution:

1. \mathcal{R} is reflexive if $\forall x \in S, x\mathcal{R}x$
Example: *Equality* is reflexive, but *is Less than*($<$) is not reflexive
2. \mathcal{R} is symmetric if $\forall x, y \in S, x\mathcal{R}y \Rightarrow y\mathcal{R}x$ Example: marriage is symmetric, but *is the parent of* is not symmetric.
3. \mathcal{R} is antisymmetric if $\forall x, y \in S, x\mathcal{R}y \wedge y\mathcal{R}x \Rightarrow x = y$
Example: Subset (\subseteq) is antisymmetric, but *is parent of* not.
4. \mathcal{R} is transitive if $\forall x, y, z \in S, x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$
Example: been an ancestor is transitive, but been a parent is not
5. \mathcal{R} is an equivalence relation if it is reflexive, symmetric and transitive
Example: *is equal to* on the set of real number is an equivalence relation. *is less or equal* (\leq) is not an equivalence relation.
6. \mathcal{R} is a partial order if it is reflexive, antisymmetric and transitive
Example: The relation *divides* ($|$) on the set of positive integers (the natural numbers \mathbb{N}), certainly reflexive, antisymmetric, and transitive. But the same relation applied to the set of integers \mathbb{Z} , is not antisymmetric since $-2 | 2$ and $2 | -2$ but $2 \neq -2$.

Question 2.

Let $S = \{a, b, c\}$ and $A = \{(c, c), (a, b), (b, b), (b, c), (c, b)\}$.

Define a relation R on S by “ x is related to y whenever $(x, y) \in A$ ”.

1. Draw the relationship digraph.
2. The relation R is not reflexive. What pair (x, y) should be added to A to make R reflexive?
3. The relation R is not symmetric. What pair (x, y) should be added to A to make R symmetric?
4. The relation R is not anti-symmetric. What pair (x, y) should be removed to make R anti-symmetric?
5. The relation R is not transitive. What pair (x, y) should be added to A to make R transitive?

Solution:

- 1.
2. (a, a)
3. (b, a)
4. (b, c) or (c, b)
5. (a, c)

Question 3.

The following relations are defined on a set $S = \{a, b, c\}$.

R_1 is the relation given by $\{(a, a), (a, b), (a, c), (b, a), (b, b), (c, a), (c, c)\}$

R_2 is given by $\{(a, a), (a, b), (b, a), (b, b), (c, c)\}$

R_3 is given by $\{(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)\}$

R_4 is given by $\{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$

Complete the table below. If the relation is an equivalence relation give the equivalence classes. Also state whether any of the relations is a partial order, justifying your answer.

	reflexive	symmetric	antisymmetric	transitive	equivalence rel.	
\mathcal{R}_1						
\mathcal{R}_2						
\mathcal{R}_3						
\mathcal{R}_4						

Solution:

	reflexive	symmetric	antisymmetric	transitive	equivalence rel.	Partial order
\mathcal{R}_1	yes	yes	no	no	no	no
\mathcal{R}_2	yes	yes	no	yes	yes	no
\mathcal{R}_3	no	yes	no	no	no	no
\mathcal{R}_4	yes	no	yes	yes	no	yes

\mathcal{R}_\in is an equivalence relation with equivalence classes $T = \{[a], [c]\}$

\mathcal{R}_Δ is the only relation which is a partial order.

Question 4.

Let $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let \mathcal{P} be the partition on \mathcal{S} given by

$$\{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}.$$

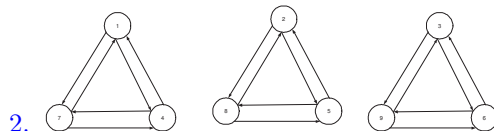
Define \mathcal{R} to be the equivalence relation associated to \mathcal{P} .

1. Give two conditions for \mathcal{P} to be a partition.
2. Draw the relationship digraph.
3. Write down the equivalence class $[5]$ as a set.

Solution:

1. \mathcal{P} is a partition if

- all the elements of \mathcal{S} are in the partition.
- each element of \mathcal{S} appears in one partition only.



3. $[5] = \{2, 5, 8\}$

Question 5.

Let $S = \mathbb{Z} \times \mathbb{N}^+$ and Let \mathcal{R} be relation on S defined as follows:

$$(a, b) \mathcal{R} (c, d) \text{ whenever } ad = bc$$

1. Show that \mathcal{R} is an equivalence relation
2. Define the equivalence class generated by (a, b) , for $a \in \mathbb{Z}$ and $b \in \mathbb{N}^+$

Solution:

1. \mathcal{R} is reflexive as $\forall (a, b) \in S. (a, b) \mathcal{R} (a, b)$
 \mathcal{R} is symmetric as $\forall (a, b), (c, d) \in S. (a, b) \mathcal{R} (c, d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c, d) \mathcal{R} (a, b)$
 \mathcal{R} is transitive as $\forall (a, b), (c, d), (e, f) \in S. (a, b) \mathcal{R} (c, d)$ and $(c, d) \mathcal{R} (e, f) \Rightarrow ad = bc$ and $cf = de \Rightarrow cbde = dacf \Rightarrow be = af \Rightarrow (a, b) \mathcal{R} (e, f)$
2. $[(a, b)] = \{(m, n) : (a, b) \in \mathbb{Z} \times \mathbb{N}^+ \text{ and } \frac{a}{b} = \frac{m}{n}\}$

Question 6.

Let A and B be two sets where:

$A = \{France, Germany, Switzerland, England, Morocco\}$ and

$B = \{French, German, English, Arabic\}$. Let \mathcal{R} be relation defined from A to B , given by $a\mathcal{R}b$ when b is a national language of a . The national language of each of these countries is as follows: French for France, German for Germany, English for England, Arabic for Morocco, whereas, Switzerland has two national languages, French and German. Find the logical matrix for the relation \mathcal{R} .

Solution:

$$M_{\mathcal{R}} = \begin{matrix} & \begin{matrix} French & German & English & Arabic \end{matrix} \\ \begin{matrix} France \\ Germany \\ England \\ Morocco \\ Switzerland \end{matrix} & \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array} \right] \end{matrix}$$

Question 7.

For each of the following relations on the set of all people, state if it is an equivalence relation. Explain your answer.

1. $\mathcal{R}_1 = \{(x, y) | x \text{ and } y \text{ are the same height}\}$.
2. $\mathcal{R}_2 = \{(x, y) | x \text{ and } y \text{ have, at some time, lived in the same country}\}$.
3. $\mathcal{R}_3 = \{(x, y) | x \text{ and } y \text{ have the same first name}\}$.
4. $\mathcal{R}_4 = \{(x, y) | x \text{ is taller than } y\}$.
5. $\mathcal{R}_5 = \{(x, y) | x \text{ and } y \text{ have the same colour hair}\}$.

Solution:

1. \mathcal{R}_1 is reflexive as for all x , x has the same height as x . \mathcal{R}_1 is symmetric as for all x, y , if x has the same height as y , then y has the same height as x . \mathcal{R}_1 is transitive as for all x, y, z , if x has the same height as y **AND** y has the same height as z , then x has the same height as z . Thus, \mathcal{R}_1 is an equivalence relation.

2. \mathcal{R}_2 is reflexive as for all $x, x\mathcal{R}_2x$. \mathcal{R}_2 is symmetric as for all x, y , if x and y have at some time lived in the same country, then y and x have at some time lived in the same country. \mathcal{R}_2 is not transitive as for all x, y, z , if x and y have at some time lived in the same country, **AND** y and z have at some time lived in the same country doesn't necessarily mean that x and z have at some time lived in the same country. \mathcal{R}_2 is then not an equivalence relation.
3. \mathcal{R}_3 is reflexive as for all $x, x\mathcal{R}_3x$. \mathcal{R}_3 is symmetric as for all x, y , if $x\mathcal{R}_3y$ then x and y have the same first name, hence, $y\mathcal{R}_3x$. \mathcal{R}_3 is transitive as for x, y, z if x has the same first name as y and y has the same first name as z then x and z have the same first name as well. \mathcal{R}_3 is then an equivalence relation.
4. \mathcal{R}_4 is not reflexive as for all x , x is not taller than itself. \mathcal{R}_4 is not symmetric either as if x is taller than y then y is not taller than x . Hence, \mathcal{R}_4 is not an equivalence relation.
5. \mathcal{R}_5 is reflexive as for all $x, x\mathcal{R}_5x$. \mathcal{R}_5 is symmetric as for all x, y , if $x\mathcal{R}_5y$ then x and y have the same colour hair, hence, $y\mathcal{R}_5x$. \mathcal{R}_5 is transitive as for x, y, z if x has the same colour hair as y and y has the same colour hair as z then x and z have the same colour hair as well. \mathcal{R}_5 is then an equivalence relation.

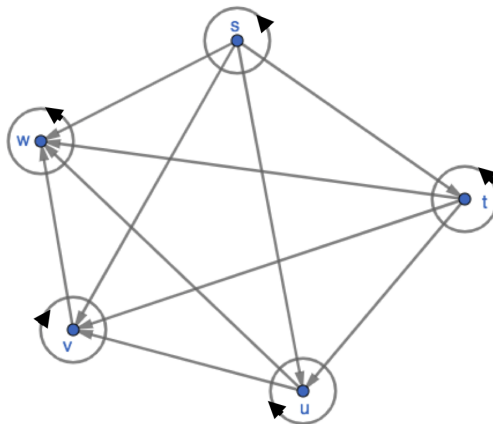
Question 8.

Let $\mathcal{S} = \{\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,4,5\}\}$. Define a relation \mathcal{R} between the elements of \mathcal{S} by

X is related to Y whenever $X \subseteq Y$.

1. Draw the relationship digraph.
2. Determine whether or not \mathcal{R} is reflexive, symmetric, antisymmetric or transitive. Give a brief justification for each of your answers.
3. State, with reasons, whether or not \mathcal{R} is an equivalence relation, whether or not it is a partial order and whether or not it is a total order.

Solution:



1.

$$s = \{1\}, \quad t = \{1, 2\}, \quad u = \{1, 2, 3\}, \quad v = \{1, 2, 3, 4\}, \quad w = \{1, 2, 3, 4, 5\}$$

2. \mathcal{R} is reflexive, $\forall x \in S, x\mathcal{R}x$.

\mathcal{R} is not symmetric, $\{1\} \subseteq \{1, 2\}$ but $\{1, 2\} \not\subseteq \{1\}$.

\mathcal{R} is antisymmetric, $\forall X, Y \in S, X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$.

\mathcal{R} is transitive, $\forall X, Y, Z \in S, X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z$.

3. \mathcal{R} is not an equivalence relation since \mathcal{R} is not symmetric.

\mathcal{R} is reflexive, antisymmetric and transitive then it is a partial order.

\mathcal{R} is total order as it is a partial order and every two element of S are comparable with respect to the relation \mathcal{R} .

Question 9.

Let $\mathcal{S} = \{a, b, c, d\}$ and let $A \subseteq \mathcal{S} \times \mathcal{S}$ be given by

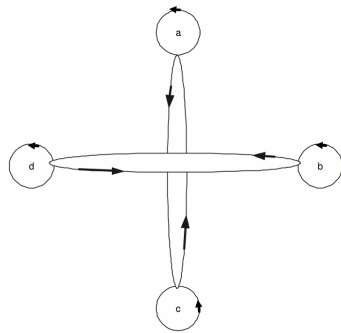
$$\{(a, a), (a, c), (b, b), (b, d), (c, a), (c, c), (d, b), (d, d)\}.$$

A relation \mathcal{R} on \mathcal{S} is defined by

x is related to y whenever $(x, y) \in A$.

1. Draw the relationship digraph.
2. Determine whether or not \mathcal{R} is reflexive, symmetric, antisymmetric or transitive, giving a brief justification for your answer.
3. State, with reasons, whether or not \mathcal{R} is an equivalence relation, whether or not it is a partial order and whether or not it is a total order.

Solution:



1.

2. \mathcal{R} is reflexive for $x \in S$, $x\mathcal{R}x$
 \mathcal{R} is symmetric, $\forall x, y \in S, x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
 \mathcal{R} is not antisymmetric, $a\mathcal{R}b$ but $b \not\mathcal{R}a$.
 \mathcal{R} is transitive if $\forall x, y, z \in S, x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$
3. \mathcal{R} is an equivalence relation. \mathcal{R} is not a partial order as \mathcal{R} is not antisymmetric.
 \mathcal{R} is not a total order as it is not a partial order

Question 10.

Let \mathcal{R} be a relation from a set A to a set B . The inverse of \mathcal{R} , denoted \mathcal{R}^{-1} , is the relation from B to A defined by $\mathcal{R}^{-1} = \{(y, x) : (x, y) \in \mathcal{R}\}$.

Given a relation \mathcal{R} from $A = \{2, 3, 4\}$ to $B = \{3, 4, 5, 6, 7\}$ defined by $(x, y) \in \mathcal{R}$ if x divides y .

1. List the elements of \mathcal{R} and write down the matrix, $M_{\mathcal{R}}$, of \mathcal{R} .
2. List the elements of \mathcal{R}^{-1} and write down the matrix, $M_{\mathcal{R}^{-1}}$, of \mathcal{R}^{-1} .

Solution:

1. $\mathcal{R} = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$

$$M_{\mathcal{R}} = \begin{matrix} & \begin{matrix} 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

2. $\mathcal{R}^{-1} = \{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}$. The relation \mathcal{R}^{-1} can be defined from B to A by $(x, y) \in \mathcal{R}^{-1}$ if x is divisible by y

$$M_{\mathcal{R}^{-1}} = \begin{matrix} & \begin{matrix} 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Question 11.

Let \mathcal{R}_1 and \mathcal{R}_2 be the relations on a set $S = \{1, 2, 3, 4\}$ given by:

$$\mathcal{R}_1 = \{(1, 1), (1, 2), (3, 4), (4, 2), (2, 4)\} \quad \mathcal{R}_2 = \{(1, 1), (3, 2), (4, 4), (2, 2), (4, 2)\}.$$

1. Find the matrix representation \mathcal{R}_1 and that of \mathcal{R}_2 .
2. Find the matrix of the intersection of both matrices in (1).
3. Find the matrix of the union both matrices in (1).

4. list the element of $\mathcal{R}_1 \cap \mathcal{R}_2$.

5. list the element of $\mathcal{R}_1 \cup \mathcal{R}_2$.

Solution:

1.

$$M_{\mathcal{R}_1} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$M_{\mathcal{R}_2} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

2.

$$M_{\mathcal{R}_1 \cap \mathcal{R}_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

3.

$$M_{\mathcal{R}_1 \cup \mathcal{R}_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

4. $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(1, 1), (4, 2)\}$.

5. $\mathcal{R}_1 \cup \mathcal{R}_2 = \{(1, 1), (1, 2), (3, 4), (4, 2), (2, 4), (3, 2), (4, 4), (2, 2)\}$.

Question 12.

Let \mathcal{R} be a relation on set A .

1. How can we quickly determine whether a relation R is reflexive by examining the matrix of \mathcal{R} ?
2. How can we quickly determine whether a relation R is symmetric by examining the matrix of \mathcal{R} ?
3. How can we quickly determine whether a relation R is anti-symmetric by examining the matrix of \mathcal{R} ?

Solution:

Let \mathcal{R}_1 and \mathcal{R}_2 be two relations

1. A relation is reflexive if and only if its matrix has 1's on the main diagonal.
2. A relation \mathcal{R} is symmetric if and only if its matrix $M_{\mathcal{R}}$ satisfies the following: For all i and j , the $(ij)^{th}$ entry of $M_{\mathcal{R}}$ is equal to the j $(ji)^{th}$ entry of $M_{\mathcal{R}}$.
3. A relation \mathcal{R} is anti-symmetric if and only if its matrix $M_{\mathcal{R}}$ satisfies the following: For all i and j with $i \neq j$, if the $(ij)^{th}$ entry of $M_{\mathcal{R}}$ is non-zero then the $(ji)^{th}$ entry of $M_{\mathcal{R}}$ is equal to zero.

Question 13.

For each of following relations on a set $A = \{a, b, c\}$ defined by their corresponding Matrices, say whether is reflexive, symmetric or anti-symmetric.

$$1. M_{\mathcal{R}_1} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$2. M_{\mathcal{R}_2} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$3. M_{\mathcal{R}_3} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$4. M_{\mathcal{R}_4} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

Solution:

$$1. M_{\mathcal{R}_1} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

\mathcal{R}_1 is not reflexive because the leading diagonal is not all 1's as $m_{33} = 0$ which means that c is not related to itself.

\mathcal{R}_1 is not symmetric as $m_{12} = 1$ whereas $m_{21} = 0$. This matrix is not symmetric.

\mathcal{R}_1 is anti-symmetric as if $m_{ij} = 1$ then $m_{ji} = 0$ for all $i \neq j$.

$$2. M_{\mathcal{R}_2} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

\mathcal{R}_2 is reflexive as the leading diagonal is all 1's.

\mathcal{R}_2 is not symmetric as $m_{12} = 1$ whereas $m_{21} = 0$. This matrix is not symmetric.

\mathcal{R}_2 is anti-symmetric as if $m_{ij} = 1$ then $m_{ji} = 0$ for all $i \neq j$.

$$3. M_{\mathcal{R}_3} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

\mathcal{R}_3 is reflexive as the leading diagonal is all 1's.

\mathcal{R}_3 is symmetric as this matrix is symmetric.

\mathcal{R}_3 is anti-symmetric as if $m_{ij} = 1$ then $m_{ji} = 0$ for all $i \neq j$.

$$4. M_{\mathcal{R}_4} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

\mathcal{R}_4 is reflexive as the leading diagonal is all 1's.

\mathcal{R}_4 is symmetric as this matrix is symmetric.

\mathcal{R}_4 is not anti-symmetric as for example $m_{i2} = m_{21} = 1$ or $m_{23} = m_{32} = 1$

Question 14.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on A defined by $x\mathcal{R}y$ if 3 divides $x - y$.

1. Show that the relation \mathcal{R} is an equivalence relation on A .
2. List all the equivalence classes of \mathcal{R} .

Solution:

1. To show that \mathcal{R} is an equivalence relation on a set A , we need to show that it is reflexive, symmetric and transitive.

Reflexive: for all x in A 3 divides $x - x = 0$, hence $x\mathcal{R}x$. Thus \mathcal{R} is a reflexive relation.

Symmetric: for all $x, y \in A$ if 3 divides $x - y$ then 3 divides $y - x$. Hence, \mathcal{R} is a symmetric relation.

Transitive: for all $x, y, z \in A$ if 3 divides $x - y$ and 3 divides $y - z$ then 3 divides $x - y + y - z = x - z$. Hence, for all $x, y, z \in A$ if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$. Thus, \mathcal{R} is a transitive relation.

2. $[1] = \{x \in A : 3 \text{ divides } x - 1\} = \{1, 4, 7, 10\}$.
 $[2] = \{x \in A : 3 \text{ divides } x - 2\} = \{2, 5, 8\}$.
 $[3] = \{x \in A : 3 \text{ divides } x - 3\} = \{3, 6, 9\}$.

Question 15.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on A defined by $x\mathcal{R}y$ if $x \bmod 2 = y \bmod 2$.

1. Show that the relation \mathcal{R} is an equivalence relation on A .
2. List all the equivalence classes of \mathcal{R} .
3. is \mathcal{R} a partial or a total order?

Solution:

1. To show that \mathcal{R} is an equivalence relation on a set A , we need to show that it is reflexive, symmetric and transitive.

Reflexive: for all x in A $x \bmod 2 = x \bmod 2$, hence $x\mathcal{R}x$. Thus \mathcal{R} is a reflexive relation.

Symmetric: for all $x, y \in A$ if $x \bmod 2 = y \bmod 2$ then $y \bmod 2 = x \bmod 2$ as the equal is commutative. Hence, \mathcal{R} is a symmetric relation.

Transitive: for all $x, y, z \in A$ if $x \bmod 2 = y \bmod 2$ and $y \bmod 2 = z \bmod 2$ then $x \bmod 2 = z \bmod 2$. Hence, for all $x, y, z \in A$ if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$. Thus, \mathcal{R} is a transitive relation.

2. $[1] = \{x \in A : x \bmod 2 = 1 \bmod 2 = 1\} = \{1, 3, 5, 7, 9\}$.
 $[2] = \{x \in A : x \bmod 2 = 2 \bmod 2 = 0\} = \{2, 4, 6, 8, 10\}$.
3. partial order: \mathcal{R} is not a partial order as it is not antisymmetric: $2\mathcal{R}4$ and $4\mathcal{R}2$ but $2 \neq 4$. Hence, \mathcal{R} is not a total order.

Question 16.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on A defined by $x\mathcal{R}y$ if $x + y \bmod 2 = 0$.

1. Show that the relation \mathcal{R} is an equivalence relation on A .
2. List all the equivalence classes of \mathcal{R} .
3. is \mathcal{R} a partial or a total order?

Solution:

1. To show that \mathcal{R} is an equivalence relation on a set A , we need to show that it is reflexive, symmetric and transitive.

Reflexive: for all x in A $x + x = 2x \bmod 2 = 0$, hence $x\mathcal{R}x$. Thus \mathcal{R} is a reflexive relation.

Symmetric: for all $x, y \in A$ if $x + y \bmod 2 = 0$ then $y + x \bmod 2 = 0$ as the addition is commutative. Hence, \mathcal{R} is a symmetric relation.

Transitive: for all $x, y, z \in A$ if $x + y = 2m$ and $y + z = 2b$ then $x + y + y + z = 2(n + m)$, which leads to $x + z = 2(n + m - y)$, hence $x + z \bmod 2 = 0$. Hence, for all $x, y, z \in A$ if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$. Thus, \mathcal{R} is a transitive relation.

2. $[1] = \{x \in A : 1 + x \pmod 2 = 0\} = \{1, 3, 5, 7, 9\}.$
 $[2] = \{x \in A : 2 + x \pmod 2 = 0\} = \{2, 4, 6, 8, 10\}.$
3. partial order: \mathcal{R} is not a partial order as it is not antisymmetric: $2\mathcal{R}4$ and $4\mathcal{R}2$ but $2 \neq 4$. Hence, \mathcal{R} is not a total order.

Question 17.

Let \mathcal{R} be a relation on the set $A = \{1, 2, 3, 4, 5\}$ defined by the rule $x\mathcal{R}y$ if $x = y - 1$. Is this relation reflexive, symmetric, antisymmetric, transitive, equivalence, and/or a partial order?

Solution:

Reflexive: $x \neq x - 1$ for all $x \in A$. Thus this relation is not reflexive.

Symmetric: $2\mathcal{R}3$ as $2 = 3 - 1$ however, $3 \not\mathcal{R}2$ as $3 \neq 2 - 1$. Thus \mathcal{R} is not symmetric.

Antisymmetric: we can show this by contradiction: Assume there exists $x, y \in A$ with $x \neq y$ and $x = y - 1$ and $y = x - 1$. This implies that $x = x - 2$, hence, $0 = -2$ (contradiction). Thus \mathcal{R} is antisymmetric.

Transitive: it suffices to give a counter-example to show that this relation is not transitive. $2\mathcal{R}3(2 = 3 - 1)$ and $3\mathcal{R}4(3 = 4 - 1)$, however, $2 \not\mathcal{R}4$ as $2 \neq 4 - 1$. Hence, \mathcal{R} is not a transitive relation.

equivalence: for a relation to be an equivalence relation, it has to be reflexive, symmetric and transitive. However, \mathcal{R} is neither symmetric nor transitive. Hence, \mathcal{R} is not an equivalence relation.

partial order: for a relation to be a partial order, it has to be reflexive, antisymmetric and transitive. However, \mathcal{R} is neither reflexive nor transitive. Hence, \mathcal{R} is not a partial order.

Question 18.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on $A \times A$ defined by $(a, b)\mathcal{R}(c, d)$ if $a + d = b + c$. Show that \mathcal{R} is an equivalence relation on $A \times A$.

Solution:

To show this relation is an equivalence relation, we need to show that it is reflexive, symmetric, and transitive. Reflexive: for all $(a, b) \in A \times A$ we have $(a, b)\mathcal{R}(a, b)$ as $a + b = a + b$.

Symmetric: for all $(a, b), (c, d) \in A \times A$ if $(a, b)\mathcal{R}(c, d)$ then $a + d = b + c$. This implies $c + b = d + a$ as both $=$ and $+$ are commutative. Hence, $(c, d)\mathcal{R}(a, b)$. Thus \mathcal{R} is symmetric.

Transitive: for all $(a, b), (c, d), (e, f) \in A \times A$ we need to show that if $(a, b)\mathcal{R}(c, d)$ and $(c, d)\mathcal{R}(e, f)$ then $(a, b)\mathcal{R}(e, f)$.

$(a, b)\mathcal{R}(c, d)$ and $(c, d)\mathcal{R}(e, f)$ implies that:

(1) $a + d = b + c$ and (2) $c + f = d + e$, hence

(1) $a + d = b + c$ and (2) $c = d + e - f$. If we substitute c from (2) in (1), we get:

$a + d = b + d + e - f$ this implies that $a + e = b + e$, hence, $(a, b)\mathcal{R}(e, f)$. Thus, \mathcal{R} is a transitive relation.

Therefore, \mathcal{R} is an equivalence relation.

Question 19.

Let $X = \{1, 2, 3, 4\}$, $Y = \{3, 4\}$, and $C = \{1, 3\}$ and let \mathcal{R} be a relation on $\mathcal{P}(X)$, the set of all subsets of X , defined as

$$\forall A, B \in \mathcal{P}(X), \quad A\mathcal{R}B \text{ if } A \cup Y = B \cup Y$$

1. Show that \mathcal{R} is an equivalence relation.

2. List the elements of $[C]$, the equivalence class containing C .

Solution:

Writing down the $\mathcal{P}(X)$ is not necessary but we can do it as it has only 16 elements.

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

1. We need to show that \mathcal{R} is reflexive, symmetric and transitive.

Reflexive: for all $A \in \mathcal{P}(X)$ we have $A \cup Y = A \cup Y$. Hence, \mathcal{R} is reflexive.

Symmetric: for all $A, B \in \mathcal{P}(X)$ if $A \mathcal{R} B$ then $A \cup Y = B \cup Y$. This implies that $B \cup Y = A \cup Y$ as equality is commutative. hence, $B \mathcal{R} A$. Thus \mathcal{R} is symmetric.

transitive: for all $A, B, C \in \mathcal{P}(X)$ we need to show if $A \mathcal{R} B$ and $B \mathcal{R} C$ then $A \mathcal{R} C$.

$A \mathcal{R} B$ and $B \mathcal{R} C$ implies that $A \cup Y = B \cup Y$ and $B \cup Y = C \cup Y$. This implies that $A \cup Y = C \cup Y$, hence, $A \mathcal{R} C$. Thus \mathcal{R} is an equivalence relation.

2. $Y = \{3, 4\}$, and $C = \{1, 3\}$

$$[C] = \{B : B \in \mathcal{P}(X) \text{ AND } B \cup Y = C \cup Y = \{1, 3, 4\}\}$$

$$[C] = \{\{1\}, \{1, 3\}, \{1, 4\}, \{1, 3, 4\}\}$$