Discrete Mathematics

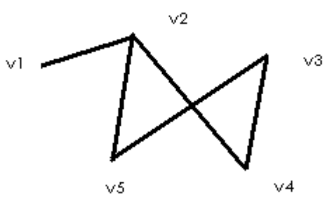
Tutorial solutions | Topic-7 | Graph Theory

**Question 1.**
Given the following graph $G := (V, E)$

$$\begin{aligned} V &= \{v_1, v_2, v_3, v_4, v_5, v_6\} \\ E &= \{\{v_1v_2\}, \{v_2v_5\}, \{v_5v_3\}, \{v_3v_4\}, \{v_2v_4\}\{v_6v_6\}\}. \end{aligned}$$

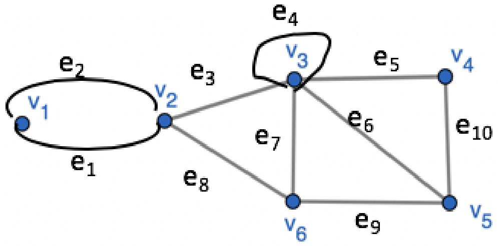
1. Draw the graph $G$.
2. List the set of vertices adjacent to $v_2$.
3. List the set of edges incident with $v_3$.
4. Give an example of a path of length 3 starting at the vertex $v_2$ and ending at the vertex $v_5$.
5. Give an example of a cycle of length 4.

Solution:




1. 
2. $v_1$; $v_4$; $v_5$
3. $v_3v_4$ and $v_3v_5$
4. $v_2v_4v_3v_5$.
5. $v_2v_4v_3v_5v_2$.

**Question 2.**
Given the following graph:




Determine which of the following walks are trails, paths or circuits.

1. $v_1e_1v_2e_3v_3e_4v_3e_5v_4$

2. $e_1e_3e_5e_5e_6$

3. $v_2v_3v_4v_5v_3v_6v_2$

4. $v_2v_3v_4v_5v_6v_2$

5. $v_1e_1v_2e_1v_1$

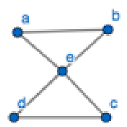
Solution:
A walk: is a sequence of vertices and edges of a graph were vertices and edges can be repeated.
A trail: is a walk in which no edge is repeated. In a trail, vertices can be repeated but no edge is ever repeated.
A circuit: is a is a closed trail. Circuits can have repeated vertices only.
A path: is a trail in which neither vertices nor edges are repeated.

1. $v_1e_1v_2e_3v_3e_4v_3e_5v_4$: this walk has a repeated vertex but it doesn't have a repeated edge. Thus it is trail from $v_1$ to $v_4$ but not a path.
2. $e_1e_3e_5e_5e_6$ : this is a walk from $v_1$ to $v_5$. It is not a trail as it contains a repeated edge
3. $v_2v_3v_4v_5v_3v_6v_2$ : This walk starts and ends at $v_2$, it contains at least one edge and it doesn't have repeated edge, thus it is a circuit
4. $v_2v_3v_4v_5v_6v_2$: This walk starts and ends at $v_2$, it contains at least one edge and it doesn't have repeated edge, thus it is a circuit.
5. $v_1e_1v_2e_1v_1$: this is just a close walk starting and ending at $v_1$ it is not a circuit as edge $e_1$ is repeated.
6. $v_1e_1v_2e_2v_1$: this is just a close walk starting and ending at $v_1$it doesn't have repeated edge, thus it is a circuit.

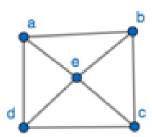
## Question 3.

**ScreenCast**

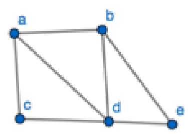
Which of the following undirected graphs have an Euler circuit? Which of those that do not an Euler circuit have an Euler path.




$G_1$




$G_2$

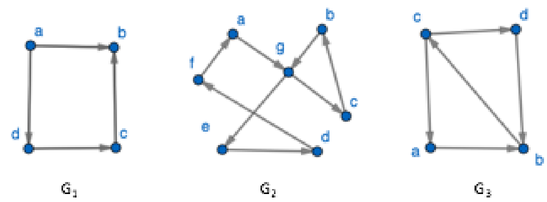



$G_3$

Solution:
An Eulerian path: in a graph is a path that uses each edge of the graph exactly once. it this path exists, the graph is called traversable.
An Eulerian circuit: in a graph is a circuit containing all the graph edges that starts and ends on the same vertex.
The graph $G_1$ contains a circutt $a, e, c, d, e, b, a$, which passes through each edge of of $G_1$ once and once only. Hence, is an Eulerian circuit
The graph $G_2$ doesn't contain an Eulerian path.
$G_3$ doesn't contain an Eulerian circuit, however, it contain a path which uses all the edges of the graph once and once only. This path is $a, cd, e, b, d, a, b$ which is an Eulerian path but not an Eulerian circuit as it doesn't start and ends on the same vertex.
We can also use this theorem although it wasn't covered in the lectures: "**If a graph has an Euler circuit, then every vertex of the graph has a positive even integer**".
we can see that $G_2$ and $G_3$ have vertices of a degree 3, which is an odd number. Thus $G_2$ and $G_3$ don't not contain an Euler circuit.

**Question 4.**

Which of the following directed graphs have an Euler circuit? Which of those that do not an Euler circuit have an Euler path.




Solution:
An Eulerian path: in a graph is a path that uses each edge of the graph exactly once. it this path exists, the graph is called traversable.
An Eulerian circuit: in a graph is a circuit containing all the graph edges that starts and ends on the same vertex.
The graph $G_1$ doesn't contain an Eulerian path.
The graph $G_2$ contains a circuit $a, g, c, b, g, e, d, f, a$, which passes through each edge of of $G_1$ once and once only. Hence, is an Eulerian circuit.
$G_3$ doesn't contain an Eulerian circuit, however, it contain a path which uses all the edges of the graph once and once only. This path is $c, a, b, c, d, b$ which is an Eulerian path but not an Eulerian circuit as it doesn't start and ends on the same vertex.

**Question 5.**

In each of the following either construct a graph with the specified properties or say why it is not possible to do so.

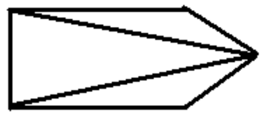
1. A graph with degree sequence 4,3,3,1.

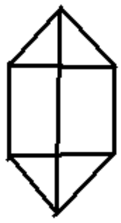
2. A simple graph with degree sequence 4,3,3,2,2.

3. A simple 3 regular graph with 6 vertices.

Solution:

1. The sum of the degrees is 11 (odd). It is impossible to construct a graph with this degree sequences.

2. 

3. 

**Question 6.**

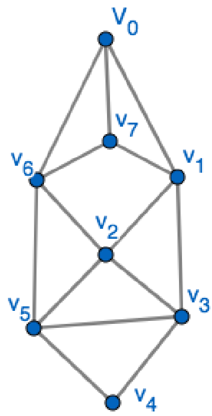
In a group of 25 people, is it possible to each shake hands with exactly 3 other people? Explain your your answer.

Solution:

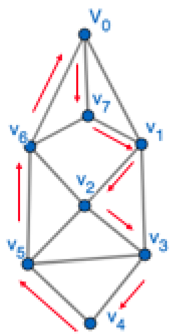
The answer is No. If the people were represented by vertices of a graph and each handshake were represented by an edge joining two vertices, the result would be a graph with 25 vertices and the degree of each vertex is 3, the sum of the degree sequence of this care would be equal $3 \times 25 = 75$, which is odd. This is impossible because the total degree of a graph must be even.

**Question 7.**

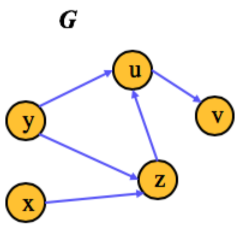
Find a Hamiltonian circuit in the following graph:




Solution:
This graph contains a hamiltonian circuit, $v_0v_7v_1v_2v_3v_4v_5v_6v_0$, shown in red on the graph below.

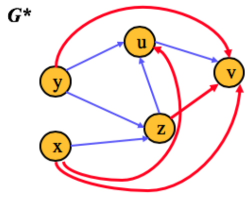



**Question 8.**
Given he following directed graph:




Find the transitive closure, $G^*$, of the graph $G$.
Solution:
To construct the transitive closure of $G$, we only need to add missing edges if there are any on the graph. The transitive closure, $G^*$, of graph $G$ i constructed as follow: We take

the starting point as the graph $G$, we then check if here is a directed path between any two vertices of $G$, for example a directed path from the vertex $u$ to the vertex $v$, then a direct edge is added from $u$ to $v$ if it's not already in the graph. the transitive closure, $G^*$, of the graph $G$ is given below:




### Question 9.

Suppose that 7 sites are connected in a network. The number of other sites to which each site has a direct connection is given by the following sequence

$$1, 2, 2, 3, 3, , 4, 7$$

1. Describe how a communications network such as this may be modeled by a graph, saying what the vertices and edges represent and what it means when two vertices are adjacent.
2. Say how many vertices it has.
3. Find how many connections there are between pairs of sites, giving a brief explanation of your method.
4. Say why it is impossible to construct a *simple* graph with this degree sequence.
5. Say why it is impossible to construct a network with 9 sites, in which each site has a direct connection to exactly 5 of the other sites.

Solution:

1. The vertices represent sites and the edges represent connections. Two vertices are joined by an edge when the corresponding sites have a connection.
2. it has 7 vertices.
3. The sum of the degrees is twice the number of edges.

$$No\ of\ edges = \frac{7+4+3+3+2+1}{2} = \frac{22}{2} = 11$$

4. There are 7 vertices or sites but one of them has 7 incident edges. As there are only 6 other vertices to connect to, one at least must be a parallel edge or a loop.

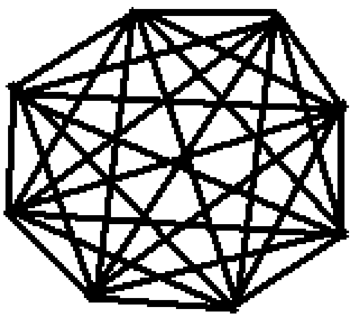
5. The sum of the degree sequence $= 9 \times 5 = 45$ (odd number). Hence, it is impossible to construct network with 9 sites and each site has exactly 5 direct connections.

**Question 10.**

1. What is meant by a **complete** graph?
2. What is the degree of each vertex of the complete graph $K_8$? Calculate the number of edges in $K_8$. Draw $K_8$.
3. What is the degree of each vertex of the complete graph $K_n$? How many edges will it have?

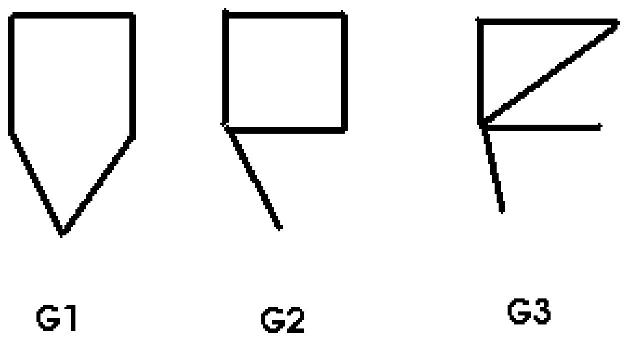
Solution:

1. A complete graph is a graph where every pair of vertices is joined by exactly one edge.
2. each vertex of $K_8$ has a degree 7. The number of edges is $\frac{8\times 7}{2} = 28$.



3. The degree of each vertex of a complete graph $K_n$ is $n-1$. The number of edges if $\frac{n\times(n-1)}{2}$

**Question 11.**
Construct 3 non isomorphic graphs with 5 vertices and 5 edges. Give one property for each graph that neither of the others has, which makes it non-isomorphic.
Solution:
G1, G2 and G3 are isomorphic: G1 has a cycle of length 5, G2 has a cycle of length 4 and G3




has a cycle of length 3.

**Question 12.**

Draw the two graphs with adjacency lists

- $v_1 : v_2, v_5$
- $v_2 : v_1, v_3, v_4, v_5$
- $v_3 : v_2, v_4, v_5$
- $v_4 : v_2, v_3, v_5$
- $v_5 : v_1, v_2, v_3, v_4$

and

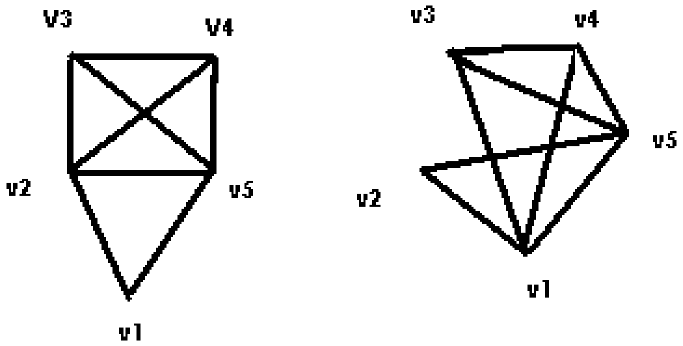
- $v_1 : v_2, v_3, v_4, v_5$
- $v_2 : v_1, v_5$
- $v_3 : v_1, v_4, v_5$
- $v_4 : v_1, v_3, v_5$
- $v_5 : v_1, v_2, v_3, v_4$

Are these graphs isomorphic? If so, show the correspondence between them.

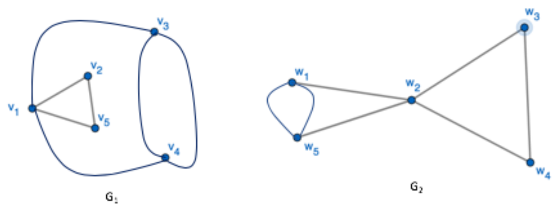
Solution:

The following two graph are isomorphic. the function showing the correspondence is

| v | v1 | v2 | v3 | v4 | v5 |
|---|---|---|---|---|---|
| f(v) | v2 | v1 | v3 | v4 | v5 |




**Question 13.**
Show that the following graphs are isomorphic:




Solution:
Both graphs have the same degree sequence, $4, 3, 3, 2, 2$, as they can't be isomorphic if they have different degree sequence.
To solve this problem, we must find a bijection $f : V(G_1) \to V(G_2)$ so that for any $v, w \in V(G_1)$, the number of edges connecting $v$ to $w$ is the same as the number of edges connecting $f(v)$ to $f(w)$. We can clearly see that such a function exists and it is defined as follows:

| $v \in V(G-1)$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|---|
| $f(v) \in V(G_2)$ | $w_2$ | $w_3$ | $w_1$ | $w_5$ | $w_4$ |

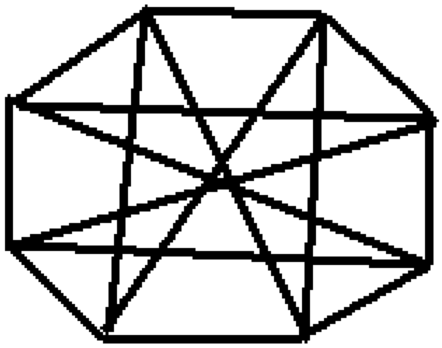
**Question 14.**

1. A simple connected graph has 7 vertices, all having the same degree $d$. Give the possible values of $d$ and for each value of $d$ give the number of edges of the graph.

2. Another simple connected graph has 8 vertices, all of the same degree sequence $d$. Draw this graph when $d = 4$ and give the other possible. values of $d$.

Solution:

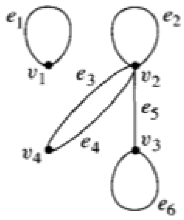
1. d= 6, 4, 2, 0 for d=6 the number of edges $= \frac{7\times 6}{2} = 21$
   for d=4 the number of edges $= \frac{7\times 4}{2} = 14$
   for d=2 the number of edges $= \frac{7\times 2}{2} = 7$
   for d=0 the number of edges $= \frac{7\times 0}{2} = 0$



2. The other values of d are 0,2,3,4,,5,,6,7.

**Question 15.**
Find adjacency matrices for the following undirected graphs

1. $G_1$




2. $K_4$, the complete graph with 4 vertices.

3. $K_{2,3}$, the complete bipartite graph on $(2,3)$ vertices.

Solution:

1.

$$A_{G_1} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

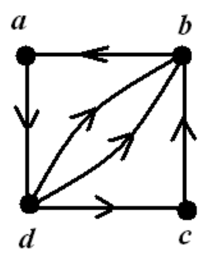
2.

$$A_{K_4} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

3.

$$A_{K_{2,3}} = \begin{array}{c} \\ a_0 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & b_a & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

**Question 16.**
Given the following digraph G




1. Write down the adjacency matrix M of G
2. Compute $M^2$
3. Find the number of paths of length 2 starting from the vertex a and ending in b.
4. What information does $M^2$ contain?
5. What information does $M^3$ contain?

Solution:

1. $M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix}$

2. $M^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$

3. there are two paths of length 2 starting from the vertex a and ending in b.
4. $M^2$ contains information about paths of length 2
5. $M^3$ contains information about paths of length 3.

**Question 17.**
Given the following adjacency matrix of a graph $G$:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

1. Find $A^2$ and $A^3$.
2. How many walks of length 2 are there from $v_2$ to $v_3$.
3. How many walks of length 2 are there from $v_3$ to $v_4$.
4. How many walks of length 3 are there from $v_1$ to $v_4$.
5. How many walks of length 3 are there from $v_2$ to $v_3$.

Solution:

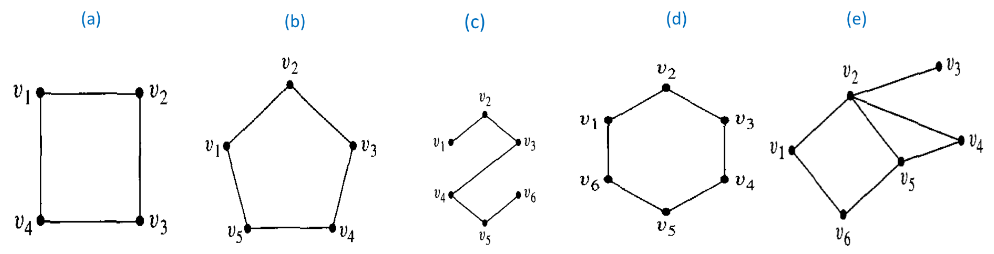
1.

$$A^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 3 \\ 2 & 2 & 6 & 3 \\ 2 & 3 & 3 & 3 \end{pmatrix} \end{array}$$

$$A^3 = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 6 & 2 & 3 \\ 2 & 2 & 6 & 3 \\ 2 & 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 4 & 8 & 8 & 6 \\ 8 & 9 & 17 & 11 \\ 8 & 17 & 9 & 11 \\ 6 & 11 & 11 & 9 \end{pmatrix} \end{array}$$

2. $(A^2)_{23} = 2$, hence, there are 2 walks of length 2 from $v_2$ to $v_3$
3. $(A^2)_{34} = 3$, hence, there are 3 walks of length 2 from $v_3$ to $v_4$
4. $(A^3)_{14} = 6$, hence, there are 6 walks of length 3 from $v_1$ to $v_4$
5. $(A^3)_{23} = 17$, hence, there are 17 walks of length 3 from $v_2$ to $v_3$

**Question 18.**
**<u>creenCast:</u>**
Find which of the following graphs are bipartite, redraw the bipartite graphs so that their bipartite nature is evident.




<u>Solution:</u>
(a) (c) and (d) are bipartite graphs and their bipartite nature is showing in Figure 1

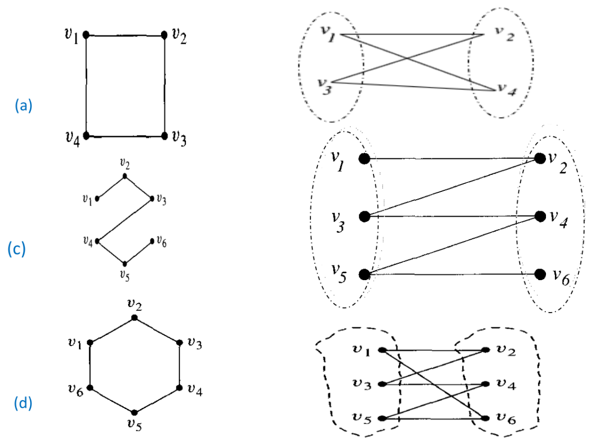



Figure 1: (a) (c) and (d)

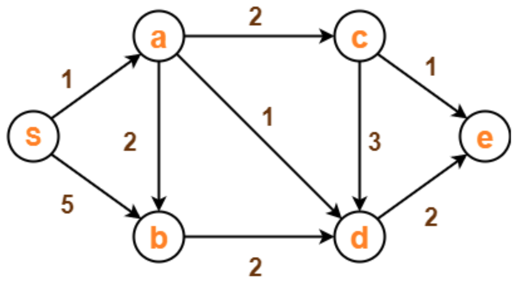
(b). Suppose the graph were bipartite with disjoint vertex sets $V_1$, and $V_2$, where no vertices within either $V_1$ or $V_2$ are connected by edges. Then $v_1$ would be in one of the sets, say $V_1$ , and so $v_2$ and $v_5$ would be in $V_2$ (because each is connected by an edge to v1). Furthermore, $v_3$ and $v_4$ would be in $V_1$, (because $v_3$ is connected by an edge to $v_2$ and $v_4$ is connected by an edge to $v_5$ ). But $v_3$ is connected by an edge to $v_4$, and so both cannot be in the same set, $V_1$. This contradiction shows that the supposition is false, and so the graph is not bipartite. Another way to look at it is that the graph $(b)$ has a cycle of an odd length, hence, it can't be a bipartite graph.
(e) Suppose the graph were bipartite with disjoint vertex sets $V_1$ and $V_2$, where no vertices within either $V_1$ or $V_2$ are connected by edges. Then the vertex $v_1$ would be in one of the sets, say $V_1$, and so $v_2$ and $v_6$ would be in $V_2$ (because each is connected by an edge to v1). Furthermore, $v_3, v_4$, and $v_5$ would be in $V_1$, (because all are connected by edges to $v_2$).

But $v_4$ is connected by an edge to $v_5$ , and so both cannot be in $V_1$. This contradiction shows that the supposition is false, and so the graph is not bipartite.
Another way to look at it is that the graph $(e)$ has a cycle of an odd length, hence, it can't be a bipartite graph.

**Question 19.**

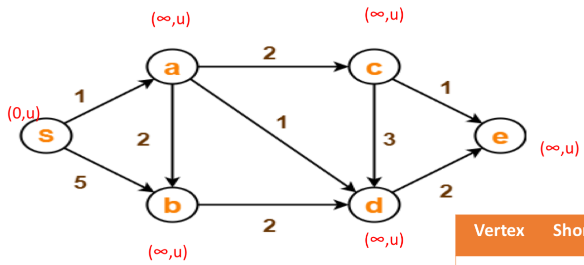
**ScreenCast**

Using Dijkstra's Algorithm, find the shortest distance from source vertex ÔSÕ to remaining vertices in the following graph:



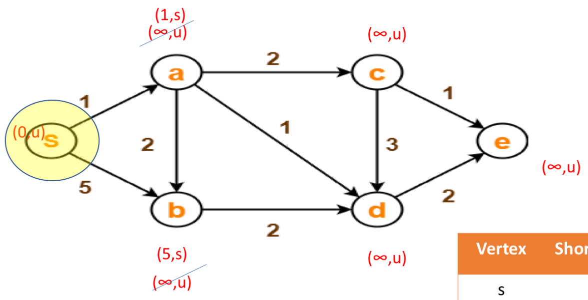

Solution:

Step 1: initialisation:
The following is created created-
Unvisited set $= \{s, a, b, c, d, e\}$
$Prev(a) = Prev(a) = Prev(b) = Prev(c) = Prev(d) = Prev(e) = NIL = undefined = u$
$dist(S) = 0$
$dist(a) = dist(b) = dist(c) = dist(d) = dist(e) = \infty$




| Vertex | Shortest distance from S | Previous vertex |
|---|---|---|
| s | 0 | undefined |
| a | ∞ | undefined |
| d | ∞ | undefined |
| c | ∞ | undefined |
| d | ∞ | undefined |
| e | ∞ | undefined |

Step 2: 1st iteration
Unvisited set $= \{a, b, c, d, e\}$
$Prev(a) = s, Prev(b) = s$
$dist(a) = 1 < \infty$, dist(b) $= 5 < \infty$,
The distance of a and b are updated as well as their previous vertex.




| Vertex | Shortest distance from S | Previous vertex |
|---|---|---|
| s | 0 | undefined |
| a | 1 | s |
| d | 5 | s |
| c | ∞ | undefined |
| d | ∞ | undefined |
| e | ∞ | undefined |

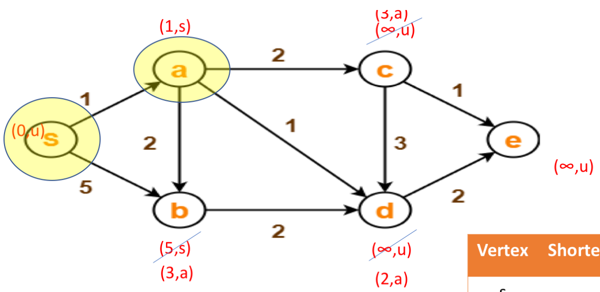
<u>Step 3: 2nd iteration</u>

Unvisited set $= \{b, c, d, e\}$

$dist(c) = 1 + 2 = 3 < \infty$, and $Prev(c) = a$

$dist(b) = 1 + 2 = 3 < 5$, and $Prev(b) = a$

$dist(d) = 1 + 1 = 2 < \infty$, and $Prev(d) = a$



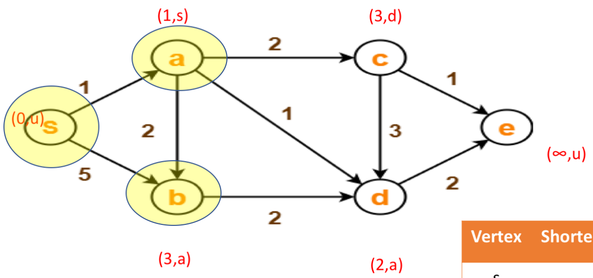

| Vertex | Shortest distance from S | Previous vertex |
|---|---|---|
| s | 0 | undefined |
| a | 1 | s |
| b | 3 | a |
| c | 3 | a |
| d | 2 | a |
| e | ∞ | undefined |

<u>Step 4: 3rd iteration</u>

The selected nod is $b$.

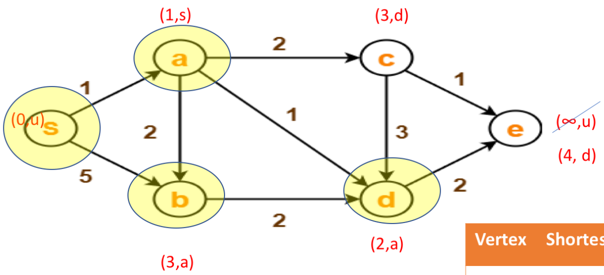
Unvisited set $= \{c, d, e\}$

$dist(d) = dist(b) + 2 = 5 > 2$, non change



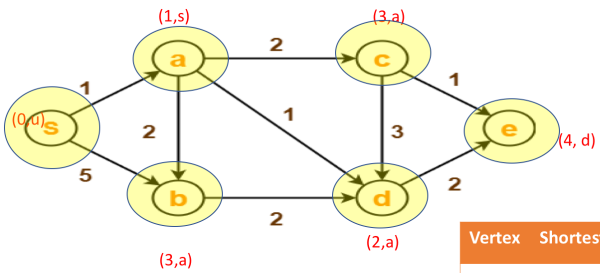

| Vertex | Shortest distance from S | Previous vertex |
|---|---|---|
| s | 0 | undefined |
| a | 1 | s |
| b | 3 | a |
| c | 3 | a |
| d | 2 | a |
| e | ∞ | undefined |

Step 5: 4th iteration
The selected nod is $d$.
Unvisited set $= \{c, e\}$
$dist(e) = dist(d) + 2 = 2 + 2 = 4 < \infty$, and $Prev(e) = d$.

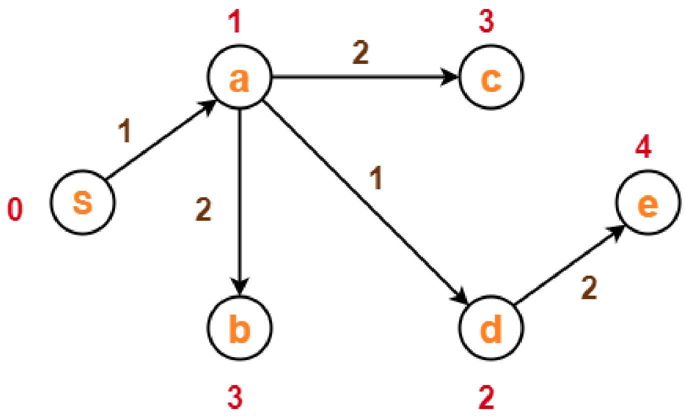



| Vertex | Shortest distance from S | Previous vertex |
|---|---|---|
| s | 0 | undefined |
| a | 1 | s |
| b | 3 | a |
| c | 3 | a |
| d | 2 | a |
| e | 4 | d |

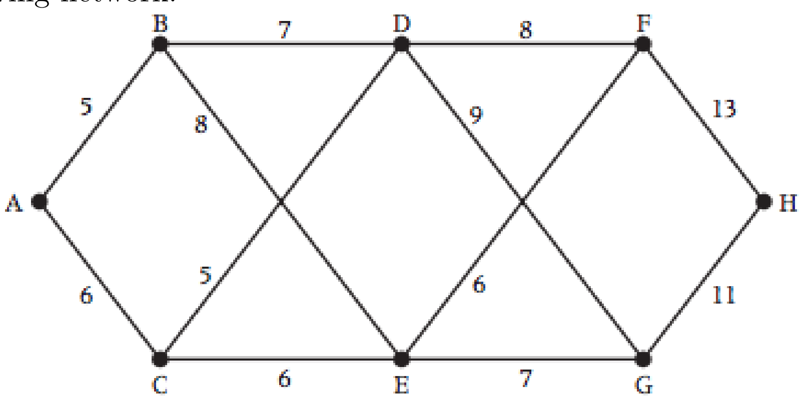
Step 6: 4th iteration
The selected nod is $e$.
Unvisited set $= \{\}$
This is because shortest path estimate for vertex $e$ is last. There are no outgoing edges for vertex $e$. So, our shortest path tree remains the same




| Vertex | Shortest distance from S | Previous vertex |
|---|---|---|
| s | 0 | undefined |
| a | 1 | s |
| b | 3 | a |
| c | 3 | a |
| d | 2 | a |
| e | 4 | d |




**Question 20.**
Use Dijkstra's algorithm to find the shortest distance from $A$ to $H$ in the following network:




Solution:

By using Dijkstra's algorithm, the shortest distance from $A$ to $H$ is 30 as is shown on the following graph.