THIS PAPER IS NOT TO BE REMOVED FROM THE EXAMINATION HALL

UNIVERSITY OF LONDON

**CM1020**

**BSc EXAMINATION**

**COMPUTER SCIENCE**

**Discrete Mathematics – Sample Exam Paper**

Time allowed: 2 hours

**DO NOT TURN OVER UNTIL TOLD TO BEGIN**

**INSTRUCTIONS TO CANDIDATES:**

This examination paper is in two parts: Part A and Part B. You should answer **ALL** of question 1 in Part A and **TWO** questions from Part B. Part A carries 40 marks, and each question from Part B carries 30 marks. If you answer more than **TWO** questions from **Part B** only your first **TWO** answers will be marked.

All answers must be written in the answer books, answers written on the question paper will not be marked. You may write notes in your answer book. Any notes or additional answers in the answer book(s) should be crossed out.

The marks for each part of a question are indicated at the end of the part in [.] brackets. There are 100 marks available on this paper.

Graph Paper is provided at the end of this question paper. If used, it must be detached and fastened securely inside the answer book.

A handheld calculator may be used when answering questions on this paper but it must not be pre-programmed or able to display graphics, text or algebraic equations. The make and type of machine must be stated clearly on the front cover of the answer book.

## PART A

Candidates should answer **ALL** of Question 1 in Part A.

## Question 1

(a) Let $A$ be a set with $5$ elements. What is the number of subsets which can be formed from $A$?

Choose ONE option. [4]

i. $10$

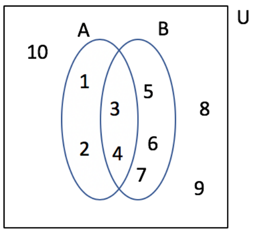
ii. $25$

iii. $32$

iv. $64$

*iii*

(b) Given the following Venn diagram representing two sets $A$ and $B$, which are subsets of the universal set $U$:




Which one of the following sets represents $(\overline{A \cap B})$?

Choose ONE option. [4]

i. $\{1, 2, 5, 6, 7, 8, 9, 10\}$

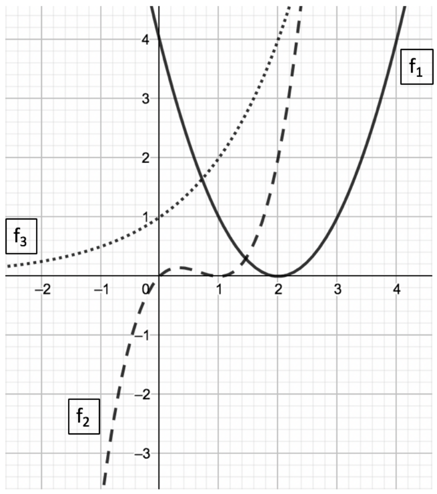
ii. $\{3, 4, 8, 9, 10\}$

iii. $\{8, 9, 10\}$

iv. none of the other options is correct

*i*

(c) The following graph shows the curves of three functions, $f_1, f_2$ and $f_3$:




Which of these functions are not invertible?
Choose ONE option. [4]

i. $f_3$
ii. $f_1$ and $f_3$
iii. $f_2$ and $f_3$
iv. $f_1$ and $f_2$

*iv*

(d) Let $p$ and $q$ be the following propositions concerning a positive integer $n$ where $p$ means '$n$ is less than or equal to 7 ' and $q$ means '$n$ cannot be represented using 3 binary digits' . Which one of the following logical expressions is equivalent to a correct formalisation of the sentence below?

' if $n$ is less than or equal to 7 then $n$ can be represented using 3 binary digits'

Choose ONE option. [4]

i. $p \wedge \neg q$
ii. $p \vee \neg q$
iii. $p \rightarrow \neg q$
iv. $p \rightarrow q$

*iii*

(e) Given the statement $S(x)$: ' $x^2 + 1 = 5$', select the correct statement from the following.
Choose ONE option. [4]

i. $S$ can be expressed using propositional logic
ii. $S$ is not a proposition, as its truth value is a function depending on $x$
iii. The truth value of $S(2)$ is False
iv. The truth value of $S(3)$ is True

*ii*

(f) The number of ways $k$ objects can be selected from $n$ objects where the ordering of the outputs is not important can be calculated using which one of the following formulations?
Choose ONE option. [4]

i. $\frac{n!}{k!(n-k)!}$
ii. $\frac{n!}{(n-k)!}$
iii. $\frac{n!}{k!}$
iv. $n(n-k)$

*i*

(g) Which one of the following degree sequences cannot represent a simple graph? [4]

i. $5, 3, 3, 2, 2$
ii. $4, 2, 2, 2, 2$
iii. $2, 2, 2, 2, 2$
iv. $4, 3, 3, 2, 2$

*i*

(h) Which one the following is a correct definition of a Hamiltonian path? [4]

i. A Hamiltonian path in a graph $G$ is a path that uses each edge in $G$ precisely once
ii. A Hamiltonian path in graph $G$ is a path that visits each vertex in $G$ exactly once
iii. A Hamiltonian path is a trail in which neither vertices nor edges are repeated
iv. A Hamiltonian path is a walk in which no edge is repeated

*ii*

(i) Let $S = \{1, 2, 3\}$ and $\mathcal{R}$ be a relation on elements in $S$ with

$$\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

Which one of the following statements is correct about the relation $\mathcal{R}$?

[4]

i. $\mathcal{R}$ is reflexive, is symmetric and is transitive
ii. $\mathcal{R}$ is NOT reflexive, is symmetric and is transitive
iii. $\mathcal{R}$ is reflexive, is symmetric and is NOT transitive
iv. $\mathcal{R}$ is NOT reflexive, NOT symmetric and NOT transitive

*ii*

(j) Let $S = \{a, b, c\}$ and $\mathcal{R}$ be a relation on elements in $S$ with

$$\mathcal{R} = \{(c, b), (a, a), (b, c)\}$$

Which one of the following statements is correct about the relation $\mathcal{R}$?

[4]

i. $\mathcal{R}$ is NOT reflexive, is NOT symmetric, and is NOT transitive.
ii. $\mathcal{R}$ is NOT reflexive, is NOT symmetric, and is transitive.
iii. $\mathcal{R}$ is NOT reflexive, is symmetric, and is NOT transitive.
iv. $\mathcal{R}$ is reflexive, is symmetric, and is transitive.

*iii*

## PART B

Candidates should answer any **TWO** questions from Part B.

**Question 2** Set theory, Propositional and Predicate logic & Combinatorics

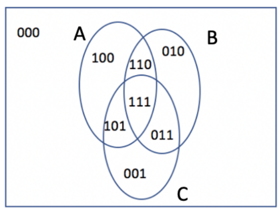
(a) i. Rewrite the following three sets using the listing method:

- $A = \{n^2 + (-1)^n : n \in \mathbb{Z} \text{ and } 0 \leq n < 5\}$
- $B = \{n + \frac{1}{n} : n \in \mathbb{Z}^+ \text{ and } n < 6\}$
- $C = \{(-2)^{-n} : n \in \mathbb{Z}^+ \text{ and } n < 5\}$

[3]

- $A = \{n^2 + (-1)^n : n \in \mathbb{Z} \textit{ and } 0 \leq n < 5\} = \{0, 1, 5, 8\}$
- $B = \{n - \frac{1}{n} : n \in \mathbb{Z}^+ \textit{ and } n < 5\} = \{1 - 1, 2 - \frac{1}{2}, 3 - \frac{1}{3}, 4 - \frac{1}{4}, 5 - \frac{1}{5} = \{0, \frac{3}{2}, \frac{8}{3}, \frac{15}{4}\}$
- $c = \{(-2)^{-n} : n \in \mathbb{Z}^+\} = \{-1/2, 1/4, -1/8, 1/16\}$

ii. Given the following Venn diagram representing three sets $A$, $B$ and $C$ intersecting in the most general way. Three binary digits are used to refer to each one of the 8 region in this diagram. In terms of $A$, $B$ and $C$, Write the set representing the area comprising the regions 011, 101 and 111. The answer must be written in its simplest




[3]

$(A \cup B) \cap C$

iii. Given three sets $A, B$ and $C$. Using set identities, prove that the expression $(A - B) - (B - C)$ is equivalent to $A - B$. [4]

$$
\begin{aligned}
(A - B) - (B - C) &= (A - B) \cap \overline{(B - C)} && \textit{De Morgan's law} \\
&= (A - B) \cap \overline{(B \cap \overline{C})} && \textit{double complement} \\
&= (A - B) \cap (\overline{B} \cup C) && \textit{De Morgan's law} \\
&= (A \cap \overline{B}) \cap \overline{B} \cap (A \cap \overline{B}) \cap C) && \textit{distributivity} \\
&= A \cap \overline{B} \\
&= A - B
\end{aligned}
$$

*[ 3 marks if the names of the algebraic laws are missing]*

(b) i. Let $p$ and $q$ be two propositions. Assume that $p$ is false and $q$ is true. Determine the truth value of for each of the following: $p \to q$; $\neg p \vee q$; $q \oplus \neg p$ and $\neg q \to p$. [4]

*$p \to q = T$; $\neg p \vee q = T$; $q \oplus \neg p = F$ and $\neg q \to p = T$.. [1 mark each]*

ii. Let $p$, $q$ and $r$ denote the following statements:

$p$ : 'I finish my home work'

$q$ : 'I will go to the gym'

$r$ : ' it's raining '

1. Write the following the following statements into their corresponding symbolic forms.
   - if i finish my homework and it is raining then i will go to the gym.
   - i will go to the gym or finish my homework but not both.

[4]

- *If i finish my homework and it is raining then i will go to the gym. = $(p \wedge r) \to q$ [2 marks]*
- *i will go to the gym or finish my homework but not both. $(p \oplus q)$ [2 marks]*

2. Write in words the contrapositive of the the following statement:
   'if i finish my homework and it is raining then i will go to the gym.' [2]
   *if don't go to the gym then it is not raining or i didn't finish writing my homework*

(c) Let $A$ be the set of all students, $p(x)$ be the proposition: 'x is enrolled in Discrete Mathematics module' and $S(x)$ is the set of all students in the same year as $x$' where x is an element of $A$. Use rules of inference with quantifiers to formalise the three following statements:

i. None of the students are enrolled in the Discrete Mathematics module. [2]

ii. Not every student is enrolled in Discrete the Mathematics module.

[2]

iii. There exists a student where no other student in their year are enrolled in the Discrete Mathematics module

[2]

*i. None of the students are enrolled in the Discrete Mathematics module. = $\forall x \in A, \neg p(x)$. [2 marks]*

ii. *Not every student is enrolled in Discrete the Mathematics module.* *=* $\exists x \in A, \neg p(x)$. *[2 marks]*

iii. *There exists a student where no other student in their year are enrolled in the Discrete Mathematics module =* $\exists x \in A, \forall y \in S(x) \neg p(y)$. *[2 marks]*

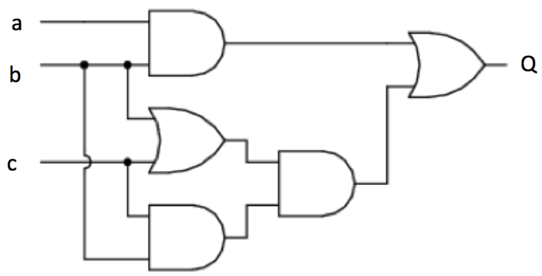
(d) Suppose that a salesman has to visit eight different cities. He must begin his trip in a specified city, but can visit the other seven cities in any order he wishes. How many possible orders can the salesman use when visiting these cities?

[4]

$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! = .5040$ *possible ways to visit the city.*

**Question 3** Boolean Algebra, Functions & Recursion

(a) Given the following logical circuit with three inputs $a, b$ and $c$:




i. Identify the logical gates used in this circuit. [2]

ii. What is the the logical expression of the output of this circuit? [4]

iii. Simplify the logical expression in (ii). Explain your answer. [4]

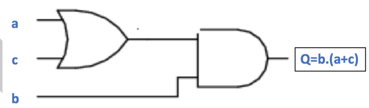
iv. Draw the resulting simplified circuit. [2]

*i. OR, AND gates [2 marks]*

*ii.* $a.b + b.c(b + c)$ *[4 marks]*

*iii.* $a.b + b.c(b + c) = a.b + b.c.b + b.c.c = a.b + bc + bc = a.b + b.c = b.(a + c)$. *[4 marks]*

*iv.*




*[2 marks]*

(b) i. Name two properties a function has to satisfy to be a an invertible function. [2]

ii. Show that the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x - 5$ is a bijection. [2]

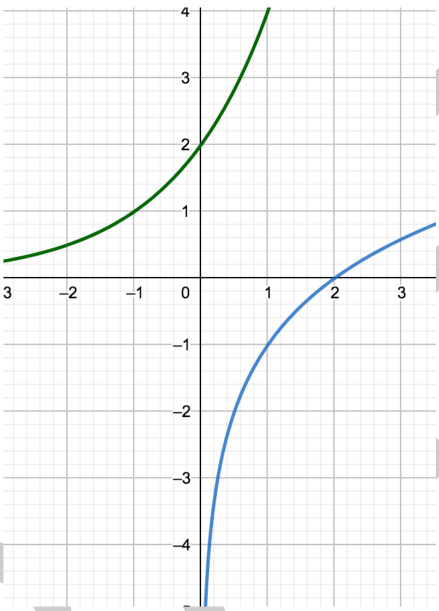
iii. Consider the following function invertible $f : \mathbb{R} \to \mathbb{R}^+$ with $f(x) = 2^{x+1}$. Find its inverse function, $f^{-1}$. [2]

iv. . Plot the curve of f and $f^{-1}$ in the graph. [4]

*i. the function has to be one to one (injective) and onto (surjective). [1 marks each]*

ii. *We $f$ need to show that f is injective or surjective.*
*Injective: let $a, b \in \mathbb{R} = D_f$, assume that $f(a) = f(b) \implies 3a - 5 = 3b - 5 \implies a = b$, hence, $f$ in an injection. [1 marks] surjective: let $y \in \mathbb{R} = co - D_f$, we need to show there exists $x \in D_f$ such that $f(x) = y$. $f(x) = y \implies 3x - 5 = y \implies x = \frac{y+5}{3} \in \mathbb{R} = D - f$ hence, $f$ is a surjection. [1 marks] . Therefore, $f$ is a bijection.*

iii. $f^{-1} : \mathbb{R}^+ \to \mathbb{R} with f^{-1}(x) = log_2 x - 1$ *[ 2 marks]*




iv.

(c) i. Let's consider the recursive relation:

$f_n = n * f_{n-1}$ for all integer $n \geq 1$ with $f_1 = 1$.

What are the values of $f_2$ and $f_3$? [2]

ii. Let $S_n = \sum_{i=1}^{i=n} (4i - 1)$ for all $n \in \mathbb{Z}^+$.

1. Find $S_1$ and $S_2$.

[1]

2. Prove by induction that $S_n = 2n^2 + n$ for all $n \in \mathbb{Z}^+$.

[5]

i. *LetÕs consider the recursive relation:*

$f_2 = 2 * f_1 = 2 * 1 = 2$ *[1 mark] and* $f_3 = 3 * f_2 = 3 * 2 * f_1 = 3! = 6$ *[1 mark]*

ii. Let $S_n = \sum_{i=1}^{i=n}(4i-1)$ for all $n \in \mathbb{Z}^+$.

1. $S_1 = 4*1-1 = 3$ [0.5 mark] and $S_2 = 4*1-1+4*2-1 = 10$ [0.5 mark].

2. Prove by induction that $S_n = 2n^2+n$ for all $n \in \mathbb{Z}^+$.
   base case:. $S_1 = 1 = 1^2$, this true. [1 mark]
   Induction hypothesis: Assume that for $n = k, S_k = 2k^2+k$. [0.5 mark]
   Induction step: we need to show that $S_{k+1} = 2(k+1)^2+(k+1)$. [0.5 mark]

   $$\begin{aligned} S_{k+1} &= S_k + 4(k+1) - 1 && \text{by definition}[1mark] \\ &= 2k^2 + k + 4(k+1) - 1 && \text{induction hypothesis} \\ &= 2(k+1)^2 + (k+1).[1mark] \end{aligned}$$

   Hence, $S_n = n^2$ for all $n \in \mathbb{Z}^+$. [ 1 marks]

This question has a mark count mismatch: expected , got **30**.

**Question 4** Graph Theory, Trees & Relations

(a) i. Give a definition of a simple graph. [2]

ii. Is it possible to draw a simple graph with a degree sequence, 4, 3, 3, 2? If yes draw the graph and if no, explain why. [2]

iii. Draw the two graphs with adjacency lists

- $a_1 : a_2, a_5$
- $a_2 : a_1, a_3, a_4, a_5$
- $a_3 : a_2, a_4, a_5$
- $a_4 : a_2, a_3, a_5$
- $a_5 : a_1, a_2, a_3, a_4$

and

- $b_1 : b_2, b_3, b_4, b_5$
- $b_2 : b_1, b_5$
- $b_3 : b_1, b_4, b_5$
- $b_4 : b_1, b_3, b_5$
- $b_5 : b_1, b_2, b_3, b_4$

[2]

1. Write down the degree sequence for each graph above. [2]
2. Are these graphs isomorphic? If so, show the correspondence between them. [2]

*i. A simple graph is a graph with no loops and no parallel edges [2 marks]*

*ii. It is not possible to draw a simple graph with the degree sequence 4, 3, 3, 2. This graph has 4 vertices with one vertex having a degree 5, however, there are only 3 other vertices to be connected to. therefore the 4th connection can only a result of parallel edge or a loop. Hence, you cannot draw a graph with this degree sequence but not simple. [3 marks]*

*iii. correct graphs [2 marks*

*iv. 1. correct degree sequence [ 2 marks]*

*2. yes there is bijection which maps vertices of one graph to the vertices of the other.*

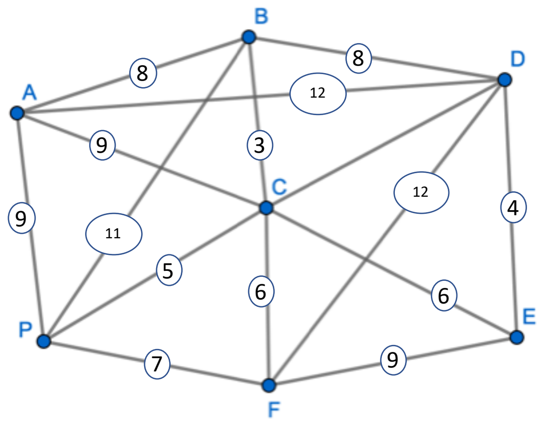
| $a$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $f(a)$ | $b_2$ | $b_1$ | $b_3$ | $b_4$ | $b_5$ |

*[2 marks]*

(b) i. What is the number of vertices in a tree with n edges? [2]

ii. Explain how to find the minimum spanning tree in a weighted graph using Prim's algorithm. [2]

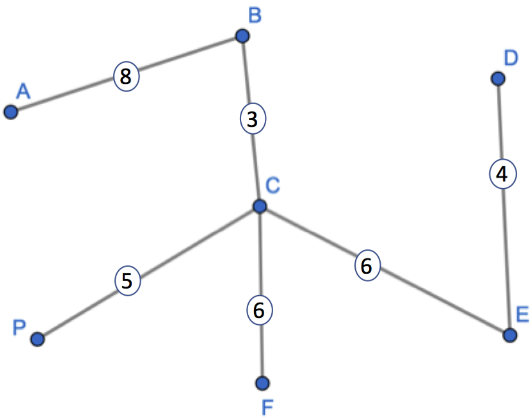
iii. A television company are setting up a new cable television network. Their system has a base station in the local business part (P) and 6 hubs (A - F) which then serve the local areas. The company wants to find the cheapest way to connect all hubs. The following graph shows the cost of each connection (in the unit $\pounds 10000$). Which cables should they lay and how much will it cost them. [6]




*i. n +1?*

*ii. 1 : Select a start vertex and make this part of your solution. 2 consider all the edges which connect any vertex already in your solution to one not in the solution, and choose the cheapest or smallest, additing it to the solution. 3 repeat sep 2 until all the vertices have been added.*

*iii. this means to find the minimum spanning tree for this graph. We can use Prim's algorithm to find the minimum spanning and get the followoing.*




*[4 marks]*

*the sum of the selected edges is 32. hence the total cost is 320 000. [2 marks]*

(c) Given $S$ be the set of integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $\mathcal{R}$ be a relation defined on $S$ by the following condition such that,
for all $x, y \in S$, $x\mathcal{R}y$ if $x \mod 3 = y \mod 3$ .

i. Draw the digraph of $\mathcal{R}$. [2]

ii. Say with reason whether or not $\mathcal{R}$ is

- reflexive;
- symmetric;
- anti-symmetric;
- transitive.

In the cases where the given property does not hold provide a counter example to justify this. [4]

iii. is $\mathcal{R}$ a partial order? Explain your answer. [1]

iv. is $\mathcal{R}$ an equivalence relation? If the answer is yes, write down the equivalence classes for this relation and if the answer is no, explain why. [3]

*i. correct digraph of $\mathcal{R}$. [2 marks]*

*ii.*
- *reflexive: yes [.5 mark] ;*
- *symmetric: yes [.5 mark]*
- *anti-symmetric: no [.5 mark];*
- *transitive: yes [.5 mark]*

*iii. $\mathcal{R}$ is not a partial order: as not Anti-Symmetric [1 mark]*

*iv. $\mathcal{R}$ an equivalence relation : yes [1 mark]*
$\{1, 4, 7\}, \{3, 6, 9\}, \{2, 5, 8\}$ *[3 marks]*

This part has a question count mismatch: expected , got **3**.

END OF PAPER

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