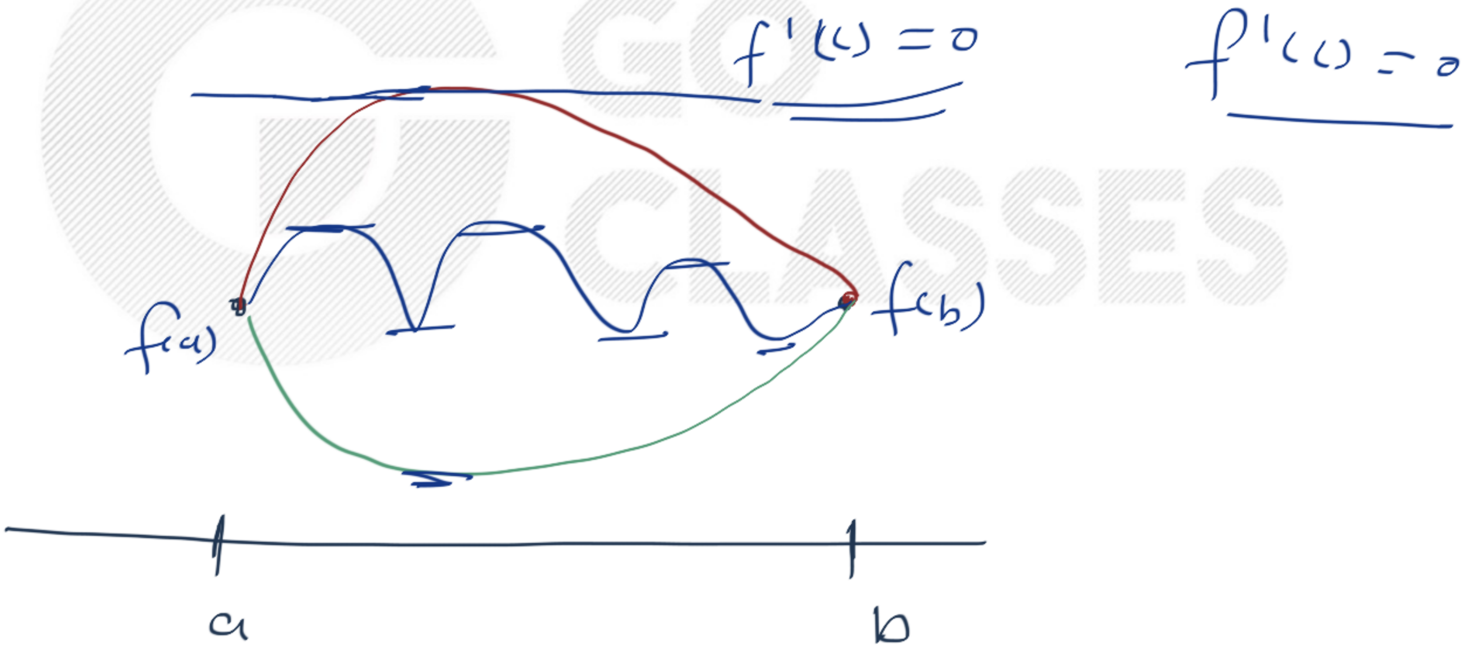
Theorem 1 (Intermediate Value Thoerem). *If f is a continuous function on the closed interval $[a, b]$, and if d is between $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ with $f(c) = d$.*

Theorem 2 (Rolle's Theorem). *Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then there is a number $c \in [a, b]$ with $f'(c) = 0$.*

Theorem 3 (The Mean Value Theorem). *Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there is a number $c \in [a, b]$ with*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Theorem 2 (Rolle's Theorem). *Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then there is a number $c \in [a, b]$ with $f'(c) = 0$.*



Determine whether Rolle's Theorem can be applied.

If so, find *c*. If not, explain why.

$$f(x)=x^4-2x^2 \qquad [-2\ ,\ 2]$$

$x^3 - 4x = 0$

$x(x^2-4)=0 \Rightarrow \begin{cases} x=0 \\ x=+2 \\ x=-2 \end{cases}$

$f(-2) = (-2)^4 - 2(-2)^2$

$f(2) = 2^4 - 2(2)^2$

$f(a) = f(b)$

$f'(x) = 0 \quad x \in (-2, 2)$

$(-2, 2)$

$c = 0$

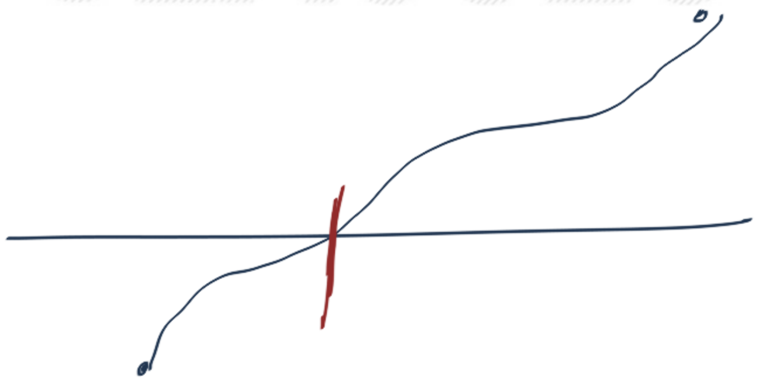
$c \in (-2, 2)$

Show that $f(x) = 2x^3 + 3x - 3$ has exactly one root.

$f'(x) = 6x^2 + 3 > 0 \implies f(x)$ is always increasing

$f(-10) < 0$

$f(10) > 0$



Show that $f(x) = 2x^3 + 3x - 3$ has exactly one root.

Proof: Since $f(0) = -3 < 0$ and $f(1) = 2 > 0$ the Intermediate Value Theorem tells us that there is at least one root in the interval $(0, 1)$. Since $f'(x) = 6x^2 + 3 > 0$ can never be zero, the existence of two roots contradicts the Corollary.

show that $x^5 + 4x = 1$ has exactly one solution.

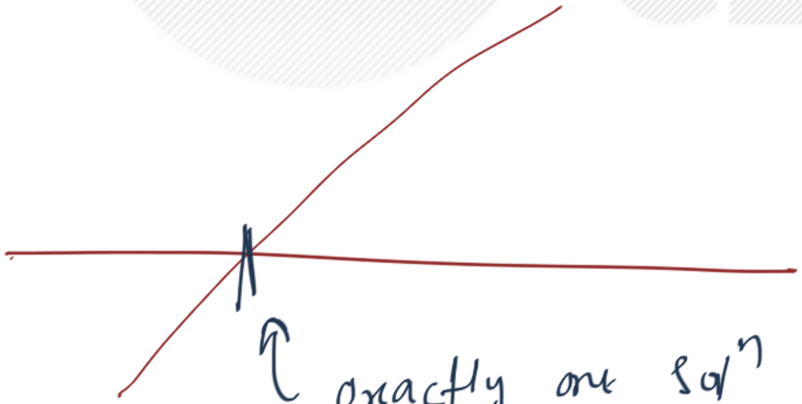
$f(x) = x^5 + 4x - 1$

$f'(x) = 5x^4 + 4 > 0 \Rightarrow f(x)$ is strictly increasing

Show the equation $x^3 + e^x = 0$ *has exactly one real solution.*

$f(x) = x^3 + e^x$

$f'(x) = 3x^2 + e^x > 0 \Rightarrow f(x)$ is increasing



exactly one sol^n

The number of roots of $e^x + 0.5x^2 - 2 = 0$ in the range $[-5, 5]$ is

A. 0

B. 1

C. 2

D. 3

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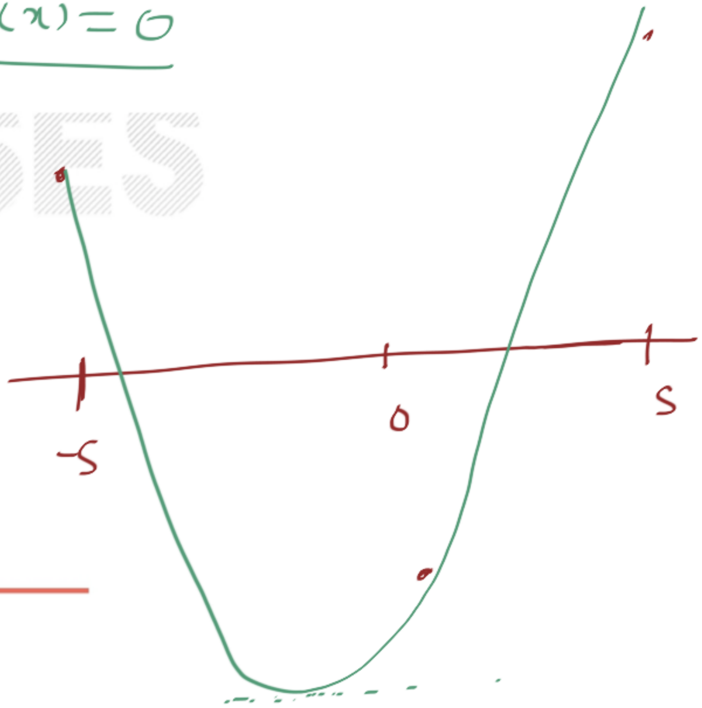
$f'(x) = 0$

$f(x) = e^x + 0.5x^2 - 2$

$f(-5) > 0$

$f(5) > 0$

$f(0) < 0$

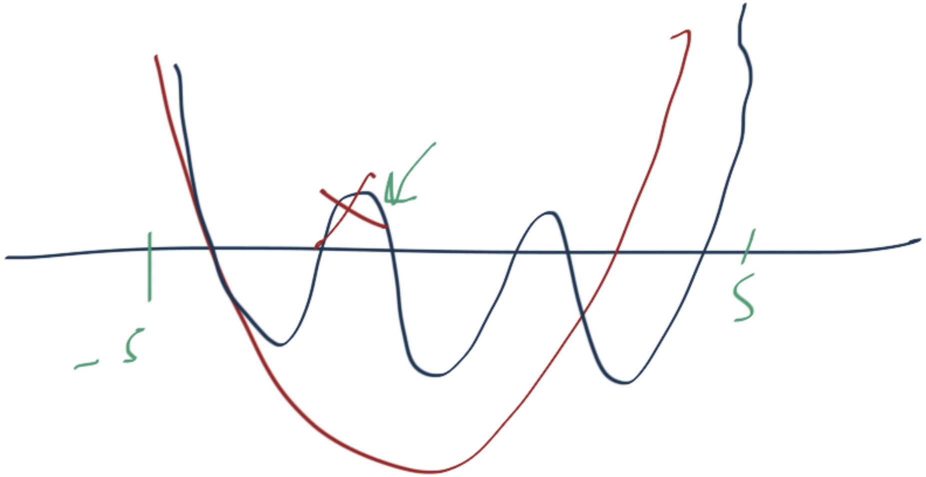


$f'(x) = e^x + x$ ⟵ $f'(x)$ is an increasing f^n

⟹ exactly one root of $f'(x)$

No. of sol^{n}s of $f'(x)$

$f''(x) = e^x > 0$



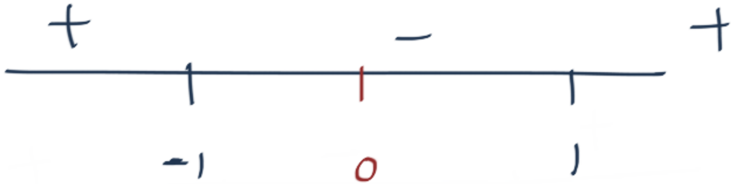
Prove that $x^3 - 3x + c$ has at most one root in $[0, 1]$, no matter what c may be.

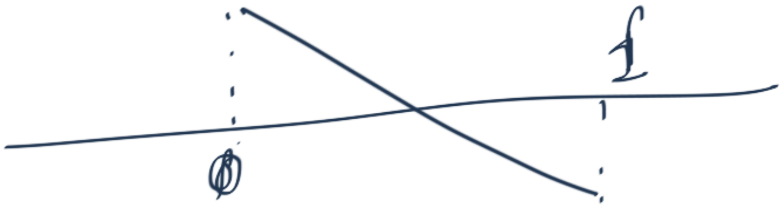
$f(x) = x^3 - 3x + c \longrightarrow f'(x) = 3x^2 - 3 \quad \underline{\underline{< 0}}$

$= 3(x-1)(x+1)$

decreasing in $[0,1]$

$$\left[\begin{aligned} f(0) &= c \\ f(1) &= c-2 \end{aligned}\right]$$





$f(0) = c$

$f(1) = c-2$