PHY 324: Pendulum Project

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Abstract

This report describes the results of experiments with a simple pendulum to investigate the validity of the damped harmonic oscillator model. For this, the mass of the pendulum bob, the length of the string connecting it to pivot, and the initial amplitude were all allowed to be independent variables. Their effects on the time period of the pendulum (T) and the decay constant (τ) were analyzed. The time period was found to be dependent on length and independent of mass as expected, but was found to also depend on amplitude which contradicts the model of damped SHM. Functional fits for the time decay constants were found.

1 Introduction and Theory

The aim of the experiments was to verify to what extent the motion of a simple pendulum follows the model described by a damped harmonic oscillator. According to the damped harmonic oscillator model, if a simple pendulum is attached at a fixed pivot and is swinging back and forth in a 2D plane, then its angular position from the equilibrium position (hanging straight down, $\theta = 0°$) is given by:

$$\theta(t) = \theta_0 e^{\frac{-t}{\tau}} \cos 2\pi \frac{t}{T} + \phi_0 \tag{1}$$

where θ_0 is the initial amplitude given to the pendulum in radians, T is its time period, τ is the time decay constant, which is the time it takes for θ to decrease by a factor of e, and ϕ_0 is the phase constant. $\phi_0 = 0$ if the pendulum is released from a specific amplitude θ_0. This assumes that the pendulum's motion in either direction is symmetric. This means that even if the pendulum were released from $\theta_0 < 0$, it would have the same T and τ dependence on the various parameters and hence would behave identically to the $\theta_0 > 0$ case.

This means that the amplitude of the θ vs t graph is

$$A(t) = \theta_0 e^{\frac{-t}{\tau}} \tag{2}$$

which should decrease exponentially.

Moreover, according to this model, if the distance between the centre of mass and the point where the pendulum mass is connected to the string is D and the length of string between pivot and point of attachment is L, then the time period of the pendulum is:

$$T = \frac{2\pi}{\sqrt{g}} \sqrt{L + D} \tag{3}$$

Thus, this predicts that the time period T is independent of the mass m and the initial amplitude θ_0 of the pendulum. Note that this assumes that $D^2 \gg L^2$. But, if it were to depend on the mass of the pendulum and the previous relationship were true, then we would have:

$$\text{mass dependence} = \frac{T}{2\pi} \sqrt{\frac{g}{L + D}} \tag{4}$$

However, here τ is predicted to not depend on any parameter of the system.

Thus, in this report I recorded the motion of a simple pendulum with controlled changes in the length of the string L, the mass of the pendulum m, and the initial amplitude θ_0 to observe effects in the time period of the pendulum and the time decay constant τ. To separate the effects of multiple variables changing together, 2 of them were maintained as constant while the other one was altered. θ_0 was inferred from data and m, L were measured. Both T and τ for a given setup were extracted from fitting the experimental data to equation 1 and were plotted against the relevant independent variable to deduce a functional fit. Fits used are:

$$y(x) = a \cdot \sqrt{x} + b \tag{5}$$

$$y(x) = m \cdot x + b \tag{6}$$

2 Method

2.1 Materials and Apparatus Used

The pendulum setup included 2 stands with clamps, cotton strings, and 3 cylindrical masses with hooks. For data collection and tracking, I used a protractor, white sheet of paper for screen, blue tape, measuring tape, a phone for video recording, and a computer with Tracker Software [1]

2.2 Procedure

Firstly, to make sure the pendulum moved in a single plane, a single piece of cotton string was looped through the mass in a triangular shape and tied to a common support as shown in figure 1. Then, to track the position of the mass, I decided to wrap a piece of blue tape at a height just below the protractor at the level of the phone camera to avoid a perspective effect to enter the angle measurements. To allow the Tracker to easily identify the marker, I also attached a white paper on the stand to have a strong contrast. The effective vertical length of the string connecting the mass was found using trignometry and measuring the distance between the two pivots on the rod and that from one of the pivots and the hook of the mass with a tape. Further, the length of the mass's hook and its height were measured to calculate D.

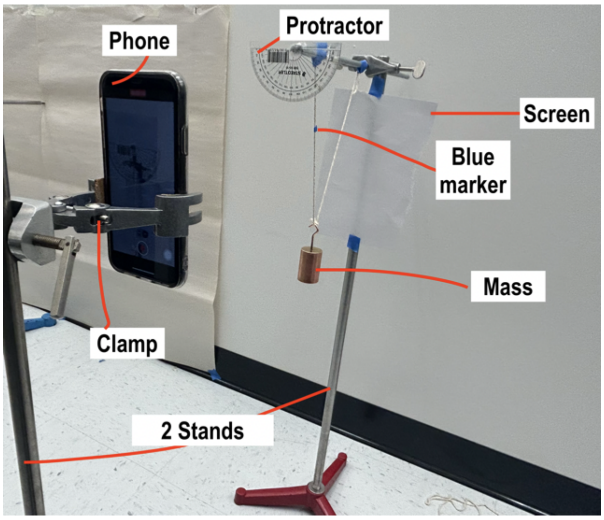

Figure 1: *Experimental Setup of simple pendulum. The angular position θ of the mass is determined from that of the a blue marker on the string. The string is looped through the mass to restrict the pendulum's motion in a single plane. Video recorded through phone is then analyzed using the Tracker software.*

The angle measurements were to be made via a protractor tool in Tracker, yet a physical protractor was still attached to the tip of the rod clamped to the pendulum stand because it aided in easily orienting the virtual protractor tool. It had a random uncertainty of about $\pm 2°$ because the tracker was sometimes off the centre of the marker by that amount. Then, I measured the dimensions of 3 cylindrical masses: 100g, 200g, 500g along with the length of their hooks when fully extended upwards (see Table 1 in Appendix) as it will be extended under the mass of the pendulum. To check effect of pendulum mass, using a string of random length, I recorded the videos of the each mass oscillating. For $m_1 = 100g$, I gave it a large initial amplitude $\approx \frac{\pi}{2}$ rad so that its data could also be used for analyzing dependence of T, τ on θ_0 ($m, L + D$ maintained constant). This is because we can treat it as having performed multiple experiments with amplitudes $\theta_{0,i^{\text{th}}} = \theta_0 e^{-t_i/\tau}$ where θ_0 is the initial amplitude set by me after starting the video recording. For the other two masses, $\theta_0 \approx 45°$to reduce length of video to be analyzed. In these videos, I calculated T and τ at $\theta_0 \approx 43\ °$(θ_0 maintained constant). This was done as the slightly larger θ_0 values were more noisy and the latter contained irregular data because the time step had to be increased to make tracking faster. Moreover, in analyzing the videos, the arm of the protractor tool was attached to the blue tape marker and the virtual protractor was displaced from the position of the physical protractor as the pivot of the pendulum was below its origin (see figure 8 in Appendix). The data files of $\theta(t)$ vs t were generated for further analysis.

Finally, to check effect of L on T, τ, I just used different lengths of strings tied in the same manner with mass $m_2 = 200g$ and collected relevant data using Tracker.

[1]Tracker. "Tracker Video Analysis and Modeling Tool for Physics Education." Physlets.org, 2009, physlets.org/tracker/.

3 Results

First, the dependence of T and τ on $L+D$ was analyzed. For this experiment, I decided to start my analysis from about $\theta_0 \approx 20°$ in the collected $\theta(t)$ data. Since fitting τ over few oscillations using 2 gave large uncertainties due to a small shift in the fit parameters dramatically changing τ as τ is generally 20 times greater than T, I decided to calculate it by taking the average of the decay constants found by fitting exponential functions to the peaks and troughs of the $\theta(t)$ data over 50 oscillations for three different lengths[2]. Then, using the mean values of the corresponding τ, eq.1 was used to fit the model of a damped harmonic oscillator over 5 oscillations. The decay constant wasn't fitted again because of above mentioned reasons[3]. Now, to verify the eq. 3, T vs $\sqrt{L+D}$ was plotted and fitted to eq. 3 as can be seen in figure 2 with the residual plot. The fitted coefficient is $2.0190 \pm 0.0009 \frac{s}{\sqrt{m}}$ is about 14σ from the accepted value of $\frac{2\pi}{\sqrt{g}} \frac{s}{\sqrt{m}} \approx 2.0064 \frac{s}{\sqrt{m}}$ ($g \approx 9.80665\ ms^{-2}$). This is noted by $\chi^2_{red} = 349.25 \gg 1$. This shows that the relationship between the time period and $L+D$ as stated in eq. 3 is not exact when calculated at $\theta_0 \approx 20°$. This should be the case because it is only valid for very small angles.

Also, the plot of τ vs $L+D$ was fitted with a quadratic fit (see figure 3) of form y $= a(x-b)^2 + 200$ where the constant 200 was added to show a better fit. Thus, the fit has a very good $\chi^2_{red} = 0.96$ value which shows the the fit is a good one. Moreover, since it's unlikely that τ would suddenly decrease with increasing $L+D$, a more reasonable approximation to the true fit for higher $L+D$ seems to be of form $a \cdot \sqrt{x-b}$. However, this wasn't fitted here as the quadratic fit is better in the given ranges of $L+D$.

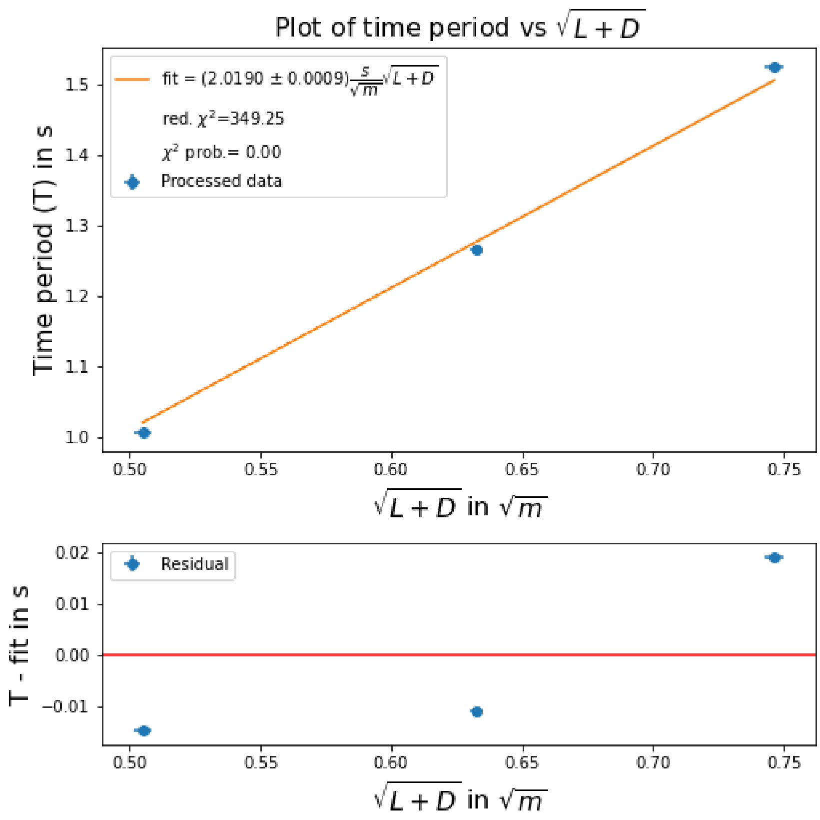

Figure 2: *Top: Plot of time period of simple pendulum vs* $\sqrt{L+D}$. *Uncertainties in* T *are* ≈ 0.001 s *and hence are quite small to be observed. The theoretical fit should be* $y_{fit} \approx 2.0064 \frac{s}{\sqrt{m}} \cdot \sqrt{L+D}$. *The empirical fit is not exact; the the fitted coefficient* $2.0190 \pm 0.0009 \frac{s}{\sqrt{m}}$ *is* 14σ *away from the accepted value. The* $\chi^2_{red} = 349.25 \gg 1$ *means that the fit is an underfit. Bottom: Residual plot of fit. The maximum deviations are* 0.02 s *which is acceptable*

[2]see A.1.1 in Appendix
[3]see A.1.2 in Appendix

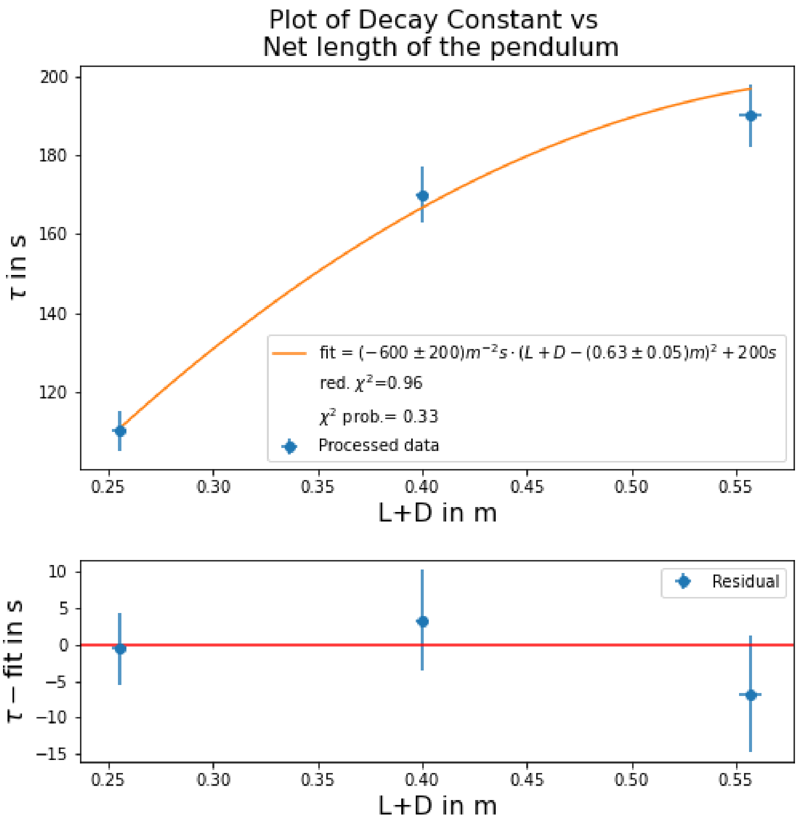

Figure 3: *Top: Plot of time decay constant of simple pendulum vs its effective string length* $L + D$*. The* $\chi^2_{red} = 0.96$ *which means the fit is quite accurate to the true relationsip between* τ *and* $L + D$*. Bottom: Residual Plot. This indicates the fit line passes through all 3 points within their error bars.*

So, I got the following fits:

$$T(L + D) = (2.0190 \pm 0.0009)\frac{s}{\sqrt{m}}\sqrt{L + D} \tag{7}$$

$$\tau(L + D) = (-600 \pm 200)\frac{\text{s}}{m^2} \cdot (L + D - (0.63 \pm 0.05)m)^2 + 200 \text{ s} \tag{8}$$

Next, I analyzed the effect of mass of the pendulum on T and τ in a similar manner[4] . Here, using a different mass would change the centre of mass (D) due to its size. Thus, I subtracted the time period due to the length of the pendulum from the time period calculated using eq.7. This is plotted in figure 4 against m with residuals. The fit function is eq. 5. The fit's really good $\chi^2_{red} = 0.98$ and has an acceptable χ^2 probability $=$ 0.32. I conclude that since the deviations from 0 s are only upto 0.05 s, which is negligible when eq. 7 is only an approximation, T doesn't depend on the mass as claimed by the Damped SHM. Similarly, the difference in time decay constants was also plotted ($\tau(m) - \tau(L + D)$ vs m) as shown in figure 5 with residuals. Here, the $\chi^2_{red} = 0.02$ which shows extreme overfitting due to the high uncertainties in $\tau(m)$. However, the trend clearly implies that τ does depend on m as:

$$\tau(m, L + D) = \tau(L + D) + (2000\pm)\text{kg}^{-2}s \cdot (m - (0.26 \pm 0.02) \text{ kg})^2 + 100 \text{ s} \tag{9}$$

[4]see A.2.2, A.2.1 in appendix

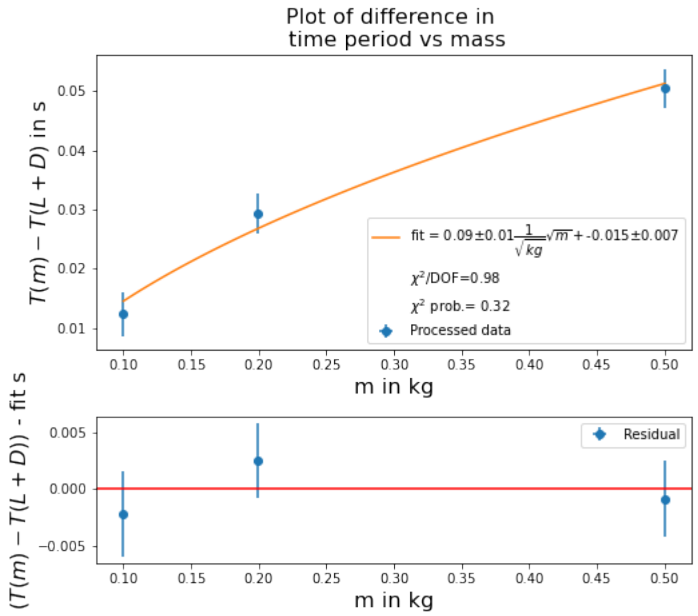

Figure 4: *Top: Plot of difference in time period of simple pendulum vs its mass m. The fit function's red.* $\chi^2 = 0.98$ *and* χ^2 *probability of 0.32 shows that it is a really good fit. Since the deviations are small, and* $T(L + D)$ *is an approximation, the time period of a simple pendulum does not depend on its mass verifying Damped SHM model. Bottom: Residual Plot. All data points pass through horizontal axis within 1 errorbar which means best fit passes through data points within their uncertainty and is quite reliable.*

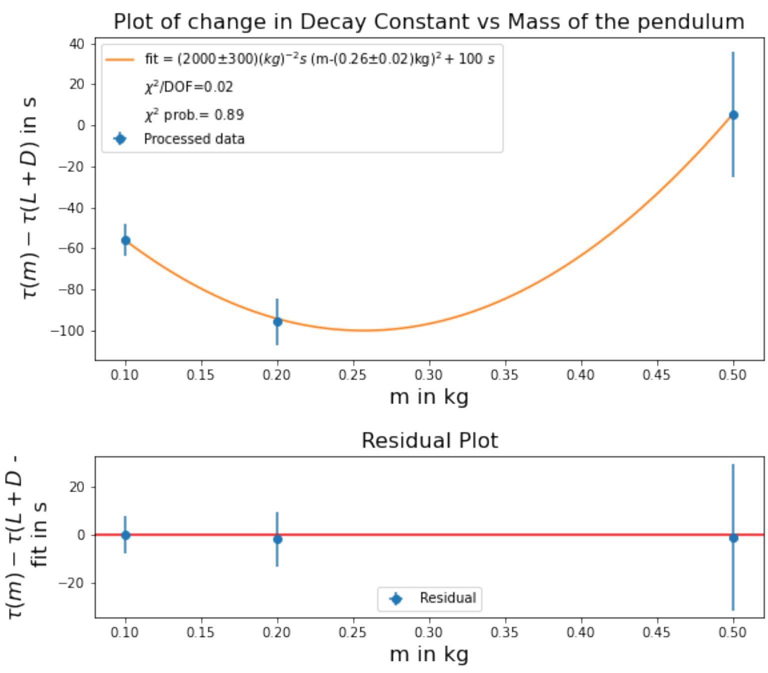

Figure 5: *Top: Plot of time decay constant* τ *of simple pendulum vs its mass m. The fit function's red.* $\chi^2 = 0.02$ *is quite low indicating overfitting. Its trend does indicate that locally* τ *depends on* m *in a quadratic manner. Bottom: Residual Plot. All data points pass through horizontal axis within 1 errorbar which means best fit passes through data points within their uncertainty.*

Finally, I analyzed the dependence of t, τ on θ_0[5]. The plot of time period vs amplitude is shown in figure 6 with a quadratic fit. Thus, it seems that the time period T of a simple pendulum does have a dependence on the initial amplitude θ_0. The $\chi^2_{red} = 0.30$ shows that there is overfitting. The χ^2 probability being 0.58

[5] see A.3.2, A.3.1 in appendix

however means that there is indeed substantial chance for this data to be collected thus verifying that the fit is not an extreme overfit. This altogether refutes the claim of Damped SHM model that the time period of a simple pendulum does not depend on its amplitude. It's dependence is given by:

$$T(\theta_0) = (6 \pm 6) \cdot 10^{-4} \frac{s}{\circ} \theta_0 + (1.48 \pm 0.02)s \tag{10}$$

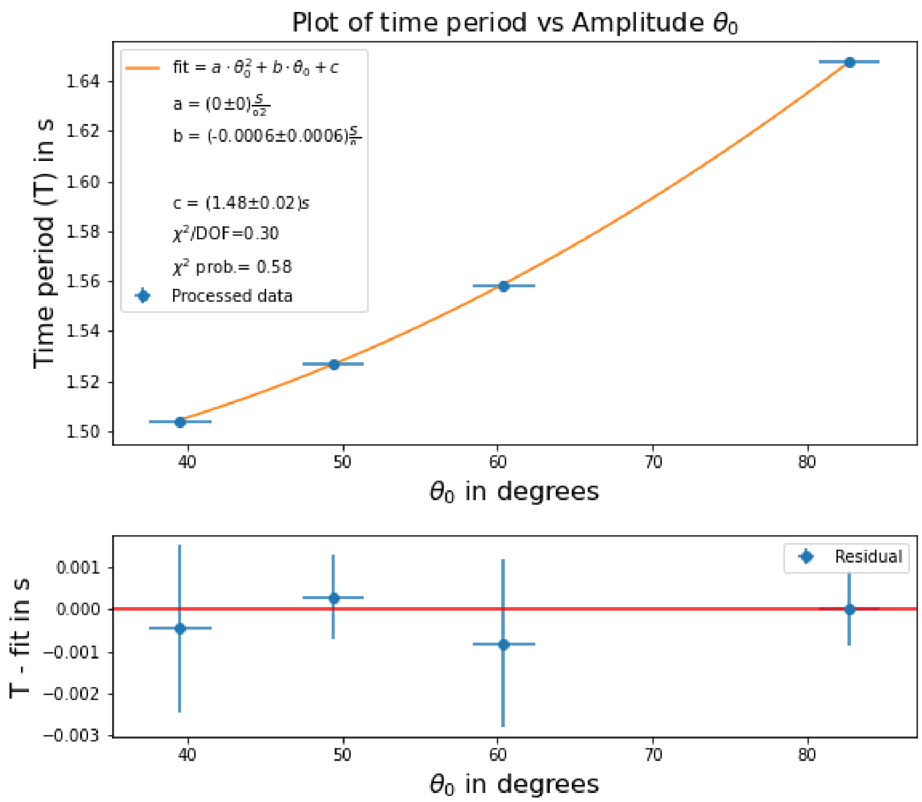

Figure 6: *Top: Plot of time period of simple pendulum vs its initial amplitude θ_0. The fit function is a quadratic equation. As can be observed, there is clearly a quadratic dependence of T on θ_0 with good χ^2 probability which contradicts the claim that a damped oscillator's time period is independent of θ_0. Bottom: Residual plot. All points pass through horizontal axis because best fit line passes through them within 1 errorbar which is good.*

Next, I also plotted the decay constant τ against θ_0 which is shown in figure 7 with residuals. The best fit function to fit the data locally was chosen to be a gaussian as shown in the figure. The $\chi^2_{red} = 0.92$ is indicative of a very good fit. A downward trend makes sense because for a higer amplitude, the pendulum would have a higher velocity and hence a higher drag force. This would slow down the pendulum faster than when it is oscillating at lower amplitudes. This then proves that the decay of a damped pendulum is not exponential because the time constant changes as θ_0 decays.

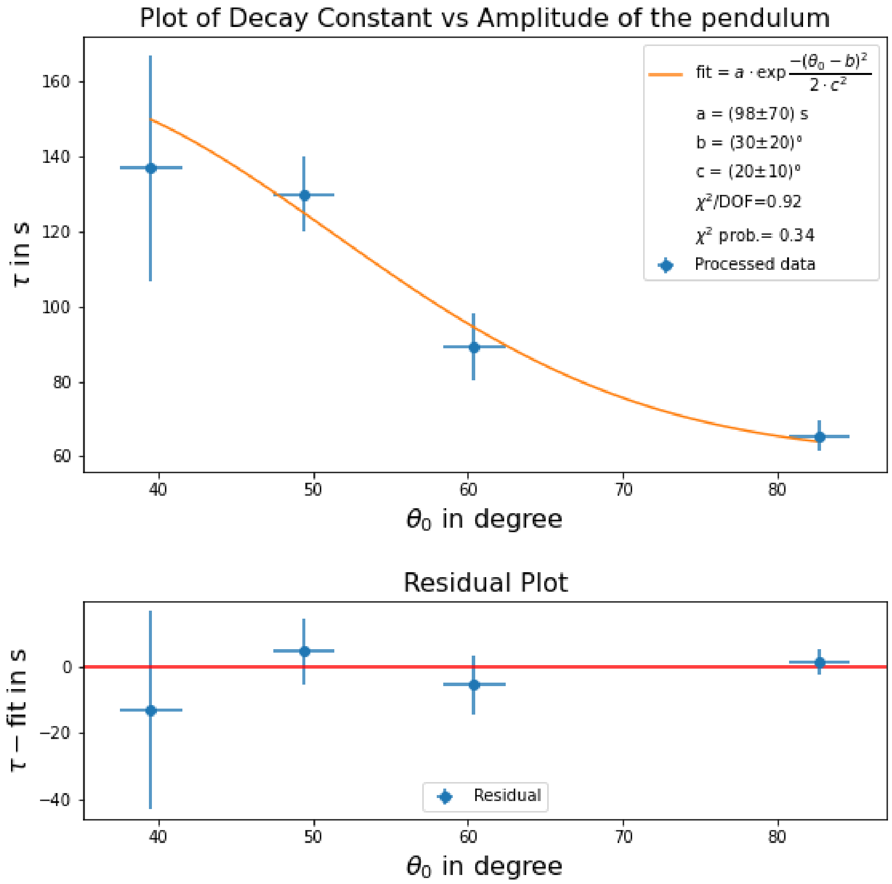

Figure 7: *Plot of decay constant of simple pendulum vs its initial amplitude θ_0. The fit function is a quadratic equation. As can be observed, there is clearly some dependence of τ on θ_0 which contradicts the claim that a damped oscillator's time decay constant is independent of θ_0. Bottom: Reidual Plot. All points pass through $x = 0$ which is good as best fit line passes through them within one errorbar.*

4 Conclusion

In this report, relationship between T, τ and $m, L + D, \theta_0$ was analyzed for a simple pendulum. It was found that the length of a simple damped pendulum approximately followed the known relation achieved from small angle approximation although deviated when $\theta_0 \approx 20°$. Next, I confirmed that T doesn't depend on m under experimental uncertianty. However, the decay of the pendulum was found to be not exponential as the τ turned out to depend on θ_0 and time period was also found to be dependent on θ_0 refuting damped SHM.

5 References

1. Tracker. "Tracker Video Analysis and Modeling Tool for Physics Education." Physlets.org, 2009, physlets.org/tracker/.

A Raw Data and Extra Plots

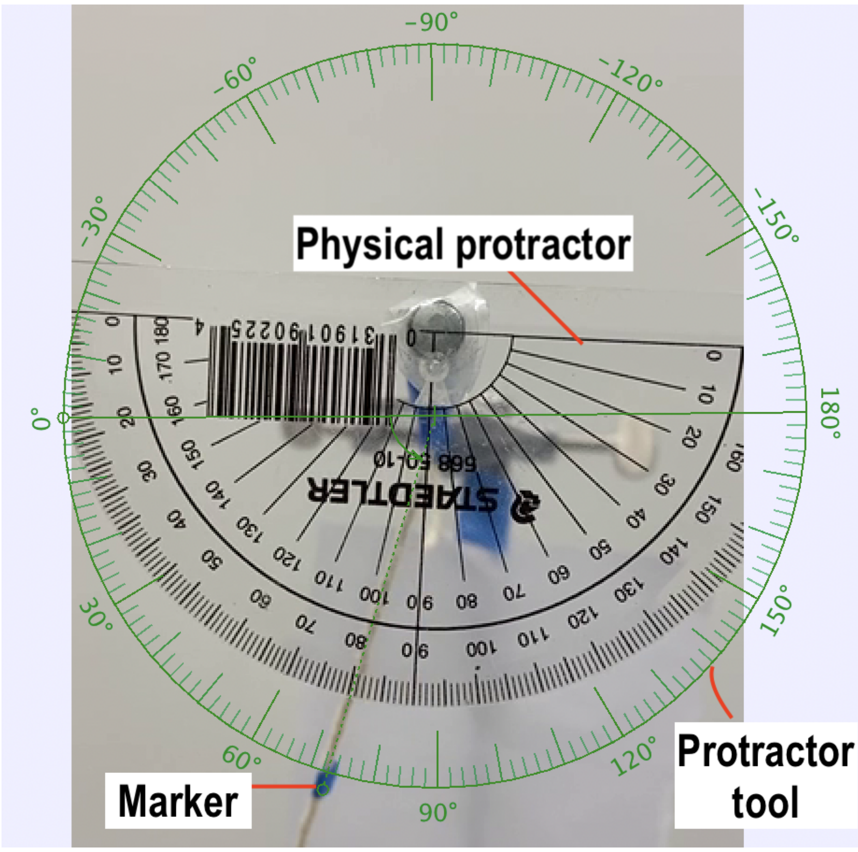

Figure 8: *Video analysis in tracker software. The physical pendulum is used to calibrate the size of the virtual protractor which is displaced to the exact position and orientating that corresponds to the pivot of the pendulum and its equilibrium position respectively. This makes the angle measurements more accurate.*

Mass (g)	Length of cylindrical mass (mm)	Length of hook (mm)
99.9 ± 0.1	35.0 ± 0.5	35 ± 2
199.6 ± 0.1	44 ± 0.5	30 ± 2
500.3 ± 0.1	56.0 ± 0.2	15 ± 2

Table 1: Data for the dimensions of the 3 masses

A.1 Length Plots

A.1.1 Plots for τ measurements

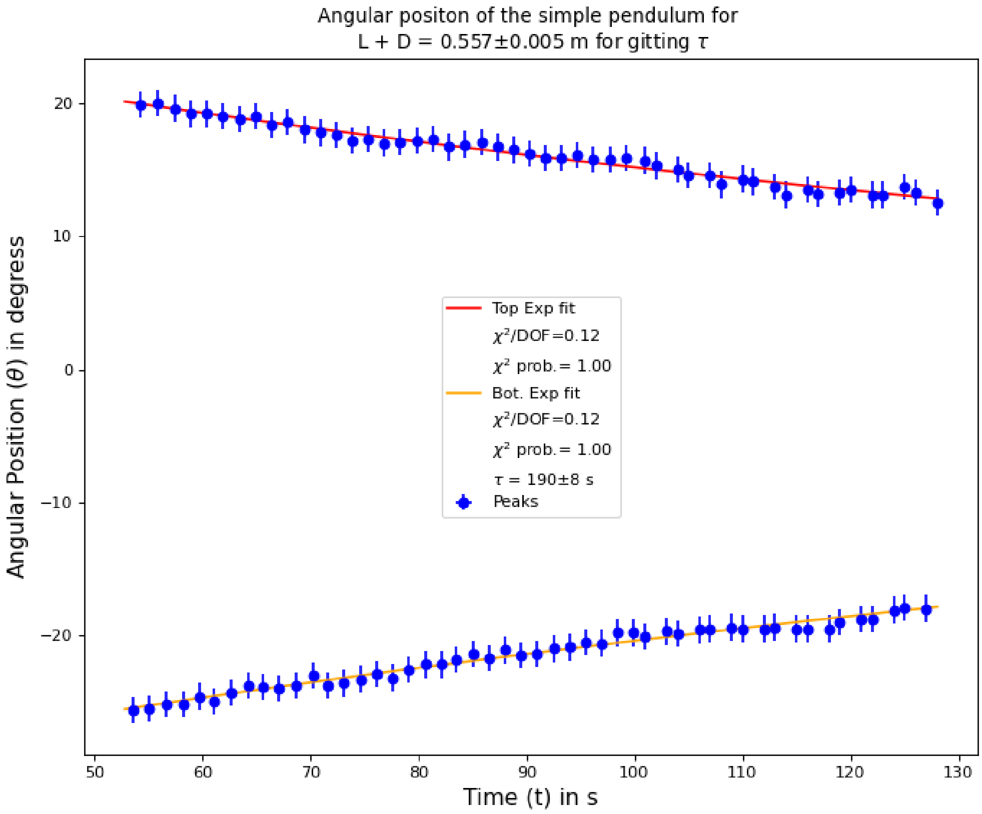

Figure 9: *Plot of peaks and troughs of θ vs t of simple pendulum and exponential fits to them to find time decay constant τ over 50 oscillations. Here, $L + D = 0.557 \pm 0.005$ m, $\theta_0 = 20.0°$, and $m = 0.1996 \pm 0.0001$ kg. The time decay constant was found $\tau = 190 \pm 8$ s. The low reduced $\chi_s$quared $= 0.12$ shows overfitting, but it is quite reasonable as the pendulum's motion smoothes out in long term giving an accurate and non-changing τ. Also, the uncertainties in θ are large enough to accommodate different fits.*

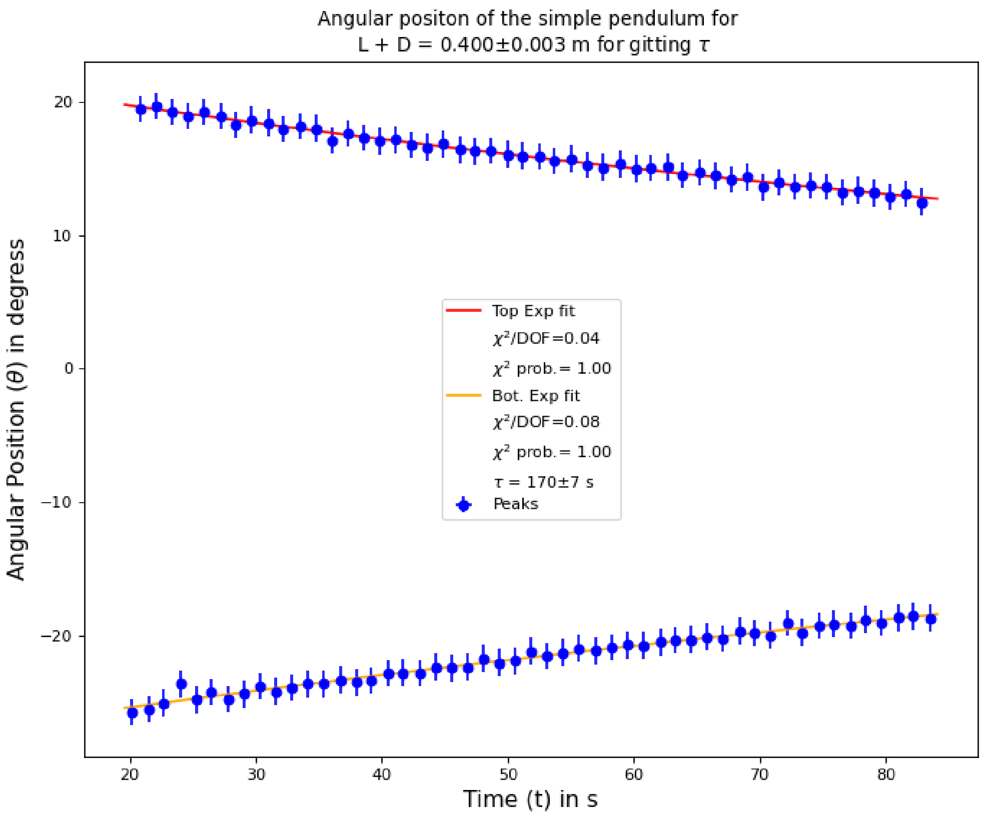

Figure 10: Plot of peaks and troughs of θ vs t of simple pendulum and exponential fits to them to find time decay constant τ over 50 oscillations. Here, $L + D = 0.400 \pm 0.003$ m, $\theta_0 = 19.9°$, and $m = 0.1996 \pm 0.0001$ kg. The time decay constant was found $\tau = 170 \pm 7$ s. The low reduced $\chi_s quared = 0.04$ shows extreme overfitting, but it is quite reasonable as the pendulum's motion smoothes out in long term giving an accurate and non-changing τ. Also, the uncertainties in θ are large enough to accommodate different fits.

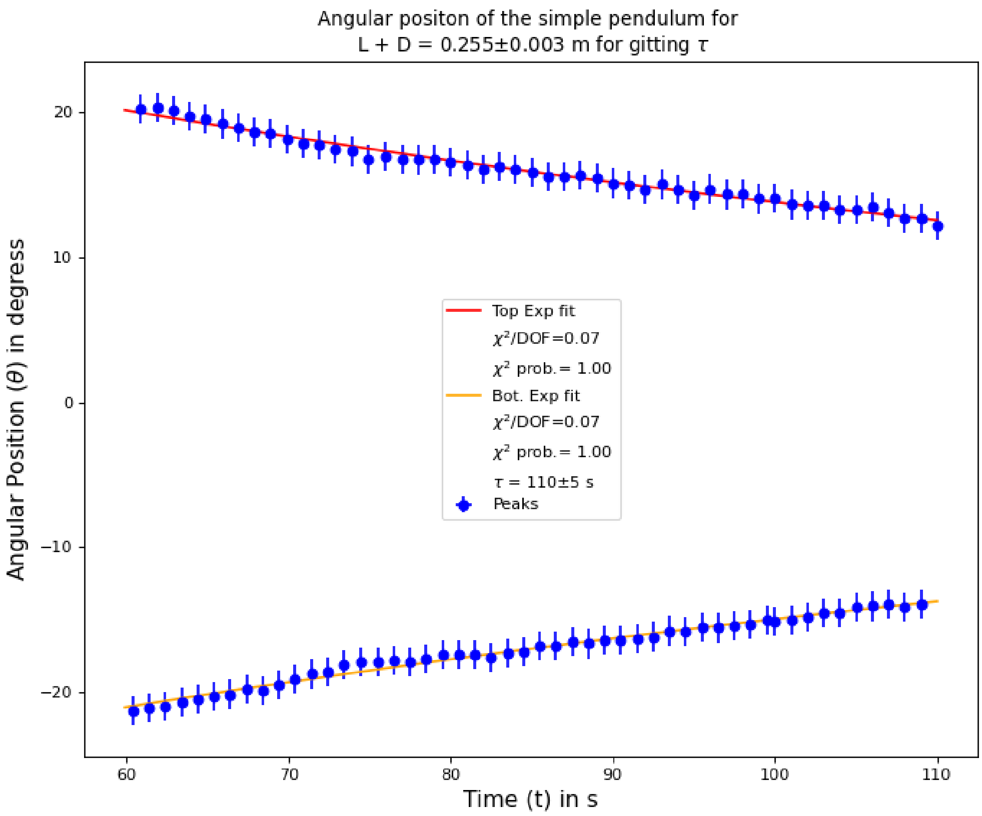

Figure 11: *Plot of peaks and troughs of* θ *vs* t *of simple pendulum and exponential fits to them to find time decay constant* τ *over 50 oscillations. Here,* $L + D = 0.255 \pm 0.003$ $m, \theta_0 = 20.6°$*, and* $m = 0.1996 \pm 0.0001$ *kg. The time decay constant was found* $\tau = 110 \pm 5$ *s. The low reduced* χ_s*quared* $= 0.07$ *shows overfitting, but it is quite reasonable as the pendulum's motion smoothes out in long term giving an accurate and non-changing* τ*. Also, the uncertainties in* θ *are large enough to accommodate different fits.*

A.1.2 Plots for T measurements

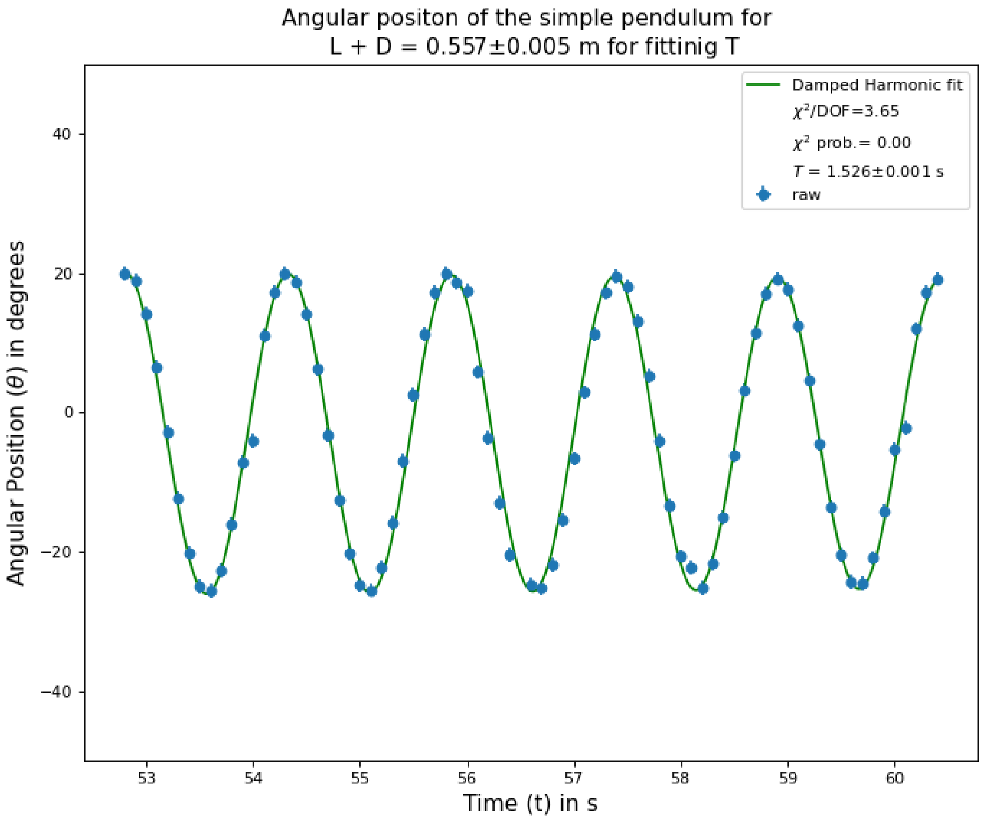

Figure 12: *Plot and fit of angular position of simple pendulum versus time to find time period T over 5 oscillations. Uncertainties are quite small to be seen. Here,* $L + D = 0.557 \pm 0.005$ *m,* $\theta_0 = 20.0°$*, and* $m = 0.1996 \pm 0.0001$ *kg. The time period was found* $T = 1.526 \pm 0.001$ *s. The reduced* χ^2 *= 3.65 shows underfitting. This can be contributed by some points in the graph which are off the clear sinusoidal pattern because of an inaccurate angle measurement made by tracker.*

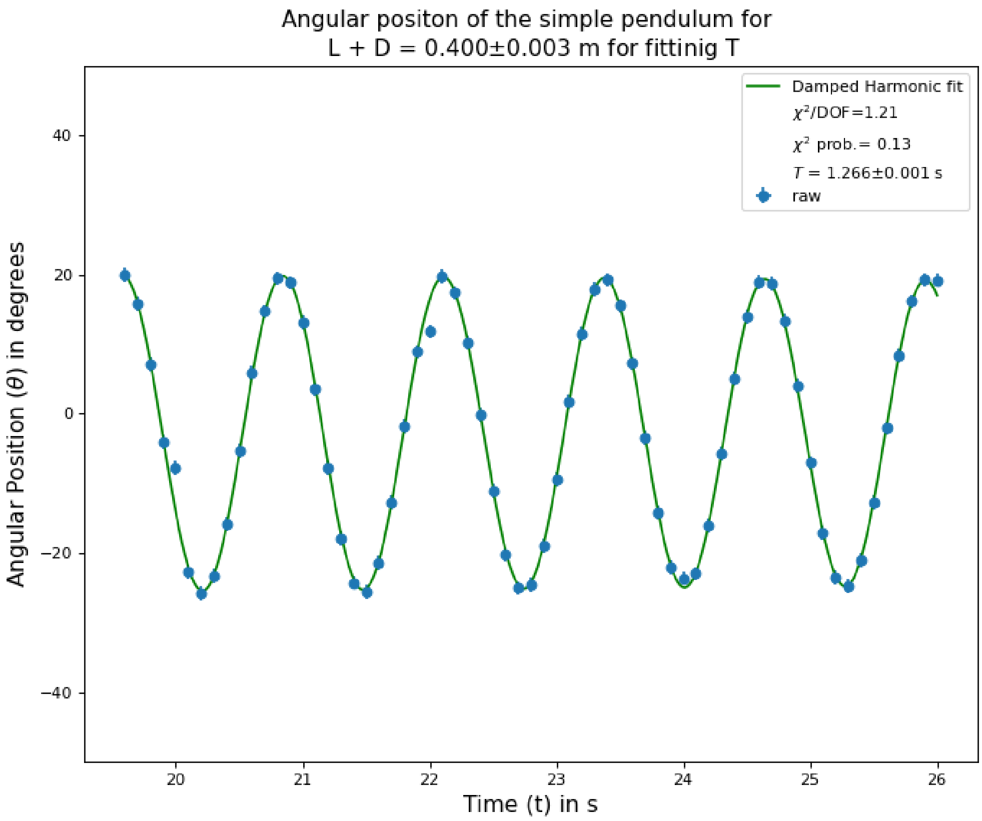

Figure 13: *Plot and fit of angular position of simple pendulum versus time to find time period T over 5 oscillations. Uncertainties are quite small to be seen. Here, $L + D = 0.400 \pm 0.003$ m, $\theta_0 = 19.9°$, and $m = 0.1996 \pm 0.0001$ kg. The time period was found $T = 1.266 \pm 0.001$ s. The reduced χ^2 = 1.21 shows that this is a very good fit. The χ^2 probability of 0.13 also suggests that there is reasoable chance of getting similar results.*

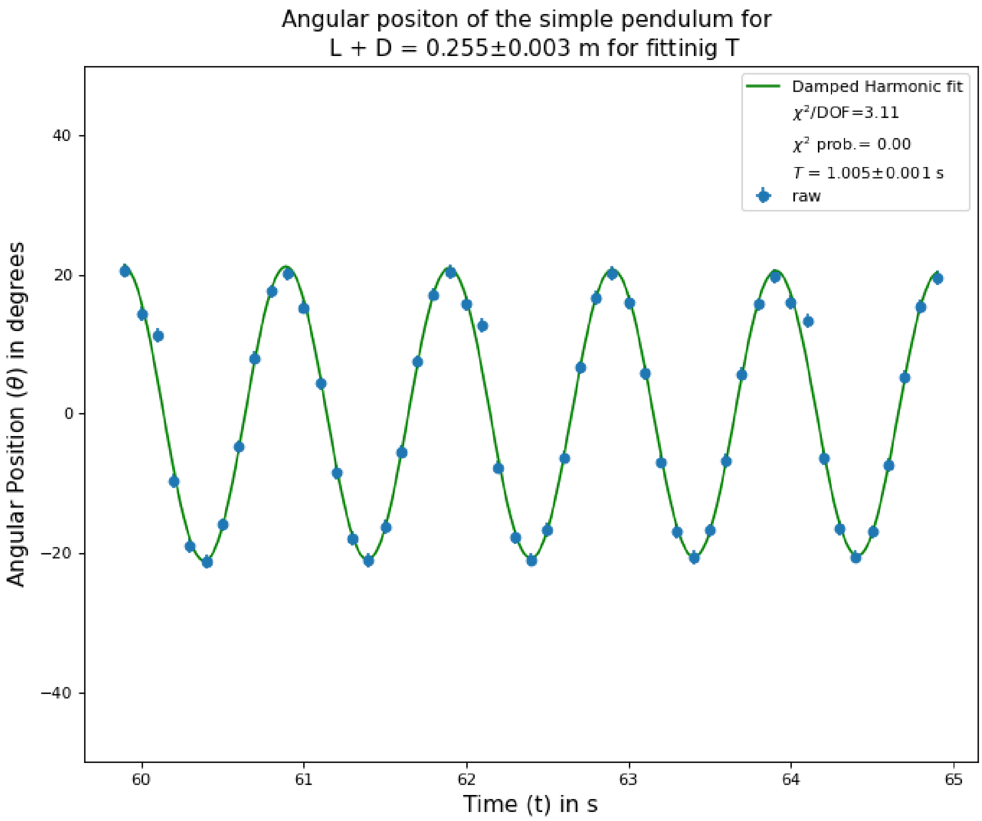

Figure 14: *Plot and fit of angular position of simple pendulum versus time to find time period T over 5 oscillations. Uncertainties are quite small to be seen. Here,* $L + D = 0.255 \pm 0.003$ *m,* $\theta_0 = 20.6°$*, and* $m = 0.1996 \pm 0.0001$ *kg. The time period was found* $T = 1.005 \pm 0.001$ *s. The reduced* χ^2 *= 3.11 shows underfitting. This can be contributed by some points in the graph which are off the clear sinusoidal pattern because of an inaccurate angle measurement made by tracker.*

A.2 Mass (m) plots

A.2.1 Plots for τ measurements

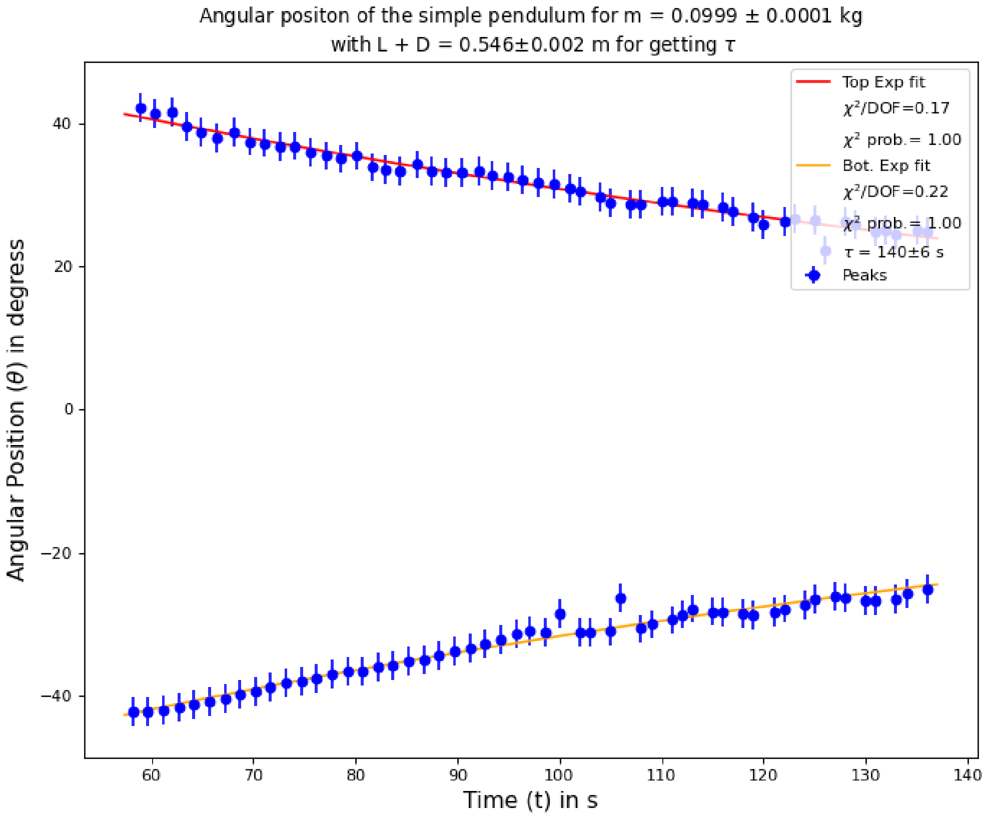

Figure 15

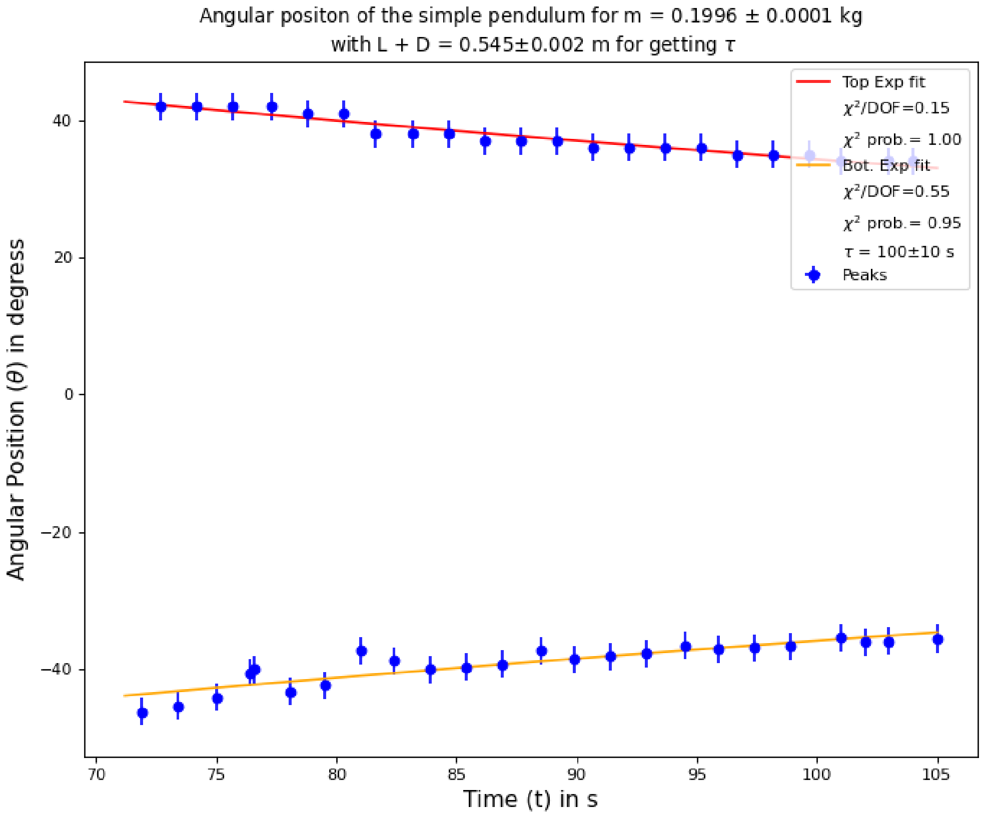

Figure 16

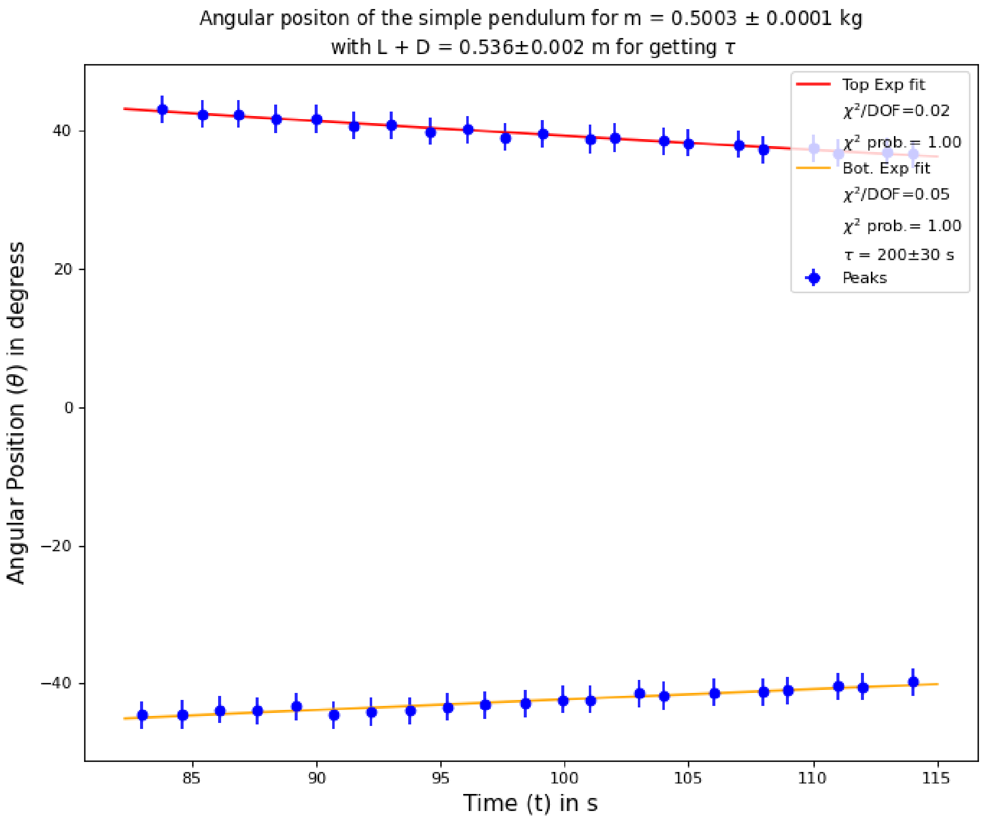

Figure 17

A.2.2 Plots for T measurements

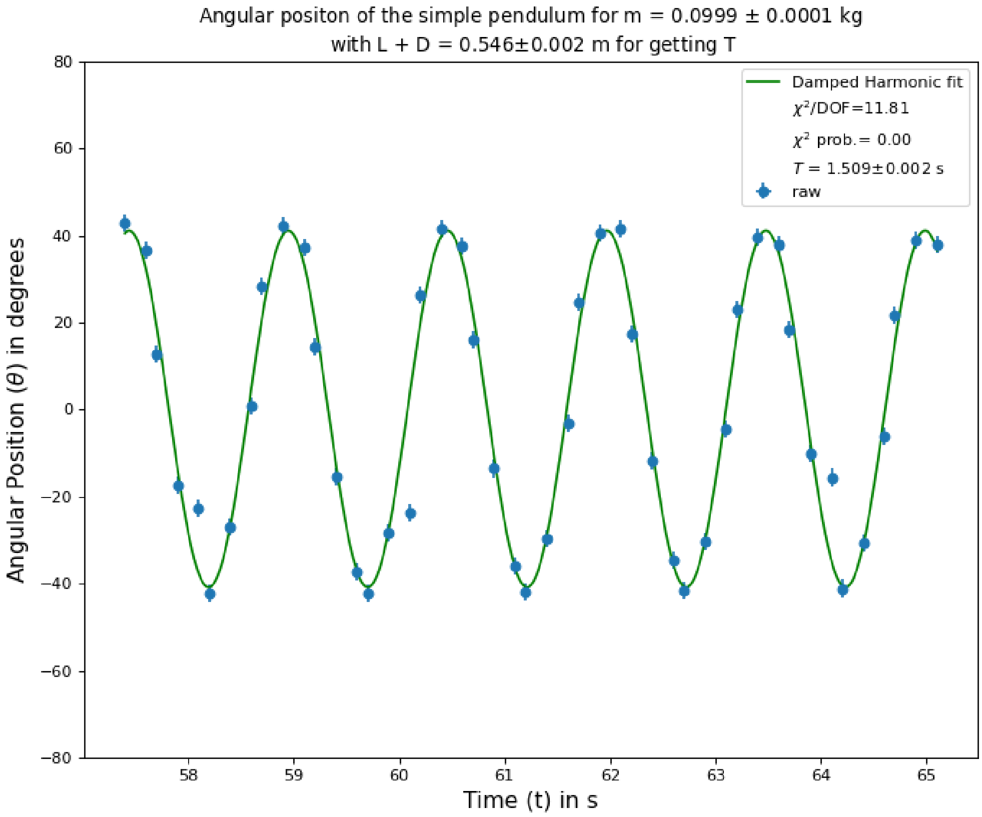

Figure 18

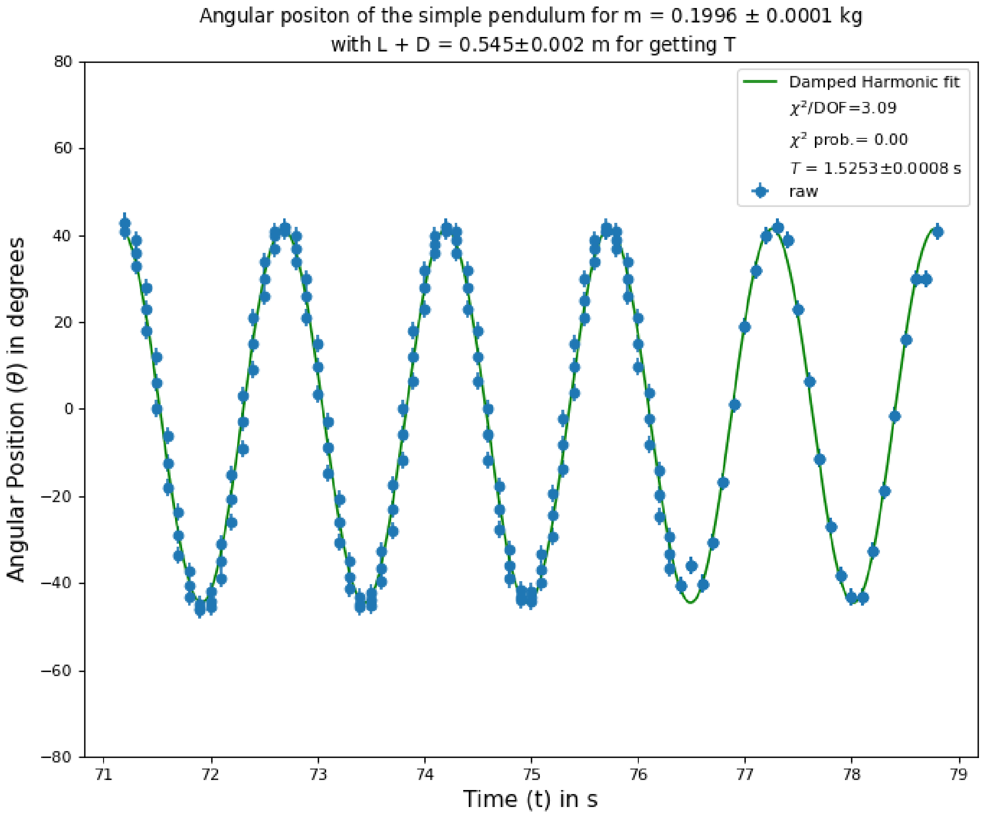

Figure 19

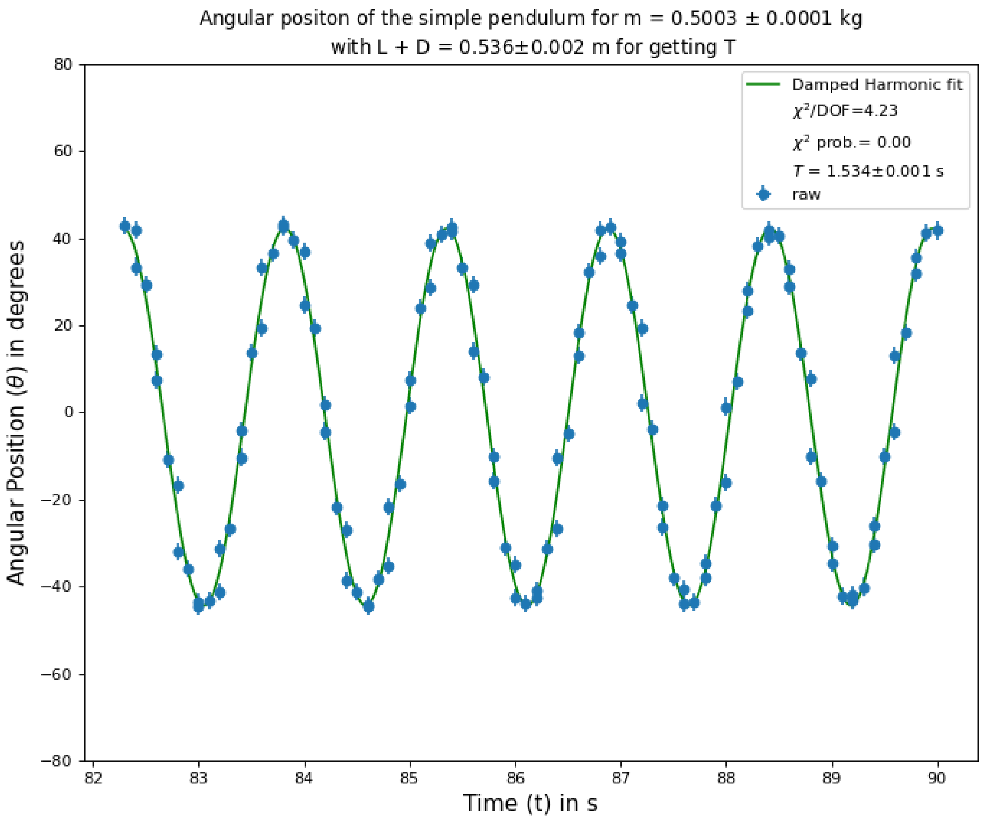

Figure 20

A.3 Amplitude (θ_0) Plots

A.3.1 Plots for τ measurements

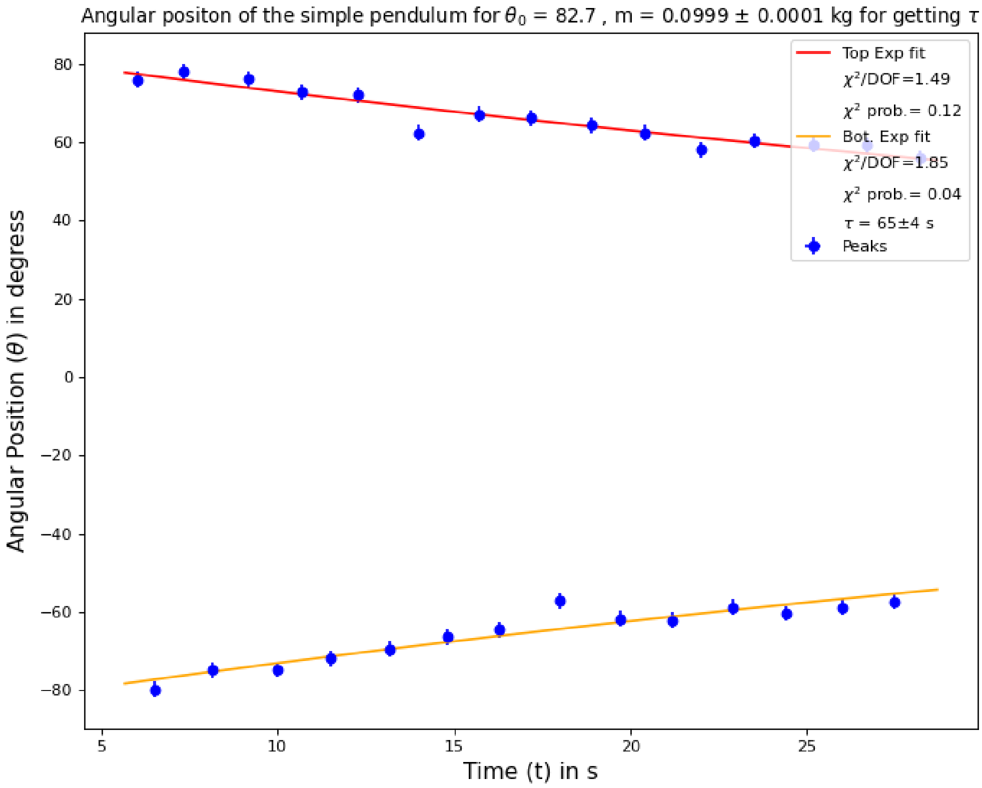

Figure 21

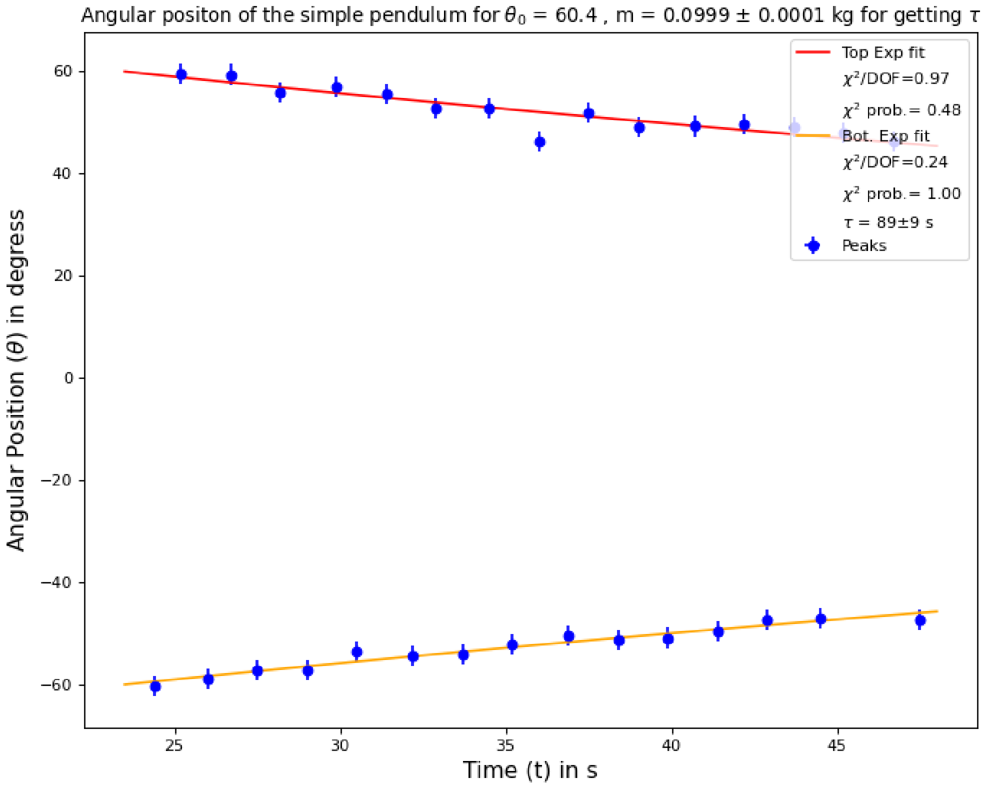

Figure 22

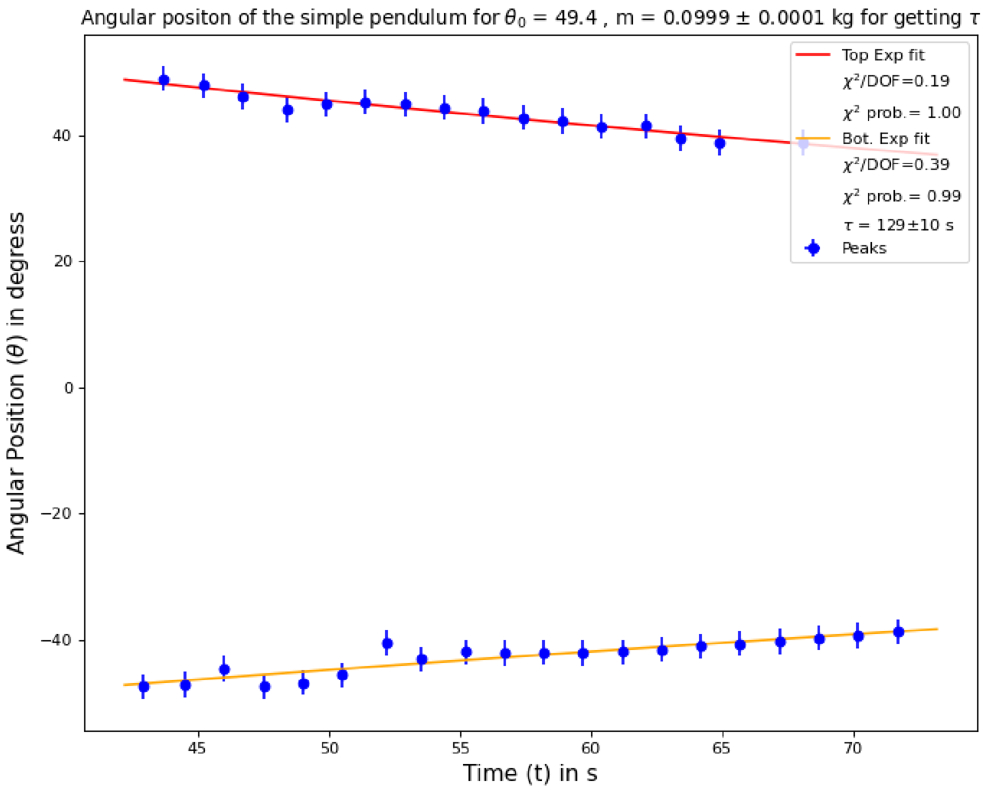

Figure 23

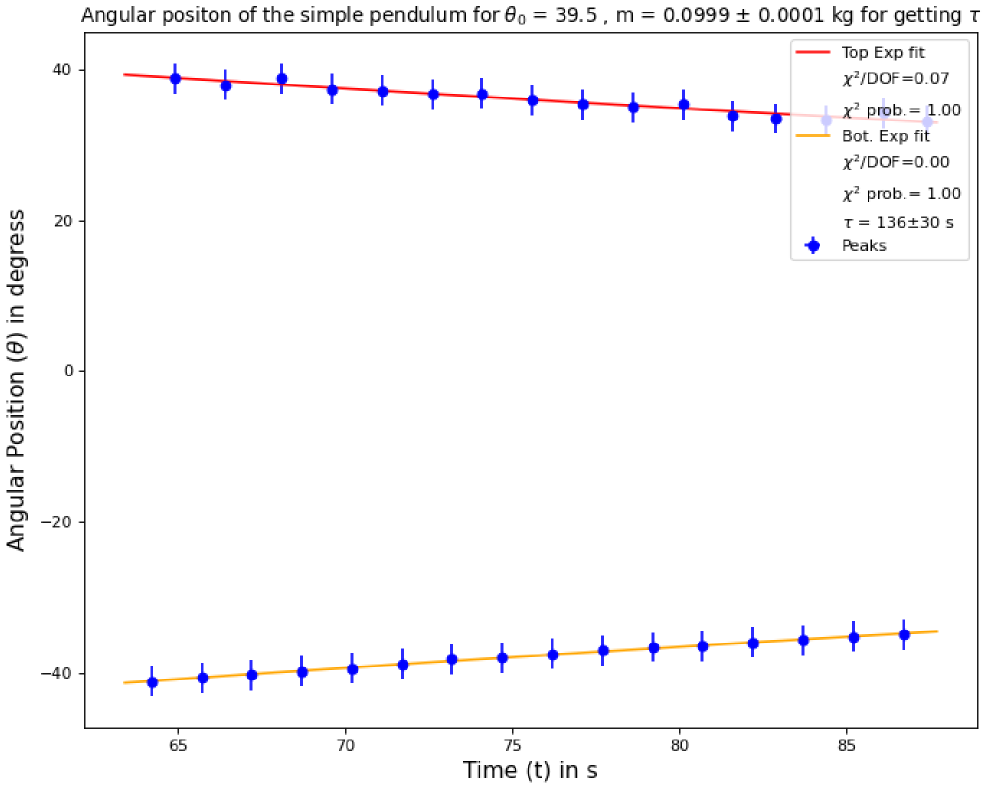
Angular positon of the simple pendulum for θ_0 = 39.5 , m = 0.0999 ± 0.0001 kg for getting τ
Top Exp fit
χ^2/DOF=0.07
χ^2 prob.= 1.00
Bot. Exp fit
χ^2/DOF=0.00
χ^2 prob.= 1.00
τ = 136±30 s
Peaks
Angular Position (θ) in degress
Time (t) in s

Figure 24

A.3.2 Plots for T measurements

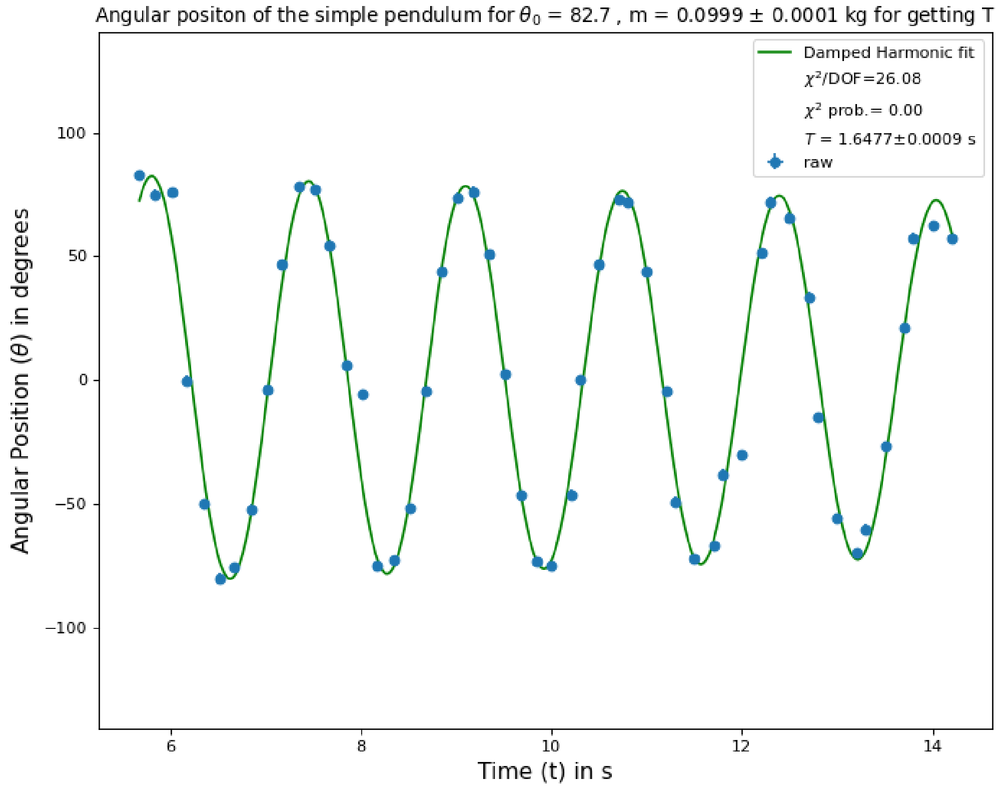

Figure 25

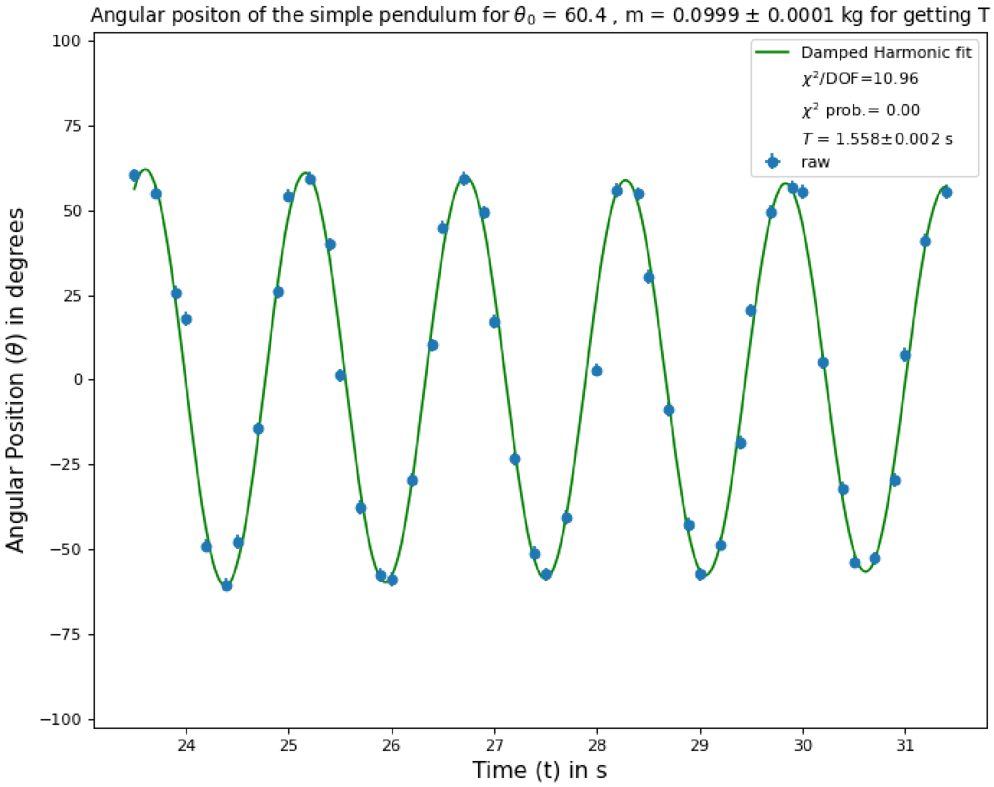

Figure 26

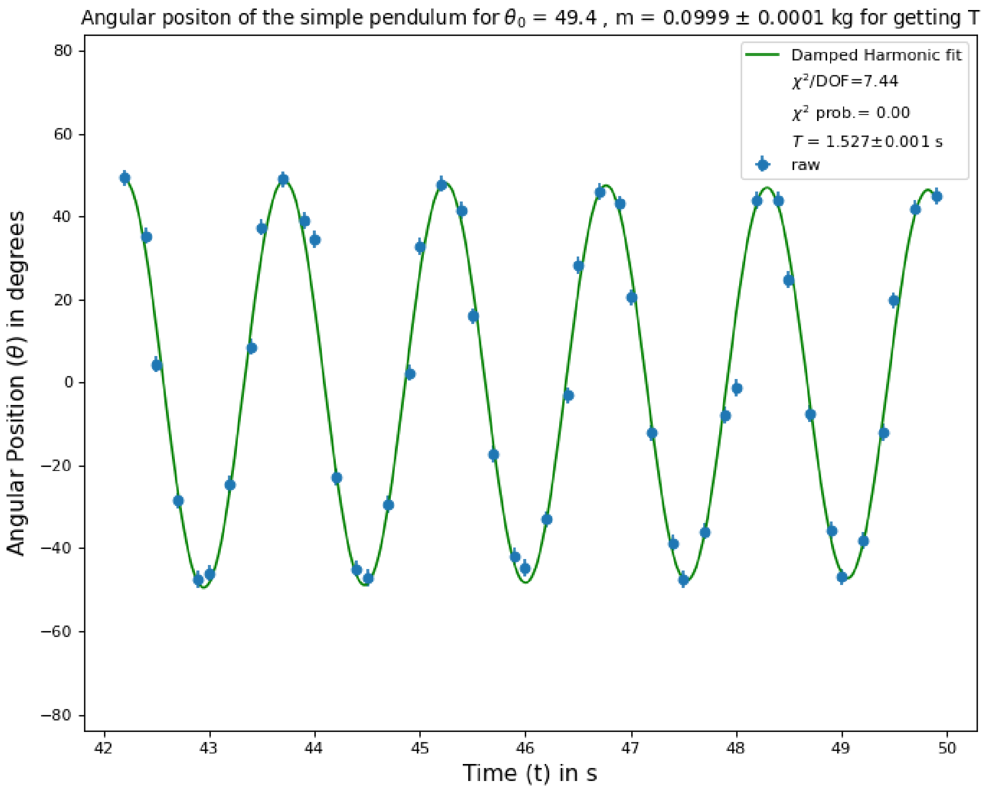

Figure 27

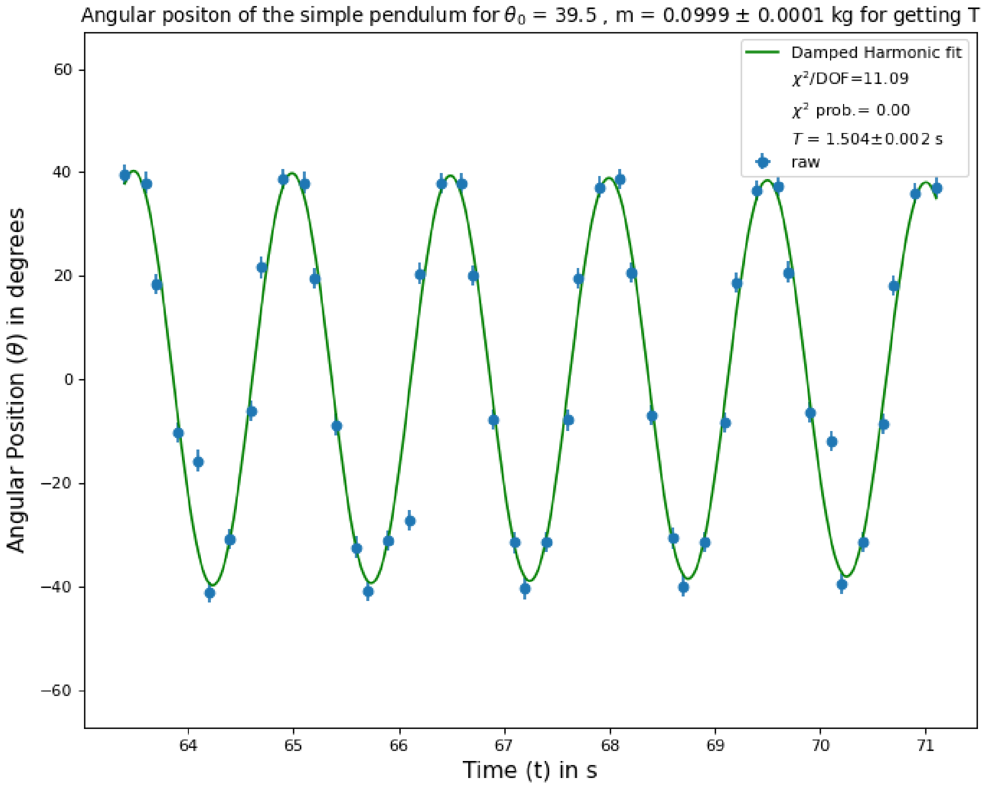
Angular positon of the simple pendulum for θ_0 = 39.5 , m = 0.0999 ± 0.0001 kg for getting T
Damped Harmonic fit
χ^2/DOF=11.09
χ^2 prob.= 0.00
T = 1.504±0.002 s
raw
Angular Position (θ) in degrees
Time (t) in s
60
40
20
0
−20
−40
−60
64
65
66
67
68
69
70
71

Figure 28