Algorithms for Model Order Reduction of Differential Algebraic Systems with Quadratic Output

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Consider a differential-algebraic system with quadratic output (DAE-QO)

$$\mathcal{S} : \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0, \\ y(t) = x(t)^T Mx(t) \end{cases} \tag{1}$$

$$\hat{\mathcal{S}} : \begin{cases} E_r\dot{x}_r(t) = A_r x(t) + B_r u(t), \qquad x_r(0) = 0, \\ y(t) = x_r(t)^T M_r x_r(t) \end{cases} \tag{2}$$

Where $\mathcal{S}$ is original system and $\hat{\mathcal{S}}$ is reduced order system.

$$\mathcal{S} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0 \\ y(t) = Cx(t) \end{cases} \tag{3}$$

Controllability and Observability Gramian equation for above system is given below

$$P = \int_0^\infty e^{At} BB^T e^{A^T t} \mathrm{d}t \tag{4a}$$

$$Q = \int_0^\infty e^{A^T t} C^T C e^{At} \mathrm{d}t \tag{4b}$$

For minimal stable system both P and Q are Uniquely satisfy below Lyapunov equation.

$$AP + PA^T + BB^T = 0 \tag{5a}$$

$$A^T Q + QA + C^T C = 0 \tag{5b}$$

Solution of above Lyapunov equation is symmetric positive definite matrices.

Algorithm 1: Balanced truncation MOR for LTI systems

Input: $A, B, C, r \leq n$
Output: A_r, B_r, C_r

1 Compute Cholesky factors $P = UU^*, \ Q = VV^*$ of the solution of Lyapunov equations

$$AP + PA^T + BB^T = 0$$

$$A^T Q + QA + C^T C = 0$$

2 Compute singular values decomposition of $U^* V$

$$U^* V = W \Sigma R^* = \begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} R_1^* \\ R_2^* \end{bmatrix}$$

where, Σ_1 =diag$(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$

3 Compute projection matrices as

$$T_1 = \Sigma_1^{-\frac{1}{2}} R_1^* V^*, \quad T_{1i} = UW_1 \Sigma_1^{-\frac{1}{2}}$$

4 Compute reduced order matrices

$$A_r = T_1 A T_{1i}, \quad B_r = T_1 B, \quad C_r = C T_{1i}$$

$$\mathcal{S} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = x(t)^T Mx(t) \end{cases} \tag{6}$$

Controllability and Observability Gramian equation for above system is given below

$$P = \int_0^\infty e^{At} BB^T e^{A^T t} \mathrm{d}t \tag{7}$$

We need to make use of adjoint theory for nonlinear systems.

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ &\dot{z}(t) = -Az(t) - Mx(t)u_d(t), \quad z(\infty) = 0, \\ &y_d(t) = Bz(t), \end{aligned}$$

from 2nd eqution we can write

$$z(t) = \int_\infty^0 \int_0^{\sigma_1 + t} e^{A^T \sigma_1} Me^{A\sigma_2} Bu\left(t - \sigma_2\right) u_d\left(t + \sigma_1\right) \mathrm{d}\sigma_2 \, \mathrm{d}\sigma_1$$

$$\begin{aligned} Q &= \int_0^\infty \int_0^\infty e^{A^T \tau_1} Me^{A\tau_2} BB^T e^{A^T \tau_2} Me^{A\tau_1} d\tau_1 d\tau_2 \\ &= \int_0^\infty e^{A^T \tau_1} MPMe^{A\tau_1} d\tau_2 \end{aligned} \tag{8}$$

P and Q satisfies below Lyapunov equations

$$AP + PA^T + BB^T = 0 \tag{9a}$$

$$A^T Q + QA + MPM = 0 \tag{9b}$$

Algorithm 2: Balanced truncation MOR for LD-QO systems

Input: $A, B, M, r \leq n$

Output: A_r, B_r, M_r

1 Compute Cholesky factors $P = UU^*$, $Q = VV^*$ of the solution of Lyapunov equations

$$AP + PA^T + BB^T = 0$$

$$A^T Q + QA + MPM = 0$$

2 Compute singular values decomposition of $U^* V$

$$U^* V = W \Sigma R^* = \begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} R_1^* \\ R_2^* \end{bmatrix}$$

where, Σ_1 =diag$(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_r)$

3 Compute projection matrices as

$$T_1 = \Sigma_1^{-\frac{1}{2}} R_1^* V^*, \quad T_{1i} = U W_1 \Sigma^{-\frac{1}{2}}$$

4 Compute reduced order matrices

$$A_r = T_1 A T_{1i}, \quad B_r = T_1 B, \quad M_r = T_{1i}^T M T_{1i}$$

Consider continuous-time descriptor system given below

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = 0 \\ y(t) &= Cx(t) \end{aligned} \tag{10}$$

For regular matrix pencil there always exists U_1 and U_2 such that we can reduce $sE - A$ into Weierstrass canonical form such that

$$U_1 E U_2 = \begin{bmatrix} I_{i_1} & 0 \\ 0 & N \end{bmatrix}, \quad U_1 A U_2 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{i_2} \end{bmatrix}$$

So after applying Weierstrass canonical form we can write descriptor system 10 as

$$\begin{aligned} \begin{bmatrix} I_{i_1} & 0 \\ 0 & N \end{bmatrix} \dot{\hat{x}}(t) &= \begin{bmatrix} A_1 & 0 \\ 0 & I_{i_2} \end{bmatrix} \hat{x}(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \quad \hat{x}(0) = 0 \\ y(t) &= [C_1, C_2]\hat{x}(t) \end{aligned} \tag{11}$$

Where $N^k = 0$, k is called index of descriptor system. Eigenvalues of A_1 are the finite eigenvalues of $sE - A$ and eigenvalues of $sN - I_{i_2}$ are at infinity only.

We can decoupled above descriptor system into
slow subsystem

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\ y_1(t) &= C_1 x_1(t) \end{aligned} \tag{12}$$

and fast subsystem

$$\begin{aligned} N\dot{x}_2(t) &= x_2(t) + B_2 u(t) \\ y_2(t) &= C_2 x_2(t) \end{aligned} \tag{13}$$

Complete solution of state vector is given below

$$x(t) = \mathcal{F}(t)Ex^0 + \int_0^t \mathcal{F}(t-\tau)Bu(\tau)d\tau + \sum_{j=0}^{k-1} F_{-j-1}Bu^{(j)}(t) \tag{14}$$

where
$\mathcal{F}(t) = U_2^{-1} \begin{bmatrix} e^{tA_1} & 0 \\ 0 & 0 \end{bmatrix} U_1^{-1}$ and
$F_j = U_2^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -N^{-j-1} \end{bmatrix} U_1^{-1}, \quad j = -1, -2, \ldots$
Clearly $F_j = 0$ for $j < -k$
$\mathcal{F}(t)$ is called fundamental solution matrix of descriptor system.

C-Controllablity

- The slow subsystem (12) is C-controllable if and only if

$$rankQ_c[A_1, B_1] = i_1.$$

or equivalently,

$$[sI - A_1 \; B_1] = i_1, \quad \forall s \in \mathbb{C}.$$

- The fast subsystem (13) is C-controllable if and only if

$$rankQc[N, B_2] = i_2.$$

or equivalently,

$$rank[N \; B_2] = i_2.$$

- The descriptor system (10) is C-controllable if and only if both slow subsystem and fast subsystem are C-controllable.

C-Observability

- The slow subsystem (12) is C-observable if and only if

$$rankQ_o[A_1, C_1] = i_1.$$

or equivalently,

$$rank \begin{bmatrix} sI - A_1 \\ C_1 \end{bmatrix} = i_1, \quad \forall s \in \mathbb{C}.$$

- The fast subsystem (13) is C-observable if and only if

$$rankQc[N, C_2] = i_2.$$

or equivalently,

$$rank \begin{bmatrix} N \\ C_2 \end{bmatrix} = i_2.$$

- The descriptor system (10) is C-observable if and only if both slow subsystem and fast subsystem are C-observable.

Proper Gramians

$$G_{pc} = \int_0^{\infty} \mathcal{F}(t)BB^T\mathcal{F}^T(t)\mathrm{d}t$$

$$G_{po} = \int_0^{\infty} \mathcal{F}^T(t)C^TC\mathcal{F}(t)\mathrm{d}t$$

Proper Gramians uniquely satisfies below generalized continous-time Lyapunov equations.

$$\begin{aligned} EG_{pc}A^T + AG_{pc}E^T &= -P_lBB^TP_l^T, \\ G_{pc} &= P_rG_{pc} \end{aligned}$$

$$\begin{aligned} E^TG_{po}A + A^TG_{po}E &= -P_r^TC^TCP_r, \\ G_{po} &= G_{po}P_l \end{aligned}$$

Improper Gramians

$$G_{ic} = \sum_{j=-k}^{-1} F_jBB^TF_j^T$$

$$G_{io} = \sum_{j=-k}^{-1} F_j^TC^TCF_j$$

improper Gramians uniquely satisfies below generalized discrete-time Lyapunov equations.

$$\begin{aligned} AG_{ic}A^T - EG_{ic}E^T &= (I - P_l)\, BB^T\, (I - P_l)^T, \\ P_rG_{ic} &= 0 \end{aligned}$$

$$\begin{aligned} A^TG_{io}A - E^TG_{io}E &= (I - P_r)^T\, C^TC\, (I - P_r), \\ G_{io}P_l &= 0 \end{aligned}$$

The matrices P_r and P_l are the spectral projection onto the right and left deflating subspaces of $sE - A$ corresponding to the finite eigenvalues.

For LTI system it's just square root of eigenvalue of product of Controllability Gramian (P) and observability Gramian (Q).

Proper Hankel Singular Values

$$\Phi_c := G_{pc} E^T G_{po} E$$

Cholesky decomposition of proper Gramians

$$G_{pc} = R_p R_p^T,$$

$$G_{po} = L_p^T L_p$$

$$\zeta_i^2 = \sigma_i^2 (L_p E R_p)$$

Improper Hankel Singular Values

$$\psi_c := G_{ic} A^T G_{io} A$$

Cholesky decomposition of improper Gramians

$$G_{ic} = R_i R_i^T,$$

$$G_{po} = L_i^T L_i$$

$$\theta_i^2 = \sigma_i^2 (L_i A R_i)$$

The proper and improper Hankel singular values of descriptor system are the standard singular values of the matrices $L_p E R_p$ and $L_i A R_i$, respectively.

Here we are using block structure of the system so that it's not required to find projection matrices P_l and P_r. After converting the system into block diagonalization form it's looks like this

$$\begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} \begin{bmatrix} \dot{x}_f(t) \\ \dot{x}_\infty(t) \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \begin{bmatrix} x_f(t) \\ x_\infty(t) \end{bmatrix} + \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} u(t)$$
$$y(t) = [C_f \; C_\infty] \begin{bmatrix} x_f(t) \\ x_\infty(t) \end{bmatrix} \tag{15}$$

solution of below continuous-time Lyapunov equations corresponds to proper controllability Gramian and proper observability Gramian respectively.

$$\begin{aligned} E_f X_{pc} A_f^T + A_f X_{pc} E_f^T + B_f B_f^T &= 0 \\ E_f^T X_{po} A_f + A_f^T X_{po} E_f + C_f^T C_f &= 0 \end{aligned} \tag{16}$$

Similarly solution of below discrete-time Lyapunov equations corresponds to improper controllability Gramian and improper observability Gramian respectively

$$\begin{aligned} A_\infty X_{ic} A_\infty^T - E_\infty X_{ic} E_\infty^T - B_\infty B_\infty^T &= 0 \\ A_\infty^T X_{io} A_\infty - E_\infty^T X_{io} E_\infty - C_\infty^T C_\infty &= 0 \end{aligned} \tag{17}$$

Algorithm 3: Balanced truncation MOR for descriptor systems

Input: $E, A, B, C, l_f \leq i_1$

Output: E_r, A_r, B_r, C_r

1 Compute block diagonalization structure of original system, i.e (15)

2 Find cholesky factorization of solution of the continuous-time Lyapunov equation (16), $X_{pc} = R_f R_f^T,\ X_{po} = L_f^T L_f$

3 Find cholesky factorization of solution of the discrete-time Lyapunov equation (17), $X_{ic} = R_\infty R_\infty^T,\ X_{io} = L_\infty^T L_\infty$

4 Compute thin SVD of $L_f E_f R_f$

$$L_f E_f R_f = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$$

where $\Sigma_1 = \mathrm{diag}(\zeta_1, ..., \zeta_{l_f})$, $\Sigma_2 = \mathrm{diag}(\zeta_{l_f+1}, ..., \zeta_{r_1})$, with $\zeta_1 \geq ... \geq \zeta_{l_f} \geq \zeta_{l_f+1} \geq ... \geq \zeta_{r_1} > 0$, $r_1 = \mathrm{rank}(L_p E R_p)$

5 Compute thin SVD of $L_\infty A_\infty R_\infty$

$$L_\infty A_\infty R_\infty = U_3 \Theta_3 V_3^T$$

where $\Theta_3 = \mathrm{diag}(\theta_1, ..., \theta_{r_2})$, with $\theta_1 \geq ... \geq \theta_{r_2} > 0$, $r_2 = \mathrm{rank}(L_\infty A_\infty R_\infty)$

6 Find $W_r = \begin{bmatrix} L_f^T U_1 \Sigma_1^{-1/2}, & L_\infty^T U_3 \Theta_3^{-1/2} \end{bmatrix}$, and $T_r = \begin{bmatrix} R_f V_1 \Sigma_1^{-1/2}, & R_\infty V_3 \Theta_3^{-1/2} \end{bmatrix}$

7 Find reduced order matrices $E_r = W_r^T E T_r, A_r = W_r^T A T_r, B_r = W_r^T B, C_r = C T_r$.

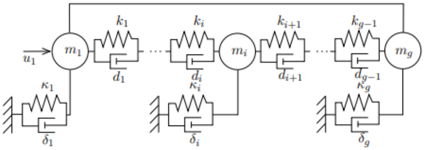

FIGURE: example setup

The i^{th} mass of weight m_i is connected to the $(i+1)^{st}$ mass by a spring and a damper with constants k_i and d_i, respectively, and also to the ground by a spring and a damper with constants κ_i and δ_i, respectively. Additionally, the first mass is connected to the last one by a rigid bar and it is influenced by the control $u(t)$. The position of the 1^{st}, 2^{nd} and $(g-1)^{st}$ masses is measured. The equation of motion for this system is given below

$$\begin{aligned} \dot{\mathbf{p}}(t) &= \mathbf{v}(t), \\ M\dot{\mathbf{v}}(t) &= K\mathbf{p}(t) + D\mathbf{v}(t) - G^T\boldsymbol{\lambda}(t) + B_2u(t), \\ 0 &= G\mathbf{p}(t), \\ y(t) &= C\mathbf{p}(t) \end{aligned} \tag{18}$$

We take $g = 600$ so that order of descriptor system n becomes 1201 with $i_1 = 1198,\ i_2 = 3$. We approximate reduced system of order 22 using Algorithm-3.

RESULTS

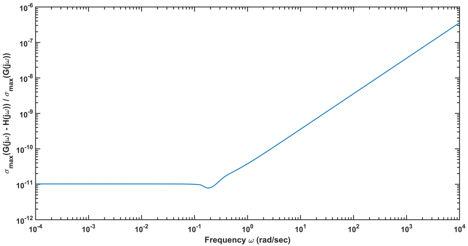

FIGURE: Sigma plot of example-2

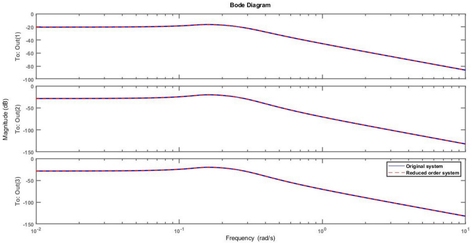

FIGURE: Frequency response of example-2

Again consider

$$\mathcal{S} : \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0, \\ y(t) = x(t)^T Mx(t) \end{cases} \tag{19}$$

Here if $M \geq 0$ then we can write $M = \tilde{C}^T \tilde{C}$. Hence we can write above equation as

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ \tilde{y}(t) &= \tilde{C}x(t) \\ y(t) &= \|\tilde{y}(t)\|_2^2 \end{aligned} \tag{20}$$

Algorithm 4: BT-MOR for DAE-QO Syetem with $M \geq 0$.

Input: $E, A, B, M, l_f \leq i_1$

Output: E_r, A_r, B_r, M_r

1 Compute Cholesky factor of $M = \tilde{C}^T \tilde{C}$

2 Now find $E_r, A_r, B_r, \tilde{C}_r$ for system (20) using algorithm-3.

3 find $M_r = \tilde{C}_r^T \tilde{C}_r$

Consider Descriptor system with quadratic output given below

$$\mathcal{S} : \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0 \\ y(t) = x^T(t)Mx(t) \end{cases} \tag{21}$$

After applying Weierstrass canonical form

$$\begin{aligned} \hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t), \\ y(t) &= \hat{x}^T(t)\hat{M}\hat{x}(t) \end{aligned} \tag{22}$$

The system matrices are

$$\hat{E} = U_1 E U_2 = \begin{bmatrix} I_{i_1} & 0 \\ 0 & N \end{bmatrix}, \quad \hat{A} = U_1 A U_2 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{i_2} \end{bmatrix}$$
$$\hat{B} = U_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \hat{M} = U_2^T M U_2 = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$$

$$\hat{x} = U_2^{-1} x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For index-1 system we get

$$\begin{aligned} &\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t), \quad x_2(t) = -B_2 u(t) \\ &y(t) = x_1^T(t) M_{11} x_1(t) - 2x_1^T(t) M_{12} B_2 u(t) + u^T(t) B_2^T M_{22} B_2 u(t) \end{aligned}$$

We can write above equation as

$$\begin{aligned}\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t)\\ y(t) &= x_1^T(t) M_{11} x_1(t) - 2x_1^T(t) M_{12} B_2 u(t)\end{aligned} \tag{23}$$

it's well known that controllability Gramian is defined as follow

$$P = \int_0^\infty e^{A_1 t} B_1 B_1^T e^{A_1^T t} \mathrm{d}t \tag{24}$$

If matrix pencil $sE - A$ is c-stable then A_1 is Hurwitz after applying Weierstass canonical form. So controllabiity Gramian P uniquely satisfies below Lyapunov equation

$$A_1 P + P A_1^T + B_1 B_1^T = 0 \tag{25}$$

For given system Hamiltonian is defined as

$$
\begin{aligned}
H &= z^T(A_1 x_1(t) + B_1 u(t)) + \left(\frac{u_a(t)}{2}^T\right) y \\
&= z^T(A_1 x_1 + B_1 u) + \frac{1}{2} u_a x_1^T(t) M_{11} x_1(t) - 2x_1^T(t) M_{12} B_2 u(t)
\end{aligned}
$$

Here u_a is a scalar. Now,

$\dot{x}_1(t) = \frac{\partial H}{\partial z}^T = A_1 x_1(t) + B_1 u(t),$

$\dot{z}(t) = -\frac{\partial H}{\partial x_1}^T = -A_1^T z(t) - c_1 x_1(t) u_a(t) + c_2 B_2 u(t) u_a(t),$

$y_a(t) = \frac{\partial H}{\partial u}^T = B_1^T z(t) + x_1(t)^T K u_a(t),$

where $K := -C_2 B_2$

So, We can write down the state-space realization of the nonlinear Hilbert adjoint operator of an DAE-QO system as follows

$$
\begin{aligned}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) && \text{(26a)} \\
\dot{z}(t) &= -A_1^T z(t) - M_{11} x_1(t) u_a - K u(t) u_a(t) && \text{(26b)} \\
y_a(t) &= B_1^T z(t) + x_1(t)^T K u_a(t) && \text{(26c)}
\end{aligned}
$$

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t)$$
$$\dot{z}(t) = -A_1^T z(t) - M_{11} x_1(t) u_a - Ku(t) u_a(t)$$

from above equation we obtain

$$z(t) = \int_\infty^0 \int_0^{t+\sigma_1} e^{A_1^T \sigma_1} M_{11} e^{A_1 \sigma_2} B_1 u\,(t - \sigma_2)\, u_a(t + \sigma_1) \mathrm{d}\sigma_2 \mathrm{d}\sigma_1 + \int_\infty^0 e^{A_1^T \sigma_1} Ku(t + \sigma_1) u_a(t + \sigma_1) \mathrm{d}\sigma_1 \quad (27)$$

So for the above system observability Gramian is defined as

$$\begin{aligned} Q &= \int_0^\infty \left[\int_0^\infty (e^{A_1^T \sigma_1} M_{11} e^{A_1 \sigma_2} B_1)(e^{A_1^T \sigma_1} M_{11} e^{A_1 \sigma_2} B_1)^T \mathrm{d}\sigma_2 + (e^{A_1^T \sigma_1} K)(e^{A_1^T} \sigma_1)^T \right] \mathrm{d}\sigma_1 \\ &= \int_0^\infty e^{A_1^T \sigma_1} M_{11} \left[\int_0^\infty (e^{A_1 \sigma_2} B_1)(e^{A_1 \sigma_2} B_1)^T \mathrm{d}\sigma_2 \right] M_{11}^T e^{A_1 \sigma_1} + (e^{A_1^T \sigma_1} K)(e^{A_1^T \sigma_1} K)^T \mathrm{d}\sigma_1 \\ &= \int_0^\infty \left(e^{A_1^T \sigma_1} M_{11} P M_{11} e^{A_1 \sigma_1} + e^{A_1^T \sigma_1} K K^T e^{A_1 \sigma_1} \right) \mathrm{d}\sigma_1 \\ &= \int_0^\infty e^{A_1^T \sigma_1} \left(M_{11} P M_{11} + K K^T \right) e^{A_1 \sigma_1} \mathrm{d}\sigma_1 \end{aligned}$$

$$Q = \int_0^\infty e^{A_1^T \sigma_1} \left(M_{11} P M_{11} + K K^T \right) e^{A_1 \sigma_1} \mathrm{d}\sigma_1$$

Now using the same argument as used for controllability Gramian, it's easy to show that observability Gramian satisfies below Lyapunov equation

$$A_1^T Q + Q A_1 + M_{11} P M_{11} + K K^T = 0$$

where $K = M_{12} B_2$

Energy functions

The controllability energy function is defined as minimum input energy required to drive the state from non-zero initial condition to zero,

$$\mathcal{E}_c = \min_{\substack{x(-\infty)=x_0,\\ x(0)=0}} \|u\|_{L_2}^2.$$

We can write the controllability energy function as

$$\mathcal{E}_c = \frac{1}{2} x_{1_0}^T P^{-1} x_{1_0}$$

The observability energy function $\mathcal{E}_o$ is defines as the output energy produce by nonzero initial condition x_0 follows

$$\mathcal{E}_o = \int_0^\infty \|y(t)\|_2^2 \mathrm{d}t$$

Theorem

For positive definite controllabiliy Gramian P and observability Gramian Q as define above. Furthermore, let us assume that the state trajectory $x(t)$, generated from a non-zero initial condition x_o with zero input, lies in $\mathcal{W}_\delta$, where $\mathcal{W}_\delta$ is the balls of radius δ centered around zero. Then, the output energy function can be bounded as follows:

$$\mathcal{E}_o \leq x_{1_0}^T Q x_{1_0}.$$

THEOREM

For Controllability Gramian (P) and Observability Gramian (Q) satisfying given Lyapunov equations. Then, following result hold:

(A) *If the system has to be driven from zero to x_{1_0}, with $x_{1_0} \in kerP$, then $\mathcal{E}_c = \infty$; hence it's unreachable.*

(B) *If P is positive definite and $x_{1_0} \in kerQ$,then $\mathcal{E}_o = 0$, thus making the state x_{1_0} unobservable.*

So far, we have proposed Gramians for DAE-QO systems and have shown how these Gramians relate to the energy functionals of the systems, under required conditions. We show that Gramians, in general case, encode controllability and observability subspaces informantion.
Having had all this discussion between the energy functions and gramians, it's clear that these Gramians allow us to determine the states which are hard to reach and hard to observe.

After applying proper balancing transformation T_b, we can write

$$P = Q = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_{i_1})$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{i_1}$ and σ_k is the k-th singular value of the system.

$$\begin{aligned} A_1\Sigma + \Sigma A_1^T + B_1 B_1^T &= 0 \\ A_1^T\Sigma + \Sigma A_1 + M_{11}\Sigma M_{11} + M_{12}B_2B_2^T M_{12}^T &= 0 \end{aligned} \tag{28}$$

where

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad M_{11} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix},$$

$$M_{12}B_2 = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

After partitioning above equation we can write block (1,1) as

$$\begin{aligned} A_{11}\Sigma_1 + \Sigma_1 A_{11}^T + B_{11}B_{11}^T &= 0 \\ A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + C_{11}\Sigma_1 C_{11} + C_{12}\Sigma_2 C_{12}^T + D_1 D_1^T &= 0 \end{aligned} \tag{29}$$

Reduced order system is balanced in generalized sense because we can write

$$A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + C_{11}\Sigma_1 C_{11} + D_1 D_1^T \leq 0$$

Algorithm 5: Square root BT-MOR for index-1 DAE-QO

Input: $E, A, B, M, r \leq i_1$
Output: E_r, A_r, B_r, M_r

1 Find Weierstrass canonnical form of matrix pencil $sE - A$ such that

$$\hat{E} = U_1 E U_2 = \begin{bmatrix} I_{i_1} & 0 \\ 0 & N \end{bmatrix}, \quad \hat{A} = U_1 A U_2 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{i_1} \end{bmatrix},$$

$$\hat{B} = U_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \hat{M} = U_2^T M U_2 = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

2 Find low-factors of Gramians P and Q such that $P = u^T u$ and $Q = v^T v$.

$$A_1^T Q + Q A_1 + M_{11} P M_{11} + M_{12} B_2 B_2^T M_{12}^T = 0$$

$$A_1 P + P A_1 + B_1 B_1^T = 0$$

3 Find SVD of $u^T v$ such that $u^T v = \begin{bmatrix} W1 & W2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} R1 & R2 \end{bmatrix}^T$; where $\Sigma_1 = R^{r \times r}$

4 Construct a projection matrices $T_1 = \Sigma_1^{-\frac{1}{2}} R_1^T v^*, \quad T_{1i} = u W_1 \Sigma^{-\frac{1}{2}}$

5 $\hat{E}_r = \hat{T}_1 \hat{E} \hat{T}_{1i}, \ \hat{A}_r = \hat{T}_1 \hat{A} \hat{T}_{1i}, \ \hat{B}_r = \hat{T}_1 \hat{B}, \ \hat{M}_r = \hat{T}_{1i}^T \hat{M} \hat{T}_{1i}$; where $\hat{T}_1 = \begin{bmatrix} T_1 & 0 \\ 0 & I_{i_2} \end{bmatrix}$ and $\hat{T}_{1i} = \begin{bmatrix} T_{1i} & 0 \\ 0 & I_{i_2} \end{bmatrix}$

Here we will try to apply our algorithm on a small-scale index-1 DAE-QO system with $i_1 = 15$ and $i_2 = 5$.

Considered input : $u(t) = e^{-\frac{1}{4}t}$ for $t \geq 0 \;\Rightarrow\; \|u \otimes u\|_{L_2} = 1$

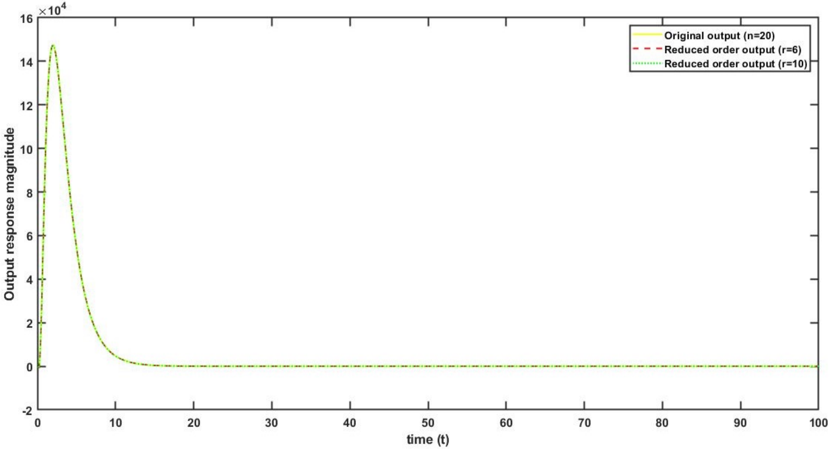

FIGURE: Output magnitude response of small scale example

ABSOLUTE ERROR

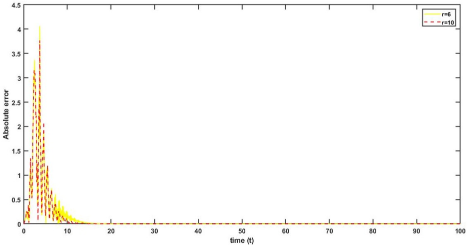

FIGURE: Absolute error comparison

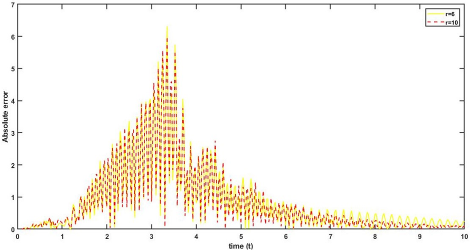

FIGURE: Absolute error comparison (Enlarged)

RELATIVE ERROR

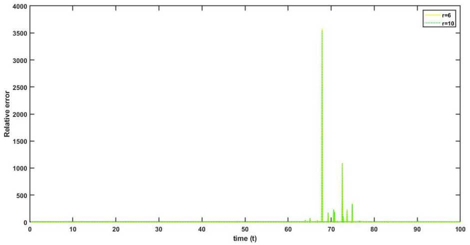

FIGURE: Relative error comparison

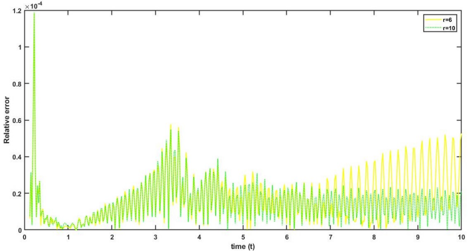

FIGURE: Relative error comparison (Enlarged)

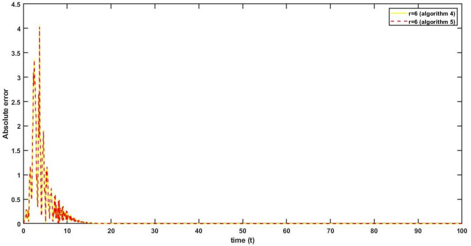

FIGURE: Absolute error comparison between two algorithms

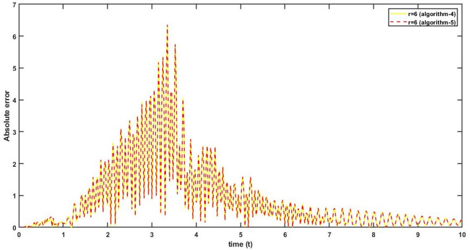

FIGURE: Absolute error comparison between two algorithms (Enlarged)

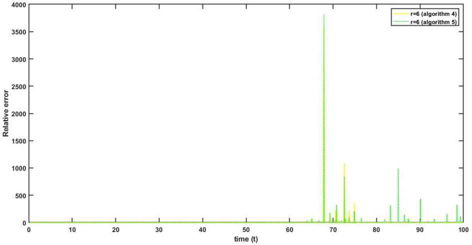

FIGURE: Relative error comparison between two algorithms

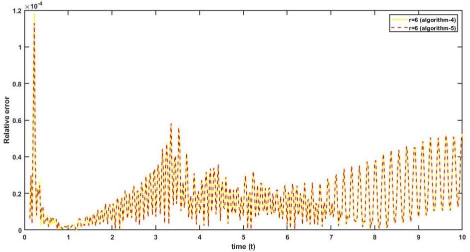

FIGURE: Relative error comparison between two algorithms (Enlarged)

FUTUTE WORK

- We observed during our experiment that numerically it's not efficient to find Weierstrass canonical form for a large-scale system. So we still need to develop a practical algorithm for a large-scale index-1 DAE-QO systems using block diagonal structute of system.
- later on, we will try to develop a general algorithm that is applicable for any DAE-QO system not just for index-1 DAE-QO systems.

THANK YOU