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SC - Homework - 3

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2020320

Q:1 Given: $A \in \mathbb{R}^{n \times n}$ is triangular matrix.

To prove: Eigenvalue of A are a_{ii} where $i \in [1, n]$.

Proof:

Case-1: Let A be an upper-triangular matrix

$$\therefore \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}_{n \times n} \quad \text{where } a_{ij} \in \mathbb{R}$$

For, eigenvalues,

$$\det(A - \lambda I) = 0$$

Thus,
$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & & & a_{2n} \\ \vdots & \vdots & a_{33}-\lambda & & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn}-\lambda \end{bmatrix}_{n \times n}$$

$$\det(A - \lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\cdots(a_{nn}-\lambda) = \text{char}(A)$$

Thus, $\text{char}(A) \Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = 0$

$\Rightarrow \lambda = a_{ii}$ where $i \in \{1, 2, \ldots, n\}$

Hence, roots of char(A) are a_{ii} i.e diagonal elemnts of A.

Thus, the eigenvalues of an upper-triangular matrix A, are diagonal elements of A.

Case-II: Let A be a lower triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}_{n \times n}$$ where $a_{ij} \in \mathbb{R}$.

Eigenvalues of A are roots of char polynomial

$$\therefore A - \lambda I = \begin{bmatrix} a_{11} - \lambda & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} - \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & a_{33} - \lambda & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

$$\text{char}(A) = \det(A - \lambda I)$$
$$= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

For eigenvalue, $\text{char}(A) = 0$

$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0.$

$\Rightarrow \lambda = a_{ii} \quad i \in \{1, 2, \ldots n\}$

↳ diagonal elemnts of A.

Thus, the eigenvalues of a lower triangular matrix are the diagonal elements of A.

$\therefore$ From, case-I & II if A is a triangular matrix, then the eigenvalues of A are the diagonal elements of A.

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Q:2 Given: $A \in \mathbb{R}^{n \times n}$ is non defective matrix.

To proof: Rank of a non-defective matrix $A \in \mathbb{R}^{n \times n}$ is equal to the number of non-zero eigenvalues of the matrix

Proof: A non-defective $\Rightarrow$ eigenvalue decomposition exists. & also diagonalizable.

$\Rightarrow$ A diagonizable matrix 'D' exists & an invertible matrix 'R' s.T

$$R^{-1} A R = D \qquad [R \in \mathbb{R}^{n \times n}, D \in \mathbb{R}^{n \times n}]$$

So, $A = RDR^{-1}$

$rank(A) = rank(RDR^{-1})$ (rank of product of matrices rule)

$rank(A) = \min(rank(R), rank(D), rank(R^{-1}))$

- R is an invertible matrix $\Rightarrow$ R has full rank.

Hence,

$rank(R) = rank(R^{-1}) = n$

But, $rank(D) \leq n$

$\Rightarrow rank(A) \leq n.$

$\therefore rank(A) = rank(D)$

So, $rank(A)$ = non-zero diagonal elements of D (rank of diagonal matrix)

As the diagonal elements of 'D' are the eigenvalues of A.

$\therefore$ Rank(A) = non-zero eigenvalues of A.

Hence,
The rank of a non-defective matrix $A \in \mathbb{R}^{n \times n}$ is equal to the no. of non zero eigenvalues of this matrix.

Q:3 Given: $u, v \in \mathbb{R}^n$, $u^T v = 1$ & $A = uv^{-1}$

To find: Eigenvalues of A.

Soln: Now, as

$$A^T A = (uv^T)^T uv^T$$

$$A^T A = v u^T u v^T$$

$$AA^T = uv^T (uv^T)^T = uv^T v u^T \quad (uv^T \neq 0)$$

as $A^T A = v u^T u v^T = v \|u\|^2 v^T = \|u\|^2 v v^T$

($\|u\|^2$ is scalar)

$$A^T A = \|u\|^2 \|v\|^2$$

and $AA^T = u v^T v u^T = u \|v\|^2 u^T = \|v\|^2 u u^T$

($\|v\|^2$ is scalar)

$$AA^T = \|v\|^2 \|u\|^2$$

So, as $A^T A = AA^T$, A is normal & hence diagonalizable

& $\text{rank}(A) = \text{rank}(uv^T)$

(By rule of mul of matrices rank)

$\text{rank}(A) = \min(\text{rank}(u), \text{rank}(v^T))$

So, $\text{rank}(u), \text{rank}(v) \leq 1$.

as $u^T v = 1$, hence u & v cannot be null vectors & hence,

$\text{rank}(u) = \text{rank}(v) = 1$

&

$\text{rank}(A) = 1$.

As A is not a full rank matrix, hence, it must have 0 as its eigenvalues.

as rank(A) = no. of non-zero eigen values of A
(as A is a non defective $\Leftrightarrow$ diagonalizable)
(Previous questⁿ)

so, we get no. of non-zero eigen values as 1
so, no. of zero eigen values are $n-1$
(as there are total n eigen values)

Now let the eigenvalue (non zero) be λ,
so,

$$Ax = \lambda x \quad (x \text{ is eigenvector})$$
(λ is scalar)

$(A = uv^T)$

$$uv^T x = x\lambda$$

(1) we take $x = u$ & $\lambda = v^T u$)
$\Rightarrow uv^T u = uv^T u$
(LHS = RHS)

$\therefore$ we get eigenvector as u & eigenvector as $v^T u$

So, the eigenvalues of A are $v^T u$ with 1, multiplicity and 0 with $n-1$ multiplicity.

To find no. of iterations used by power method to converge to the dominant eigenvalue eigenvector pair $(v^T u, u)$, we have

$$u_k = \lambda_1^k \left[c_1 v_1 + c_2 v_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k + \dots + c_n v_n \left(\frac{\lambda_n}{\lambda_1} \right)^k \right]$$

where u_k is the estimated eigenvector in k^{th} iteration and λ is the largest eigen value and v_i is the eigenvector at i^{th} iteration

Hence as $\lambda = v^T u$, so,
$\lambda = ((v^T u)^T)^T = (u^T v)^T = 1^T = 1$

so, $\lambda = 1$.

& so we get

$u_k = 1^k [c_1 v_1 + 0 + 0 \ldots 0] = 1$

so, for this power iteration method will converge in $\underline{\underline{1}}$ iteration.

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Q:4 Given: $A \in \mathbb{R}^{n \times n}$ is a symmmetrix, positive definite matrix.

To prove: All eigenvalues of A are real & $\lambda_i > 0$

Proof:

a). Let (λ, u) be the eigenvalve, eigenvector pair for A & $\lambda \in \mathbb{C}$, $u \in \mathbb{C}^n$.

$\Rightarrow$. $\langle Au, u \rangle = \langle \lambda u, u \rangle$

$\Rightarrow$ $u^H A u = u^H (\lambda u)$

$\Rightarrow$ $u^H A u = \lambda u^H u$ — ①.

Taking Hermitian transpose on both sides

$\Rightarrow$ $(u^H A u)^H = (\lambda u^H u)^H$

$\Rightarrow$ $u^H A^H (u^H)^H = \lambda^* u^H (u^H)^H$

$\Rightarrow$ $u^H A^H u = \lambda^* u^H u$

But $A = A^T$ (Symmmetric matrix)

$\therefore A = A^H$ (Since A is real valued)

$\Rightarrow$ $u^H A u = \lambda^* u^H u$ — ②.

From ① & ② $\lambda = \lambda^*$ $\Rightarrow$ λ is real.

b. To prove: $\lambda_i > 0$ where $i \in \{1, 2, \ldots n\}$

Let (λ, u) be the eigenvalue, eigenvector pair

We know, $Au = \lambda u$ (u = eigenvector corresponding to λ)

Multiplying it by u^T on both sides

$\Rightarrow u^T A u = u^T \lambda u$

$\Rightarrow u^T A u = \lambda u^T u$

$\Rightarrow u^T A u = \lambda \, ||u||^2$

Now, since A is +ve definite,

$u^T A u > 0$.

$\therefore \lambda ||u||^2 > 0$

$||u||^2$ is always positive since it is the sum of sq elements (all +ve).

So, $\therefore$ $\lambda > 0$

Hence, proved.

From (a) & (b). we can say that if $A \in \mathbb{R}^{n \times n}$ it is a symmmetric positive definite matrix then all the eigenvalues of A are real & greater than zero.

Hence, proved.

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Q:5 Given : $A \in \mathbb{R}^{n \times n}$ is symmmetric, positive definite matrix & non defective.

To find : e^A in terms of eigenvalues of A

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Answer : A is non-defective matrix, hence it is diagonizable

so, $A = RDR^{-1}$ where R is invertible & diagonlized of A

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & & \vdots \\ \vdots & - & - & \lambda_n \end{bmatrix}_{n \times n}$$

λ_i is eigenvalue of A.

Multiply $A = RDR^{-1}$ to $A = RDR^{-1}$

$A^2 = (RDR^{-1})(RDR^{-1})$

$A^2 = RD^2R^{-1}$

$\vdots$

Multiply k times

$\Rightarrow A^k = RD^kR^{-1}$ —— (1)

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} (RD^kR^{-1})$$

$$e^A = R\left(\sum_{k=0}^{\infty} \frac{1}{k!} D^k\right) R^{-1}$$ — using (1). (R independent of k)

$\therefore \boxed{e^A = Re^DR^{-1}}$

R^{-1} is computable as R is invertible.

D diagonal matrix of eigen value of A

R containtaing eigenvector of each eigenvalues of A as column vector in the matrix.

```
    a1 = np.linalg.norm(y, ord=np.inf)
```

Q6 Output SS

```
    return [a1, x]
```

They are almost equal when ignored the computing error

```
def reverse_iteration(A, x):
    for i in range(1, 101):
        y = np.linalg.solve(A, x)
        x = y / np.linalg.norm(y, ord=np.inf)
    a2 = 1 / np.linalg.norm(y, ord=np.inf)
    return [a2, x]


A = np.array([[2, 3, 2], [10, 3, 4], [3, 6, 1]])
b = np.array([0, 0, 1])
X = b.T

l1 = power_iteration(A, X)

print("Normalised Power Iteration Eigenvalue", l1[0])
print("Normalised Power Iteration Eigenvector", l1[1])

l2 = reverse_iteration(A, X)

print("\nInverse Iteration Eigenvalue is ", l2[0])
print("Inverse IterationEigenvector is ", l2[1])

print("\nActual Eigenvalues is", np.linalg.eig(A)[0])
print("Actual Eigenvectors are \n", np.linalg.eig(A)[1])
```

```
Normalised Power Iteration Eigenvalue 11.0
Normalised Power Iteration Eigenvector [0.5  1.   0.75]

Inverse Iteration Eigenvalue is  2.0000000000000004
Inverse IterationEigenvector is  [-0.2 -0.4  1. ]

Actual Eigenvalues is [11. -2. -3.]
Actual Eigenvectors are
 [[ 3.71390676e-01  1.82574186e-01 -5.26283806e-16]
 [ 7.42781353e-01  3.65148372e-01 -5.54700196e-01]
 [ 5.57086015e-01 -9.12870929e-01  8.32050294e-01]]
```

Q7 Output SS

[1]:
```python
import numpy as np
import numpy.linalg as npla

A = np.array([[6, 2, 1], [2, 3, 1], [1, 1, 1]])
I = np.zeros([3,3])
I[0][0] = I[1][1] = I[2][2] = 1

X = np.array([1, 1, 1])

i = 1
Y = 0
while i <= 99:
    Y = np.linalg.solve((A-2*I), X)
    X = Y/np.linalg.norm(Y, ord=np.inf)
    i += 1

a = 2 + 1/npla.norm(Y,ord=np.inf)

print("Eigenvalue is : ", a)
print("Eigenvector is : ", X)

print("\nAll Actual Eigenvalue is :", npla.eigh(A, 'U')[0])
print("All Actual Eigenvector are : \n", npla.eigh(A,'L')[1])
```

```
Eigenvalue 2.133074475348525
Eigenvector [-0.60692002  1.          0.34691451]

All Actual Eigenvalue [0.57893339 2.13307448 7.28799214]
All Actual Eigenvector
 [[-0.0431682  -0.49742503 -0.86643225]
 [-0.35073145  0.8195891  -0.45305757]
 [ 0.9354806   0.28432735 -0.20984279]]
```

main.py Q8 Output SS

Run

```python
import numpy as np

A = np.array([[2, 3, 2], [10, 3, 4], [3, 6, 1]])
x = np.array([[1], [0], [1]])

k = 0
I = np.identity(3)

while True:
    if k == 1000:
        break
    k = k + 1
    ans = np.matmul(np.matmul(x.T, A), x) / np.matmul(x.T, x)
    y = np.linalg.solve(A - ans * I, x)
    x = np.linalg.norm(y, ord=np.inf)
    x = y / x

print("Rayleigh Quotient:")
print('Eigenvalue:>', ans[0][0])
print('Eigenvalue:', x)

print("Library Routine :")
print(np.linalg.eigh(A))
```

Shell Clear

```
Rayleigh Quotient:
Eigenvalue:> 11.0
Eigenvalue: [[-0.5 ]
 [-1.  ]
 [-0.75]]
Library Routine :
(array([-8.05778972, -1.34550956, 15.40329928]), array([[-0.63829307, -0.47912245
    ,  0.60251443],
       [ 0.72204791, -0.1012749 ,  0.6843904 ],
       [-0.26688721,  0.87188593,  0.41059243]]))
>
```

Q9 Output SS

```
[2]: import numpy as np
import numpy.linalg as npla
import scipy.linalg as spla

A = np.array([[2, 3, 2], [10, 3, 4], [3, 6, 1]])
mu = A[2,2]
I = np.zeros([3,3])
I[0][0] = 1
I[1][1] = 1
I[2][2] = 1


k=1
while(k<1000):
    Q,R = spla.qr(A-mu*I)
    A = np.matmul(R, Q) + mu*I
    k=k+1
print("Eigenvalue from Q6 is :", [A[0,0], A[1,1], A[2,2]])
print("Actual Eigenvalue from Q6 is :", npla.eig(A)[0])

A = np.array([[6, 2, 1], [2, 3, 1], [1, 1, 1]])
mu = A[2,2]
I = np.identity(3)

j=1
while(j<100):
    Q, R = spla.qr(A-mu*I)
    A = np.matmul(R, Q) + mu*I
    j=j+1
print("\nEigenvalue from Q7 is :", [A[0,0], A[1,1], A[2,2]])
print("Actual Eigenvalue from Q7 is :",npla.eig(A)[0])
```

```
Eigenvalue from Q6 is : [10.999999999999993, -3.000000000000009, -2.0]
Actual Eigenvalue from Q6 is : [11. -3. -2.]

Eigenvalue from Q7 is : [7.28799213896042, 2.1330744753485242, 0.5789333856910526]
Actual Eigenvalue from Q7 is : [7.28799214 2.13307448 0.57893339]
```