# PHY3320 ASSIGNMENT 4

MOHAMMED CHAMMA 6379153
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**Problem 1.** For a hollow rectangular waveguide with dimensions $a = 0.03$m and $b = 0.02$m determine the cutoff frequencies for all modes up to 20GHz and show them schematically. Over what frequency range will the guide support the propagation of a single dominant mode?

The angular cutoff frequency is given by $\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$, so the cutoff frequency is $f_{mn} = \frac{\omega}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

$$
\begin{aligned}
f_{01} &= \frac{c}{2}\sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \frac{3\times 10^8}{2}\sqrt{\left(\frac{1}{0.02}\right)^2} = 7.5\text{GHz} \\
f_{10} &= \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = 5.0\text{GHz} \\
f_{11} &= \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 9.0\text{GHz} \\
f_{12} &= \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{2}{b}\right)^2} = 15.8\text{GHz} \\
f_{21} &= \frac{c}{2}\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 12.5\text{GHz} \\
f_{22} &= \frac{c}{2}\sqrt{\left(\frac{2}{a}\right)^2 + \left(\frac{2}{b}\right)^2} = 18.0\text{GHz} \\
f_{32} &= \frac{c}{2}\sqrt{\left(\frac{3}{a}\right)^2 + \left(\frac{2}{b}\right)^2} = 21.2\text{GHz}
\end{aligned}
$$

If you wanted the 10 mode to be the single dominant mode then a frequency range of $|f_{01} - f_{10}| = 7.5 - 5 = 2.5$GHz would allow for the propagation of a single mode.

In general the range would be given by $|f_{mn} - f_{nm}|$ since the $nm$ mode is always the closest in frequency to the $mn$ mode.

**Problem 2.** A TM wave propagating in a dielectric filled waveguide of unknown permittivity has a magnetic field with y-component given by $B_y = 6\cos(25\pi x)\sin(100\pi y)\sin(1.5\pi\times 10^{10}t - 109\pi z)$. The guide dimensions are $a = 2$cm and $b = 4$cm.

**(a)** Determine the mode numbers.

$$
\begin{aligned}
k_x &= 25\pi = \frac{m\pi}{a} \\
m &= 25a = 25(2) = 50
\end{aligned}
$$

$$\begin{aligned} k_y &= 100\pi = \frac{n\pi}{b} \\ n &= 100b = 100(4) = 400 \end{aligned}$$

**(b)** Determine the relative permittivity of the material in the guide.

Since $k = \frac{\omega}{c}\sqrt{\varepsilon_r}$, $\varepsilon_r = \frac{k^2c^2}{\omega^2}$.

$$\begin{aligned} \varepsilon_r &= \frac{k^2c^2}{\omega^2} = \frac{(109\pi)^2(3\times 10^8)^2}{(1.5\pi\times 10^{10})^2} \\ \varepsilon_r &= 4.75 \end{aligned}$$

**(c)** Determine the phase velocity

$$v = \frac{\omega}{k} = \frac{1.5\pi\times 10^{10}}{109\pi} = 1.38\times 10^8$$

**(d)** Obtain an expression for $E_x$

We know that the tangential components are related

$$\begin{aligned} E_t^0 &= -\frac{ck_g}{k_0}(\hat{z}\times B_t^0) \\ E_x^0\hat{x} + E_y^0\hat{y} &= -\frac{c^2k_g}{\omega}(B_x^0\hat{y} - B_y^0\hat{x}) \end{aligned}$$

Since this is a TM wave, $E_y^0 = B_x^0 = 0$. So

$$\begin{aligned} E_x &= -\frac{c^2k_g}{\omega}B_y \\ &= -\frac{(3\times 10^8)^2(109\pi)}{1.5\pi\times 10^{10}}B_y \\ &= -6.54\times 10^8 B_y \end{aligned}$$

**Problem 3.** Work out the theory of TM modes for a rectangular waveguide. Find all the field components, cut-off frequencies, and wave group velocities. Find the ratio of the lowest TM cut-off frequency to the lowest TE cut-off frequency for a given wave guide.

We know that $E_z^0(x,y)$ completely determines the field. We need to solve

$$(\nabla_t^2 + k_c^2)E_z^0 = 0$$

under the boundary conditions $E_z^0|_s = 0$. Let $E_z^0(x,y) = X(x)Y(y)$. Substituting,

$$\begin{aligned} Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + k_c^2XY &= 0 \\ \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 &= 0 \end{aligned}$$

Let $k_c^2 = k_x^2 + k_y^2$. We have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \qquad \text{and} \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

The general solution for $X$ is

$$X(x) = A\sin(k_xx) + B\cos(k_xx)$$

Since the boundary condition is $E_z^0|_s = 0$, $X(a) = 0$ and $X(0) = 0$. For $X(0) = 0$, we need $B = 0$ since $\cos(0) = 1$. For $X(a) = 0$, we need

$$\begin{aligned} k_x a &= m\pi \qquad m = 0, 1, 2... \\ k_x &= \frac{m\pi}{a} \end{aligned}$$

Similarly, for $Y$, (which has the same general solution) we need $Y(0) = 0$ and $Y(b) = 0$. So

$$Y(y) = A' \sin(k_y y)$$

$$k_y = \frac{n\pi}{b} \qquad n = 0, 1, 2...$$

So the solution for $E_z^0$ writing $AA' \equiv E_0$ is

$$\begin{aligned} E_z^0(x, y) &= X(x)Y(y) \\ E_z^0(x, y) &= E_0 \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \end{aligned}$$

Which is the $\text{TM}_{mn}$ mode.

To find $k_g$ we use $k_c^2 = k_0^2 - k_g^2$ and $k_c^2 = k_x^2 + k_y^2$.

$$\begin{aligned} k_g^2 &= \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \\ &= \frac{\omega^2}{c^2} - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \\ k_g &= \sqrt{\frac{\omega^2}{c^2} - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \end{aligned}$$

The cutoff frequency is the same as the TE mode.

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

This means the wave velocity and the group velocity are the same as the TE mode.

$$\begin{aligned} v &= \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}} \\ v_g &= \frac{1}{dk/d\omega} = c\sqrt{1 - (\omega_{mn}/\omega)^2} \end{aligned}$$

We want to find the other components. Since this is TM mode, $B_x = E_y = 0$ so we don't need to find those. To get $E_x$ we use the fact that $\nabla E_z^0 = -\frac{ik_c^2}{k_g} E_t$ and $E_t = E_x\hat{x} + E_y\hat{y} = E_x\hat{x}$.

$$\begin{aligned} E_x &= -\frac{k_g}{ik_c^2}(\nabla E_z^0)_x = \frac{ik_g}{k_c^2}\frac{\partial E_z^0}{\partial x} \\ &= \frac{ik_g}{k_c^2} E_0 \frac{m\pi}{a} \sin(\frac{m\pi x}{a}) \cos(\frac{m\pi x}{a}) \sin(\frac{n\pi x}{b}) \\ &= \frac{ik_g}{k_c^2}\frac{m\pi}{a} \cos(\frac{m\pi x}{a}) E_z \end{aligned}$$

To get $B_y$

$$\begin{aligned} E_t &= -\frac{ck_g}{k_0}(\hat{z}\times B_t) \\ E_x &= -\frac{ck_g}{k_0}(-B_y) \\ &= \frac{ck_g}{k_0}B_y \end{aligned}$$

so

$$B_y = \frac{k_0}{ck_g}E_x$$

Now to find the ratio of the lowest TM cut-off frequency ($TM_{11}$) to the lowest TE cut-off frequency ($TE_{10}$) suppose we have a waveguide with $a > b$. The ratio is

$$\begin{aligned} \frac{\omega_{11}}{\omega_{10}} &= \frac{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}}{\sqrt{\left(\frac{1}{a}\right)^2}} \\ &= \sqrt{1 + a^2/b^2} \end{aligned}$$

**Problem 4.** Show that the $TE_{00}$ mode cannot occur in a rectangular waveguide by showing that $B_z = 0$.

In a TE mode $k$ is given by

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}$$

If we're talking about $TE_{00}$ then $k$ is

$$k = \frac{\omega}{c}$$

Now we look at Maxwell's equations (9.179 in Griffiths) and we apply the fact that $E_z = 0$ (since TE mode) and $k = \omega/c$.

From $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ we have

$$\begin{aligned} \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= i\omega B_z \\ -ikE_y &= i\omega B_x \implies E_y = -cB_x \\ ikE_x &= i\omega B_y \implies E_x = cB_y \end{aligned}$$

From $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ we have

$$\begin{aligned} \frac{\partial B_y}{\partial x} &= \frac{\partial B_x}{\partial y} \\ \frac{\partial B_z}{\partial y} - ikB_y &= -\frac{i\omega}{c^2}E_x = -i\frac{\omega}{c}B_y = -ikB_y \implies \frac{\partial B_z}{\partial y} = 0 \\ ikB_x - \frac{\partial B_z}{\partial x} &= -\frac{i\omega}{c^2}E_y = i\frac{\omega}{c}B_x = ikB_x \implies \frac{\partial B_z}{\partial x} = 0 \end{aligned}$$

This shows that $B_z$ is constant over the $x - y$ plane.

Now, Faraday's law in integral form is

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int \mathbf{B}\cdot d\mathbf{a}$$

If we choose an area transverse to the rectangular waveguide (on the x-y plane) and just inside the metal where $E = 0$, then we get

$$0 = -ab\frac{\partial B_z}{\partial t} = -ab(-i\omega B_z) = i\omega ab B_z \implies B_z = 0$$

If $B_z$ is zero, then it is a TEM mode, but we already know that TEM waves cannot propagate in a waveguide.

**Problem 5.** A hollow rectangular waveguide is to be used to transmit signals at a carrier frequency of 6GHz. Choose its dimensions so that the cutoff frequency of the dominant TE mode is lower than the carrier by 25% and that of the next mode is at least 25% higher than the carrier.

Suppose that $a > b$ and assume we're dealing with the lowest two modes, $f_{10}$ and $f_{01}$ where $f_{10} < f_{01}$ since $a > b$.

$$f_{10} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{c}{2a}$$
$$f_{01} = \frac{c}{2}\sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \frac{c}{2b}$$

Since we want the cutoff of the dominant mode to be 25% less, and that of the next mode to be 25% higher,

$$\begin{aligned} f_{10} &= 6 \times 10^9(0.75) = 4.5 \times 10^9\text{Hz} \\ f_{01} &= 6 \times 10^9(1.25) = 7.5 \times 10^9\text{Hz} \end{aligned}$$

So the required dimensions are

$$a = \frac{c}{2f_{10}} = \frac{3 \times 10^8}{2(4.5 \times 10^9)} = 0.033\text{m}$$

$$b = \frac{c}{2f_{01}} = \frac{3 \times 10^8}{2(7.5 \times 10^9)} = 0.02\text{m}$$