MAT3155
ASSIGNMENT 4

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Problem 1. Suppose S_1 and S_2 are surfaces and $f : S_1 \to S_2$ is a diffeomorphism. Let (σ, U, W_1) be an r.s.p. for S_1, with $\sigma(0,0) = P \in W_1$, and $(f\sigma, U, W_2)$ be an r.s.p for S_2, with $f(P) \in W_2$. Define maps $J_\sigma : \mathbf{R}^2 \to T_P S_1$ and $J_{f\sigma} : \mathbf{R}^2 \to T_{f(P)} S_2$ defined respectively by

$$J_\sigma(v) = J(\sigma) \cdot v \qquad \text{and} \qquad J_{f\sigma}(v) = J(f\sigma) \cdot v$$

Let $T : T_P S_1 \to T_{f(P)} S_2$ denote the tangent map of f at P $T_P f$ defined by $T_P f = \frac{d(f\gamma)}{dt}(0)$ for some curve $\gamma : \mathbf{R} \to S_1$.

a) Show that $T = J_{f\sigma} J_\sigma^{-1}$.

Let γ be a curve on S_1 such that $\gamma = \sigma \circ \alpha$ where $\alpha : \mathbf{R} \to U \subset \mathbf{R}^2$ is another curve in U. Pick α so that $\alpha(0) = \sigma^{-1}(P)$. This means $\gamma(0) = \sigma(\alpha(0)) = \sigma(\sigma^{-1}(P)) = P$.

Let $\dot\gamma(0) = w$. This vector is a tangent vector at P on S_1 so $w \in T_P S_1$. Take this vector and apply the tangent map T to it

$$\begin{aligned} T(w) &= \frac{d(f \circ \gamma)}{dt}(0) = \frac{d(f \circ \sigma \circ \alpha)}{dt}(0) \\ &= J(f\sigma)\Big|_{\alpha(0)} \cdot \dot\alpha(0) \end{aligned}$$

using the chain rule. Note that $J(f\sigma)\big|_{\alpha(0)} = J(f\sigma)\big|_{\sigma^{-1}(P)} = J_{f\sigma}$ since $J(f\sigma)\big|_{\sigma^{-1}(P)} \cdot v$ is a map from $\mathbf{R}^2 \to T_{f(P)} S_2$ for some vector v. This means

$$T(w) = J(f\sigma)\big|_{\alpha(0)} \cdot \dot\alpha(0) = J_{f\sigma}(\dot\alpha(0))$$

This matrix product makes sense becase $J(f\sigma)$ is a 3x2 matrix (since $f\sigma : \mathbf{R}^2 \to S_2 \subset \mathbf{R}^3$) and $\dot\alpha(0)$ is a 2x1 column vector. The product then gives a 3x1 column vector which is the mapped tangent in $T_{f(P)} S_2$ as expected.

Note that $w = \dot\gamma(0) = (\dot{\overline{\sigma \circ \alpha}}) = J(\sigma)\big|_{\alpha(0)} \cdot \dot\alpha(0)$. Similarly, $J(\sigma)\big|_{\alpha(0)} = J(\sigma)\big|_{\sigma^{-1}(P)} = J_\sigma$ since $J(\sigma)\big|_{\sigma^{-1}(P)} \cdot v$ is a map from $\mathbf{R}^2 \to T_P S_1$ for some vector v. So $w = J(\sigma)\big|_{\sigma^{-1}(P)} \cdot \dot\alpha(0) = J_\sigma(\dot\alpha(0))$. Assuming the inverse J_σ^{-1} exists, whatever it may be, knowing that it isn't matrix multiplication by the inverse matrix of J_σ, we at least know then that $J_\sigma^{-1}(w) = J_\sigma^{-1}(J_\sigma(\dot\alpha(0))) = \dot\alpha(0)$. Substituting this into the previous expression for $T(w)$ we see

$$T(w) = J_{f\sigma}(J_\sigma^{-1}(w)) = (J_{f\sigma} \circ J_\sigma^{-1})(w)$$

Or, compactly, $T = J_{f\sigma} J_\sigma^{-1}$.

b) Prove that f is an isometry iff $Tw_1 \cdot Tw_2 = w_1 \cdot w_2$ for all $w_1, w_2 \in T_P S_1$.

Pick some curve γ on S_1 such that $\gamma = \sigma \circ \alpha$. Similarly, let there be another curve β on S_2 st. $\beta = f \circ \gamma$.

The length of γ between endpoints $\gamma(t_0)$ and $\gamma(t_1)$ is

$$\int_{t_0}^{t_1} ||\dot{\gamma}||dt = \int_{t_0}^{t_1} \sqrt{\dot{\gamma}(t)\cdot\dot{\gamma}(t)}dt$$

The corresponding length of β between the same endpoints is

$$\int_{t_0}^{t_1} \sqrt{\dot{\beta}(t)\cdot\dot{\beta}(t)}dt$$

Since $\beta = f\circ\gamma$, $\dot{\beta}(t) = \frac{d}{dt'}(f\circ\gamma)(t)$ is the tangent map T at $P=\gamma(t)\in S_1$. The integral for the length of β involves constructing the tangent map at the point $\gamma(t)$ for each step of t between t_0 and t_1, mapping the tangent vector $\dot{\gamma}(t)$, and computing the integrand. That is $\dot{\beta}(t) = T(w) = T(\dot{\gamma}(t))$ and

$$\int_{t_0}^{t_1} \sqrt{\dot{\beta}(t)\cdot\dot{\beta}(t)}dt = \int_{t_0}^{t_1} \sqrt{T\dot{\gamma}(t)\cdot T\dot{\gamma}(t)}dt$$

If we assume $Tw_1\cdot Tw_2 = w_1\cdot w_2$ for all $w_1, w_2\in T_PS_1$, then, noting that $\dot{\gamma}(t)\in T_PS_1$, we have that $T\dot{\gamma}\cdot T\dot{\gamma} = \dot{\gamma}\cdot\dot{\gamma}$ which means

$$= \int_{t_0}^{t_1} \sqrt{T\dot{\gamma}(t)\cdot T\dot{\gamma}(t)}dt = \int_{t_0}^{t_1} \sqrt{\dot{\gamma}(t)\cdot\dot{\gamma}(t)}dt$$

implying that the length of a curve γ in S_1 is preserved by f. This means f is an isometry.

Conversely, assume that f is an isometry and that the integrals are the same. We know that $T\dot{\gamma}\cdot T\dot{\gamma} = \dot{\gamma}\cdot\dot{\gamma}$. Setting $w_1 = \dot{\gamma}$ we've got $Tw_1\cdot Tw_1 = w_1\cdot w_1$.

Let $w_2\in T_PS_1$ be any other tangent vector at P, like w_1. There could be a different curve δ whose tangent is w_2, so this means that $Tw_2\cdot Tw_2 = w_2\cdot w_2$ as well. We can add these vectors together to make w_1+w_2. There could be another curve whose tangent vector is w_1+w_2 (indeed that curve is $\gamma+\delta$ since $(\dot{\gamma+\delta}) = \dot{\gamma}+\dot{\delta}$) so we also have $T(w_1+w_2)\cdot T(w_1+w_2) = (w_1+w_2)\cdot(w_1+w_2)$. Now calculate the dot product of w_1 and w_2. Expressing it as a quadratic form we get and using the fact that T is linear:

$$\begin{aligned}
w_1\cdot w_2 &= \frac{1}{2}\Big((w_1+w_2)\cdot(w_1+w_2) - w_1\cdot w_1 - w_2\cdot w_2\Big)\\
w_1\cdot w_2 &= \frac{1}{2}\Big(T(w_1+w_2)\cdot T(w_1+w_2) - Tw_1\cdot Tw_1 - Tw_2\cdot Tw_2\Big)\\
w_1\cdot w_2 &= \frac{1}{2}\Big((Tw_1+Tw_2)\cdot(Tw_1+Tw_2) - Tw_1\cdot Tw_1 - Tw_2\cdot Tw_2\Big)\\
w_1\cdot w_2 &= Tw_1\cdot Tw_2
\end{aligned}$$

Which was what we wanted.

c) Prove that f is an equiareal map iff $||Tw_1\times Tw_2|| = ||w_1\times w_2||$, for all $w_1, w_2\in T_PS_1$.

Assume that $||Tw_1\times Tw_2|| = ||w_1\times w_2||$, for all $w_1, w_2\in T_PS_1$. Since $\sigma_u(0,0)$ and $\sigma_v(0,0)$ are basis vectors of T_PS_1, they are elements of T_PS_1 and we can set $w_1 = \sigma_u(0,0)$ and $w_2 = \sigma_v(0,0)$. The map T will map basis vectors to basis vectors, and the corresponding basis vectors of $T_{f(P)}S_2$ are $(f\sigma)_u(0,0)$ and $(f\sigma)_v(0,0)$, that is, $Tw_1 = T\sigma_u(0,0) = (f\sigma)_u(0,0)$ and $Tw_2 = T\sigma_v(0,0) = (f\sigma)_v(0,0)$.

Letting the point $\sigma^{-1}(P)$ vary over some region $R\subseteq U$ we can construct the tangent space at each point P of $\sigma(R)$ and, using $\sigma_u(u,v)$ and $\sigma_v(u,v)$ as basis vectors of the tangent space $T_{\sigma(u,v)}S_1$, find the area of the region on S_1 given by

$$\mathcal{A}_\sigma(R) = \int_R ||\sigma_u\times\sigma_v||dudv$$

Since the tangent map is defined on each point of $\sigma(R)$, setting $w_1 = \sigma_u(u,v)$ and $w_2 = \sigma_v(u,v)$, we have

$$\begin{aligned}\mathcal{A}_\sigma(R) &= \int_R ||\sigma_u \times \sigma_v|| dudv \\ &= \int_R ||T\sigma_u \times T\sigma_v|| dudv \\ &= \int_R ||(f\sigma)_u \times (f\sigma)_v|| dudv = \mathcal{A}_{f\sigma}(R)\end{aligned}$$

since $(f\sigma)_u(u,v)$ and $(f\sigma)_v(u,v)$ are basis vectors of $T_{f(\sigma(R))}S_2$. Since f preserved the area of the region, it is equiareal.

There's another way to do this by doing

$$w_1 \times w_2 = \lambda_1\lambda_2\sigma_u \times \sigma_v....$$

Problem 2. *(Lambert azimuthal equal-area projection: 1772)* Let $H = \{(x,0,z)|x \leq 0\}$, S the plane with cartesian equation $z = -1$, and define a map $\ell : \mathbf{S}^2\backslash H \to S$ by

$$\ell(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi) = \sqrt{2+2\cos\varphi}\Big(\cos\theta, \sin\theta, -\frac{1}{\sqrt{2+2\cos\varphi}}\Big)$$

prove that the determinant of the first fundamental forms of $\mathbf{S}^2\backslash H$ and $\mathbf{S}^2$ are equal.

The surface patch of $\mathbf{S}^2\backslash H$ is

$$\sigma(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$$

We want to compute the first fundamental forms of both surfaces using the surface patches σ for $\mathbf{S}^2\backslash H$ and $\ell\sigma$ for S.

$$\begin{aligned}\sigma_\theta &= (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0) \\ \sigma_\varphi &= (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi) \\ (\ell\sigma)_\theta &= \sqrt{2+2\cos\varphi}(-\sin\theta, \cos\theta, 0) \\ (\ell\sigma)_\varphi &= \Big(\frac{-\cos\theta\sin\varphi}{\sqrt{2+2\cos\varphi}}, \frac{-\sin\theta\sin\varphi}{\sqrt{2+2\cos\varphi}}, 0\Big) \\ &= \frac{-\sin\varphi}{\sqrt{2+2\cos\varphi}}(\cos\theta, \sin\theta, 0)\end{aligned}$$

The first fundamental form of $\mathbf{S}^2\backslash H$ is

$$\begin{aligned}E_1 &= ||\sigma_\theta|| = \sin^2\theta\sin^2\varphi + \cos^2\theta\sin^2\varphi = \sin^2\varphi \\ F_1 &= \sigma_\theta \cdot \sigma_\varphi = -\sin\theta\cos\theta\sin\varphi\cos\varphi + \sin\theta\cos\theta\sin\varphi\cos\varphi + 0 = 0 \\ G_1 &= ||\sigma_\varphi|| = \cos^2\theta\cos^2\varphi + \sin^2\theta\cos^2\varphi + \sin^2\varphi = 1\end{aligned}$$

and it's determinant is

$$E_1G_1 - F_1^2 = \sin^2\varphi$$

The first fundamental form of S is

$$\begin{aligned}E_2 &= ||(\ell\sigma)_\theta|| = 2 + 2\cos\varphi \\ F_2 &= (\ell\sigma)_\theta \cdot (\ell\sigma)_\varphi = \sin\theta\cos\theta\sin\varphi - \sin\theta\cos\theta\sin\varphi = 0 \\ G_2 &= \frac{\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta}{2+2\cos\varphi} = \frac{\sin^2\varphi}{2+2\cos\varphi}\end{aligned}$$

and it's determinant is

$$E_2G_2 - F_2^2 = \sin^2\varphi = E_1G_1 - F_1^2$$

If ℓ were a diffeo this would imply that it is equiareal.

Problem 3. A regular curve γ on a surface S is a *line of curvature* if its tangent is everywhere a principal vector. If N_σ is the unit normal of a surface patch σ, show that γ is a line of curvate iff $\dot{N_\sigma} = -\lambda\dot{\gamma}$, for some scalar function λ with the same domain as γ, and that in this case the corresponding principal curvature is λ.

Let $\gamma = \sigma \circ \alpha$ for some curve α in $U \subseteq \mathbf{R}^2$. Let $N_\sigma = N \circ \sigma$.

If γ is a line of curvature then $\dot{\gamma}$ is in the principal direction. Since $\dot{\gamma} = J(\sigma)\cdot\dot{\alpha}$, $\dot{\alpha}$ is an eigenvector of $(\mathcal{F}_{II} - \lambda\mathcal{F}_I)\dot{\alpha} = 0$ where $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are the first and second fundamental forms, and we see

$$\mathcal{F}_{II}\cdot\dot{\alpha} = \lambda\mathcal{F}_I\cdot\dot{\alpha}$$

Now, since $\gamma = \sigma \circ \alpha$ we can form the composition $N_\sigma = N \circ \sigma \circ \alpha$ which is a function of t and represents the unit normal along the curve γ. Taking the derivative:

$$\begin{aligned} \dot{N_\sigma} &= J(N\circ\sigma)\dot{\alpha} \\ &= -J(\sigma)\mathcal{F}_I^{-1}\mathcal{F}_{II}\dot{\alpha} \end{aligned}$$

Substituting $\mathcal{F}_{II}\cdot\dot{\alpha} = \lambda\mathcal{F}_I\cdot\dot{\alpha}$ we get

$$\begin{aligned} \dot{N_\sigma} &= -J(\sigma)\mathcal{F}_I^{-1}\lambda\mathcal{F}_I\cdot\dot{\alpha} \\ &= -\lambda J(\sigma)\mathcal{F}_I^{-1}\mathcal{F}_I\cdot\dot{\alpha} \\ &= -\lambda J(\sigma)\cdot\dot{\alpha} \\ \dot{N_\sigma} &= -\lambda\dot{\gamma} \end{aligned}$$

Conversely, assuming simply that $\dot{N_\sigma} = -\lambda\dot{\gamma}$, we have

$$\begin{aligned} \dot{N_\sigma} &= -\lambda\dot{\gamma} \\ -J(\sigma)\mathcal{F}_I^{-1}\mathcal{F}_{II}\dot{\alpha} &= -\lambda J(\sigma)\cdot\dot{\alpha} \end{aligned}$$

Multiplication by $J(\sigma)$ is injective so we can drop it from both sides:

$$\begin{aligned} \mathcal{F}_I^{-1}\mathcal{F}_{II}\dot{\alpha} &= \lambda\dot{\alpha} \\ \mathcal{F}_{II}\dot{\alpha} &= \lambda\mathcal{F}_I\dot{\alpha} \\ \implies (\mathcal{F}_{II} - \lambda\mathcal{F}_I)\dot{\alpha} &= 0 \end{aligned}$$

This means that $J(\sigma)\dot{\alpha} = \dot{\gamma}$ is a principal direction at P.

Problem 4. Suppose S is smooth oriented surface and that $N : S \to \mathbf{S}^2$ is the smooth unit normal.

a) Show that $T_PS = T_{N(P)}\mathbf{S}^2$

For a smooth oriented surface we have a regular surface patch σ. For every point $P \in S$ there is a smooth unit normal N_σ that is perpendicular to the tangent plane at P on S, T_PS. The map N takes the unit normal N_σ and places it's tail at the origin of the sphere $\mathbf{S}^2$. Thus, $N(P) \in \mathbf{S}^2$ is parralel to $N_\sigma \in S$. For every point $Q = N(P) \in \mathbf{S}^2$, the tangent plane goes through the origin and the normal is Q, which is parralel to N_σ. Since these are both unit normals these are infact the same vectors: $N_\sigma = N(P) = Q$. Since a plane is uniquely determined by it's normal, and the tangent plane at Q on $\mathbf{S}^2$ ($T_Q\mathbf{S^2}$) shares the same normal as the tangent plane at P on S (T_PS), they are the same plane.

$$T_PS = T_Q\mathbf{S}^2 = T_{N(P)}\mathbf{S}^2$$

This means that the tangent map for N is a map from $T_PS \to T_PS$.

b) By (a), the tangent map $G = T_PN$ for N satisfies $G : T_PS \to T_PS$. Show that $\det G$ is the Gaussian curvature at P.

$$\begin{aligned}\frac{d(N\circ\gamma)}{dt}(0) &= \frac{d(N\circ\sigma\circ\alpha)}{dt}(0)\\ &= J(N\circ\sigma)\dot{\alpha}\\ &= -J(\sigma)\mathcal{F}_I^{-1}\mathcal{F}_{II}\dot{\alpha}\end{aligned}$$

The G map is the matrix $-\mathcal{F}_I^{-1}\mathcal{F}_{II}$, and the determinant is

$$\det G = \det -\mathcal{F}_I^{-1}\mathcal{F}_{II} = (-1)^2 \det \mathcal{F}_I^{-1}\mathcal{F}_{II} = \det \mathcal{F}_I^{-1}\mathcal{F}_{II}$$

which is the Gaussian curvature at P.

This would be more convincing if I could write the last line as

$$-J(\sigma)\mathcal{F}_I^{-1}\mathcal{F}_{II}\dot{\alpha} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}J(\sigma)\dot{\alpha} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}\dot{\gamma} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}w$$

where w is the argument of the map G.