PHY4311 LAST ASSIGNMENT

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APRIL 2ND 2015

Problem 1. Show that $\varepsilon(r) = \frac{A}{r}\exp(ikr)$ is a solution of the Helmholtz Equation.

The Helmholtz equation is $\nabla^2\varepsilon(r) + k^2\varepsilon(r) = 0$ where $k^2 = \omega^2/c^2$. We need to use the Laplacian in spherical coordinates (r, θ, ϕ). Since our function has no θ or ϕ dependance, the θ and ϕ derivatives are zero and we neglect them.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)$$

$$\begin{aligned}
\nabla^2\varepsilon(r) &= \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\varepsilon(r)\Big) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\Big(-\frac{A}{r^2}\exp(ikr) + \frac{A}{r}ik\exp(ikr)\Big)\Big) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\Big(-A\exp(ikr) + ikrA\exp(ikr)\Big) \\
&= \frac{1}{r^2}\Big(-Aik\exp(ikr) + Aik\exp(ikr) - k^2rA\exp(ikr)\Big) \\
&= -k^2\frac{A}{r}\exp(ikr) \\
&= -k^2\varepsilon(r)
\end{aligned}$$

This means that

$$\nabla^2\varepsilon(r) + k^2\varepsilon(r) = -k^2\varepsilon(r) + k^2\varepsilon(r) = 0$$

as required.

Problem 2. A $\lambda = 500$nm Gaussian beam emerges from a laser with a beam waist of $w_0 = 1$mm and with a plane wave front.

(a) How far does the beam wave propagate before its area increases by a factor of 2?

The radius of the spot size at z is $w(z) = w_0\sqrt{1 + z^2/z_0^2}$. The area of the spot size at the waist is πw_0^2 and the area at z is $\pi w_0^2(1 + z^2/z_0^2)$. We want z such that

$$\begin{aligned}
2\pi w_0^2 &= \pi w_0^2(1 + z^2/z_0^2) \\
2 &= 1 + \frac{z^2}{z_0^2} \\
z^2 &= z_0^2 \\
z &= z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi(1\times 10^{-3})^2}{500\times 10^{-9}} = 6.28\text{m}
\end{aligned}$$

(b) How does this distance relate to z_0? They're the same!

(c) If there's a plane 1km away how big is the spot size on the plane when you shine a laser pointer at it?

$z = 1000\text{m}$

$$\begin{aligned}
\pi w^2 &= \pi w_0^2(1 + z^2/z_0^2) \\
&= \pi(10^{-6})(1 + 10^6/(6.28)^2) \\
&= 0.08\text{m}^2
\end{aligned}$$

(d) Same question, with $w_0 = 3\text{mm}$:

The Rayleigh range changes to

$$z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi(9)(10^{-6})}{500 \times 10^{-9}} = 56.5\text{m}$$

The area is now

$$\begin{aligned}
\pi w^2 &= \pi w_0^2(1 + z^2/z_0^2) \\
&= \pi(9)(10^{-6})(1 + 10^6/(56.5)^2) \\
&= 0.009\text{m}^2
\end{aligned}$$

Problem 3. Gouy Phase

(a) Show that a beam experiences a phase shift of 180° as it passes through it's focus at $z = 0$.

Suppose the beam starts at $z_i = -\infty$ and passes through $z = 0$ on its way to $z_f = \infty$. The phase is given by $\phi(z) = \tan^{-1}(z/z_0)$. The difference is

$$\begin{aligned}
\phi(z_f) - \phi(z_i) &= \tan^{-1}(\infty) - \tan^{-1}(-\infty) \\
&= \frac{\pi}{2} - (-\frac{\pi}{2}) \\
&= \pi = 180^\circ
\end{aligned}$$

(b) How much Gouy phase is accumulated by propagating from the position where the beam has its minimum waist to where its waist is increased by $\sqrt{2}$?

The position where the beam has its minimum waist is $z_i = 0$. The position where it's waist is increased by $\sqrt{2}$ is $z_f = z_0$ since $w(z_0) = w_0\sqrt{1 + z_0^2/z_0^2} = w_0\sqrt{2}$. So the phase difference is

$$\begin{aligned}
\phi(z_f) - \phi(z_i) &= \tan^{-1}(1) - \tan^{-1}(0) \\
&= \frac{\pi}{4} - 0 = \frac{\pi}{4} = 90^\circ
\end{aligned}$$

Problem 4. If we double the frequency of a Gaussian pulse (beam?) the waist gets smaller. Assume a plane wavefront.

(a) By what ratio does the beam waist decrease?

Since we double the frequency, we halve the wavelength. Let w_{0i} denote the waist before it's doubled and w_{0f} denote the waist after it's doubled.

$$z_0 = \frac{\pi w_0^2}{\lambda} \implies w_0^2 = \frac{\lambda z_0}{\pi}$$

$$\begin{aligned}
\frac{w_{0i}^2}{w_{0f}^2} &= \frac{\lambda z_{0i}}{\pi}\frac{2\pi}{\lambda z_{0f}} = 2\frac{z_{0i}}{z_{0f}} \\
\frac{w_{0i}}{w_{0f}} &= \sqrt{2}\sqrt{\frac{z_{0i}}{z_{0f}}}
\end{aligned}$$

(b) Show that the Rayleigh range of both beams are the same

I don't know how to show this. I know that if $w_{0i} = \sqrt{2} w_{0f}$ then

$$z_{0i} = \frac{\pi w_{0i}^2}{\lambda} = \frac{2\pi w_{0f}^2}{\lambda} = \frac{\pi w_{0f}^2}{\lambda/2} = z_{0f}$$

I keep going in circles

Problem 5. A beam with wavelength of $\lambda = 1$micron emerges from a laser with $R = \infty$ and with $w_0 = 0.1$cm. The beam propagates to $z = 3$m before encountering a lens with $f = 50$cm.

(a) What is the Rayleigh range of the beam leaving the laser?

$$z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi (1 \times 10^{-3})^2}{1 \times 10^{-6}} = \pi \text{m}$$

(b) What is $w(z)$ at the lens?

$$\begin{aligned} w(3) &= w_0 \sqrt{1 + 9/z_0^2} \\ &= 10^{-3} \sqrt{1 + 9/\pi^2} \\ &= 1.4 \times 10^{-3} \text{m} \end{aligned}$$

(c) After passing through the lens what is the new beam waist?

$$\begin{aligned} w_0' &= \frac{\lambda f}{\pi w_0} \frac{1}{\sqrt{1 + f^2/z_0^2}} \\ &= \frac{10^{-6}(5)(10^{-1})}{\pi (10^{-3})} \frac{1}{\sqrt{1 + (5 \times 10^{-1})^2/\pi^2}} \\ &= 1.6 \times 10^{-4} \text{m} \end{aligned}$$

Problem 6. Consider a Gaussian beam at its waist. Show that only 86.5% of the beam's power passes through an aperture with radius $r = w_0$.

The intensity of the beam is

$$I(x,y,z) = \frac{\frac{1}{2}c\epsilon_0|A|^2}{1+z^2/z_0^2}e^{-2(x^2+y^2)/w^2(z)}$$

Since we're at the beam waist $z = 0$

$$I(x,y) = \frac{1}{2}c\epsilon_0|A|^2 e^{-2(x^2+y^2)/w_0^2} = I_0 e^{-2(x^2+y^2)/w_0^2}$$

The power through the hole is the integral of the intensity over the area of the aperture

$$P_h = \int I da$$

In cylindrical coordinates, $da = rdrd\phi$ and $r^2 = x^2 + y^2$. Since the radius of the aperture is w_0 we integrate r from 0 to w_0:

$$P_h = \int_0^{w_0} Irdr \int_0^{2\pi} d\phi = 2\pi I_0 \int_0^{w_0} re^{-2r^2/w_0^2}dr$$

Similarly the total beam power is just the integral over the entire area of the plane

$$P_t = 2\pi I_0 \int_0^{\infty} re^{-2r^2/w_0^2}dr$$

The ratio is

$$\frac{P_h}{P_t} = \frac{\int_0^{w_0} re^{-2r^2/w_0^2}dr}{\int_0^{\infty} re^{-2r^2/w_0^2}dr}$$

To solve $\int re^{-\frac{2}{w_0^2}r^2}dr$, substitute $u = r^2$. This means $dr = \frac{1}{2r}du$. So

$$\begin{aligned}\int re^{-\frac{2}{w_0^2}r^2}dr &= \frac{1}{2}\int e^{-\frac{2}{w_0^2}u}du \\ &= \frac{1}{2}\Big(-\frac{w_0^2}{2}e^{-\frac{2}{w_0^2}u}\Big) = -\frac{w_0^2}{4}e^{-\frac{2}{w_0^2}r^2}\end{aligned}$$

Now,

$$\frac{P_h}{P_t} = \frac{e^{-\frac{2}{w_0^2}r^2}\Big|_0^{w_0}}{e^{-\frac{2}{w_0^2}r^2}\Big|_0^{\infty}} = \frac{e^{-2}-e^0}{0-e^0} = 1 - e^{-2} \doteq 0.865 = 86.5\%$$