# PHY3320 ASSIGNMENT 2

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**Problem 1.** Problem 8.1. A long coaxial cable of length $L$ carries current $I$, which flows down the surface of the inner cylinder of radius $a$, and back along the outer cylinder of radius $b$ as shown. Calculate the power (energy per unit time) transported down the cables. Assume the two conductors are held at a potential difference $V$.

Since the Poynting vector $\mathbf{S}$ is the energy per unit time per unit area, the power through an area $d\mathbf{a}$ is

$$P = \int \mathbf{S} \cdot d\mathbf{a}$$

Since we're interested in power down the wire, $d\mathbf{a}$ points down the wire and it's magnitude is the area of the crossection of a cylindrical shell (in between the two conducting cylinders). The shell spans from a radius $s$ to $s + ds$. So

$$\begin{aligned} d\mathbf{a} &= \pi(s+ds)^2 - \pi s^2 = \pi(s^2 + 2sds + ds^2 - s^2) \\ &= \pi(2sds + ds^2) \\ &= 2\pi s ds \hat{\mathbf{z}} \qquad (ds^2 \approx 0) \end{aligned}$$

The Poynting vector is given by $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$. The magnetic field between the cylinders is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

(In this case the z-axis points down the cable). The electric field is given by

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$$

Where $\hat{\mathbf{s}}$ points in the radial direction, and $\lambda$ is the uniform charge per unit length. So

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0}(\frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \times \frac{\mu_0 I}{2\pi s}\hat{\phi}) \\ &= \frac{1}{\mu_0}\frac{\mu_0 \lambda I}{4\pi^2\epsilon_0 s^2}\hat{\mathbf{z}} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\hat{\mathbf{z}} \end{aligned}$$

Now the power is just

$$\begin{aligned} P &= \int \mathbf{S} \cdot d\mathbf{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int \frac{1}{s^2}2\pi s ds \\ &= \frac{\lambda I}{2\pi\epsilon_0}\int_a^b \frac{1}{s}ds \\ &= \frac{\lambda I}{2\pi\epsilon_0}\ln(\frac{b}{a}) \end{aligned}$$

**Problem 2.** Consider two equal point charges $q$, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other. Do the same for charges that are opposite in sign.

Place one charge $q$ on the z-axis at $z = a$ and the other at $z = -a$. The force is given by

$$\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$$

Since the charges are on the z-axis the force only has a z-component. So we only need to find

$$\mathbf{F}_z = \oint_S (\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z$$

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = T_{xz} d\mathbf{a}_x + T_{yz} d\mathbf{a}_y + T_{zz} d\mathbf{a}_z$$

Say we want to find the force on the upper charge. The force comes from the second charge, which is below the plane. The surface we consider must face the source of the force, so the surface element should point downwards. So we put a negative sign in front of $d\mathbf{a}$:

$$d\mathbf{a} = -r dr d\phi \hat{\mathbf{z}}$$

$$\begin{aligned} (\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z &= T_{zz} d\mathbf{a}_z \\ &= -\epsilon_0 \Big( E_z E_z - \frac{1}{2} E^2 \Big) r dr d\phi \end{aligned}$$

The electric field at a point on the plane is given by a superposition of the fields from each charge. Because any non-radial components of the field from one charge are cancelled out by the other charge, only the radial component of the field from each charge contributes. So the total electric field on the plane is radial.

Let $\mathbf{E}_1$ be the field from the top charge and $\mathbf{E}_2$ be the field from the bottom charge. The radial component of each field is $\mathbf{E}_r = E \cos\theta \hat{\mathbf{r}}$

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{1r} + \mathbf{E}_{2r} \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \cos\theta \hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \cos\theta \hat{\mathbf{r}} = \frac{1}{2\pi\epsilon_0} \frac{q}{d^2} \cos\theta \hat{\mathbf{r}} \end{aligned}$$

The distance from one of the charges to a point on the plane is

$$d = \sqrt{r^2 + a^2}$$

The angle between $d$ and $r$ is

$$\cos\theta = \frac{r}{d}$$

So the field is

$$\begin{aligned} \mathbf{E} &= \frac{1}{2\pi\epsilon_0} \frac{q}{d^2} \cos\theta \hat{\mathbf{r}} \\ &= \frac{1}{2\pi\epsilon_0} \frac{rq}{d(r^2 + a^2)} \hat{\mathbf{r}} \\ &= \frac{1}{2\pi\epsilon_0} \frac{rq}{(r^2 + a^2)^{3/2}} \hat{\mathbf{r}} \end{aligned}$$

Which means $E_z = 0$.

$$\begin{aligned} (\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z &= -\epsilon_0 \Big( -\frac{1}{2} E^2 \Big) r dr d\phi \\ &= \frac{\epsilon_0}{2} E^2 r dr d\phi \\ &= \frac{\epsilon_0}{2} \Big( \frac{1}{2\pi\epsilon_0} \Big)^2 \frac{r^3 q^2}{(r^2 + a^2)^3} dr d\phi \end{aligned}$$

$$
\begin{aligned}
\mathbf{F}_z &= \oint_S (\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z \\
&= \frac{q^2}{8\pi^2\epsilon_0}\int_0^{2\pi} d\phi \int_0^\infty \frac{r^3}{(r^2+a^2)^3} dr \\
&= \frac{q^2}{4\pi\epsilon_0}\int_0^\infty \frac{r^3}{(r^2+a^2)^3} dr \\
&= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a^2+2r^2}{4(a^2+r^2)^2}\right]_0^\infty \\
&= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{r^2}{r^4}+\frac{a^2}{4(a^2)^2}\right] \\
&= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{\infty^2}+\frac{1}{4a^2}\right] \\
&= \frac{q^2}{4\pi\epsilon_0}\frac{1}{4a^2} = \frac{q^2}{16\pi\epsilon_0 a^2}
\end{aligned}
$$

For a charge of opposite sign we expect the force to be equal and opposite to what we found. The field at a point in the plane now has only a z-component. The z-component of the field is given by $\mathbf{E}_z = E\sin\theta\hat{\mathbf{z}}$.

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_{1z}\hat{\mathbf{z}} + \mathbf{E}_{2z}\hat{\mathbf{z}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\sin\theta\hat{\mathbf{z}} - \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\sin\theta\hat{\mathbf{z}} = -\frac{1}{2\pi\epsilon_0}\frac{q}{d^2}\sin\theta\hat{\mathbf{z}}
\end{aligned}
$$

With

$$\sin\theta = \frac{a}{d}$$

$$
\begin{aligned}
E_z &= -\frac{1}{2\pi\epsilon_0}\frac{q}{d^2}\sin\theta \\
&= -\frac{1}{2\pi\epsilon_0}\frac{qa}{(r^2+a^2)^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z &= -\epsilon_0\Big(E_zE_z - \frac{1}{2}E^2\Big) r dr d\phi \\
&= -\epsilon_0\Big(\frac{1}{2}E_z^2\Big) r dr d\phi \\
&= -\frac{\epsilon_0}{2}\Big(\frac{qa}{2\pi\epsilon_0}\Big)^2 \frac{r}{(r^2+a^2)^3} dr d\phi
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_z &= -\frac{\epsilon_0}{2}\Big(\frac{qa}{2\pi\epsilon_0}\Big)^2 \int_0^{2\pi} d\phi \int_0^\infty \frac{r}{(r^2+a^2)^3} dr \\
&= -\frac{q^2a^2}{4\pi\epsilon_0}\int_0^\infty \frac{r}{(r^2+a^2)^3} dr \\
&= -\frac{q^2a^2}{4\pi\epsilon_0}\frac{1}{4a^4} = -\frac{q^2}{16\pi\epsilon_0 a^2}
\end{aligned}
$$

**Problem 3.** A fat wire of radius $a$ carries a constant $I$, uniformly distributed over its cross section. A narrow gap in the wire, of width $w << a$, forms a parallel plate capacitor.

(a) Find the electric and magnetic field in the gap, as functions of distance s from the axis and time $t$. (assume charge is 0 at $t = 0$).

Place the z-axis so that it is pointing along the wire in the direction of flowing current. The gap forms a capacitor so the electric field does not depend on $s$. It is given by

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{z}}$$

Where $\sigma$ is the surface charge per unit area over the crossection. This means

$$\sigma = \frac{q}{\pi a^2}$$

But since $q$ depends on the time, $q(t) = It$. So

$$\sigma = \frac{It}{\pi a^2}$$

Therefore the field is

$$\mathbf{E} = \frac{It}{\pi a^2 \epsilon_0}\hat{\mathbf{z}}$$

There is no current density, but there is a displacement current density in the gap, which by Ampere's law means there is a $\mathbf{B}$ field.

$$\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{I}{\pi a^2}$$

Drawing a circle of radius $s$ from the axis in between the gap, and integrating over it, we have

$$\begin{aligned}\oint \mathbf{B} \cdot d\mathbf{l} &= B 2\pi s = \mu_0 I_{enc} = \mu_0 I \Big(\frac{\pi s^2}{\pi a^2}\Big) \\ &= \mu_0 I \frac{s^2}{a^2}\end{aligned}$$

Now

$$\begin{aligned}B &= \mu_0 I \frac{s^2}{a^2} \frac{1}{2\pi s} \\ &= \frac{\mu_0 I s}{2\pi a^2}\end{aligned}$$

The $\mathbf{B}$ field goes in circles around the wire so

$$\mathbf{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}$$

The $\mathbf{B}$ field doesn't seem to depend on time.

(b) Find the energy density $u_{em}$ and Poynting vector $\mathbf{S}$ in the gap. Check if the continuity equation for energy is satisfied or not.

The energy density $u_{em}$ is given by

$$\begin{aligned}
u_{em} &= \frac{1}{2}\Big(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\Big) \\
&= \frac{1}{2}\Big(\epsilon_0 \frac{I^2 t^2}{\pi^2 a^4 \epsilon_0^2} + \frac{1}{\mu_0}\frac{\mu_0^2 I^2 s^2}{4\pi^2 a^4}\Big) \\
&= \frac{1}{2}\Big(\frac{I^2 t^2}{\pi^2 a^4 \epsilon_0} + \frac{\mu_0 I^2 s^2}{4\pi^2 a^4}\Big) \\
&= \frac{I^2}{2\pi^2 a^4}\Big(\frac{t^2}{\epsilon_0} + \frac{\mu_0 s^2}{4}\Big)
\end{aligned}$$

Now the Poynting vector is given by

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \\
&= \frac{1}{\mu_0}\Big(\frac{It}{\pi a^2 \epsilon_0}\hat{\mathbf{z}}\Big) \times \Big(\frac{\mu_0 I s}{2\pi a^2}\hat{\phi}\Big) \\
&= -\frac{I^2 t s}{2\pi^2 a^4 \epsilon_0}\hat{\mathbf{s}}
\end{aligned}$$

The Poynting vector points inward radially! Energy flows inward to the center.

We want to check that $\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\nabla \cdot \mathbf{S}$ is satisfied, with $u_{mech} = 0$.

$$\frac{\partial u_{em}}{\partial t} = \frac{2tI^2}{2\pi^2 a^4 \epsilon_0} = \frac{tI^2}{\pi^2 a^4 \epsilon_0}$$

$$\begin{aligned}
\nabla \cdot \mathbf{S} &= \frac{\partial \mathbf{S}}{\partial s} + \frac{\partial \mathbf{S}}{\partial \phi} + \frac{\partial \mathbf{S}}{\partial z} = \frac{\partial \mathbf{S}}{\partial s} \\
&= -\frac{tI^2}{2\pi^2 a^4 \epsilon_0}
\end{aligned}$$

We see that

$$\frac{\partial u_{em}}{\partial t} = -\frac{1}{2}\nabla \cdot \mathbf{S}$$

Where did that 1/2 come from?

(c) Determine the total energy in the gap as a function of time. Calculate the total power flowing into the gap by integrating the Poynting vector over the appropriate surface.

The total power is

$$P = -\frac{d}{dt}\int_V u_{em} d\tau - \oint_S \mathbf{S} \cdot d\mathbf{a}$$

The volume of the gap is $\tau = \pi s^2 w$. So $d\tau = 2\pi s w ds$.

$$\begin{aligned}
\int_V u_{em} d\tau &= \int_0^a u_{em} 2\pi s w ds \\
&= 2\pi w \int_0^a u_{em} s ds \\
&= 2\pi w \int_0^a \frac{I^2}{2\pi^2 a^4}\Big(\frac{t^2}{\epsilon_0} + \frac{\mu_0 s^2}{4}\Big) s ds \\
&= 2\pi w \frac{I^2}{2\pi^2 a^4} \int_0^a (\frac{st^2}{\epsilon_0} + \frac{\mu_0 s^3}{4}) ds \\
&= \frac{wI^2}{\pi a^4}\Big(\frac{t^2}{\epsilon_0}[\frac{1}{2}s^2]_0^a + \frac{\mu_0}{4}[\frac{1}{4}s^4]_0^a\Big) \\
&= \frac{wI^2}{\pi a^4}\Big(\frac{a^2 t^2}{2\epsilon_0} + \frac{\mu_0 a^4}{16}\Big) \\
&= \frac{wI^2}{2\pi a^2 \epsilon_0}\Big(t^2 + \frac{\mu_0 \epsilon_0 a^2}{8}\Big) = U_{em}
\end{aligned}$$

Now for the time derivative

$$\frac{dU_{em}}{dt} = \frac{2twI^2}{2\pi a^2 \epsilon_0} = \frac{twI^2}{\pi a^2 \epsilon_0}$$

Since the Poynting vector is pointing radially we have to integrate over the curved surface of a cylinder. Let the cylinder have a radius $b$.

$$d\mathbf{a} = 2\pi b w \hat{\mathbf{s}}$$

$$\begin{aligned}
P_{in} &= \oint_S \mathbf{S} \cdot d\mathbf{a} = -\frac{I^2 ts}{2\pi^2 a^4 \epsilon_0}\hat{\mathbf{s}} \cdot 2\pi b w \hat{\mathbf{s}} \\
&= -\frac{2\pi b w I^2 ts}{2\pi^2 a^4 \epsilon_0} \\
&= -\frac{bwI^2 ts}{\pi a^4 \epsilon_0}
\end{aligned}$$

So the total power is

$$\begin{aligned}
P &= -\frac{dU_{em}}{dt} - P_{in} \\
&= -\frac{twI^2}{\pi a^2 \epsilon_0} + \frac{bwI^2 ts}{\pi a^4 \epsilon_0} \\
&= -\frac{twI^2}{\pi a^2 \epsilon_0}(1 - bs)
\end{aligned}$$

**Problem 4.** Problem 8.12 P. 362 Magnetic Monopole Page 328

Place the charge $q_e$ at the origin and the magnetic monopole $q_m$ a distance $d$ away on the z-axis. A vector $\mathbf{r}$ points from the origin to an arbitrary observation point. A vector $\mathbf{r}'$ points from the origin (where $q_e$ is) to the monopole $q_m$ with a length $d$ along the z-axis. The angle between $\mathbf{r}$ and $\mathbf{r}'$ is $\theta$. The vector $\mathbf{r} - \mathbf{r}'$ points from the monopole to the observation point.

The density of angular momentum is

$$\ell_{\mathbf{em}} = \epsilon_0(\mathbf{r} \times (\mathbf{E} \times \mathbf{B}))$$

$$
\begin{aligned}
\mathbf{E}\times\mathbf{B} &= \frac{1}{4\pi\epsilon_0}\frac{q_e}{r^2}\hat{\mathbf{r}}\times\frac{\mu_0}{4\pi}\frac{q_m}{|\mathbf{r}-\mathbf{r}'|^2}(\mathbf{r}\hat{-}\mathbf{r}') \\
&= \frac{\mu_0}{(4\pi)^2\epsilon_0}\frac{q_e q_m}{r^3|\mathbf{r}-\mathbf{r}'|^3}[\mathbf{r}\times(\mathbf{r}-\mathbf{r}')]
\end{aligned}
$$

Note that $\mathbf{r}\times(\mathbf{r}-\mathbf{r}') = \mathbf{r}\times\mathbf{r} - \mathbf{r}\times\mathbf{r}' = -\mathbf{r}\times\mathbf{r}'$ so

$$
\mathbf{E}\times\mathbf{B} = -\frac{\mu_0}{(4\pi)^2\epsilon_0}\frac{q_e q_m}{r^3|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}\times\mathbf{r}')
$$

$$
\begin{aligned}
\ell_{\mathbf{em}} &= \epsilon_0(\mathbf{r}\times(\mathbf{E}\times\mathbf{B})) \\
&= -\frac{\mu_0\epsilon_0}{(4\pi)^2\epsilon_0}\frac{q_e q_m}{r^3|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}\times(\mathbf{r}\times\mathbf{r}'))
\end{aligned}
$$

Use the identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$
\begin{aligned}
\mathbf{r}\times(\mathbf{r}\times\mathbf{r}') &= \mathbf{r}(\mathbf{r}\cdot\mathbf{r}') - \mathbf{r}'(\mathbf{r}\cdot\mathbf{r}) \\
&= \mathbf{r}(rd\cos\theta) - \mathbf{r}'(r^2) = rd\cos\theta\mathbf{r} - r^2\mathbf{r}'
\end{aligned}
$$

So now we have

$$
\begin{aligned}
\ell_{\mathbf{em}} &= -\frac{\mu_0}{(4\pi)^2}\frac{q_e q_m}{r^3|\mathbf{r}-\mathbf{r}'|^3}[rd\cos\theta\mathbf{r} - r^2\mathbf{r}'] \\
&= \frac{\mu_0 q_e q_m}{(4\pi)^2}\frac{[r^2\mathbf{r}' - rd\cos\theta\mathbf{r}]}{r^3|\mathbf{r}-\mathbf{r}'|^3}
\end{aligned}
$$

The distance between the monopole and the observation point is found by the law of cosines

$$
|\mathbf{r}-\mathbf{r}'| = (r^2 + r'^2 - 2|\mathbf{r}||\mathbf{r}'|\cos\theta)^{\frac{1}{2}} = (r^2 + d^2 - 2rd\cos\theta)^{\frac{1}{2}}
$$

Plugging this back into $\ell_{\mathbf{em}}$:

$$
\begin{aligned}
\ell_{\mathbf{em}} &= \frac{\mu_0 q_e q_m}{(4\pi)^2}\frac{[r^2\mathbf{r}' - rd\cos\theta\mathbf{r}]}{r^3(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} \\
&= \frac{\mu_0 q_e q_m}{(4\pi)^2}\frac{[r^2 d\hat{\mathbf{z}} - rd\cos\theta(r\hat{\mathbf{r}})]}{r^3(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} \\
&= \frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{[r^2\hat{\mathbf{z}} - r^2\cos\theta\hat{\mathbf{r}}]}{r^3(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} \\
&= \frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{r^2[\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{r}}]}{r^3(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} \\
&= \frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{[\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{r}}]}{r(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}}
\end{aligned}
$$

The angular momentum density is a vector. So to get the total angular momentum, we add up all the density vectors over every point in space. We do this infinite addition through an integral. Since ultimately we're just adding up vectors, we can solve for the total angular momentum componentwise. First lets work out the components from the vector part of the density vector:

$$
\begin{aligned}
\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{r}} &= (1)\hat{\mathbf{z}} - \cos\theta(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}) \\
&= -\cos\theta\sin\theta\cos\phi\hat{\mathbf{x}} - \cos\theta\sin\theta\sin\phi\hat{\mathbf{y}} + (1-\cos^2\theta)\hat{\mathbf{z}}
\end{aligned}
$$

$$\begin{aligned}
\mathbf{L}_x &= \int_V \ell_{\mathbf{em}x} dV \\
&= \frac{\mu_0 q_e q_m d}{(4\pi)^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{-r^2 \cos\theta \sin\theta \cos\phi}{r(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} \sin\theta dr d\theta d\phi \\
&= 0
\end{aligned}$$

Because $\int_0^{2\pi} \cos\phi d\phi = 0$. Similarly, we have, since $\int_0^{2\pi} \sin\phi d\phi = 0$,

$$\mathbf{L}_y = 0$$

The last component $\mathbf{L}_z$ has no $\phi$ dependance so it's not immediately 0:

$$\begin{aligned}
\mathbf{L}_z &= \frac{\mu_0 q_e q_m d}{(4\pi)^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{r(1-\cos^2\theta)\sin\theta}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} dr d\theta d\phi \\
&= \frac{\mu_0 q_e q_m d}{(4\pi)^2} \int_0^{2\pi} d\phi \int_0^{\pi} \int_0^{\infty} \frac{r(1-\cos^2\theta)\sin\theta}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} dr d\theta
\end{aligned}$$

Let $u = \cos\theta$. So $d\theta = \frac{-1}{\sin\theta} du$. When $\theta = 0$, $u = \cos(0) = 1$. When $\theta = \pi$, $u = \cos(\pi) = -1$.

$$\begin{aligned}
\mathbf{L}_z &= \frac{\mu_0 q_e q_m d}{(4\pi)^2} (2\pi) \int_1^{-1} \int_0^{\infty} -\frac{r(1-u^2)}{(r^2+d^2-2rdu)^{\frac{3}{2}}} dr du \\
&= \frac{\mu_0 q_e q_m d}{8\pi} \int_{-1}^{1} \int_0^{\infty} \frac{r(1-u^2)}{(r^2+d^2-2rdu)^{\frac{3}{2}}} dr du \\
&= \frac{\mu_0 q_e q_m d}{8\pi} \int_{-1}^{1} (1-u^2) du \int_0^{\infty} \frac{r}{(r^2+d^2-2rdu)^{\frac{3}{2}}} dr
\end{aligned}$$

The $r$-integral is of the form

$$\int_0^{\infty} \frac{r}{(r^2+d^2-2rdu)^{\frac{3}{2}}} dr = \int_0^{\infty} \frac{r}{(r^2+(-2du)r+d^2)^{\frac{3}{2}}} dr$$

Looking it up...

$$\int \frac{x}{(ax^2+bx+c)^{3/2}} dx = \frac{2bx+4c}{(b^2-4ac)\sqrt{x(ax+b)+c}}$$

So

$$\begin{aligned}
\int_0^{\infty} \frac{r}{(r^2+(-2du)r+d^2)^{\frac{3}{2}}} dr &= \left. \frac{2(-2du)r+4d^2}{(4d^2u^2-4d^2)\sqrt{r(r-2du)+d^2}} \right|_0^{\infty} \\
&= \left. \frac{-4d(ru-d)}{4d^2(u^2-1)\sqrt{r^2-2rdu+d^2}} \right|_0^{\infty} \\
&= \left. \frac{-(ru-d)}{d(u^2-1)\sqrt{r^2-2rdu+d^2}} \right|_0^{\infty} \\
&= \left. \frac{ru-d}{d(1-u^2)\sqrt{r^2-2rdu+d^2}} \right|_0^{\infty}
\end{aligned}$$

Now we'll evaluate this at the limits

$$\lim_{r\to\infty} \frac{ru-d}{d(1-u^2)\sqrt{r^2-2rdu+d^2}} = \frac{ru}{d(1-u^2)\sqrt{r^2}} = \frac{u}{d(1-u^2)}$$

$$\lim_{r\to 0}\frac{ru-d}{d(1-u^2)\sqrt{r^2-2rdu+d^2}} = \frac{-d}{d(1-u^2)\sqrt{d^2}} = \frac{-1}{d(1-u^2)}$$

So

$$\begin{aligned}
\int_0^\infty \frac{r}{(r^2+d^2-2rdu)^{\frac{3}{2}}}dr &= \frac{u}{d(1-u^2)} + \frac{1}{d(1-u^2)} = \frac{u+1}{d(1-u^2)} \\
&= \frac{u+1}{d(1+u)(1-u)} \\
&= \frac{1}{d(1-u)}
\end{aligned}$$

Moving back to the angular momentum calculation:

$$\begin{aligned}
\mathbf{L}_z &= \frac{\mu_0 q_e q_m d}{8\pi}\int_{-1}^{1}(1-u^2)\frac{1}{d(1-u)}du \\
&= \frac{\mu_0 q_e q_m}{8\pi}\int_{-1}^{1}\frac{1-u^2}{1-u}du \\
&= \frac{\mu_0 q_e q_m}{8\pi}\int_{-1}^{1}(1+u)du = \frac{\mu_0 q_e q_m}{8\pi}(u+\frac{1}{2}u^2)\Big|_{-1}^{1} = \frac{\mu_0 q_e q_m}{8\pi}(2) \\
&= \frac{\mu_0 q_e q_m}{4\pi}
\end{aligned}$$

There is no dependence on $d$!

**Problem 5.** A charged parralel plate capacitor with uniform electric field $\mathbf{E} = E(z)$ is placed in a uniform magnetic field $\mathbf{B} = B(x)$ as shown in the figure on the question sheet.

(a) Find the EM Momentum in the space between the plates

The density of momentum is

$$\begin{aligned}
\wp_{\mathbf{em}} &= \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}) \\
&= \epsilon_0 EB\hat{\mathbf{y}}
\end{aligned}$$

So the total EM momentum is the density times the volume

$$\begin{aligned}
P_{em} &= \wp_{\mathbf{em}}V \\
&= \epsilon_0 EBAd\hat{\mathbf{y}}
\end{aligned}$$

where $A$ is the area of the plate and $d$ is the space between the plates.

(b) Now a resistive wire is connected between the plates along the z-axis so that the capacitor slowly discharges. The current through the wire will experience a magnetic force. What is the total impulse delivered to the system during the discharge?

The wire has a resistivity $\rho$. The current flowing through it is $I$. The current flows downward so $\mathbf{I} = -Id\hat{\mathbf{z}}$. It experiences a magnetic force equal to

$$\mathbf{F} = \mathbf{I}\times\mathbf{B}$$

The total impulse is given by a time integral of the force from 0 to infinity:

$$
\begin{aligned}
\mathbf{J} &= \int_0^\infty F dt \\
&= \int_0^\infty \mathbf{I} \times \mathbf{B} dt \\
&= \int_0^\infty -IB(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) dt \\
&= B\hat{\mathbf{y}} \int_0^\infty -I dt \\
&= B\hat{\mathbf{y}} \int_0^\infty \Big( -\frac{dQ}{dt} \Big) dt \\
&= -B\hat{\mathbf{y}} Q|_0^\infty \\
&= Q_0 B\hat{\mathbf{y}} \qquad \text{(since } Q(\infty) = 0\text{)}
\end{aligned}
$$

QED