PHY4346
ASSIGNMENT 3
MOHAMMED CHAMMA 6379153

Problem 1. Recent Hubble measurements of Sirius' white dwarf companion show that the white dwarf plausibly has a mass of $M = 1.02 M_\odot$ and a radius of $r_E = 5640 km$ (with uncertainties of 2%) and that spectral features of its light have a fractional redshift of about $\frac{\Delta\lambda}{\lambda_E} = 2.68 \times 10^{-4} \pm 6\%$, where $\frac{\Delta\lambda}{\lambda_E} = \frac{\lambda_D - \lambda_E}{\lambda_E} = \frac{\lambda_D}{\lambda_E} - 1$. Are these results consistent with the redshift of general relativity?

GR predicts a fractional redshift of $\frac{\Delta\lambda}{\lambda_E} = \frac{\lambda_D}{\lambda_E} - 1 = \frac{\sqrt{1-2GM/r_D}}{\sqrt{1-2GM/r_E}} - 1$. The distance from Earth to Sirius is $r_D = 8.14 \times 10^{13} km$. Then the predicted gravitational redshift would be:

$$\frac{\Delta\lambda}{\lambda_E} = \frac{\sqrt{1-2(1477)(1.02)/8.14\times 10^{16}}}{\sqrt{1-2(1477)(1.02)/5640\times 10^{3}}} - 1 = 2.67 \times 10^{-4}$$

The uncertainty on the experimental values give the following range:

$$r_E = 5640 km \pm 2\% = 5640 \pm 113 km \implies r_E \textit{ is between } 5527 km \textit{ and } 5753 km$$

$$M = 1.02 M_\odot \pm 2\% = 1.02 M_\odot \pm 0.02 M_\odot \implies M \textit{ is between } 1.00 M_\odot \textit{ and } 1.04 M_\odot$$

So the theoretical redshift falls in the range:
$\frac{\Delta\lambda}{\lambda_E} = 8.24 \times 10^{-6} \textit{ and } 8.41 \times 10^{-6}$ using $(5527 km,\ 1.00 M_\odot)$ for the minimum value and $(5753 km,\ 1.04 M_\odot)$ for the maximum value.
The measurement falls in the range:

$$\frac{\Delta\lambda}{\lambda_E} = 2.68 \times 10^{-4} \pm 6\% = 2.68 \times 10^{-4} \pm 0.16 \times 10^{-4} \implies \frac{\Delta\lambda}{\lambda_E} \textit{ is between } 2.52 \times 10^{-4} \textit{ and } 2.84 \times 10^{-4}$$

The theoretical value easily falls into the range of the measurement.

Problem 2. An advanced civilisation constructs two concentric massless shells around a neutron star of mass M. The inner shell has a circumference of $6\pi GM$ and the outer shell has a circumference of $20\pi GM$. What is the exact physical distance between the shells?

We can get the radii of the two shells with $C = 2\pi r$:

$$\begin{aligned} 6\pi GM &= 2\pi r_1 \implies r_1 = 3GM \\ 20\pi GM &= 2\pi r_2 \implies r_2 = 10GM \end{aligned}$$

Now we use the metric to find the distance, noting that $dt = d\theta = d\phi = 0$

$$\begin{aligned} ds^2 &= \frac{dr^2}{1 - \frac{2GM}{r}} \\ ds &= \frac{dr}{\sqrt{1 - \frac{2GM}{r}}} \end{aligned}$$

And the distance is given by

$$\Delta s = \int ds = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{2GM}{r}}}$$

Using the change of coordinates $u = \frac{2GM}{r}$, $du = \frac{-2GM}{r^2}dr \implies dr = \frac{-r^2}{2GM} = \frac{-2GM}{u^2}du$, and noting that $u_1 = \frac{2}{3}$, $u_2 = \frac{1}{5}$, we get

$$\begin{aligned}
\Delta s &= \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{2GM}{r}}} = \int_{u_1}^{u_2} \frac{-2GM}{u^2\sqrt{1-u}}du \\
&= -2GM \int_{u_1}^{u_2} \frac{1}{u^2\sqrt{1-u}}du = -2GM\Big(\frac{-\sqrt{1-u}}{u} - \tanh^{-1}(\sqrt{1-u})\Big)\Bigg|_{u_1}^{u_2} \\
&= -2GM\Big(\frac{-\sqrt{4/5}}{1/5} - \tanh^{-1}(\sqrt{4/5}) + \frac{\sqrt{1/3}}{2/3} + \tanh^{-1}(\sqrt{1/3})\Big) \\
&= -2GM\Big(-4.47 - 1.44 + 0.87 + 0.66\Big) \\
\Delta s &= 8.76GM
\end{aligned}$$

If the space were flat, this distance would be $7GM$.

Problem 3. Derive equations 10.32 and 10.33. Then do problem 10.9.

A.

Derive an expression for ℓ^2 using equation 10.11:

$$r_c = \frac{6GM}{1 \pm \sqrt{1 - 12(GM/\ell)^2}}$$

Starting with the upper + sign first:

$$\begin{aligned}
r_c &= \frac{6GM}{1 + \sqrt{1 - 12(GM/\ell)^2}} \\
1 + \sqrt{1 - 12(GM/\ell)^2} &= \frac{6GM}{r_c} \\
\sqrt{1 - 12(GM/\ell)^2} &= \frac{6GM}{r_c} - 1 \\
1 - 12(GM/\ell)^2 &= \Big(\frac{6GM}{r_c} - 1\Big)^2 = \frac{36G^2M^2}{r_c^2} - \frac{12GM}{r_c} + 1 \\
-\frac{12G^2M^2}{\ell^2} &= \frac{36G^2M^2}{r_c^2} - \frac{12GM}{r_c} \\
-\frac{G^2M^2}{\ell^2} &= \frac{3G^2M^2}{r_c^2} - \frac{GM}{r_c} = \frac{GM}{r_c}\Big(\frac{3GM}{r_c} - 1\Big) \\
\frac{G^2M^2}{\ell^2} &= \frac{GM}{r_c}\Big(1 - \frac{3GM}{r_c}\Big) = \frac{GM}{r_c}\Big(\frac{r_c - 3GM}{r_c}\Big) \\
\ell^2 &= \frac{r_c^2 GM}{r_c - 3GM}
\end{aligned}$$

as required. For the lower $-$ sign:

$$\begin{aligned}
r_c &= \frac{6GM}{1-\sqrt{1-12(GM/\ell)^2}} \\
1-\sqrt{1-12(GM/\ell)^2} &= \frac{6GM}{r_c} \\
-\sqrt{1-12(GM/\ell)^2} &= \frac{6GM}{r_c} - 1 \\
1-12(GM/\ell)^2 &= \Big(\frac{6GM}{r_c} - 1\Big)^2
\end{aligned}$$

And the argument proceeds exactly the same way.

B.

The effective energy per unit mass $\tilde{E}$ is given by

$$\tilde{E} = \frac{1}{2}\Big(\frac{\partial r}{\partial \tau}\Big)^2 - \frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}$$

For a circular orbit, $r = r_c$ and is constant. That is, $\frac{\partial r}{\partial \tau} = 0$. Using the expression from part A for ℓ^2,

$$\begin{aligned}
\tilde{E} &= -\frac{GM}{r_c} + \frac{\ell^2}{2r_c^2} - \frac{GM\ell^2}{r_c^3} \\
&= -\frac{GM}{r_c} + \frac{GMr_c^2}{2r_c^2(r_c-3GM)} - \frac{G^2M^2r_c^2}{r_c^3(r_c-3GM)} \\
&= -\frac{GM}{r_c} + \frac{GM}{2(r_c-3GM)} - \frac{G^2M^2}{r_c(r_c-3GM)} \\
&= -GM\Big(\frac{1}{r_c} - \frac{1}{2(r_c-3GM)} - \frac{GM}{r_c(r_c-3GM)}\Big) \\
&= -GM\Big(\frac{2(r_c-3GM) - r_c + 2GM}{2r_c(r_c-3GM)}\Big) \\
&= \frac{-GM}{2r_c(r_c-3GM)}\Big(2r_c - 6GM - r_c + 2GM\Big) \\
&= \frac{-GM}{2r_c(r_c-3GM)}\Big(r_c - 4GM\Big) \\
&= \frac{-GMr_c}{2r_c(r_c-3GM)}\Big(1 - \frac{4GM}{r_c}\Big) \\
\tilde{E} &= \frac{-GM}{2r_c}\Big(1 - \frac{3GM}{r_c}\Big)^{-1}\Big(1 - \frac{4GM}{r_c}\Big)
\end{aligned}$$

as required.

C.

A spaceship is in a stable circular orbit at a Schwarzschild radial coordinate of $r = 10GM$ around a supermassive black hole whose mass is 10^6 solar masses.

- What is this orbit's circumference in km?

$$C = 2\pi r = 2\pi(10)GM = 20\pi(1477)(10^6) = 9.28\times10^{10}m = 9.28\times10^7 km$$

- What are $\tilde{E}$ and ℓ for this orbit?

$$\ell = \sqrt{\frac{GMr_c^2}{r_c - 3GM}} = \sqrt{\frac{G^3M^3(100)}{10GM - 3GM}} = \sqrt{\frac{100G^2M^2}{7}} = \frac{10(1477)(10^6)}{\sqrt{7}} = 5.58 \times 10^9 m/kg$$

$$\tilde{E} = \frac{-GM}{2r_c}\left(1 - \frac{3GM}{r_c}\right)^{-1}\left(1 - \frac{4GM}{r_c}\right) = \frac{-GM}{20GM}\left(1 - \frac{3GM}{10GM}\right)^{-1}\left(1 - \frac{4GM}{10GM}\right) = \frac{-1}{20}\left(\frac{10}{7}\right)\left(\frac{6}{10}\right) = -0.043$$

- What is the period of the spaceship's orbit according to its own clock?
  The period as seen from an observer far away is given by

$$T = \frac{2\pi}{\sqrt{GM}}\sqrt{r_c^3} = \frac{2\pi}{\sqrt{1477(10^6)}} = \sqrt{1000(1477)^3(10^6)^3} = 2.93 \times 10^{11} m$$

We should be able to transform this result to the reference frame of the spaceship. We need to know what $\frac{\partial t}{\partial \tau}$ is.

$$\tilde{E} = \frac{1}{2}(e^2 - 1) \implies e = \sqrt{2\tilde{E} + 1} = \sqrt{2(-0.043) + 1} = 0.96$$

Now,

$$\begin{aligned} e &= \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \\ \frac{dt}{d\tau} &= \frac{e}{1 - \frac{2GM}{r}} = \frac{0.96}{1 - 1/5} = 1.2 \implies \frac{d\tau}{dt} = \frac{1}{1.2} \end{aligned}$$

So,

$$\Delta\tau = \frac{d\tau}{dt}\Delta T = \frac{1}{1.2}(2.93 \times 10^{11}) = 2.44 \times 10^{11} m$$

The rest is incomplete.