PHY4346
ASSIGNMENT 5

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Problem 1. $T^{\alpha\beta} = (\rho_0 + P_0)u^\alpha u^\beta + P_0 g^{\alpha\beta}$.

a) Starting from $\nabla_\mu T^{\mu\nu} = 0$, derive $P_0 \nabla_\mu u^\mu + \nabla_\mu(\rho_0 u^\mu) = 0$. Use an LIF.
In an LIF the metric $g^{\mu\nu} = \eta^{\mu\nu}$. Using $u_\nu u^\nu = -1$ and $u_\nu \nabla_\mu u^\nu = 0$:

$$
\begin{aligned}
0 &= \nabla_\mu T^{\mu\nu} \\
&= \nabla_\mu\Big((\rho_0 + P_0)u^\mu u^\nu\Big) + \nabla_\mu(P_0 g^{\mu\nu}) \\
&= u^\mu u^\nu \nabla_\mu(\rho_0 + P_0) + (\rho_0 + P_0)u^\nu \nabla_\mu u^\mu + (\rho_0 + P_0)u^\mu \nabla_\mu u^\nu + P_0 \nabla_\mu g^{\mu\nu} + g^{\mu\nu}\nabla_\mu P_0 \\
&= u^\mu u^\nu \nabla_\mu(\rho_0 + P_0) + (\rho_0 + P_0)u^\nu \nabla_\mu u^\mu + (\rho_0 + P_0)u^\mu \nabla_\mu u^\nu + g^{\mu\nu}\nabla_\mu P_0 \\
&= u_\nu(u^\mu u^\nu \nabla_\mu(\rho_0 + P_0) + (\rho_0 + P_0)u^\nu \nabla_\mu u^\mu + (\rho_0 + P_0)u^\mu \nabla_\mu u^\nu + g^{\mu\nu}\nabla_\mu P_0) \\
&= -u^\mu \nabla_\mu(\rho_0 + P_0) - (\rho_0 + P_0)\nabla_\mu u^\mu + u^\mu \nabla_\mu P_0 \\
&= -u^\mu \nabla_\mu \rho_0 - u^\mu \nabla_\mu P_0 - \rho_0 \nabla_\mu u^\mu - P_0 \nabla_\mu u^\mu + u^\mu \nabla_\mu P_0 \\
&= -u^\mu \nabla_\mu \rho_0 - \rho_0 \nabla_\mu u^\mu - P_0 \nabla_\mu u^\mu \\
&= u^\mu \nabla_\mu \rho_0 + \rho_0 \nabla_\mu u^\mu + P_0 \nabla_\mu u^\mu \\
0 &= P_0 \nabla_\mu u^\mu + \nabla_\mu(\rho_0 u^\mu)
\end{aligned}
$$

b) Starting from $\nabla_\alpha T^{\alpha\beta} = 0$, derive $(u^\mu u^\nu + g^{\mu\nu})\nabla_\mu P_0 + (P_0 + \rho_0)u^\mu \nabla_\mu u^\nu = 0$.
Note that $\nabla_\mu g^{\mu\nu} = 0$.

$$
\begin{aligned}
0 &= \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho_0 + P_0)u^\mu u^\nu + \nabla_\mu(P_0 g^{\mu\nu}) \\
&= \nabla_\mu \rho_0 u^\mu u^\nu + \nabla_\mu P_0 u^\mu u^\nu + P_0 \nabla_\mu g^{\mu\nu} + g^{\mu\nu}\nabla_\mu P_0 \\
&= \nabla_\mu \rho_0 u^\mu u^\nu + \nabla_\mu P_0 u^\mu u^\nu + g^{\mu\nu}\nabla_\mu P_0 \\
&= u^\mu u^\nu \nabla_\mu \rho_0 + \rho_0 u^\nu \nabla_\mu u^\mu + \rho_0 u^\mu \nabla_\mu u^\nu \\
&\quad + u^\mu u^\nu \nabla_\mu P_0 + P_0 u^\nu \nabla_\mu u^\mu + P_0 u^\mu \nabla_\mu u^\nu + g^{\mu\nu}\nabla_\mu P_0 \\
&= (u^\mu u^\nu + g^{\mu\nu})\nabla_\mu P_0 + (P_0 + \rho_0)u^\mu \nabla_\mu u^\nu + (P_0 + \rho_0)u^\nu \nabla_\mu u^\mu + u^\mu u^\nu \nabla_\mu \rho_0 \\
&= (u^\mu u^\nu + g^{\mu\nu})\nabla_\mu P_0 + (P_0 + \rho_0)u^\mu \nabla_\mu u^\nu \\
&\quad + P_0 u^\nu \nabla_\mu u^\mu + \rho_0 u^\nu \nabla_\mu u^\mu + u^\mu u^\nu \nabla_\mu \rho_0 \\
&= (u^\mu u^\nu + g^{\mu\nu})\nabla_\mu P_0 + (P_0 + \rho_0)u^\mu \nabla_\mu u^\nu + u^\nu(P_0 \nabla_\mu u^\mu + \rho_0 \nabla_\mu u^\mu + u^\mu \nabla_\mu \rho_0) \\
&= (u^\mu u^\nu + g^{\mu\nu})\nabla_\mu P_0 + (P_0 + \rho_0)u^\mu \nabla_\mu u^\nu + u^\nu(P_0 \nabla_\mu u^\mu + \nabla_\mu(\rho_0 u^\mu)) \\
0 &= (u^\mu u^\nu + g^{\mu\nu})\nabla_\mu P_0 + (P_0 + \rho_0)u^\mu \nabla_\mu u^\nu
\end{aligned}
$$

Where the last step uses the result from (a).

c) Consider the non-relativistic case when $\rho_0 >> P_0$ and $u^\mu \approx (1, v^i)$.
Let's see how the continuity equation reduces in this case:

$$\begin{aligned}
0 &= P_0 \nabla_\mu u^\mu + \nabla_\mu(\rho_0 u^\mu) \\
&= P_0\Big(\frac{d1}{dt} + \frac{dv^i}{dx^i}\Big) + \Big(\frac{d\rho_0}{dt} + \frac{d(\rho_0 v^i)}{dx^i}\Big) \\
&= P_0 \partial_i v^i + \partial_t \rho_0 + \partial_i(\rho_0 v^i) \\
&= \partial_t \rho_0 + P_0 \partial_i v^i + \rho_0 \partial_i v^i + v^i \partial_i \rho_0 \\
&= \partial_t \rho_0 + (P_0 + \rho_0)\partial_i v^i + v^i \partial_i \rho_0
\end{aligned}$$

Since $\rho_0 >> P_0$, we can approximate $(P_0 + \rho_0) \approx \rho_0$. So,

$$0 = \partial_t \rho_0 + \rho_0 \partial_i v^i + v^i \partial_i \rho_0 = \partial_t \rho_0 + \partial_i(\rho_0 v^i)$$

Now let's see how the Euler equation reduces in the non-relativistic case:

$$\begin{aligned}
0 &= (u^\mu u^\nu + g^{\mu\nu})\nabla_\mu P_0 + (P_0 + \rho_0) u^\mu \nabla_\mu u^\nu \\
&= (\eta^{0k}\partial_t P_0 + \eta^{ik}\partial_i P_0 + v^0 v^k \partial_t P_0 + v^i v^k \partial_i P_0) + \rho_0(v^0 \partial_t v^k + v^i \partial_i v^k) \\
&= -\partial_t P_0 + \partial_i P_0 + \partial_t P_0 + (v^i)^2 \partial_i P_0 + \rho_0 \partial_t v^i + \rho_0 v^i \partial_i v^i \\
&= (1 + (v^i)^2)\partial_i P_0 + \rho_0 \partial_t v^i + \rho_0 v^i \partial_i v^i
\end{aligned}$$

Problem 2. Moore, p. 241 Problem 20.5

a) The Electromagnetic field tensor $F^{\mu\nu}$ and it's covariant form $F_{\mu\nu} = \eta_{\mu\alpha} F^{\alpha\beta} \eta_{\beta\nu}$ is

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}, \qquad F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

We first want to prove that $F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2)$ where $B^2 = B_x^2 + B_y^2 + B_z^2$ and $E^2 = E_x^2 + E_y^2 + E_z^2$.
Now,

$$\begin{aligned}
F^{\mu\nu}F_{\mu\nu} &= F^{00}F_{00} + F^{01}F_{01} + ... + F^{10}F_{10} + F^{11}F_{11} + ... + F^{20}F_{20} + F^{21}F_{21} + ... + F^{30}F_{30} + F^{31}F_{31} + ... \\
&= -(E_x^2 + E_y^2 + E_z^2) + (-E_x^2 + B_z^2 + B_y^2) + (-E_y^2 + B_z^2 + B_x^2) + (-E_z^2 + B_y^2 + B_x^2) \\
&= -E^2 - (E_x^2 + E_y^2 + E_z^2) + (2B_z^2 + 2B_y^2 + 2B_x^2) \\
&= -E^2 - E^2 + 2B^2 \\
F^{\mu\nu}F_{\mu\nu} &= 2(B^2 - E^2)
\end{aligned}$$

We're aiming for a second-rank symmetric tensor $T^{\mu\nu}$ whose tt component is

$$T^{00} = \rho_E = \frac{1}{8\pi k}(E^2 + B^2) \qquad \text{with } k = \frac{1}{4\pi\epsilon_0}$$

We can derive some useful expressions for the $(E^2 + B^2)$ term. Note that $E^2 = F^{0\nu}F_{\nu 0}$. Isolating for B^2 from the expression we proved at the beginning we get

$$B^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + E^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + F^{0\nu}F_{\nu 0}$$

So we write an expression for $E^2 + B^2$ in terms of tensor products and we get

$$E^2 + B^2 = 2F^{0\nu}F_{\nu 0} + \frac{1}{2}F^{\mu\nu}F_{\mu\nu}$$

The first term of the RHS of this expression can be seen as th tt component of a second-rank rank tensor but the second term is a scalar. We can make it a second-rank tensor by multiplying it by $-\eta^{00} = 1$ to get

$$E^2 + B^2 = 2F^{0\nu}F_{\nu 0} - \frac{1}{2}F^{\mu\nu}F_{\mu\nu}\eta^{00}$$

Proposing that this expression is the tt component of a second rank tensor $8\pi k T^{\alpha\beta}$ and placing the free labels we get

$$T^{\alpha\beta} = \frac{1}{8\pi k}\Big(2F^{\alpha\nu}F_{\nu\beta} - \frac{1}{2}F^{\mu\nu}F_{\mu\nu}\eta^{\alpha\beta}\Big)$$

b) Calculate $T^{tx} = T^{01}$

$$\begin{aligned} T^{01} &= \frac{1}{8\pi k}\Big(2F^{0\nu}F_{\nu 1} - \frac{1}{2}F^{\mu\nu}F_{\mu\nu}\eta^{01}\Big) \\ &= \frac{1}{8\pi k}\Big(2F^{0\nu}F_{\nu 1}\Big) \\ &= \frac{1}{8\pi k}(2(F^{02}F_{21} + F^{03}F_{31})) \\ &= \frac{1}{8\pi k}(2(E_zB_y - E_yB_z)) = \frac{1}{4\pi k}(E_zB_y - E_yB_z) \end{aligned}$$

Note that $T^{01} = \frac{1}{4\pi k}(E_zB_y - E_yB_z) = \epsilon_0(\mathbf{E}\times\mathbf{B})_x$. This is the x-component of the Poynting vector.

Problem 3. First derive $R_{\mu\nu} = K(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + \Lambda g_{\mu\nu}$. Then do Moore p. 251 Problem 21.1.

a) Starting from Einstein's equation with cosmological constant $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = KT_{\mu\nu}$ and multiplying by $g^{\nu\alpha}$:

$$\begin{aligned} R_{\mu\nu}g^{\nu\alpha} - \frac{1}{2}g_{\mu\nu}g^{\nu\alpha}R + \Lambda g_{\mu\nu}g^{\nu\alpha} &= KT_{\mu\nu}g^{\nu\alpha} \\ R^{\alpha}_{\mu} - \frac{1}{2}\delta^{\alpha}_{\mu}R + \Lambda\delta^{\alpha}_{\mu} &= KT^{\alpha}_{\mu} \end{aligned}$$

Summing over α ($\mu = \alpha$) we get

$$\begin{aligned} R - \frac{1}{2}(4)R + \Lambda(4) &= KT \\ R - 2R + 4\Lambda &= KT \\ -R + 4\Lambda &= KT \end{aligned}$$

Multiplying by $-\frac{1}{2}g_{\mu\nu}$ we get $\frac{1}{2}g_{\mu\nu}R - 2g_{\mu\nu}\Lambda = -\frac{1}{2}g_{\mu\nu}KT$. Adding this to Einstein's equation we have

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R - 2g_{\mu\nu}\Lambda &= KT_{\mu\nu} - \frac{1}{2}g_{\mu\nu}KT \\ R_{\mu\nu} - \Lambda g_{\mu\nu} &= K(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \\ R_{\mu\nu} &= K(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + \Lambda g_{\mu\nu} \end{aligned}$$

as required.

b) Moore p. 251 Problem 21.1

(i) Assume there is a gravitational potential Φ about an arbitrary origin in empty space such that it is spherically symmetric and $\Phi(r = 0) = 0$. The Laplacian of the potential in this case is

$$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{d\Phi}{dr}\Big)$$

Prove that the potential in the Newtonian limit is

$$\vec{g} = -\frac{d\Phi}{dr} = \frac{\Lambda}{3}r$$

In the Newtonian limit Einstein's equation reduces to

$$\nabla^2\Phi = \frac{1}{2}K\rho - \Lambda$$

Since we're in empty space, the mass-density $\rho = 0$. So we expect that $\nabla^2\Phi = -\Lambda$. Plugging $\vec{g}$ into the Laplacian:

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r^2}\frac{d}{dr}\Big(r^2(-\frac{\Lambda}{3})r\Big) \\
&= -\frac{\Lambda}{3r^2}\frac{d}{dr}(r^3) \\
&= -\frac{\Lambda}{3r^2}(3r^2) = -\Lambda
\end{aligned}$$

as expected.
(ii) $GM_\odot \approx 1500m$ and $r \sim 10^{12}m$. Assuming that $\frac{1}{3}\Lambda r << \frac{GM_\odot}{r^2}$ we see that

$$\begin{aligned}
\Lambda &<< \frac{3GM_\odot}{r^3} \\
\Lambda &<< \frac{3(1500m)}{(10^{12}m)^3} << \frac{10^3}{10^{36}}\mathrm{m}^{-2} = 10^{-33}\mathrm{m}^{-2} \\
\frac{\Lambda}{8\pi G} &<< \frac{10^{-33}}{8\pi G}\mathrm{m}^{-2}\mathrm{m}^{-1}\mathrm{kg} \\
\frac{\Lambda}{8\pi G} &<< \frac{10^{-33}}{10\times 10^{-28}}\frac{\mathrm{kg}}{\mathrm{m}^3} = \frac{10^{-33}}{10^{-27}}\frac{\mathrm{kg}}{\mathrm{m}^3} = 10^{-6}\frac{\mathrm{kg}}{\mathrm{m}^3}
\end{aligned}$$

Which is kinda close to 10^{-7}...

Problem 4. Moore, p. 251 Problem 21.5

The only independent component of the Riemann tensor in two dimensions of spacetime is the R_{0101} component. We know that $R^{\mu\nu} = g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}R_{\alpha\beta\gamma\sigma}$. Since there are only two dimensions and $R^{\mu\nu}$ is symmetric, the components of the Ricci tensor R^{00}, R^{10}, and R^{11} can be written in terms of R_{0101}. We have

$$R^{\mu\nu} = g^{\mu\beta}g^{\nu\sigma}g^{\alpha\gamma}R_{\alpha\beta\gamma\sigma}$$

Knowing that R_{0101} is the only independent component we can go ahead and set $\alpha = \gamma$ and $\beta = \sigma$. This gives us:

$$\begin{aligned}
R^{\mu\nu} &= g^{\mu\sigma}g^{\nu\sigma}g^{\gamma\gamma}R_{\gamma\sigma\gamma\sigma} \\
&= g^{\mu\sigma}g^{\nu\sigma}(g^{00}R_{0\sigma0\sigma} + g^{11}R_{1\sigma1\sigma}) \\
&= g^{\mu\sigma}g^{\nu\sigma}g^{00}R_{0\sigma0\sigma} + g^{\mu\sigma}g^{\nu\sigma}g^{11}R_{1\sigma1\sigma} \\
&= g^{\mu0}g^{\nu0}g^{00}R_{0000} + g^{\mu1}g^{\nu1}g^{00}R_{0101} + g^{\mu0}g^{\nu0}g^{11}R_{1010} + g^{\mu1}g^{\nu1}g^{11}R_{1111}
\end{aligned}$$

Using the symmetry relations we have that $R_{0000} = 0$ and $R_{1111} = 0$. We also have $R_{1010} = -R_{0110} = R_{0101}$. So

$$R^{\mu\nu} = (g^{\mu1}g^{\nu1}g^{00} + g^{\mu0}g^{\nu0}g^{11})R_{0101}$$

So the components of the Ricci tensor are (noting the the $g^{\mu\nu}$ is symmetric)

$$\begin{aligned}
R^{00} &= R^{tt} = (g^{01}g^{01}g^{00} + g^{00}g^{00}g^{11})R_{0101} = g^{tt}((g^{tx})^2 + g^{tt}g^{xx})R_{txtx} \\
R^{10} &= R^{xt} = (g^{11}g^{01}g^{00} + g^{10}g^{00}g^{11})R_{0101} = 2g^{tt}g^{xx}g^{tx}R_{txtx} \\
R^{11} &= R^{xx} = (g^{11}g^{11}g^{00} + g^{10}g^{10}g^{11})R_{0101} = g^{xx}((g^{tx})^2 + g^{tt}g^{xx})R_{txtx} = g^{xx}R^{tt}
\end{aligned}$$

Since the components of the Ricci tensor are 0 $R^{\mu\nu} = 0$ we have

$$0 = g^{tt}((g^{tx})^2 + g^{tt}g^{xx})R_{txtx} \quad \text{and} \quad 0 = 2g^{tt}g^{xx}g^{tx}R_{txtx}$$

From Einstein's equation we have $R^{\mu\nu} = 0 = K(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T)$ ($\Lambda = 0$ since there is no vacuum energy) which implies that

$$g^{\mu\nu} = \frac{2T^{\mu\nu}}{T}$$

This implies that in general the metric can be non-diagonal and non-zero. In order for the condition $R^{\mu\nu} = 0$ to hold we must require then that $R_{txtx} = 0$. Since this is the only independent component of the Riemann tensor we can say that the entire Riemann tensor is 0, which means that the space is always flat regardless of mass and energy distribution.

Problem 5. Moore p. 251 Problem 21.8
Consider the metric

$$ds^2 = -e^{2gx}dt^2 + dx^2 + dy^2 + dz^2$$

a) The geodesic equation is

$$\frac{\partial}{\partial\tau}\Big(g_{\alpha\nu}\frac{dx^\nu}{d\tau}\Big) - \frac{1}{2}(\partial_\alpha g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Letting $\alpha = 1 = x$ we get (noting that $g_{\mu\nu}$ is diagonal so $\mu = \nu$):

$$\begin{aligned}
0 &= \frac{\partial}{\partial\tau}\Big(g_{11}\frac{dx}{d\tau}\Big) - \frac{1}{2}\Big(\frac{dg_{00}}{dx}\frac{dt^2}{d\tau^2} + \frac{dg_{11}}{dx}\frac{dx^2}{d\tau^2} + \frac{dg_{22}}{dx}\frac{dy^2}{d\tau^2} + \frac{dg_{33}}{dx}\frac{dz^2}{d\tau^2}\Big) \\
&= \frac{d^2x}{d\tau^2} - \frac{1}{2}\Big(-\frac{de^{2gx}}{dx}\frac{dt^2}{d\tau^2} + \frac{d1}{dx}\frac{dx^2}{d\tau^2} + \frac{d1}{dx}\frac{dy^2}{d\tau^2} + \frac{d1}{dx}\frac{dz^2}{d\tau^2}\Big) \\
&= \frac{d^2x}{d\tau^2} + \frac{1}{2}(2g)e^{2gx}\frac{dt^2}{d\tau^2} \\
\frac{d^2x}{d\tau^2} &= -ge^{2gx}\frac{dt^2}{d\tau^2}
\end{aligned}$$

Now, assume there is a particle initially at rest so that $dx = dy = dz = 0$. Using $d\tau = \sqrt{-ds^2}$ we find that $d\tau = \sqrt{-(-e^{2gx})dt^2} = e^{gx}dt$ for this particle and so

$$\frac{dt}{d\tau} = \frac{dt}{e^{gx}dt} = e^{-gx}$$

This means that

$$\frac{d^2x}{d\tau^2} = -ge^{2gx}(e^{-2gx}) = -g$$

This is saying that a particle initially at rest at any point in this space-time will experience an x-acceleration of $-g$.

b) Show that the only nonzero Christoffel symbols for this metric are $\Gamma^t_{tx} = \Gamma^t_{xt} = g$ and $\Gamma^x_{tt} = ge^{2gx}$.
The Christoffel symbols are given by

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2}g^{\gamma\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$$

Since the metric is diagonal we must have $\alpha = \gamma$ otherwise the symbol is 0. So

$$\Gamma^{\gamma}_{\mu\nu} = \frac{1}{2} g^{\gamma\gamma}(\partial_\mu g_{\gamma\nu} + \partial_\nu g_{\gamma\mu} - \partial_\gamma g_{\mu\nu})$$

For the symbol to be nonzero, either μ or ν must be equal to γ. If each label is different from the other then the symbol is 0 since the metric terms $g_{\gamma\nu}$, $g_{\gamma\mu}$, and $g_{\mu\nu}$ will simultaneously be 0. One of the labels must be an x label since the t, y, and z derivatives will always be zero for this metric. There must also be a t label since it is the only metric term with an x-dependence and will not disappear under differentiation w.r.t. x. The last label must therefore be either a t or an x since the labels cannot be all different from each other. This leaves three possible nonzero and independent components:

$$\begin{aligned}
\Gamma^t_{xx} &= \frac{1}{2} g^{tt}(\partial_x g_{tx} + \partial_x g_{tx} - \partial_t g_{xx}) = 0 \\
\Gamma^t_{tx} &= \frac{1}{2} g^{tt}(\partial_t g_{tx} + \partial_x g_{tt} - \partial_t g_{tx}) = \frac{1}{2}(-e^{-2gx})\Big(\frac{d(-e^{2gx})}{dx}\Big) \\
&= \frac{1}{2} 2ge^{-2gx}e^{2gx} = g = \Gamma^t_{xt} \\
\Gamma^x_{tt} &= \frac{1}{2} g^{xx}(\partial_t g_{xt} + \partial_t g_{xt} - \partial_x g_{tt}) = -\frac{1}{2}\frac{d(-e^{2gx})}{dx} = ge^{2gx}
\end{aligned}$$

c) Argue that R_{txtx} and its permutations are the only nonzero Riemann tensor components.
The Riemann tensor is given by

$$R_{\alpha\beta\mu\nu} = g_{\alpha\gamma} R^{\gamma}_{\beta\mu\nu} = g_{\alpha\gamma}(\partial_\mu \Gamma^{\gamma}_{\beta\nu} - \partial_\nu \Gamma^{\gamma}_{\beta\mu} + \Gamma^{\gamma}_{\mu\sigma}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\gamma}_{\nu\delta}\Gamma^{\delta}_{\beta\mu})$$

Right away $\gamma = \alpha$ for nonzero terms.

$$R_{\alpha\beta\mu\nu} = g_{\alpha\alpha}(\partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\nu\delta}\Gamma^{\delta}_{\beta\mu})$$

The components must only involve the nonzero Christoffel symbols, so the labels can only be x or t. If three of the labels are x then there will always be a Christoffel symbol with two x labels in each term, which are zero, and the last term will be Γ^t_{xx} which is zero. Similar argument for if three of the labels are t. What's left are R_{txtx} and the components related to it by the symmetries.

$$R_{txtx} = g_{tt}(\partial_t \Gamma^t_{xx} - \partial_x \Gamma^t_{xt} + \Gamma^t_{t\sigma}\Gamma^{\sigma}_{xx} - \Gamma^t_{x\delta}\Gamma^{\delta}_{xt})$$

Consider the sum over σ. We already know that the terms with y or z labels are zero. If the label is t, then the first term of the sum is $\Gamma^t_{tt} = 0$. If the label is x then the second term of the sum is $\Gamma^x_{xx} = 0$. So the sum over σ term is zero.
Now consider the sum over δ. If the label is x then the first term of the sum is $\Gamma^t_{xx} = 0$. If the label is t then the whole term is $\Gamma^t_{xt}\Gamma^t_{xt} = g^2$. Thus

$$\begin{aligned}
R_{txtx} &= g_{tt}(-\partial_x \Gamma^t_{xt} - g^2) \\
&= -e^{2gx}(-\partial_x g - g^2) \\
R_{txtx} &= g^2 e^{2gx}
\end{aligned}$$

d) Show R^{tt} and R^{xx} are the only components.
The Ricci tensor components are given by

$$R^{\mu\nu} = g^{\mu\beta} g^{\nu\sigma} g^{\alpha\gamma} R_{\alpha\beta\gamma\sigma}$$

R_{txtx} is the only independent component, implying

$$R^{\mu\nu} = g^{\mu x} g^{\nu x} g^{tt} R_{txtx}$$

Since the metric is symmetric $\mu = \nu = x$ and making sure to use the value after raising the label of $g_{\mu\nu}$:

$$\begin{aligned} R^{xx} &= (g^{xx})^2 g^{tt} R_{txtx} \\ &= -e^{-2gx}(g^2 e^{2gx}) = -g^2 \end{aligned}$$

By symmetry we have $R_{txtx} = R_{xtxt}$ so

$$R^{\mu\nu} = g^{\mu t} g^{\nu t} g^{xx} R_{xtxt}$$

Now $\mu = \nu = t$ so

$$\begin{aligned} R^{tt} &= (g^{tt})^2 g^{xx} R_{xtxt} \\ &= e^{-4gx} g^2 e^{2gx} = g^2 e^{-2gx} \end{aligned}$$

There are no other components arising from the other symmetries because of the term $g^{\alpha\gamma}$. The first and third labels need to be the same.
e) Claim: If there is no vacuum energy then $g = 0$, and there is no gravitational field.
Einstein's vacuum equation is

$$R^{\mu\nu} = K(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T)$$

What does equation say for empty space with no stress-energy? $T^{\mu\nu} = T = 0$. Then we'd expect $R^{\mu\nu} = 0$. Using R^{tt} we have

$$\begin{aligned} 0 &= R^{tt} \\ 0 &= g^2 e^{-2gx} \end{aligned}$$

But e^{6gx} is never 0. This condition will fail unless $g = 0$.
f) Can g be nonzero inside some kind of fluid?
Assume it is nonzero. Inside a fluid $T^{\mu\nu} \neq 0$ and $T = g_{\alpha\beta} T^{\alpha\beta} = -e^{2gx} T^{tt} + T^{xx} + T^{yy} + T^{zz}$. Using Einstein's equation $R^{\mu\nu} = K(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T)$ we get

$$\begin{aligned} R^{tt} &= g^2 e^{-2gx} = K(T^{tt} - \frac{1}{2} g^{tt} T) = K(T^{tt} + \frac{1}{2} e^{-2gx} T) \\ R^{xx} &= -g^2 = K(T^{xx} - \frac{1}{2} g^{xx} T) = K(T^{xx} - \frac{1}{2} T) \\ R^{yy} &= 0 = K(T^{yy} - \frac{1}{2} T) \\ R^{zz} &= 0 = K(T^{zz} - \frac{1}{2} T) \end{aligned}$$

Since $R^{yy} = R^{zz}$ we have

$$\begin{aligned} K(T^{yy} - \frac{1}{2} T) &= K(T^{zz} - \frac{1}{2} T) \\ T^{yy} &= T^{zz} = \frac{T}{2} \end{aligned}$$

Plugging this into our equation for T we get

$$\begin{aligned} T &= -e^{2gx} T^{tt} + T^{xx} + T^{yy} + T^{zz} \\ T &= -e^{2gx} T^{tt} + T^{xx} + T \\ T^{xx} - e^{2gx} T^{tt} &= 0 \end{aligned}$$

From the expression for R^{tt} we have $g^2 e^{-2gx} = K(T^{tt} + \frac{1}{2}e^{-2gx}T)$. Using the expression for R^{xx} we have $g^2 = -K(T^{xx} - \frac{1}{2}T)$. Substituting we get

$$\begin{aligned}
-K(T^{xx} - \frac{1}{2}T)e^{-2gx} &= K(T^{tt} + \frac{1}{2}e^{-2gx}T) \\
-e^{-2gx}(T^{xx} - \frac{1}{2}T) &= T^{tt} + \frac{1}{2}e^{-2gx}T \\
-e^{-2gx}T^{xx} + \frac{1}{2}e^{-2gx}T &= T^{tt} + \frac{1}{2}e^{-2gx}T \\
T^{tt} &= -e^{-2gx}T^{xx}
\end{aligned}$$

Combining this with $T^{xx} - e^{2gx}T^{tt} = 0$ we get

$$\begin{aligned}
T^{xx} &= e^{2gx}T^{tt} \\
&= -e^{2gx}e^{-2gx}T^{xx} \\
T^{xx} &= -T^{xx} \\
\implies T^{xx} &= 0
\end{aligned}$$

Since $T^{xx} = 0$ we also have $T^{tt} = 0$. Now

$$\begin{aligned}
R^{xx} &= -g^2 = K(-\frac{1}{2}T) \\
\implies T &= \frac{2g^2}{K}
\end{aligned}$$

This means

$$T^{zz} = T^{yy} = \frac{T}{2} = \frac{g^2}{K} = \frac{g^2}{8\pi G}$$

This should have been negative! If it were it wouldn't make sense because the $T^{zz} = T^{yy}$ components of a fluid correspond to the pressure of the fluid, meaning that the pressure would be negative.

Problem 6. Moore, p. 263 Problem 22.2

Take the stress-energy tensor in a vacuum to be $T^{\mu\nu} = -\frac{\Lambda}{8\pi G}g^{\mu\nu}$. This means

$$\begin{aligned}
T_{\mu\nu} &= g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta} = -\frac{\Lambda}{8\pi G}g_{\mu\alpha}g_{\nu\beta}g^{\alpha\beta} \\
&= -\frac{\Lambda}{8\pi G}g_{\mu\alpha}\delta^{\alpha}_{\nu} \\
T_{\mu\nu} &= -\frac{\Lambda}{8\pi G}g_{\mu\nu}
\end{aligned}$$

We find that T is

$$\begin{aligned}
T &= g_{\mu\nu}T^{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}g^{\mu\nu} = -\frac{\Lambda}{8\pi G}\delta^{\mu}_{\mu} = -\frac{\Lambda}{8\pi G}(4) \\
T &= -\frac{\Lambda}{2\pi G}
\end{aligned}$$

For the gravitoelectric energy density we have

$$\rho_g = 2T_{tt} - \eta_{tt}T$$

For T_{tt} we have $T_{tt} = -\frac{\Lambda}{8\pi G} g_{tt}$ so

$$\begin{aligned}
\rho_g &= -2\frac{\Lambda}{8\pi G} g_{tt} + \frac{-\Lambda}{2\pi G} \\
&= -\frac{\Lambda}{4\pi G} g_{tt} - \frac{\Lambda}{2\pi G} \\
&= -\frac{\Lambda}{2\pi G}(\frac{1}{2} g_{tt} + 1) \\
&= -\frac{\Lambda}{2\pi G}(\frac{1}{2}(\eta_{tt} + h_{tt}) + 1) \\
&= -\frac{\Lambda}{2\pi G}(-\frac{1}{2} + \frac{h_{tt}}{2} + 1)
\end{aligned}$$

For the gravitomagnetic current density we have

$$\begin{aligned}
\Pi_i &= -T_{ti} + \frac{1}{2}\eta_{ti} T \\
&= \frac{\Lambda}{8\pi G} g_{ti} - \frac{1}{2}\eta_{ti}\frac{\Lambda}{2\pi G} \\
&= \frac{\Lambda}{4\pi G}(\frac{1}{2} g_{ti} - \eta_{ti}) \\
&= \frac{\Lambda}{4\pi G}(\frac{1}{2}(\eta_{ti} + h_{ti}) - \eta_{ti}) \\
&= \frac{\Lambda}{4\pi G}(-\frac{1}{2}\eta_{ti} + \frac{h_{ti}}{2})
\end{aligned}$$

For the curvature energy density we have

$$\begin{aligned}
\rho_c &= 2T_{ii} - \eta_{ii} T \\
&= -\frac{\Lambda}{4\pi G} g_{ii} - \eta_{ii}(-\frac{\Lambda}{2\pi G}) \\
&= -\frac{\Lambda}{4\pi G} g_{ii} + \eta_{ii}\frac{\Lambda}{2\pi G} \\
&= -\frac{\Lambda}{2\pi G}(\frac{1}{2} g_{ii} - \eta_{ii}) \\
&= -\frac{\Lambda}{2\pi G}(-\frac{1}{2}\eta_{ii} + \frac{h_{ii}}{2})
\end{aligned}$$

Problem 7. Moore, p. 263 Problem 22.6

a) In the stationary weak-field approximation we have the metric perturbation is

$$h_{\mu\nu} = 2\int_{src} \frac{G(2T_{\mu\nu} - \eta_{\mu\nu} T)}{|\vec{r} - \vec{r}_s|} d\vec{r}_s$$

Since the wire is on the z-axis and is infinitely long we have $\vec{r} = (x, y, 0)$ and $\vec{r}_s = (0, 0, z)$ so $|\vec{r} - \vec{r}_s| = \sqrt{x^2 + y^2 + z^2}$. Also $dV = dz$ and the integral is from $z = -\infty$ to $z = \infty$. So

$$h_{\mu\nu} = 2\int_{-\infty}^{\infty} \frac{G(2T_{\mu\nu} - \eta_{\mu\nu} T)}{\sqrt{x^2 + y^2 + z^2}} dz$$

For $\Phi_G = -h_{00}/2$,$T_{00} - \eta_{00} T = \rho_g = \lambda$ and the relevant integral is

$$h_{00} = 2G\lambda \int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dz$$

For $F_{kj} = \partial_k h_{tj} - \partial_j h_{tk}$. To first order $u_i = (1, 0, 0, V)$ and $\Pi_i = \lambda u_i$. The relevant integral is then

$$h_{tj} = 4G\lambda u_j \int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2+z^2}} dz$$

b) For the infinitely long wire on the z-axis, all points on a circle in a plane at some fixed z will experience the same $h_{\mu\nu}$, so $h_{\mu\nu}$ can't depend on z.
The gravitoelectric energy density is $\rho_g \approx \rho_0 + 3P_0 \approx \rho_0 = \lambda$. So, Φ_G is

$$\Phi_G = -G\lambda \int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2+z^2}} dz$$

The gravitomagnetic aspect F_{kj} is

$$F_{kj} = \partial_k(h_{tj}) - \partial_j(h_{tk})$$

Because of the u_j term in h_{tj}, the only nonzero components are when $j = t$ or $j = z$. Let $D = \sqrt{x^2+y^2+z^2}$. Let's compute the derivatives $\partial_k(h_{tt})$.

$$\begin{aligned}
\partial_t(h_{tt}) &= 0 \\
\partial_x(h_{tt}) &= 4G\lambda \frac{\partial}{\partial x}\Big(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2+z^2}} dz\Big) = 4G\lambda \frac{x}{D(D+z)} \\
\partial_y(h_{tt}) &= 4G\lambda \frac{\partial}{\partial y}\Big(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2+z^2}} dz\Big) = 4G\lambda \frac{y}{D(D+z)} \\
\partial_z(h_{tt}) &= 4G\lambda \frac{\partial}{\partial y}\Big(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2+z^2}} dz\Big) = \frac{4G\lambda}{D}
\end{aligned}$$

And the derivatives $\partial_k(h_{tz})$ are

$$\begin{aligned}
\partial_t(h_{tz}) &= 0 \\
\partial_x(h_{tz}) &= V\partial_x(h_{tt}) \\
\partial_y(h_{tz}) &= V\partial_y(h_{tt}) \\
\partial_z(h_{tz}) &= V\partial_z(h_{tt})
\end{aligned}$$

Since $\Phi_G = \frac{-h_{tt}}{2}$, $\partial_k(\Phi_G) = -\frac{1}{2}\partial_k(h_{tt})$, we have

$$\begin{aligned}
\partial_x(\Phi_G) &= -2G\lambda \frac{x}{D(D+z)} \\
\partial_y(\Phi_G) &= -2G\lambda \frac{y}{D(D+z)} \\
\partial_z(\Phi_G) &= -\frac{2G\lambda}{D}
\end{aligned}$$

Integrating the derivatives we get

$$\Phi_G = -2G\lambda \ln(D+z)$$

But this is not supposed to depend on z... And the z-derivative should always be 0...
For $F_{kj} = \partial_k(h_{tj}) - \partial_j(h_{tk})$, the only nonzero terms are F_{zz}, F_{zt}, and F_{tz}

$$\begin{aligned}
F_{zz} &= \partial_z(h_{tz}) - \partial_z(h_{tz}) = 0 \\
F_{zt} &= \partial_z(h_{tt}) - \partial_t(h_{tz}) = \frac{4G\lambda}{D} - 0 = \frac{4G\lambda}{D} \\
F_{tz} &= \partial_t(h_{tz}) - \partial_z(h_{tt}) = -F_{zt} = -\frac{4G\lambda}{D}
\end{aligned}$$

c) Find the $\eta^{ik}\partial_k\Phi_G$ and $\eta^{ik}F_{kj}v^j$ contributions to a test particle's acceleration moving with $\vec{v}$ where $v << 1$
Given the values above

$$\begin{aligned}
\eta^{1k}\partial_k\Phi_G &= \eta^{11}\partial_1\Phi_G = -2G\lambda\frac{x}{D(D+z)} \\
\eta^{2k}\partial_k\Phi_G &= \eta^{22}\partial_2\Phi_G = -2G\lambda\frac{y}{D(D+z)} \\
\eta^{3k}\partial_k\Phi_G &= \eta^{33}\partial_3\Phi_G = -\frac{2G\lambda}{D}
\end{aligned}$$

For the $\eta^{ik}F_{kj}v^i$ contributions $i = z$ for nonzero terms

$$\begin{aligned}
j = z \implies \quad \eta^{zk}F_{kz}v^z &= \eta^{zz}F_{zz}v^z = 0 \\
j = t \implies \quad \eta^{zk}F_{kt}v^z &= \eta^{zz}F_{zt}v^z = \frac{4G\lambda}{D}v^z
\end{aligned}$$

d) ...

Problem 8. Moore, p. 276 Problem 23.3
In empty space $R_{\mu\nu} = \Lambda g_{\mu\nu}$ which implies $R_{tt} = -A\Lambda$, $R_{rr} = B\Lambda$ and $R_{\theta\theta} = r^2\Lambda$.

a) Show $A = \frac{1}{B} = 1 + \frac{C}{r} - \frac{\Lambda}{3}r^2$

$$\frac{1}{A}\frac{\partial A}{\partial r} = -\frac{1}{B}\frac{\partial B}{\partial r}$$

Since $R_{\theta\theta} = r^2\Lambda$

$$\begin{aligned}
r^2\Lambda &= -\frac{r}{2AB}\frac{\partial A}{\partial r} + \frac{r}{2B^2}\frac{\partial B}{\partial r} + 1 - \frac{1}{B} \\
r^2\Lambda &= \frac{r}{2B^2}\frac{\partial B}{\partial r} + \frac{r}{2B^2}\frac{\partial B}{\partial r} + 1 - \frac{1}{B} = \frac{r}{B^2}\frac{\partial B}{\partial r} + 1 - \frac{1}{B} \\
\implies 1 &= r^2\Lambda - \frac{r}{B^2}\frac{\partial B}{\partial r} + \frac{1}{B} = r^2\Lambda + \frac{\partial}{\partial r}\left(\frac{r}{B}\right)
\end{aligned}$$

Integrating we get

$$\begin{aligned}
\int 1 dr &= \Lambda \int r^2 dr + \frac{r}{B} \\
r + C &= \frac{\Lambda r^3}{3} + \frac{r}{B} \\
1 + \frac{C}{r} &= \frac{\Lambda}{3}r^2 + \frac{1}{B} \\
\implies \frac{1}{B} &= 1 + \frac{C}{r} - \frac{\Lambda}{3}r^2
\end{aligned}$$

b) Assume that there is a large range of r such that $1 >> \frac{|C|}{r} >> \Lambda r^2$

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{\partial A}{\partial r}$$