PHY4370 ASSIGNMENT 3

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Problem 1. Problem 7.1. Consider a particle of mass μ confined within a box with impenetrable walls of sides L_1, L_2, and L_3. If $L_1 = L_2$, obtain the allowed energies and discuss the degeneracy of the first few energy levels.

The wavefunction of the particle confined in the box is (page 332)

$$\psi_{n_x n_y n_z}(x,y,z) = \Big(\frac{8}{L_1 L_2 L_3}\Big)^{1/2} \sin\Big(\frac{n_x \pi x}{L_1}\Big) \sin\Big(\frac{n_y \pi y}{L_2}\Big) \sin\Big(\frac{n_z \pi z}{L_3}\Big)$$

The allowed energies are

$$E_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2\mu}\Big(\frac{n_x^2}{L_1^2} + \frac{n_y^2}{L_2^2} + \frac{n_z^2}{L_3^2}\Big)$$

Now if two of the sides are the same $L_1 = L_2$ then the allowed energies become

$$\begin{aligned} E_{n_x n_y n_z} &= \frac{\hbar^2 \pi^2}{2\mu}\Big(\frac{n_x^2}{L_1^2} + \frac{n_y^2}{L_1^2} + \frac{n_z^2}{L_3^2}\Big) \\ &= \frac{\hbar^2 \pi^2}{2\mu}\Big(\frac{1}{L_1^2}(n_x^2 + n_y^2) + \frac{n_z^2}{L_3^2}\Big) \end{aligned}$$

This introduces degeneracy through the $n_x^2 + n_y^2$ term. The ground state energy is

$$E_{111} = \frac{\hbar^2 \pi^2}{2\mu}\Big(\frac{2}{L_1^2} + \frac{1}{L_3^2}\Big)$$

Another energy level is

$$\begin{aligned} E_{121} &= \frac{\hbar^2 \pi^2}{2\mu}\Big(\frac{1}{L_1^2}(1+4) + \frac{1}{L_3^2}\Big) \\ &= \frac{\hbar^2 \pi^2}{2\mu}\Big(\frac{5}{L_1^2} + \frac{1}{L_3^2}\Big) \\ &= E_{211} \end{aligned}$$

So this energy level is two-fold degenerate. Looking at a few more:

$E(n_x, n_y, n_z)$	Degeneracy
(1,1,1)	1
(1,1,2)	1
(1,2,1), (2,1,1)	2
(1,2,2), (2,1,2)	2
(2,2,1)	1
(2,2,2)	1
(1,1,3)	1
(1,3,1), (3,1,1)	2
(3,1,3), (1,3,3)	2
(3,3,1)	1
(1,2,3), (2,1,3)	2
(2,3,1), (3,2,1)	2
(3,1,2), (1,3,2)	2

I cannot find an example where the degeneracy is more than 2.

Problem 2. Problem 7.16

Using the generating function

$$U_P(\rho, s) = \frac{(-s)^P}{(1-s)^{P+1}} \exp\Big(-\rho \frac{s}{1-s}\Big) = \sum_{q=P}^{\infty} \frac{s^q}{q!} L_q^P(\rho) \qquad |s| < 1$$

and the radial wave function of the hydrogenic atom

$$R_{nl}(r) = -\left[\Big(\frac{2Z}{na_\mu}\Big)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}\right]^{1/2} e^{-\rho/2}\rho^l L_{n+l}^{2l+1}(\rho)$$

where $\rho = \frac{2Z}{na_\mu} r$ and $a_\mu = \frac{(4\pi\epsilon_0)\hbar^2}{\mu e^2}$, find the average values of

$$\langle r^k \rangle_{nlm} = \int \psi_{nlm}^*(\mathbf{r}) r^k \psi_{nlm} d\mathbf{r} = \int_0^\infty |R_{nl}(r)|^2 r^{k+2} dr$$

for $k = 1, -1, -2$ and -3.

Start with

$$\langle r^k \rangle = \int_0^\infty |R_{nl}(\rho)|^2 r^{k+2} dr$$

Since

$$r = \frac{na_\mu}{2Z}\rho \implies dr = \frac{na_\mu}{2Z} d\rho$$

and

$$r^{k+2} = \Big(\frac{na_\mu}{2Z}\Big)^{k+2} \rho^{k+2}$$

The average value of r^k is

$$\begin{aligned}
\langle r^k \rangle &= \Big(\frac{na_\mu}{2Z}\Big)^{k+3} \int_0^\infty |R_{nl}(\rho)|^2 \rho^{k+2} d\rho \\
&= \Big(\frac{na_\mu}{2Z}\Big)^{k+3} \Big(\frac{2Z}{na_\mu}\Big)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3} \int_0^\infty e^{-\rho}\rho^{2l+k+2} \Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
&= \Big(\frac{na_\mu}{2Z}\Big)^{k} \frac{(n-l-1)!}{2n[(n+l)!]^3} \int_0^\infty \rho^{2l+k+2} e^{-\rho} \Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho
\end{aligned}$$

Now we're going to need to solve the integral. We don't do it directly; we use the generating function to investigate in general integrals of the form $\int \rho^{2l+k+2} e^{-\rho} L_q^{2l+1}(\rho) L_{\tilde{q}}^{2l+1}(\rho) d\rho$ and then look at our special case of $q = \tilde{q} = n + l$.

Let

$$\begin{aligned}
U_{2l+1}(\rho, s) &= \frac{(-s)^{2l+1}}{(1-s)^{2l+2}} \exp\Big(-\rho \frac{s}{1-s}\Big) = \sum_{q=2l+1}^{\infty} \frac{s^q}{q!} L_q^{2l+1}(\rho) \\
U_{2l+1}(\rho, t) &= \frac{(-t)^{2l+1}}{(1-t)^{2l+2}} \exp\Big(-\rho \frac{t}{1-t}\Big) = \sum_{\tilde{q}=2l+1}^{\infty} \frac{t^{\tilde{q}}}{\tilde{q}!} L_{\tilde{q}}^{2l+1}(\rho)
\end{aligned}$$

Now consider the integral

$$\int_0^\infty \rho^{2l+k+2} e^{-\rho} U_{2l+1}(\rho, s) U_{2l+1}(\rho, t) d\rho = \sum_{q=2l+1}^{\infty} \sum_{\tilde{q}=2l+1}^{\infty} \frac{s^q t^{\tilde{q}}}{q!\tilde{q}!} \int_0^\infty \rho^{2l+k+2} e^{-\rho} L_q^{2l+1}(\rho) L_{\tilde{q}}^{2l+1}(\rho) d\rho$$

Instead of substituting the series we can substitute the exponentials and get

$$\begin{aligned}\int_0^\infty \rho^{2l+k+2}e^{-\rho}U_{2l+1}(\rho,s)U_{2l+1}(\rho,t)d\rho &= \frac{(st)^{2l+1}}{[(1-s)(1-t)]^{2l+2}}\int_0^\infty \rho^{2l+k+2}e^{-\rho}\exp\Big(-\rho(\frac{s}{1-s}+\frac{t}{1-t})\Big)d\rho \\ &= \frac{(st)^{2l+1}}{[(1-s)(1-t)]^{2l+2}}\int_0^\infty \rho^{2l+k+2}\exp\Big(-\rho\Big(1+\frac{s}{1-s}+\frac{t}{1-t}\Big)\Big)d\rho\end{aligned}$$

Simplifying the term in the exponential,

$$\begin{aligned}1+\frac{s}{1-s}+\frac{t}{1-t} &= 1+\frac{(1-t)s+(1-s)t}{(1-s)(1-t)} = 1+\frac{s+t-2ts}{(1-s)(1-t)} \\ &= \frac{(1-s)(1-t)+s+t-2ts}{(1-s)(1-t)} = \frac{1-t-s+st+s+t-2ts}{(1-s)(1-t)} \\ &= \frac{(1-st)}{(1-s)(1-t)}\end{aligned}$$

So the integral is

$$\int_0^\infty \rho^{2l+k+2}e^{-\rho}U_{2l+1}(\rho,s)U_{2l+1}(\rho,t)d\rho = \frac{(st)^{2l+1}}{[(1-s)(1-t)]^{2l+2}}\int_0^\infty \rho^{2l+k+2}\exp\Big(-\rho\frac{(1-st)}{(1-s)(1-t)}\Big)d\rho$$

Now we notice that this is of the form of the Laplace transform $F(x)=\int_0^\infty \rho^n e^{-\rho x}d\rho = \frac{n!}{x^{n+1}}$ with $x=\frac{(1-st)}{(1-s)(1-t)}$. So

$$\int_0^\infty \rho^{2l+k+2}\exp\Big(-\rho\frac{(1-st)}{(1-s)(1-t)}\Big)d\rho = \frac{(2l+k+2)![(1-s)(1-t)]^{2l+k+3}}{(1-st)^{2l+k+3}}$$

Substituting this back in

$$\begin{aligned}\int_0^\infty \rho^{2l+k+2}e^{-\rho}U_{2l+1}(\rho,s)U_{2l+1}(\rho,t)d\rho &= \frac{(st)^{2l+1}}{[(1-s)(1-t)]^{2l+2}}\frac{(2l+k+2)![(1-s)(1-t)]^{2l+k+3}}{(1-st)^{2l+k+3}} \\ &= \frac{(2l+k+2)![(1-s)(1-t)]^{k+1}(st)^{2l+1}}{(1-st)^{2l+k+3}}\end{aligned}$$

In class we showed that $\frac{1}{(1-x)^{m+1}} = \sum_{q=0}^\infty \frac{(m+q)!}{q!m!}x^q$. So we'll use this to expand the term in the denominator into a series:

$$\begin{aligned}\int_0^\infty \rho^{2l+k+2}e^{-\rho}U_{2l+1}(\rho,s)U_{2l+1}(\rho,t)d\rho &= (2l+k+2)![(1-s)(1-t)]^{k+1}(st)^{2l+1}\frac{1}{(1-st)^{2l+k+3}} \\ &= (2l+k+2)![(1-s)(1-t)]^{k+1}(st)^{2l+1}\sum_{q=0}^\infty \frac{(2l+k+2+q)!}{q!(2l+k+2)!}(st)^q \\ &= [(1-s)(1-t)]^{k+1}(st)^{2l+1}\sum_{q=0}^\infty \frac{(2l+k+2+q)!}{q!}(st)^q \\ &= [(1-s)(1-t)]^{k+1}\sum_{q=0}^\infty \frac{(2l+k+2+q)!}{q!}(st)^{2l+1+q}\end{aligned}$$

Now equate the two series expansions of the integral and we see

$$[(1-s)(1-t)]^{k+1}\sum_{q=0}^\infty \frac{(2l+k+2+q)!}{q!}(st)^{2l+1+q} = \sum_{q=2l+1}^\infty\sum_{\tilde q=2l+1}^\infty \frac{s^q t^{\tilde q}}{q!\tilde q!}\int_0^\infty \rho^{2l+k+2}e^{-\rho}L_q^{2l+1}(\rho)L_{\tilde q}^{2l+1}(\rho)d\rho$$

We only care about term where $\Big(L_{n+l}^{2l+1}(\rho)\Big)^2$ shows up, which means, on the right side, $q=\tilde q=n+l$. On the left side, q is any value that makes a $(st)^{n+l}$ term.

For $k=1$

$$[(1-s)(1-t)]^2\frac{(2l+3+q)!}{q!}(st)^{2l+1+q} \quad = \quad (1-2t+t^2-2s+4st-2st^2+s^2-2ts^2+(st)^2)\frac{(2l+3+q)!}{q!}(st)^{2l+1+q}$$

To get a $(st)^{n+l}$ term, q can be $(n+l)-2l-1$, or $(n+l)-2l-2$, or $(n+l)-2l-3$.

When $q=n+l-2l-1=n-l-1$ the coefficient of $(st)^{n+l}$ is

$$\frac{(2l+3+q)!}{q!}=\frac{(n+l+2)!}{(n-l-1)!}$$

When $q=n+l-2l-2=n-l-2$ the coefficient is

$$4\frac{(2l+3+q)!}{q!}=4\frac{(n+l+1)!}{(n-l-2)!}$$

When $q=n+l-2l-3=n-l-3$ the coefficient is

$$\frac{(2l+3+q)!}{q!}=\frac{(n+l)!}{(n-l-3)!}$$

So the coefficient of $(st)^{n+l}$ on the left side is

$$\begin{aligned}
\frac{(n+l+2)!}{(n-l-1)!}+4\frac{(n+l+1)!}{(n-l-2)!}+\frac{(n+l)!}{(n-l-3)!} & \\
&= \tfrac{(n+l)!}{(n-l-3)!}\left(\tfrac{(n+l+2)(n+l+1)}{(n-l-1)(n-l-2)}+4\tfrac{(n+l+1)}{(n-l-2)}+1\right)\\
&= \tfrac{(n+l)!}{(n-l-3)!}\left(\tfrac{(n+l+1)}{(n-l-2)}\left[\tfrac{(n+l+2)}{(n-l-1)}+4\right]+1\right)\\
&= \tfrac{(n+l)!}{(n-l-3)!}\left(\tfrac{(n+l+1)}{(n-l-2)}\left[\tfrac{(n+l+2+4n-4l-4)}{(n-l-1)}\right]+1\right)\\
&= \tfrac{(n+l)!}{(n-l-3)!}\left(\tfrac{(n+l+1)}{(n-l-2)}\tfrac{(5n-3l-2)}{(n-l-1)}+1\right)\\
&= \tfrac{(n+l)!}{(n-l-1)!}\Big((n+l+1)(5n-3l-2)+(n-l-2)(n-l-1)\Big)\\
&= \tfrac{(n+l)!}{(n-l-1)!}\Big(5n^2+2ln+3n-3l^2-5l-2+n^2-3n-2ln+l^2+3l+2\Big)\\
&= \tfrac{(n+l)!}{(n-l-1)!}(6n^2-2l^2-2l)\\
&= \tfrac{(n+l)!}{(n-l-1)!}(6n^2-2l(l+1))
\end{aligned}$$

Now, equating powers on both sides

$$\begin{aligned}
\frac{(n+l)!}{(n-l-1)!}(6n^2-2l(l+1))(st)^{n+l} &= \frac{(st)^{n+l}}{[(n+l)!]^2}\int_0^\infty \rho^{2l+3}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2d\rho\\
\int_0^\infty \rho^{2l+3}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2d\rho &= \frac{[(n+l)!]^3}{(n-l-1)!}(6n^2-2l(l+1))
\end{aligned}$$

So

$$
\begin{aligned}
\langle r\rangle &= \Big(\frac{na_\mu}{2Z}\Big)\frac{(n-l-1)!}{2n[(n+l)!]^3}\int_0^\infty \rho^{2l+3}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
&= \Big(\frac{na_\mu}{2Z}\Big)\frac{(n-l-1)!}{2n[(n+l)!]^3}\frac{[(n+l)!]^3}{(n-l-1)!}(6n^2-2l(l+1)) \\
&= \Big(\frac{a_\mu}{2Z}\Big)\frac{(6n^2-2l(l+1))}{2} \\
&= \Big(\frac{a_\mu}{2Z}\Big)(3n^2-l(l+1)) \\
&= \frac{a_\mu n^2}{Z}\Big(\frac{3}{2}-\frac{l(l+1)}{2n^2}\Big) \\
&= \frac{a_\mu n^2}{Z}\Big(1+\frac{1}{2}\Big(1-\frac{l(l+1)}{n^2}\Big)\Big) \qquad (\frac{3}{2}=1+\frac{1}{2})
\end{aligned}
$$

For $k=-1$

$$
[(1-s)(1-t)]^0\sum_{q=0}^\infty\frac{(2l+1+q)!}{q!}(st)^{2l+1+q} = \sum_{q=2l+1}^\infty\sum_{\tilde q=2l+1}^\infty \frac{s^q t^{\tilde q}}{q!\tilde q!}\int_0^\infty \rho^{2l+1}e^{-\rho}L_q^{2l+1}(\rho)L_{\tilde q}^{2l+1}(\rho)d\rho
$$

Equate powers and focus on $(st)^{n+l}$

$$
\frac{(2l+1+q)!}{q!}(st)^{2l+1+q} = \frac{(st)^{n+l}}{[(n+l)!]^2}\int_0^\infty \rho^{2l+1}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho
$$

$q=n+l-2l-1=n-l-1$.

$$
\begin{aligned}
\frac{(2l+1+n-l-1)!}{(n-l-1)!}(st)^{n+l} &= \frac{(st)^{n+l}}{[(n+l)!]^2}\int_0^\infty \rho^{2l+1}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
\frac{(n+l)!}{(n-l-1)!}(st)^{n+l} &= \frac{(st)^{n+l}}{[(n+l)!]^2}\int_0^\infty \rho^{2l+1}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
\int_0^\infty \rho^{2l+1}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho &= \frac{[(n+l)!]^3}{(n-l-1)!}
\end{aligned}
$$

Now

$$
\begin{aligned}
\langle r^{-1}\rangle &= \Big(\frac{na_\mu}{2Z}\Big)^{-1}\frac{(n-l-1)!}{2n[(n+l)!]^3}\int_0^\infty \rho^{2l+1}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
&= \frac{2Z}{na_\mu}\frac{(n-l-1)!}{2n[(n+l)!]^3}\frac{[(n+l)!]^3}{(n-l-1)!} \\
&= \frac{2Z}{na_\mu}\frac{1}{2n} \\
&= \frac{Z}{a_\mu n^2}
\end{aligned}
$$

For $k=-2$

$$
\begin{aligned}
[(1-s)(1-t)]^{-2+1}\sum_{q=0}^\infty\frac{(2l-2+2+q)!}{q!}(st)^{2l+1+q} &= \sum_{q=2l+1}^\infty\sum_{\tilde q=2l+1}^\infty \frac{s^q t^{\tilde q}}{q!\tilde q!}\int_0^\infty \rho^{2l-2+2}e^{-\rho}L_q^{2l+1}(\rho)L_{\tilde q}^{2l+1}(\rho)d\rho \\
[(1-s)(1-t)]^{-1}\sum_{q=0}^\infty\frac{(2l+q)!}{q!}(st)^{2l+1+q} &= \sum_{q=2l+1}^\infty\sum_{\tilde q=2l+1}^\infty \frac{s^q t^{\tilde q}}{q!\tilde q!}\int_0^\infty \rho^{2l}e^{-\rho}L_q^{2l+1}(\rho)L_{\tilde q}^{2l+1}(\rho)d\rho
\end{aligned}
$$

Find the coefficient of $(st)^{n+l}$ on the left side. $q = n+l-2l-1 = n-l-1$

$$\begin{aligned}
\frac{(2l+n-l-1)!}{(n-l-1)!}(st)^{n+l} &= \frac{(st)^{n+l}}{[(n+l)!]^2}\int_0^\infty \rho^{2l}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
\frac{(n+l-1)![(n+l)!]^2}{(n-l-1)!} &= \int_0^\infty \rho^{2l}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho
\end{aligned}$$

$$\begin{aligned}
\langle r^{-2}\rangle &= \left(\frac{na_\mu}{2Z}\right)^k \frac{(n-l-1)!}{2n[(n+l)!]^3}\int_0^\infty \rho^{2l+k+2}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
&= \left(\frac{na_\mu}{2Z}\right)^{-2} \frac{(n-l-1)!}{2n[(n+l)!]^3}\int_0^\infty \rho^{2l}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
&= \frac{4Z^2}{a_\mu^2 n^2}\frac{(n-l-1)!}{2n[(n+l)!]^3}\frac{(n+l-1)![(n+l)!]^2}{(n-l-1)!} \\
&= \frac{2Z^2}{a_\mu^2 n^3}\frac{(n+l-1)!}{(n+l)!} \\
&= \frac{2Z^2}{a_\mu^2 n^3}\frac{1}{(n+l)}
\end{aligned}$$

Which is not quite right.

<u>For $k=-3$</u>

$$\begin{aligned}
[(1-s)(1-t)]^{-3+1}\sum_{q=0}^{\infty}\frac{(2l-3+2+q)!}{q!}(st)^{2l+1+q} &= \sum_{q=2l+1}^{\infty}\sum_{\tilde q=2l+1}^{\infty}\frac{s^q t^{\tilde q}}{q!\tilde q!}\int_0^\infty \rho^{2l-3+2}e^{-\rho}L_q^{2l+1}(\rho)L_{\tilde q}^{2l+1}(\rho)d\rho \\
[(1-s)(1-t)]^{-2}\sum_{q=0}^{\infty}\frac{(2l-1+q)!}{q!}(st)^{2l+1+q} &= \sum_{q=2l+1}^{\infty}\sum_{\tilde q=2l+1}^{\infty}\frac{s^q t^{\tilde q}}{q!\tilde q!}\int_0^\infty \rho^{2l-1}e^{-\rho}L_q^{2l+1}(\rho)L_{\tilde q}^{2l+1}(\rho)d\rho
\end{aligned}$$

$q = n-l-1$.

$$\begin{aligned}
\frac{(2l-1+n-l-1)!}{(n-l-1)!}(st)^{n+l} &= \frac{(st)^{n+l}}{[(n+l)!]^2}\int_0^\infty \rho^{2l-1}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
\frac{(n+l-2)![(n+l)!]^2}{(n-l-1)!} &= \int_0^\infty \rho^{2l-1}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho
\end{aligned}$$

$$\begin{aligned}
\langle r^{-3}\rangle &= \left(\frac{na_\mu}{2Z}\right)^{-3} \frac{(n-l-1)!}{2n[(n+l)!]^3}\int_0^\infty \rho^{2l-3+2}e^{-\rho}\Big(L_{n+l}^{2l+1}(\rho)\Big)^2 d\rho \\
&= \frac{8Z^3}{n^3 a_\mu^3}\frac{(n-l-1)!}{2n[(n+l)!]^3}\frac{(n+l-2)![(n+l)!]^2}{(n-l-1)!} \\
&= \frac{8Z^3}{n^3 a_\mu^3}\frac{(n+l-2)!}{2n(n+l)!} \\
&= \frac{8Z^3}{n^3 a_\mu^3}\frac{1}{2n(n+l)(n+l-1)} \\
&= \frac{4Z^3}{n^4 a_\mu^3}\frac{1}{(n+l)(n+l-1)}
\end{aligned}$$

which is not quite right either.

Problem 3. Problem 7.20 A two dimensional harmonic oscillator has the Hamiltonian

$$H = -\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2$$

(a) Find the energy levels

The Hamiltonian can be written as

$$\begin{aligned} H &= \Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k_1x^2\Big) + \Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}k_2y^2\Big) \\ &= H_x + H_y \end{aligned}$$

Substituting into Schrödinger we see

$$\begin{aligned} H\psi &= E\psi \\ \left[\Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k_1x^2\Big) + \Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}k_2y^2\Big)\right]\psi &= E\psi \end{aligned}$$

Since each term in brackets on the left side is just a function of x or y, respectively, and the right side E is a constant, E must be a sum of constants

$$\left[\Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k_1x^2\Big) + \Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}k_2y^2\Big)\right]\psi = (E_x + E_y)\psi$$

The solution is a product of functions of a single variable $\psi(x,y) = \psi_x(x)\psi_y(y)$:

$$\begin{aligned} \left[\Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k_1x^2\Big) + \Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}k_2y^2\Big)\right]\psi_x\psi_y &= (E_x + E_y)\psi_x\psi_y \\ \Big(-\frac{\hbar^2}{2\mu}\psi_y\frac{\partial^2\psi_x}{\partial x^2} + \frac{1}{2}k_1x^2\psi_x\psi_y\Big) + \Big(-\frac{\hbar^2}{2\mu}\psi_x\frac{\partial^2\psi_y}{\partial y^2} + \frac{1}{2}k_2y^2\Big) &= E_x\psi_x\psi_y + E_y\psi_x\psi_y \end{aligned}$$

Divide both sides by ψ

$$\Big(-\frac{\hbar^2}{2\mu}\frac{1}{\psi_x}\frac{\partial^2\psi_x}{\partial x^2} + \frac{1}{2}k_1x^2\Big) + \Big(-\frac{\hbar^2}{2\mu}\frac{1}{\psi_y}\frac{\partial^2\psi_y}{\partial y^2} + \frac{1}{2}k_2y^2\Big) = E_x + E_y$$

This gives us two ordinary differential equations

$$\begin{aligned} \Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k_1x^2\Big)\psi_x &= E_x\psi_x \qquad \text{(multiply by } \psi_x) \\ \Big(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}k_2y^2\Big)\psi_y &= E_y\psi_y \qquad \text{(multiply by } \psi_y) \end{aligned}$$

Each of these are linear harmonic oscillators in one dimension, whose solution is given by Section 4.7

$$\begin{aligned} \psi_x &= \psi_{n_x}(x) = \Big(\frac{\alpha_1}{\sqrt{\pi}2^{n_x}n_x!}\Big)^{1/2} e^{-\alpha_1^2x^2}H_{n_x}(\alpha_1 x) \\ \psi_y &= \psi_{n_y}(y) = \Big(\frac{\alpha_2}{\sqrt{\pi}2^{n_y}n_y!}\Big)^{1/2} e^{-\alpha_2^2y^2}H_{n_y}(\alpha_2 y) \end{aligned}$$

Where n_x and n_y are integers and

$$\alpha_1 = \Big(\frac{\mu k_1}{\hbar^2}\Big)^{1/4} \qquad \alpha_2 = \Big(\frac{\mu k_2}{\hbar^2}\Big)^{1/4}$$

The energy levels corresponding to ψ_{n_x} and ψ_{n_y} are also given:

$$E_x = (n_x + \frac{1}{2})\hbar\omega_1 \qquad E_y = (n_y + \frac{1}{2})\hbar\omega_2$$

So the energy levels of the 2d system are

$$\begin{aligned} E &= E_x + E_y \\ &= (n_x + \frac{1}{2})\hbar\omega_1 + (n_y + \frac{1}{2})\hbar\omega_2 \end{aligned}$$

(b) Assuming the oscillator is isotropic ($k_1 = k_2 = k$) what is the degeneracy of each energy level?

Since $k_1 = k_2 = k$, $\omega_1 = \omega_2 = \omega$. The energy levels are

$$\begin{aligned} E_{n_x n_y} &= (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega \\ &= \hbar\omega(n_x + n_y + 1) \end{aligned}$$

If $n = n_x + n_y$ then when $n = 0$, $(n_x, n_y) = (0, 0)$ and the energy level is non-degenerate. When $n = 1$, the combinations $(n_x, n_y) = (1, 0)$, $(0, 1)$ would give the same energy level so the energy level is 2-fold degenerate. When $n = 2$, $(n_x, n_y) = (2, 0)$, $(0, 2)$, $(1, 1)$ give the same energy level and it is 3-fold degenerate. When $n = 3$, $(n_x, n_y) = (3, 0)$, $(0, 3)$, $(2, 1)$, $(1, 2)$ give the same energy level so it is 4-fold degenerate.

In general, $E_{n_x} E_{n_y}$ is $(n + 1) = (n_x + n_y + 1)$-fold degenerate.

(c) Solve the Schrödinger equation for the two-dimensional isotropic oscillator in plane polar coordinates (r, φ).

The Laplacian in these coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

The potential is

$$V(r) = \frac{1}{2}kr^2 = \frac{1}{2}\mu\omega^2 r^2$$

Where $\omega = (k/\mu)^{1/2}$. The Hamiltonian is

$$\begin{aligned} H &= -\frac{\hbar^2}{2\mu}\nabla^2 + V(r) = -\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega^2 r^2 \\ &= -\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right] + \frac{1}{2}\mu\omega^2 r^2 \end{aligned}$$

So the Schrödinger equation for the two dimensional isotropic oscillator in these coordinates is

$$\left(-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right] + \frac{1}{2}\mu\omega^2 r^2 \right)\psi(r, \varphi) = E\psi(r, \varphi)$$

According to the book the solution to this is separable so

$$\psi(r, \varphi) = R_{El}(r)\Phi(\varphi)$$

The $\Phi(\varphi)$ function is the same as the one from the spherical harmonic function $Y_{lm}(\theta, \varphi) = \Theta_{lm}(\theta)\Phi_m(\phi)$ so

$$\Phi(\varphi) = \Phi_m(\varphi) = e^{im\varphi}$$

with some normalization constant. So the solution $\psi(r, \varphi)$ has the form

$$\psi_{lm} = R_{El}(r)e^{im\varphi}$$

The radial part R_{El} of the wavefunction must satisfy (page 337)

$$\left(- \frac{\hbar^2}{2\mu}\Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\Big) + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega^2 r^2 \right) R_{El}(r) = E R_{El}(r)$$

Let $u_{El}(r) = rR_{El}(r)$. The radial equation for u_{El} is

$$\begin{aligned}
-\frac{\hbar^2}{2\mu}\frac{d^2 u_{El}}{dr^2} + \Big(\frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega^2 r^2\Big)u_{El} &= E u_{El} \\
\Big(- \frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + \frac{1}{2}\mu\omega^2 r^2 - E\Big)u_{El} &= 0 \\
\Big(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{\mu^2\omega^2 r^2}{\hbar^2} + \frac{2\mu E}{\hbar^2}\Big)u_{El} &= 0
\end{aligned}$$

Let $\rho = \alpha r$, $\lambda = \frac{2E}{\hbar\omega}$, and $\alpha = \left(\frac{\mu k}{\hbar^2}\right)^{1/4} = \left(\frac{\mu\omega}{\hbar}\right)^{1/2}$. Then

$$\begin{aligned}
d\rho &= \alpha dr \implies dr = \frac{d\rho}{\alpha} \implies dr^2 = \frac{d\rho^2}{\alpha^2} \\
r &= \frac{\rho}{\alpha}
\end{aligned}$$

So

$$\begin{aligned}
\Big(\alpha^2\frac{d^2}{d\rho^2} - \frac{l(l+1)\alpha^2}{\rho^2} - \frac{\mu^2\omega^2\rho^2}{\hbar^2\alpha^2} + \frac{2\mu E}{\hbar^2}\Big)u_{El} &= 0 \\
\Big(\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \frac{\mu^2\omega^2\rho^2}{\hbar^2\alpha^4} + \frac{2\mu E}{\hbar^2\alpha^2}\Big)u_{El} &= 0 \\
\Big(\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \frac{\mu^2\omega^2\rho^2\hbar^2}{\hbar^2\mu k} + \frac{2\mu E\hbar}{\hbar^2\sqrt{\mu k}}\Big)u_{El} &= 0 \\
\Big(\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \frac{\mu\omega^2\rho^2}{k} + \frac{2E}{\hbar\omega}\Big)u_{El} &= 0 \\
\Big(\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \rho^2 + \lambda\Big)u_{El} &= 0
\end{aligned}$$

As $\rho \to \infty$, two of the terms become very small compared to ρ^2 and we get

$$\Big(\frac{d^2}{d\rho^2} - \rho^2\Big)u_{El} = 0$$

The solutions to this are of the form $u_{El}(\rho) = e^{\pm\rho^2/2}f(\rho)$, but we don't want u_{El} to explode so we take the negative sign. $u_{El}(\rho) = e^{-\rho^2/2}f(\rho)$.

Substitute this back into the unitless radial equation for u_{El}:

$$\Big(\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \rho^2 + \lambda\Big)e^{-\rho^2/2}f(\rho) = 0$$

???

Suppose $f(\rho)$ is of the form $f(\rho) = \rho^{|m|+1}g(\rho)$ and $g(\rho)$ must satisfy

$$\frac{d^2 g}{d\rho^2} + \Big(\frac{2|m|+1}{\rho} - 2\rho\Big)\frac{dg}{d\rho} - [2(|m|+1) - \lambda]g = 0$$

Then $u_{El} = e^{-\rho^2/2}\rho^{|m|+1}g(\rho)$ and so

$$\begin{aligned} R_{El}(\rho) &= \frac{u_{El}}{r} = \alpha\frac{u_{El}}{\rho} = \alpha e^{-\rho^2/2}\rho^{|m|}g(\rho) \\ \psi_{lm} &= R_{El}(\rho)e^{im\varphi} \\ &= \alpha e^{-\rho^2/2}\rho^{|m|}g(\rho)e^{im\varphi} \end{aligned}$$

Let $v = \rho^2$. Then $dv = 2\rho d\rho$ and $d\rho = \frac{dv}{2\rho} = \frac{dv}{2\sqrt{v}}$. Substitute into the equation on $g(\rho)$:

$$\begin{aligned} 4v\frac{d^2g}{dv^2} + \Big(\frac{2|m|+1}{v^{1/2}} - 2v^{1/2}\Big)2v^{1/2}\frac{dg}{dv} - [2(|m|+1) - \lambda]g &= 0 \\ 4v\frac{d^2g}{dv^2} + 2(2|m|+1-2v)\frac{dg}{dv} - [2(|m|+1) - \lambda]g &= 0 \\ 4v\frac{d^2g}{dv^2} + 4(|m|+\frac{1}{2}-v)\frac{dg}{dv} - 2[(|m|+1) - \frac{\lambda}{2}]g &= 0 \\ v\frac{d^2g}{dv^2} + (|m|+\frac{1}{2}-v)\frac{dg}{dv} - \frac{1}{2}[(|m|+1) - \frac{\lambda}{2}]g &= 0 \end{aligned}$$

The radial quantization condition is

$$\begin{aligned} n_r &= -\frac{1}{2}\Big((|m|+1) - \frac{\lambda}{2}\Big) = 0, 1, 2... \\ -2n_r &= |m| + 1 - \frac{\lambda}{2} \\ \frac{\lambda}{2} &= 2n_r + |m| + 1 \end{aligned}$$

Let $n \equiv 2n_r + |m|$

$$\lambda = 2(n+1)$$

Since $\lambda = \frac{2E}{\hbar\omega}$

$$\begin{aligned} \frac{2E}{\hbar\omega} &= 2(n+1) \\ E_n &= \hbar\omega(n+1) \end{aligned}$$

These energy levels are $(n+1)(n+2)/2$-fold degenerate.

(d) The overall wave function is

$$\psi_{nm}(r,\varphi) = N\rho^{|m|}e^{-\rho^2/2}e^{im\varphi}L^{|m|}_{n_r+|m|}(\rho^2) = N\rho^{|m|}e^{-\rho^2/2}e^{im\varphi}L^{|m|}_{(n+|m|)/2}(\rho^2)$$

We want to find the normalization constant N.

$$1 = \int |\psi(r,\varphi)|^2 d\mathbf{r}$$

Ignore the radial part.

$$\begin{aligned} 1 &= \int |N|^2\rho^{2|m|}e^{-\rho^2}\left(L^{|m|}_{(n+|m|)/2}(\rho^2)\right)^2 r^2 dr \\ &= \frac{|N|^2}{\alpha^3}\int_0^\infty \rho^{2|m|}e^{-\rho^2}\left(L^{|m|}_{(n+|m|)/2}(\rho^2)\right)^2\rho^2 d\rho \\ &= \frac{|N|^2}{\alpha^3}\int_0^\infty \rho^{2|m|+2}e^{-\rho^2}\left(L^{|m|}_{(n+|m|)/2}(\rho^2)\right)^2 d\rho \end{aligned}$$

Let

$$
\begin{aligned}
U_{|m|}(\rho^2, s) &= \frac{(-s)^{|m|}}{(1-s)^{|m|+1}} \exp\Big(- \rho^2 \frac{s}{1-s}\Big) = \sum_{q=|m|}^{\infty} \frac{s^q}{q!} L_q^{|m|}(\rho^2) \\
U_{|m|}(\rho^2, t) &= \frac{(-t)^{|m|}}{(1-t)^{|m|+1}} \exp\Big(- \rho^2 \frac{t}{1-t}\Big) = \sum_{\tilde{q}=|m|}^{\infty} \frac{t^{\tilde{q}}}{\tilde{q}!} L_{\tilde{q}}^{|m|}(\rho^2)
\end{aligned}
$$

Now

$$
\int_0^{\infty} \rho^{2|m|+2} e^{-\rho^2} U_{|m|}(\rho^2, s) U_{|m|}(\rho^2, t) d\rho
$$

Then do the same thing as before goodnight