PHY4370 ASSIGNMENT 2

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Problem 1. Problem 6.22 $\chi_{0,0} = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) - \beta(1)\alpha(2)]$ for singlet and $\chi_{1,1} = \alpha(1)\alpha(2)$, $\chi_{1,0} = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) + \beta(1)\alpha(2)]$, $\chi_{1,-1} = \beta(1)\beta(2)$ for triplets.

First the singlet state $|0,0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$. Since $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ and $\mathbf{S}_1$ and $\mathbf{S}_2$ commute,

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2)$$

So

$$\begin{aligned}
\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle &= \langle 0,0|\mathbf{S}_1 \cdot \mathbf{S}_2|0,0\rangle \\
&= \frac{1}{2}\langle 0,0|\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2|0,0\rangle \\
&= \frac{1}{2}\langle 0,0|\mathbf{S}^2|0,0\rangle - \frac{1}{2}\langle 0,0|\mathbf{S}_1^2|0,0\rangle - \frac{1}{2}\langle 0,0|\mathbf{S}_2^2|0,0\rangle
\end{aligned}$$

Looking at each part individually

$$\begin{aligned}
\mathbf{S}_1^2|0,0\rangle &= \frac{1}{\sqrt{2}}\Big(\mathbf{S}_1^2|\uparrow\downarrow\rangle - \mathbf{S}_1^2|\downarrow\uparrow\rangle\Big) \qquad (\mathbf{S}_1^2 \text{ only affects first particle}) \\
&= \frac{1}{\sqrt{2}}\Big(\frac{3}{4}\hbar^2|\uparrow\downarrow\rangle - \frac{3}{4}\hbar^2|\downarrow\uparrow\rangle\Big) \\
&= \frac{3}{4}\hbar^2|0,0\rangle \\
\langle 0,0|\mathbf{S}_1^2|0,0\rangle &= \frac{3}{4}\hbar^2\langle 0,0|0,0\rangle = \frac{3}{4}\hbar^2
\end{aligned}$$

Cool. Rinse, repeat

$$\langle 0,0|\mathbf{S}_2^2|0,0\rangle = \frac{3}{4}\hbar^2\langle 0,0|0,0\rangle = \frac{3}{4}\hbar^2$$

Last one. The total spin number is $S = 0$. So

$$\langle 0,0|\mathbf{S}^2|0,0\rangle = S(S+1)\hbar^2\langle 0,0|0,0\rangle = 0$$

Therefore

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \frac{1}{2}(0) - \frac{1}{2}(\frac{3}{4}\hbar^2) - \frac{1}{2}(\frac{3}{4}\hbar^2) = -\frac{3}{4}\hbar^2$$

Now we do the triplet states. The total spin number is $S = 1$. The states are

$$\begin{aligned}
|1,1\rangle &= |\uparrow\uparrow\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] \\
|1,-1\rangle &= |\downarrow\downarrow\rangle
\end{aligned}$$

The first one we'll calculate is $\langle 1,1|\mathbf{S}_1\cdot\mathbf{S}_2|1,1\rangle$:

$$\begin{aligned}
\langle 1,1|\mathbf{S}_1\cdot\mathbf{S}_2|1,1\rangle &= \frac{1}{2}\langle 1,1|\mathbf{S}^2|1,1\rangle - \frac{1}{2}\langle 1,1|\mathbf{S}_1^2|1,1\rangle - \frac{1}{2}\langle 1,1|\mathbf{S}_2^2|1,1\rangle \\
&= \frac{1}{2}\langle 1,1|\mathbf{S}^2|1,1\rangle - \frac{1}{2}(2)(\frac{3}{4}\hbar^2) \\
&= \frac{1}{2}\langle 1,1|\mathbf{S}^2|1,1\rangle - \frac{3}{4}\hbar^2 \\
&= \frac{1}{2}S(S+1)\hbar^2 - \frac{3}{4}\hbar^2 = \frac{1}{2}(2)\hbar^2 - \frac{3}{4}\hbar^2 \\
&= \frac{1}{4}\hbar^2
\end{aligned}$$

Now we'll do $\langle 1,0|\mathbf{S}_1\cdot\mathbf{S}_2|1,0\rangle$:

$$\begin{aligned}
\langle 1,0|\mathbf{S}_1\cdot\mathbf{S}_2|1,0\rangle &= \frac{1}{2}\langle 1,0|\mathbf{S}^2|1,0\rangle - \frac{1}{2}\langle 1,0|\mathbf{S}_1^2|1,0\rangle - \frac{1}{2}\langle 1,0|\mathbf{S}_2^2|1,0\rangle \\
&= \frac{1}{4}\hbar^2
\end{aligned}$$

They're all the same

$$\begin{aligned}
\langle 1,-1|\mathbf{S}_1\cdot\mathbf{S}_2|1,-1\rangle &= \frac{1}{2}\langle 1,-1|\mathbf{S}^2|1,-1\rangle - \frac{1}{2}\langle 1,-1|\mathbf{S}_1^2|1,-1\rangle - \frac{1}{2}\langle 1,-1|\mathbf{S}_2^2|1,-1\rangle \\
&= \frac{1}{4}\hbar^2
\end{aligned}$$

Problem 2. Problem 6.23

Treat the two particle system as a single system with spin $S' = 0, 1$. With a third spin $\frac{1}{2}$ particle, the system's total spin number is $S = S' + \frac{1}{2}$.

If the two-particle system is in the singlet state, the spin number for the whole system is $S = 0 + \frac{1}{2} = \frac{1}{2}$.

Conversely if it's in the triplet state then the spin number for the whole system is $S = 1 + \frac{1}{2} = \frac{3}{2}$.

The possible states for $S = \frac{1}{2}$ are

$$\chi_{\frac{1}{2},\frac{1}{2}}, \chi_{\frac{1}{2},-\frac{1}{2}}$$

The possible states for $S = \frac{3}{2}$ are

$$\chi_{\frac{3}{2},\frac{3}{2}}, \chi_{\frac{3}{2},\frac{1}{2}}, \chi_{\frac{3}{2},-\frac{1}{2}}, \chi_{\frac{3}{2},-\frac{3}{2}}$$

Problem 3. "Quantum Mechanics" by Claude Cohen-Tannoudji.

(a) Assume $\langle L_x\rangle = \langle L_y\rangle = 0$. Find the states for which $(\Delta L_x)^2 + (\Delta L_y)^2 + (\Delta L_z)^2$ is minimal. $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$

$$\begin{aligned}
(\Delta L_x)^2 + (\Delta L_y)^2 + (\Delta L_z)^2 &= \langle L_x^2\rangle - \langle L_x\rangle^2 + \langle L_y^2\rangle - \langle L_y\rangle^2 + \langle L_z^2\rangle - \langle L_z\rangle^2 \\
&= \langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle - \langle L_z\rangle^2
\end{aligned}$$

Since $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$, $\langle \mathbf{L}^2 \rangle = \langle L_x^2 + L_y^2 + L_z^2 \rangle = \langle L_x^2 \rangle + \langle L_y^2 \rangle + \langle L_z^2 \rangle$, so

$$\begin{aligned}
(\Delta L_x)^2 + (\Delta L_y)^2 + (\Delta L_z)^2 &= \langle \mathbf{L}^2 \rangle - \langle L_z \rangle^2 \\
&= \langle lm|\mathbf{L}^2|lm\rangle - \langle lm|L_z|lm\rangle^2 \\
&= l(l+1)\hbar^2 \langle lm|lm\rangle - (m\hbar\langle lm|lm\rangle)^2 \\
&= l(l+1)\hbar^2 - m^2\hbar^2 \\
&= \hbar^2(l(l+1) - m^2)
\end{aligned}$$

Since l is fixed, we see that $\hbar^2(l(l+1) - m^2)$ is minimal when m^2 is maximal, so when $m = -l, l$ (since $-l \leq m \leq l$)

$$\begin{aligned}
\hbar^2(l(l+1) - m^2) &= \hbar^2(l(l+1) - l^2) \\
&= \hbar^2(l^2 + l - l^2) \\
&= l\hbar^2
\end{aligned}$$

So the states for which the sum is minimal are $|l, -l\rangle$, and $|l, l\rangle$.

Imagine a unit vector defined by polar coordinates $\hat{\mathbf{n}}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. A unit vector at an angle α to the z-axis is

$$\hat{\mathbf{n}}(\alpha, 0) = (\sin\alpha, 0, \cos\alpha)$$

The component of the angular momentum $\mathbf{L}$ along the unit vector at an angle α to the z-axis would be

$$L_\alpha = \mathbf{L} \cdot \hat{\mathbf{n}} = \sin\alpha L_x + \cos\alpha L_z$$

Now, for $m = -l, l$,

$$\begin{aligned}
(\Delta L_\alpha)^2 &= \langle L_\alpha^2 \rangle - \langle L_\alpha \rangle^2 \\
&= \langle lm|L_\alpha^2|lm\rangle - \langle lm|L_\alpha|lm\rangle^2 \\
&= \langle lm|\sin^2\alpha L_x^2 + \sin\alpha\cos\alpha L_x L_z + \sin\alpha\cos\alpha L_z L_x + \cos^2\alpha L_z^2|lm\rangle \\
&\quad -\langle lm|\sin\alpha L_x + \cos\alpha L_z|lm\rangle^2 \\
&= \sin^2\alpha\langle lm|L_x^2|lm\rangle + \sin\alpha\cos\alpha\langle lm|L_x L_z + L_z L_x|lm\rangle + \cos^2\alpha\langle lm|L_z^2|lm\rangle \\
&\quad -(\sin^2\alpha\langle lm|L_x|lm\rangle^2 + \cos^2\alpha\langle lm|L_z|lm\rangle^2) \\
&= \sin^2\alpha\langle lm|L_x^2|lm\rangle + \sin\alpha\cos\alpha\langle lm|L_x L_z + L_z L_x|lm\rangle + \cos^2\alpha\langle lm|L_z^2|lm\rangle \\
&\quad - \cos^2\alpha\langle lm|L_z|lm\rangle^2
\end{aligned}$$

We need to simplify. We already know $\langle lm|L_z|lm\rangle^2 = \hbar^2 l^2$. Further,

$$\langle lm|L_x^2|lm\rangle = \langle L_x^2 \rangle = \frac{\hbar^2}{2}(l(l+1) - m^2) = \frac{\hbar^2 l}{2}$$

also,

$$\begin{aligned}
\langle L_x^2 \rangle &= \frac{1}{2}\langle \mathbf{L}^2 - L_z^2 \rangle \implies \langle L_z^2 \rangle = \langle \mathbf{L}^2 \rangle - 2\langle L_x^2 \rangle \\
\langle lm|L_z^2|lm\rangle &= \langle L_z^2 \rangle \\
&= l(l+1)\hbar^2 - 2(\frac{\hbar^2 l}{2}) = \hbar^2(l(l+1) - l) \\
&= \hbar^2 l^2
\end{aligned}$$

Need to deal with the $\langle lm|L_x L_z + L_z L_x|lm\rangle$ term. First notice:

$$\begin{aligned}
[L_z, L_x] &= i\hbar L_y \\
L_x L_z &= L_z L_x - i\hbar L_y
\end{aligned}$$

And now we get

$$\begin{aligned}\langle lm|L_xL_z+L_zL_x|lm\rangle &= \langle lm|L_zL_x-i\hbar L_y+L_zL_x|lm\rangle \\ &= \langle lm|2L_zL_x|lm\rangle - i\hbar\langle lm|L_y|lm\rangle \\ &= 2\langle lm|L_zL_x|lm\rangle\end{aligned}$$

Use the identity $L_x=\frac{1}{2}(L_++L_-)$ and we get

$$\begin{aligned}\langle lm|L_xL_z+L_zL_x|lm\rangle &= \langle lm|L_z(L_++L_-)|lm\rangle \\ &= \langle lm|L_zL_+|lm\rangle+\langle lm|L_zL_-|lm\rangle \\ &= \hbar\sqrt{l(l+1)-m(m+1)}\langle lm|L_z|l(m+1)\rangle \\ &\quad +\hbar\sqrt{l(l+1)-m(m-1)}\langle lm|L_z|l(m-1)\rangle \\ &= m\hbar^2\sqrt{l(l+1)-m(m+1)}\langle lm|l(m+1)\rangle \\ &\quad +m\hbar^2\sqrt{l(l+1)-m(m-1)}\langle lm|l(m-1)\rangle \\ &= 0\end{aligned}$$

Ok, now putting this back into $(\Delta L_\alpha)^2$:

$$\begin{aligned}(\Delta L_\alpha)^2 &= \sin^2\alpha\frac{\hbar^2 l}{2}+\cos^2\alpha h^2l^2-\cos^2\alpha h^2l^2 \\ &= \hbar^2\frac{l}{2}\sin^2\alpha \\ \Delta L_\alpha &= \hbar\sqrt{\frac{l}{2}}\sin\alpha\end{aligned}$$

(b)

(i) Show that the state $|\psi_0\rangle$of the system for which $(\Delta L_x)^2+(\Delta L_y)^2+(\Delta L_z)^2$ is minimal is such that $(L_x+iL_y)|\psi_0\rangle=0$ and $L_z|\psi_0\rangle=l\hbar|\psi_0\rangle$.

The state we're dealing with is $|\psi_0\rangle=|ll\rangle$ where $m=l$.

$$L_z|\psi_0\rangle=m\hbar|\psi_0\rangle=l\hbar|\psi_0\rangle$$

$$\begin{aligned}(L_x+iL_y)|\psi_0\rangle &= L_+|\psi_0\rangle \\ &= L_+|l,l\rangle \\ &= |l,l+1\rangle=0 \qquad \text{(cuz can't raise } m\text{past } l)\end{aligned}$$

(ii) Let θ_0 be the angle between Oz and OZ and ϕ_0 be the angle between Oy and OY, prove the relations

$$\begin{aligned}L_X+iL_Y &= \cos^2\frac{\theta_0}{2}e^{-i\phi_0}L_+-\sin^2\frac{\theta_0}{2}e^{i\phi_0}L_--\sin\theta_0L_z \\ L_Z &= \sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}e^{-i\phi_0}L_++\sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}e^{i\phi_0}L_-+\cos\theta_0L_z\end{aligned}$$

Okay we'll use $L_x=\frac{1}{2}(L_++L_-)$ and $L_y=\frac{1}{2i}(L_+-L_-)$

$$OZ=\hat{\mathbf{n}}(\theta_0,\phi_0)=(\sin\theta_0\cos\phi_0,\sin\theta_0\sin\phi_0,\cos\theta_0)$$

$$\begin{aligned}
L_Z &= \mathbf{L}\cdot OZ = \sin\theta_0\cos\phi_0 L_x + \sin\theta_0\sin\phi_0 L_y + \cos\theta_0 L_z \\
&= \frac{1}{2}\sin\theta_0\cos\phi_0(L_+ + L_-) + \frac{1}{2i}\sin\theta_0\sin\phi_0(L_+ - L_-) + \cos\theta_0 L_z \\
&= \frac{1}{2}\sin\theta_0\Big((\cos\phi_0 + \frac{1}{i}\sin\phi_0)L_+ + (\cos\phi_0 - \frac{1}{i}\sin\phi_0)L_-\Big) + \cos\theta_0 L_z \\
&= \frac{1}{2}\sin\theta_0\Big((\cos\phi_0 - i\sin\phi_0)L_+ + (\cos\phi_0 + i\sin\phi_0)L_-\Big) + \cos\theta_0 L_z \\
&= \frac{1}{2}\sin\theta_0(e^{-i\phi_0}L_+ + e^{i\phi_0}L_-) + \cos\theta_0 L_z \\
&= \sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}e^{-i\phi_0}L_+ + \sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2}e^{i\phi_0}L_- + \cos\theta_0 L_z
\end{aligned}$$

Now find OY. OY is like $\hat{\varphi}$ in spherical coords:

$$OY = (-\sin\phi_0, \cos\phi_0, 0)$$

$$L_Y = \mathbf{L}\cdot OY = -\sin\phi_0 L_x + \cos\phi_0 L_y$$

$$\begin{aligned}
OX &= OY \times OZ \\
&= (\cos\phi_0\cos\theta_0, \sin\phi_0\cos\theta_0, -\sin^2\phi_0\sin\theta_0 - \sin\theta_0\cos^2\phi_0) \\
&= (\cos\phi_0\cos\theta_0, \sin\phi_0\cos\theta_0, -\sin\theta_0)
\end{aligned}$$

$$L_X = \mathbf{L}\cdot OX = \cos\phi_0\cos\theta_0 L_x + \sin\phi_0\cos\theta_0 L_y - \sin\theta_0 L_z$$

Now

$$L_X + iL_Y = (\cos\phi_0\cos\theta_0 - i\sin\phi_0)L_x + (\sin\phi_0\cos\theta_0 + i\cos\phi_0)L_y - \sin\theta_0 L_z$$

Expanding the L_x term:

$$\begin{aligned}
(\cos\phi_0\cos\theta_0 - i\sin\phi_0)L_x &= \frac{1}{2}(\cos\phi_0\cos\theta_0 - i\sin\phi_0)(L_+ + L_-) \\
&= \frac{1}{2}(\cos\phi_0\cos\theta_0 - i\sin\phi_0)L_+ + \frac{1}{2}(\cos\phi_0\cos\theta_0 - i\sin\phi_0)L_-
\end{aligned}$$

Expanding the L_y term:

$$\begin{aligned}
(\sin\phi_0\cos\theta_0 + i\cos\phi_0)L_y &= \frac{1}{2i}(\sin\phi_0\cos\theta_0 + i\cos\phi_0)(L_+ - L_-) \\
&= \frac{1}{2i}(\sin\phi_0\cos\theta_0 + i\cos\phi_0)L_+ - \frac{1}{2i}(\sin\phi_0\cos\theta_0 + i\cos\phi_0)L_- \\
&= \frac{1}{2}(-i\sin\phi_0\cos\theta_0 + \cos\phi_0)L_+ + \frac{1}{2}(i\sin\phi_0\cos\theta_0 - \cos\phi_0)L_-
\end{aligned}$$

Grouping the L_+ terms:

$$\begin{aligned}
\frac{1}{2}L_+(\cos\phi_0\cos\theta_0 - i\sin\phi_0 - i\sin\phi_0\cos\theta_0 + \cos\phi_0) &= \frac{1}{2}L_+(\cos\theta_0(\cos\phi_0 - i\sin\phi_0) + \cos\phi_0 - i\sin\phi_0) \\
&= \frac{1}{2}L_+(\cos\theta_0(e^{-i\phi_0}) + e^{-i\phi_0}) \\
&= \frac{1}{2}L_+e^{-i\phi_0}(\cos\theta_0 + 1) \\
&= \frac{1}{2}L_+e^{-i\phi_0}(2\cos^2\frac{\theta_0}{2} - 1 + 1) \\
&= \cos^2\frac{\theta_0}{2}e^{-i\phi_0}L_+
\end{aligned}$$

Group the L_- terms:

$$\begin{aligned}
\frac{1}{2}L_-(\cos\phi_0\cos\theta_0 - i\sin\phi_0 + i\sin\phi_0\cos\theta_0 - \cos\phi_0) &= \frac{1}{2}L_-(\cos\theta_0(\cos\phi_0 + i\sin\phi_0) - (\cos\phi_0 + i\sin\phi_0)) \\
&= \frac{1}{2}L_-(\cos\theta_0(e^{i\phi_0}) - e^{i\phi_0}) \\
&= \frac{1}{2}L_-e^{i\phi_0}(\cos\theta_0 - 1) \\
&= \frac{1}{2}L_-e^{i\phi_0}(1 - 2\sin^2\frac{\theta_0}{2} - 1) \\
&= -\sin^2\frac{\theta_0}{2}e^{i\phi_0}L_-
\end{aligned}$$

And we get

$$L_X + iL_Y = \cos^2\frac{\theta_0}{2}e^{-i\phi_0}L_+ - \sin^2\frac{\theta_0}{2}e^{i\phi_0}L_- - \sin\theta_0 L_z$$

Set

$$|\psi_0\rangle = \sum_{m=-l}^{l} d_m|l,m\rangle \text{ and show that } d_m = \tan\frac{\theta_0}{2}e^{i\phi_0}\sqrt{\frac{l+m+1}{l-m}}d_{m+1}$$

Problem 4. Work out $\mathbf{r}' = \exp(\alpha\hat{\mathbf{n}} \times \mathbf{r})$.