PHY4370 ASSIGNMENT 1

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Problem 1. Problem 6.4

a) Prove that $[L_i, x_j] = i\hbar\varepsilon_{ijk}x_k$ where $(i, j, k = 1, 2, 3)$ and ε_{ijk} is the Levi-Cevita antisymmetric symbol.

$$\begin{aligned}
[L_x, x] &= [yp_z - zp_y, x] = [yp_z, x] - [zp_y, x] \\
&= (yp_z x - xyp_z)\Psi - (zp_y x - xzp_y)\Psi \\
&= -i\hbar yx\frac{\partial\Psi}{\partial z} + i\hbar yx\frac{\partial\Psi}{\partial z} + i\hbar zx\frac{\partial\Psi}{\partial y} - i\hbar zx\frac{\partial\Psi}{\partial y} = 0
\end{aligned}$$

Similarly, $[L_y, y] = [L_z, z] = 0$.

$$[L_x, y] = [yp_z - zp_y, y] = [yp_z, y] - [zp_y, y] = -[zp_y, y] = -z[p_y, y] = -z(-i\hbar) = i\hbar z$$

$$[L_y, z] = [zp_x - xp_z, z] = [zp_x, z] - [xp_z, z] = -[xp_z, z] = -[xp_z, z] = -x(-i\hbar) = i\hbar x$$

Similarly, $[L_z, x] = i\hbar y$.

$$[L_y, x] = [zp_x - xp_z, z, x] = [zp_x, x] - [xp_z, x] = [zp_x, x] = z[p_x, x] = -i\hbar z = -[L_x, y]$$

Similarly, $[L_z, y] = -[L_y, z]$ and $[L_x, z] = -[L_z, x]$. Summarizing these results in a matrix, we see it is antisymmetric:

$$\begin{bmatrix} [L_x, x] & [L_y, x] & [L_z, x] \\ [L_x, y] & [L_y, y] & [L_z, y] \\ [L_x, z] & [L_y, z] & [L_z, z] \end{bmatrix} = \begin{bmatrix} 0 & -[L_x, y] & [L_z, x] \\ [L_x, y] & 0 & -[L_y, z] \\ -[L_z, x] & [L_y, z] & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i\hbar z & i\hbar y \\ i\hbar z & 0 & -i\hbar x \\ -i\hbar y & i\hbar x & 0 \end{bmatrix}$$

This can be summarized as $[L_i, x_j] = i\hbar\varepsilon_{ijk}x_k$.

b) Prove $[L_i, p_j] = i\hbar\varepsilon_{ijk}p_k$.

$$\begin{aligned}
[L_x, p_x] &= [yp_z - zp_y, p_x] = [yp_z, p_x] - [zp_y, p_x] = 0 \\
[L_y, p_y] &= [zp_x - xp_z, p_y] = 0 \\
[L_z, p_z] &= [xp_y - yp_x, p_z] = 0
\end{aligned}$$

$$[L_x, p_y] = [yp_z - zp_y, p_y] = [yp_z, p_y] - z[p_y, p_y] = p_z[y, p_y] = i\hbar p_z$$

Similarly, $[L_y, p_z] = i\hbar p_x$ and $[L_z, p_x] = i\hbar p_y$.

$$[L_y, p_x] = [zp_x - xp_z, p_x] = [zp_x, p_x] - [xp_z, p_x] = -p_z[x, p_x] = -i\hbar p_z = -[L_x, p_y]$$

Similarly, $[L_z, p_y] = -i\hbar p_x = -[L_y, p_z]$ and $[L_x, p_z] = -i\hbar p_y = -[L_z, p_x]$. Analogous to part (a), these results are summarized by $[L_i, p_j] = i\hbar\varepsilon_{ijk}p_k$.

c) $\mathbf{L} \cdot \mathbf{r} = 0 = \mathbf{L} \cdot \mathbf{p}$.

$$\begin{aligned}
\mathbf{L} \cdot \mathbf{r} &= L_x x + L_y y + L_z z = (yp_z - zp_y)x + (zp_x - xp_z)y + (xp_y - yp_x)z \\
&= yp_z x - zp_y x + zp_x y - xp_z y + xp_y z - yp_x z \\
&= yp_z x - xp_z y - zp_y x + xp_y z - zp_x y - yp_x z
\end{aligned}$$

Since $p_i = -i\hbar\partial_i$ we can do $p_i x_j = x_j p_i$ so long as $i \neq j$

$$\mathbf{L}\cdot\mathbf{r} = yp_z x - yp_z x - zp_y x + zp_y x - zp_x y - zp_x y = 0$$

$$\begin{aligned}
\mathbf{L}\cdot\mathbf{p} &= L_x p_x + L_y p_y + L_z p_z \\
&= [yp_z p_x - zp_y p_x] + [zp_x p_y - xp_z p_y] + [xp_y p_z - yp_x p_z] \\
&= yp_z p_x - yp_x p_z - zp_y p_x + zp_x p_y - xp_z p_y + xp_y p_z \\
&= 0
\end{aligned}$$

Since $p_i p_j = -\hbar\partial_i\partial_j = -\hbar\partial_j\partial_i = p_j p_i$ when $i \neq j$.

Problem 2. Problem 6.5

$\hat{\mathbf{u}} = (u_x, u_y, u_z)$, $\hat{\mathbf{v}} = (v_x, v_y, v_z)$, $\hat{\mathbf{w}} = (w_x, w_y, w_z)$. $\mathbf{L} = (L_x, L_y, L_z)$.

$$\begin{aligned}
L_u &= \hat{\mathbf{u}}\cdot\mathbf{L} = u_x L_x + u_y L_y + u_z L_z \\
L_v &= \hat{\mathbf{v}}\cdot\mathbf{L} = v_x L_x + v_y L_y + v_z L_z \\
L_w &= \hat{\mathbf{w}}\cdot\mathbf{L} = w_x L_x + w_y L_y + w_z L_z
\end{aligned}$$

$\hat{\mathbf{w}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$ so

$$w_x = u_y v_z - u_z v_y, \qquad w_y = u_z v_x - u_x v_z \qquad w_z = u_x v_y - u_y v_x$$

$$\begin{aligned}
[L_u, L_v] &= [u_x L_x + u_y L_y + u_z L_z, v_x L_x + v_y L_y + v_z L_z] \\
&= [u_x L_x, v_x L_x] + [u_x L_x, v_y L_y] + [u_x L_x, v_z L_z] \\
&\quad +[u_y L_y, v_x L_x] + [u_y L_y, v_y L_y] + [u_y L_y, v_z L_z] \\
&\quad +[u_z L_z, v_x L_x] + [u_z L_z, v_y L_y] + [u_z L_z, v_z L_z] \\
&= [u_x L_x, v_y L_y] + [u_x L_x, v_z L_z] + [u_y L_y, v_x L_x] + [u_y L_y, v_z L_z] + [u_z L_z, v_x L_x] + [u_z L_z, v_y L_y] \\
&= u_x v_y [L_x, L_y] + u_x v_z [L_x, L_z] + u_y v_x [L_y, L_x] + u_y v_z [L_y, L_z] + u_z v_x [L_z, L_x] + u_z v_y [L_z, L_y] \\
&= i\hbar u_x v_y L_z - i\hbar u_x v_z L_y - i\hbar u_y v_x L_z + i\hbar u_y v_z L_x + i\hbar u_z v_x L_y - i\hbar u_z v_y L_x \\
&= i\hbar\Big((u_y v_z - u_z v_y)L_x + (u_z v_x - u_x v_z)L_y + (u_x v_y - u_y v_x)L_z\Big) \\
&= i\hbar(w_x L_x + w_y L_y + w_z L_z) \\
&= i\hbar L_w
\end{aligned}$$

$$\begin{aligned}
[L_v, L_w] &= [v_x L_x + v_y L_y + v_z L_z, w_x L_x + w_y L_y + w_z L_z] \\
&= [v_x L_x, w_y L_y] + [v_x L_x, w_z L_z] + [v_y L_y, w_x L_x] + [v_y L_y, w_z L_z] + [v_z L_z, w_x L_x] + [v_z L_z, w_y L_y] \\
&= v_x w_y [L_x, L_y] + v_x w_z [L_x, L_z] + v_y w_x [L_y, L_x] + v_y w_z [L_y, L_z] + v_z w_x [L_z, L_x] + v_z w_y [L_z, L_y] \\
&= i\hbar(v_x w_y L_z - v_x w_z L_y - v_y w_x L_z + v_y w_z L_x + v_z w_x L_y - v_z w_y L_x) \\
&= i\hbar\Big((v_y w_z - v_z w_y)L_x + (v_z w_x - v_x w_z)L_y + (v_x w_y - v_y w_x)L_z\Big) \\
&= i\hbar(u_x L_x + u_y L_y + u_z L_z) \\
&= i\hbar L_u
\end{aligned}$$

Since $\hat{\mathbf{u}} = \hat{\mathbf{v}} \times \hat{\mathbf{w}}$. Last one:

$$\begin{aligned}
[L_w, L_u] &= [w_x L_x + w_y L_y + w_z L_z, u_x L_x + u_y L_y + u_z L_z] \\
&= w_x u_y [L_x, L_y] + w_x u_z [L_x, L_z] + w_y u_x [L_y, L_x] + w_y u_z [L_y, L_z] + w_z u_x [L_z, L_x] + w_z u_y [L_z, L_y] \\
&= i\hbar\Big((w_y u_z - w_z u_y) L_x + (w_z u_x - w_x u_z) L_y + (w_x u_y - w_y u_x) L_z\Big) \\
&= i\hbar(v_x L_x + v_y L_y + v_z L_z) \\
&= i\hbar L_v
\end{aligned}$$

Since $\hat{\mathbf{v}} = \hat{\mathbf{w}} \times \hat{\mathbf{u}}$. QED

Problem 3. Problem 6.14

The eigenvalues belonging to the eigenvector $|jm\rangle$ of $\mathbf{J}^2$ and J_z are $j(j+1)\hbar^2$ and $m\hbar$ respectively. If $j = \frac{3}{2}$, then $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$.

$$\begin{aligned}
(\mathbf{J}^2)_{j'm',jm} &= j(j+1)\hbar^2 \delta_{jj'} \delta_{mm'} \\
&= \frac{3}{2}\Big(\frac{3}{2}+1\Big)\hbar^2 I_4 = \frac{15}{4}\hbar^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
(J_z)_{j'm',jm} &= m\hbar \delta_{jj'} \delta_{mm'} \\
&= \frac{\hbar}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}
\end{aligned}$$

We need the matrix representations of J_+ and J_- to find J_x and J_y.

$$\begin{aligned}
(J_+)_{j'm',jm} &= \hbar\sqrt{j(j+1) - m(m+1)}\delta_{jj'}\delta_{m'm+1} \\
&= \hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
(J_-)_{j'm',jm} &= \hbar\sqrt{j(j+1) - m(m-1)}\delta_{jj'}\delta_{m'm-1} \\
&= \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}
\end{aligned}$$

$$J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$J_y = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2i} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}$$

Problem 4. Problem 6.20

$$\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$S_n = \hat{\mathbf{n}}\cdot\mathbf{S} = \sin\theta\cos\phi S_x + \sin\theta\sin\phi S_y + \cos\theta S_z$$

Since $s = 1$, $m_s = 1, 0, -1$. The eigenvectors of $\mathbf{S}^2$ are given by $s(s+1)\hbar|sm_s\rangle$ and are

$$2\hbar^2|1,1\rangle, \qquad 2\hbar^2|1,0\rangle, \qquad 2\hbar^2|1,-1\rangle$$

Use these as basis vectors (they are also basis vectors for S_z):

$$\chi_{1,1} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \chi_{1,0} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \chi_{1,-1} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

We can write $\mathbf{S}^2$ as this matrix:

$$\mathbf{S}^2 = 2\hbar^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The eigenvectors of S_z are given by $m_s\hbar|sm_s\rangle$ and are

$$\hbar|1,1\rangle, \qquad 0|1,0\rangle, \qquad -\hbar|1,-1\rangle$$

As a matrix (where the columns are the eigenvectors) this is

$$S_z = \hbar\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}$$

From page 300 of Bransden and Joachain, S_x and S_y for a particle of spin 1 are given by

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix}$$

So

$$\begin{aligned}
S_n &= \frac{\hbar}{\sqrt{2}}\sin\theta\cos\phi\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} + \frac{\hbar}{\sqrt{2}}\sin\theta\sin\phi\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix} + \hbar\cos\theta\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\\
&= \frac{\hbar}{\sqrt{2}}\left(\sin\theta\cos\phi\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} + \sin\theta\sin\phi\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix} + \sqrt{2}\cos\theta\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\right)\\
&= \frac{\hbar}{\sqrt{2}}\begin{bmatrix}\sqrt{2}\cos\theta & e^{-i\phi}\sin\theta & 0\\ e^{i\phi}\sin\theta & 0 & e^{-i\phi}\sin\theta\\ 0 & e^{i\phi}\sin\theta & -\sqrt{2}\cos\theta\end{bmatrix}
\end{aligned}$$

To get the eigenvalues and eigenvectors of S_n we solve $S_n\chi = \frac{\hbar}{\sqrt{2}}\lambda\chi$. Let $\chi = \begin{pmatrix}a\\b\\c\end{pmatrix}$.

$$S_n\chi = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}a\sqrt{2}\cos\theta + be^{-i\phi}\sin\theta\\ ae^{i\phi}\sin\theta + ce^{-i\phi}\sin\theta\\ be^{i\phi}\sin\theta - c\sqrt{2}\cos\theta\end{pmatrix}$$

$$\begin{aligned}
S_n\chi &= \frac{\hbar}{\sqrt{2}}\lambda\chi \\
\frac{\hbar}{\sqrt{2}}\begin{pmatrix} a\sqrt{2}\cos\theta + be^{-i\phi}\sin\theta \\ ae^{i\phi}\sin\theta + ce^{-i\phi}\sin\theta \\ be^{i\phi}\sin\theta - c\sqrt{2}\cos\theta \end{pmatrix} &= \frac{\hbar}{\sqrt{2}}\begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} a(\sqrt{2}\cos\theta - \lambda) + be^{-i\phi}\sin\theta \\ ae^{i\phi}\sin\theta + ce^{-i\phi}\sin\theta - b\lambda \\ be^{i\phi}\sin\theta - c(\sqrt{2}\cos\theta - \lambda) \end{pmatrix} = \begin{bmatrix} \sqrt{2}\cos\theta - \lambda & e^{-i\phi}\sin\theta & 0 \\ e^{i\phi}\sin\theta & -\lambda & e^{-i\phi}\sin\theta \\ 0 & e^{i\phi}\sin\theta & \sqrt{2}\cos\theta - \lambda \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The determinant of this matrix must be 0 for non-trivial solutions.

$$\begin{aligned}
\begin{vmatrix} \sqrt{2}\cos\theta - \lambda & e^{-i\phi}\sin\theta & 0 \\ e^{i\phi}\sin\theta & -\lambda & e^{-i\phi}\sin\theta \\ 0 & e^{i\phi}\sin\theta & \sqrt{2}\cos\theta - \lambda \end{vmatrix} &= 0 \\
\sqrt{2}\cos\theta - \lambda\begin{vmatrix} -\lambda & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & \sqrt{2}\cos\theta - \lambda \end{vmatrix} - e^{-i\phi}\sin\theta\begin{vmatrix} e^{i\phi}\sin\theta & e^{-i\phi}\sin\theta \\ 0 & \sqrt{2}\cos\theta - \lambda \end{vmatrix} &= 0 \\
(\sqrt{2}\cos\theta - \lambda)(-\lambda(\sqrt{2}\cos\theta - \lambda) - \sin^2\theta) - e^{-i\phi}\sin\theta(e^{i\phi}\sin\theta(\sqrt{2}\cos\theta - \lambda)) &= 0 \\
(\sqrt{2}\cos\theta - \lambda)(-\lambda(\sqrt{2}\cos\theta - \lambda) - \sin^2\theta) - \sin^2\theta(\sqrt{2}\cos\theta - \lambda) &= 0 \\
(\sqrt{2}\cos\theta - \lambda)(-\lambda(\sqrt{2}\cos\theta - \lambda) - \sin^2\theta - \sin^2\theta) &= 0 \\
(\sqrt{2}\cos\theta - \lambda)(-\lambda(\sqrt{2}\cos\theta - \lambda) - 2\sin^2\theta) &= 0 \\
-\lambda(\sqrt{2}\cos\theta - \lambda)^2 - 2(\sqrt{2}\cos\theta - \lambda)\sin^2\theta &= 0 \\
-\lambda(2\cos^2\theta - 2\sqrt{2}\lambda\cos\theta + \lambda^2) - 2\sqrt{2}\cos\theta\sin^2\theta + 2\lambda\sin^2\theta &= 0 \\
-2\lambda\cos^2\theta + 2\sqrt{2}\lambda^2\cos\theta - \lambda^3 - 2\sqrt{2}\cos\theta\sin^2\theta + 2\lambda\sin^2\theta &= 0 \\
-2\lambda(\cos^2\theta - \sin^2\theta) + 2\sqrt{2}\cos\theta(\lambda^2 - \sin^2\theta) - \lambda^3 &= 0 \\
-2\lambda(1 - 2\sin^2\theta) + 2\sqrt{2}\cos\theta(\lambda^2 - \sin^2\theta) - \lambda^3 &= 0
\end{aligned}$$

unfinished.