MAT3155
ASSIGNMENT 3

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Problem 1. Let T_PS denote the tangent space to a smooth surface S at $P \in S$. Suppose S_1 and S_2 are smooth surfaces and $f : S_1 \to S_2$ is smooth. The *tangent map of f at $P \in S_1$* is the map

$$T_{P_1}(f) : T_PS_1 \to T_{f(P)}S_2$$

defined by

$$T_P(f)(w) = \frac{d(f\gamma)}{dt}(0)$$

where $\gamma : (-1,1) \to S_1$ is any smooth curve with $\gamma(0) = P$ and $\dot{\gamma}(0) = w$.

a) Show that $T_{P_1}(f)$ is well-defined.

Let γ and η be smooth curves $(-1,1) \to S_1$. Let $\sigma : U_1 \to S_1$ be an r.s.p (σ, U_1, S_1) covering P and let $\tau : U_2 \to S_2$ be an r.s.p (τ, U_2, S_2) covering $f(P)$. Consider the composition $f\gamma : (-1,1) \to S_2$. Note that $f\gamma = \tau(\tau^{-1}f\sigma)\sigma^{-1}\gamma$. Since f is smooth, we know that the map $\tau^{-1}f\sigma : U_1 \to U_1$ is smooth. Now, using the chain rule

$$\begin{aligned}
\frac{d(f\gamma)}{dt} &= \frac{d(\tau(\tau^{-1}f\sigma)\sigma^{-1}\gamma)}{dt} \\
&= J(\tau)J((\tau^{-1}f\sigma)\sigma^{-1}\gamma) \\
&= J(\tau)J(\tau^{-1}f\sigma)\frac{d(\sigma^{-1}\gamma)}{dt}
\end{aligned}$$

Evaluated at $\gamma(0) = P$, we get the tangent map

$$T_P(f)(w) = \frac{d(f\gamma)}{dt}(0) = \Big(J(\tau)(\tau^{-1}(f(P))\Big)\Big(J(\tau^{-1}f\sigma)(\sigma^{-1}(P))\Big)\frac{d(\sigma^{-1}\gamma)}{dt}(0)$$

Note that

$$\frac{d(\sigma^{-1}\gamma)}{dt}(0) = J(\sigma^{-1})(\gamma(0))\dot{\gamma}(0) = \Big(J(\sigma^{-1})(P)\Big)w$$

Let η be a curve distinct from γ but such that, locally they are the same. That is, $\eta(0) = P$ and $\dot{\eta}(0) = \dot{\gamma}(0) = w$. If we want to construct $T_p(f)(w)$ using η, we expect that $\frac{d(\sigma^{-1}\gamma)}{dt}(0) = \frac{d(\sigma^{-1}\eta)}{dt}(0)$. Indeed we have:

$$\frac{d(\sigma^{-1}\gamma)}{dt}(0) = \Big(J(\sigma^{-1})(\gamma(0))\Big)\dot{\gamma}(0) = \Big(J(\sigma^{-1})(\eta(0))\Big)\dot{\eta}(0) = \frac{d(\sigma^{-1}\eta)}{dt}$$

This implies

$$\frac{d(f\gamma)}{dt}(0) = \frac{d(f\eta)}{dt}(0) \equiv T_P(f)(w)$$

Conversely, we can require that the tangent map be unique at a point P and a tangent vector w given two curves γ and η and only knowing that $\gamma(0) = \eta(0) = P$ and $\dot{\gamma}(0) = w$, we can deduce that $\dot{\eta}(0) = \dot{\gamma}(0) = w$. Assuming $\frac{d(f\gamma)}{dt}(0) = \frac{d(f\eta)}{dt}(0)$, we have

$$\begin{aligned} \frac{d(\sigma^{-1}\gamma)}{dt}(0) &= \frac{d(\sigma^{-1}\eta)}{dt} \\ J(\sigma^{-1})(P)\dot{\gamma}(0) &= J(\sigma^{-1})(P)\dot{\eta}(0) \\ \dot{\gamma}(0) &= \dot{\eta}(0) \end{aligned}$$

The point of all this is that only the local properties of the chosen curve matter when constructing the tangent map. Thus for any curve and any two surface patches covering P and $f(P)$, $T_P(f)(w)$ is uniquely defined by P, f, and w only.

b) Let (σ, U_1, W_1) be an r.s.p. for S_1 with $\sigma(0,0) = P \in W$, and (τ, U_2, W_2) be an r.s.p. for S_2, with $\tau(0,0) = f(P) \in W_2$. If $a, b \in \mathbb{R}$ and $v = a\sigma_u(0,0) + b\sigma_v(0,0) \in T_P S_1$, show that

$$T_{P_1}(f)(v) = \Big(J(\tau)(0,0)\Big)\Big(J(\tau^{-1}f\sigma)(0,0)\Big)\begin{bmatrix} a \\ b \end{bmatrix}$$

Let $\gamma : (-1,1) \to S_1$ be a smooth curve st. $\gamma(0) = P$ and $\dot{\gamma}(0) = v$. From part (a) we know

$$T_{P_1}(f)(v) = \Big(J(\tau)(\tau^{-1}(f(P))\Big)\Big(J(\tau^{-1}f\sigma)(\sigma^{-1}(P))\Big)\frac{d(\sigma^{-1}\gamma)}{dt}(0)$$

Substituting $\sigma^{-1}(P) = (0,0)$ and $\tau^{-1}(f(P)) = (0,0)$ we get

$$T_{P_1}(f)(v) = \Big(J(\tau)(0,0)\Big)\Big(J(\tau^{-1}f\sigma)(0,0)\Big)\frac{d(\sigma^{-1}\gamma)}{dt}(0)$$

Now let $\gamma(t) = \sigma(at, bt)$. Note that $\gamma(0) = \sigma(0,0) = P$ as required and also that

$$\dot{\gamma}(0) = \Big(J(\sigma)(0,0)\Big)\frac{d(at,bt)}{dt} = \begin{bmatrix} \sigma_u(0,0) & \sigma_v(0,0) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = a\sigma_u(0,0) + b\sigma_v(0,0) = v$$

by the chain rule. We can use this curve to construct the tangent map because it fits the local properties and beyond that the choice of γ is arbitrary, as shown in part (a). Now consider the composition of σ^{-1} and γ:

$$\sigma^{-1}\gamma = \sigma^{-1}\sigma(at, bt) = (at, bt)$$

So the derivative is then

$$\frac{d(\sigma^{-1}\gamma)}{dt}(0) = \frac{d(at,bt)}{dt}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$$

and we get the result

$$T_{P_1}(f)(v) = \Big(J(\tau)(0,0)\Big)\Big(J(\tau^{-1}f\sigma)(0,0)\Big)\begin{bmatrix} a \\ b \end{bmatrix} \qquad \square$$

c) Show that if f is a diffeomorphism, then $T_{P_1}(f)$ is invertible for all $P_1 \in S_1$.

Since f is a diffeomorphism, it's inverse $f^{-1} : S_2 \to S_1$ exists and is smooth.
Let $\sigma : U_1 \to S_1$ be an r.s.p (σ, U_1, S_1) covering P and let $\tau : U_2 \to S_2$ be an r.s.p (τ, U_2, S_2) covering $f(P)$.

Let $\beta : (-1, 1) \to S_2$ be a curve in S_2 such that $\beta(0) = f(P)$ and $\dot{\beta}(0) = v$. Consider the composition $f^{-1}\beta : (-1, 1) \to S_1$:

$$f^{-1}\beta = \sigma(\sigma^{-1}f^{-1}\tau)\tau^{-1}\beta$$

Now calculate the map $\frac{d(f^{-1}\beta)}{dt}(0)$ analagously to part (a):

$$\begin{aligned}
\frac{d(f^{-1}\beta)}{dt} &= \frac{d(\sigma(\sigma^{-1}f^{-1}\tau)\tau^{-1}\beta)}{dt} \\
&= J(\sigma)J((\sigma^{-1}f^{-1}\tau)\tau^{-1}\beta) \\
&= J(\sigma)J(\sigma^{-1}f^{-1}\tau)\frac{d(\tau^{-1}\beta)}{dt}
\end{aligned}$$

Let's evaulate this at $\beta(0) = f(P)$:

$$\begin{aligned}
\frac{d(f^{-1}\beta)}{dt}(0) &= \Big(J(\sigma)(\sigma^{-1}(P))\Big)\Big(J(\sigma^{-1}f^{-1}\tau)(\tau^{-1}(f(P))\Big)\frac{d(\tau^{-1}\beta)}{dt}(0) \\
&= \Big(J(\sigma)(\sigma^{-1}(P))\Big)\Big(J(\sigma^{-1}f^{-1}\tau)(\tau^{-1}(f(P)))\Big)\Big(J(\tau^{-1})(f(P))\Big)v
\end{aligned}$$

Consider the composition of $T_P(f)(w)$ with $\frac{d(f^{-1}\beta)}{dt}(0)$ (leaving out the evaluated point P):

$$\begin{aligned}
T_P(f)\Big(\frac{d(f^{-1}\beta)}{dt}(0)\Big) &= T_P(f)\Big(J(\sigma)J(\sigma^{-1}f^{-1}\tau)J(\tau^{-1})v\Big) \\
&= J(\tau)J(\tau^{-1}f\sigma)J(\sigma^{-1})J(\sigma)J(\sigma^{-1}f^{-1}\tau)J(\tau^{-1})v
\end{aligned}$$

Since $J(\psi^{-1}) = J(\psi)^{-1}$ for any smooth map ψ that has an inverse, we have

$$\begin{aligned}
T_P(f)\Big(\frac{d(f^{-1}\beta)}{dt}(0)\Big) &= J(\tau)J(\tau^{-1}f\sigma)J(\sigma^{-1})J(\sigma)J(\sigma^{-1}f^{-1}\tau)J(\tau^{-1})v \\
&= J(\tau)J(\tau^{-1}f\sigma)J(\sigma)^{-1}J(\sigma)J(\tau^{-1}f\sigma)^{-1}J(\tau)^{-1}v \\
&= J(\tau)J(\tau^{-1}f\sigma)IJ(\tau^{-1}f\sigma)^{-1}J(\tau)^{-1}v \\
&= J(\tau)J(\tau)^{-1}v \\
&= v = \dot{\beta}(0)
\end{aligned}$$

Indeed we see that $\frac{d(f^{-1}\beta)}{dt}(0)$ is a map from $T_{f(P)}S_2 \to T_PS_1$ and is the inverse of $T_P(f)$ and we can write

$$\begin{aligned}
T_{f(P)}(f^{-1})(v) &= \frac{d(f^{-1}\beta)}{dt}(0) \\
T_P(f)(T_{f(P)}(f^{-1})(v)) &= v
\end{aligned}$$

for tangent vectors $v \in T_{f(P)}S_2$. Since this inverse exists, $T_P(f)$ is invertible.

d) Is the converse to (c) true? Give a proof or counterexample.

The converse to (c) is not true.
Let S_1 be the open disc in the xy-plane $S_1 = \{(x, y, 0)|x^2 + y^2 < 1\}$ and S_2 be the xy-plane. Note that $S_1 \subset S_2$.
Let $f : S_1 \to S_2$ be the map defined by $f(x, y, 0) = (x, y, 0)$. Then f is not surjective, since it does not map points on the plan that are outside of the open disc. This means f is not a diffeomorpism.

The tangent map $T_p(f) : T_PS_1 \to T_PS_2$ from P to $f(P) = P$ of f at P is $T_P(f)(w) = w$ for some tangent vector w of some curve γ in S_1. Since $T_p(f)(w)$ is an identity map, it has an inverse and is invertible. Thus we have an invertible tangent map of f even though is not a diffeomorphism. Indeed, f is a local diffeomorphism on the open disc and by Proposition 4.4.6 in the book, the tangent map is invertible.

Problem 2. Compute the first fundamental form of $\mathbf{S}^2\backslash\{N\}$ using the r.s.p. given by the inverse of the stereographic projection.

The r.s.p given by the inverse of the stereographic projection is

$$\sigma(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

The derivatives of the r.s.p are given by

$$\begin{aligned}\sigma_u(u,v) &= \left(\frac{2}{u^2+v^2+1} - \frac{4u^2}{(u^2+v^2+1)^2}, -\frac{4uv}{(u^2+v^2+1)^2}, \frac{2u}{u^2+v^2+1} - \frac{2u(u^2+v^2-1)}{(u^2+v^2+1)^2}\right)\\ &= \left(\frac{2(v^2-u^2+1)}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}\right)\end{aligned}$$

$$\sigma_v(u,v) = \left(\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2(u^2-v^2+1)}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2}\right)$$

The first fundamental form is given by $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$, and $G = \sigma_v \cdot \sigma_v$:

$$\begin{aligned}E &= \frac{4(v^2-u^2+1)^2}{(u^2+v^2+1)^4} + \frac{16u^2v^2}{(u^2+v^2+1)^4} + \frac{16u^2}{(u^2+v^2+1)^4}\\ &= \frac{4}{(u^2+v^2+1)^2}\end{aligned}$$

$$\begin{aligned}F &= \frac{-8uv(v^2-u^2+1)}{(u^2+v^2+1)^4} - \frac{8uv(u^2-v^2+1)}{(u^2+v^2+1)^4} + \frac{16uv}{(u^2+v^2+1)^4}\\ &= \frac{-8uv(v^2-u^2+1+u^2-v^2+1-2)}{(u^2+v^2+1)^4} = 0\end{aligned}$$

$$G = \frac{4}{(u^2+v^2+1)^2}$$

So the first fundamental form is

$$\begin{bmatrix} \frac{4}{(u^2+v^2+1)^2} & 0 \\ 0 & \frac{4}{(u^2+v^2+1)^2} \end{bmatrix}$$

b) Prove that $\mathbf{S}^2$ is orientable using the definition.

Using stereographic projection, an atlas for the northern hemisphere (σ, U, W) and for the southern hemisphere $(\tilde{\sigma}, \tilde{U}, \tilde{W})$ can be obtained. The union of the two atlases is connected, so $J(\phi)$ is nonsingular where ϕ is defined. This means its determinant is either always positive or always negative. If it is always negative, the parameters of one of the surface patches can be reversed $(u,v) \mapsto (v,u)$

and the determinant will become positive. Since $det(J(\phi)) > 0$, $\mathbf{S}^2$ is orientable.

c) Use a theorem from class