PHY3320 ASSIGNMENT 1

MOHAMMED CHAMMA 6379153
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Problem 1. A parallel plate capacitor is immersed in sea water and driven by a voltage $V = V_0 \cos(2\pi\nu t)$ where $\nu = 4 \times 10^8$ Hz. The sea water has permittivity $\varepsilon = 81\varepsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\,\Omega \cdot \text{m}$. What is the ratio of conduction current to displacement current?

Assume the plates of the capacitor are separated by a distance d. The plates have an area A. Since water is conductive there is some current that flows across the plates. The electric field between the plates is

$$E(t) = \frac{V(t)}{d} = \frac{V_0}{d}\cos(2\pi\nu t)$$

The current density is related to the resistivity of the sea water

$$J(t) = \frac{E(t)}{\rho} = \frac{V_0}{d\rho}\cos(2\pi\nu t)$$

The conduction current is then

$$I_c = \int J(t) \cdot da = AJ(t) = \frac{AV_0}{d\rho}\cos(2\pi\nu t)$$

The displacement current density is

$$\begin{aligned} J_D(t) &= \varepsilon\frac{\partial E}{\partial t} = \varepsilon\frac{V_0}{d}(-2\pi\nu)\sin(2\pi\nu t) \\ &= -\frac{2\pi\nu\varepsilon V_0}{d}\sin(2\pi\nu t) \end{aligned}$$

So the displacement current is

$$I_D = AJ_D(t) = \frac{-2\pi\nu A\varepsilon V_0}{d}\sin(2\pi\nu t)$$

The ratio is

$$\begin{aligned} \frac{I_c}{I_D} &= \frac{AV_0\cos(2\pi\nu t)d}{d\rho(-2\pi\nu A\varepsilon V_0)\sin(2\pi\nu t)} = -\cot(2\pi\nu t)\frac{1}{2\pi\nu\rho\varepsilon} \\ &= -\cot(2\pi\nu t)\frac{1}{2\pi\nu\rho 81\varepsilon_0} \\ &= -\cot(2\pi\nu t)\frac{1}{2\pi(0.4\times 10^9 Hz)(0.23\Omega m)(81)(8.85\times 10^{-12}F/m)} \\ &= -2.41\cot(2\pi\nu t) \end{aligned}$$

The ratio of the amplitudes is 2.41. This is unitless because

$$\frac{\Omega m}{s}\frac{F}{m} = \frac{V}{A}\frac{F}{s} = \frac{VC}{AVs} = \frac{C}{As} = \frac{Cs}{Cs} = 1$$

Problem 2. Two large metal plates are held a distance d apart, one at $V = 0$ and the other at $V = V_0$. A metal sphere of radius $a << d$ is placed on the grounded $V = 0$ plate (and so is also grounded). The region between the plates is filled with a weakly conducting material of uniform conductivity σ. What current flows to the hemisphere?

We want to find the potential $V(r,\theta)$ between the plates. We'll use spherical coordinates where $r=0$ is the center of the hemisphere. We need to solve $\nabla^2 V = 0$ (why not Poisson's equation since there is charge in between the plates?). In spherical coordinates, without a dependence on ϕ, this is:

$$\frac{\partial}{\partial r}\Big(r^2\frac{\partial V}{\partial r}\Big) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial V}{\partial\theta}\Big) = 0$$

The general solution to this is

$$V(r,\theta) = \sum_{l=0}^{\infty}\Big(A_l r^l + \frac{B_l}{r^{l+1}}\Big)P_l(\cos\theta)$$

To find the constants A_l and B_l we note these boundary conditions:

$$\begin{aligned} V &\to 0 \qquad \text{when } r=a \text{ or } \theta=\pi/2 \\ V &\to V_0 \qquad \text{when } r\cos\theta = d \iff r = \frac{d}{\cos\theta} \end{aligned}$$

Firstly,

$$\begin{aligned} V(a,\theta) &= \sum_{l=0}^{\infty}\Big(A_l a^l + \frac{B_l}{a^{l+1}}\Big)P_l(\cos\theta) = 0 \\ 0 &= A_l a^l + \frac{B_l}{a^{l+1}} \\ B_l &= -A_l a^{2l+1} \end{aligned}$$

Secondly,

$$V(r,\pi/2) = \sum_{l=0}^{\infty}\Big(A_l r^l + \frac{B_l}{r^{l+1}}\Big)P_l(0) = 0$$

The Legendre polynomial $P_l(0)$ is a constant when l is even and is 0 when l is odd. If l is odd, then the condition is satisfied. But what if l is even? Denote $P_l(0) = k \neq 0$ as a constant in this case and we see

$$\begin{aligned} 0 &= \sum_{l=0}^{\infty}\Big(A_l r^l + \frac{B_l}{r^{l+1}}\Big)k \\ B_l &= -A_l r^{2l+1} \end{aligned}$$

But if we are free to pick r, then B_l (and A_l) cannot be constants to satisfy this condition. We can never satisfy this condition, so l must always be <u>odd</u>. So

$$\begin{aligned} V(r,\theta) &= \sum_{l=1,3,5..}^{\infty}\Big(A_l r^l - \frac{A_l a^{2l+1}}{r^{l+1}}\Big)P_l(\cos\theta) \\ &= \sum_{l=1,3,5..}^{\infty} A_l\Big(r^l - \frac{a^{2l+1}}{r^{l+1}}\Big)P_l(\cos\theta) \end{aligned}$$

Lastly,

$$V(\frac{d}{\cos\theta},\theta) = \sum_{l=1,3,5..}^{\infty} A_l\Big(\frac{d^l}{\cos^l\theta} - \frac{a^{2l+1}\cos^{l+1}\theta}{d^{l+1}}\Big)P_l(\cos\theta) = V_0$$

This must hold for all $\theta \in [0, \frac{\pi}{2})$. So lets pick $\theta = 0$ so that $\cos\theta = 1$ and simplify

$$\begin{aligned} V_0 &= \sum_{l=1,3,5}^{\infty} A_l\Big(d^l - \frac{a^{2l+1}}{d^{l+1}}\Big) \qquad (P_l(1) = 1) \\ &= \sum_{l=1,3,5}^{\infty} A_l d^l \qquad (d >> a) \end{aligned}$$

Since V_0 is constant pick A_1 and set all the other coefficients to 0

$$A_1 = \frac{V_0}{d}$$

So finally the potential is given by

$$\begin{aligned} V(r,\theta) &= A_1(r^1 - \frac{a^{2+1}}{r^{1+1}})P_1(\cos\theta) \\ &= \frac{V_0}{d}\Big(r - \frac{a^3}{r^2}\Big)\cos(\theta) \end{aligned}$$

Now we use this to find the electric field $\mathbf{E}$:

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= -\Big(\frac{\partial V}{\partial r}, \frac{\partial V}{\partial \theta}\Big) \\ &= -\frac{V_0}{d}\Big(\cos\theta(1 - \frac{a^3(-2)}{r^3}), -\sin\theta\Big(r - \frac{a^3}{r^2}\Big)\Big) \\ &= \Big(-\frac{V_0}{d}\cos\theta\Big(1 + \frac{2a^3}{r^3}\Big), \frac{V_0}{d}\sin\theta\Big(r - \frac{a^3}{r^2}\Big)\Big) \end{aligned}$$

The current density is simply $\mathbf{J} = \sigma\mathbf{E}$. To find the current that flows to the hemisphere, we integrate over its surface:

$$\begin{aligned} I &= \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot a^2\sin\theta d\theta d\phi\hat{\mathbf{r}} \\ &= \sigma\int -\frac{V_0}{d}\cos\theta\Big(1 + \frac{2a^3}{r^3}\Big)\cdot a^2\sin\theta d\theta d\phi \\ &= -\frac{3a^2\sigma V_0}{d}\int_0^{\pi/2}\cos\theta\sin\theta d\theta\int_0^{2\pi} d\phi \\ &= -\frac{6\pi a^2\sigma V_0}{d}(\frac{1}{2})\int_0^{\pi/2}\sin 2\theta d\theta \\ &= -\frac{3\pi a^2\sigma V_0}{d}(-\frac{1}{2}\cos 2\theta)\Big|_0^{\pi/2} \\ &= -\frac{3\pi a^2\sigma V_0}{d}(-\frac{1}{2}(-1) + \frac{1}{2}) \\ I &= -\frac{3\pi a^2\sigma V_0}{d} \end{aligned}$$

Problem 3. In a perfect conductor the conductivity is infinite, so $\mathbf{E} = 0$ and any net charge resides on the surface.

(a) Show that the magnetic field is constant inside a perfect conductor

Since $\mathbf{E} = 0$, $\nabla \times \mathbf{E} = 0$. But Faraday's law says $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, so

$$\frac{\partial \mathbf{B}}{\partial t} = 0$$

Since $\frac{\partial \mathbf{B}}{\partial t} = 0$, the magnetic field is constant.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

Faraday's law in integral form is $\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi}{\partial t}$. Since $\mathbf{E} = 0$,

$$\frac{\partial \Phi}{\partial t} = -\oint \mathbf{E} \cdot d\mathbf{l} = -\oint 0 \cdot d\mathbf{l} = 0$$

So the magnetic flux through a perfectly conducting loop is constant.

(c) Show that the current in a superconductor is confined to the surface. A superconductor is a perfect conductor where $E = B = 0$ inside.

Ampere's Law says $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$. Inside the superconductor, we have $\mathbf{B} = \mathbf{E} = 0$ so $\nabla \times \mathbf{B} = 0 = \frac{\partial \mathbf{E}}{\partial t}$. Therefore, inside the superconductor

$$\mathbf{J} = 0$$

Since the current density is always 0 inside the superconductor, there is no current inside the superconductor.

Problem 4. Derive the continuity equation starting from Maxwell's equations.

Start from Ampere's Law with Maxwell's Correction and take the divergence of both sides:

$$\begin{aligned}
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot (\nabla \times \mathbf{B}) &= \nabla \cdot (\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t})
\end{aligned}$$

Use the vector identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ to get a 0 on the left side

$$\begin{aligned}
0 &= \nabla \cdot (\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \\
&= \nabla \cdot \mu_0 \mathbf{J} + \nabla \cdot \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\
&= \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} \\
&= \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E}
\end{aligned}$$

Use Gauss's Law: $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$

$$\begin{aligned} 0 &= \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} \\ &= \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{1}{\epsilon_0} \frac{\partial \rho}{\partial t} \\ &= \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \frac{\partial \rho}{\partial t} \\ \mu_0 \nabla \cdot \mathbf{J} &= -\mu_0 \frac{\partial \rho}{\partial t} \\ \nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \end{aligned}$$

Done!

Problem 5. For a conductive material the conduction current density is much greater than the displacement current density.

(a) Show that Maxwell's equations imply a sort of diffusion equation for $\mathbf{B}$.

Start from Ampere's law.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Ignoring the displacement current term...

$$\begin{aligned} \nabla \times \mathbf{B} &\approx \mu_0 \mathbf{J} \\ &= \mu_0 \sigma \mathbf{E} \end{aligned}$$

where σ is the conductivity of the material. Take the curl of both sides

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= \mu_0 \sigma \nabla \times \mathbf{E} \\ &= -\mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law)} \\ \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} &= -\mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Vector identity on LS)} \\ -\nabla^2 \mathbf{B} &= -\mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} \qquad (\nabla \cdot \mathbf{B} = 0) \\ \nabla^2 \mathbf{B} &= \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} \\ \frac{\partial \mathbf{B}}{\partial t} &= \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \end{aligned}$$

The last line is a diffusion equation.

(b) Solve the diffusion equation for the case of the magnetic flux density $B_x(z,t)$ near a planar vacuum-copper interface, assuming for copper $\mu = \mu_0$ and $\sigma = 5.8 \times 10^7 S/m$. Assume that a 60Hz time-harmonic EM signal is applied.

Let the positive z-direction be normal to the boundary and increasing towards the vacuum. Plug $B_x(z,t)$ into the diffusion equation

$$\begin{aligned} \nabla^2 B_x(z,t) &= \mu_0 \sigma \frac{\partial B_x(z,t)}{\partial t} \\ \frac{\partial^2 B_x(z,t)}{\partial x^2} + \frac{\partial^2 B_x(z,t)}{\partial y^2} + \frac{\partial^2 B_x(z,t)}{\partial z^2} &= \mu_0 \sigma \frac{\partial B_x(z,t)}{\partial t} \end{aligned}$$

Since B_x only depends on z and t, the other derivatives are 0

$$\frac{\partial^2 B_x(z,t)}{\partial z^2} = \mu_0 \sigma \frac{\partial B_x(z,t)}{\partial t}$$

Let $\nu = 60Hz$ and $\omega = 2\pi\nu$. The harmonic time variation can be represented by $e^{i\omega t}$ and we can separate variables:

$$B_x(z,t) = B_x(z) e^{i\omega t}$$

Substituting into out differential equation,

$$\begin{aligned}
\frac{\partial^2 B_x(z) e^{i\omega t}}{\partial z^2} &= \mu_0 \sigma \frac{\partial B_x(z) e^{i\omega t}}{\partial t} \\
e^{i\omega t} \frac{\partial^2 B_x(z)}{\partial z^2} &= \mu_0 \sigma B_x(z) \frac{\partial e^{i\omega t}}{\partial t} \\
e^{i\omega t} \frac{\partial^2 B_x(z)}{\partial z^2} &= i\omega \mu_0 \sigma B_x(z) e^{i\omega t} \\
\frac{\partial^2 B_x(z)}{\partial z^2} &= i\omega \mu_0 \sigma B_x(z)
\end{aligned}$$

A solution to this is of the form $B_x(z) = B_0 e^{-\gamma z}$ since $\frac{\partial^2 B_0 e^{-\gamma z}}{\partial z^2} = B_0 \gamma^2 e^{-\gamma z} = \gamma^2 B_x(z)$. This solution requires that

$$\begin{aligned}
\gamma^2 &= i\omega\mu_0\sigma \\
\gamma &= \sqrt{i\omega\mu_0\sigma}
\end{aligned}$$

The positive square root of i is $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$. So

$$\begin{aligned}
\gamma &= \sqrt{i} \cdot \sqrt{\omega\mu_0\sigma} \\
&= \frac{1}{\sqrt{2}}(1+i)\sqrt{\omega\mu_0\sigma} \\
&= \frac{1}{\sqrt{2}}\sqrt{\omega\mu_0\sigma} + i\sqrt{\omega\mu_0\sigma} \\
&= \sqrt{\frac{\omega\mu_0\sigma}{2}} + i\sqrt{\omega\mu_0\sigma}
\end{aligned}$$

We can only use the real part in physics so we'll denote the real part of γ by $\alpha = \sqrt{\frac{\omega\mu_0\sigma}{2}} = \sqrt{2\pi\nu\mu_0\sigma/2} = \sqrt{\pi\nu\mu_0\sigma}$ and the solution is

$$B_x(z) = B_0 e^{-\alpha z}$$

With

$$\alpha = \sqrt{\pi(60s^{-1})(4\pi \times 10^{-7} N/A^2)(5.8 \times 10^7 \Omega^{-1}/m)} = 117m^{-1}$$