PHY4346
ASSIGNMENT 4

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Problem 1. Prove that $\frac{\partial \mathbf{e}^\alpha}{\partial x^\beta} = -\Gamma^\alpha_{\beta\gamma}\mathbf{e}^\gamma$.

We know that by definition $\frac{\partial \mathbf{e}_\nu}{\partial x^\beta} \equiv \Gamma^\delta_{\beta\nu}\mathbf{e}_\delta$. So,

$$\begin{aligned}
\frac{\partial \mathbf{e}^\alpha}{\partial x^\beta} &= \frac{\partial g^{\alpha\nu}\mathbf{e}_\nu}{\partial x^\beta} = g^{\alpha\nu}\frac{\partial \mathbf{e}_\nu}{\partial x^\beta} \qquad \because (\partial_\beta g^{\alpha\nu} = 0) \\
&= g^{\alpha\nu}\Gamma^\delta_{\beta\nu}\mathbf{e}_\delta \\
&= g^{\alpha\nu}\Gamma^\delta_{\beta\nu}g_{\delta\gamma}\mathbf{e}^\gamma \qquad \because (\mathbf{e}_\delta = g_{\delta\gamma}\mathbf{e}^\gamma) \\
&= g^{\alpha\gamma}\Gamma^\delta_{\beta\gamma}g_{\delta\gamma}\mathbf{e}^\gamma \qquad \because (\text{set } \nu = \gamma) \\
&= g^{\alpha\gamma}g_{\delta\gamma}\Gamma^\delta_{\beta\gamma}\mathbf{e}^\gamma \\
&= \delta^\alpha_\delta\Gamma^\delta_{\beta\gamma}\mathbf{e}^\gamma \\
\frac{\partial \mathbf{e}^\alpha}{\partial x^\beta} &= \Gamma^\alpha_{\beta\gamma}\mathbf{e}^\gamma
\end{aligned}$$

But where is the negative sign?

Problem 2.

a) Prove the covariant derivative of a vector A^α is a second-rank tensor.

$$\begin{aligned}
\nabla_{\mu'}A^{\nu'} &= \frac{\partial A^{\nu'}}{\partial x^{\mu'}} + \Gamma^{\nu'}_{\mu'\delta'}A^{\delta'} \\
&= \frac{\partial}{\partial x^{\mu'}}\Big(\frac{\partial x^{\nu'}}{\partial x^\beta}A^\beta\Big) + \Gamma^{\nu'}_{\mu'\delta'}\frac{\partial x^{\delta'}}{\partial x^\gamma}A^\gamma
\end{aligned}$$

Note that the Christoffel Symbol $\Gamma^{\nu'}_{\mu'\delta'}$ and the vector $A^{\delta'}$ transforms as:

$$\Gamma^{\nu'}_{\mu'\delta'} = \frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\partial_\gamma\frac{\partial x^\beta}{\partial x^{\mu'}} + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma} \qquad \text{and} \qquad A^{\delta'} = \frac{\partial x^{\delta'}}{\partial x^\tau}A^\tau$$

Let's transform each term separately...

$$\begin{aligned}
\frac{\partial A^{\nu'}}{\partial x^{\mu'}} &= \frac{\partial}{\partial x^{\mu'}}\Big(\frac{\partial x^{\nu'}}{\partial x^\beta}A^\beta\Big) \\
&= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial}{\partial x^\alpha}\Big(\frac{\partial x^{\nu'}}{\partial x^\beta}A^\beta\Big) \\
&= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\alpha\partial x^\beta}A^\beta + \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial A^\beta}{\partial x^\alpha}
\end{aligned}$$

and the other term:

$$\begin{aligned}
\Gamma^{\nu'}_{\mu'\delta'}A^{\delta'} &= \Big(\frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\partial_\gamma\frac{\partial x^\beta}{\partial x^{\mu'}} + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma}\Big)\Big(\frac{\partial x^{\delta'}}{\partial x^\tau}A^\tau\Big)\\
&= \frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^{\delta'}}{\partial x^\tau}A^\tau\partial_\gamma\frac{\partial x^\beta}{\partial x^{\mu'}} + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma}\frac{\partial x^{\delta'}}{\partial x^\tau}A^\tau\\
&= \frac{\partial x^{\nu'}}{\partial x^\beta}A^\gamma\partial_\gamma\frac{\partial x^\beta}{\partial x^{\mu'}} + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma}A^\gamma \qquad (\text{set } \tau=\gamma)\\
&= \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial x^\gamma\partial x^{\mu'}}A^\gamma + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma}A^\gamma\\
&= \frac{\partial x^{\nu'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^\beta\partial x^{\mu'}}A^\beta + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma}A^\gamma \qquad (\text{rename indices in first term})
\end{aligned}$$

Now, combining the two

$$\begin{aligned}
\frac{\partial A^{\nu'}}{\partial x^{\mu'}} + \Gamma^{\nu'}_{\mu'\delta'}A^{\delta'} &= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\alpha\partial x^\beta}A^\beta + \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial A^\beta}{\partial x^\alpha} + \frac{\partial x^{\nu'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^\beta\partial x^{\mu'}}A^\beta + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma}A^\gamma\\
&= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\alpha\partial x^\beta}A^\beta + \frac{\partial x^{\nu'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^\beta\partial x^{\mu'}}A^\beta + \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial A^\beta}{\partial x^\alpha} + \frac{\partial x^{\nu'}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\mu'}}\Gamma^\beta_{\alpha\gamma}A^\gamma\\
&= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\alpha\partial x^\beta}A^\beta + \frac{\partial x^{\nu'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^\beta\partial x^{\mu'}}A^\beta + \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\Big(\frac{\partial A^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma}A^\gamma\Big)\\
&= \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\alpha\partial x^\beta}A^\beta + \frac{\partial x^{\nu'}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^\beta\partial x^{\mu'}}A^\beta + \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\beta}\Big(\nabla_\alpha A^\beta\Big)
\end{aligned}$$

If the second-order terms were zero, then we would have the required transformation rule.

b) ...

Problem 3. Prove that $\nabla_\alpha g_{\mu\nu} = 0$.

Start from the general definition of the absolute gradient and then use the fact that $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}]$ and the fact that $g_{\mu\nu}$ is symmetric:

$$\begin{aligned}
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^\beta_{\alpha\mu}g_{\beta\nu} - \Gamma^\delta_{\alpha\nu}g_{\mu\delta}\\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}(g^{\beta\sigma}[\partial_\alpha g_{\mu\sigma} + \partial_\mu g_{\sigma\alpha} - \partial_\sigma g_{\alpha\mu}])g_{\beta\nu} - \frac{1}{2}(g^{\delta\tau}[\partial_\alpha g_{\nu\tau} + \partial_\nu g_{\tau\alpha} - \partial_\tau g_{\alpha\nu}])g_{\mu\delta}\\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}\delta^\sigma_\nu[\partial_\alpha g_{\mu\sigma} + \partial_\mu g_{\sigma\alpha} - \partial_\sigma g_{\alpha\mu}] - \frac{1}{2}\delta^\tau_\mu[\partial_\alpha g_{\nu\tau} + \partial_\nu g_{\tau\alpha} - \partial_\tau g_{\alpha\nu}]\\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}[\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}] - \frac{1}{2}[\partial_\alpha g_{\nu\mu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}]\\
&= \partial_\alpha g_{\mu\nu} - \partial_\alpha g_{\mu\nu} - \frac{1}{2}\partial_\mu g_{\nu\alpha} + \frac{1}{2}\partial_\mu g_{\nu\alpha} + \frac{1}{2}\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\nu g_{\alpha\mu}\\
\nabla_\alpha g_{\mu\nu} &= 0\\
&\quad \square
\end{aligned}$$

Problem 4. Prove that the Riemann tensor $R_{\alpha\beta\mu\nu}$ has 20 independent components.

Since $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$ and $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$, the only nonzero terms are when $\alpha \neq \beta$ and $\mu \neq \nu$. The only unique terms are when the pairs $\alpha\beta$ and $\mu\nu$ take on one of the six numerically distinct pairs 01, 02, 03, 12, 13, or 23 (the 10 term is related to 01, 20 related to 02, etc.). This means there are no more than $6 \times 6 = 36$ independent values.

Now imagine a 6×6 matrix of these 36 values. The symmetry $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$ implies that this 6×6 matrix is symmetric. This means that all the components above the diagonal, not including the diagonal, are not independent. There $5+4+3+2+1 = 15$ components above the diagonal, so that means there are no more than $36 - 15 = 21$ independent components of the Riemann tensor.

The final symmetry equation $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$ eliminates one of the 21 components where all the indices are unique so we're left with 20 independent components.

In 2 dimensions, there are a total of $4^2 = 16$ components. The first and second symmetry $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$ and $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$ imply that the only non-zero terms are where $\alpha \neq \beta$ and $\mu \neq \nu$ and so, analogous to above, the only unique non-zero term is R_{0101}, since R_{1001}, R_{0110}, and R_{1010} are all related to R_{0101} through the symmetry relations. You can't have less than one independent component so the Riemann tensor's only independent component in 2 dimensions is R_{0101}. There is only 1 independent component in this case.

The rest is incomplete.