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Stimulated Emission of Bremsstrahlung in a Periodic Magnetic Field

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The Weizsäcker–Williams method is used to calculate the gain due to the induced emission of radiation into a single electromagnetic mode parallel to the motion of a relativistic electron through a periodic transverse dc magnetic field. Finite gain is available from the far-infrared through the visible region raising the possibility of continuously tunable amplifiers and oscillators at these frequencies with the further possibility of partially coherent radiation sources in the ultraviolet and x-ray regions to beyond 10 keV. Several numerical examples are considered.

At least two authors have considered in detail the process of induced bremsstrahlung at radio and optical frequencies due to the scattering of an electron beam by the ion cores in neutral and ionized matter concluding that appreciable gain was available under favorable conditions.[1–3] This analysis deals with the radiation emitted by a relativistic electron beam moving through a periodic transverse dc magnetic field. We will consider the process as the scattering of virtual photons using the Weizsäcker–Williams method[4] to relate the transition rates to the more easily calculable rates for Compton scattering. As shall be seen, finite gain is available under the appropriate conditions from the far-infrared through the visible region raising the possibility of laser-type amplifiers and oscillators at these frequencies with the further possibility of partially coherent radiation sources in the ultraviolet and x-ray regions to beyond 10 keV.

In geometry, the devices suggested herein resemble the undulator structures proposed by Motz in 1950[5] and subsequently developed by him[6] as sources of millimeter wave and infrared radiation. However, Motz' proposal was entirely classical requiring a bunched beam harmonically related to the magnet period. These restrictions do not apply to the process of stimulated emission.

A closer resemblance is to be found between this paper and that of Pantell, Soncini, and Puthoff[7] who proposed the use of stimulated inverse Compton scattering but were restricted in gain to the infrared by the low microwave photon densities obtainable at present as compared to the number of virtual photons in a strong dc magnetic or electric field.

I. TRANSITION RATES

A vector potential

$$\mathbf{A}(z)=\operatorname{Re}[(B_0/q)e^{iqz}]\hat{\mathbf{y}}$$

in the lab frame yields a magnetic field

$$\mathbf{B}=\nabla\times\mathbf{A}=\operatorname{Re}[(-i)B_0e^{iqz}]\hat{\mathbf{x}}.$$

In the rest frame of a relativistic electron moving in the direction $\hat{\mathbf{z}}$ the potential appears as a traveling wave $\mathbf{A}'=\operatorname{Re}\{(B_0/q)\exp[iq\gamma(z'+\beta ct')]\}\hat{\mathbf{y}}$ yielding the fields:

$$\mathbf{B}'=\operatorname{Re}\{(-i)\gamma B_0\exp[iq\gamma(z'+\beta ct')]\}\hat{\mathbf{x}},$$

$$\boldsymbol{\mathcal{E}}'=\operatorname{Re}\{(-i)\gamma\beta B_0\exp[iq\gamma(z'+\beta ct')]\}\hat{\mathbf{y}},\qquad(1)$$

where

$$\beta\equiv v_z/c$$

and

$$\gamma\equiv1/(1-\beta^2)^{1/2}.$$

In the limit $\beta\rightarrow1$ these fields approach in detail those associated with a wave packet of real photons moving in the direction $-\hat{\mathbf{z}}'$. Assuming that the matrix element for scattering is a continuous and slowly changing function of the initial photon mass, in the limit that the photon energy in the electron rest frame is large in comparison with the photon mass excess, the mass zero expressions (for Compton scattering) can be used to obtain the desired transition rates.

The virtual photon mass excess can be computed in the laboratory frame since it is a scalar invariant:

$$m_\nu^2c^4=E^2-p^2c^2=(\hbar\omega)^2-(q\hbar)^2c^2.\qquad(2)$$

Since $\omega=0$ in the lab frame:

$$|m_\nu c^2|=q\hbar c.\qquad(3)$$

The Weizsäcker–Williams method would then be expected to be applicable in the limit:

$$h\nu=\gamma\hbar qc\gg|m_\nu c^2|=q\hbar c,\qquad(4)$$

which reduces to $\gamma\gg1$. In this limit the same scattering transition rates would be obtained in passage of the electron beam through a pulse of radiation with lab frame wavelength $\lambda_0=(1+\beta)\lambda_q$ and peak magnetic field $B=B_0/(1+\beta)$ as with an alternating dc field of period λ_q and peak field B_0. This approximation, with β set equal to one, will be used throughout the remainder of this analysis.

With a magnet period of 1 mm and a 20-GeV electron beam the wavelength of the equivalent radiation pulse in the electron rest frame is

$$\lambda'=(1/\gamma)\lambda_q=2.5\ \text{Å}\gg h/m_ec.\qquad(5)$$

Since these are the highest beam energy and shortest magnet period likely to be used, scattering in the electron rest frame can be regarded to be governed by the Thompson formula:

$$d\sigma/d\Omega=(1/2)r_0^2(1+\cos^2\theta)\approx r_0^2\quad\text{for}\quad\theta=\pi,\qquad(6)$$

where r_0 is the classical electron radius. Furthermore, the backscattered photon can be regarded as having the same electron rest frame wavelength as the incident photon.

Returning to the lab frame, for $\theta \approx \pi$ the scattering cross section is transformed to

$$(d\sigma/d\Omega)\,(\text{lab}) \approx \gamma^2(1+\beta)^2(d\sigma/d\Omega)\,(\text{electron rest frame})$$
$$\approx \gamma^2(1+\beta)^2 r_0^2. \tag{7}$$

The backscattered final-state photon wavelength is reduced by a factor of $(1+\beta)\gamma$ to

$$\lambda_f \approx \lambda_q/(1+\beta)\gamma^2 = \lambda_0/(1+\beta)^2\gamma^2 \approx \lambda_0/4\gamma^2. \tag{8}$$

Multiplied by the photon flux, the lab frame cross section yields the scattering transition rate from a definite initial state to a closely spaced set of final states:

Γ_{Compton} = transition rate from a definite initial state into a solid angle $d\Omega$,

$$= (d\sigma/d\Omega)\,\text{photon flux}\cdot d\Omega,$$
$$= (2\pi/\hbar)|\langle f \mid H' \mid i\rangle|^2 \rho_f, \tag{9}$$

where ρ_f is the density of final states.

The gain available due to stimulated emission is proportional to the spontaneous transition rate into a definite, if unspecified final photon state (typically, one of the resonant modes of an optical cavity), taking into account all possible combinations of initial photon and electron momenta leading to that state. The quantity required is

Γ_d = spontaneous transition rate into a definite final photon state

$$= (2\pi/\hbar)|\langle f \mid H' \mid i\rangle|^2 \rho_i, \tag{10}$$

where ρ_i is the appropriate density of initial states. The matrix element is the same in both cases so that Γ_d can be obtained by multiplying the Compton rate by the ratio of initial-to-final state densities:

$$\Gamma_d = \Gamma_{\text{Compton}} \cdot \rho_i/\rho_f. \tag{11}$$

Using Feynman's formula for the density of final states of a two-particle system[8]

$$dN = (1/hc)^3 \cdot k_{f0} p_{f0} \cdot c \mid \mathbf{k}_f \mid^3 d\Omega_f$$
$$\cdot [E\mathbf{k}_f^2 - k_{f0}(\mathbf{p}_f + \mathbf{k}_f)\cdot \mathbf{k}_f]^{-1}, \tag{12}$$

where the electron and photon wavefunctions have been normalized to 1 per cm³ and where k_{f0}, $\mathbf{k}_f$, and $d\Omega_f$ are the energy, momentum, and differential solid angle defining the backscattered photon, p_{f0} and $\mathbf{p}_f$ are the final-state electron energy and momentum, and $E = p_{f0} + k_{f0}$.

In the limit $\lambda_0/2\gamma \gg h/mc$, dN reduces to

$$dN \approx \gamma(1+\beta) k_{f0}^2 p_{f0} d\Omega_f / mc^2 (hc)^3. \tag{13}$$

Calculation of the density of initial states is facilitated by reference to its definition as the number of distinct combinations of states per increment of error in energy conservation which lead to the desired final state and precisely conserve momentum. If the final-state photon momentum k_f and the initial-state electron momentum p_i are specified, selection of the equivalent initial photon momentum k_i determines p_f through momentum conservation:

$$p_i - k_i = p_f + k_f \qquad (\mathbf{k}_f, \mathbf{k}_i, \mathbf{p}_f, \mathbf{p}_i \parallel \hat{\mathbf{z}}). \tag{14}$$

The density of states with p_i fixed is then determined by the density of initial photon states

$$\rho_i' dE' = \rho_\nu(k_{i0})\, dk_{i0}, \tag{15}$$

where E' is the error in energy conservation

$$E' = k_{i0} + p_{i0} - (k_{f0} + p_{f0}),$$
$$= (c^2 p_i^2 + m^2c^4)^{1/2} + k_{i0}$$
$$-[c^2(p_i - k_i - k_f)^2 + m^2c^4]^{1/2} - k_{f0}, \tag{16}$$

so that

$$dE'/dk_{i0} = 1 + c(p_i - k_i - k_f)/[c^2(p_i - k_i - k_f)^2 + m^2c^4]^{1/2}$$
$$\simeq 2 \quad \text{for} \quad \gamma \gg 1, \tag{17}$$

and

$$\rho_i' = \rho_\nu(k_{i0})\,(dk_{i0}/dE')$$
$$\approx (1/2)\rho_\nu(k_{i0}). \tag{18}$$

The total number of combinations of states leading to $\mathbf{k}_f$ is obtained by integration over p_{i0}:

$$\rho_i = \int dp_{i0} \rho_e(p_{i0}) \cdot \rho_i'$$
$$\approx \tfrac{1}{2} \int dp_{i0} \rho_e(p_{i0}) \rho_\nu(k_{i0}). \tag{19}$$

We shall assume Gaussian distributions for the normalized initial photon and electron states

$$\rho_e(p_{i0}) = (1/\pi^{1/2})(1/\Delta p_{i0}) \exp[-(p_{i0} - p_{i0}^*)^2/\Delta p_{i0}^2],$$
$$= (1/\pi^{1/2})(1/\Delta\gamma mc^2) \exp[-(\gamma-\gamma_0)^2/\Delta\gamma^2], \tag{20}$$

where

$p_{i0} = \gamma mc^2$,

$\gamma_0 \equiv$ center of the electron distribution $(p_{i0}^* \equiv \gamma_0 mc^2)$,

$\Delta\gamma \equiv$ width of the electron distribution,

and

$$\rho_\nu(k_{i0}) = (1/\pi^{1/2})(1/\Delta k_{i0}) \exp[-(k_{i0} - k_{i0}^*)^2/\Delta k_{i0}],$$
$$= (1/\pi^{1/2})(1/h\Delta\nu) \exp[-(\nu-\nu_0)^2/\Delta\nu^2], \tag{21}$$

where

$$k_{i0} = h\nu,$$

$\nu_0 \equiv$ center of the equivalent (real) photon distribution $(k_{i0}^* \equiv h\nu_0)$, and $\Delta\nu \equiv$ width of the equivalent photon distribution. Then, defining $\delta\nu$ through the choice of k_{f0} by the equation $k_{f0} = h(1+\beta)^2\gamma_0^2(\nu_0 + \delta\nu)$, the density of initial states is given by

$$\rho_i = (1/2)\int d\gamma (1/\pi^{1/2})(1/\Delta\gamma)\exp[-(\gamma-\gamma_0)^2/\Delta\gamma^2] \cdot (1/\pi^{1/2})(1/h\Delta\nu) \cdot \exp(-(1/\Delta\nu)^2\{[(\gamma_0/\gamma)^2-1]\nu_0+(\gamma_0/\gamma)^2\delta\nu\}^2) \quad (22)$$

which in the limit $\Delta\gamma \ll \gamma_0$, $\Delta\nu \ll \nu_0$, and $\delta\nu \ll \nu_0$ reduces upon completion of the integral to

$$\rho_i \approx (1/2\pi^{1/2})(1/\Delta\gamma)(1/h\Delta\nu)\cdot[(1/\Delta\gamma)^2+(2\nu_0/\gamma_0\Delta\nu)^2]^{-1/2} \cdot \exp(-2\{(\Delta\gamma/\gamma_0)(\delta\nu/2\nu_0)/[(\Delta\gamma/\gamma_0)^2+(\Delta\nu/2\nu_0)^2]\}^2). \quad (23)$$

Equation (11) can now be used to compute the spontaneous transition rate per cm³ to a definite final photon state for initial photon and electron densities of one per cm³:

$$\Gamma_d = \text{photon flux}\cdot(d\sigma/d\Omega)d\Omega\cdot\rho_i/\rho_f \approx c(2/\pi^{1/2})r_0^2\lambda_f^2(1/\Delta\gamma)(c/\Delta\nu) \cdot[(1/\Delta\gamma)^2+(2\nu_0/\gamma_0\Delta\nu)^2]^{-1/2} \cdot\exp(-2\{(\Delta\gamma/\gamma_0)(\delta\nu/2\nu_0)/[(\Delta\gamma/\gamma_0)^2+(\Delta\nu/2\nu_0)^2]\}^2). \quad (24)$$

The product of Γ_d, the actual electron and equivalent photon densities, ρ_{e0} and $\rho_{\nu 0}$, and the volume of the interaction region in cm³ per meter yields the spontaneous transition rate per meter along the magnet array.

The gain per meter is the number of excess scattered photons per meter due to the passage of a stimulating photon wave packet containing one photon. Noting that a stimulating photon density of 1 per cm³ doubles the transition rate, the gain is equal to the number of photons emitted spontaneously during the time in which the stimulating photon traverses 1 cm³ of interaction volume. Thus

$$G \equiv \text{gain in dB per meter} = 4.34\Gamma_d\cdot\rho_{\nu 0}\rho_{e0}\cdot 100\ \text{cm}\cdot(1/c). \quad (25)$$

Competing with stimulated emission and acting to depopulate the final photon state is stimulated absorption[7] wherein a photon with energy k_{f0} is backscattered from an electron with energy p_{f0} to yield a photon with energy k_{i0} and an electron of energy $p_{i0}=p_{f0}+k_{f0}-k_{i0}$. The process is stimulated by the photons already present in the dc field which are identical to the scattered photon.

The transition rate for this process can be calculated using Eq. (10) requiring only an appropriately defined density of states for absorption, ρ_{abs}. With the high-energy photon and the initial electron energy fixed as before, the induced transition rate is proportional to the product of the density of modes available to the low-energy photon multiplied by the average number of stimulating photons per mode, i.e., to $\rho_\nu(k_{i0})$. Including in ρ'_{abs} the dependence of the transition rate upon the stimulating radiation the density of states for fixed p_{f0} is

$$\rho'_{\text{abs}}dE' = \rho_\nu(k_{i0})dk_{i0}. \quad (26)$$

Utilizing Eqs. (16) and (17) to obtain dk_{i0}/dE',

$$\rho'_{\text{abs}} \approx \tfrac{1}{2}\rho_\nu(k_{i0}) \quad \text{for} \quad \gamma \gg 1. \quad (27)$$

Integration over all possible initial-state electron energies yields

$$\rho_{\text{abs}} = (1/2)\int dp_{f0}\rho_e(p_{f0})\rho_\nu(k_{i0}). \quad (28)$$

Using the same distributions for ρ_e and ρ_ν as in Eqs. (20) and (21), the integral becomes

$$\rho_{\text{abs}} = (1/2)\int d\gamma(1/\pi^{1/2})(1/\Delta\gamma) \cdot\exp\{-[(\gamma-\gamma_0)-k_{f0}/mc^2]^2/\Delta\gamma^2\}\cdot(1/\pi^{1/2})(1/h\Delta\nu) \cdot\exp(-(1/\Delta\nu)^2\{[(\gamma_0/\gamma)^2-1]\nu_0+(\gamma_0/\gamma)^2\delta\nu\}^2) \quad (29)$$

which reduces in the limit $\Delta\gamma \ll \gamma_0$, $\Delta\nu \ll \nu_0$, and $\delta\nu \ll \nu_0$ to

$$\rho_{\text{abs}} = (1/2\pi^{1/2})(1/\Delta\gamma)(1/h\Delta\nu)[(1/\Delta\gamma)^2+(2\nu_0/\gamma_0\Delta\nu)^2]^{-1/2} \cdot\exp(-2\{(\Delta\gamma/\gamma_0)[(k_{f0}/\gamma_0 mc^2)-\delta\nu/2\nu_0] \cdot[(\Delta\gamma/\gamma_0)^2+(\Delta\nu/2\nu_0)^2]^{-1}\}^2) \quad (30)$$

which is identical to the density of initial states for emission given in Eq. (23) except for the exponential factor.

The average gain per meter is then given by the difference between the gain due to stimulated emission and the loss due to absorption:

$$\bar{G} \equiv \text{average gain per meter} = (8.69/\pi^{1/2})r_0^2\lambda_f^2\cdot\rho_{\nu 0}\rho_{e0}(1/\Delta\gamma)(c/\Delta\nu) \cdot 100\ \text{cm}\cdot[(1/\Delta\gamma)^2+(2\nu_0/\gamma_0\Delta\nu)^2]^{-1/2} \cdot\left(\exp\left\{-2\left[\frac{(\Delta\gamma/\gamma_0)(\delta\nu/2\nu_0)}{(\Delta\gamma/\gamma_0)^2+(\Delta\nu/2\nu_0)^2}\right]^2\right\} - \exp\left\{-2\left[\frac{(\Delta\gamma/\gamma_0)[(k_{f0}/\gamma_0 mc^2)-(\delta\nu/2\nu_0)]}{(\Delta\gamma/\gamma_0)^2+(\Delta\nu/2\nu_0)^2}\right]^2\right\}\right). \quad (31)$$

In the limit in which absorption can be neglected ($k_{f0} \gtrsim \Delta\gamma mc^2$), $\bar{G}$ equals G and has the following reductions with $\delta\nu=0$: For a broad electron beam, $\Delta\gamma/\gamma_0 \gg \Delta\nu/\nu_0$:

$$\bar{G} = G = 245\lambda_0\lambda_f^2r_0^2\rho_{\nu 0}\rho_{e0}(\Delta\gamma/\gamma_0)^{-1}\ \text{dB m}^{-1}, \quad (32)$$

for a broad photon beam, $\Delta\nu/2\nu_0 \gg \Delta\gamma/\gamma_0$:

$$\bar{G} = G = 490(c/\Delta\nu)\lambda_f^2r_0^2\rho_{\nu 0}\rho_{e0}\ \text{dB m}^{-1}, \quad (33)$$

for photon and electron beams of equal width, $\Delta\nu/2\nu_0 = \Delta\gamma/\gamma_0$:

$$\bar{G} = G = 173\lambda_0\lambda_f^2r_0^2\rho_{\nu 0}\rho_{e0}(\Delta\gamma/\gamma_0)^{-1}\ \text{dB m}^{-1}. \quad (34)$$

Whereas if absorption is significant ($k_{f0} < \Delta\gamma mc^2$), the following reductions apply: For a broad electron beam, $\Delta\gamma/\gamma_0 \gg \Delta\nu/2\nu_0$:

$$\bar{G}=490\lambda_0\lambda_f{}^2r_0{}^2\rho_{\nu 0}\rho_{e0}(\Delta\gamma/\gamma_0)^{-1}\cdot(k_{f0}/\Delta\gamma mc^2)\cdot[(k_{f0}/\Delta\gamma mc^2)-(\delta\nu/\nu_0)/(\Delta\gamma/\gamma_0)]\ \mathrm{dB\ m^{-1}}, \quad (35)$$

For a broad photon beam, $\Delta\nu/2\nu_0 \gg \Delta\gamma/\gamma_0$:

$$\bar{G}=1.57\times10^4\frac{c}{\Delta\nu}\lambda_f{}^2r_0{}^2\rho_{\nu 0}\rho_{e0}\left[\frac{(\Delta\gamma/\gamma_0)}{(\Delta\nu/\nu_0)}\right]^2 \cdot\left[\frac{(k_{f0}/\gamma_0 mc^2)}{(\Delta\nu/\nu_0)}\right]\cdot\left[\frac{(k_{f0}/\gamma_0 mc^2)}{(\Delta\nu/\nu_0)}-\frac{(\delta\nu/\nu_0)}{(\Delta\nu/\nu_0)}\right]\mathrm{dB\ m^{-1}}, \quad (36)$$

while for photon and electron distributions of equal widths, $\Delta\gamma/\gamma_0=\Delta\nu/2\nu_0$:

$$\bar{G}=86.6\lambda_0\lambda_f{}^2r_0{}^2\rho_{\nu 0}\rho_{e0}(\Delta\gamma/\gamma_0)^{-1}\cdot(k_{f0}/\Delta\gamma mc^2)\cdot[(k_{f0}/\Delta\gamma mc^2)-(\delta\nu/\nu_0)/(\Delta\gamma/\gamma_0)]\ \mathrm{dB\ m^{-1}}. \quad (37)$$

II. DISCUSSION

Limits on the Validity of the Weak Field Approximation

In Eq. (6) it was assumed that the electron remained nearly stationary during the scattering process. However, as the field strength is increased, the electron is subject to an increasingly violent acceleration in its own rest frame. Clearly, the Doppler shift caused by this acceleration must at some point manifest itself in the angular dependence and spectral characteristics of the scattered radiation.

In the limit of very large fields, the classical expressions for synchrotron radiation[9] can be used to calculate the energy scattered per magnet semi-period into a differential range of frequencies about $\omega_{f0}\equiv 4\pi c\gamma^2/\lambda_q$ and a differential solid angle about $\theta=\pi$, a result which will be proportional to the spontaneous transition rate per mode in the frequency range of interest:

$$\frac{d^2E}{d\omega d\Omega}\approx\frac{e^2}{3\pi^2c}\left(\frac{\omega_{f0}\rho}{c}\right)^2\frac{1}{\gamma^2}K_{2/3}{}^2\left[\left(\frac{1}{\gamma^3}\right)\left(\frac{\omega_{f0}\rho}{3c}\right)\right], \quad (38)$$

an expression valid for $B\gg(4mc^2/e\lambda_q)$. The argument of the modified Bessel function is given by

$$(1/\gamma^3)(\omega_{f0}\rho/3c)=(4\pi/3)(c/\lambda_q)(mc/eB), \quad (39)$$

so that in the range of validity of Eq. (38), the Bessel function can be replaced by its small argument approximation:

$$\frac{d^2E}{d\omega d\Omega}\approx\left(\frac{2\sqrt{3}}{\pi^2}\right)^{2/3}\Gamma^2(2/3)\left(\frac{\gamma^4e^2}{c}\right)\left(\frac{mc^2}{\lambda_q eB}\right)^{2/3} \quad (40)$$

which is proportional to $B^{-2/3}$ as opposed to the B^2 dependence displayed by Eq. (31) through its dependence on the density of photons in the equivalent radiation pulse.

Landau and Pomeranchuk[10] have established that the probability of emission of a bremsstrahlung photon of wavelength λ_f is substantially diminished when multiple scattering within a distance $2\gamma^2\lambda_f$ of the point of emission of the photon becomes a significant factor. In matter, this effect leads to the dependence of the bremsstrahlung cross section upon the density of the absorber, the so-called density effect. In the present case, $2\gamma^2\lambda_f=\lambda_q$, the magnet period, and Landau's criterion for the validity of the weak field approximation requires that the probability of scattering two or more photons in a magnet period be small. The average number of scattered photons per magnet period is $2\sigma_T\lambda_q\rho_{\nu 0}$, where σ_T is the total Compton cross section and $\rho_{\nu 0}$ the density of photons in the equivalent radiation pulse. In units of the field strength B_0 and the classical electron radius r_0:

$$2\sigma_T\lambda_q\rho_{\nu 0}=(r_0B_0\lambda_q)^2/(3hc)=1.34\times10^{-10}B_0{}^2\lambda_q{}^2. \quad (41)$$

We may consider the value of the product $\lambda_qB_0=86$ kG cm for which an average of one photon per electron would be scattered every magnet period as setting an upper limit to the applicability of the formulas in the preceding section.

A quantitative estimate of the effect of a given strength field upon the frequency of the emitted radiation can be obtained using the classical picture of the emission process. While the radiation emitted by the electron in the forward direction propagates along the z axis at the speed of light, the electron moves at an average velocity determined by its energy and its path length per magnet period. After one period, the electron lags behind the radiation emitted at the corresponding point in the preceding period by a distance corresponding to the wavelength of the final-state photon.

If the transverse field is assumed to vary abruptly between $\pm B\hat{\mathbf{x}}$ between magnet semi-periods, the path length l per period is given by

$$l=(\lambda_q\phi/2)/\sin(\phi/2)\approx\lambda_q[1+\tfrac{1}{6}(\phi/2)^2] \quad \text{for} \quad \phi\ll1, \quad (42)$$

where

$$\phi=2\sin^{-1}(1/\gamma)(\lambda_q/4c)(eB/mc)\approx\gamma^{-1}(\lambda_q/2c)(eB/mc).$$

The final-state wavelength is then given by

$$\lambda_f=\lambda_q[(l/\beta\lambda_q)-1]\approx(\lambda_q/2\gamma^2)[1+\tfrac{1}{3}(\lambda_q/4c)^2(eB/mc)^2]. \quad (43)$$

For $(\lambda_q/4c)(eB/mc)\ll1$, Eq. (8) for the wavelength of the final-state photon applies while larger fields and/or magnet periods produce a finite shift in the final-state wavelength.

For a Fourier component at the fundamental wavelength of the equivalent radiation pulse of 5 kG, $B=(5\pi/4)$ kG which with $\lambda_q=1$ cm would produce a fractional change in wavelength:

$$\Delta\lambda_f/\lambda_f\approx\tfrac{1}{3}(\lambda_q/4c)^2(eB/mc)^2=0.102. \quad (44)$$

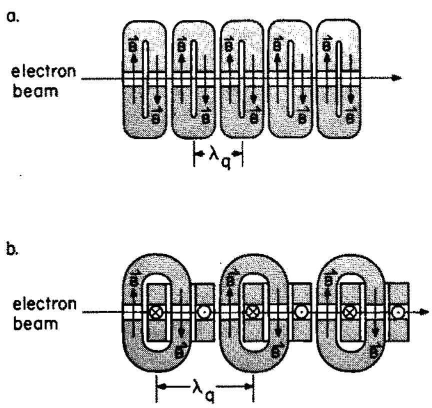

FIG. 1. Some possible configurations producing periodic magnetic fields: (a) Array using toroidal iron magnets to produce linearly polarized radiation. (b) Array of magnets producing circularly polarized radiation.

Our primary concern is, of course, the gain, and it seems unlikely that any serious deviation from the weak field result would occur in this case considering the limit for the field inferred from Eq. (41) amounting to 86 kG in this example. A 10% shift in frequency nonetheless represents a significant departure from the weak field picture. Accordingly, no higher fields will be considered in the examples presented in the succeeding sections.

Photon Beam Homogeneity

The use of a periodic magnetic field of finite length sets an upper limit to the homogeneity of the equivalent radiation pulse. Assuming a sinusoidal field n periods in length, the square of the Fourier transform of the amplitude of the equivalent radiation pulse $|f(\omega)|^2$ is proportional to

$$|f(\omega)|^2 \propto \left(\frac{\sin[2\pi n(\omega-\omega_0)/2\omega_0]}{(\omega-\omega_0)} - \frac{\sin[2\pi n(\omega+\omega_0)/2\omega_0]}{(\omega+\omega_0)}\right)^2. \quad (45)$$

Approximating $|f(\omega)|^2$ in the neighborhood of ω_0 by the Gaussian form assumed in Eq. (21) we obtain

$$(\Delta\nu/\nu) \approx 0.53/n = 0.53\lambda_q/L. \quad (46)$$

Several possible configurations for the magnet array are shown in Fig. 1. The use of toroidal magnets bored along a diameter to permit passage of the beam would reduce fringing fields albeit at the cost of increased demagnetizing fields within the magnets. In terms of the equivalent radiation pulse, either linear or circular polarization could be obtained [Figs. 1(a) and 1(b)]. Assuming a beam hole of the order of 5 mm, a 1–2-cm period might be realizable producing photon beam homogeneities of the order of 10^{-3} to 10^{-4} for magnet arrays of the order of ten meters in length.

Alternatively, the individual magnets could be split into distinct and symmetric halves with the electron beam passing through the gap between the opposed pole faces. This geometry has the advantage of permitting the introduction of an alternating transverse gradient in the magnetic field by tapering the width of the gaps. The focusing effect associated with the alternating gradient would greatly reduce the transverse excursions of electrons in the beam due to space-charge effects or to an initial component of transverse momentum. Although the introduction of such a gradient also degrades the homogeneity of the equivalent radiation pulse, the effect seems to be acceptably small for typical fields and gradients.

A separate set of deflection magnets would have to be employed at the beginning and end of the magnet array to separate the electron beam from the incident and emergent photons. An alternative in the far-infrared and millimeter regions would be to employ a waveguide to confine the radiation with the guide bent and pierced at the ends to permit the injection and extraction of the electron beam.

Electron Beam Homogeneity

The mean energy of the scattered photons can be calculated by integrating the product of the lab frame final-state photon energy and the normalized Thompson cross section in the electron rest frame:

$$\begin{aligned}\bar{k}_{f0} &= (\sigma_T^{-1}) \int_{\text{electron rest frame}} d\Omega k_{f0}(\theta) \left(\frac{d\sigma}{d\Omega}\right), \quad (47)\\ &= \tfrac{1}{2} k_{f0}(\theta=\pi),\\ &\approx 2\gamma^2 k_{i0} \quad \text{for} \quad \gamma \gg 1.\end{aligned}$$

Since the energy lost by an electron in scattering is proportional to γ^2, the more energetic electrons will tend to lose energy more rapidly than those in the lower half of the distribution. Assuming that each electron in the beam has scattered the same number of photons, the mean change in $\Delta\gamma$ per scattered photon is

$$d(\Delta\gamma)/dn|_{\text{scattering}} = -(k_{f0}/mc^2)(\Delta\gamma/\gamma). \quad (48)$$

An account must also be taken of the statistical variations in the number of scattered photons. After an average of n scattering events per electron the square of the standard deviation of the energy lost per electron is $nk_{f0}^{*2} = (7/20)nk_{f0}^2$, where k_{f0}^* is the root mean-squared final-state photon energy. Assuming a Gaussian initial distribution, the convolution of the statistical broadening with the initial distribution yields for the square of the modified distribution width $\Delta\gamma'$:

$$(\Delta\gamma')^2 = \Delta\gamma^2 + 0.7n[k_{f0}^2/(mc^2)^2] \quad (49)$$

and for $d(\Delta\gamma)/dn$:

$$d(\Delta\gamma)/dn\,|_{\text{statistics}} = 0.35k_{f0}^2/\Delta\gamma(mc^2)^2. \quad (50)$$

In the limit in which other sources of broadening can be neglected the beam homogeneity then evolves according

TABLE I. Values of gain (G) and gain minus absorption ($\bar{G}$) for some selected values of electron beam energy and current. For all the cases cited, the electron beam cross section is 1.0 mm², the magnet period is 1.0 cm, and the array length 10 m.

γ	λ_f	$(\Delta\gamma/\gamma)$	$(\Delta\nu/\nu)$	I	B_0	$\bar{G}$	G
10	50μ	2.6×10^{-4}	5.3×10^{-4}	5 mA	5×10^3 G	7.7 dB m^{-1}	6.8×10^5 dB m^{-1}
10^2	5000 Å	1.3×10^{-4}	5.3×10^{-4}	0.5 A	5×10^3 G	0.7 dB m^{-1}	8.2×10^3 dB m^{-1}
10^3	50 Å	4.1×10^{-4}	5.3×10^{-4}	1 A	5×10^3 G	1.0×10^{-3} dB m^{-1}	1.0 dB m^{-1}
7.7×10^3	0.85 Å	1.1×10^{-3}	5.3×10^{-4}	1 A	5×10^3 G	4.5×10^{-7} dB m^{-1}	1.2×10^{-4} dB m^{-1}

to

$$d(\Delta\gamma)/dz=[d(\Delta\gamma)/dn](dn/dz)$$
$$=2\rho_{\nu0}\sigma_T\left[\frac{0.35k_{f0}^2}{\Delta\gamma(mc^2)^2}-\frac{k_{f0}}{mc^2}\frac{\Delta\gamma}{\gamma}\right], \quad (51)$$

where $\rho_{\nu0}$ is the density of photons in the equivalent radiation pulse and σ_T is the total Compton cross section.

Ideally, the electron beam would be prepared with a width $\Delta\gamma mc^2 \ll k_{f0}$ so that absorption could be neglected. With this condition, statistical broadening is seen to dominate with $\Delta\gamma/\gamma$ increasing according to Eq. (50). The feasibility of this approach evidently requires that the change in $\Delta\gamma mc^2$ in passage through the interaction region be small in comparison with k_{f0}:

$$L\cdot(d/dz)(\Delta\gamma mc^2/k_{f0})=\rho_{\nu0}\sigma_T L(0.7k_{f0}/\Delta\gamma mc^2)\ll1, \quad (52)$$

where L is the length of the interaction region. Unfortunately, $\rho_{\nu0}\sigma_T L$ will typically be greater than one with $(k_{f0}/\Delta\gamma mc^2)>1$ by assumption so that the condition cannot generally be met.

At $(\Delta\gamma/\gamma)=(0.35k_{f0}/\gamma mc^2)^{1/2}$ the broadening effects of statistical fluctuations are canceled by the narrowing due to energy-dependent scattering. This value represents the equilibrium state of the beam in the absence of other sources of broadening and would be observed after passage through a very long interaction region or in a recirculating electron beam device. For a fixed value of λ_q, the equilibrium homogeneity $(\Delta\gamma/\gamma)$ is proportional to $\gamma^{1/2}$ since the final-state photon energy is proportional to γ^2.

At very low final-state photon energies the equilibrium value is likely to be significantly smaller than the photon beam homogeneity. Since the gain per unit length is optimized when the electron and photon beams have the same homogeneity, it would be desirable in this situation to prepare the electron beam with a width exceeding the equilibrium value. A suitable condition in this case is that the fractional change in $\Delta\gamma$ be small in comparison with one:

$$L\cdot(\Delta\gamma)^{-1}[d(\Delta\gamma)/dz]=2\rho_{\nu0}\sigma_T L(k_{f0}/\gamma mc^2)\ll1. \quad (53)$$

This criterion will be satisfied under a fairly broad range of conditions.

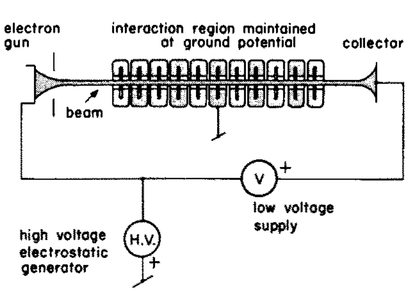

FIG. 2. Experimental setup.

Available Gain

Some theoretical values for the gain calculated using Eq. (31) are listed in Table I for several values of beam energy and current and of the magnetic field strength and the length and number of field periods. The homogeneity of the equivalent radiation pulse was calculated using Eq. (46) while that of the electron beam was set equal to the equilibrium value $(\Delta\gamma/\gamma)=(0.35k_{f0}/\gamma mc^2)^{1/2}$ except for the case $\gamma=10$ in which it was set equal to one-half the photon beam homogeneity (a factor of 7 greater than the equilibrium value) to maximize the gain per unit length. The electron and photon densities were calculated using the formulas:

$$\rho_{\nu0}=5.008B_0^2\lambda_0\times10^{13}\ \text{photons/cm}^3\ (\text{with } B_0 \text{ in gauss and } \lambda_0 \text{ in cm}),$$
$$\rho_{e0}=2.082I\times10^{10}\ \text{electrons/cm}^3 \text{ for a 1-mm}^2 \text{ beam cross section with } I \text{ in amperes.} \quad (54)$$

The electron beam currents required for finite gain above $\gamma=10^2$ would evidently require a recirculating electron beam device of some form for continuous operation. But in the low-energy limit, assuming that the beam loss in the interaction region could be reduced to negligible proportions, it is interesting to note that the accelerating potential could be supplied by an electrostatic generator. With the beam current recovered a few hundred volts above the cathode potential and returned to the cathode, the high-voltage generator would perform no net work on an electron in the beam reducing the high-voltage power requirements to the level of the mechanical and electrical losses in the generator (Fig. 2).

Spontaneously Emitted Radiation

The dependence of the gain on the square of the final-state wavelength probably precludes the development of steady-state oscillation in the region beyond the ultraviolet. However, it remains possible that, considering its intensity, the spontaneously emitted radiation might itself find applications. For example, with the parameters listed in Table I for $\gamma=10^3$ and using Eq. (25) to relate the spontaneous scattering rate to the gain, the spontaneous flux would itself be of the order of 40 W $(\text{mHz})^{-1}$ $(\text{stearadian})^{-1}$ at the line center at 50 Å. The statistical properties of the radiation emitted in this regime are also worthy of note.

Assuming that the number of stimulated photons per unit length obeys Poisson statistics while absorption is governed by an independent binomial distribution law, it can be shown that while the mean number of photons originating spontaneously and passing through a medium in which absorption equals stimulated emission is unaffected by the medium, the square of the standard deviation in the number n counted in an interval of time τ is

$$\langle n^2\rangle-\langle n\rangle^2=\langle n\rangle[1+(GL/20\cdot\log e)], \tag{55}$$

where G is the gain in the absence of absorption and L the length of the interaction region. For the case just considered at 50 Å this would represent nearly a 50% increase in the standard deviation over what would have been calculated if n had been assumed to obey Poisson statistics. The presence of the medium can be interpreted as increasing the joint probability of emission of two or more photons in excess of what would have been observed if each scattering event were completely independent. Although the difference would not be significant in processes involving the interaction and detection of single photons, it could have an important effect on the counting rates in photon correlation experiments.

A particularly interesting possibility is the use of glancing angle reflection from a series of Mössbauer absorbers[11] to produce a high-intensity source of partially coherent radiation at x-ray wavelengths. Considering as an example ^{57}Fe, the cross section for resonant absorption of a 14.4-keV (0.85-Å) incident photon is $\sigma_A=0.6\times10^{-18}$ cm^2 at 770°C[12] with a linewidth of 10^{-8} eV (fwhm) corresponding to an excited state half-life of 100 nsec.[13] Operation above the Curie point is required to suppress the internal magnetic field which would otherwise split the degenerate nuclear levels responsible for absorption. Assuming a Lorentz line shape, the linear absorption coefficient κ due to recoilless absorption is

$$\kappa=\rho\sigma_A(\Delta\omega)^2/[(\omega-\omega_0)^2+(\Delta\omega)^2]\ \text{cm}^{-1}, \tag{56}$$

where

$$\Delta\omega=(1/2)\times\text{linewidth (fwhm)}$$

and

$$\rho=\text{no. of atoms per cm}^3.$$

Identifying κ as $(2\pi/\lambda_f)$ times the imaginary part of the index of refraction $n(\omega)$, the contribution of the absorption line to the real part of $n(\omega)$ can be obtained from the dispersion integral:

$$\begin{aligned}\text{Re}[n(\omega)]&=(1/\pi)P\int_{-\infty}^{+\infty}d\omega\ \text{Im}[n(\omega)/(\omega-\omega_0)],\\ &=(1/\pi)(\lambda_f/2\pi)\rho\sigma_A\\ &\quad\times P\int_{-\infty}^{+\infty}\frac{d\omega}{(\omega-\omega_0)}\frac{(\Delta\omega)^2}{(\omega-\omega_0)^2+(\Delta\omega)^2},\\ &=0.676\times10^{-4}\{(\omega-\omega_0)\Delta\omega[(\omega-\omega_0)^2+(\Delta\omega^2)]^{-1}\},\end{aligned} \tag{57}$$

where a 100% enriched ^{57}Fe absorber has been assumed. The measured value of $n(\omega)$ will include the contributions of the various atomic absorption edges as well as the local modulation near ω_0. The energy involved is considerably greater than the binding energy of the most tightly bound electrons so that the contribution of the atomic levels will be small and negative. Assuming the index of refraction due to these levels can be adequately represented by the formula[14]:

$$\begin{aligned}n^2&=1-(4\pi e^2\rho Z/m_e\omega^2)\\ &=1-1.59\times10^{-5}\quad\text{at}\quad 0.85\ \text{Å},\end{aligned} \tag{58}$$

the critical angle for total external reflection is 0.52° at $(\omega-\omega_0)=\Delta\omega$ falling to 0.22° at $\omega=\omega_0$ and 0.47° at $(\omega-\omega_0)=2\Delta\omega$. Thus, for an angle of incidence in excess of 0.22°, only those photons within a neighborhood of the Mössbauer line will be totally reflected with the attenuation of those above and below the line determined by the value of the angle of incidence. At a critical angle of 0.47°, the reflection coefficient for a photon outside the linewidth would be 3.5×10^{-3}. After successive reflections at this angle the beam would consist primarily of photons within a linewidth of ω_0 with a coherence length of approximately $2\pi c/\Delta\omega=$ 125 m.

In estimating the available power we note that the spacing of the cavity modes describing the final-state radiation for a 10-m interaction length would be $\Delta\nu=c/2L=15$ mHz, a factor of 6 greater than the Mössbauer linewidth, so that only one axial mode needs to be considered. If all of the radial modes lying within the electron beam are included (3×10^5 modes for a 1 mm^2 beam at 0.85 Å) a total of 5×10^8 photons per second would be emitted into the Mössbauer linewidth with the parameters listed in Table I for $\gamma=7.7\times10^3$ (3.9 GeV). This amounts to about 1 μW of power into a solid angle of 5×10^{-9} sr. To produce a beam of the equivalent luminosity using a ^{57}Co source would require an activity in excess of 10^7 Ci.

We have omitted from this analysis the consequences of the absorption occurring in the nuclei lying just inside the surface of the absorbers. The effect of this process is to reduce the reflection coefficient within the Mössbauer line from $R=1$ to $R\approx 0.22$ for an angle of incidence of 0.47° while leaving the reflection coefficient outside the line essentially unchanged. While this would substantially reduce the number of reflected photons, the estimated flux still appears to be large enough to be of experimental interest provided that the angular spread in the reflected beam due to imperfections in the mirrors could be kept at a minimum.

Note Added in Proof. The classical result for the radiated energy calculated using the synchrotron radiation formula [Eq. (38)] is also proportional to the square of the magnetic field strength provided that an account is taken of the shift in wavelength of the emitted radiation. If ω_{f0} in Eq. (38) is understood to be the frequency of the backscattered photons at a beam energy of $\gamma_0 mc^2$ in the weak field limit and if the actual beam energy is selected to compensate for the wavelength shift due to the magnetic field [Eqs. (43) and (44)], then in the region of validity of Eq. (38) the radiated energy per magnet semi-period is given by

$$\frac{d^2E}{d\omega d\Omega} \approx [(2\pi^2)^{-2/3}\Gamma^2(2/3)/3]\left(\frac{\gamma_0{}^4e^2}{c}\right)\left(\frac{\lambda_q eB}{4mc^2}\right)^2.$$

This result remains valid for arbitrarily large magnetic fields. Evidently the increase in the scattering transition rate with γ compensates for the effects of multiple scattering above the Landau limit.

With respect to the available gain this result indicates that a significant improvement could be obtained through the use of very large magnetic fields. Extrapolation of the weak field results in Table I to 100 kG indicates that useful gain would be available to beyond 100 Å.

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