Prerequisites

Find the rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}$

Transformations :

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 3R_1$
$R_4 \rightarrow R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{bmatrix}$$

$R_4 \rightarrow R_4 - 3R_2$
$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank of $A = 2$

Test for consistency and solve

$$x + y - z = 0$$
$$2x - y + z = 3$$
$$4x + 2y - 2z = 2$$

note:

$\rho(A:B) = \rho(A) = n \rightarrow$ has unique sol^n

$\rho(A:B) = \rho(A) < n \rightarrow$ has ∞ sol^n

$\rho(A:B) \neq \rho(A) \rightarrow$ no sol^n

The corresponding augumented matrix is :-

$$[A:B] = \begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 2 & -1 & 1 & : & 3 \\ 4 & 2 & -2 & : & 2 \end{bmatrix}$$

Transformations :-

$R_3 \rightarrow R_3/2$
$$\begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 2 & -1 & 1 & : & 3 \\ 2 & 1 & -1 & : & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 2R_1$
$$\begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 0 & -3 & 3 & : & 3 \\ 0 & -1 & 1 & : & 1 \end{bmatrix}$$

after $R_3 \leftrightarrow R_2$
$R_3 \rightarrow R_3 - 3R_2$
$$\begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 0 & -1 & 1 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\therefore \rho(A:B) = \rho(A) = 2$

$\therefore$ it has ∞ solutions

solving this :-

$$x + y - z = 0$$
$$-y + z = 1$$

let $z = k$

$\Rightarrow -y = 1 - k \Rightarrow y = k - 1$

$\Rightarrow x = k - (k-1)$

$x = 1$

$$\boxed{\therefore (x, y, z) = (1, k-1, k)}$$

if a column has a leading entry its called a pivotal column.

if a column isn't a pivotal column, it is considered as a free variable (ie we put $z = k$)

leading entry $\Rightarrow$ the first non-zero element in a given row

Natural nos : $\mathbb{N}$: $\{1, 2, 3, 4 \dots\}$

Whole nos : W : $\{0, 1, 2, 3 \dots\}$

Integers : $\mathbb{Z}$: $\{0, \pm 1, \pm 2 \dots\}$

Rational : $\mathbb{Q}$: $\{x \mid x = p/q,\ p \& q \in \mathbb{Z},\ q \neq 0\}$

Irrational : $\mathbb{R} - \mathbb{Q}$

Real nos : $\mathbb{R}$

complex : $\mathbb{C}$

C^*, R^*, Q^* : non zero complex / Real / irrational

Consider a non empty set G

$$* : G \times G \rightarrow G$$

$(a, b) \in G \times G$

$a * b \in G$

$(G, *) \rightarrow$ binary structure

The binary structure $(G, *)$ in group is

I) $\forall\ a, b, c \in G$

$a * (b * c) = (a * b) * c$

II) $\forall\ a \in G$ there exists ($\exists$) an element

$e \in G \ni a * e = a = e * a$

e is called the identity element

III) $\forall\ a \in G$ $\exists\ b \in G \ni a * b = e = b * a$

then b is called the iverse of a group

IV) $\forall\ a, b \in G$ $a * b = b * a$ abelian group

closure, commutative, associative, identity, inverse ← satisfies set of integers and operation of addition

ex : consider Z with b.o usual addition '+'

$\forall\ a, b \in Z$. $a + b \in Z$

closure property is satisfied in Z

$\forall\ a, b, c \in Z$

$(a+b)+c = a+(b+c)$

0 which belongs to Z is identity element

$\because \forall a \in Z$, $a + 0 = a$

$\forall a \in Z \quad \exists\ -a \in Z$

$\Rightarrow \quad a + (-a) = 0$

$\forall\ a, b \in Z$

$a + b = b + a$

$\therefore (Z, +)$ is an abelian group

ST $Q - \{1\}$ is an abelian group under the binary operation $*$ defined by

$$a * b = \underbrace{a+b}_{\text{rational}} - \underbrace{ab}_{\text{rational}}$$

- closure property is satisfied on the set $Q - \{1\}$
- associative

$$(a*b)*c$$
$$= (a+b-ab)*c$$
$$= a*c + b*c - ab*c$$
$$= a+c-ac + b+c-bc - (ab+c-abc)$$
$$= a+c+b-ab-bc-ac+abc \quad (1)$$

$$a*(b*c)$$
$$= a*(b+c-bc)$$
$$= a*b + a*c - a*bc$$
$$= a+b-ab+a+c-ac-(a+bc-abc)$$
$$= a+b+c-ab-bc-ac+abc \quad (2)$$

$\therefore$ as (1) = (2)

RHS = LHS, hence proved

Q) Binary operation $*$ is defined by

$$a*b = a+b-1 \; ; \; a,b \in Z$$

compute the identity element e and inverse of a ie a^{-1}

For identity element :-

$$a*e = a$$

$$a+e-1 = a$$

$$e = 1$$

For inverse element :-

$$a * a^{-1} = e$$

$$a+a^{-1}-1 = 1$$

$$a + a^{-1} = 2$$

$$a^{-1} = 2-a$$

Q) Binary operation $*$ is defined by

$$a*b = ab/3 \, , \; a,b \in R^* \text{ (non zero reals)}$$

compute the identity element e and inverse of a ie a^{-1}

$$a*e = a$$

$$\frac{ae}{3} = a$$

$$\Rightarrow e = 3 \qquad \text{(identity element)}$$

$$a * a^{-1} = e$$

$$\frac{a a^{-1}}{3} = 3$$

$$a^{-1} = \frac{9}{a} \quad \text{(inverse of a)}$$

Field : The algebraic structure $(F, +, \cdot)$ is said to be a field if :-

i) $(F, +)$ is an abelian group
ii) $(F, \circ)$ is an abelian group
iii) Distributive property

$$a \cdot (b+c) = ab + ac$$
$$(a+b) \cdot c = ac + bc$$

Q) $(Z, +, \cdot)$ where $Z \rightarrow$ integers

i) $(Z, +) \rightarrow$ it is abelian
ii) $(Z, \cdot) \rightarrow$ talk about non zero elements

∴ it is not abelian

(multiplicative inverse doesnt exist for all)

∴ It is not a field

Q) $(R, +, \cdot)$ where $R \rightarrow$ real nos

i) $(R, +) \rightarrow$ It is abelian
ii) $(R, \cdot) \rightarrow$ it is abelian (don't consider zero)
iii) Distributive :

$$a \cdot (b+c) = ab + ac$$
$$(a+b) \cdot c = ac + bc$$

(show working of all 5 conditions for each pair in exam)

∴ It is a field

The set of real numbers R, rational numbers Q and complex numbers C are fields with the binary operations + (addition) and $\circ$ (multiply)

VECTOR SPACES :

Let F be a field an V be a non empty set of vectors for every ordered pair $(u, v) \in V$ let there be defined uniquely a sum $u+v$ and for every $u \in V$ and for every $c \in F$, a scalar product $c \circ u \in V$. The set V is called the vector space over the field F if the following axioms are satisfied for every $u, v, w \in V$ and $c, c' \in F$

i) closure property :

$$\left.\begin{array}{r} u+v \in V \\ \text{and} \quad c \circ u \in V \end{array}\right\} \begin{array}{l} \forall\, u, v \in V \\ \forall\, c \in F \end{array}$$

ii) Associative property :

$$u+(v+w) = (u+v)+w$$

iii) Identity :

The element e is called the identity element $e \in V$ if $\forall u$, $u+e = u = e+u$

iv) Inverse :

$\forall u \in V$, There exists a unique element

$u^{-1} \in V$ such that

$$u + u^{-1} = e = u^{-1} + u$$

v) commutative :

$\forall\ u, v \in V$

$$u + v = v + u$$

vi) $c \cdot (u + v) = cu + cv$

vii) $(c + c')u = cu + c'u$

} distributivity

viii) $c \cdot (c'u) = (cc') \cdot u$

ix) $1 \cdot u = u$

where, 1 is the multiplicative identity element

Examples :

1) Let F be a field and n be the positive integer

2) Let $V_n(F)$ be a set of all ordered n tupples of the elements of the field F

ie $V_n(F) = \{(x_1, x_2 \ldots x_n) \mid x_i \in F\}$

Defined vector addition & scalar multiplication as below

i) $u + v = (x_1, x_2, \ldots x_n) + (y_1, y_2, \ldots y_n)$

$= (x_1 + y_1, x_2 + y_2 + \ldots, x_n + y_n)$

ii) $c \cdot u = c \cdot (x_1, x_2, \ldots, x_n)$

$= (cx_1, cx_2, \ldots, cx_n) \quad \forall c \in F$

For all $c \in F$ show that $V_n(F)$ is a vector space over the field F

Let $u = (x_1, x_2, \dots x_n)$
$v = (y_1, y_2, \dots y_n)$
also, $c, c' \in F$

consider all the properties
i) closure :
$u + v = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in V_n(F)$

ii) associative:
$(u+v)+w = (x_1+y_1+z_1, x_2+y_2+z_2, \dots)$
$u+(v+w) = (x_1+y_1+z_1, x_2+y_2+z_2, \dots)$

iii) identity element :
$\exists$ a zero vector is $0 = (0, 0, \dots 0) \forall V_n(F)$
$\ni u+0 = u = 0+u \quad \forall u \in V_n(F)$
$\therefore$ 0 is the identity element

iv) inverse :
For every $u = (x_1, x_2, \dots, x_n)$
$\exists \quad -u = (-x_1, -x_2, \dots, -x_n) \in V_n(F)$
$\ni u + (-u) = 0 = (-u) + u$
$\Rightarrow (-u)$ is the inverse of u

v) commutative
$u + v = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$ — (1)

$v + u = (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n)$ — ②

$\therefore$ it is commutative as ① = ②

vi) $c \cdot (u + v) = cu + cv$

LHS: $c(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$= (cx_1 + cy_1, cx_2 + cy_2, \dots)$

RHS: $c(x_1, x_2, \dots) + c(y_1, y_2, \dots)$

$= (cx_1, cx_2, \dots) + (cy_1, cy_2, \dots)$

$= (cx_1 + cy_1, cx_2 + cy_2, \dots)$

$\therefore$ it satisfies it

vii) $(c + c')u = cu + c'u$

LHS: $(c + c')(x_1, x_2, \dots, x_n)$

$= ((c + c')x_1, (c + c')x_2, \dots)$

RHS: $c(x_1, x_2, \dots) + c'(x_1, x_2, \dots)$

$= (cx_1, cx_2, \dots) + (c'x_1, c'x_2, \dots)$

$= (cx_1 + c'x_1, cx_2 + c'x_2, \dots)$

$= ((c + c')x_1, (c + c')x_2, \dots)$

$\therefore$ it satisfies it

viii) $c(c'u) = cc'(u)$

LHS: $c(c'x_1, c'x_2, \dots)$

$(cc'x_1, cc'x_2, \dots)$

RHS: $cc'(x_1, x_2, \dots)$

$(cc'x_1, cc'x_2, \dots)$

$\therefore$ it satisfies it

ix) $1 \cdot u = u$

$1 \cdot (x_1, x_2, x_3) = (x_1, x_2, x_3)$

$\therefore$ it satisfies it

Hence, $V_n(F)$ is a vector space over F

Note:

For $F = R$ set of real numbers, then we get

$V_n(F) = V_n(R)$

$\hookrightarrow$ vector tupple of n dimension, each being a real number

now, $V(R) = R$ - Real number

$V_2(R) = R^2$ - like $((x_1, y_1), (x_2, y_2), \dots)$ theyre plane vectors

$V_3(R) = R^3$ - like $((x_1, y_1, z_1), \dots)$ theyre space vectors

Show that the set

$V = \{ a + \sqrt{2} b \mid a, b \in Q \}$ where Q - set of rational numbers is a vector space over Q under usual addition & scalar multiplication

$u = a_1 + b_1\sqrt{2}$, $v = a_2 + b_2\sqrt{2}$, $w = a_3 + b_3\sqrt{2} \in V$

$c, c' \in Q$

i) closure (✓)

$u + v = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} \in V$

ii) associative (✓)

$(u+v)+w = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2}$

$u+(v+w) = a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2} + a_1 + b_1\sqrt{2}$

iii) identity element (✓)

$u + 0 = u = 0 + u$

$a_1 + b_1\sqrt{2} + 0 = a_1 + b_1\sqrt{2} = 0 + a_1 + b_1\sqrt{2}$

iv) inverse (✓)

$u + (-u) = 0$

$a_1 + b_1\sqrt{2} - a_1 - b_1\sqrt{2} = 0$

v) commutative (✓)

$u + v = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2}$

$v + u = a_2 + b_2\sqrt{2} + a_1 + b_1\sqrt{2}$

vi) $c \circ (u+v) = cu + cv$ (✓)

LHS: $c(a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2})$

$= ca_1 + cb_1\sqrt{2} + ca_2 + cb_2\sqrt{2}$

RHS: $c(a_1 + b_1\sqrt{2}) + c(a_2 + b_2\sqrt{2})$

$= ca_1 + cb_1\sqrt{2} + ca_2 + cb_2\sqrt{2}$

vii) $(c+c')u = cu + c'u$ (✓)

LHS: $(c+c')(a_1 + b_1\sqrt{2})$

$= c(a_1 + b_1\sqrt{2}) + c'(a_1 + b_1\sqrt{2})$

RHS: $c(a_1 + b_1\sqrt{2}) + c'(a_1 + b_1\sqrt{2})$

viii) $c(c'u) = cc'(u)$ $(\checkmark)$

LHS : $c(c'a_1 + c'b_1\sqrt{2})$

$= cc'a_1 + cc'b_1\sqrt{2}$

RHS : $cc'(a_1 + b_1\sqrt{2})$

$= cc'a_1 + cc'b_1\sqrt{2}$

ix) $1 \cdot u = u$ $(\checkmark)$

$1 \circ (a_1 + b_1\sqrt{2}) = a_1 + b_1\sqrt{2} = u$

Let V be a polynomial of degree $\leq n$ with coefficients in the field F together with zero polynomial then show that V is a vector space under addition of defined by

$c \circ u = c(a_0 + a_1x + a_2x^2 + \dots + a_nx^n)$

$= ca_0 + ca_1x + ca_2x^2 + \dots + ca_nx^n$

Let
$$\left.\begin{aligned} u &= a_0 + a_1x + \dots + a_nx^n \\ v &= b_0 + b_1x + \dots + b_nx^n \\ w &= c_0 + c_1x + \dots + c_nx^n \end{aligned}\right\} \text{belong to set } V$$

consider $c, c' \in F$

Now, consider all axioms :-

i) closure $(\checkmark)$

$u + v \in V$

$\because$ sum of 2 polynomials is a polynomial

ii) associative $(\checkmark)$

$(u+v)+w = u+(v+w)$

iii) identity element (✓)

There exists a zero polynomial $0 = 0 + 0x + 0x^2 + \ldots \in V$

$\ni\ u + 0 = u = 0 + u$

iv) multiplicative inverse (✓)

$u + (-u) = 0$

for every $u = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in V$

There exists $-u = -a_0 - a_1 x - a_2 x^2 - \ldots - a_n x^n \in V$

$-u$ is the inverse of u

v) commutative property (✓)

$u + v = v + u$

vi) $c(u+v) = cu + cv$ (✓)

vii) $(c + c')u = cu + c'u$ (✓)

viii) $c(c'u) = cc'(u)$ (✓)

ix) $1 \cdot u = u$ (✓)

$\therefore$ V is a vector space over the field F

Let $V = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \;\middle|\; x, y \in C \right\}$

under usual addition & multiplication with field C of complex numbers. Show that V is a vector space

Let $u = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}$

$v = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$

consider k & k' belonging $\in C$ be the scalars.

i) closure (✓)

$u + v = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{bmatrix} \in V$

ii) associative (✓)

$(u + v) + w = u + (v + w) \in V$

iii) identity element (✓)

$u + 0 = u = 0 + u$ ($0 \Rightarrow$ null matrix)

iv) multiplicative inverse (✓)

$u + (-u) = 0$

v) commucative property (✓)

$u + v = v + u$

vi) $c(u+v) = cu + cv$

LHS: $c\begin{bmatrix} x_1+x_2 & y_1+y_2 \\ -y_1-y_2 & x_1+x_2 \end{bmatrix} = \begin{bmatrix} cx_1+cx_2 & cy_1+cy_2 \\ -cy_1-cy_2 & cx_1+cx_2 \end{bmatrix}$

RHS $= \begin{bmatrix} cx_1 & cy_1 \\ -cy_1 & cx_1 \end{bmatrix} + \begin{bmatrix} cx_2 & cy_2 \\ -cy_2 & cx_2 \end{bmatrix} = \begin{bmatrix} cx_1+cx_2 & cy_1+cy_2 \\ -cy_1-cy_2 & cx_1+cx_2 \end{bmatrix}$

vii) $(c+c')u = cu + c'u$ ($\checkmark$)

LHS : $\begin{bmatrix} cx_1+c'x_1 & cy_1+c'y_1 \\ -cy_1-c'y_1 & cx_1+c'x_1 \end{bmatrix}$

RHS : $\begin{bmatrix} cx_1 & cy_1 \\ -cy_1 & cx_1 \end{bmatrix} + \begin{bmatrix} c'x_1 & c'y_1 \\ -c'y_1 & c'x_1 \end{bmatrix} = \begin{bmatrix} cx_1+c'x_1 & cy_1+c'y_1 \\ -cy_1-c'y_1 & cx_1+c'x_1 \end{bmatrix}$

viii) $c(c'u) = cc'(u)$ ($\checkmark$)

ix) $1 \circ u = u$ ($\checkmark$)

Let R^+ be a set of all +ve real nos define the operations of addition & scalar multiplication as below

$\alpha + \beta = \alpha\beta$ and

$c \circ \alpha = \alpha^c$

ST R^+ is a vector space over the real field

i) closure:

$\alpha+\beta = \alpha\beta \in R^+$ ∴ holds good as $\alpha\beta \in R^+$ as $\alpha, \beta \in R^+$

ii) associative :

$(\alpha+\beta)+\gamma = (\alpha\beta)+\gamma = \alpha\beta\gamma$

$\alpha+(\beta+\gamma) = \alpha+(\beta\gamma) = \alpha\beta\gamma$

$\therefore$ Holds good

iii) identity element :

it can be 1

$\Rightarrow \alpha+1 = \alpha = 1+\alpha$

$\therefore$ as $1 \in R^+$ it holds good

iv) multiplicative inverse :

$\alpha+\left(\frac{1}{\alpha}\right) = \alpha\times\frac{1}{\alpha} = 1$

and as $1/\alpha \in R$, it holds good

v) commutative property :

$\left.\begin{array}{l}\alpha+\beta = \alpha\beta \\ \beta+\alpha = \beta\alpha\end{array}\right\}$ $\therefore$ LHS = RHS holds good

vi) $c(u+v) = cu+cv$:-

LHS : $c(\alpha\beta) = (\alpha\beta)^c$

RHS : $\alpha^c+\beta^c = (\alpha\beta)^c$

$\therefore$ LHS = RHS holds good

vii) $(c+c')u = cu+c'u$:-

LHS : $\alpha^{(c+c')}$

RHS : $\alpha^c+\alpha^{c'} = (\alpha)^{c+c'}$

$\therefore$ LHS = RHS holds good

viii) $c(c'u) = cc'(u)$:-

LHS : $(\alpha^{c'})^{c} = c \cdot \alpha^{c'} = \alpha^{cc'}$

RHS : $cc'(\alpha) = \alpha^{cc'}$

$\therefore$ LHS = RHS holds good

ix) $1 \cdot u = u$:-

LHS : $1 \cdot u = u^{1} = u$

$\therefore$ LHS = RHS holds good

Hence, R^{+} is a vector space over R.

Q) The set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; x_1 + x_2 = 0 ; x_1, x_2 \in R \right\}$ is a vector space over R

Q) The set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; x_1 + x_2 + x_3 = 0 ; x_1, x_2, x_3 \in R \right\}$ is a vector space over R

SUBSPACES

A non empty subset W of a vector space V over a field F is called a subspace of V if W itself is a vector space over F under the same operations of vector addition & scalar multiplication as defined in V.

note: the set $\{0\}$ (consisting of zero vector) is a subspace of V and the entire vector space V itself is a subspace of V. These two subspaces are called trivial or improper subspaces of V.
Any other subspace different from these two is called a proper subspace of V.

Theorem:-

A non-empty subset W of a vector space V over a field F is a subspace of V iff

i) $\forall\ \alpha, \beta \in W$, $\alpha + \beta \in W$

ii) $\forall\ \alpha, \beta \in W$ & $c \in F$, $c\alpha \in W$

This implies that W is closed under vector addition and scalar multiplication, then the set becomes a subspace of V.

iii) $\{0\}$ must be a part of the vector space

1. R^2 R^3

$\begin{pmatrix} x \\ y \end{pmatrix}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow$ if $z = 0$ then it becomes R^2

$R^2 \rightarrow$ set of all vectors lying on a plane passing thru the origin is a subspace of the vector space $\mathbb{R}^3$.

2. consider the vector space $\mathbb{R}^2$ defined over field $\mathbb{R}$

i) consider $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \in \mathbb{Z} \right\}$

To check if S is a subspace of $\mathbb{R}^2$

Let $\alpha = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\beta = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ belonging to S

where, $x_1, y_1, x_2, y_2 \in \mathbb{Z}$

$$\alpha + \beta = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in S$$

Let $c = \sqrt{2} \in \mathbb{R}$

consider $\sqrt{2} \cdot \alpha = \sqrt{2} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}x_1 \\ \sqrt{2}y_1 \end{pmatrix} \notin S$

$\therefore$ S is not closed under scalar multiplication

$\therefore$ S is not a subspace of $\mathbb{R}^2$

ii) consider $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x = 0 \text{ or } y = 0 \right\}$

$$\alpha + \beta = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \notin S$$

$\Rightarrow$ S is not closed under vector addition

$\therefore$ S is not a subspace of $\mathbb{R}^2$

Q) Let S be the set of all 4 dimensional vectors of the form $S = \{(x, 2x, -3x, x) \mid x \in R\}$ in V_4. Show that S is a subspace of V_4

Let $\left.\begin{aligned} \alpha &= (x_1, 2x_1, -3x_1, x_1) \\ \beta &= (x_2, 2x_2, -3x_2, x_2) \end{aligned}\right\} \in S$

$\alpha + \beta = (x_1 + x_2, 2(x_1 + x_2), -3(x_1 + x_2), (x_1 + x_2))$

$\therefore \alpha + \beta \in S$ ✓

consider $c\alpha$

$c \cdot (x_1, 2x_1, -3x_1, x_1)$

$\Rightarrow (cx_1, 2cx_1, -3cx_1, cx_1)$

$\therefore c\alpha \in S$ ✓

$\therefore$ S is a subspace of V_4

Q) Prove that the set of all solutions (a, b, c) of the equation $a + b + 2c = 0$ is a subspace of R^3

$\alpha + \beta = (a_1, b_1, c_1) + (a_2, b_2, c_2)$

$= (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

also, $a_1 + b_1 + 2c_1 = 0$ and $a_2 + b_2 + 2c_2 = 0$

we've to take $(a_1 + a_2) + (b_1 + b_2) + 2(c_1 + c_2)$

$\Rightarrow a_1 + b_1 + 2c_1 + a_2 + b_2 + 2c_2$

$= 0 + 0 \Rightarrow 0$ ✓

consider $k\alpha$

$\Rightarrow k(a, b, c)$

$\Rightarrow (ka, kb, kc)$ — ①

now wkt $a+b+2c=0$

multiply BS by $k \Rightarrow ka+kb+2kc=0$

$\therefore$ consider ①

$\Rightarrow ka+kb+2kc=0$ ✓

$\therefore$ S is a subspace of R^3

Is the set $\{(x, y, z) \mid x^2+y^2+z^2 <= 1\}$ is a subspace of R^3 or not

$\alpha+\beta = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$(x_1+x_2, y_1+y_2, z_1+z_2)$

let $x_1 = 0.5$, $y_1 = 0$, $z_1 = 0$

let $x_2 = 0.75$, $y_2 = 0$, $z_1 = 0$

now,

$(x_1+x_2)^2 + (y_1+y_2)^2 + (z_1+z_2)^2$

$(0.5+0.75)^2 + 0 + 0$

$\Rightarrow (1.25)^2$ its not $<= 1$

$\therefore$ it isn't closed under vector addition & its not a subspace of R^3.

Determine which of the following are subspaces of M_{22}, set of all 2×2 matrices with real entries (over field $\mathbb{R}$)

i) The set of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are integers

ii) The set of 2×2 matrices such that $A = A^T$

iii) The set of all 2×2 matrices such that $|\det A| = 0$

i) $\alpha + \beta = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

$\therefore$ ($\checkmark$)

$$c \cdot \alpha = \sqrt{2}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}a_1 & \sqrt{2}b_1 \\ \sqrt{2}c_1 & \sqrt{2}d_1 \end{bmatrix}$$

$\therefore$ ($\times$)

Not a subspace

ii) $\alpha = \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}$ $\quad \beta = \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix}$

$$\alpha + \beta = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & c_1+c_2 \end{bmatrix}$$

$\therefore$ ($\checkmark$)

$$(A+B)^T = A^T + B^T$$
$$= A + B \quad \text{(hence proved)}$$

$$k \cdot \alpha = k\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ kb_1 & kc_1 \end{bmatrix}$$

$\therefore$ ($\checkmark$)

is a subspace

$$(k \cdot A)^T = k(A^T)$$
$$= k(A) \quad \text{(hence proved)}$$

$\therefore$ its a subspace

iii) $|A| = 0 \Rightarrow ad - bc = 0$ or $ad = bc$

$$\alpha = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad \beta = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$A = \alpha + \beta = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$|A| = ((a_1+a_2)(d_1+d_2) - (b_1+b_2)(c_1+c_2))$$
$$= a_1d_1 + a_1d_2 + a_2d_1 + a_2d_2 - b_1c_1 - b_1c_2$$
$$- b_2c_1 - b_2c_2$$
$$= a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1$$

take example & disprove

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

but $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore$ NO

Verify $W = \{ p(x) \in P_3 \mid p'(0) = 0 \}$ is a subspace of P_3

$P_3 = \{ a_0 x^3 + a_1 x^2 + a_2 x + a_3 \mid a_i \in \mathbb{R} \}$

$$\alpha(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$$
$$\beta(x) = b_0 x^3 + b_1 x^2 + b_2 x + b_3$$

$$\alpha'(0) = 0 \cdot x^2 + 0 \cdot x + 0 + 0 \quad \Rightarrow \quad \alpha'(0) = 0$$

i) $\alpha + \beta = (a_0 + b_0) x^3 + (a_1 + b_1) x^2 + (a_2 + b_2) x + (a_3 + b_3)$

it $\in P_3$ $(\therefore \checkmark)$

$\Rightarrow p'(0) \Rightarrow (A_0)\, 0 \cdot (A_1) \cdot 0 + (A_2) \to 0$

$p'(0) = 0$

ii) $c \cdot \alpha = c(a_0 x^3 + a_1 x^2 + a_2 x + a_3)$

$= c a_0 x^3 + c a_1 x^2 + c a_2 x + c a_3$

$p'(0) = (A_0) \cdot 0 + (A_1) \cdot 0 + (A_2) \to 0$

$p'(0) = 0$

Hence, W is a subspace of P_3

Verify whether $W = \{ f(x) \mid 2f(0) = f(1) \}$ over $0 \le x \le 1$ is a subspace of V which is the set of all functions over the field $\mathbb{R}$.

Let $f_1, f_2 \in W$

of the form where only $x+1$ has some effect

i) $f_1 + f_2$

$$2f_1(0) = f_1(1)$$
$$2f_2(0) = f_2(1)$$
$$2(f_1 + f_2(0))$$
$$2(f_1(0) + f_2(0)) = f_1(1) + f_2(1)$$
$$2(F(0)) = F(1) \quad (\checkmark)$$

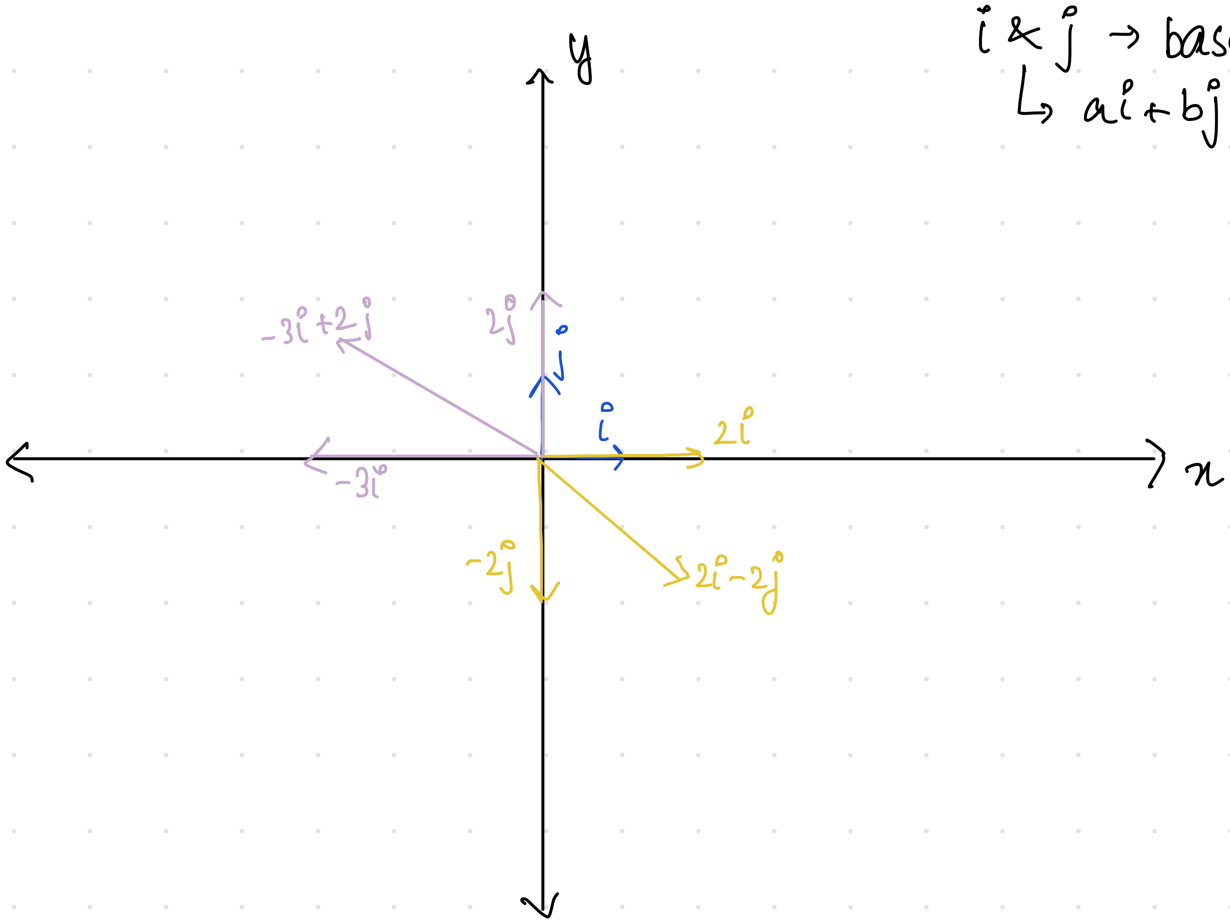
ii) $\alpha \cdot f$

$$\alpha \cdot 2(f(0)) = \alpha \cdot f(1) \quad (\checkmark)$$

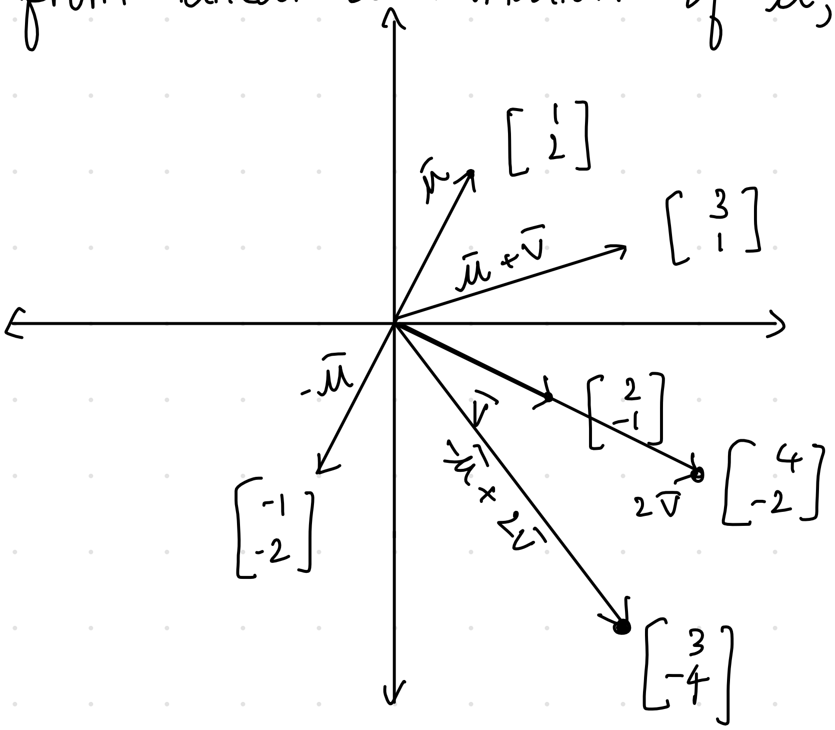
$\therefore$ W is a subspace of V

LINEAR COMBINATION

$\hat{i}$ & $\hat{j}$ → bases
↳ $a\hat{i} + b\hat{j}$



many vectors from linear combination of u, v



definition :-

Let V be a vector space over the field F and $u_1, u_2, \ldots u_n$ be any n vectors of $V(F)$ the vector of the form $c_1u_1 + c_2u_2 + \ldots + c_nu_n$ which can also be written as $\sum_{i=0}^{n} c_i u_i$ where $c_1, c_2 \ldots c_n$ belongs to F is called the linear combination of the vectors $u_1, u_2 \ldots, u_n$

eg: $(2,3,-1) = 2(1,0,0) + 3(0,1,0) + 1(0,0,-1)$

linear span

$S = \{u_1, u_2, \ldots, u_n\}$ of $V(F)$

$L[S] = \{u \mid u = c_1u_1 + c_2u_2 + \ldots + c_nu_n, c_i \in F\}$

Let $S = \{u_1, u_2 \ldots u_n\}$ be n vectors of R^n then the set of all linear combination of the elements of S is called the linear span of S denoted by $L[S]$

ie $L[S] = \{u / u = c_1u_1 + c_2u_2 + \ldots + c_nu_n, c_i \in F\}$

$L[S]$ is a subset of $V(F)$

and $L[S]$ is a subspace of $V(F)$

Theorem :

Let S be a non empty subset of $V(F)$ then, :-

1. $L[S]$ is a subspace of $V(F)$
2. S is a subset of $L[S]$

3. $L[S]$ is the smallest subspace of V containing S

Q) Show that $(3,7)$ belongs to $L[(1,2),(0,1)]$ in $V_2(\mathbb{R})$

consider $(3,7)$ as a linear combination of the two.

$$(3,7) = c_1(1,2) + c_2(0,1)$$
$$= (c_1, 2c_1) + (0, c_2)$$
$$= (c_1, 2(c_1)+c_2)$$

$\Rightarrow c_1 = 3 \quad \Rightarrow \quad 2(3)+(c_2) = 7 \quad \Rightarrow \quad c_2 = 1$

$\therefore$ yes

Q) determine whether $(8,0,5)$ is a linear combo of $\{(1,2,3),(0,1,4),(2,-1,1)\}$

$$(8,0,5) = (c_1+0+2c_3,\ 2c_1+c_2-c_3,\ 3c_1+4c_2+c_3\}$$

$$\Rightarrow \quad c_1+0+2c_3 = 8$$
$$2c_1+c_2-c_3 = 0$$
$$3c_1+4c_2+c_3 = 5$$

$$\boxed{\begin{aligned} c_1 &= 2 \\ c_2 &= -1 \\ c_3 &= 3 \end{aligned}}$$

Q) $h = 4x^2+3x-7$ lies in span $\{f, g\}$ where $f = 2x^2-5$, $g = x+1$

$$4x^2+3x-7 = 2ax^2-5a+xb+b$$
$$4x^2 = 2ax^2 \quad \Rightarrow \quad a = 2$$
$$3x = bx \quad \Rightarrow \quad b = 3$$

$-7 = -5a + b \Rightarrow -7 = -10 + 3 = -7$

$\therefore$ it does ✓

Q) Find the subspace spanned by $S = \{(2,0,0), (0,0,-2)\}$ in $V_3(R)$

The linear span is a subspace

given, we have S and $L[S]$ is the linear span of S and $L[S]$ is a subspace of $V_3(R)$

$$L[S] = \{u / u = c_1(2,0,0) + c_2(0,0,-2)\}$$
$$L[S] = \{u/u = (2c_1, 0, -2c_2)\}, \quad c_1, c_2 \in R$$

In $V_3(R)$, show that the plane $x_3 = 0$ can be spanned by a pair of vectors $\{(2,2,0), (4,1,0)\}$

Prove that any vector spanned by $\{(2,2,0), (4,1,0)\}$ lies in the plane $x_3 = 0$. Show that any vector u can be expressed as a linear combination of these vectors.

$$u = c_1(2,2,0) + c_2(4,1,0)$$
$$= (2c_1 + 4c_2, 2c_1 + c_2, 0)$$

$\therefore x_3 = 0$

Consider these vectors $c_1u_1 + c_2u_2 + c_3u_3 = 0$

ie $c_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + c_3 \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0$

we will have either 1. ∞ sol^n : if $\rho(A) < n$

2. trivial sol^n $(0,0,0)$: if $\rho(A) = n$

LINEAR dependence :

if $u_1, u_2, \ldots n$ vectors of a vector space $V(T)$, then these vectors are said to be linearly dependent if :

$$c_1 u_1 + c_2 u_2 + \ldots + c_n u_n = 0$$

implies atleast one $c_i \neq 0$

LINEAR independence

The vectors $u_1, u_2, \ldots u_n$ are linearly independent if $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n = 0$ implies all $c_i = 0$

note:

all determinants (minors) of the matrix of order less than the rank is zero

$\therefore$ If the determinant of $u_1, u_2, \ldots, u_n = 0$, they are linearly dependent else they're independent

Q) show that $s = \{(1,0,1), (1,1,0), (-1,0,-1)\}$ is linearly dependent

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = 0$$

$$(c_1, 0, c_1) + (c_2, c_2, 0) + (-c_3, 0, -c_3) = 0$$

$$c_1 + c_2 - c_3 = 0$$
$$c_2 = 0$$
$$c_1 - c_3 = 0$$

$\therefore c_1, c_2, c_3 = 0$

$\therefore$ linearly dependent

Note :-

1. The set $\{(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\}$ of the vector space $V(R)$ is linearly dependent only if
$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

2. 2 vectors $\in V_2(R)$ are linearly dependent iff one vector can be expressed as a scalar multiple of another
$\Rightarrow u = kv$

3. A set of vectors of V containing the zero vector is linearly dependent

4. The set containing a single vector u of $V(F)$ is linearly dependent if $u \neq 0$

Q) check dependency

i) $\{(3,0,0), (4,1,0), (2,5,2)\}$ independent

ii) $\{(1,1,2), (1,2,1), (3,1,1)\}$ independent

i) $\begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 5 & 2 \end{vmatrix} = 3(2) - 0 - 0$
$= 6$ $\therefore$ independent

ii) $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(2-1) - 1(1-3) + 2(1-6)$
$-1 + 2 - 10$
$= -9$ $\therefore$ independent

Q) check dependency

$$V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = 0$$

$$\therefore (c_1, c_2, c_3) = (0, 0, 0)$$

Q) ST $e_1 = (1, 0, 0 \dots 0)$, $e_2 = (0, 1, \dots 0)$ $\dots$, $e_n = (0, 0 \dots, 1)$ of the vector space $V_n(R)$ are linearly independent

$$c_1 e_1 + c_2 e_2 + \dots + c_n e_n = 0$$

$$\Rightarrow (c_1, c_2, \dots, c_n) = 0$$

$\therefore$ each $c_i = 0$ $\therefore$ linearly independent

or can also write in det form (rank $= n$) (identity matrix) so $|A| = |I| = 1$

$\therefore$ linearly independent

basis

A subset B of a vector space $V(F)$ is said to be a basis for V if

i) B is linearly independent

ii) $L[B] = V(F)$

dimensions

The number of elements in a basis is called the dimension of the vector space & it is denoted by $\dim(V)$

ex: $V_n(R)$ is an n-dimensional vector space
$V_3(R)$ is a 3 dimensional space

STANDARD BASIS

The basis $S = \{e_1, e_2, \ldots, e_n\}$ of the vector space $V_n(R)$ is called the standard basis where $e_1 = (1, 0, 0 \ldots, 0)$
$e_2 = (0, 1, 0 \ldots 0) \ldots, e_n = (0, 0, \ldots, 1)$

for 2 elements:

$\{(1, 0), (0, 1)\}$

$S = \{e_1, e_2, e_3\}$ is the standard basis of $V_3(R)$

finite dimensional space

A vector space $V(F)$ is considered finite dimensional space if it has a finite basis.

NOTE:

1. any 2 basis of a finite dimensional vector space V have the same finite number of elements

2. a vector space which is not finitely generated

aybe called an infinite dimensional space

3. in an n-dimensional vector space :-
   a. any n+1 elements of V are linearly dependent
   b. no set of n-1 elements can span V

4. in an n-dimensional vector space V(F), any n linearly independent vetors is a basis

5. any linearly independent set of elements of a finite dimensional vector space V is part of basis

6. for n-vectors of n-dimensional vector space V to be a basis, it is sufficient that they span V or that they are linearly independent

Q) check the set { (1, 2, 3), (3, 1, 0), (-2, 1, 3) } for linear dependence in $V_3(R)$. Do they form basis?

if $V_n(R)$ you need n linearly independent vectors

consider det if $\det(A) = 0$ ↳ linearly dependent

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{vmatrix}$$

$$= 1(3-0) - 2(9-0) + 3(3+2)$$
$$= 0$$

∴ vectors are linearly dependent and hence, they cannot form the basis of $V_3(R)$.

Q) check for linear dependence for $\{(1,0,1), (0,2,2), (3,7,1)\}$ do they form basis of $V_3(R)$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 7 & 1 \end{vmatrix} = \begin{aligned} &1(2-14) + 0 + 1(0-6) \\ &-12-6 \\ &= -18 \neq 0 \end{aligned}$$

$\therefore$ they're linearly independent and hence they form the basis of $V_3(R)$.

Q) Let $A = \{(1,-2,5), (2,3,1)\}$ be a linearly independent basis of $V_3(R)$. Extend this to form a basis of $V_3(R)$

wkt, no of vectors in the basis of $V_3(R)$ is 3.
Let us find the 3rd vector given 2 linearly independent vectors.

let $u_1 = (1,-2,5)$, $u_2 = (2,3,1)$

consider $L[A] = \{c_1u_1 + c_2u_2 \mid c_1, c_2 \in R\}$

$L[A] = \{c_1+2c_2, -2c_1+3c_2, 5c_1+c_2 \mid c_1, c_2 \in R\}$

consider,

the 3rd, $u_3 = (1,0,0)$

let us check if $u_3 \in L[A]$

$\Rightarrow$ $c_1 + 2c_2 = 1$

$-2c_1 + 3c_2 = 0 \quad \Rightarrow \quad 3c_2 = 2c_1$

$5c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -5c_1$

$c_1 - 10c_1 = 1 \quad \Rightarrow \quad -9c_1 = 1 \Rightarrow c_1 = -1/9$

$\Rightarrow C_2 = \frac{2}{3} \times \frac{-1}{9} \times 2 \quad = \frac{-4}{27}$

$\frac{3}{3} \times 5 \times \frac{-1}{9} \quad - \frac{4}{27}$

$-\frac{15}{27} - \frac{4}{27} \quad \neq 0$

$\therefore u_3 \notin L[A]$ (u_3 independent of u_1, u_2)

$\therefore u_3 = (1,0,0)$ is valid to form basis of $V_3(\mathbb{R})$

Q) given 2 linearly independent vectors $\{(2,1,4,3)$ & $(2,1,2,0)\}$ basis of $V_4(\mathbb{R})$ that includes these 2 vectors

$u_3 = (0,0,0,1)$

$u_4 = (1,0,0,0)$

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 2 & 1 & 4 & 3 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \quad \begin{aligned} &1(\text{det inner}) \\ &= 2(0-0) - 1(-2) + 4(-1) \\ &= 2 - 4 = -2 \\ &= \underline{-2} \quad \neq 0 \end{aligned}$$

$\therefore$ linearly independent

hence u_3 & u_4 valid

note: the non zero rows of the row reduced echelon form of a matrix are linearly independent

ST the vectors $\{(2,-1,-5,2), (1,-1,-4,0), (2,1,1,6)\}$ $(1,1,2,4)$ in R^4 & extract a linearly independent. Also find the dimension & basis of the subspace spanned by them

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$R_2 \to -1/3\, R_2$
$R_3 \to -1/2\, R_3$

$$\text{Ans} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 2$$

There are 2 non zero rows & these form a linearly independent set OR
The rows of mat (A) corresponding to these non zero rows forms the linearly independent set, thus the set consisting of $A = \{(1,1,2,4), (0,1,3,2)\}$ are linearly independent or ①

$A = \{(1,1,2,4), \underbrace{(2,-1,-5,2)}_{②}\}$

$L[A] = \{c_1(1,1,2,4) + c_2(2,-1,-5,2)\}$ is a subspace of $V_4(R)$.

↳ cause only 2 vectors.

dimension of $L[A] = 2$

and the basis of $L[A]$ are ① & ②

Q) Let S be the subspace of R^3 defined by $S = \{(a,b,c) / a+b+c = 0\}$. Find basis & dimension of S.

Let us check if α & β are linearly independent

$\alpha = (1,-1,0)$ & $\beta = (1,0,-1) \in S$

consider their linear combination

$$c_1\alpha + c_2\beta = 0$$

$$c_1 + c_2 = 0$$

$$-c_1 = 0$$

$$-c_2 = 0$$

$$\therefore c_1 \;\&\; c_2 = 0$$

$\Rightarrow$ α & β are linearly independent, hence a set consisting of α & β forms a basis of S

Q) Show that the field C of complex numbers is a vector space over the field R of reals. What is the dimension.

Let: $u = a_1 + ib_1$, $v = a_2 + ib_2$, $w = a_3 + ib_3 \in C$

$a_1, a_2, a_3, \in R$

b_1, b_2, b_3

consider $u+v = (a_1+a_2) + i(b_1+b_2) \in C$

$c_i \cdot u = c_1 a_1 + i b_1 = c_1 a_1 + i c_1 b_1 \in C$

C is a vector space

$C = \{ c_1(1) + c_2(i) \mid c_1, c_2 \in R \}$

The set $\{1, i\}$ is linearly independent & every element of C can be expressed as a linear combination of $\{1, i\}$, thus $\{1, i\}$ forms the basis of C

$\dim(C) = 2$

Q) find the basis & dimension of the subspace spanned by the subset

$$S = \left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} \right\}$$

of the vector space of all 2×2 matrices over R

a 2×2 matrix can be represented by a 4 dim vector, hence we shall write the given set of matrices as

$$\text{mat } A = \begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$

$R_4 \to 3R_4 + R_3$ $\begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The matrices corresponding to the non-zero rows $\left\{ \begin{pmatrix} 1 & -5 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 5 \end{pmatrix} \right\}$ forms the basis of the subspace spanned by S

$\dim(L[S]) = 2$

Q) find the dimension, basis & the spanning set for the 4 fundamental subspaces of mat A

$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

Reduced row echolon form

$$A \Rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = C$$

now, $A \equiv C$

pivotal column

(pivot) x_1 & x_3 are basic variables and
(non pivot) x_2 & x_4 are free variables

Null (A) $\Rightarrow$ $Cx = 0$
consider $Cx = 0$ $\qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$\Rightarrow Cx = 0$
$\Rightarrow x_1 + 3x_2 + 3x_3 + 2x_4 = 0$
$3x_3 + 3x_4 = 0$

non pivotal colum vars take free values
(treated as free variables)

Let us express the basic vars in terms of the free variables

$\Rightarrow x_3 = -x_4$
$x_1 + 3x_2 - 3x_4 + 2x_4 = 0$
$x_1 = -3x_2 + x_4$

$$\text{Null}(A) = X = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$$

$$= \left\{ x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \;/\; x_2, x_4 \in R \right\}$$

Thus the basis of Null (A) = {(-3, 1, 0, 0), (1, 0, -1, 1)}
∴ dimension of Null (A) = 2

Null (A) is a subspace of R^4

column space col(A) :

In the above q we have 1^{st} & 3^{rd} columns as pivotal col. cols of A corresponding to these pivotal cols are (1, 2, -1) & (3, 9, 3) & they form the basis of col A.

$$\text{col } A = \{ c_1(1,2,-1) + c_2(3,9,3) \mid c_1, c_2 \in R \}$$

$$\dim(\text{col } A) = 2$$

row space row(A):

non zero rows of echolon A or of A forms the basis of row (A)

$$\therefore \text{basis}(\text{row}(A)) = \{(1,3,3,2), (0,0,3,3)\} \text{ or}$$
$$= \{(1,3,3,2), (2,6,9,7)\}$$

$$\text{row}(A) = \{ c_1(1,3,3,2) + c_2(2,6,9,7) \}$$

row (A) is a subspace of R^4.

Left null space Nul(A^T):

consider $A^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}_{4\times3}$

op 1

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 4R_1$

op 2

$R_2 \longleftrightarrow R_4$

op 3

$R_3 \to R_3 - R_2$

$\therefore$ we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

A^T ka echolon form

y_3 is non a pivotal col

now $A^T y = 0$ or $By = 0$

$\Rightarrow$ $x + 2y - z = 0$

$3y + 6z = 0$

let $z = k$

$y = -\frac{6k}{3} = -2k$

$x = k - 2(-2k)$

$x = k + 4k = 5k$

$(5k, -2k, k)$

$Nul(A^T) = \{(5, -2, 1)\}$

$dim(Nul(A^T)) = 1$

$\therefore$ $Nul(A^T)$ is a subspace of R^3

rank nullity theorem

nullity : $dim(Nul(A))$

$rank(A) + nullity(A) = n$ no of cols of A

for the given matrix, $rank(A) = 2$

nullity $= 1$ & n = num of cols $= 3$

$\therefore$ $2 + 1 = 3 = 3$ hence, verified.

nul space is set of all sol^n s of $Ax = 0$ and nul space of an $n \times n$ matrix is a subspace of R^n.

col space : is set of all linear combinations of cols A
it is a subspace of R^m (given mat : $m \times n$)

row space : set of all linear combination of rows of mat A. it is a subspace of R

left null space : set of all sol^n of $A^Ty = 0$

Find the basis & dim for 4 fundamental subspaces of A also verify rank nullity theorem.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$\rho(A) = 2$

i) $\text{Null}(A) = Bx = 0$

$$x_1 - 2x_2 + 2x_3 + 3x_4 + x_5 = 0$$
$$x_3 + 2x_2 - 2x_5 = 0$$
$$\Rightarrow x_3 = 2x_5 - 2x_2$$
$$x_1 = 2x_2 + 2(2x_5 - 2x_2) - 2x_3 - 3x_4 - x_5$$
$$\Rightarrow x_1 = -2x_2 + 3x_5 - 2x_3 - 3x_4$$

$$\left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

dim = 3

basis = $\{(-2,1,-2,0,0), (-3,0,0,1,0), (-1,0,-1,0,1)\}$

finish ques later

TRANSFORMATION:

Q) Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. Find the image of the square whose vectors $(0,0)$ $(2,0)$ $(2,2)$ $(0,2)$ under the transformation $T(x) = Ax$

Consider the vertices of the square

$x_1 = (0,0)$, $x_2 = (2,0)$, $x_3 = (2,2)$ $x_4 = (0,2)$

given transformation is

$T(x) = Ax$

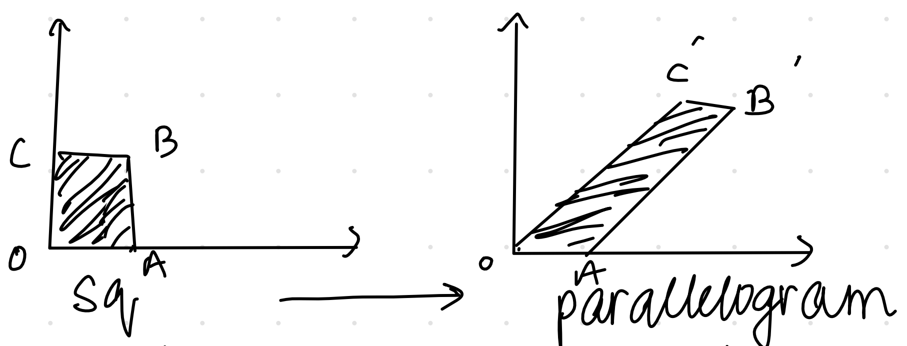
$\Rightarrow T(x) = Ax_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= Ax_2 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$Ax_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$Ax_4 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

now sketch graph :-



base is kept intact

Q) Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

and define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$Tx = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 3x_2 \end{bmatrix}$$

a) Find $T(u)$, the image of u under T

b) Find an x in $\mathbb{R}^2$ whose image under T is b

c) If there are more than one x whose image under T is b

d) determine if c is in the range of the tranformation of T

a) $T(u) = Au = \begin{bmatrix} 5 \\ 1 \\ 9 \end{bmatrix}$

b) $Ax = B$

$$[A : B] = \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \qquad x = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$\rho(A : B) = \rho(A) = 2 =$ no of unknowns

d) $Ax = C$

$$[A : C] = \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 0 & 70 \end{array}\right]$$

$\rho(A) = 2 \qquad \rho(A : C) = 3$

$Ax = C$ has no sol^n

LINEAR TRANSFORMATION:

Let U & V be two vector spaces under the same field F. The mapping $T: U \to V$ is said to be a linear transformation (LT) if:

i) $T(\alpha+\beta) = T(\alpha) + T(\beta) \quad \forall \alpha, \beta \in U$

ii) $T(c\alpha) = c \cdot T(\alpha) \quad \forall c \in F$ & $\alpha \in U$

A linear transformation is also called a linear map on U.

note:

i) every matrix transformation is a LT

ii) LT preserves the operation of vector addn & scalar mul

Theorem:

A mapping $T: U \to V$ from the vector space $U(F)$ into $V(F)$ is a LT iff

$$T(c_1\alpha + c_2\beta) = c_1 T(\alpha) + c_2(T(\beta)) \quad \forall c_1, c_2 \in F$$
$$\& \ \alpha, \beta \in U$$

Q) If T is a mapping from $V_3(R)$ into $V_3(R)$ defined by $T(x_1, x_2, x_3)$. ST T is a LT

consider $T(\alpha+\beta) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$

$$= T((x_1+y_1), (x_2+y_2), (x_3+y_3))$$
$$= (0, x_2+y_2, x_3+y_3)$$
$$= (0, x_2, x_3) + (0, y_2, y_3)$$
$$= T(x_1, x_2, x_3) + T(y_1, y_2 y_3)$$
$$= T(\alpha) + T(\beta)$$

consider $T(c \cdot \alpha) = T(c(x_1, x_2, x_3))$

$$= T(cx_1, cx_2, cx_3)$$
$$= (0, cx_2, cx_3)$$
$$= c(0, x_2, x_3)$$
$$= cT(x_1, x_2, x_3)$$
$$= cT(\alpha)$$