Method of variation of parameters

- There is another general technique to solving linear differential equations — the method of variation of parameters.
- Consider again the general first order linear equation

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = g(t)}$$

- The method works in three steps.
- <u>Step 1</u>: solve this equation for the right-hand side $g(t) = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = 0 \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -p(t)y \Longrightarrow \frac{\mathrm{d}y}{y} = -p(t)\mathrm{d}t$$

$$\Longrightarrow \int \frac{\mathrm{d}y}{y} = -\int p(t)\mathrm{d}t \Longrightarrow \ln|y| = -\int p(t)\,\mathrm{d}t + C_1$$

$$\Longrightarrow |y| = e^{-\int p(t)\,\mathrm{d}t}\, e^{C_1} \Longrightarrow \boxed{y = \pm e^{C_1}\, e^{-\int p(t)\,\mathrm{d}t} = C\, e^{-\int p(t)\,\mathrm{d}t}}$$

where we set $C = \pm e^{C_1}$

Method of variation of parameter

- Step 2: Now assume that C is not a constant but a function of t, i.e., $C = C(t)$. Then $y(t) = C(t)\, e^{-\int p(t)\,\mathrm{d}t}$, and by differentiating this expression we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}t}\, e^{-\int p(t)\,\mathrm{d}t} + C(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\int p(t)\,\mathrm{d}t}\right)$$

$$\Longrightarrow \boxed{\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}t}\, e^{-\int p(t)\,\mathrm{d}t} - C(t)p(t)\, e^{-\int p(t)\,\mathrm{d}t}}$$

- Step 3: By substituting the established expressions for $y(t)$ and $\mathrm{d}y/\mathrm{d}t$ into the original equation, $\mathrm{d}y/\mathrm{d}t + p(t)y = g(t)$, we will get

$$\frac{\mathrm{d}C}{\mathrm{d}t}\, e^{-\int p(t)\,\mathrm{d}t} - C(t)p(t)\, e^{-\int p(t)\,\mathrm{d}t} + p(t)C(t)e^{-\int p(t)\,\mathrm{d}t} = g(t)$$

$$\Longrightarrow \frac{\mathrm{d}C}{\mathrm{d}t}\, e^{-\int p(t)\,\mathrm{d}t} = g(t) \Longrightarrow \boxed{\frac{\mathrm{d}C}{\mathrm{d}t} = g(t)\, e^{\int p(t)\,\mathrm{d}t}}$$

- By integrating it we find $C(t)$ and thus complete the solution.

Example

- Let us solve the following first order linear differential equation

$$(y + t^2)\mathrm{d}t = t\mathrm{d}y$$

- Before applying the method of variation of parameters, it is necessary to convert this equation to the standard form $(\mathrm{d}y/\mathrm{d}t + p(t)y = g(t))$, which is obtained as

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{y}{t} = t}$$

  so, $p(t) = -1/t$ and $g(t) = t$.
- Step 1: solve this equation with right-hand side set to zero.

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{y}{t} = 0 \Longrightarrow \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}t}{t} \Longrightarrow \ln|y| = \ln|t| + C_1$$

$$\Longrightarrow |y| = |t|\, e^{C_1} \Longrightarrow \boxed{y = \pm e^{C_1}\,|t| = C\,t}$$

Example

- Step 2: Now assume that C is not a constant but a function of t, that is $C = C(t)$. Then $y(t) = C(t)\,t$, and by differentiating it we have

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}t}\,t + C(t)}$$

- Step 3: By substituting the established expressions for $y(t)$ and $\mathrm{d}y/\mathrm{d}t$ into the original equation, $\mathrm{d}y/\mathrm{d}t - y/t = t$, we will get

$$\frac{\mathrm{d}C}{\mathrm{d}t}\,t + C(t) - tC(t)/t = t \Longrightarrow \boxed{\frac{\mathrm{d}C}{\mathrm{d}t} = 1}$$

- By integrating it we will find $C(t) = t + c$, where c (small c) is the integration constant. Thus, the solution of the original differential equation is $\boxed{y(t) = t^2 + ct}$.

The Bernoulli equation

- Consider the following nonlinear differential equation

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = b(t)y^n}$$

- Assume that $n \neq 1$. If $n = 1$ (also if $n = 0$) this equation is linear.
- Define $\boxed{z = y^{1-n}}$ and find

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}t} \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t}\frac{y^n}{1-n}$$

- Substitute it in the original equation and convert it to

$$\frac{\mathrm{d}z}{\mathrm{d}t}\frac{y^n}{1-n} + a(t)y = b(t)y^n \Longrightarrow \boxed{\frac{\mathrm{d}z}{\mathrm{d}t}\frac{1}{1-n} + a(t)z = b(t)}$$

- This equation is linear. Find $z(t)$ and then $\boxed{y(t) = z^{\frac{1}{1-n}}}$

Example

- Let us solve the following nonlinear differential equation

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{t}y = -y^2}$$

- Here, $n = 2, a(t) = -1/t, b(t) = -1$
- Define $\boxed{z = y^{1-n}} = y^{-1}$ and convert to linear equation

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}\frac{1}{1-n} + a(t)z = b(t)} \Longrightarrow \frac{\mathrm{d}z}{\mathrm{d}t} + \frac{1}{t}z = 1$$

- Note that t is integrating factor, i.e.,

$$\frac{\mathrm{d}z}{\mathrm{d}t}t + z = t \Longrightarrow \frac{\mathrm{d}(zt)}{\mathrm{d}t} = t \Longrightarrow zt = \int t\mathrm{d}t = \frac{1}{2}t^2 + C$$

$$zt = \frac{t^2 + 2C}{2} \Longrightarrow z = \frac{t^2 + 2C}{2t} \Longrightarrow \boxed{y = z^{-1} = \frac{2t}{t^2 + 2C}}$$

Homogeneous equation

- A function $f(x, y)$ is called homogeneous of order m if $f(tx, ty) = t^m f(x, y)$.
- A function $f(y/x)$ is clearly homogeneous of order 0: $f(ty/tx) = f(y/x)$.
- Consider the following differential equation

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = f(y/x)}$$

- Set $y = x\,z(x)$, where $z(x)$ is a new function (to be found) $\Longrightarrow \boxed{y/x = z(x)}$
- Differentiate $y = x\,z(x)$ and note that $\mathrm{d}y/\mathrm{d}x = f(z)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z(x) + x\frac{\mathrm{d}z}{\mathrm{d}x} = f(z) \Longrightarrow x\frac{\mathrm{d}z}{\mathrm{d}x} = f(z) - z \Longrightarrow \boxed{\frac{\mathrm{d}z}{f(z) - z} = \frac{\mathrm{d}x}{x}}$$

Homogeneous equation

- We arrived at a separable equation $\boxed{\dfrac{\mathrm{d}z}{f(z)-z} = \dfrac{\mathrm{d}x}{x}}$ that can be integrated as

$$\int_{z_0}^{z} \frac{\mathrm{d}z}{f(z)-z} = \int_{x_0}^{x} \frac{\mathrm{d}x}{x} = \ln|x| + C_1 \Longrightarrow \boxed{x = C\, e^{\int_{z_0}^{z} \frac{\mathrm{d}z}{f(z)-z}}}$$

- Having established $x(z)$ we then find $z(x)$ and finally $y = xz(x)$
  - In the above formula $C \neq 0$ and $f(z_0) \neq z_0$
  - If z_0 is a root of $f(z) = z$ then $y = z_0 x$ is a solution of our equation. Indeed, then $\mathrm{d}y/\mathrm{d}x = f(z) = z_0 \Longrightarrow \mathrm{d}y = z_0 \mathrm{d}x \Longrightarrow y = z_0 x$.
  - If $y = \varphi(x)$ is a solution then $y = \alpha^{-1}\varphi(\alpha x)$ is also a solution. Indeed, $\mathrm{d}y/\mathrm{d}x = f(y/x) = f(\varphi(x))/x) = f(\varphi(\alpha x))/\alpha x) = f(\alpha^{-1}\varphi(\alpha x))/x)$.

Example 1

- Let us solve the following first order linear differential equation

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x}}$$

- Here, $f(y/x) = (y+x)/x = 1 + y/x$
- Set $y = x\,z(x) \Longrightarrow \boxed{y/x = z(x)} \Longrightarrow f(z) = 1 + y/x = 1 + z$
- Differentiate $y = x\,z(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z(x) + x\frac{\mathrm{d}z}{\mathrm{d}x} = f(z) = 1 + z(x) \Longrightarrow x\frac{\mathrm{d}z}{\mathrm{d}x} = 1 \Longrightarrow \boxed{\mathrm{d}z = \frac{\mathrm{d}x}{x}}$$

- Solution:

$$z(x) = \ln|x| + C \Longrightarrow \boxed{y(x) = x(\ln|x| + C)}$$

Example 2

- Let us solve the following first order linear differential equation

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}}$$

- Here, $f(y/x) = (y+x)/(x-y) = \frac{1+y/x}{1-y/x}$
- Set $y = x\,z(x) \implies \boxed{y/x = z(x)} \implies f(z) = \frac{1+y/x}{1-y/x} = \frac{1+z}{1-z}$
- Differentiate $y = xz(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z(x) + x\frac{\mathrm{d}z}{\mathrm{d}x} = f(z) = \frac{1+z}{1-z} \implies x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1+z}{1-z} - z = \frac{1+z^2}{1-z}$$

$$\implies \frac{(1-z)\mathrm{d}z}{1+z^2} = \frac{\mathrm{d}x}{x} \implies \int \frac{(1-z)\mathrm{d}z}{1+z^2} = \int \frac{\mathrm{d}x}{x}$$

$$\implies \underbrace{\int \frac{(\mathrm{d}z}{1+z^2}}_{\arctan z} - \underbrace{\int \frac{z\mathrm{d}z}{1+z^2}}_{\frac{1}{2}\ln(1+z^2)} = \underbrace{\int \frac{\mathrm{d}x}{x}}_{\ln|x| - C}$$

Example 2

- Simplification

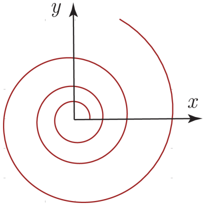
$$\arctan z = \ln\sqrt{1+z^2} + \ln\sqrt{x^2} - C = \ln\left(\sqrt{x^2+y^2}\right) - C$$

- Solution

$$\boxed{\arctan\left(\frac{y}{x}\right) = \ln\left(\sqrt{x^2+y^2}\right) - C}$$

- Transformation to polar coordinates, $\theta = \arctan(y/x)$ and $r = \sqrt{x^2+y^2}$, and use of new constant $C = \ln C_1$

$$\theta = \ln r - \ln C_1 \Longrightarrow \boxed{r = C_1 e^{\theta}}$$



Remark about the relation to autonomous systems

- A system (described by scalar or vector differential equation) is called autonomous if the right hand side does not explicitly depend on the independent variable. Examples: $\mathrm{d}y/\mathrm{d}x = f(y)$, $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(\boldsymbol{x})\dots$
- Autonomous system of two linear differential equations with constant coefficients

$$\mathrm{d}x/\mathrm{d}t = \alpha x + \beta y$$
$$\mathrm{d}y/\mathrm{d}t = \gamma x + \delta y$$

can be converted to a homogeneous equation

$$\boxed{\mathrm{d}y/\mathrm{d}x = \frac{\gamma x + \delta y}{\alpha x + \beta y} = \frac{\gamma + \delta(y/x)}{\alpha + \beta(y/x)} = f(y/x)}$$