Ch 2.6:Exact Equations and Integrating Factors

- Consider a first order ODE of the form

$$M(x,y)+N(x,y)y'=0$$

- Suppose there is a function $\psi(x,y)$ such that

$$\psi_x \triangleq \frac{\partial \psi(x,y)}{\partial x} = M(x,y), \quad \text{and} \quad \psi_y \triangleq \frac{\partial \psi(x,y)}{\partial x} = N(x,y)$$

and such that $\psi(x,y)=c$ defines $y=\varphi(x)$ implicitly. Then

$$M(x,y)+N(x,y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x,\varphi(x))$$

and hence the original ODE becomes

$$\frac{d}{dx}\psi(x,\varphi(x)) = 0$$

- Thus $\psi(x,y)=c$ defines a solution implicitly.
- In this case, the ODE is said to be an **exact differential equation**.

Example 1: Exact Equation

- Consider the equation:

$$2x + y^2 + 2xyy' = 0$$

- It is neither linear nor separable, but there is a function φ such that

$$2x + y^2 = \frac{\partial \psi}{\partial y} \text{ and } 2xy = \frac{\partial \psi}{\partial x}$$

- The function that works is $\psi(x,y) = x^2 + xy^2$
- Thinking of y as a function of x and calling upon the chain rule, the differential equation and its solution become

$$\frac{d\psi}{dx} = \frac{d}{dx}(x^2 + xy^2) = 0 \Rightarrow \psi(x,y) = x^2 + xy^2 = c$$

Theorem 2.6.1

- Suppose an ODE can be written in the form

$$M(x, y) + N(x, y)y' = 0 \qquad (1)$$

where the functions M, N, M_y and N_x are all continuous in the rectangular region R: $\alpha < x < \beta,\ \gamma < y < \delta$. Then Eq. (1) is an **exact** differential equation if and only if

$$\boxed{M_y(x, y) = N_x(x, y),\ \forall (x, y) \in R \qquad (2)}$$

- That is, there exists a function ψ satisfying the conditions

$$\psi_x(x, y) = M(x, y),\ \psi_y(x, y) = N(x, y) \qquad (3)$$

if and only if M and N satisfy Equation (2).

Practical construction of solutions

- Assume that $M(x,y)+N(x,y)y'=0$ is exact, which implies that

$$M_y = \frac{\partial M(x,y)}{\partial y} = N_x = \frac{\partial N(x,y)}{\partial x}$$

where $M(x,y)=\partial\psi(x,y)/\partial x$ and $N(x,y)=\partial\psi(x,y)/\partial y$

- To find $\psi(x,y)$ (and establish the implicit solution $\psi(x,y)=c$), two integration steps are necessary.
- **Step 1**: by integrating $\partial\psi(x,y)/\partial x = M(x,y)$ with y set as fixed, one obtains

$$\boxed{\psi(x,y)=\int M(x,y)\,\mathrm{d}x+h(y)=Q(x,y)+h(y)}$$

where h is not a constant but function of y (so $\partial h(y)/\partial x=0$)

- Note that $Q(x,y)$ is already computed and the unknown function $h(y)$ is yet to be established (at Step 2)

Practical construction of solutions

- **Step 2**: to find out $h(y)$, we use the definition $N(x,y) = \partial\psi(x,y)/\partial y$ and therefore

$$\frac{\partial\psi(x,y)}{\partial y} = N(x,y) = \frac{\partial Q(x,y)}{\partial y} + \frac{\mathrm{d}h(y)}{\mathrm{d}y}$$

- From here we have $$\boxed{\frac{\mathrm{d}h(y)}{\mathrm{d}y} = N(x,y) - \frac{\partial Q(x,y)}{\partial y}}$$

  and by solving this differential equation (with x set as fixed parameter) we will establish h(y)

- The final solution is then $\psi(x,y) = Q(x,y) + h(y) = c$

- Note that in the scheme above we first used the condition $\partial\psi(x,y)/\partial x = M(x,y)$ and then $\partial\psi(x,y)/\partial y = N(x,y)$. The order (which condition to use at Step 1 and which at Step 2 can different (it is totally up to us). Sometimes, it is more convenient to use $\partial\psi(x,y)/\partial y = N(x,y)$ at Step1 and $\partial\psi(x,y)/\partial x = M(x,y)$ Please refer to Problems 5,6 in the additional Sample Problems).

Example 2: Exact Equation (1 of 3)

- Consider the following differential equation.

$$(y\cos x + 2xe^{y}) + (\sin x + x^{2}e^{y} - 1)y' = 0$$

- Then

$$M(x,y) = y\cos x + 2xe^{y},\ N(x,y) = \sin x + x^{2}e^{y} - 1$$

and hence

$$M_{y}(x,y) = \cos x + 2xe^{y} = N_{x}(x,y) \ \Rightarrow\ \text{ODE is exact}$$

- From Theorem 2.6.1,

$$\psi_{x}(x,y) = M = y\cos x + 2xe^{y},\ \ \psi_{y}(x,y) = N = \sin x + x^{2}e^{y} - 1$$

- Thus

$$\psi(x,y) = \int \psi_{x}(x,y)\,dx = \int \left(y\cos x + 2xe^{y}\right)dx = y\sin x + x^{2}e^{y} + h(y)$$

Example 2: Solution (2 of 3)

- We have

$$\psi_x(x,y) = M = y\cos x + 2xe^y, \ \ \psi_y(x,y) = N = \sin x + x^2e^y - 1$$

and

$$\psi(x,y) = \int \psi_x(x,y)dx = \int \left(y\cos x + 2xe^y\right)dx = y\sin x + x^2e^y + h(y)$$

- It follows that

$$\psi_y(x,y) = \sin x + x^2e^y - 1 = \sin x + x^2e^y + h'(y)$$

$$\Rightarrow \ h'(y) = -1 \ \Rightarrow \ h(y) = -y - c$$

- Thus

$$\psi(x,y) = y\sin x + x^2e^y - y - c$$

- By Theorem 2.6.1, the solution is given implicitly by

$$y\sin x + x^2e^y - y = c$$

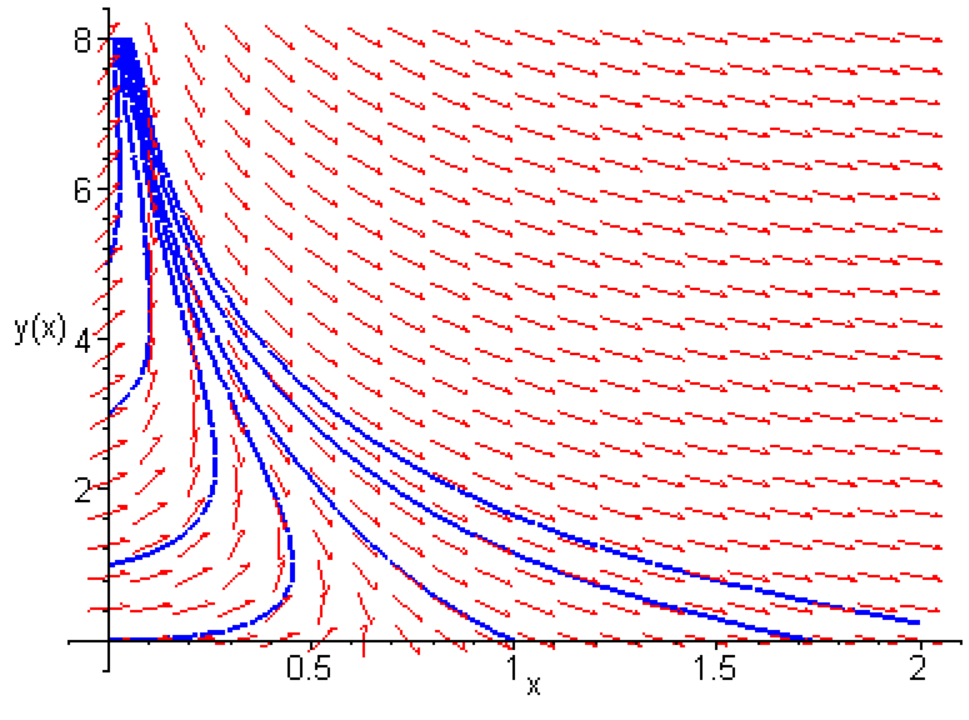
Example 2: Direction Field & Solution Curves (3 of 3)

- Our differential equation and solutions are given by

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0,$$

$$y\sin x + x^2 e^y - y = c$$

- Note that an explicit solution in the form $y = \varphi(x)$ is difficult to establish in the analytical form.
  However, a graph of the direction field for this differential equation, along with several solution curves, can be obtained numerically from the established implicit solution. They are shown in this fugure.



Example 3: Non-Exact Equation (1 of 2)

- Consider the following differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Then

$$M(x, y) = 3xy + y^2,\ N(x, y) = x^2 + xy$$

and hence

$$M_y(x, y) = 3x + 2y \neq 2x + y = N_x(x, y) \Rightarrow \text{ODE is not exact}$$

- To show that our differential equation cannot be solved by this method, let us seek a function ψ such that

$$\psi_x(x, y) = M = 3xy + y^2,\ \psi_y(x, y) = N = x^2 + xy$$

- Thus

$$\psi(x,y) = \int \psi_x(x,y)dx = \int \left(3xy + y^2\right)dx = \frac{3}{2}x^2y + xy^2 + h(y)$$

Example 3: Non-Exact Equation (2 of 2)

- We seek ψ such that

$$\psi_x(x,y) = M = 3xy + y^2, \;\; \psi_y(x,y) = N = x^2 + xy$$

and

$$\psi(x,y) = \int \psi_x(x,y)dx = \int \left(3xy + y^2\right)dx = 3x^2y/2 + xy^2 + C(y)$$

- Then

$$\psi_y(x,y) = x^2 + xy = \frac{3}{2}x^2 + 2xy + h'(y)$$

$$\Rightarrow \; h'(y) \overset{?}{=} -\frac{1}{2}\;x^2 - xy$$

- Because $h'(y)$ depends on x as well as y, there is no such function $\psi(x, y)$ such that

$$\frac{d\psi}{dx} = (3xy + y^2) + (x^2 + xy)y'$$

Integrating Factors

- It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x,y)$:

$$M(x,y) + N(x,y)y' = 0$$

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

- For this equation to be exact, we need

$$(\mu M)_y = (\mu N)_x \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

- This partial differential equation may be difficult to solve. If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

provided right side is a function of x only. Similarly if μ is a function of y alone. See text for more details.

Example 4: Non-Exact Equation

- Consider the following non-exact differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Seeking an integrating factor, we solve the linear equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \quad \Leftrightarrow \quad \frac{d\mu}{dx} = \frac{\mu}{x} \quad \Rightarrow \quad \mu(x) = x$$

- Multiplying our differential equation by μ, we obtain the exact equation

$$(3x^2 y + xy^2) + (x^3 + x^2 y)y' = 0,$$

which has its solutions given implicitly by

$$x^3 y + \frac{1}{2}x^2 y^2 = c$$

Ch 2.4: Differences Between Linear and Nonlinear Equations

- Recall that a first order ODE has the form $y' = f(t,y)$, and is linear if f is linear in y, and nonlinear if f is nonlinear in y.
- Examples: $y' = ty - e^t$, $y' = ty^2$.
- In this section, we will see that first order linear and nonlinear equations differ in a number of ways, including:
  - The theory describing existence and uniqueness of solutions, and corresponding domains, are different.
  - Solutions to linear equations can be expressed in terms of a general solution, which is not usually the case for nonlinear equations.
  - Linear equations have explicitly defined solutions while nonlinear equations typically do not, and nonlinear equations may or may not have implicitly defined solutions.
- For both types of equations, numerical and graphical construction of solutions are important.

Theorem 2.4.1

- Consider the linear first order initial value problem:

$$y' + p(t)y = g(t), \;\; y(t_0) = y_0$$

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the IVP for each t in I.

- **Proof outline:** Use Ch 2.1 discussion and results:

$$y = \frac{\int_{t_0}^{t} \mu(t)g(t)dt + y_0}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s)ds}$$

Theorem 2.4.2

- Consider the nonlinear first order initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$

- Let the functions *f* and $\partial f / \partial y$ be continuous in some rectangle $\alpha < t < \beta, \ \gamma < y < \delta$ containing the point (t_0, y_0).
- Then in some interval $t_0 - h < t < t_0 + h$ in the rectangle there is a unique solution $y = \phi(t)$ of the initial value problem.

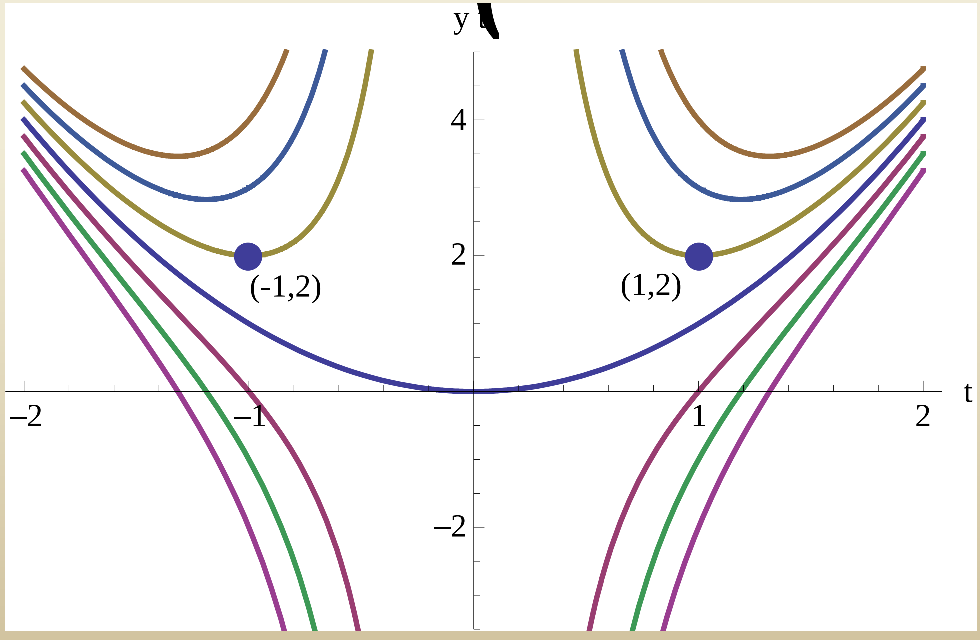
- **Proof discussion:** Since there is no general formula for the solution of arbitrary nonlinear first order IVPs, this proof is difficult, and is beyond the scope of this course.
- It turns out that conditions stated in Theorem 2.4.2 are sufficient but not necessary to guarantee existence of a solution, and continuity of *f* ensures existence but not uniqueness of $y = \phi(t)$.

Example 1: Linear IVP

- Recall the initial value problem from Chapter 2.1 slides:

$$ty' + 2y = 4t^2, \quad y(1) = 2 \quad \Rightarrow \quad y = t^2 + \frac{1}{t^2}$$

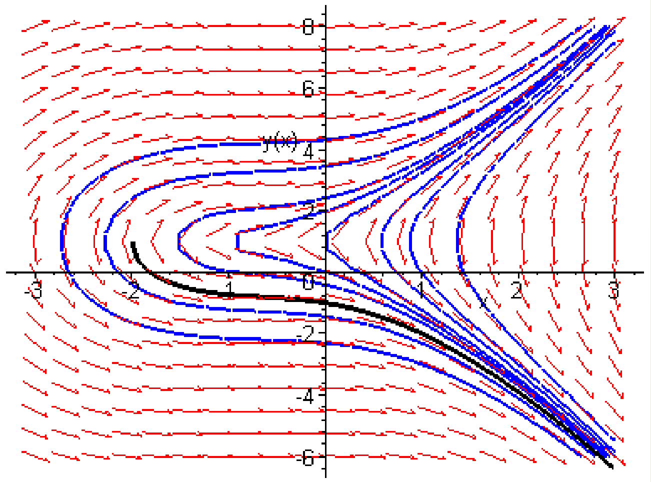
- The solution to this initial value problem is defined for $t > 0$, the interval on which $p(t) = 2/t$ is continuous.
- If the initial condition is $y(-1) = 2$, then the solution is given by same expression as above, but is defined on $t < 0$.
- In either case, Theorem 2.4.1 guarantees that solution is unique on corresponding interval.



Example 2: Nonlinear IVP (1 of 2)

- Consider nonlinear initial value problem from Ch 2.2:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$



- The functions f and $\partial f / \partial y$ are given by

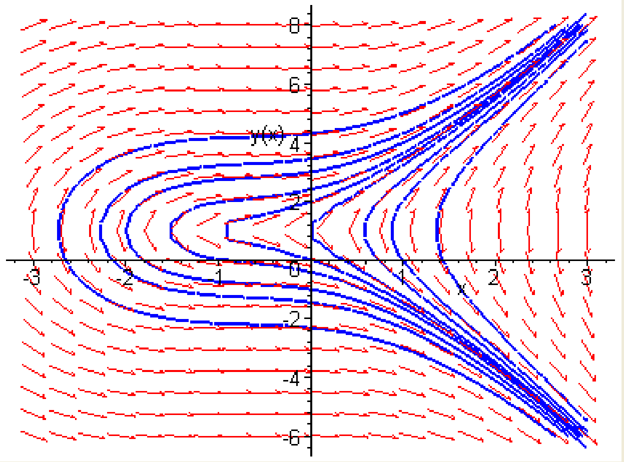
$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$

and are continuous except on line $y = 1$.

- Thus we can draw an open rectangle about (0, –1) in which f and $\partial f / \partial y$ are continuous, as long as it doesn't cover $y = 1$.
- How wide is the rectangle? Recall solution defined for $x > -2$, with

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Example 2: Change Initial Condition (2 of 2)



- Our nonlinear initial value problem is

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

with

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$

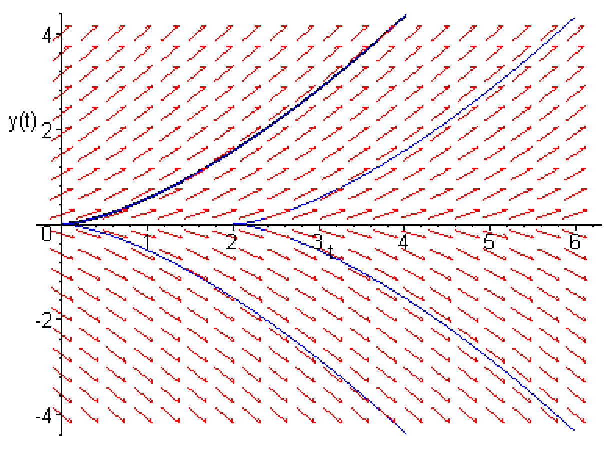
which are continuous except on line $y = 1$.

- If we change initial condition to $y(0) = 1$, then Theorem 2.4.2 is not satisfied. Solving this new IVP, we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}, \quad x > 0$$

- Thus a solution exists but is not unique.

Example 3: Nonlinear IVP



- Consider nonlinear initial value problem

$$y' = y^{1/3},\ y(0) = 0 \qquad (t \ge 0)$$

- The functions f and $\partial f / \partial y$ are given by

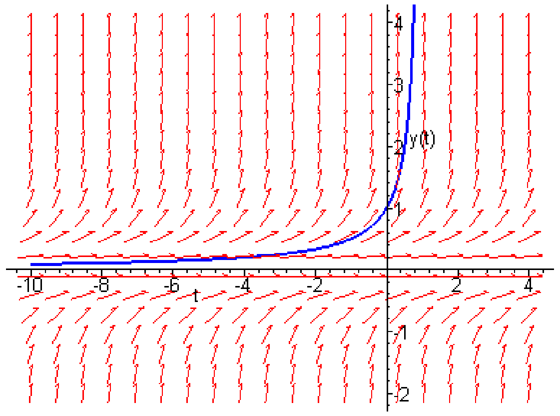
$$f(t, y) = y^{1/3},\ \frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{-2/3}$$

- Thus f continuous everywhere, but $\partial f / \partial y$ doesn't exist at $y = 0$, and hence Theorem 2.4.2 does not apply. Solutions exist but are not unique. Separating variables and solving, we obtain

$$y^{-1/3} dy = dt \ \Rightarrow\ \frac{3}{2} y^{2/3} = t + c \ \Rightarrow\ y = \pm\left(\frac{2}{3}t\right)^{3/2},\ t \ge 0$$

- If initial condition is not on t-axis, then Theorem 2.4.2 does guarantee existence and uniqueness.

Example 4: Nonlinear IVP



- Consider nonlinear initial value problem

$$y' = y^2, \; y(0) = 1$$

- The functions f and $\partial f / \partial y$ are given by

$$f(t, y) = y^2, \; \frac{\partial f}{\partial y}(t, y) = 2y$$

- Thus f and $\partial f / \partial y$ are continuous at $t = 0$, so Theorem 2.4.2 guarantees that solutions exist and are unique.
- Separating variables and solving, we obtain

$$y^{-2} dy = dt \;\Rightarrow\; -y^{-1} = t + c \;\Rightarrow\; y = -\frac{1}{t + c} \;\Rightarrow\; y = \frac{1}{1 - t}$$

- The solution $y(t)$ is defined on $(-\infty, 1)$. Note that the singularity at $t = 1$ is not obvious from original IVP statement.

Interval of Existence: Linear Equations

- By Theorem 2.4.1, the solution of a linear initial value problem

$$y' + p(t)y = g(t), \;\; y(0) = y_0$$

  exists throughout any interval about $t = t_0$ on which p and g are continuous.
- Vertical asymptotes or other discontinuities of solution can only occur at points of discontinuity of p or g.
- However, solution may be differentiable at points of discontinuity of p or g. See Chapter 2.1: Example 3 of text.
- Compare these comments with Example 1 and with previous linear equations in Chapter 1 and Chapter 2.

Interval of Existence: Nonlinear Equations

- In the nonlinear case, the interval on which a solution exists may be difficult to determine.
- The solution $y = \phi(t)$ exists as long as $[t, \phi(t)]$ remains within a rectangular region indicated in Theorem 2.4.2. This is what determines the value of *h* in that theorem. Since $\phi(t)$ is usually not known, it may be impossible to determine this region.
- In any case, the interval on which a solution exists may have no simple relationship to the function *f* in the differential equation $y' = f(t, y)$, in contrast with linear equations.
- Furthermore, any singularities in the solution may depend on the initial condition as well as the equation.
- Compare these comments to the preceding examples.

General Solutions

- For a first order linear equation, it is possible to obtain a solution containing one arbitrary constant, from which all solutions follow by specifying values for this constant.
- For nonlinear equations, such general solutions may not exist. That is, even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by specifying values for this constant.
- Consider Example 4: The function $y = 0$ is a solution of the differential equation, but it cannot be obtained by specifying a value for c in solution found using separation of variables:

$$\frac{dy}{dt} = y^2 \quad \Rightarrow \quad y = -\frac{1}{t+c}$$

Explicit Solutions: Linear Equations

- By Theorem 2.4.1, a solution of a linear initial value problem
$$y' + p(t)y = g(t), \quad y(0) = y_0$$
exists throughout any interval about $t = t_0$ on which p and g are continuous, and this solution is unique.
- The solution has an explicit representation,
$$y = \frac{\int_{t_0}^{t} \mu(t) g(t) dt + y_0}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s) ds},$$
and can be evaluated at any appropriate value of t, as long as the necessary integrals can be computed.

Explicit Solution Approximation

- For linear first order equations, an explicit representation for the solution can be found, as long as necessary integrals can be solved.
- If integrals can't be solved, then numerical methods are often used to approximate the integrals.

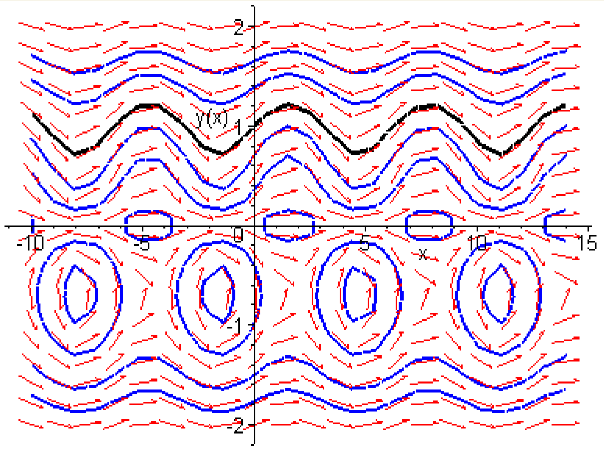
$$y = \frac{\int_{t_0}^{t} \mu(t) g(t) dt + C}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s) ds}$$

$$\int_{t_0}^{t} \mu(t) g(t) dt \approx \sum_{k=1}^{n} \mu(t_k) g(t_k) \Delta t_k$$

Implicit Solutions: Nonlinear Equations

- For nonlinear equations, explicit representations of solutions may not exist.
- As we have seen, it may be possible to obtain an equation which implicitly defines the solution. If equation is simple enough, an explicit representation can sometimes be found.
- Otherwise, numerical calculations are necessary in order to determine values of *y* for given values of *t*. These values can then be plotted in a sketch of the integral curve.
- Recall the examples from earlier in the chapter and consider the following example

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1 \;\Rightarrow\; \ln y + y^3 = \sin x + 1$$



Direction Fields

- In addition to using numerical methods to sketch the integral curve, the nonlinear equation itself can provide enough information to sketch a direction field.
- The direction field can often show the qualitative form of solutions, and can help identify regions in the *ty*-plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigations.