Introduction to Differential Equations
Assignment # 11

Date Given: June 20, 2022
Date Due: June 27, 2022

P1. (1 point) Find the Laplace transform of $f(t) = \displaystyle\int_0^t \sin(t-\tau)\cos\tau\,\mathrm{d}\tau$.

Solution: The function $f(t)$ can be expressed explicitly as $f(t) = \frac{1}{2}t\sin(t)$. However we do not need this expression if we recognize that $f(t)$ is in the form of a convolution integral.

We have $f(t) = (g * h)(t)$, in which $g(t) = \sin t$ and $h(t) = \cos t$. Based on Theorem 6.6.1, the transform of the convolution integral is

$$\mathcal{L}[f(t)] = \mathcal{L}\left[\int_0^t g(t-\tau)h(\tau)\,\mathrm{d}\tau\right] = \frac{1}{s^2+1}\,\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}.$$

P2. (1 point) By using the convolution theorem, find (express in terms of a convolution integral) the inverse Laplace transform of $F(s) = \dfrac{1}{(s+1)^2(s^2+4)}$.

Solution: $\mathcal{L}^{-1}[1/(s+1)^2] = te^{-t}$ and $\mathcal{L}^{-1}[1/(s^2+4)] = \frac{1}{2}\sin 2t$. Therefore, based on Theorem 6.6.1,

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2}\int_0^t (t-\tau)\,e^{-(t-\tau)}\sin 2\tau\,\mathrm{d}\tau.$$

Note that we can also write it as

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2}\int_0^t \tau\,e^{-\tau}\sin(2(t-\tau))\,\mathrm{d}\tau.$$

Note also that $f(t)$ can be computed explicitly[1].

P3. (2 points) Express in terms of a convolution integral the solution of the following initial value problem: $y'' + 2y' + 2y = \sin\alpha t$; $y(0) = 0, y'(0) = 0$.

Solution: Applying the Laplace transform to the equation, we have

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] + 2Y(s) = \frac{\alpha}{s^2+\alpha^2}.$$

Applying the initial conditions, we get

$$(s^2+2s+2)Y(s) = \frac{\alpha}{s^2+\alpha^2}.$$

Therefore

$$Y(s) = \frac{\alpha}{(s^2+\alpha^2)(s^2+2s+2)} = \frac{1}{(s+1)^2+1}\,\frac{\alpha}{s^2+\alpha^2}.$$

Therefore

$$y(t) = \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin(\alpha\tau)\,\mathrm{d}\tau.$$

Note that this integral can be computed explicitly[2].

[1] The explicit form is $f(t) = \frac{1}{50}e^{-t}\left(10t - 3e^t\sin(2t) - 4e^t\cos(2t) + 4\right)$

[2] The solution is $y(t) = \dfrac{\alpha e^{-t}\left\{\alpha^2\sin(t) + 2\cos(t)\right\}}{\alpha^4+4} - \dfrac{\left(\alpha^2-2\right)\sin(\alpha t) + 2\alpha\cos(\alpha t)}{\alpha^4+4}$

P4. (2 points) Express in terms of a convolution integral the solution of the following initial value problem: $y'' + 4y' + 4y = g(t);\;\; y(0) = 2, y'(0) = -3.$

Solution: Applying the Laplace transform to the equation, we have

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4\left[sY(s) - y(0)\right] + 4Y(s) = G(s).$$

Applying the initial conditions and solving for the transform, we get

$$Y(s) = \frac{2s+5}{(s+2)^2)} + \frac{G(s)}{(s+2)^2}.$$

We can write

$$\frac{2s+5}{(s+2)^2} = \frac{2}{(s+2)} + \frac{1}{(s+2)^2}.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+2)}\right] = 2e^{-2t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}.$$

Therefore, based on the convolution theorem, the solution of the initial value problem is

$$y(t) = 2e^{-2t} + te^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}g(\tau)\,\mathrm{d}\tau.$$

P5. (1 point) Transform the differential equation $u^{(4)} - u = 0$ with initial conditions $u(0) = 0, u'(0) = 1, u''(0) = 2, u'''(0) = 3$, into a system of first order equations corresponding to this initial value problem.

Solution: Set $x_1 = u$, $x_2 = u'$, $x_3 = u''$, and $x_4 = u'''$. Then

$$x_1' = x_2,\; x_2' = x_3,\; x_3' = x_4,\; x_4' = x_1, \qquad x_1(0) = 0,\; x_2(0) = 1,\; x_1(3) = 2,\; x_3(0) = 3.$$

In the matrix notation we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

P6. (3 points)

(a) Transform the system $x_1' = -0.5x_1 + 2x_2$, $x_2' = -2x_1 - 0.5x_2$ into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the initial conditions $x_1(0) = -2$, $x_2(0) = 2$.

(c) Sketch the graph of the solution in the x_1x_2-plane.

Solution:

(a) Solving the first equation for x_2 gives $x_2 = x_1'/2 + x_1/4$. Substituting this into second differential equation we obtain $x_1''/2 + x_1'/4 = -2x_1 - (x_1'/2 + x_1/4)/2$, that is

$$x_1'' + x_1' + \frac{17}{4}x_1 = 0.$$

(b) The general solution of the 2nd order differential equation in part (a) is $x_1 = e^{-t/2}\left(c_1\cos 2t + c_2\sin 2t\right)$. With x_2 given in terms of x_1, it follows that $x_2 = e^{-t/2}\left(c_2\cos 2t - c_1\sin 2t\right)$. Imposing the specified initial conditions, we obtain $c_1 = -2$ and $c_2 = 2$. Hence

$$x_1(t) = 2e^{-t/2}\left(-\cos 2t + \sin 2t\right), \quad \text{and} \quad x_2(t) = 2e^{-t/2}\left(\cos 2t + \sin 2t\right).$$

(c) The graph of the solution in the x_1x_2-plane is shown in red in Figure 1. Also shown there are the direction filed and, in green, some other curves corresponding to different initial values $x_1(0)$, $x_2(0)$.

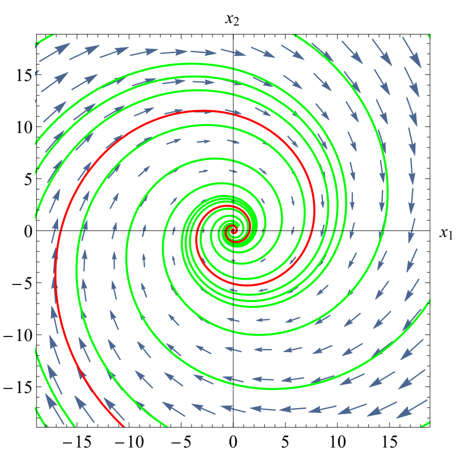

Figure 1: Illustration to problem P6.