Introduction to Differential Equations
Assignment # 11

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P1.

$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$

$(f * g)(t) = \mathcal{L}\{F(s)G(s)\}$

$\int_0^t \sin(t - \tau) \cos \tau \, d\tau$

$f(t) = \sin t \,, g(t) = \cos t$

$(f * g)(t) = \int_0^t \sin(t - \tau) \cos \tau \, d\tau$

$\mathcal{L}\left\{\int_0^t \sin(t - \tau) \cos \tau \, d\tau\right\} = \mathcal{L}\{f(t) = \sin t\}\mathcal{L}\{g(t) = \cos t\}$

$\mathcal{L}\{f(t) = \sin t\} = \frac{1}{s^2+1^2} = \frac{1}{s^2+1}$

$\mathcal{L}\{g(t) = \cos t\} = \frac{s}{s^2+1^2} = \frac{s}{s^2+1}$

$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \frac{1}{s^2+1} \times \frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$

$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\left\{\int_0^t \sin(t - \tau) \cos \tau \, d\tau\right\} = \frac{s}{(s^2+1)^2}$

P2.

$F(s) = \frac{1}{(s+1)^2(s^2+4)}$

$F(s) = \frac{1}{(s+1)^2} \times \frac{1}{s^2+4}$

$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2} \times \frac{1}{2}\right\}$

$= \frac{1}{2}\sin 2t$

$\mathcal{L}^{-1}\{F(s)\} = (f * g)(t)$

$(f * g)(t)$
$= \int_0^t (t-\tau)e^{-(t-\tau)}\frac{1}{2}\sin 2\tau\, d\tau$

p3.
$y'' + 2y' + 2y = \sin \alpha t\,; y(0) = 2, y'(0) = 0$
$\mathcal{L}\{y'' + 2y' + 2y = \sin \alpha t\}$
$(s^2Y(s) - sy(0) - y'(0)) + 2\big(sY(s) - y(0)\big) + 2Y(s) = \frac{\alpha}{s^2+\alpha^2}$
$s^2Y(s) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = \frac{\alpha}{s^2+\alpha^2}$
$Y(s)(s^2 + 2s + 2) = \frac{\alpha}{s^2+\alpha^2}$
$Y(s) = \frac{\alpha}{(s^2+\alpha^2)(s^2+2s+2)}$
$\frac{\alpha}{(s^2+\alpha^2)(s^2+2s+2)}$
$= \frac{\alpha}{s^2+\alpha^2} \times \frac{1}{s^2+2s+2}$
$= \frac{\alpha}{s^2+\alpha^2} \times \frac{1}{(s+1)^2+1^2}$

$\mathcal{L}^{-1}\{\frac{\alpha}{s^2+\alpha^2} \times \frac{1}{(s+1)^2+1^2}\}$
$y(t) = \sin \alpha t\; e^{-t}\sin t$
$y(t) = \int_0^t \sin(\alpha(t-\tau))\, e^{-\tau}\sin\tau\, d\tau$

P4.
$y'' + 4y' + 4y = g(t); y(0) = 2, y'(0) = -3$
$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = G(s)$
$Y(s)(s^2 + 4s + 4) - 2s - (-3) - 4 \times 2 = G(s)$
$Y(s)(s^2 + 4s + 4) = G(s) + 2s + 5$
$Y(s) = \frac{G(s)}{s^2+4s+4} + \frac{2s}{s^2+4s+4} + \frac{5}{s^2+4s+4}$

$\frac{5}{s^2+4s+4} = \frac{5}{2} \times \frac{2}{(s+2)^2}$
$\mathcal{L}^{-1}\left\{5 \times \frac{1}{(s+2)^2}\right\}$

$= 5te^{-2t}$

$\frac{2s}{s^2+4s+4} = 2\frac{s}{(s+2)^2}$
$\mathcal{L}^{-1}\left\{2\frac{s}{(s+2)^2}\right\}$
$= 2(1-2t)e^{-2t}$

$\frac{G(s)}{s^2+4s+4} = \frac{G(s)}{(s+2)^2} = \frac{1}{(s+2)^2}G(s)$
$g(t) = \mathcal{L}^{-1}\{G(s)\}$
$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$
$= te^{-2t}$

$y(t) = 5te^{-2t} + 2(1-2t)e^{-2t} + \int_0^t (t-\tau)e^{-2(t-\tau)}\mathrm{g}(\tau)\mathrm{d}\tau$

P5.
$u^{(4)} - u = 0; u(0) = 0, u'(0) = 1, u''(0) = 2, u'''(0) = 3$
$x_1 = u, x_2 = u', x_3 = u'', x_4 = u'''$
$x_1' = x_2, x_2' = x_3, x_3' = x_4$
$x_4' - x_1 = 0$
$x_4' = x_1$
$x_1(0) = 0, x_2(0) = 1, x_3(0) = 2, x_4(0) = 3$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$