Introduction to Differential Equations
Assignment # 6

Date Given: May 16, 2022
Date Due: May 23, 2022

P1. (1 point) Use the method of undetermined coefficients to find the general solution of the differential equation $y'' - 2y' + 5y = e^{2t}(\cos t - 3\sin t)$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 - 2r + 5 = 0$, which has two roots $r = 1 + 2\imath$, $r = 1 - 2\imath$. Therefore, the solution of the homogeneous problem is $y_c(t) = e^t(c_1 \cos 2t + c_2 \sin 2t)$.

(b) To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p = Ae^{2t}\cos t + Be^{2t}\sin t$. Substituting a function of this form into the differential equation, and equating the terms, we have $4A + 2B = 1, -2A + 4B = -3$. Therefore $A = 1/2, B = -1/2$ and the general solution of the nonhomogeneous problem is

$$y(t) = e^t(c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{2}e^{2t}(\cos t - \sin t).$$

P2. (1 point) Use the method of undetermined coefficients to find the general solution of the differential equation $y'' + 2y' + y = \cos t + 3\sin 2t$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 + 2r + 1 = 0$, which has the repeated (double) root $r = -1$. Therefore, the solution of the homogeneous problem is $y_c(t) = c_1e^{-t} + c_2te^{-t}$.

(b) To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p = A\cos t + B\sin t + C\cos 2t + D\sin 2t$. Substituting a function of this form into the differential equation, and equating the terms, we have $2B = 0, -2A = 1, -3C + 4D = 3, -4C - 3D = 0$. Therefore $A = -1/2, B = 0, C = -9/25, D = 12/25$ and the general solution of the nonhomogeneous problem is

$$y(t) = c_1e^{-t} + c_2te^{-t} - \frac{1}{2}\cos t - \frac{9}{25}\cos 2t + \frac{12}{25}\sin 2t.$$

P3. (2 points) Find the solution of the initial value problem $y'' - 2y' - 3y = 3te^{2t}$, $y(0) = 1$, $y'(0) = 0$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, which has the roots $r = 3$ and $r = -1$. Therefore, the solution of the homogeneous problem is $y_c(t) = c_1e^{3t} + c_2e^{-t}$.

(b) To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p = Ae^{2t} + Bte^{2t}$. Substituting a function of this form into the differential equation, and equating the terms, we find $A = -2/3$ and $B = -1$. Therefore the general solution of the nonhomogeneous problem is

$$y(t) = -2e^{2t}/3 - te^{2t} + c_1e^{3t} + c_2e^{-t}.$$

(c) The initial conditions imply $c_1 + c_2 - 2/3 = 1$ and $3c_1 - c_2 - 4/3 - 1 = 0$. Therefore $c_1 = 1$ and $c_2 = 2/3$. Thus the solution of the initial value problem is

$$y(t) = -2e^{2t}/3 - te^{2t} + e^{3t} + 2e^{-t}/3.$$

P4. (2 points) Find the solution of the initial value problem $y'' + 2y' + 5y = 4e^{-t}\cos 2t$, $y(0) = 1$, $y'(0) = 0$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2+2r+5=0$, which has complex roots $r = -1 \pm 2\imath$. Therefore, the solution of the homogeneous problem is $y_c(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$.

(b) To find a solution of the nonhomogeneous problem, based on the form of $g(t)$ we look for a solution of the form $y_p = Ate^{-t}\cos 2t + Bte^{-t}\sin 2t$. Substituting a function of this form into the differential equation, and equating the terms, we find $A = 0$ and $B = 1$. Therefore the general solution of the nonhomogeneous problem is

$$y(t) = te^{-t}\sin 2t + c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t.$$

(c) The initial conditions imply $c_1 = 1$ and $-c_1 + 2c_2 = 0$. Therefore $c_1 = 1$ and $c_2 = 1/2$. Thus the solution of the initial value problem is

$$y(t) = te^{-t}\sin 2t + e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t.$$

P5. (2 points) Use the method of variation of parameters to find the general solution of the differential equation $y'' - 2y' + y = e^t/(1+t^2)$.

Solution:

(a) The solution of the homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 te^t.$$

The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions. The Wronskian of these functions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = e^{2t}.$$

(b) Here $g(t) = e^t/(1+t^2)$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$u_1'(t) = -\frac{te^te^t}{e^{2t}(1+t^2)} = -\frac{t}{1+t^2}, \quad \text{and} \quad u_2'(t) = \frac{e^te^t}{e^{2t}(1+t^2)} = \frac{1}{1+t^2}.$$

Therefore,

$$u_1(t) = -\int \frac{t}{1+t^2}\mathrm{d}t = -\frac{1}{2}\ln(1+t^2), \quad \text{and} \quad u_2(t) = \int \frac{1}{1+t^2}\mathrm{d}t = \arctan t$$

Hence the particular solution is $Y(t) = -\frac{1}{2}e^t\ln(1+t^2) + te^t\arctan t$. The general solution is given by

$$y(t) = c_1 e^t + c_2 te^t + e^t \ln\frac{1}{\sqrt{1+t^2}} + te^t\arctan t.$$

P6. (2 points) Use the method of variation of parameters to find the general solution of the differential equation $y'' + 2y' + y = e^{-t}\ln t$.

Solution:

(a) The solution of the homogeneous equation is

$$y_c(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

The functions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = e^{-2t}.$$

(b) Here $g(t) = e^{-t} \ln t$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$u_1'(t) = -\frac{te^{-t}e^{-t}\ln t}{e^{-2t}} = -t\ln t, \quad \text{and} \quad u_2'(t) = \frac{e^{-t}e^{-t}\ln t}{e^{3t}} = \ln t.$$

Therefore,

$$u_1(t) = -\int t \ln t \mathrm{d}t = -\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2, \quad \text{and} \quad u_2(t) = \int \ln t \mathrm{d}t = t\ln t - t$$

Hence the particular solution is $Y(t) = e^{-t}\left(-\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2\right) + te^{-t}(t\ln t - t)$. The general solution is given by

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2}t^2 e^{-t} \ln t - \frac{3}{4}t^2 e^{-t}.$$