Introduction to Differential Equations
Assignment # 4

Date Given: May 2, 2022
Date Due: May 9, 2022

P1. (1 point) Find the general solution of the differential equation $y'' + 5y' = 0$.

Solution: The characteristic equation is $r^2 + 5r = 0$, with the roots $r = 0, -5$. Therefore the general solution is $y = c_1 + c_2 e^{-5t}$.

P2. (2 points) Find the solution of the initial value problem $y'' + 4y' + 3y = 0$, $y(0) = 2$, $y'(0) = -1$. Sketch the graph of the solution and describe its behavior as t increases.

Solution: The characteristic equation is $r^2 + 4r + 3 = 0$, with the roots $r = -1, -3$. Therefore the general solution is $y(t) = c_1 e^{-t} + c_2 e^{-3t}$ and $y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$. From $y(0) = 2$ we get $c_1 + c_2 = 2$, and from $y'(0) = -1$ we get $-c_1 - 3c_2 = -1$. Solving for the constants, one obtains $c_1 = 5/2$ and $c_2 = -1/2$. Hence the specific solution is

$$y(t) = 5e^{-t}/2 - e^{-3t}/2.$$

The solution clearly converges to 0 as $t \to \infty$ (see Figure 1).

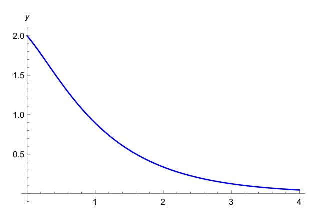

Figure 1: Illustration to problem P2.

P3. (1 point) Find a differential equation whose general solution is $y = c_1 e^{2t} + c_2 e^{-3t}$.

Solution: An algebraic equation with roots 2 and -3 is $(r-2)(r+3) = r^2 + r - 6 = 0$. This is the characteristic equation for the differential equation $y'' + y' - 6y = 0$.

P4. (2 points) Determine the values of α, if any, for which all solutions of the differential equation $y'' + (3-\alpha)y' - 2(\alpha-1)y = 0$ tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

Solution: The characteristic equation is $r^2 + (3-\alpha)r - 2(\alpha-1) = 0$, with the roots $r = \alpha - 1, -2$. Therefore the general solution is $y = c_1 e^{(\alpha-1)t} + c_2 e^{-2t}$.

- In order for all solutions to tend to zero, we need $\alpha - 1 < 0$. Therefore, the solutions will all tend to zero as long as $\alpha < 1$.
- Due to the term $c_2 e^{-2t}$, we can never guarantee that all solutions will become unbounded as $t \to \infty$ (that is, solutions for which $c_1 = 0$ are always bounded)

P5. (2 points) Verify that the functions $y_1(x) = x$ and $y_2(x) = xe^x$ are solutions of the differential equation $x^2 y'' - x(x+2)y' + (x+2)y = 0$, $x > 0$. Do they constitute a fundamental set of solutions?

Solution:

- For $y_1 = x$, $y_1' = 1$, and $y_1'' = 0$. Therefore $x^2y_1'' - x(x+2)y_1' + (x+2)y_1 = -x(x+2) + (x+2)x = 0$. Next, for $y_2 = xe^x$, $y_2' = (1+x)e^x$, and $y_2'' = (2+x)e^x$. Therefore $x^2y_2'' - x(x+2)y_2' + (x+2)y_2 = x^2(2+x)e^x - x(x+1)(x+2)e^x + x(x+2)e^x = 0$.
- Further, $W(x, xe^x) = x^2e^x \neq 0$ for $x > 0$. Therefore, the solutions form a fundamental set.

P6. (2 points) If the differential equation $ty'' + 2y' + te^ty = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2) = 2$ at $t = 1$, find the value of $W(y_1, y_2)$ at $t = 5$.

Solution: The expression for Wronskian (see textbook, Chapter 3.2, Abel's Theorem), we have $W(y_1, y_2)(t) = ce^{-\int p(t)\mathrm{d}t}$, where $p(t) = 2/t$ from the differential equation. Thus $W(y_1, y_2)(t) = c/t^2$. We identify $c = 2$ from $W(y_1, y_2)(1) = 2$ and then find

$$W(y_1, y_2)(5) = 2/25.$$