Ch 4.3: The Method of Undetermined Coefficients

- The method of undetermined coefficients can be used to find a particular solution Y of an nth order linear, constant coefficient, nonhomogeneous ODE

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t),$$

provided g is of an appropriate form.

- As with 2nd order equations, the method of undetermined coefficients is typically used when g is a sum or product of polynomial, exponential, and sine or cosine functions.
- Section 4.4 discusses the more general variation of parameters method.

Undetermined Coefficients

- The particular solution of $a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$t^s\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_n\right)$
$P_n(t)e^{\alpha t}$	$t^s\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_n\right)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\cos\beta t\\ \sin\beta t\end{cases}$	$t^s\left\{\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_n\right)e^{\alpha t}\cos\beta t + \right.$ $\left. + \left(B_0 t^n + B_1 t^{n-1} + \cdots + B_n\right)e^{\alpha t}\sin\beta t\right\}$

- 1st row: s is the number of times when 0 is a root of the characteristic equation
- 2nd row: s is the number of times when α is a root of the characteristic equation
- 3rd row: s is the number of times when $\alpha + i\beta$ is a root of the characteristic equation

Example 1

- Consider the differential equation

$$y''' - 3y'' + 3y' - y = 4e^t$$

- For the homogeneous case,

$$y(t) = e^{rt} \quad \Rightarrow \quad r^3 - 3r^2 + 3r - 1 = 0 \Leftrightarrow \left(r-1\right)^3 = 0$$

- Thus the general solution of homogeneous equation is

$$y_C(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

- For nonhomogeneous case, keep in mind the form of homogeneous solution. Thus begin with

$$Y(t) = At^3 e^{2t}$$

- As in Chapter 3, it can be shown that

$$Y(t) = \frac{2}{3} t^3 e^{2t} \quad \Rightarrow \quad y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^{2t}$$

Example 2

- Consider the equation $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$
- For the homogeneous case,
$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 + 2r^2 + 1 = 0 \Leftrightarrow \left(r^2+1\right)\left(r^2+1\right) = 0$$
- Thus the general solution of the homogeneous equation is
$$y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos(t) + c_4 t \sin(t)$$
- For the nonhomogeneous case, because of the form of the solution for the homogeneous equation, we need
$$Y(t) = t^2 (A\sin t + B\cos t)$$
- As in Chapter 3, it can be shown that $Y(t) = -\frac{3}{8}\sin t + \frac{5}{8}\cos t$
- Thus, the general solution for the nonhomgeneous equation is
$$y(t) = y_c(t) + Y(t)$$

Example 3

- Consider the equation

$$y''' - 4y' = t + 3\cos t + e^{-2t}$$

- For the homogeneous case,

$$y(t) = e^{rt} \;\Rightarrow\; r^3 - 4r = 0 \Leftrightarrow r\left(r^2 - 4\right) \Leftrightarrow r(r-2)(r+2) = 0$$

- Thus the general solution of homogeneous equation is

$$y_C(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$$

- For nonhomogeneous case, keep in mind form of homogeneous solution. Thus we have two subcases:

$$Y_1(t) = (A + Bt)t,\; Y_2(t) = C\cos t + D\sin t,\; Y_3(t) = Ete^{2t},$$

- As in Chapter 3, can be shown that $Y(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}t\,e^{-2t}$
- The general solution is $y(t) = y_c(t) + Y(t)$

Ch 4.4: The Method of Variation of Parameters

- The variation of parameters method can be used to find a particular solution of the nonhomogeneous *n*th order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t),$$

  provided g is continuous.
- As with 2nd order equations, begin by assuming $y_1, y_2 \ldots, y_n$ are fundamental solutions to homogeneous equation.
- Next, assume the particular solution Y has the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$$

  where $u_1, u_2, \ldots u_n$ are functions to be solved for.
- In order to find these n functions, we need n equations.

Variation of Parameters Derivation (2 of 5)

- First, consider the derivatives of Y:

$$Y' = \left(u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n\right) + \left(u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'\right)$$

- If we require

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0$$

then

$$Y'' = \left(u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n'\right) + \left(u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n''\right)$$

- Thus we next require

$$u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0$$

- Continuing in this way, we require

$$u_1' y_1^{(m)} + u_2' y_2^{(m)} + \cdots + u_n' y_n^{(m)} = 0, \quad m = 1, \ldots, n-2$$

and hence

$$Y^{(m)} = u_1 y_1^{(m)} + \cdots + u_n y_n^{(m)}, \; m = 2, 3, \ldots, n-1$$

Variation of Parameters Derivation

- From the previous slide,

$$Y^{(m)} = u_1 y_1^{(m)} + \cdots + u_n y_n^{(m)}, \; m = 2,3,\ldots,n-1$$

- Finally,

$$Y^{(n)} = \left(u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right) + \left(u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}\right)$$

- Next, substitute these derivatives into our equation

$$y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = g(t)$$

- Recalling that y_1, y_2 …, y_n are solutions to homogeneous equation, and after rearranging terms, we obtain

$$u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

Variation of Parameters Derivation (4 of 5)

- The n equations needed in order to find the n functions $u_1, u_2, \ldots u_n$ are

$$
\begin{aligned}
u_1' y_1 + \quad \cdots \quad + u_n' y_1 \quad &= 0 \\
u_1' y_1' + \quad \cdots \quad + u_n' y_n' \quad &= 0 \\
&\vdots \\
u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= g
\end{aligned}
$$

- Using Cramer's Rule, for each $m = 1, \ldots, n$,

$$u_m'(t) = \frac{g(t) W_m(t)}{W(t)}, \text{ where } W(t) = W(y_1, \ldots, y_n)(t)$$

and W_m is determinant obtained by replacing mth column of W with $(0, 0, \ldots, 1)$.

Variation of Parameters Derivation (5 of 5)

- From the previous slide,

$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}, \; m = 1, \ldots, n$$

- Integrate to obtain u_1, u_2,… u_n:

$$u_m(t) = \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)} ds, \; m = 1, \ldots, n$$

- Thus, a particular solution Y is given by

$$Y(t) = \sum_{m=1}^{n} \left[\int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)} ds \right] y_m(t)$$

where t_0 is arbitrary.

Example 1 (1 of 3)

- Consider the equation below, along with the given solutions of corresponding homogeneous solutions y_1, y_2, y_3:

$$y''' - y'' - y' + y = g(t), \;\; y_1(t) = e^t, \; y_2(t) = te^t, \; y_3(t) = e^{-t}$$

- Then a particular solution of this ODE is given by

$$Y(t) = \sum_{m=1}^{3} \left[\int_{t_0}^{t} \frac{e^{2s} W_m(s)}{W(s)} ds \right] y_m(t)$$

- It can be shown that

$$W(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} = 4e^t$$

Example 1 (2 of 3)

- Also,

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix} = -2t-1$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = 2$$

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = e^t$$

Example 1 (3 of 3)

- Thus a particular solution in integral form is

$$
\begin{aligned}
Y(t) &= \sum_{m=1}^{3}\left[\int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)}\,ds\right] y_m(t) \\
&= e^{t}\int_{t_0}^{t} \frac{g(s)\left(-2s-1\right)}{4e^{s}}\,ds + te^{t}\int_{t_0}^{t} \frac{g(s)2}{4e^{s}}\,ds + e^{-t}\int_{t_0}^{t} \frac{g(s)e^{2s}}{4e^{s}}\,ds \\
&= \frac{1}{4}\int_{t_0}^{t}\left[e^{t-s}\left(-1+2(t-s)\right)+e^{-(t-s)}\right]g(s)\,ds
\end{aligned}
$$