Introduction to Differential Equations
Sample problems # 14

Date Given: July 11, 2022

P1. (a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \boldsymbol{x}$$

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 3-r & -4 \\ 1 & -1-r \end{vmatrix} = r^2 - 2r + 1 = 0 \quad \Longrightarrow \quad r_1 = 1, r_2 = 1.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, one solution is

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}_1 e^t$$

For $r = r_2 = r_1$, solving $(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \boldsymbol{0}$ does not give new eigenvectors. In order to obtain a second linearly independent solution, we assume that $\boldsymbol{x} = \boldsymbol{\xi}_1 te^t + \boldsymbol{\eta} e^t$, where $\boldsymbol{\eta}$ is a generalized eigenvector. Then $\boldsymbol{\eta}$ must satisfy

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}_1 \quad \Longrightarrow \quad \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

These equations reduce to $\eta_1 - 2\eta_2 = 1$. Set $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = 1 + 2k$. Therefore, a second solution is

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1+2k \\ k \end{bmatrix} e^t = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + k \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t.$$

Since the last term is a multiple of the first solution, it can be dropped. The general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \right).$$

(b) From the general solution we have $\dfrac{x_2(t)}{x_1(t)} = \dfrac{c_1 + c_2 t}{2c_1 + 2c_2 t + c_2}$, so that $\lim\limits_{t\to\infty} \dfrac{x_2(t)}{x_1(t)} = 1/2$. Thus, all solutions (except the trivial one) diverge to infinity along lines of slop 1/2, which can be seen in the trajectories shown in Figure 1. The direction field and a few trajectories of the system are shown in Figure 1.

P2. (a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \boldsymbol{x}$$

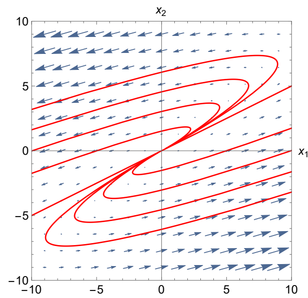

Figure 1: Illustration to problem P1.

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 4-r & -2 \\ 8 & -4-r \end{vmatrix} = r^2 = 0 \quad \Longrightarrow \quad r_1 = 0, r_2 = 0.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore, one solution is

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}_1,$$

which is a constant vector.

For $r = r_2 = r_1$, solving $(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \mathbf{0}$ does not give new eigenvectors. In order to obtain a second linearly independent solution, we assume that $\boldsymbol{x} = \boldsymbol{\xi}_1 t + \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is a generalized eigenvector. Then $\boldsymbol{\eta}$ must satisfy

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}_1 \quad \Longrightarrow \quad \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

These equations reduce to $2\eta_1 - \eta_2 = 1/2$. Set $\eta_1 = k$, some arbitrary constant. Then $\eta_2 = 2k - 1/2$. Therefore, a second solution is

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} k \\ 2k - 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} + k \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since the last term is a multiple of the first solution, it can be dropped. The general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} t - \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \right).$$

(b) From the general solution we have $x_2(t) = 2c_1 + 2c_2 t - c_2/2$ and $x_1(t) = c_1 + c_2 t$, so that $x_2(t) = 2x_1(t) - c_2/2$, that is solutions in the phase plane are straight lines. All of the points on the line $x_2 = 2x_1$ are equilibrium points. Solutions starting at all other points become unbounded. The direction field and a few trajectories of the system are shown in Figure 2.

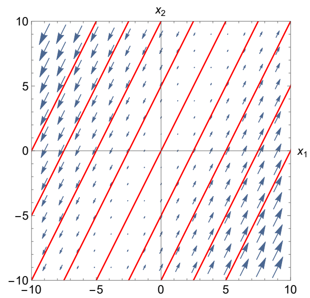

Figure 2: Illustration to problem P2.

P3. (a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \boldsymbol{x}$$

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} -\frac{3}{2} - r & 1 \\ -\frac{1}{4} & -\frac{1}{2} - r \end{vmatrix} = r^2 + 2r + 1 = 0 \quad \Longrightarrow \quad r_1 = -1, r_2 = -1.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, one solution is

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}_1 e^{-t}$$

For $r = r_2 = r_1$, solving $(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \boldsymbol{0}$ does not give new eigenvectors. In order to obtain a second linearly independent solution, we assume that $\boldsymbol{x} = \boldsymbol{\xi}_1 te^{-t} + \boldsymbol{\eta} e^{-t}$, where $\boldsymbol{\eta}$ is a generalized eigenvector. Then $\boldsymbol{\eta}$ must satisfy

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}_1 \quad \Longrightarrow \quad \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

These equations reduce to $-\eta_1 + 2\eta_2 = 4$. Set $\eta_1 = 2k$, where k is some arbitrary constant. Then $\eta_2 = 2 + k$. Therefore, a second solution is

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 2k \\ 2+k \end{bmatrix} e^{-t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-t} + k \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}.$$

Since the last term is a multiple of the first solution, it can be dropped. The general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^{-t} \right).$$

(b) The origin is attracting. From the general solution we have $\dfrac{x_2(t)}{x_1(t)} = \dfrac{c_1 + c_2 t + 2c_2}{2c_1 + 2c_2 t}$, so that $\lim_{t\to\infty} \dfrac{x_2(t)}{x_1(t)} = 1/2$. Thus, all solutions (except the trivial one) approaches the origin tangent to the line $x_2 = x_1/2$, which can be seen in the trajectories shown in Figure 3. The direction field and a few trajectories of the system are shown in Figure 3.

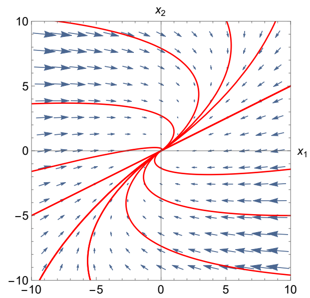

Figure 3: Illustration to problem P3.

P4. Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \boldsymbol{x}$$

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Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{vmatrix} = -r^3 + 3r^2 - 4 = 0 \quad \Longrightarrow \quad r_1 = -1, r_2 = r_3 = 2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system reduces to two equations (because the second equation is a linear combination of the first and third ones):

$$\begin{aligned} 2\xi_1 + \xi_2 + \xi_3 &= 0, \\ \xi_2 - 2\xi_3 &= 0 \end{aligned}$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_1 = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}.$$

Therefore, one solution is

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}_1 e^{-t}$$

For $r = r_2 = r_3$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system reduces to two equations (because the second equation is a linear combination of the first and third ones):

$$\begin{aligned} -\xi_1 + \xi_2 + \xi_3 &= 0, \\ \xi_2 + \xi_3 &= 0. \end{aligned}$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore, the second solution is

$$\boldsymbol{x}^{(2)} = \boldsymbol{\xi}_2 e^{2t}$$

There is no more eigenvectors corresponding to the double eigenvalue, and we have to look for a generalized eigenvector. The third linearly independent solution will have the form

$$\boldsymbol{x}^{(3)} = \boldsymbol{\xi}_2 t e^{2t} + \boldsymbol{\eta} e^{2t}$$

Substituting this into the given system, we find that

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}_2 \quad \Longrightarrow \quad \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Using row reduction, we find that $\eta_1 = 1$ and $\eta_2 + \eta_3 = 1$. If we choose $\eta_2 = 0$, then

$$\boldsymbol{\eta} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

and thus

$$\boldsymbol{x}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}.$$

Therefore, the general solution nay be written as

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} \right).$$

P5. Solve the initial value problem

$$\boldsymbol{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = r^2 + 6r + 9 = 0 \quad \Longrightarrow \quad r_1 = r_2 = -3.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, one solution is

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}_1 e^{-3t}$$

For $r = r_2 = r_1$, solving $(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \boldsymbol{0}$ does not give new eigenvectors. In order to obtain a second linearly independent solution, we assume that $\boldsymbol{x} = \boldsymbol{\xi}_1 t e^{-3t} + \boldsymbol{\eta} e^{-3t}$, where $\boldsymbol{\eta}$ is a generalized eigenvector. Then $\boldsymbol{\eta}$ must satisfy

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}_1 \quad \Longrightarrow \quad \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

These equations reduce to $4\eta_1 - 4\eta_2 = 1$. Set $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. Therefore, a second solution is

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} k + 1/4 \\ k \end{bmatrix} e^{-3t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} + k \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}.$$

Since the last term is a multiple of the first solution, it can be dropped. The general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right).$$

Imposing the initial conditions, we require that $c_1 + c_2/4 = 3$, $c_1 = 2$, which results in $c_1 = 2$ and $c_2 = 4$. Therefore the solution of the initial value problem is

$$\boldsymbol{x}(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-3t} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{-3t}.$$

P6. Solve the initial value problem

$$\boldsymbol{x}' = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 3 - r & 9 \\ -1 & -3 - r \end{vmatrix} = r^2 = 0 \quad \Longrightarrow \quad r_1 = r_2 = 0.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Therefore, one solution is

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}_1$$

For $r = r_2 = r_1$, solving $(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \boldsymbol{0}$ does not give new eigenvectors. In order to obtain a second linearly independent solution, we assume that $\boldsymbol{x} = \boldsymbol{\xi}_1 t + \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is a generalized eigenvector. Then $\boldsymbol{\eta}$ must satisfy

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}_1 \quad \Longrightarrow \quad \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

These equations reduce to $\eta_1 + 3\eta_2 = -1$. Set $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = -3k - 1$. Therefore, a second solution is

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -3k - 1 \\ k \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Since the last term is a multiple of the first solution, it can be dropped. The general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right).$$

Imposing the initial conditions, we require that $-3c_1 - c_2 = 2$, $c_1 = 4$, which results in $c_1 = 4$ and $c_2 = -14$. Therefore the solution of the initial value problem is

$$\boldsymbol{x}(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 42 \\ -14 \end{bmatrix} t.$$

P7. Solve the initial value problem

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} -1 \\ 2 \\ -30 \end{bmatrix}.$$

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Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1-r & 0 & 0 \\ -4 & 1-r & 0 \\ 3 & 6 & 2-r \end{vmatrix} = (r-1)^2(r-2) = 0 \quad \Longrightarrow \quad r_1 = 2, r_2 = r_3 = 1.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, one solution is

$$\boldsymbol{x}^{(1)} = \boldsymbol{\xi}_1 e^{2t}$$

For $r = r_2 = r_3$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system reduces to two equations (because the second equation is a linear combination of the first and third ones):

$$\begin{aligned} \xi_1 &= 0, \\ 6\xi_2 + \xi_3 &= 0 \end{aligned}$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}.$$

Therefore, the second solution is

$$\boldsymbol{x}^{(2)} = \boldsymbol{\xi}_2 e^{t}$$

There is no more eigenvectors corresponding to the double eigenvalue, and we have to look for a generalized eigenvector. The third linearly independent solution will have the form

$$\boldsymbol{x}^{(3)} = \boldsymbol{\xi}_2 t e^{t} + \boldsymbol{\eta} e^{t}$$

Substituting this into the given system, we find that

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{\xi}_2 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}.$$

Using row reduction, we find that $\eta_1 = -1/4$ and $6\eta_2 + \eta_3 = -21/4$. If we choose $\eta_2 = 0$, then

$$\boldsymbol{\eta} = \begin{bmatrix} -1/4 \\ 0 \\ -21/4 \end{bmatrix},$$

and thus

$$\boldsymbol{x}^{(3)} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} te^t + \begin{bmatrix} -1/4 \\ 0 \\ -21/4 \end{bmatrix} e^t.$$

Therefore, the general solution nay be written as

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} te^t + \begin{bmatrix} -1/4 \\ 0 \\ -21/4 \end{bmatrix} e^t \right).$$

The initial conditions then yield $c_1 = 3$, $c_2 = 2$, and $c_3 = 4$ and hence

$$\boldsymbol{x}(t) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} e^{2t} + 4 \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} te^t + \begin{bmatrix} -1 \\ 2 \\ -33 \end{bmatrix} e^t.$$