Introduction to Differential Equations
Sample problems # 3

Date Given: April 25, 2022

P1. (Existence and uniqueness.)
State where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied for $y' = \sqrt{1 - t^2 - y^2}$.

Solution: Theorem 2.4.2 guarantees a unique solution to the differential equation through any point (t_0, y_0) such that $t_0^2 + y_0^2 < 1$ since $\partial f/\partial y = -y/\sqrt{1 - t^2 - y^2}$ is defined and continuous only for $1 - t^2 - y^2 > 0$. Note also that $f = \sqrt{1 - t^2 - y^2}$ is defined and continuous in this region as well as on the boundary $t^2 + y^2 = 1$. The boundary cannot be included in the final region due to the discontinuity of $\partial f/\partial y$ there.

P2. (Existence and uniqueness.)
State where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied for $y' = (1+t^2)/(3y - y^2)$.

Solution: In this case $f = (1 + t^2)/(3y - y^2)$ and

$$\frac{\partial f}{\partial y} = \frac{1 + t^2}{y(3 - y)^2} - \frac{1 + t^2}{y^2(3 - y)},$$

which are both continuous everywhere except for $y = 0$ and $y = 3$.

P3. (Existence and uniqueness.)
Solve the initial value problem $y' = -4t/y$, $y(0) = y_0$, and determine how the interval in which the solution exists depends on the initial value y_0.

Solution: The differential equation can be written as $y\mathrm{d}y = -4t\mathrm{d}t$, so $y^2/2 = -2t^2+c$ or $y^2 = c-4t^2$. The initial condition yields $y_0^2 = c$, so that $y^2 = y_0^2 - 4t^2$ which implicitly defines an ellipse (with semi-axes $|y_0|/2$ and $|y_0|$). In the explicit form, $y = \pm\sqrt{y_0^2 - 4t^2}$, which is defined for $4t^2 < y_0^2$ or $|t| < |y_0|/2$. Note that $y_0 \neq 0$ since Theorem 2.4.2 does not hold there.

P4. (Existence and uniqueness.)
In this problem:

(a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1.$$

Where are these solutions valid?

(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.

(c) Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

Solution:

(a) For $y_1(t) = 1 - t$, $y_1'(t) = -1$, so the substitution into the differential equation gives

$$-1 = \frac{-t + \sqrt{t^2 + 4(1 - t)}}{2} = \frac{-t + \sqrt{(t - 2)^2}}{2} = \frac{-t + |t - 2|}{2}.$$

By the definition of the absolute value, the right side is -1 if $t-2 \geq 0$. Setting $t=2$ in $y_1(t)$ we get $y_1(2)=-1$, as required by the initial condition.
For $y_2(t)=-t^2/4$, $y_2'(t)=-t/2$, so the substitution into the differential equation gives

$$-\frac{t}{2}=\frac{-t+\sqrt{t^2+4(-t^2/4)}}{2}=-\frac{t}{2}.$$

which is valid for all values of t. Also, $y_2(2)=-1$

(b) By Theorem 2.4.2 we are guaranteed a unique solution only where $f(t,y)=(-t+\sqrt{t^2+4(1-t)})/2$ and $\partial f(t,y)/\partial y=1/\sqrt{t^2+4y}$ are continuous. In this case the initial point $(2,-1)$ lies in the region $t^2+4y \leq 0$, so $\partial f/\partial y$ is not continuous and hence the theorem is not applicable and there is no contradiction.

(c) In this case $y(t)=ct+c^2$, $y'(t)=c$, so the substitution into the differential equation gives

$$c=\frac{-t+\sqrt{t^2+4(ct+c^2)}}{2}=\frac{-t+\sqrt{(t+2c)^2}}{2}=\frac{-t+|t+2c|}{2}.$$

By the definition of the absolute value, the right side is -1 if $t+2c \geq 0$ that is $t \geq -2c$. If $c=-1$ the solution $y=y_1(t)$ is obtained.
If $y=y_2(t)$ then we must have $ct+c^2=-t^2/4$ for all t, which is not possible since c is a constant.

P5. (Exact equations.)
Determine whether the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{ax+by}{bx+cy}$$

is exact or not. If it is exact, find the solution.

Solution: Here $M(x,y)=ax+by$ and $N(x,y)=bx+cy$. Since $\partial M/\partial y=b=\partial N/\partial x$, the equation is exact.

- Since $\partial\psi/\partial x=M=ax+by$, to solve for ψ, we integrate M with respect to x and obtain $\psi=ax^2/2+bxy+h(y)$.
- Then $\partial\psi/\partial y=bx+h'(y)=N=bx+cy$ implies that $h'(y)=cy$. Therefore $h(y)=cy^2/2$ and $\psi(x,y)=ax^2/2+bxy+cy^2/2$. Thus, the solution of the equation, written in the implicit form, can be represented as $ax^2/2+bxy+cy^2/2=C$.

P6. (Exact equations.)
Determine whether the equation $(e^x\sin y-2y\sin x)+(e^x\cos y+2\cos x)y'=0$ is exact or not. If it is exact, find the solution.

Solution: Here $M(x,y)=e^x\sin y-2y\sin x$ and $N(x,y)=e^x\cos y+2\cos x$. Since $\partial M/\partial y=e^x\cos y-2\sin x=\partial N/\partial x$, the equation is exact.

- Since $\partial\psi/\partial x=M=e^x\sin y-2y\sin x$, to solve for ψ, we integrate M with respect to x and obtain $\psi=e^x\sin y+2y\cos x+h(y)$.
- Then $\partial\psi/\partial y=-e^x\cos y+2\cos x+h'(y)=N=e^x\cos y+2\cos x$ implies that $h'(y)=0$. Therefore $h(y)=\text{const}$ and $\psi(x,y)=e^x\sin y+2y\cos x+\text{const}$. Thus, the solution of the equation, written in the implicit form, can be represented as $e^x\sin y+2y\cos x=C$.

P7. (Exact equations.)
Determine whether the equation

$$(y/x+6x)+(\ln x-2)\,y'=0, \quad x>0$$

is exact or not. If it is exact, find the solution.

Solution: Here $M(x,y) = y/x + 6x$ and $N(x,y) = \ln x - 2$. Since $\partial M/\partial y = 1/x = \partial N/\partial x$, the equation is exact.

- Finding $\psi(x,y)$ by integrating $M(x,y)$ with respect to x, as in the conventional scheme, leads to longer (but still correct) computations. Instead, we can employ an alternative scheme in which the roles of x and y are interchanged. Specifically, we first find $\psi(x,y)$ by integrating $N(x,y)$ with respect to y.
- Since $\partial\psi/\partial y = N = \ln x - 2$, to solve for ψ, we integrate N with respect to y and obtain $\psi = (\ln x - 2)\,y + h(x)$. Then we find $h(x)$ by differentiating $\psi(x,y)$ with respect to x and setting it equal to $M(x,y)$.
- Since $\partial\psi/\partial x = y/x + h'(x) = M = y/x + 6x$ implies that $h'(x) = 6x$. Therefore $h(x) = 3x^2$ and $\psi(x,y) = (\ln x - 2)\,y + 3x^2$. Thus, the solution of the differential equation, written in the implicit form, can be represented as $\psi(x,y) = (\ln x - 2)\,y + 3x^2 = C$.

P8. (Exact equations: integrating factor.)
Show that the equation $x^2y^3 + x(1+y^2)y' = 0$ is not exact but becomes exact when multiplied by the integrating factor $\mu(x,y) = 1/xy^3$. Then solve this equation.

Solution: Here $M(x,y) = x^2y^3$ and $N(x,y) = x(1+y^2)$. Since $\partial M/\partial y = 3x^2y^2 \neq \partial N/\partial x = 1+y^2$, the equation is not exact. Now, multiplying the equation by $\mu(x,y) = 1/xy^3$, the equation becomes $\tilde{M}(x,y)\mathrm{d}x + \tilde{N}(x,y)\mathrm{d}y$, where $\tilde{M}(x,y) = x$ and $\tilde{N}(x,y) = (1+y^2)/y^3$. Now we see that for this equation $\partial\tilde{M}\partial y = 0 = \partial\tilde{N}/\partial x$, so the transformed equation is exact.

- Since $\partial\psi/\partial x = \tilde{M} = x$, to solve for ψ, we integrate $\tilde{M}$ with respect to x and obtain $\psi = x^2/2 + h(y)$.
- Then $\partial\psi/\partial y = h'(y) = \tilde{N} = (1+y^2)/y^3$. Therefore $h'(y) = 1/y^3 + 1/y$. Therefore $h(y) = -1/2y^2 + \ln|y|$ and $\psi(x,y) = x^2/2 - 1/2y^2 + \ln|y|$. Thus, the solution of the equation, written in the implicit form, can be represented as $x^2 - 1/y^2 + 2\ln|y| = C$.

P9. (Exact equations: integrating factor.)
Show that the equation

$$y + (2x - ye^y)\,y' = 0$$

is not exact but becomes exact when multiplied by the integrating factor $\mu(x,y) = y$. Then solve this equation.

Solution: Here $M(x,y) = y$ and $N(x,y) = 2x - ye^y$. Since $\partial M/\partial y = 1 \neq \partial N/\partial x = 2$, the equation is not exact. Now, multiplying the equation by $\mu(x,y) = y$, the equation becomes $\tilde{M}(x,y)\mathrm{d}x + \tilde{N}(x,y)\mathrm{d}y$, where $\tilde{M}(x,y) = y^2$ and $\tilde{N}(x,y) = 2xy - y^2e^y$. Now we see that for this equation $\partial\tilde{M}/\partial y = 2y = \partial\tilde{N}/\partial x$, so the transformed equation is exact.

- Since $\partial\psi/\partial x = \tilde{M} = y^2$, to solve for ψ, we integrate $\tilde{M}$ with respect to x and obtain $\psi(x,y) = xy^2 + h(y)$.
- Then $\partial\psi/\partial y = 2xy + h'(y) = \tilde{N} = 2xy - y^2e^y$. Therefore $h'(y) = -y^2e^y$. Integrating by parts[1] we obtain[2] $h(y) = -e^y(y^2 - 2y + 2)$ and $\psi(x,y) = xy^2 - e^y(y^2 - 2y + 2)$. Thus, the solution of the equation, written in the implicit form, can be represented as $xy^2 - e^y(y^2 - 2y + 2) = C$.

P10. (Exact equations: integrating factor.)
Find an integrating factor and solve the equation $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$.

Solution: Here $M(x,y) = 3x^2y + 2xy + y^3$ and $N(x,y) = x^2 + y^2$. Since $\partial M/\partial y = 3x^2 + 2x + 3y^2 \neq \partial N/\partial x = 2x$, the equation is not exact. We will first look for the integrating factor in the form $\mu = \mu(x)$. Since $(\partial M/\partial y - \partial N/\partial x)/N = 3$ is a function of x only, from $\mathrm{d}\mu/\mathrm{d}x = 3$ we find $\mu(x) = e^{3x}$.

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[1] It employs the differential of the product $\mathrm{d}(uv) = u\mathrm{d}v + v\mathrm{d}u$, from which one gets $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.

[2] The integration by parts gives $\int y^2e^y\mathrm{d}y = \int y^2\mathrm{d}(e^y) = y^2e^y - \int e^y 2y\mathrm{d}y = y^2e^y - 2\int y\mathrm{d}(e^y) = y^2e^y - 2\left(ye^y - \int e^y\mathrm{d}y\right) = y^2e^y - 2ye^y + e^y = e^y(y^2 - 2y + 2)$

- Now, multiplying the equation by $\mu(x) = e^{3x}$, the equation becomes $\tilde{M}(x, y)\mathrm{d}x + \tilde{N}(x, y)\mathrm{d}y$, where $\tilde{M}(x, y) = e^{3x}(3x^2y + 2xy + y^3)$ and $\tilde{N}(x, y) = e^{3x}(x^2 + y^2)$. Now we see that for this equation $\partial\tilde{M}/\partial y = e^{3x}(3x^2 + 2x + 3y^2) = \partial\tilde{N}/\partial x$, so the transformed equation is exact.
- Since $\partial\psi/\partial x = \tilde{M} = e^{3x}(3x^2y + 2xy + y^3)$, to solve for ψ, we integrate $\tilde{M}$ with respect to x and obtain[3] $\psi(x, y) = (x^2y + y^3/3)e^{3x} + h(y)$.
- Then $\partial\psi/\partial y = (x^2 + y^2)e^{3x} + h'(y) = \tilde{N} = e^{3x}(x^2 + y^2)$. Therefore $h'(y) = 0$ and $h(y) = \text{const}$. Then $\psi(x, y) = (x^2y + y^3/3)e^{3x} + \text{const}$. Thus, the solution of the equation, written in the implicit form, can be represented as $(3x^2y + y^3)e^{3x} = C$.

P11. (Exact equations: integrating factor.)

Find an integrating factor and solve the equation $1 + (x/y - \sin y)y' = 0$.

Solution: Here $M(x, y) = 1$ and $N(x, y) = x/y - \sin y$. Since $\partial M/\partial y = 0 \neq \partial N/\partial x = 1/y$, the equation is not exact. Let us first look for the integrating factor in the form $\mu = \mu(x)$. Since $(\partial M/\partial y - \partial N/\partial x)/N = 1/(y \sin y - x)$ is not a function of x only, let us look for the integrating factor in the form $\mu = \mu(y)$. In this case the integrating factor is defined from $\mathrm{d}\mu/\mathrm{d}y = \mu\,(\partial N/\partial x - \partial M/\partial y)/M$. Since $(\partial N/\partial x - \partial M/\partial y)/M = 1/y$ is a function of y only, from $\mathrm{d}\mu/\mathrm{d}y = \mu/y$ we establish[4] $\mu(y) = y$.

- Now, multiplying the equation by $\mu(y) = y$, the equation becomes $\tilde{M}(x, y)\mathrm{d}x+\tilde{N}(x, y)\mathrm{d}y$, where $\tilde{M}(x, y) = y$ and $\tilde{N}(x, y) = x - y \sin y$. Now we see that for this equation $\partial\tilde{M}/\partial y = 1 = \partial\tilde{N}/\partial x$, so the transformed equation is exact.
- Since $\partial\psi/\partial x = \tilde{M} = y$, to solve for ψ, we integrate $\tilde{M}$ with respect to x and obtain $\psi(x, y) = xy + h(y)$.
- Then $\partial\psi/\partial y = x + h'(y) = \tilde{N} = x - y \sin y$. Therefore $h'(y) = -y \sin y$ and[5] $h(y) = -\sin y + y \cos y$. Then $\psi(x, y) = xy - \sin y + y \cos y$. Thus, the solution of the equation, written in the implicit form, can be represented as $xy - \sin y + y \cos y = C$.

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[3] Note that $\int e^{3x}y^3\mathrm{d}x = y^3e^{3x}/3$, and $\int e^{3x}2xy\mathrm{d}x = y\int 2x\,\mathrm{d}\left(e^{3x}\right)/3 = y\left\{2xe^{3x}/3 - \int 2e^{3x}\mathrm{d}x/3\right\} = y\left\{2xe^{3x}/3 - 2e^{3x}/9\right\}$, and $\int e^{3x}3x^2y\mathrm{d}x = y\left\{x^2\mathrm{d}\left(e^{3x}\right)\right\} = y\left\{x^2e^{3x} - \int e^{3x}2x\mathrm{d}x\right\} = y\left\{x^2e^{3x} - \int 2x\mathrm{d}\left(e^{3x}\right)/3\right\} = y\left\{x^2e^{3x} - 2xe^{3x}/3 + \int 2e^{3x}\mathrm{d}x/3\right\} = y\left\{x^2e^{3x} - 2xe^{3x}/3 + 2e^{3x}/9\right\}$.

[4] The integration constant is not important here as we are interested in only one solution for $\mu(y)$.

[5] Note that $\int y \sin y\mathrm{d}y = -\int y\mathrm{d}\left(\cos y\right) = -y \cos y + \int \cos y\mathrm{d}y = -y \cos y + \sin y$.