Introduction to Differential Equations
Sample problems # 7

Date Given: May 23, 2022

P1. Verify that the functions 1, $\cos t$, $\sin t$ are solutions of the differential equation $y''' + y' = 0$, and determine their Wronskian.

Solution:

(a) For $y_1(t) = 1$ we have $y_1''' + y_1' = 0$, for $y_2(t) = \cos t$ we have $y_2''' + y_2' = 0$, and for $y_3(t) = \sin t$ we have $y_3''' + y_3' = 0$. Therefore these functions are solutions of the given differential equation.

(b) We have

$$W(1, \cos t, \sin t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1.$$

P2. Verify that the functions 1, x, x^3 are solutions of the differential equation $xy''' - y'' = 0$, and determine their Wronskian.

Solution:

(a) For $y_1(x) = 1$ we have $xy_1''' - y_1'' = 0$, for $y_2(x) = x$ we have $xy_2''' - y_2'' = 0$, and for $y_3(x) = x^3$ we have $xy_3''' - y_3'' = 0$. Therefore these functions are solutions of the given differential equation.

(b) We have

$$W(1, x, x^3) = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x.$$

P3. Verify that the functions e^x, e^{2x}, e^{3x} are solutions of the differential equation $y''' - 6y'' + 11y' - 6y = 0$, and determine their Wronskian.

Solution:

(a) For $y_1(x) = e^x$ we have $y_1''' - 6y_1'' + 11y_1' - 6y_1 = 0$, for $y_2(x) = e^{2x}$ we have $y_2''' - 6y_2'' + 11y_2' - 6y_2 = 0$ and for $y_3(x) = e^{3x}$ we have $y_3''' - 6y_3'' + 11y_3' - 6y_3 = 0$. Therefore these functions are solutions of the given differential equation.

(b) We have

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0.$$

P4. Find the general solution of the differential equation $y''' - y = 0$.

Solution: The characteristic equation is $r^3 - 1 = 0$. The roots are given by $r = (1)^{1/3}$, that is the three third roots of $1 = \cos(0) + \imath \sin(0)$. They are $e^{\imath(\frac{0+2\pi k}{3})}$, $k = 0, 1, 2$. Explicitly, $r_1 = 1$, $r_2 = (-1 + \imath\sqrt{3})/2$, $r_3 = (-1 - \imath\sqrt{3})/2$. Note that they are all distinct. Thus

$$y(t) = c_1 t + c_2 e^{-t/2} \cos(\sqrt{3}t/2) + c_3 e^{-t/2} \cos(\sqrt{3}t/2)$$

P5. Find the general solution of the differential equation $y''' - 5y'' + 3y' + 9y = 0$.

Solution: The characteristic equation is $r^3 - 5r^2 + 3r + 9 = 0$. The roots are $r_1 = -1$, $r_2 = 3$, $r_3 = 3$. Note that two roots are repeated. Thus

$$y(t) = c_1 e^{-t} + c_2 e^{3t} + c_3 t e^{3t}$$

P6. Find the general solution of the differential equation $y^{(4)} + y''' + y'' = 0$.

Solution: The characteristic equation is $r^4 + r^3 + r^2 = 0$. The roots are $r_1 = 0$, $r_2 = 0$, $r_3 = (-1 + \sqrt{3}\imath)/2$, $r_4 = (-1 - \sqrt{3}\imath)/2$. Note that two roots are repeated. Thus

$$y(t) = c_1 + c_2 t + c_3 e^{-t/2} \cos(\sqrt{3}t/2) + c_4 e^{-t/2} \sin(\sqrt{3}t/2)$$

P7. Find the general solution of the differential equation $y^{(5)} + 5y^{(4)} - 2y^{(3)} - 10y'' + y' + 5y = 0$.

Solution: The characteristic equation is $r^5 + 5r^4 - 2r^3 - 10r^2 + r + 5 = 0$. The roots are $r_1 = -1$, $r_2 = -1$, $r_3 = 1$, $r_4 = 1$, $r_5 = -5$. Note that there are repeated roots. Thus

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 e^t + c_4 t e^t + c_5 e^{-5t}$$

P8. Find the general solution of the differential equation $y^{(4)} - 4y''' + 4y'' = 0$.

Solution: The characteristic equation is $r^2(r-2)^2 = 0$. The roots are $r_1 = 0$, $r_2 = 0$, $r_3 = 2$, $r_4 = 2$. There are two repeated roots, and hence the general solution is given by $y(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$.

P9. Find the general solution of the differential equation $y^{(4)} - 8y' = 0$.

Solution: The characteristic equation is $r(r^3 - 8) = 0$. with roots $r = 0, 2e^{2k\theta\imath/3}, k = 0, 1, 2$, that is $r_1 = 0$, $r_2 = 2$, $r_3 = -1 + \sqrt{3}\imath$, $r_4 = -1 - \sqrt{3}\imath$. Hence the general solution is

$$y = c_1 + c_2 e^{2t} + e^{-t} \left(c_3 \cos \sqrt{3}t + c_4 \sin \sqrt{3}t \right).$$

P10. Find the general solution of the differential equation $y^{(8)} + 8y^{(4)} + 16y = 0$.

Solution: The characteristic equation can be written as $(r^4 + 4)^2 = 0$. The roots are $1 \pm \imath$ and $-1 \pm \imath$. Each of these roots has multiplicity two. The general solution is

$$y(t) = e^t (c_1 \cos t + c_2 \sin t) + t e^t (c_3 \cos t + c_4 \sin t) + e^{-t} (c_5 \cos t + c_6 \sin t) + t e^{-t} (c_7 \cos t + c_8 \sin t).$$

P11. Find the solution of the initial value problem $y''' - y'' + y' - y = 0$, $y(0) = 2$, $y'(0) = -1$, $y''(0) = -2$, and plot its graph. How does the solution behave as $t \to \infty$?

Solution: The characteristic equation is $r^3 - r^2 + r - 1 = 0$, with roots $r = 1$ and $r = \pm\imath$. Hence

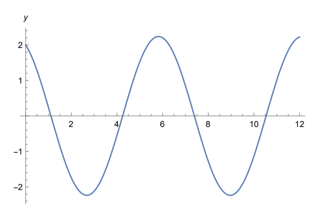

Figure 1: Illustration to problem P11.

the general solution is $y(t) = c_1 e^t + c_2 \cos t + c_3 \sin t$. Invoking the initial conditions, we obtain the system of equations $c_1 + c_2 = 2, c_1 + c_3 = -1, c_1 - c_2 = -2$, with solution $c_1 = 0, c_2 = 2, c_3 = -1$. Therefore the solution of the initial value problem is

$$y(t) = 2 \cos t - \sin t$$

Note that $y(t) = 2 \cos t - \sin t = \sqrt{5} \cos(t - \varphi)$, where $\varphi = \arctan(1/2)$. The solution oscillates as $t \to \infty$ (see Figure 1).

P12. Find a differential equation whose general solution is $y = c_1 + c_2 t + c_3 e^t$.

Solution: An algebraic equation with roots $r_1 = 0$, $r_2 = 0$ and $r_3 = 1$ is $(r-r_1)(r-r_2)(r-r_3) = r^3 - r^2 = 0$. This is the characteristic equation for the differential equation

$$y''' - y'' = 0.$$

P13. Determine the third roots of the complex number $27\imath$.

Solution: Writing $27\imath$ in the form $\rho e^{\imath\theta} = \rho(\cos\theta + \imath\sin\theta)$, we have $\rho = 27$ and $\theta = \pi/2$. Thus $27\imath = 3^3\, e^{\imath(\pi/2+2\pi k)}$ (where k is any integer), and hence

$$\sqrt[3]{27\imath} = \sqrt[3]{27}\, e^{\imath\left(\frac{\pi/2}{3} + \frac{2k\pi}{3}\right)} = 3\left\{\cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \imath\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right\}$$

We obtain the three third roots by setting $k = 0, 1, 2$. They are

$$\begin{aligned}
r_1 &= 3\left\{\cos\left(\frac{\pi}{6}\right) + \imath\sin\left(\frac{\pi}{6}\right)\right\} = \frac{3}{2}\left(\sqrt{3} + \imath\right), \\
r_2 &= 3\left\{\cos\left(\frac{5\pi}{6}\right) + \imath\sin\left(\frac{5\pi}{6}\right)\right\} = \frac{3}{2}\left(-\sqrt{3} + \imath\right), \\
r_3 &= 3\left\{\cos\left(\frac{9\pi}{6}\right) + \imath\sin\left(\frac{9\pi}{6}\right)\right\} = 3\left\{\cos\left(\frac{3\pi}{2}\right) + \imath\sin\left(\frac{3\pi}{2}\right)\right\} = -3\imath.
\end{aligned}$$

P14. Determine the square roots of the complex number $1 - \imath$.

Solution: Writing $1-\imath$ in the form $\rho e^{\imath\theta} = \rho(\cos\theta + \imath\sin\theta)$, we have $\rho = \sqrt{2}$ and $\theta = -\pi/4$. Thus $1 - \imath = \sqrt{2}\, e^{\imath(-\pi/4+2\pi k)}$ (where k is any integer), and hence

$$(1-\imath)^{1/2} = \sqrt[4]{2}\, e^{\imath(-\pi/8+k\pi)} = \sqrt[4]{2}\left\{\cos\left(-\frac{\pi}{8} + k\pi\right) + \imath\sin\left(-\frac{\pi}{8} + k\pi\right)\right\}$$

We obtain the two square roots by setting $k = 0, 1$. They are

$$r_1 = \sqrt[4]{2}\, e^{-\imath\pi/8} = \sqrt[4]{2}\left(\cos\frac{\pi}{8} - \imath\sin\frac{\pi}{8}\right) = \frac{\sqrt[4]{2}}{2}\left(\sqrt{2+\sqrt{2}} - \imath\sqrt{2-\sqrt{2}}\right),$$

and

$$r_2 = \sqrt[4]{2}\, e^{7\imath\pi/8} = \sqrt[4]{2}\left(\cos\frac{7\pi}{8} + \imath\sin\frac{7\pi}{8}\right) = \sqrt[4]{2}\left(-\cos\frac{\pi}{8} + \imath\sin\frac{\pi}{8}\right) = \frac{\sqrt[4]{2}}{2}\left(-\sqrt{2+\sqrt{2}} + \imath\sqrt{2-\sqrt{2}}\right).$$

Note that $r_2 = -r_1$.

Note also that if $r = \alpha + \imath\beta$ is the root of $1 - \imath$, then $r^2 = (\alpha^2 - \beta^2) + 2\alpha\beta\imath = 1 - \imath$, and one can find α and β from solving the system of two equations, $\alpha^2 - \beta^2 = 1$ and $2\alpha\beta = -1$. Sometimes it is easier to find square roots by using this alternative way.