Introduction to Differential Equations
Sample problems # 6

Date Given: May 16, 2022

P1. Use the method of undetermined coefficients to find the general solution of the differential equation $y'' - 16y = 2e^{4t}$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 - 16 = 0$, which has two roots $r = 4$, $r = -4$. Therefore, the solution of the homogeneous problem is $y_c(t) = c_1 e^{4t} + c_2 e^{-4t}$.

(b) To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p = Ate^{4t}$. Substituting a function of this form into the differential equation, and equating the terms, we have $8A = 2$. Therefore $A = 1/4$ and the general solution of the nonhomogeneous problem is

$$y(t) = c_1 e^{-t} + c_2 te^{-t} + \frac{1}{4} te^{4t}.$$

P2. Use the method of undetermined coefficients to find the general solution of the differential equation $y'' + y = 2t \sin t$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 + 1 = 0$, which has two roots $r = \imath$, $r = -\imath$. Therefore, the solution of the homogeneous problem is $y_c(t) = c_1 \cos t + c_2 \sin t$.

(b) To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p = t(At+B)\cos t + t(Ct+D)\sin t$. Substituting a function of this form into the differential equation, and equating the terms, we have $4C = 0, 2A + 2D = 0, -4A = 2, -2B + 2C = 0$. Therefore $A = -1/2, B = 0, C = 0, D = 1$ and the general solution of the nonhomogeneous problem is

$$y(t) = c_1 \cos t + c_2 \sin t - \frac{1}{2} t^2 \cos t + \frac{1}{2} t \sin t.$$

P3. Find the solution of the initial value problem $y'' + 4y' + 4y = (3+t)e^{-2t}$, $y(0) = 2$, $y'(0) = 5$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 + 4r + 4 = 0$, which has the repeated (double) root $r = -2$. Therefore, the solution of the homogeneous problem is $y_c(t) = c_1 e^{-2t} + c_2 te^{-2t}$.

(b) To find a solution of the nonhomogeneous problem, based on the form of $g(t)$ we look for a solution of the form $y_p = t^2(At + B)e^{-2t}$. Substituting a function of this form into the differential equation, and equating the terms, we find $A = 1/6$ and $B = 3/2$. Therefore the general solution of the nonhomogeneous problem is

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t} + \left(\frac{1}{6} t^3 + \frac{3}{2} t^2\right) e^{-2t}.$$

(c) From the initial conditions we get $c_1 = 2$ and $c_2 = 9$. Thus the solution of the initial value problem is

$$y(t) = 2e^{-2t} + 9te^{-2t} + \left(\frac{1}{6} t^3 + \frac{3}{2} t^2\right) e^{-2t}.$$

P4. Find the solution of the initial value problem $y'' + y = 4t + 10 \sin t$, $y(\pi) = 0$, $y'(\pi) = 2$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 + 1 = 0$, with complex root $r = \pm\imath$. Hence, the solution of the homogeneous problem is $y_c(t) = c_1 \cos\omega t + c_2 \sin\omega t$.

(b) To find a solution of the nonhomogeneous problem, based on the form of $g(t)$ we look for a solution of the form $y_p = (At + B) + t(C \cos t + D \sin t)$. Substituting a function of this form into the differential equation, and equating the terms, we find $A = 4, B = 0, C = -5$, and $D = 0$. Therefore the general solution of the nonhomogeneous problem is

$$y(t) = c_1 \cos t + c_2 \sin t + 4t - 5t \cos t.$$

(c) From the initial conditions we get $c_1 = 9\pi$ and $c_2 = 7$. Thus the solution of the initial value problem is

$$y(t) = 9\pi \cos t + 7 \sin t + 4t - 5t \cos t.$$

P5. Find the general solution of the differential equation $y'' + 2y' + y = 2e^{-t}$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 + 2r + 1 = 0$, which has the repeated (double) root $r = -1$. Therefore, the solution of the homogeneous problem is $y_c(t) = c_1e^{-t} + c_2te^{-t}$.

(b) To find a solution of the nonhomogeneous problem, we look for a solution of the form $y_p = At^2e^{-t}$ because the multiplicity of the root $r = -1$ is two. Substituting a function of this form into the differential equation, and equating the terms, we have $2A = 2$. Therefore $A = 1$ and the general solution of the nonhomogeneous problem is $y(t) = t^2e^{-t} + c_1e^{-t} + c_2te^{-t}$.

P6. Find the general solution of the differential equation $u'' + \omega_0^2 u = \cos\omega t$, $\omega \neq \omega_0$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 + \omega_0^2 = 0$, with complex root $r = \pm\omega_0\imath$. Hence, the solution of the homogeneous problem is $u_c(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$.

(b) Since $\omega \neq \omega_0$, we look for a particular solution of the form $u_p = A\cos\omega t + B \sin\omega t$. Substitution into the differential equation and comparing the coefficients results in the system of equations we have $(\omega_0^2 - \omega^2)A = 1$ and $(\omega_0^2 - \omega^2)B = 0$. Therefore $A = 1/(\omega_0^2 - \omega^2)$ and the general solution of the nonhomogeneous problem is

$$u(t) = \frac{1}{\omega_0^2 - \omega^2} \cos\omega t + c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

P7. Find the general solution of the differential equation $u'' + \omega_0^2 u = \cos\omega_0 t$.

Solution:

(a) The characteristic equation for the homogeneous problem is $r^2 + \omega_0^2 = 0$, with complex root $r = \pm\omega_0\imath$. Hence, the solution of the homogeneous problem is $u_c(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$.

(b) Since $\cos\omega_0 t$ is a solution of the homogeneous problem, set $u_p = At \cos\omega_0 t + Bt \sin\omega_0 t$. Substitution into the differential equation and comparing the coefficients results in the system of equations we have $A = 0$ and $B = 1/2\omega_0$. Therefore the general solution of the nonhomogeneous problem is

$$u(t) = \frac{1}{2\omega_0} t \sin\omega_0 t + c_1 \cos\omega_0 t + c_2 \sin\omega_0 t.$$

P8. Use the method of variation of parameters to find the general solution of the differential equation $y'' + y = \cos^2 t$.

Solution:

(a) The solution of the homogeneous equation is $y_c(t) = c_1 \cos t + c_2 \sin t$. The functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ form a fundamental set of solutions. The Wronskian of these functions is

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.$$

(b) Here $g(t) = \cos^2 t$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$u_1'(t) = -\frac{\sin t \cos^2 t}{W(t)} = -\sin t \cos^2 t, \quad \text{and} \quad u_2'(t) = \frac{\cos t(\cos^2 t)}{W(t)} = \cos^3 t = \cos t(1 - \sin^2 t).$$

Therefore,

$$u_1(t) = \int (-\sin t \cos^2 t)\mathrm{d}t = \frac{1}{3}\cos^3 t, \quad \text{and} \quad u_2(t) = \int (\cos t - \cos t \sin^2 t)\mathrm{d}t = \sin t - \frac{1}{3}\sin^3 t$$

Hence the particular solution is $Y(t) = \frac{1}{3}\cos^4 t + \sin^2 t - \frac{1}{3}\sin^4 t$. The general solution is given by

$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{3}\cos^4 t + \sin^2 t - \frac{1}{3}\sin^4 t = c_1 \cos t + c_2 \sin t + \frac{1}{6}(3 - \cos 2t).$$

P9. Use the method of variation of parameters to find the general solution of the differential equation $y'' + 3y' + 2y = 1/(1 + e^t)$.

Solution:

(a) The solution of the homogeneous equation is $y_c(t) = c_1 e^{-t} + c_2 e^{-2t}$. The functions $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t}.$$

(b) Here $g(t) = 1/(1 + e^t)$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$u_1'(t) = \frac{e^t}{1 + e^t}, \quad \text{and} \quad u_2'(t) = -\frac{e^{2t}}{1 + e^t} = \frac{e^t}{1 + e^t} - e^t.$$

Therefore,

$$u_1(t) = \int \frac{e^t}{1 + e^t}\mathrm{d}t = \ln(1 + e^t), \quad \text{and} \quad u_2(t) = \int \left(\frac{e^t}{1 + e^t} - e^t\right)\mathrm{d}t = \ln(1 + e^t) - e^t$$

Hence the particular solution is $Y(t) = e^{-t}\ln(1 + e^t) + e^{-2t}(\ln(1 + e^t) - e^t)$. The general solution is given by

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} - e^{-t} + (e^{-t} + e^{-2t})\ln(1 + e^t).$$

P10. Find the general solution of the differential equation $y'' + y = \tan t$, $0 < t < \pi/2$.

Solution:

(a) The solution of the homogeneous equation is $y_c(t) = c_1 \cos t + c_2 \sin t$. The two functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ form a fundamental set of solutions, with

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.$$

(b) Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{\sin t(\tan t)}{W(t)}\mathrm{d}t = \int \frac{\cos^2 t - 1}{\cos t}\mathrm{d}t = \int (\cos t - \sec t)\mathrm{d}t = \sin t - \ln(\sec t + \tan t), \\ u_2(t) &= \int \frac{\cos t(\tan t)}{W(t)}\mathrm{d}t = -\cos t. \end{aligned}$$

Hence the particular solution is $Y(t) = -\cos t \ln(\sec t + \tan t)$. The general solution is given by

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln(\sec t + \tan t).$$

P11. Find the general solution of the differential equation $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$.

Solution:

(a) The solution of the homogeneous equation is $y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}$. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is

$$\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - t2e^{-2t} \end{vmatrix} = e^{-4t}.$$

(b) Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}\mathrm{d}t = -\ln t, \quad \text{and} \quad u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}\mathrm{d}t = -1/t.$$

Hence the particular solution is $Y(t) = -e^{-2t}\ln t - e^{-2t}$. Since the second term is a solution of the homogeneous equation, the general solution is given by

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t} - e^{-2t}\ln t.$$