Ch 3.3: Complex Roots of Characteristic Equation

- Recall our discussion of the equation

$$ay'' + by' + cy = 0$$

where a, b and c are constants.

- Assuming an exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 and r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$, then complex roots: $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

Thus

$$y_1(t) = e^{(\lambda + i\mu)t}, \;\; y_2(t) = e^{(\lambda - i\mu)t}$$

Euler's Formula; Complex Valued Solutions

- Substituting *it* into Taylor series for e^t, we obtain **Euler's formula**:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} = \cos t + i \sin t$$

- Generalizing Euler's formula, we obtain

$$e^{i\mu t} = \cos \mu t + i \sin \mu t$$

- Then

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\left[\cos \mu t + i \sin \mu t\right] = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

- Therefore

$$y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

$$y_2(t) = e^{(\lambda - i\mu)t} = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

Real Valued Solutions

- Our two solutions thus far are complex-valued functions:

$$y_1(t) = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t$$
$$y_2(t) = e^{\lambda t}\cos\mu t - ie^{\lambda t}\sin\mu t$$

- We would prefer to have real-valued solutions, since our differential equation has real coefficients.
- To achieve this, recall that linear combinations of solutions are themselves solutions:

$$y_1(t) + y_2(t) = 2e^{\lambda t}\cos\mu t$$
$$y_1(t) - y_2(t) = 2ie^{\lambda t}\sin\mu t$$

- Ignoring constants, we obtain the two solutions

$$y_3(t) = e^{\lambda t}\cos\mu t, \quad y_4(t) = e^{\lambda t}\sin\mu t$$

Real Valued Solutions: The Wronskian

- Thus we have the following real-valued functions:

$$y_3(t) = e^{\lambda t}\cos \mu t, \quad y_4(t) = e^{\lambda t}\sin \mu t$$

- Checking the Wronskian, we obtain

$$W = \begin{vmatrix} e^{\lambda t}\cos \mu t & e^{\lambda t}\sin \mu t \\ e^{\lambda t}(\lambda\cos \mu t - \mu\sin \mu t) & e^{\lambda t}(\lambda\sin \mu t + \mu\cos \mu t) \end{vmatrix}$$

$$= \mu e^{2\lambda t} \neq 0$$

- Thus y_3 and y_4 form a fundamental solution set for our ODE, and the general solution can be expressed as

$$y(t) = c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t$$

Example 1 (1 of 2)

- Consider the differential equation

$$y'' + y' + 9.25y = 0$$

- For an exponential solution, the characteristic equation is

$$y(t) = e^{rt} \Rightarrow r^2 + r + 9.25 = 0 \Rightarrow r = \frac{-1 + \sqrt{1 - 37}}{2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i$$

- Therefore, separating the real and imaginary components,

$$\lambda = -\frac{1}{2}, \ \mu = 3$$

and thus the general solution is

$$y(t) = c_1 e^{-t/2} \cos(3t) + c_2 e^{-t/2} \sin(3t) = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

Example 1 (2 of 2)

- Using the general solution just determined

$$y(t) = e^{-t/2}\left\{c_1\cos(3t) + c_2\sin(3t)\right\}$$

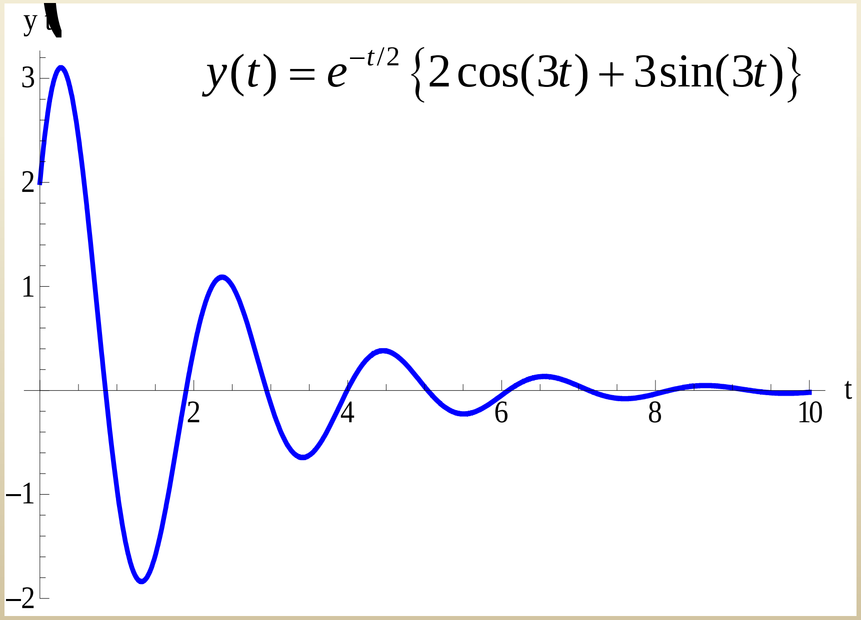
- We can determine the particular solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 8$
- So

$$\left.\begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= -\tfrac{1}{2}c_1 + 3c_2 = 8 \end{aligned}\right\} \Rightarrow c_1 = 2,\ c_2 = 3$$

- Thus the solution of this IVP is

$$y(t) = e^{-t/2}\left\{2\cos(3t) + 3\sin(3t)\right\}$$

- The solution is a decaying oscillation



Example 2

- Consider the initial value problem

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1$$

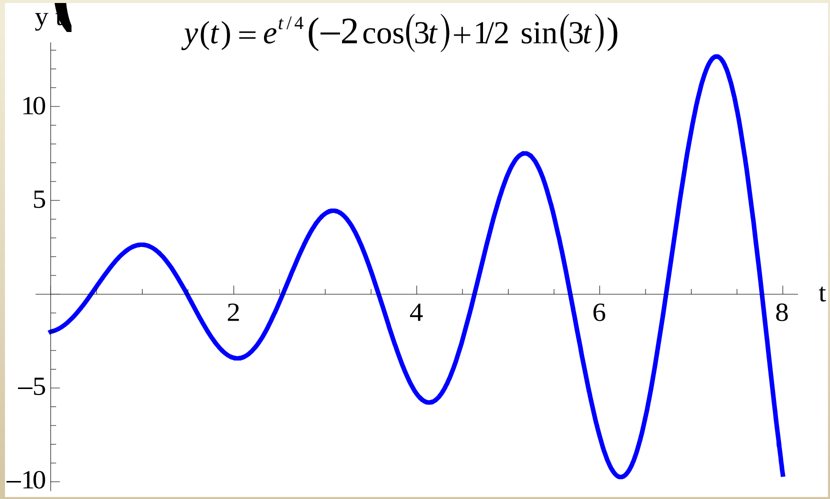
- Then $y(t) = e^{rt} \Rightarrow 16r^2 - 8r + 145 = 0 \Leftrightarrow r = \frac{1}{4} \pm 3i$
- Thus the general solution is $y(t) = c_1 e^{t/4}\cos(3t) + c_2 e^{t/4}\sin(3t)$
- And

$$\left.\begin{aligned} y(0) &= c_1 = -2 \\ y'(0) &= -\frac{1}{4}c_1 + 3c_2 = 1 \end{aligned}\right\} \Rightarrow c_1 = -2,\ c_2 = \frac{1}{2}$$

- The solution of the IVP is

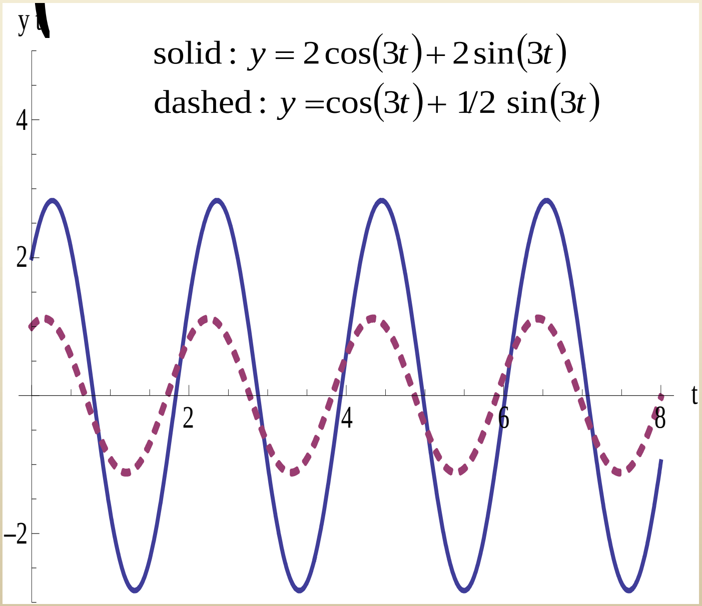
$$y(t) = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t)$$

- The solution is displays a growing oscillation



Example 3

- Consider the equation
$$y'' + 9y = 0$$
- Then $y(t) = e^{rt} \;\Rightarrow\; r^2 + 9 = 0 \;\Leftrightarrow\; r = \pm 3i$
- Therefore $\lambda = 0,\ \mu = 3$
- and thus the general solution is
$$y(t) = c_1 \cos(3t) + c_2 \sin(3t)$$
- Because $\lambda = 0,$ there is no exponential factor in the solution, so the amplitude of each oscillation remains constant. The figure shows the graph of two typical solutions



Ch 3.4: Repeated Roots; Reduction of Order

- Recall our 2nd order linear homogeneous ODE

$$ay'' + by' + cy = 0$$

- where a, b and c are constants.
- Assuming an exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 and r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac = 0$, $r_1 = r_2 = -b/(2a)$, since method only gives one solution:

$$y_1(t) = ce^{-bt/(2a)}$$

Second Solution: Multiplying Factor *v*(*t*)

- We know that

$$y_1(t) \text{ a solution} \Rightarrow y_2(t) = c y_1(t) \text{ a solution}$$

- Since y_1 and y_2 are linearly dependent, we generalize this approach and multiply by a function v, and determine conditions for which y_2 is a solution:

$$y_1(t) = e^{-bt/(2a)} \text{ a solution} \;\Rightarrow\; \text{try } \; y_2(t) = v(t)e^{-bt/(2a)}$$

- Then

$$y_2(t) = v(t)e^{-bt/(2a)}$$

$$y_2'(t) = v'(t)e^{-bt/(2a)} - \frac{b}{2a}v(t)e^{-bt/(2a)}$$

$$y_2''(t) = v''(t)e^{-bt/(2a)} - \frac{b}{2a}v'(t)e^{-bt/(2a)} - \frac{b}{2a}v'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}v(t)e^{-bt/(2a)}$$

$$ay'' + by' + cy = 0$$

Finding Multiplying Factor *v*(*t*)

- Substituting derivatives into ODE, we seek a formula for *v*:

$$e^{-bt/(2a)}\left\{a\left[v''(t)-\frac{b}{a}v'(t)+\frac{b^2}{4a^2}v(t)\right]+b\left[v'(t)-\frac{b}{2a}v(t)\right]+cv(t)\right\}=0$$

$$av''(t)-bv'(t)+\frac{b^2}{4a}v(t)+bv'(t)-\frac{b^2}{2a}v(t)+cv(t)=0$$

$$av''(t)+\left(\frac{b^2}{4a}-\frac{b^2}{2a}+c\right)v(t)=0$$

$$av''(t)+\left(\frac{b^2}{4a}-\frac{2b^2}{4a}+\frac{4ac}{4a}\right)v(t)=0 \iff av''(t)+\left(\frac{-b^2}{4a}+\frac{4ac}{4a}\right)v(t)=0$$

$$av''(t)-\left(\frac{b^2-4ac}{4a}\right)v(t)=0$$

$$v''(t)=0 \quad\Rightarrow\quad v(t)=k_3t+k_4$$

General Solution

- To find our general solution, we have:

$$\begin{aligned} y(t) &= k_1 e^{-bt/(2a)} + k_2 v(t) e^{-bt/(2a)} \\ &= k_1 e^{-bt/(2a)} + \left(k_3 t + k_4\right) e^{-bt/(2a)} \\ &= c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)} \end{aligned}$$

- Thus the general solution for repeated roots is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}$$

Wronskian

- The general solution is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}$$

- Thus every solution is a linear combination of

$$y_1(t) = e^{-bt/(2a)}, \; y_2(t) = t e^{-bt/(2a)}$$

- The Wronskian of the two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/(2a)} & t e^{-bt/(2a)} \\ -\dfrac{b}{2a} e^{-bt/(2a)} & \left(1 - \dfrac{bt}{2a}\right) e^{-bt/(2a)} \end{vmatrix}$$

$$= e^{-bt/a}\left(1 - \frac{bt}{2a}\right) + e^{-bt/a}\left(\frac{bt}{2a}\right)$$

$$= e^{-bt/a} \neq 0 \quad \text{for all } t$$

- Thus y_1 and y_2 form a fundamental solution set for equation.

Example 1 (1 of 2)

- Consider the initial value problem
$$y'' + 4y' + 4y = 0$$
- Assuming exponential soln leads to characteristic equation:
$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 4r + 4 = 0 \;\Leftrightarrow\; (r+2)^2 = 0 \;\Leftrightarrow r = -2$$
- So one solution is $y_1(t) = e^{-2t}$ and a second solution is found:
$$y_2(t) = v(t)e^{-2t}$$
$$y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$$
$$y_2''(t) = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}$$
- Substituting these into the differential equation and simplifying yields $v''(t) = 0,\ v'(t) = k_1,\ v(t) = k_1 t + k_2$ where c_1 and c_2 are arbitrary constants.

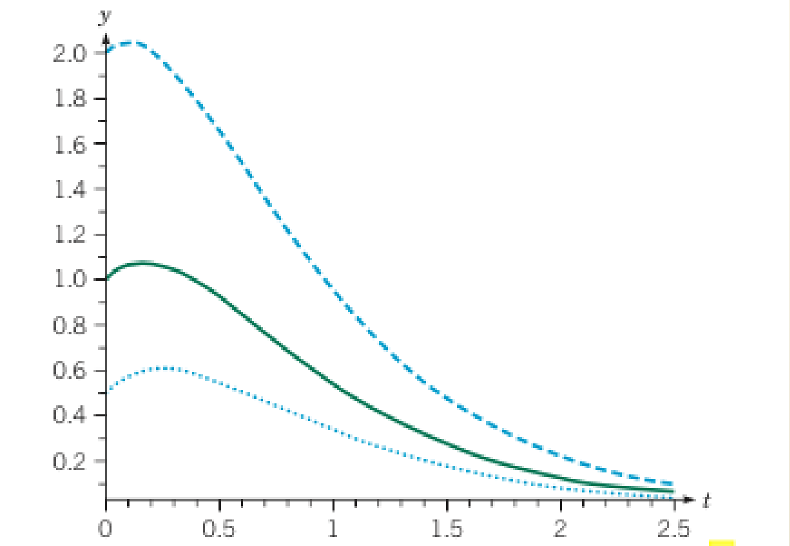
Example 1 (2 of 2)

- Letting $k_1 = 1$ and $k_2 = 0$, $v(t) = t$ and $y_2(t) = te^{-2t}$
- So the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

- Note that both y_1 and y_2 tend to 0 as $t \to \infty$ regardless of the values of c_1 and c_2

- Here are three solutions of this equation with different sets of initial conditions.
- $y(0) = 2$, $y'(0) = 1$ (top)
- $y(0) = 1$, $y'(0) = 1$ (middle)
- $y(0) = ½$, $y'(0) = 1$ (bottom)



Example 2 (1 of 2)

- Consider the initial value problem

$$y'' - y' + \frac{1}{4}y = 0, \; y(0) = 2, \; y'(0) = \frac{1}{3}$$

- Assuming exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \Rightarrow r^2 - r + \frac{1}{4} = 0 \Leftrightarrow (r - \frac{1}{2})^2 = 0 \Leftrightarrow r = \frac{1}{2}$$

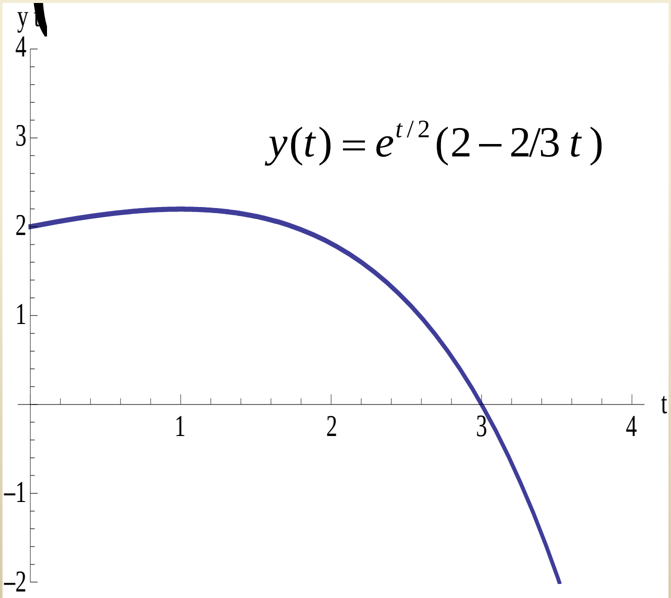
- Thus the general solution is

$$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$

- Using the initial conditions:

$$\left.\begin{array}{rcl} c_1 & = & 2 \\ \frac{1}{2}c_1 + c_2 & = & \frac{1}{3} \end{array}\right\} \Rightarrow c_1 = 2, \; c_2 = -\frac{2}{3}$$

- Thus $\quad y(t) = 2e^{t/2} - \frac{2}{3}te^{t/2}$



Example 2 (2 of 2)

- Suppose that the initial slope in the previous problem was increased

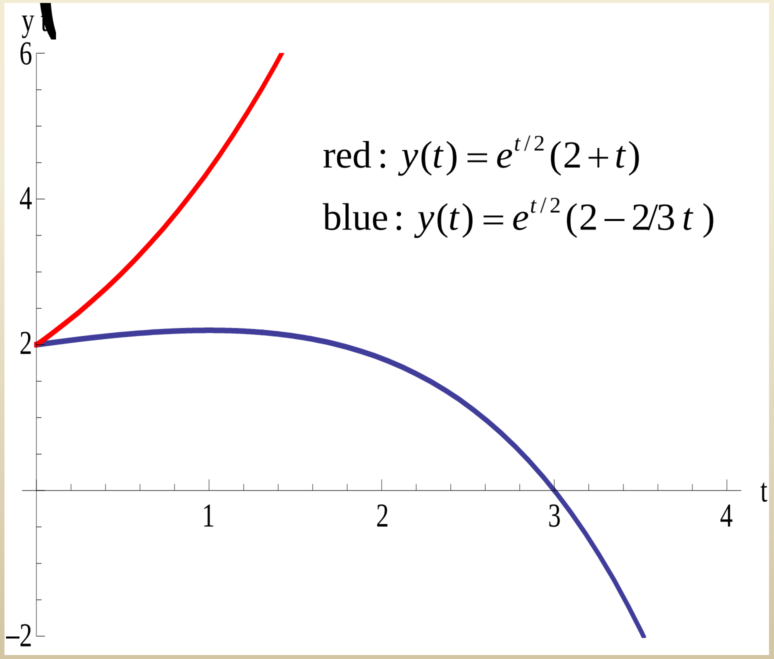
$$y(0) = 2, \;\; y'(0) = 2$$

- The solution of this modified problem is

$$y(t) = 2e^{t/2} + te^{t/2}$$

- Notice that the coefficient of the second term is now positive. This makes a big difference in the graph, since the exponential function is raised to a positive power:

$$\lambda = \frac{1}{2} > 0$$



Reduction of Order

- The method used so far in this section also works for equations with nonconstant coefficients:

$$y'' + p(t)y' + q(t)y = 0$$

- That is, given that y_1 is solution, try $y_2 = v(t)y_1$:

$$y_2(t) = v(t)y_1(t)$$
$$y_2'(t) = v'(t)y_1(t) + v(t)y_1'(t)$$
$$y_2''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$$

- Substituting these into ODE and collecting terms,

$$y_1 v'' + \left(2y_1' + py_1\right)v' + \left(y_1'' + py_1' + qy_1\right)v = 0$$

- Since y_1 is a solution to the differential equation, this last equation reduces to a first order equation in v :

$$y_1 v'' + \left(2y_1' + py_1\right)v' = 0$$

Example 3: Reduction of Order

- Given the variable coefficient equation and solution y_1,

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{-1},$$

use reduction of order method to find a second solution:

$$y_2(t) = v(t)\, t^{-1}$$

$$y_2'(t) = v'(t)\, t^{-1} - v(t)\, t^{-2}$$

$$y_2''(t) = v''(t)\, t^{-1} - 2v'(t)\, t^{-2} + 2v(t)\, t^{-3}$$

- Substituting these into the ODE and collecting terms,

$$2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} = 0$$

$$\Leftrightarrow 2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0$$

$$\Leftrightarrow \ 2tv'' - v' = 0$$

$$\Leftrightarrow \ 2tu' - u = 0, \ \text{where } u(t) = v'(t)$$

Example 3: Finding *v*(*t*) (2 of 3)

- To solve

$$2tu' - u = 0, \quad u(t) = v'(t)$$

for u, we can use the separation of variables method:

$$2t\frac{du}{dt} - u = 0 \quad \Leftrightarrow \quad \int \frac{du}{u} = \int \frac{1}{2t}\,dt \quad \Leftrightarrow \quad \ln|u| = 1/2\ \ln|\,t\,| + C$$

$$\Leftrightarrow \quad |u| = |\,t\,|^{1/2} e^{C} \quad \Leftrightarrow \quad u = ct^{1/2}, \quad \text{since } t > 0.$$

- Thus

$$v' = ct^{1/2}$$

and hence

$$v(t) = \frac{2}{3}c\,t^{3/2} + k$$

Example 3: General Solution (3 of 3)

- Since $$v(t) = \frac{2}{3}c t^{3/2} + k$$
$$y_2(t) = \left(\frac{2}{3}c t^{3/2} + k\right)t^{-1} = \frac{2}{3}c t^{1/2} + k\, t^{-1}$$
- Recall that $y_1(t) = t^{-1}$
- So we can neglect the second term of y_2 to obtain
$$y_2(t) = t^{1/2}$$
- The Wronskian of $y_1(t)$ and $y_2(t)$ can be computed
$$W[y_1, y_2](t) = \frac{3}{2}t^{-3/2} \neq 0, \quad t > 0$$
- Hence the general solution to the differential equation is
$$y(t) = c_1 t^{-1} + c_2 t^{1/2}$$

Euler's equation

- Sometimes a differential equation with variable coefficients equation

$$y'' + p(t)y' + q(t)y = 0$$

  can be transformed to a linear differential equation with constant coefficients by a change of variables.
- Consider the so-called Euler's equation

$$t^2 y'' + \alpha t\, p(t)y' + \beta\, q(t)y = 0$$

  where are some constants (real numbers)
- Let $t = e^x \Rightarrow x = \ln t$
- Since now $y = y(t(x)) = y(x)$, by the chain rule we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{t}$$

Euler's equation

- For the second derivative we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{1}{t} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}\frac{1}{t} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{t^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\frac{1}{t^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{t^2}$$

Then the original equation is transformed to

$$t^2\left\{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\right\}\frac{1}{t^2} + \alpha t\frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{t} + \beta y = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0$$

- We can solve it and find $y(x) = y(\ln t)$
- If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions in *x* then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set in *t*

Example 3 revisited

- Consider the differential equation from Example 3:

$$2t^2 y'' + 3t\,y' - y = 0 \quad \Rightarrow t^2 y'' + \frac{3}{2} t\,y' - \frac{1}{2} y = 0$$

It is the Euler's equation with $\alpha = 3/2,\ \beta = -1/2$

- Upon substitution $x = \ln t$ it is transformed to

$$\frac{d^2 y}{dx^2} + \frac{1}{2}\frac{dy}{dx} - \frac{1}{2} y = 0$$

which we know how to solve.

- The characteristic equation is $r^2 + r/2 - 1/2 = 0$ with the roots $r_1 = -1,\ r_2 = 1/2$. Therefore $y(x) = C_1 e^{-x} + C_2 e^{x/2}$
- Upon transforming back to t we finally get

$$y = C_1 e^{-\ln t} + C_2 e^{\frac{1}{2}\ln t} = C_1 e^{\ln t^{-1}} + C_2 e^{\ln t^{1/2}} = C_1 t^{-1} + C_2 t^{1/2}$$