Introduction to Differential Equations
Sample problems # 11

Date Given: June 20, 2022

P1. Find the Laplace transform of $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, \mathrm{d}\tau$.

Solution: The function $f(t)$ can be expressed explicitly as $f(t) = \frac{1}{2}(t - \sin(t)\cos(t))$. However, we do not need this expression if we recognize that $f(t)$ is in the form of a convolution integral.

Note that $\mathcal{L}[t^2] = 2/s^3$ and $\mathcal{L}[\cos(2t)] = s/(s^2+4)$. Therefore, based on Theorem 6.6.1,

$$\mathcal{L}\left[f(t)\right] = \left(\frac{2}{s^3}\right)\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2(s^2+4)}.$$

P2. By using the convolution theorem, find (express in terms of a convolution integral) the inverse Laplace transform of $F(s) = \dfrac{1}{s^4(s^2+1)}$.

Solution: $\mathcal{L}^{-1}[1/s^4] = t^3/6$ and $\mathcal{L}^{-1}[1/(s^2+1)] = \sin t$. Therefore, based on Theorem 6.6.1,

$$\mathcal{L}^{-1}\left[F(s)\right] = f(t) = \int_0^t \frac{1}{6}(t-\tau)^3 \sin\tau \, \mathrm{d}\tau.$$

Note that we can also write it as

$$\mathcal{L}^{-1}\left[F(s)\right] = f(t) = \int_0^t \frac{1}{6}\tau^3 \sin(t-\tau) \, \mathrm{d}\tau.$$

Note also that $f(t)$ can be computed explicitly[1].

P3. By using the convolution theorem, find (express in terms of a convolution integral) the inverse Laplace transform of $F(s) = \dfrac{s}{(s+1)(s^2+4)}$.

Solution: $\mathcal{L}^{-1}[1/(s+1)] = e^{-t}$ and $\mathcal{L}^{-1}[s/(s^2+4)] = \cos 2t$. Therefore, based on Theorem 6.6.1,

$$\mathcal{L}^{-1}\left[F(s)\right] = f(t) = \frac{1}{2}\int_0^t e^{-(t-\tau)} \cos 2\tau \, \mathrm{d}\tau.$$

Note that we can also write it as

$$\mathcal{L}^{-1}\left[F(s)\right] = f(t) = \int_0^t e^{-\tau} \cos(2(t-\tau)) \, \mathrm{d}\tau.$$

Note also that $f(t)$ can be computed explicitly[2].

P4. Express in terms of a convolution integral the solution of the following initial value problem: $y'' + \omega^2 y = g(t);\;\; y(0) = 0, y'(0) = 1$.

[1] The explicit form is $f(t) = \dfrac{1}{6}t\left(t^2-6\right) + \sin(t)$

[2] The explicit form is $f(t) = \frac{1}{5}\left(-e^{-t} + 2\sin(2t) + \cos(2t)\right)$

Solution: Taking the initial conditions into consideration, the transform of the differential equation is

$$s^2Y(s) - 1 + \omega^2 Y(s) = G(s)$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2+\omega^2} + \frac{G(s)}{s^2+\omega^2}$$

Let $g(t) = \mathcal{L}^{-1}[G(s)]$. Since $\mathcal{L}^{-1}[1/(s^2+\omega^2)] = \sin\omega t$, based on Theorem 6.6.1 we have,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+\omega^2}\right] = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))g(\tau)\,\mathrm{d}\tau.$$

Hence

$$y(t) = \frac{1}{\omega}\sin(\omega t) + \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))g(\tau)\,\mathrm{d}\tau.$$

P5. Express in terms of a convolution integral the solution of the following initial value problem: $y'' + 3y' + 2y = \cos\alpha t$; $y(0) = 1, y'(0) = 0$.

Solution: Applying the Laplace transform to the equation, we have

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = \frac{s}{s^2+\alpha^2}.$$

Applying the initial conditions, we get

$$(s^2+3s+2)Y(s) = s + 3 + \frac{s}{s^2+\alpha^2}.$$

Therefore

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{s}{(s^2+3s+2)(s^2+\alpha^2)}.$$

Using partial fractions, we write

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}, \quad \text{and} \quad \frac{1}{(s^2+3s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Therefore, we can conclude that

$$y(t) = 2e^{-t} - e^{-2t} + \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)\cos(\alpha\tau)\,\mathrm{d}\tau.$$

Note that this integral can be computed explicitly[3].

P6. Transform the differential equation $t^2u'' + tu' + u = 0$ with initial conditions $u(0) = 2, u'(0) = 3$ into a system of first order equations corresponding to this initial value problem.

Solution: First divide both sides of the equation by t^2, and write

$$u'' + \frac{1}{t}u' + \frac{1}{t^2}u = 0$$

[3] The solution is $y(t) = 2e^{-t} - e^{-2t} + \dfrac{2e^{-2t} - \alpha\sin(\alpha t) - 2\cos(\alpha t)}{\alpha^2+4} - \dfrac{e^{-t} - \alpha\sin(\alpha t) - \cos(\alpha t)}{\alpha^2+1}$

Set $x_1 = u$, and $x_2 = u'$. Then

$$x_1' = x_2,\ x_2' = -\frac{1}{t}x_2 - \frac{1}{t^2}x_1, \qquad x_1(0) = 2,\ x_2(0) = 3.$$

In the matrix notation we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{1}{t} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

P7. (a) Transform the system $x_1' = 3x_1 - 2x_2$, $x_2' = 2x_1 - 2x_2$ into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the initial conditions $x_1(0) = 3$, $x_2(0) = 1/2$.

(c) Sketch the graph of the solution in the x_1x_2-plane.

Solution:

(a) Solving the first equation for x_2 gives $x_2 = (3x_1 - x_1')/2$. Substituting this into second differential equation we obtain $(3x_1' - x_1'')/2 = 2x_1 - 2(3x_1 - x_1')/2$, i.e. $x_1'' = x_1' + 2x_1$, that is

$$x_1'' - x_1' - 2x_1 = 0.$$

(b) The general solution of the 2nd order differential equation in part (a) is $x_1 = c_1e^{2t} + c_2e^{-t}$. Differentiation this and substituting into $x_2 = (3x_1 - x_1')/2$ yields $x_2 = c_1e^{2t}/2 + 2c_2e^{-t}$. The initial conditions then give $c_1 + c_2 = 3$ and $c_1/2 + 2c_2 = 1/2$. This implies that $c_1 = 11/3$ and $c_2 = -2/3$. Thus

$$x_1(t) = (11e^{2t} - 2e^{-t})/3, \quad \text{and} \quad x_2(t) = (11e^{2t} - 8e^{-t})/6.$$

(c) The graph of the solution in the x_1x_2-plane is shown in red in Figure 1. Also shown there are the direction filed and, in green, some other curves corresponding to different initial values $x_1(0)$, $x_2(0)$.

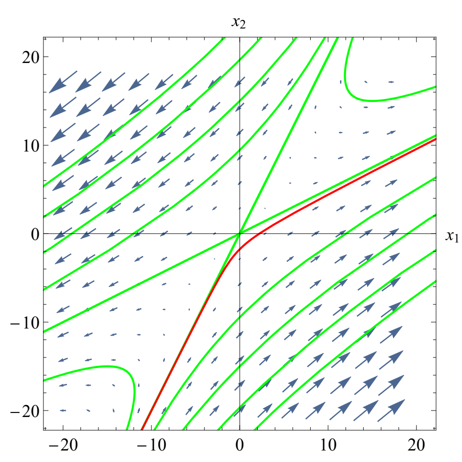

Figure 1: Illustration to problem P7.

P8. (a) Transform the system $x_1' = 2x_2$, $x_2' = -2x_1$ into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the initial conditions $x_1(0) = 3$, $x_2(0) = 4$.

(c) Sketch the graph of the solution in the x_1x_2-plane.

Solution:

(a) Solving the first equation for x_2 gives $x_2 = x_1'/2$. Substituting this into second differential equation we obtain $x_1''/2 = -2x_1$, that is

$$x_1'' + 4x_1 = 0.$$

(b) The general solution of the 2nd order differential equation in part (a) is $x_1 = c_1 \cos 2t + c_2 \sin 2t$. With x_2 given in terms of x_1 , it follows that $x_2(t) = -c_1 \sin 2t + c_2 \cos 2t$. Imposing the specified initial conditions, we obtain $c_1 = 3$ and $c_2 = 4$. Hence

$$x_1 = 3\cos 2t + 4\sin 2t, \quad \text{and} \quad x_2 = -3\sin 2t + 4\cos 2t.$$

(c) The graph of the solution[4] in the x_1x_2-plane is shown in red in Figure 2. Also shown there are the direction filed and, in green, some other curves corresponding to different initial values $x_1(0)$, $x_2(0)$.

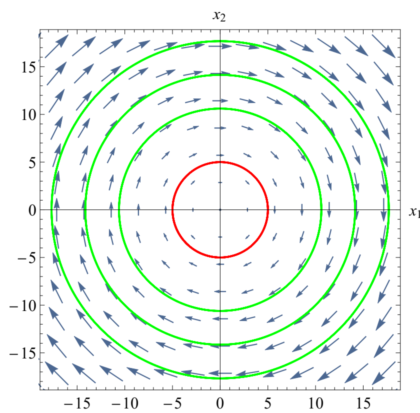

Figure 2: Illustration to problem P8.

[4]The circle $x_1^2 + x_2^2 = 5^2$.