ISSE Course, Ritsumeikan University — Spring Semester, 2022

Introduction to Differential Equations
Sample problems # 12

Date Given: June 27, 2022

P1. (a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} \boldsymbol{x}$$

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 5/4 - r & 3/4 \\ 3/4 & 5/4 - r \end{vmatrix} = r^2 + (5/2)r + 1 = 0 \quad \Longrightarrow \quad r_1 = 2, r_2 = 1/2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 3/4 & 3/4 \\ 3/4 & 3/4 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t/2}.$$

(b) If the initial condition is a multiple of $(1,1)^{\mathrm{T}}$, then the solution will go out of the origin along the eigenvector $\boldsymbol{\xi}_1$. Likewise, if the initial condition is a multiple of $(1,-1)^{\mathrm{T}}$, then the solution will go out of the origin along the eigenvector $\boldsymbol{\xi}_2$. The direction field and a few trajectories of the system are shown in Figure 1.

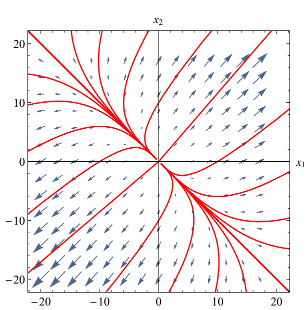

Figure 1: Illustration to problem P1.

P2. (a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} \boldsymbol{x}$$

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(b) Draw a direction field and plot a few trajectories of the system.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 3-r & 6 \\ -1 & -2-r \end{vmatrix} = r^2 + r = 0 \quad \Longrightarrow \quad r_1 = 1, r_2 = 0.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(b) The entire line along the eigenvector $\boldsymbol{\xi}_2 = (2, -1)^{\mathrm{T}}$ consists of equilibrium points. All other solutions diverge. The direction field changes across the line $x_1 + 2x_2 = 0$ (blue line). Eliminating the exponential terms in the solution, the trajectories are given by $x_1 + 3x_2 = -c_2$ (red lines). The direction field and a few trajectories of the system are shown in Figure 2.

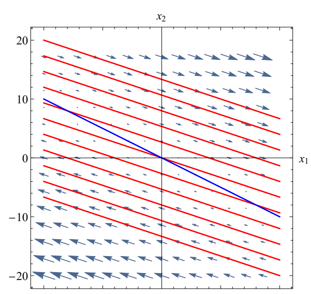

Figure 2: Illustration to problem P2.

P3. Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \boldsymbol{x}.$$

Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1-r & 0 & 1 \\ 0 & 1-r & 0 \\ 1 & 0 & 1-r \end{vmatrix} = (1-r)(r^2 - 2r) = 0 \quad \Longrightarrow \quad r_1 = 0, r_2 = 1, r_1 = 2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $r = r_3$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}.$$

P4. Solve the initial value problem

$$\boldsymbol{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

and describe the behavior of the solution as $t \to \infty$.

Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 5 - r & -1 \\ 3 & 1 - r \end{vmatrix} = r^2 - 6r + 8 = 0 \implies r_1 = 4, r_2 = 2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \implies \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations $c_1 + c_2 = 2$ and $c_1 + 3c_2 = -1$. Hence, $c_1 = 7/2$ and $c_2 = -3/2$

$$\boldsymbol{x}(t) = \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}.$$

The solution diverges from the origin (equilibrium) as $t \to \infty$.

P5. Solve the initial value problem

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1-r & 1 & 4 \\ 0 & 2-r & 0 \\ 1 & 1 & 1-r \end{vmatrix} = (2-r)(r^2 - 2r - 3) = 0 \quad \Longrightarrow \quad r_1 = 2, r_2 = 3, r_1 = -1.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -2 & 1 & 4 \\ 0 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

For $r = r_3$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-t}.$$

If

$$\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

then $c_1 = -1, c_2 = 5/2, c_3 = -1/2$ and therefore

$$\boldsymbol{x}(t) = -\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} e^{2t} + \frac{5}{2} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{3t} - \frac{1}{2} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-t}.$$