37. For each of the following initial value problems, use the results of Problem 29 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{\phi(t)\}$, where $y = \phi(t)$ is the solution of the given initial value problem.
    (a) $y'' - ty = 0; \qquad y(0) = 1, \quad y'(0) = 0$ (Airy's equation)
    (b) $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0; \qquad y(0) = 0, \quad y'(0) = 1$ (Legendre's equation)

    Note that the differential equation for $Y(s)$ is of first order in part (a), but of second order in part (b). This is due to the fact that t appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.
38. Suppose that
$$g(t) = \int_0^t f(\tau)\, d\tau.$$
    If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that
$$G(s) = F(s)/s.$$
39. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that
$$F(s) = P(s)/Q(s),$$
    where $Q(s)$ is a polynomial of degree n with distinct zeros $r_1, \dots, r_n$, and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form
$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{i}$$
    where the coefficients $A_1, \dots, A_n$ must be determined.
    (a) Show that
$$A_k = P(r_k)/Q'(r_k), \qquad k = 1, \dots, n. \tag{ii}$$
    Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.
    (b) Show that
$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii}$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones, we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing below will be assumed to be piecewise continuous and of exponential order, so that their Laplace transforms exist, at least for s sufficiently large.

To deal effectively with functions having jump discontinuities, it is very helpful to introduce a function known as the **unit step function** or **Heaviside function**. This function will be denoted by u_c and is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases} \tag{1}$$

Since the Laplace transform involves values of t in the interval $[0, \infty)$, we are also interested only in nonnegative values of c. The graph of $y = u_c(t)$ is shown in Figure 6.3.1. We have somewhat arbitrarily assigned the value one to u_c at $t = c$. However, for a piecewise continuous function such as u_c, the value at a discontinuity point is usually irrelevant. The step can also be negative. For instance, Figure 6.3.2 shows the graph of $y = 1 - u_c(t)$.

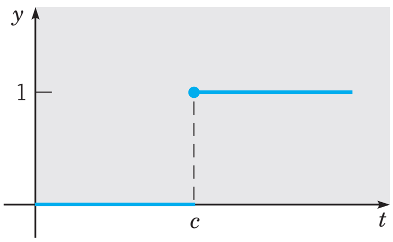

FIGURE 6.3.1 Graph of $y = u_c(t)$.

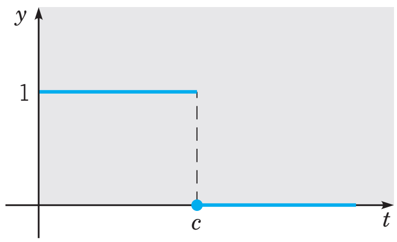

FIGURE 6.3.2 Graph of $y=1-u_c(t)$.

EXAMPLE 1

Sketch the graph of $y = h(t)$, where

$$h(t) = u_\pi(t) - u_{2\pi}(t), \qquad t \geq 0.$$

From the definition of $u_c(t)$ in Eq. (1), we have

$$h(t) = \begin{cases} 0 - 0 = 0, & 0 \leq t < \pi, \\ 1 - 0 = 1, & \pi \leq t < 2\pi, \\ 1 - 1 = 0, & 2\pi \leq t < \infty. \end{cases}$$

Thus the equation $y = h(t)$ has the graph shown in Figure 6.3.3. This function can be thought of as a rectangular pulse.

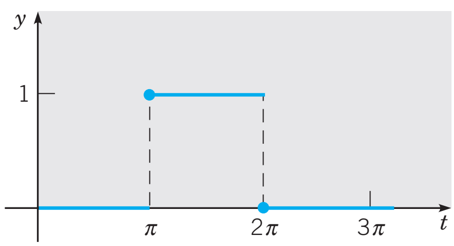

FIGURE 6.3.3 Graph of $y = u_\pi(t) - u_{2\pi}(t)$.

EXAMPLE 2

Consider the function

$$f(t) = \begin{cases} 2, & 0 \le t < 4, \\ 5, & 4 \le t < 7, \\ -1, & 7 \le t < 9, \\ 1, & t \ge 9, \end{cases} \tag{2}$$

whose graph is shown in Figure 6.3.4. Express $f(t)$ in terms of $u_c(t)$.

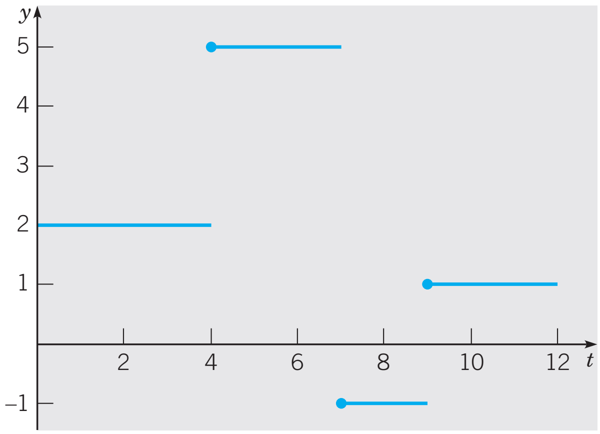

FIGURE 6.3.4 Graph of the function in Eq. (2).

We start with the function $f_1(t) = 2$, which agrees with $f(t)$ on $[0, 4)$. To produce the jump of three units at $t = 4$, we add $3u_4(t)$ to $f_1(t)$, obtaining

$$f_2(t) = 2 + 3u_4(t),$$

which agrees with $f(t)$ on $[0, 7)$. The negative jump of six units at $t = 7$ corresponds to adding $-6u_7(t)$, which gives

$$f_3(t) = 2 + 3u_4(t) - 6u_7(t).$$

Finally, we must add $2u_9(t)$ to match the jump of two units at $t = 9$. Thus we obtain

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t). \tag{3}$$

The Laplace transform of u_c for $c \ge 0$ is easily determined:

$$\begin{aligned} \mathcal{L}\{u_c(t)\} &= \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt \\ &= \frac{e^{-cs}}{s}, \qquad s > 0. \end{aligned} \tag{4}$$

For a given function f defined for $t \ge 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t - c), & t \ge c, \end{cases}$$

which represents a translation of f a distance c in the positive t direction; see Figure 6.3.5. In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t)f(t-c).$$

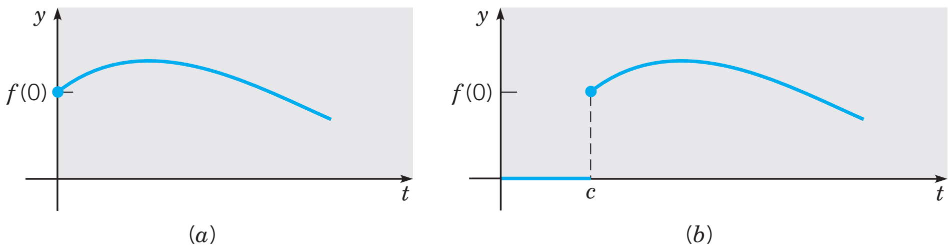

FIGURE 6.3.5 A translation of the given function. (a) $y = f(t)$; (b) $y = u_c(t)f(t-c)$.

The unit step function is particularly important in transform use because of the following relation between the transform of $f(t)$ and that of its translation $u_c(t)f(t-c)$.

Theorem 6.3.1 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > a. \tag{5}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{6}$$

Theorem 6.3.1 simply states that the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by e^{-cs}. To prove Theorem 6.3.1, it is sufficient to compute the transform of $u_c(t)f(t-c)$:

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt \\ &= \int_c^\infty e^{-st}f(t-c)\,dt.\end{aligned}$$

Introducing a new integration variable $\xi = t - c$, we have

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-(\xi+c)s}f(\xi)\,d\xi = e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)\,d\xi \\ &= e^{-cs}F(s).\end{aligned}$$

Thus Eq. (5) is established; Eq. (6) follows by taking the inverse transform of both sides of Eq. (5).

A simple example of this theorem occurs if we take $f(t) = 1$. Recalling that $\mathcal{L}\{1\} = 1/s$, we immediately have from Eq. (5) that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$. This result agrees with that of Eq. (4). Examples 3 and 4 illustrate further how Theorem 6.3.1 can be used in the calculation of transforms and inverse transforms.

EXAMPLE 3

If the function f is defined by

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4, \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4, \end{cases}$$

find $\mathcal{L}\{f(t)\}$. The graph of $y = f(t)$ is shown in Figure 6.3.6.

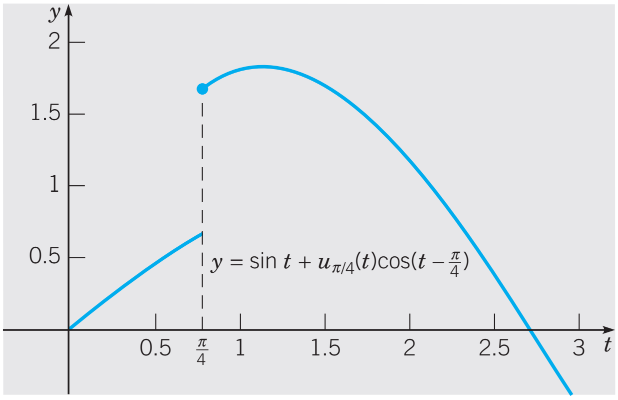

FIGURE 6.3.6 Graph of the function in Example 3.

Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & t < \pi/4, \\ \cos(t - \pi/4), & t \ge \pi/4. \end{cases}$$

Thus

$$g(t) = u_{\pi/4}(t) \cos(t - \pi/4)$$

and

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t) \cos(t - \pi/4)\} \\ &= \mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}. \end{aligned}$$

Introducing the transforms of $\sin t$ and $\cos t$, we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} = \frac{1 + se^{-\pi s/4}}{s^2+1}.$$

You should compare this method with the calculation of $\mathcal{L}\{f(t)\}$ directly from the definition.

EXAMPLE 4

Find the inverse transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

From the linearity of the inverse transform, we have

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} \\ &= t - u_2(t)(t-2). \end{aligned}$$

The function f may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & t \ge 2. \end{cases}$$

The following theorem contains another very useful property of Laplace transforms that is somewhat analogous to that given in Theorem 6.3.1.

Theorem 6.3.2 If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \ge 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad s > a + c. \tag{7}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{8}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. To prove this theorem, we evaluate $\mathcal{L}\{e^{ct}f(t)\}$. Thus

$$\begin{aligned}\mathcal{L}\{e^{ct}f(t)\} &= \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt \\ &= F(s-c),\end{aligned}$$

which is Eq. (7). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \le Ke^{at}$; hence $|e^{ct}f(t)| \le Ke^{(a+c)t}$. Equation (8) is obtained by taking the inverse transform of Eq. (7), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 5.

EXAMPLE 5

Find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

By completing the square in the denominator, we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2+1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t}\sin t.$$

The results of this section are often useful in solving differential equations, particularly those that have discontinuous forcing functions. The next section is devoted to examples illustrating this fact.

PROBLEMS

In each of Problems 1 through 6, sketch the graph of the given function on the interval $t \geq 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t)$, where $f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12:

(a) Sketch the graph of the given function.

(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

7. $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$
8. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$
9. $f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$
10. $f(t) = \begin{cases} t^2, & 0 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$
11. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 2, \\ t-2, & 2 \leq t < 3, \\ 0, & t \geq 3. \end{cases}$
12. $f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 2, & 2 \leq t < 5, \\ 7-t, & 5 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$

In each of Problems 13 through 18, find the Laplace transform of the given function.

13. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$
14. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$
15. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$
16. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
17. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$
18. $f(t) = t - u_1(t)(t-1), \qquad t \geq 0$

In each of Problems 19 through 24, find the inverse Laplace transform of the given function.

19. $F(s) = \dfrac{3!}{(s-2)^4}$
20. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$
21. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$
22. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$
23. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$
24. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$
25. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.

    (a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right), \qquad s > ca.$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right).$$

(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\left(\frac{t}{a}\right).$$

In each of Problems 26 through 29, use the results of Problem 25 to find the inverse Laplace transform of the given function.

26. $F(s) = \dfrac{2^{n+1}n!}{s^{n+1}}$

27. $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

28. $F(s) = \dfrac{1}{9s^2-12s+3}$

29. $F(s) = \dfrac{e^2e^{-4s}}{2s-1}$

In each of Problems 30 through 33, find the Laplace transform of the given function. In Problem 33, assume that term-by-term integration of the infinite series is permissible.

30. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

31. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

32. $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \displaystyle\sum_{k=1}^{2n+1}(-1)^k u_k(t)$

33. $f(t) = 1 + \displaystyle\sum_{k=1}^{\infty}(-1)^k u_k(t)$. See Figure 6.3.7.

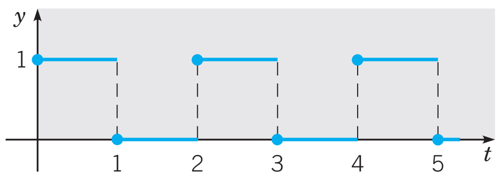

FIGURE 6.3.7 The function $f(t)$ in Problem 33; a square wave.

34. Let f satisfy $f(t+T) = f(t)$ for all $t \ge 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \le t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}.$$

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

35. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t)$.

Compare with Problem 33.

36. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t)$.

See Figure 6.3.8.

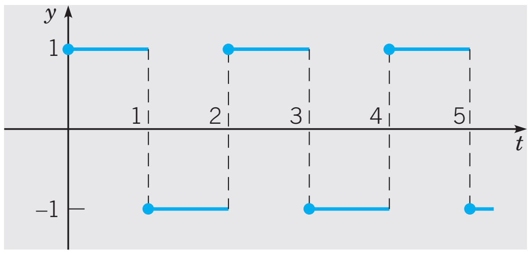

FIGURE 6.3.8 The function $f(t)$ in Problem 36; a square wave.

37. $f(t) = t, \quad 0 \le t < 1;$
$f(t+1) = f(t).$
See Figure 6.3.9.

38. $f(t) = \sin t, \quad 0 \le t < \pi;$
$f(t+\pi) = f(t).$
See Figure 6.3.10.

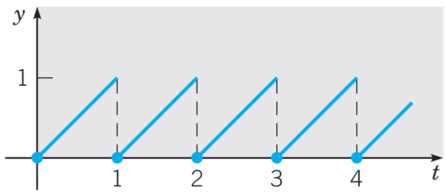

FIGURE 6.3.9 The function $f(t)$ in Problem 37; a sawtooth wave.

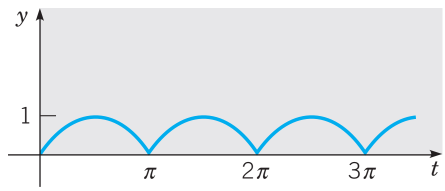

FIGURE 6.3.10 The function $f(t)$ in Problem 38; a rectified sine wave.

39. (a) If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$; compare with Problem 30. Sketch the graph of $y = f(t)$.

(b) Let $g(t) = \int_0^t f(\xi)\, d\xi$, where the function f is defined in part (a). Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$.

(c) Let $h(t) = g(t) - u_1(t)g(t-1)$, where g is defined in part (b). Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$.

40. Consider the function p defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

(a) Sketch the graph of $y = p(t)$.

(b) Find $\mathcal{L}\{p(t)\}$ by noting that p is the periodic extension of the function h in Problem 39(c) and then using the result of Problem 34.

(c) Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\, dt,$$

where f is the function in Problem 36, and then using Theorem 6.2.1.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we turn our attention to some examples in which the nonhomogeneous term, or forcing function, is discontinuous.

EXAMPLE 1

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \tag{1}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \text{and} \quad t \ge 20. \end{cases} \tag{2}$$

Assume that the initial conditions are

$$y(0) = 0, \qquad y'(0) = 0. \tag{3}$$

This problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for $5 \le t < 20$. Alternatively, y may represent the response of a damped oscillator subject to the applied force $g(t)$.

The Laplace transform of Eq. (1) is

$$\begin{aligned} 2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) &= \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} \\ &= (e^{-5s} - e^{-20s})/s. \end{aligned}$$

Introducing the initial values (3) and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{4}$$

To find $y = \phi(t)$, it is convenient to write $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \tag{5}$$

where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}. \tag{6}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{7}$$

Observe that we have used Theorem 6.3.1 to write the inverse transforms of $e^{-5s}H(s)$ and $e^{-20s}H(s)$, respectively. Finally, to determine $h(t)$, we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}. \tag{8}$$

Upon determining the coefficients, we find that $a = \frac{1}{2}$, $b = -1$, and $c = -\frac{1}{2}$. Thus

$$\begin{aligned} H(s) = \frac{1/2}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} &= \frac{1/2}{s} - \left(\frac{1}{2}\right)\frac{\left(s + \frac{1}{4}\right) + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}} \\ &= \frac{1/2}{s} - \left(\frac{1}{2}\right)\left[\frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} + \frac{1}{\sqrt{15}}\frac{\sqrt{15}/4}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right]. \end{aligned} \tag{9}$$

Then, by referring to lines 9 and 10 of Table 6.2.1, we obtain

$$h(t) = \tfrac{1}{2} - \tfrac{1}{2}\left[e^{-t/4}\cos(\sqrt{15}\,t/4) + (\sqrt{15}/15)e^{-t/4}\sin(\sqrt{15}\,t/4)\right]. \tag{10}$$

In Figure 6.4.1 the graph of $y = \phi(t)$ from Eqs. (7) and (10) shows that the solution consists of three distinct parts. For $0 < t < 5$, the differential equation is

$$2y'' + y' + 2y = 0, \tag{11}$$

and the initial conditions are given by Eq. (3). Since the initial conditions impart no energy to the system, and since there is no external forcing, the system remains at rest; that is, $y = 0$ for $0 < t < 5$. This can be confirmed by solving Eq. (11) subject to the initial conditions (3). In particular, evaluating the solution and its derivative at $t = 5$, or, more precisely, as t approaches 5 from below, we have

$$y(5) = 0, \qquad y'(5) = 0. \tag{12}$$

Once $t > 5$, the differential equation becomes

$$2y'' + y' + 2y = 1, \tag{13}$$

whose solution is the sum of a constant (the response to the constant forcing function) and a damped oscillation (the solution of the corresponding homogeneous equation). The plot in Figure 6.4.1 shows this behavior clearly for the interval $5 \le t \le 20$. An expression for this portion of the solution can be found by solving the differential equation (13) subject to the initial conditions (12). Finally, for $t > 20$ the differential equation becomes Eq. (11) again, and the initial conditions are obtained by evaluating the solution of Eqs. (13), (12) and its derivative at $t = 20$. These values are

$$y(20) \cong 0.50162, \qquad y'(20) \cong 0.01125. \tag{14}$$

The initial value problem (11), (14) contains no external forcing, so its solution is a damped oscillation about $y = 0$, as can be seen in Figure 6.4.1.

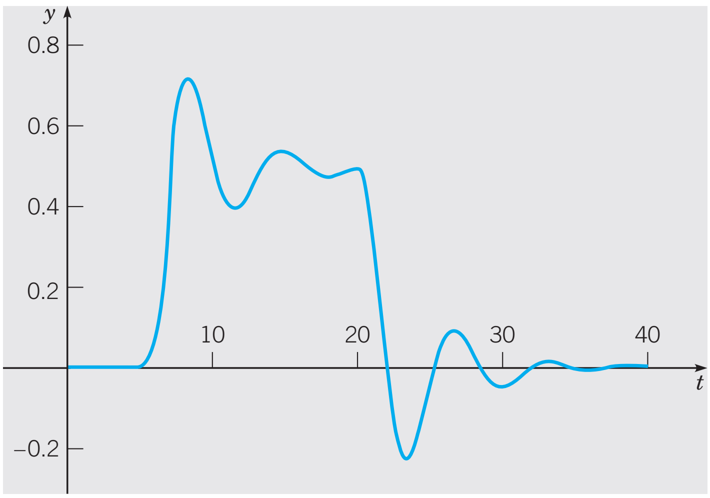

FIGURE 6.4.1 Solution of the initial value problem (1), (2), (3): $2y'' + y' + 2y = u_5(t) - u_{20}(t)$, $y(0) = 0$, $y'(0) = 0$.

Although it may be helpful to visualize the solution shown in Figure 6.4.1 as composed of solutions of three separate initial value problems in three separate intervals, it is somewhat tedious to find the solution by solving these separate problems. Laplace transform methods provide a much more convenient and elegant approach to this problem and to others that have discontinuous forcing functions.

The effect of the discontinuity in the forcing function can be seen if we examine the solution $\phi(t)$ of Example 1 more closely. According to the existence and uniqueness Theorem 3.2.1, the solution ϕ and its first two derivatives are continuous except possibly at the points $t = 5$ and $t = 20$, where g is discontinuous. This can also be seen at once from Eq. (7). One can also show by direct computation from Eq. (7) that ϕ and ϕ' are continuous even at $t = 5$ and $t = 20$. However, if we calculate ϕ'', we find that

$$\lim_{t\to 5-} \phi''(t) = 0, \qquad \lim_{t\to 5+} \phi''(t) = 1/2.$$

Consequently, $\phi''(t)$ has a jump of $1/2$ at $t = 5$. In a similar way, we can show that $\phi''(t)$ has a jump of $-1/2$ at $t = 20$. Thus the jump in the forcing term $g(t)$ at these points is balanced by a corresponding jump in the highest order term $2y''$ on the left side of the equation.

Consider now the general second order linear equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{15}$$

where p and q are continuous on some interval $\alpha < t < \beta$, but g is only piecewise continuous there. If $y = \psi(t)$ is a solution of Eq. (15), then ψ and ψ' are continuous on $\alpha < t < \beta$, but ψ'' has jump discontinuities at the same points as g. Similar remarks apply to higher order equations; the highest derivative of the solution appearing in the differential equation has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous even at those points.

EXAMPLE 2

Describe the qualitative nature of the solution of the initial value problem

$$y'' + 4y = g(t), \tag{16}$$

$$y(0) = 0, \qquad y'(0) = 0, \tag{17}$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le t < 10, \\ 1, & t \ge 10, \end{cases} \tag{18}$$

and then find the solution.

In this example the forcing function has the graph shown in Figure 6.4.2 and is known as ramp loading. It is relatively easy to identify the general form of the solution. For $t < 5$ the solution is simply $y = 0$. On the other hand, for $t > 10$ the solution has the form

$$y = c_1 \cos 2t + c_2 \sin 2t + 1/4. \tag{19}$$

The constant $1/4$ is a particular solution of the nonhomogeneous equation, while the other two terms are the general solution of the corresponding homogeneous equation. Thus the solution (19) is a simple harmonic oscillation about $y = 1/4$. Similarly, in the intermediate range $5 < t < 10$, the solution is an oscillation about a certain linear function. In an engineering context, for example, we might be interested in knowing the amplitude of the eventual steady oscillation.

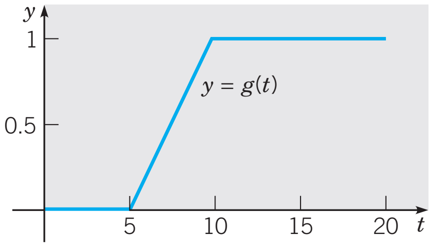

FIGURE 6.4.2 Ramp loading; $y = g(t)$ from Eq. (18) or Eq. (20).

To solve the problem, it is convenient to write

$$g(t) = \left[u_5(t)(t-5) - u_{10}(t)(t-10)\right]/5, \tag{20}$$

as you may verify. Then we take the Laplace transform of the differential equation and use the initial conditions, thereby obtaining

$$(s^2 + 4)Y(s) = (e^{-5s} - e^{-10s})/5s^2$$

or

$$Y(s) = (e^{-5s} - e^{-10s})H(s)/5, \tag{21}$$

where

$$H(s) = \frac{1}{s^2(s^2+4)}. \tag{22}$$

Thus the solution of the initial value problem (16), (17), (18) is

$$y = \phi(t) = \left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]/5, \tag{23}$$

where $h(t)$ is the inverse transform of $H(s)$. The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}, \tag{24}$$

and it then follows from lines 3 and 5 of Table 6.2.1 that

$$h(t) = \tfrac{1}{4}t - \tfrac{1}{8}\sin 2t. \tag{25}$$

The graph of $y = \phi(t)$ is shown in Figure 6.4.3. Observe that it has the qualitative form that we indicated earlier. To find the amplitude of the eventual steady oscillation, it is sufficient to locate one of the maximum or minimum points for $t > 10$. Setting the derivative of the solution (23) equal to zero, we find that the first maximum is located approximately at $(10.642, 0.2979)$, so the amplitude of the oscillation is approximately 0.0479.

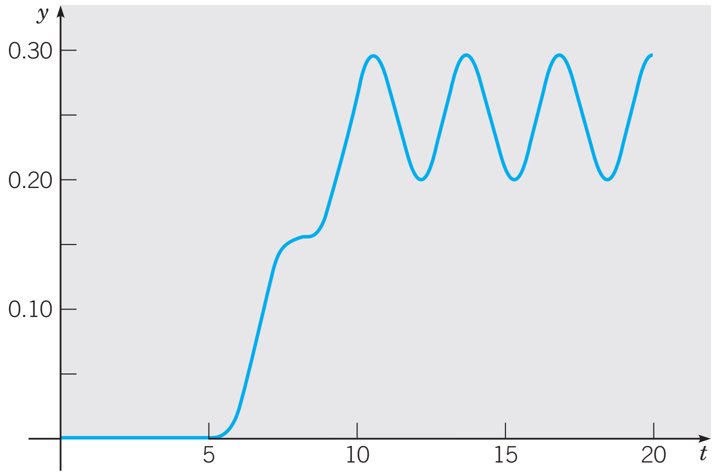

FIGURE 6.4.3 Solution of the initial value problem (16), (17), (18).

Note that in this example, the forcing function g is continuous but g' is discontinuous at $t = 5$ and $t = 10$. It follows that the solution ϕ and its first two derivatives are continuous everywhere, but ϕ''' has discontinuities at $t = 5$ and at $t = 10$ that match the discontinuities in g' at those points.

PROBLEMS

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

1. $y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}$

2. $y'' + 2y' + 2y = h(t); \quad y(0) = 0, \quad y'(0) = 1; \quad h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \quad \text{and} \quad t \ge 2\pi \end{cases}$

3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0$

4. $y'' + 4y = \sin t + u_{\pi}(t)\sin(t - \pi); \quad y(0) = 0, \quad y'(0) = 0$

5. $y'' + 3y' + 2y = f(t); \quad y(0) = 0, \quad y'(0) = 0; \quad f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$

6. $y'' + 3y' + 2y = u_2(t); \quad y(0) = 0, \quad y'(0) = 1$

7. $y'' + y = u_{3\pi}(t); \quad y(0) = 1, \quad y'(0) = 0$

8. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(t - \pi/2); \quad y(0) = 0, \quad y'(0) = 0$

9. $y'' + y = g(t); \quad y(0) = 0, \quad y'(0) = 1; \quad g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$

10. $y'' + y' + \frac{5}{4}y = g(t); \quad y(0) = 0, \quad y'(0) = 0; \quad g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

11. $y'' + 4y = u_{\pi}(t) - u_{3\pi}(t); \quad y(0) = 0, \quad y'(0) = 0$

12. $y^{(4)} - y = u_1(t) - u_2(t);\qquad y(0) = 0,\quad y'(0) = 0,\quad y''(0) = 0,\quad y'''(0) = 0$
13. $y^{(4)} + 5y'' + 4y = 1 - u_\pi(t);\qquad y(0) = 0,\quad y'(0) = 0,\quad y''(0) = 0,\quad y'''(0) = 0$
14. Find an expression involving $u_c(t)$ for a function f that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$.
15. Find an expression involving $u_c(t)$ for a function g that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$ and then ramps back down to zero at $t = t_0 + 2k$.
16. A certain spring–mass system satisfies the initial value problem

$$u'' + \tfrac{1}{4}u' + u = kg(t),\qquad u(0) = 0,\quad u'(0) = 0,$$

where $g(t) = u_{3/2}(t) - u_{5/2}(t)$ and $k > 0$ is a parameter.

(a) Sketch the graph of $g(t)$. Observe that it is a pulse of unit magnitude extending over one time unit.

(b) Solve the initial value problem.

(c) Plot the solution for $k = 1/2$, $k = 1$, and $k = 2$. Describe the principal features of the solution and how they depend on k.

(d) Find, to two decimal places, the smallest value of k for which the solution $u(t)$ reaches the value 2.

(e) Suppose $k = 2$. Find the time τ after which $|u(t)| < 0.1$ for all $t > \tau$.

17. Modify the problem in Example 2 of this section by replacing the given forcing function $g(t)$ by

$$f(t) = \left[u_5(t)(t-5) - u_{5+k}(t)(t-5-k)\right]/k.$$

(a) Sketch the graph of $f(t)$ and describe how it depends on k. For what value of k is $f(t)$ identical to $g(t)$ in the example?

(b) Solve the initial value problem

$$y'' + 4y = f(t),\qquad y(0) = 0,\quad y'(0) = 0.$$

(c) The solution in part (b) depends on k, but for sufficiently large t the solution is always a simple harmonic oscillation about $y = 1/4$. Try to decide how the amplitude of this eventual oscillation depends on k. Then confirm your conclusion by plotting the solution for a few different values of k.

18. Consider the initial value problem

$$y'' + \tfrac{1}{3}y' + 4y = f_k(t),\qquad y(0) = 0,\quad y'(0) = 0,$$

where

$$f_k(t) = \begin{cases} 1/2k, & 4-k \le t < 4+k \\ 0, & 0 \le t < 4-k \quad \text{and} \quad t \ge 4+k \end{cases}$$

and $0 < k < 4$.

(a) Sketch the graph of $f_k(t)$. Observe that the area under the graph is independent of k. If $f_k(t)$ represents a force, this means that the product of the magnitude of the force and the time interval during which it acts does not depend on k.

(b) Write $f_k(t)$ in terms of the unit step function and then solve the given initial value problem.

(c) Plot the solution for $k = 2$, $k = 1$, and $k = \frac{1}{2}$. Describe how the solution depends on k.

Resonance and Beats. In Section 3.8 we observed that an undamped harmonic oscillator (such as a spring–mass system) with a sinusoidal forcing term experiences resonance if the frequency of the forcing term is the same as the natural frequency. If the forcing frequency is slightly different from the natural frequency, then the system exhibits a beat. In Problems 19 through 23 we explore the effect of some nonsinusoidal periodic forcing functions.

19. Consider the initial value problem

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $f(t)$ on an interval such as $0 \le t \le 6\pi$.

(b) Find the solution of the initial value problem.

(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does.

(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

20. Consider the initial value problem

$$y'' + 0.1y' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $f(t)$ is the same as in Problem 19.

(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

(b) Estimate the amplitude and frequency of the steady state part of the solution.

(c) Compare the results of part (b) with those from Section 3.8 for a sinusoidally forced oscillator.

21. Consider the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$g(t) = u_0(t) + \sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $g(t)$ on an interval such as $0 \le t \le 6\pi$. Compare the graph with that of $f(t)$ in Problem 19(a).

(b) Find the solution of the initial value problem.

(c) Let $n = 15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 19.

(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

22. Consider the initial value problem

$$y'' + 0.1y' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $g(t)$ is the same as in Problem 21.

(a) Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

(b) Estimate the amplitude and frequency of the steady state part of the solution.

(c) Compare the results of part (b) with those from Problem 20 and from Section 3.8 for a sinusoidally forced oscillator.

23. Consider the initial value problem

$$y'' + y = h(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t).$$

Observe that this problem is identical to Problem 19 except that the frequency of the forcing term has been increased somewhat.

(a) Find the solution of this initial value problem.

(b) Let $n \geq 33$ and plot the solution for $0 \leq t \leq 90$ or longer. Your plot should show a clearly recognizable beat.

(c) From the graph in part (b), estimate the "slow period" and the "fast period" for this oscillator.

(d) For a sinusoidally forced oscillator, it was shown in Section 3.8 that the "slow frequency" is given by $|\omega - \omega_0|/2$, where ω_0 is the natural frequency of the system and ω is the forcing frequency. Similarly, the "fast frequency" is $(\omega + \omega_0)/2$. Use these expressions to calculate the "fast period" and the "slow period" for the oscillator in this problem. How well do the results compare with your estimates from part (c)?

6.5 Impulse Functions

In some applications it is necessary to deal with phenomena of an impulsive nature—for example, voltages or forces of large magnitude that act over very short time intervals. Such problems often lead to differential equations of the form

$$ay'' + by' + cy = g(t), \tag{1}$$

where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and is otherwise zero.

The integral $I(\tau)$, defined by

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\, dt, \tag{2}$$

or, since $g(t) = 0$ outside of the interval $(t_0 - \tau, t_0 + \tau)$, by

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\, dt, \tag{3}$$

is a measure of the strength of the forcing function. In a mechanical system, where $g(t)$ is a force, $I(\tau)$ is the total **impulse** of the force $g(t)$ over the time interval $(t_0 - \tau, t_0 + \tau)$. Similarly, if y is the current in an electric circuit and $g(t)$ is the time derivative of the voltage, then $I(\tau)$ represents the total voltage impressed on the circuit during the interval $(t_0 - \tau, t_0 + \tau)$.

In particular, let us suppose that t_0 is zero and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} 1/(2\tau), & -\tau < t < \tau, \\ 0, & t \le -\tau \quad \text{or} \quad t \ge \tau, \end{cases} \tag{4}$$

where τ is a small positive constant (see Figure 6.5.1). According to Eq. (2) or (3), it follows immediately that in this case, $I(\tau) = 1$ independent of the value of τ, as long as $\tau \neq 0$. Now let us idealize the forcing function d_τ by prescribing it to act over shorter and shorter time intervals; that is, we require that $\tau \to 0^+$, as indicated in Figure 6.5.2. As a result of this limiting operation, we obtain

$$\lim_{\tau \to 0^+} d_\tau(t) = 0, \qquad t \neq 0. \tag{5}$$

Further, since $I(\tau) = 1$ for each $\tau \neq 0$, it follows that

$$\lim_{\tau \to 0^+} I(\tau) = 1. \tag{6}$$

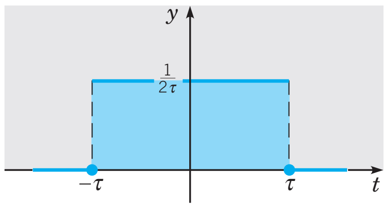

FIGURE 6.5.1 Graph of $y = d_\tau(t)$.

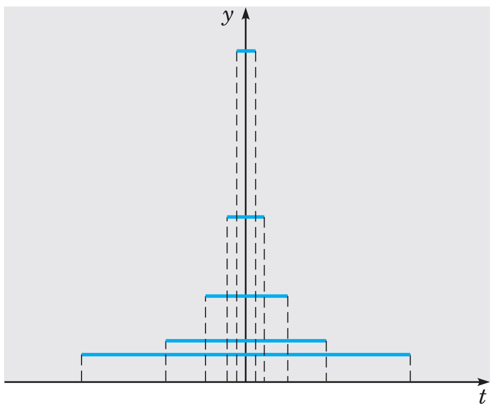

FIGURE 6.5.2 Graphs of $y = d_\tau(t)$ as $\tau \to 0^+$.

Equations (5) and (6) can be used to define an idealized **unit impulse function** δ, which imparts an impulse of magnitude one at $t = 0$ but is zero for all values of t other than zero. That is, the "function" δ is defined to have the properties

$$\delta(t) = 0, \qquad t \neq 0; \tag{7}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{8}$$

There is no ordinary function of the kind studied in elementary calculus that satisfies both Eqs. (7) and (8). The "function" δ, defined by those equations, is an example of what are known as generalized functions; it is usually called the Dirac[3] **delta function**. Since $\delta(t)$ corresponds to a unit impulse at $t = 0$, a unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$. From Eqs. (7) and (8), it follows that

$$\delta(t - t_0) = 0, \qquad t \neq t_0; \tag{9}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1. \tag{10}$$

The delta function does not satisfy the conditions of Theorem 6.1.2, but its Laplace transform can nevertheless be formally defined. Since $\delta(t)$ is defined as the limit of $d_\tau(t)$ as $\tau \to 0^+$, it is natural to define the Laplace transform of δ as a similar limit of the transform of d_τ. In particular, we will assume that $t_0 > 0$ and will define $\mathcal{L}\{\delta(t - t_0)\}$ by the equation

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\tau \to 0^+} \mathcal{L}\{d_\tau(t - t_0)\}. \tag{11}$$

To evaluate the limit in Eq. (11), we first observe that if $\tau < t_0$, which must eventually be the case as $\tau \to 0^+$, then $t_0 - \tau > 0$. Since $d_\tau(t - t_0)$ is nonzero only in the interval from $t_0 - \tau$ to $t_0 + \tau$, we have

$$\begin{aligned}
\mathcal{L}\{d_\tau(t - t_0)\} &= \int_0^{\infty} e^{-st} d_\tau(t - t_0)\, dt \\
&= \int_{t_0 - \tau}^{t_0 + \tau} e^{-st} d_\tau(t - t_0)\, dt.
\end{aligned}$$

Substituting for $d_\tau(t - t_0)$ from Eq. (4), we obtain

$$\begin{aligned}
\mathcal{L}\{d_\tau(t - t_0)\} &= \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st}\, dt = -\frac{1}{2s\tau} e^{-st} \bigg|_{t = t_0 - \tau}^{t = t_0 + \tau} \\
&= \frac{1}{2s\tau} e^{-st_0} (e^{s\tau} - e^{-s\tau})
\end{aligned}$$

or

$$\mathcal{L}\{d_\tau(t - t_0)\} = \frac{\sinh s\tau}{s\tau} e^{-st_0}. \tag{12}$$

---

[3] Paul A. M. Dirac (1902–1984), English mathematical physicist, received his Ph.D. from Cambridge in 1926 and was professor of mathematics there until 1969. He was awarded the Nobel Prize for Physics in 1933 (with Erwin Schrödinger) for fundamental work in quantum mechanics. His most celebrated result was the relativistic equation for the electron, published in 1928. From this equation he predicted the existence of an "anti-electron," or positron, which was first observed in 1932. Following his retirement from Cambridge, Dirac moved to the United States and held a research professorship at Florida State University.

The quotient $(\sinh s\tau)/s\tau$ is indeterminate as $\tau \to 0^+$, but its limit can be evaluated by L'Hôpital's[4] rule. We obtain

$$\lim_{\tau \to 0^+} \frac{\sinh s\tau}{s\tau} = \lim_{\tau \to 0^+} \frac{s \cosh s\tau}{s} = 1.$$

Then from Eq. (11) it follows that

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}. \tag{13}$$

Equation (13) defines $\mathcal{L}\{\delta(t - t_0)\}$ for any $t_0 > 0$. We extend this result, to allow t_0 to be zero, by letting $t_0 \to 0^+$ on the right side of Eq. (13); thus

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0^+} e^{-st_0} = 1. \tag{14}$$

In a similar way, it is possible to define the integral of the product of the delta function and any continuous function f. We have

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t - t_0) f(t)\, dt. \tag{15}$$

Using the definition (4) of $d_\tau(t)$ and the mean value theorem for integrals, we find that

$$\begin{aligned} \int_{-\infty}^{\infty} d_\tau(t - t_0) f(t)\, dt &= \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} f(t)\, dt \\ &= \frac{1}{2\tau} \cdot 2\tau \cdot f(t^*) = f(t^*), \end{aligned}$$

where $t_0 - \tau < t^* < t_0 + \tau$. Hence $t^* \to t_0$ as $\tau \to 0^+$, and it follows from Eq. (15) that

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = f(t_0). \tag{16}$$

The following example illustrates the use of the delta function in solving an initial value problem with an impulsive forcing function.

EXAMPLE 1

Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t - 5), \tag{17}$$

$$y(0) = 0, \qquad y'(0) = 0. \tag{18}$$

This initial value problem arises from the study of the same electric circuit or mechanical oscillator as in Example 1 of Section 6.4. The only difference is in the forcing term.

To solve the given problem, we take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

[4]Marquis Guillaume de L'Hôpital (1661–1704) was a French nobleman with deep interest in mathematics. For a time he employed Johann Bernoulli as his private tutor in calculus. L'Hôpital published the first textbook on differential calculus in 1696; in it appears the differentiation rule that is named for him.

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2} \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}. \tag{19}$$

By Theorem 6.3.2 or from line 9 of Table 6.2.1,

$$\mathcal{L}^{-1}\left\{ \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}} \right\} = \frac{4}{\sqrt{15}} e^{-t/4} \sin \frac{\sqrt{15}}{4} t. \tag{20}$$

Hence, by Theorem 6.3.1, we have

$$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin \frac{\sqrt{15}}{4} (t-5), \tag{21}$$

which is the formal solution of the given problem. It is also possible to write y in the form

$$y = \begin{cases} 0, & t < 5, \\ \dfrac{2}{\sqrt{15}} e^{-(t-5)/4} \sin \dfrac{\sqrt{15}}{4} (t-5), & t \geq 5. \end{cases} \tag{22}$$

The graph of Eq. (22) is shown in Figure 6.5.3. Since the initial conditions at $t = 0$ are homogeneous and there is no external excitation until $t = 5$, there is no response in the interval $0 < t < 5$. The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely. The response is continuous at $t = 5$ despite the singularity in the forcing function at that point. However, the first derivative of the solution has a jump discontinuity at $t = 5$, and the second derivative has an infinite discontinuity there. This is required by the differential equation (17), since a singularity on one side of the equation must be balanced by a corresponding singularity on the other side.

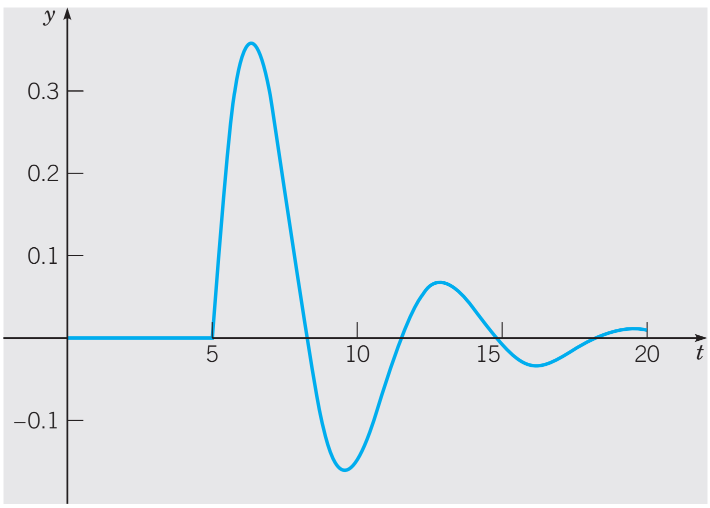

FIGURE 6.5.3 Solution of the initial value problem (17), (18): $2y'' + y' + 2y = \delta(t-5)$, $y(0) = 0$, $y'(0) = 0$.

In dealing with problems that involve impulsive forcing, the use of the delta function usually simplifies the mathematical calculations, often quite significantly.

However, if the actual excitation extends over a short, but nonzero, time interval, then an error will be introduced by modeling the excitation as taking place instantaneously. This error may be negligible, but in a practical problem it should not be dismissed without consideration. In Problem 16 you are asked to investigate this issue for a simple harmonic oscillator.

PROBLEMS

In each of Problems 1 through 12:
(a) Find the solution of the given initial value problem.
(b) Draw a graph of the solution.

1. $y'' + 2y' + 2y = \delta(t - \pi); \qquad y(0) = 1, \quad y'(0) = 0$
2. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi); \qquad y(0) = 0, \quad y'(0) = 0$
3. $y'' + 3y' + 2y = \delta(t - 5) + u_{10}(t); \qquad y(0) = 0, \quad y'(0) = 1/2$
4. $y'' - y = -20\delta(t - 3); \qquad y(0) = 1, \quad y'(0) = 0$
5. $y'' + 2y' + 3y = \sin t + \delta(t - 3\pi); \qquad y(0) = 0, \quad y'(0) = 0$
6. $y'' + 4y = \delta(t - 4\pi); \qquad y(0) = 1/2, \quad y'(0) = 0$
7. $y'' + y = \delta(t - 2\pi)\cos t; \qquad y(0) = 0, \quad y'(0) = 1$
8. $y'' + 4y = 2\delta(t - \pi/4); \qquad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t); \qquad y(0) = 0, \quad y'(0) = 0$
10. $2y'' + y' + 4y = \delta(t - \pi/6)\sin t; \qquad y(0) = 0, \quad y'(0) = 0$
11. $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2); \qquad y(0) = 0, \quad y'(0) = 0$
12. $y^{(4)} - y = \delta(t - 1); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. Consider again the system in Example 1 of this section, in which an oscillation is excited by a unit impulse at $t = 5$. Suppose that it is desired to bring the system to rest again after exactly one cycle—that is, when the response first returns to equilibrium moving in the positive direction.

    (a) Determine the impulse $k\delta(t - t_0)$ that should be applied to the system in order to accomplish this objective. Note that k is the magnitude of the impulse and t_0 is the time of its application.

    (b) Solve the resulting initial value problem, and plot its solution to confirm that it behaves in the specified manner.
14. Consider the initial value problem

$$y'' + \gamma y' + y = \delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$

    where γ is the damping coefficient (or resistance).

    (a) Let $\gamma = \frac{1}{2}$. Find the solution of the initial value problem and plot its graph.

    (b) Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.

    (c) Let $\gamma = \frac{1}{4}$ and repeat parts (a) and (b).

    (d) Determine how t_1 and y_1 vary as γ decreases. What are the values of t_1 and y_1 when $\gamma = 0$?
15. Consider the initial value problem

$$y'' + \gamma y' + y = k\delta(t - 1), \qquad y(0) = 0, \quad y'(0) = 0,$$

where k is the magnitude of an impulse at $t = 1$, and γ is the damping coefficient (or resistance).

(a) Let $\gamma = \frac{1}{2}$. Find the value of k for which the response has a peak value of 2; call this value k_1.

(b) Repeat part (a) for $\gamma = \frac{1}{4}$.

(c) Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?

16. Consider the initial value problem

$$y'' + y = f_k(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $f_k(t) = [u_{4-k}(t) - u_{4+k}(t)]/2k$ with $0 < k \le 1$.

(a) Find the solution $y = \phi(t, k)$ of the initial value problem.

(b) Calculate $\lim\limits_{k\to 0^+} \phi(t, k)$ from the solution found in part (a).

(c) Observe that $\lim\limits_{k\to 0^+} f_k(t) = \delta(t - 4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t - 4)$. Is it true that $\phi_0(t) = \lim\limits_{k\to 0^+} \phi(t, k)$?

(d) Plot $\phi(t, 1/2)$, $\phi(t, 1/4)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t, k)$ and $\phi_0(t)$.

Problems 17 through 22 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

(a) Try to predict the nature of the solution without solving the problem.

(b) Test your prediction by finding the solution and drawing its graph.

(c) Determine what happens after the sequence of impulses ends.

17. $f(t) = \sum\limits_{k=1}^{20} \delta(t - k\pi)$

18. $f(t) = \sum\limits_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi)$

19. $f(t) = \sum\limits_{k=1}^{20} \delta(t - k\pi/2)$

20. $f(t) = \sum\limits_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi/2)$

21. $f(t) = \sum\limits_{k=1}^{15} \delta[t - (2k - 1)\pi]$

22. $f(t) = \sum\limits_{k=1}^{40} (-1)^{k+1}\delta(t - 11k/4)$

23. The position of a certain lightly damped oscillator satisfies the initial value problem

$$y'' + 0.1y' + y = \sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 18.

(a) Try to predict the nature of the solution without solving the problem.

(b) Test your prediction by finding the solution and drawing its graph.

(c) Determine what happens after the sequence of impulses ends.

24. Proceed as in Problem 23 for the oscillator satisfying

$$y'' + 0.1y' + y = \sum_{k=1}^{15} \delta[t - (2k - 1)\pi], \qquad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 21.

25. (a) By the method of variation of parameters, show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t - \tau)\, d\tau.$$

(b) Show that if $f(t) = \delta(t - \pi)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t - \pi).$$

(c) Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t - \pi)$, and confirm that the solution agrees with the result of part (b).

6.6 The Convolution Integral

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other transforms $F(s)$ and $G(s)$, the latter transforms corresponding to known functions f and g, respectively. In this event, we might anticipate that $H(s)$ would be the transform of the product of f and g. However, this is not the case; in other words, the Laplace transform cannot be commuted with ordinary multiplication. On the other hand, if an appropriately defined "generalized product" is introduced, then the situation changes, as stated in the following theorem.

Theorem 6.6.1 If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a, \tag{1}$$

where

$$h(t) = \int_0^t f(t - \tau)g(\tau)\, d\tau = \int_0^t f(\tau)g(t - \tau)\, d\tau. \tag{2}$$

The function h is known as the convolution of f and g; the integrals in Eq. (2) are called convolution integrals.

The equality of the two integrals in Eq. (2) follows by making the change of variable $t - \tau = \xi$ in the first integral. Before giving the proof of this theorem, let us make some observations about the convolution integral. According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f * g)(t). \tag{3}$$