Introduction to Differential Equations
Sample problems # 9

Date Given: June 6, 2022

P1. Find the Laplace transform of the following functions:

(a) $f(t) = t$

(b) $f(t) = t^2$

(c) $f(t) = t^n$, where n is a positive integer

Solution:

(a) Since t is continuous for $0 \le t \le A$ for any positive A and since $t \le e^{at}$ for any $a \ge 0$ for t sufficiently large, it follows from Theorem 6.1.2 that the Laplace transform of $f(t) = t$ exists for $s \ge 0$. Using integration by parts [1] we obtain

$$F(s) = \int_0^\infty te^{-st}\mathrm{d}t = \lim_{M\to\infty}\int_0^M te^{-st}\mathrm{d}t = \lim_{M\to\infty}\left[\left.\frac{-te^{-st}}{s}\right|_0^M + \frac{1}{s}\int_0^M e^{-st}\mathrm{d}t\right] =$$

$$\lim_{M\to\infty}\frac{-Me^{-sM}}{s} + \frac{1}{s}\lim_{M\to\infty}\left[\left.\frac{e^{-st}}{s}\right|_0^M\right] = 0 + \frac{1}{s}\lim_{M\to\infty}\left[\frac{e^{-sM}}{s} + \frac{1}{s}\right] = \frac{1}{s^2}$$

From this point forward (in parts (b) and (c)), we do not check the assumptions of Theorem 6.1.2; they stand for all the functions given. Also, we omit the limit finding process - you should check these.

(b) For $f(t) = t^2$, the Laplace transform (using integration by parts twice) is

$$F(s) = \int_0^\infty t^2e^{-st}\mathrm{d}t = -\left.\frac{e^{-st}s^2t^2 + 2e^{-st}st + 2e^{-st}}{s^3}\right|_0^\infty = \frac{2}{s^3}$$

(c) For $f(t) = t^n$, denote the Laplace transform as $F_n(s)$. Then, using integration by parts, we get

$$F_n(s) = \int_0^\infty t^ne^{-st}\mathrm{d}t = \int_0^\infty t^n\mathrm{d}\left(-\frac{1}{s}e^{-st}\right) = -\left.\frac{t^ne^{-st}}{s}\right|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\mathrm{d}t = \frac{n}{s}F_{n-1}(s)$$

Therefore we obtain by induction

$$F_n(s) = \frac{n}{s}F_{n-1}(s) = \frac{n(n-1)}{s^2}F_{n-2}(s) = \ldots = \frac{n!}{s^{n-1}}F_1(s) = \frac{n!}{s^{n-1}}\frac{1}{s^2} = \frac{n!}{s^{n+1}}$$

P2. Use integration by parts to find the Laplace transform of $f(t) = t\sin at$, where a is a real constant.

Solution: We will use the fact that $\sin at = (e^{\imath at} - e^{-\imath at})/2\imath$. Therefore

$$\int_0^A e^{-st}\,t\sin at\,\mathrm{d}t = \frac{1}{2\imath}\left[\int_0^A te^{(\imath a-s)t}\mathrm{d}t - \int_0^A te^{(\imath a+s)t}\mathrm{d}t\right] =$$

$$\frac{1}{2\imath}\left[-\left.\frac{te^{(\imath a-s)t}}{s-\imath a}\right|_0^A + \int_0^A\frac{e^{(\imath a-s)t}}{s-\imath a}\mathrm{d}t\right] - \frac{1}{2\imath}\left[-\left.\frac{te^{(-\imath a-s)t}}{s+\imath a}\right|_0^A + \int_0^A\frac{e^{(-\imath a-s)t}}{s+\imath a}\mathrm{d}t\right] =$$

$$\frac{1}{2\imath}\left[-\frac{Ae^{(\imath a-s)A}}{s-\imath a} + \frac{Ae^{(-\imath a-s)A}}{s+\imath a} + \frac{1}{(s-\imath a)^2} - \frac{1}{(s+\imath a)^2}\right]$$

[1] Here we used $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$ with $v = -e^{-st}/s$ and $\mathrm{d}v = e^{-st}\mathrm{d}t$.

Taking the limit as $A \to \infty$, we have

$$F(s) = \int_0^\infty e^{-st}\, t \sin at \,\mathrm{d}t = \frac{1}{2\imath}\left[-\frac{(s+\imath a)^2-(s-\imath a)^2}{(s-\imath a)^2(s+\imath a)^2}\right] = \frac{2as}{(s^2+a^2)^2}$$

P3. Use integration by parts to find the Laplace transform of $f(t) = t\cosh at$, where a is a real constant.

Solution: Observe that $t\cosh at = (te^{at}at + te^{-at})/2$. For any value of c,

$$\int_0^A te^{ct}e^{-st}\mathrm{d}t = -\frac{te^{(c-s)t}}{s-c}\bigg|_0^A + \int_0^A A\frac{e^{(c-s)t}}{s-c}\mathrm{d}t = \frac{1-e^{A(c-s)}+A(c-s)e^{A(c-s)}}{(c-s)^2}$$

Taking the limit as $A \to \infty$, we have

$$\int_0^\infty te^{ct}e^{-st}\mathrm{d}t = \frac{1}{(c-s)^2}$$

Note that the limit exists as long as $s > |c|$. Therefore,

$$F(s) = \int_0^\infty t\cosh at\, e^{-st}\mathrm{d}t = \frac{1}{2}\left[\frac{1}{(s-a)^2}+\frac{1}{(s+a)^2}\right] = \frac{s^2+a^2}{(s-a)^2(s+a)^2}$$

P4. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t < \pi \\ 0, & \text{if } \pi \le t < \infty \end{cases}$$

Solution: Using the fact that $f(t) = 0$ when $t \ge \pi$ we obtain that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st}f(t)\mathrm{d}t = \int_0^1 e^{-st}\mathrm{d}t = \left[-\frac{e^{-st}}{s}\right]_0^\pi = \frac{1-e^{-\pi s}}{s}$$

P5. Find the Laplace transform of

$$f(t) = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 1, & \text{if } 1 \le t < \infty \end{cases}$$

Solution: By the definition of the Laplace transform

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st}f(t)\mathrm{d}t = \int_0^1 te^{-st}\mathrm{d}t + \int_1^\infty e^{-st}\mathrm{d}t$$

The first integral is

$$\int_0^1 te^{-st}\mathrm{d}t = \left[-\frac{e^{-st}}{s}t\right]_0^1 + \int_0^1 \frac{e^{-st}}{s}\mathrm{d}t = -\frac{e^{-s}}{s} + \left[-\frac{e^{-st}}{s^2}\right]_0^1 = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s},$$

and the second one is

$$\int_1^\infty e^{-st}\mathrm{d}t = \left[-\frac{e^{-st}}{s}\right]_1^\infty = \frac{e^{-s}}{s}.$$

Therefore

$$\mathcal{L}\left[f(t)\right] = \frac{1-e^{-s}}{s^2}$$

P6. Find the Laplace transform of the following functions:

(a) $f(t) = \cos at$, where a is a real constant

(b) $f(t) = \sin at$, where a is a real constant

Solution:

(a) $f(t) = \cos at$ satisfies the hypothesis of Theorem 6.1.2 because $|\cos at| \leq 1$ for all t. Using integration by parts[2] twice we obtain

$$F(s) = \int_0^\infty e^{-st} \cos at \, \mathrm{d}t = \left[-\frac{e^{-st} \cos at}{s} + \frac{ae^{-st} \sin at}{s^2} \right]_0^\infty - \frac{a^2}{s^2} \int_0^\infty e^{-st} \cos at \, \mathrm{d}t.$$

This implies that

$$F(s) = \frac{1}{s} - \frac{a^2}{s^2} F(s),$$

and hence (after rearrangement of terms and division by $(1 + a^2/s^2)$) we get that

$$F(s) = \frac{s}{a^2 + s^2}.$$

(b) Similar reasoning is applied for $f(t) = \sin at$. Using integration by parts twice we obtain

$$F(s) = \int_0^\infty e^{-st} \sin at \, \mathrm{d}t = \left[-\frac{e^{-st} \sin at}{s} - \frac{ae^{-st} \cos at}{s^2} \right]_0^\infty - \frac{a^2}{s^2} \int_0^\infty e^{-st} \sin at \, \mathrm{d}t.$$

This implies that

$$F(s) = \frac{a}{s^2} - \frac{a^2}{s^2} F(s),$$

and hence

$$F(s) = \frac{a}{a^2 + s^2}.$$

P7. By using the inverse Euler formulae, $\cos t = (e^{\imath t} + e^{-\imath t})/2$ and $\sin t = (e^{\imath t} - e^{-\imath t})/2\imath$, find the Laplace transform of the following functions:

(a) $f(t) = e^{at} \cos bt$, where a and b are real constants

(b) $f(t) = e^{at} \sin bt$, where a and b are real constants

Solution:

(a) First, represent

$$f(t) = e^{at} \cos bt = \frac{e^{at}(e^{\imath bt} + e^{-\imath bt})}{2} = \frac{e^{(a+\imath b)t} + e^{(a-\imath b)t}}{2}$$

Next, using the linearity of the Laplace transform, we can write

$$\mathcal{L}\left[e^{at} \cos bt\right] = \frac{1}{2}\mathcal{L}\left[e^{(a+\imath b)t}\right] + \frac{1}{2}\mathcal{L}\left[e^{(a-\imath b)t}\right]$$

Note that

$$\mathcal{L}\left[e^{(a+\imath b)t}\right] = \int_0^\infty e^{-st} e^{(a+\imath b)t} \mathrm{d}t = \int_0^\infty e^{-(s-a-\imath b)t} \mathrm{d}t = \frac{1}{s-a-\imath b}$$

and

$$\mathcal{L}\left[e^{(a-\imath b)t}\right] = \int_0^\infty e^{-st} e^{(a-\imath b)t} \mathrm{d}t = \int_0^\infty e^{-(s-a+\imath b)t} \mathrm{d}t = \frac{1}{s-a+\imath b}$$

Therefore

$$\mathcal{L}\left[e^{at} \cos bt\right] = \frac{1}{2}\left(\frac{1}{s-a-\imath b} + \frac{1}{s-a+\imath b}\right) = \frac{s-a}{(s-a)^2 + b^2}.$$

The above is valid for $s > a$.

[2] Note that here we used $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$ with $v = -e^{-st}/s$ and $\mathrm{d}v = e^{-st}\mathrm{d}t$. Also note that the alternative way, not requiring integration by parts, is to use the inverse Euler formula for $\cos at$.

(b) First, represent

$$f(t) = e^{at}\sin bt = \frac{e^{at}(e^{\imath bt} - e^{-\imath bt})}{2\imath} = \frac{e^{(a+\imath b)t} - e^{(a-\imath b)t}}{2\imath}$$

Next, using the linearity of the Laplace transform, we can write

$$\mathcal{L}\left[e^{at}\sin bt\right] = \frac{1}{2\imath}\mathcal{L}\left[e^{(a+\imath b)t}\right] - \frac{1}{2\imath}\mathcal{L}\left[e^{(a-\imath b)t}\right]$$

Therefore, by using the above established formulae for $\mathcal{L}\left[e^{(a+\imath b)t}\right]$ and $\mathcal{L}\left[e^{(a+\imath b)t}\right]$, we get

$$\mathcal{L}\left[e^{at}\sin bt\right] = \frac{1}{2\imath}\left(\frac{1}{s-a-\imath b} - \frac{1}{s-a+\imath b}\right) = \frac{b}{(s-a)^2+b^2}$$

Again, the above is valid for $s > a$.

P8. Find the inverse Laplace transform of $F(s) = \dfrac{3s}{s^2 - s - 6}$

Solution: The problem is solved by using partial fractions and algebra to manipulate the given function into a form matching one of the functions appearing in the middle column of Table 6.2.1.

Using partial fractions we write

$$\frac{3s}{s^2-s-6} = \frac{9}{5}\,\frac{1}{s-3} + \frac{6}{3}\,\frac{5}{s+2}.$$

Therefore, we establish (from line 2 in Table 6.2.1) that

$$\mathcal{L}^{-1}[Y(s)] = \frac{9}{5}\,e^{3t} + \frac{6}{5}\,e^{-2t}$$

P9. Find the inverse Laplace transform of $F(s) = \dfrac{2s+2}{s^2+2s+5}$

Solution: Note that the denominator s^2+2s+5 is irreducible over the reals. Completing the square, $s^2+2s+5 = (s+1)^2+4$. Now convert the function to a rational function of the variable $\xi = s+1$. That is,

$$\frac{2s+2}{s^2+2s+5} = \frac{2(s+1)}{(s+1)^2+4}$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\xi}{\xi^2+4}\right] = 2\cos 2t$$

Using the fact[3] that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to(s-a)}$,

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] = 2e^{-t}\cos 2t$$

P10. Find the inverse Laplace transform of $F(s) = \dfrac{2s-3}{s^2-4}$

Solution: Using partial fractions,

$$\frac{2s-3}{s^2-4} = \frac{1}{4}\left[\frac{1}{s-2} + \frac{7}{s+2}\right]$$

Hence[4]

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-4}\right] = \frac{e^{2t}+7e^{-2t}}{4}$$

[3]Refer to line 14 in Table 6.2.1.

[4]Note that we can also right $\frac{2s-3}{s^2-4} = \frac{2s}{s^2-4} - \frac{3}{2}\frac{2}{s^2-4}$, and write down the result as $2\cosh t - \frac{3}{2}\sinh 2t$.

P11. Find the inverse Laplace transform of $F(s) = \dfrac{1-2s}{s^2+4s+5}$

Solution: Note that the denominator s^2+4s+5 is irreducible over the reals. Completing the square, $s^2+4s+5 = (s+2)^2+1$. Now convert the function to a rational function of the variable $\xi = s+2$. That is,

$$\frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1}$$

We find that

$$\mathcal{L}^{-1}\left[\frac{5}{\xi^2+1} - \frac{2\xi}{\xi^2+1}\right] = 5\sin t - 2\cos t$$

Using the fact[5] that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to(s-a)}$,

$$\mathcal{L}^{-1}\left[\frac{1-2s}{s^2+4s+5}\right] = e^{-2t}\left(5\sin t - 2\cos t\right)$$

P12. Use the Laplace transform to solve the following initial value problem: $y'' - y' - 6y = 0,\ y(0) = 1, y'(0) = -1$.

Solution: Taking the Laplace transform of the differential equation, using Eq. (1) and Eq. (3) of Section 6.2, we obtain $\left[s^2Y(s) - sy(0) - y'(0)\right] - [sY(s) - y(0)] - 6Y(s) = 0$, Using the initial conditions and solving for $Y(s)$, we obtain $Y(s) = (s-2)/(s^2-s-6)$. Using partial fractions,

$$Y(s) = \frac{s-2}{s^2-s-6} = \frac{1}{5}\,\frac{1}{s-3} + \frac{4}{5}\,\frac{1}{s+2}$$

which implies that

$$y(t) = \frac{1}{5}\,e^{3t} + \frac{4}{5}\,e^{-2t}$$

P13. Use the Laplace transform to solve the following initial value problem: $y'' - 4y' + 4y = 0,\ y(0) = 1, y'(0) = 1$.

Solution: Taking the Laplace transform of the differential equation, using Eq. (1) and Eq. (3) of Section 6.2, we obtain $\left[s^2Y(s) - sy(0) - y'(0)\right] - 4\,[sY(s) - y(0)] + 4Y(s) = 0$, Using the initial conditions and solving for $Y(s)$, we obtain $Y(s) = (s-3)/(s^2-4s+4)$. Using partial fractions,

$$Y(s) = \frac{s-3}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

which implies that

$$y(t) = e^{2t} - te^{2t}$$

P14. Use the Laplace transform to solve the following initial value problem: $y'''' - 4y''' + 6y'' - 4y' + y = 0$, $y(0) = 1, y'(0) = 1, y''(0) = 0, y'''(0) = 1$.

Solution: Taking the Laplace transform of the differential equation, using (4) of Section 6.2, we obtain

$$\begin{aligned}\left[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right] - 4\left[s^3Y(s) - s^2y(0) - sy'(0)y''(0)\right]\\ + 6\left[s^2Y(s) - sy(0) - y'(0)\right] - 4\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = 0\end{aligned}$$

Applying the initial conditions

$$s^4Y(s) - 4s^3Y(s) + 6s^2Y(s) - 4sY(s) + Y(s) = s^2 - 4s + 7$$

[5] Refer to line 14 in Table 6.2.1.

Solving for the transform of the solution,

$$Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1} = \frac{s^2 - 4s + 7}{(s-1)^4}$$

Using partial fractions,

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}$$

Using lines 11 and 14 from Table 6.2.1, we obtain

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2 - 4s + 7}{(s-1)^4}\right] = \frac{2}{3}t^3e^t - t^2e^t + te^t$$

P15. Use the Laplace transform to solve the following initial value problem: $y'' + \omega^2 y = \cos 2t,\ \ \omega^2 \neq 4$, $y(0) = 1, y'(0) = 0$.

Solution: Taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + \omega^2 Y(s) = \frac{s}{s^2 + 4}$$

Applying the initial conditions

$$s^2Y(s) + \omega^2 Y(s) - s = \frac{s}{s^2 + 4}$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{(s^2 + \omega^2)(s^2 + 4)} + \frac{s}{s^2 + \omega^2}$$

Using partial fractions on the first term,

$$\frac{s}{(s^2 + \omega^2)(s^2 + 4)} = \frac{1}{4 - \omega^2}\left[\frac{s}{s^2 + \omega^2} - \frac{s}{s^2 + 4}\right]$$

First note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2 + \omega^2}\right] = \cos \omega t \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2 + 4}\right] = \cos 2t$$

Hence the solution of the initial value problem is

$$y(t) = \frac{1}{4 - \omega^2}\cos \omega t - \frac{1}{4 - \omega^2}\cos 2t + \cos \omega t = \frac{5 - \omega^2}{4 - \omega^2}\cos \omega t - \frac{1}{4 - \omega^2}\cos 2t$$

P16. Find the Laplace transform $Y(s) = \mathcal{L}[y]$ of the solution of the following initial value problem

$$y'' + y = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 2 - t, & \text{if } 1 \le t < 2 \\ 0, & \text{if } 2 \le t < \infty \end{cases}$$

$y(0) = 0, y'(0) = 0$.

Solution: First, using the fact that $f(t) = 0$ when $t \geq 2$ we obtain (after integration by parts and some simplification) that

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st}f(t)\mathrm{d}t = \int_0^1 te^{-st}\mathrm{d}t + \int_1^2 (2-t)e^{-st}\mathrm{d}t =$$

$$\left[-\frac{e^{-st}(1 + st)}{s^2}\right]_0^1 + \left[\frac{e^{-st}(1 - 2s + st)}{s^2}\right]_1^2 = \frac{e^{-2s}(e^s - 1)^2}{s^2}$$

Taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{e^{-2s}(e^s - 1)^2}{s^2}$$

Therefore, using the initial conditions and solving for $Y(s)$, we get

$$Y(s) = \frac{e^{-2s}(e^s - 1)^2}{s^2(s^2 + 1)}$$