This equation is known as the adjoint of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second order equation.

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

47. $x^2y'' + xy' + (x^2 - \nu^2)y = 0,$ Bessel's equation
48. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$ Legendre's equation
49. $y'' - xy = 0,$ Airy's equation

50. For the second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$, show that the adjoint of the adjoint equation is the original equation.
51. A second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be self-adjoint if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 47 through 49 is self-adjoint.

3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

We showed in Section 3.1 that if the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of Eq. (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq. (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]. \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from Eq. (5),

$$y_1(3) = e^{-3+6i}. \tag{6}$$

What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as Euler's formula.

Euler's Formula. To assign a meaning to the expressions in Eqs. (5), we need to give a definition of the complex exponential function. Of course, we want the definition to reduce to the familiar real exponential function when the exponent is real. There are several ways to discover how this extension of the exponential function should be defined. Here we use a method based on infinite series; an alternative is outlined in Problem 28.

Recall from calculus that the Taylor series for e^t about $t = 0$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad -\infty < t < \infty. \tag{7}$$

If we now assume that we can substitute it for t in Eq. (7), then we have

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}, \end{aligned} \tag{8}$$

where we have separated the sum into its real and imaginary parts, making use of the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so forth. The first series in Eq. (8) is precisely the Taylor series for $\cos t$ about $t = 0$, and the second is the Taylor series for $\sin t$ about $t = 0$. Thus we have

$$e^{it} = \cos t + i \sin t. \tag{9}$$

Equation (9) is known as Euler's formula and is an extremely important mathematical relationship. Although our derivation of Eq. (9) is based on the unverified assumption that the series (7) can be used for complex as well as real values of the independent variable, our intention is to use this derivation only to make Eq. (9) seem plausible. We now put matters on a firm foundation by adopting Eq. (9) as the *definition* of e^{it}. In other words, whenever we write e^{it}, we mean the expression on the right side of Eq. (9).

There are some variations of Euler's formula that are also worth noting. If we replace t by $-t$ in Eq. (9) and recall that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, then we have

$$e^{-it} = \cos t - i \sin t. \tag{10}$$

Further, if t is replaced by μt in Eq. (9), then we obtain a generalized version of Euler's formula, namely,

$$e^{i\mu t} = \cos \mu t + i \sin \mu t. \tag{11}$$

Next, we want to extend the definition of the exponential function to arbitrary complex exponents of the form $(\lambda + i\mu)t$. Since we want the usual properties of the exponential function to hold for complex exponents, we certainly want $\exp[(\lambda + i\mu)t]$ to satisfy

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t}. \tag{12}$$

Then, substituting for $e^{i\mu t}$ from Eq. (11), we obtain

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t}(\cos\mu t + i\sin\mu t) \\ &= e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t. \end{aligned} \tag{13}$$

We now take Eq. (13) as the definition of $\exp[(\lambda + i\mu)t]$. The value of the exponential function with a complex exponent is a complex number whose real and imaginary parts are given by the terms on the right side of Eq. (13). Observe that the real and imaginary parts of $\exp[(\lambda + i\mu)t]$ are expressed entirely in terms of elementary real-valued functions. For example, the quantity in Eq. (6) has the value

$$e^{-3+6i} = e^{-3}\cos 6 + ie^{-3}\sin 6 \cong 0.0478041 - 0.0139113i.$$

With the definitions (9) and (13), it is straightforward to show that the usual laws of exponents are valid for the complex exponential function. You can also use Eq. (13) to verify that the differentiation formula

$$\frac{d}{dt}(e^{rt}) = re^{rt} \tag{14}$$

holds for complex values of r.

EXAMPLE 1

Find the general solution of the differential equation

$$y'' + y' + 9.25y = 0, \tag{15}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \qquad y'(0) = 8, \tag{16}$$

and draw its graph.

The characteristic equation for Eq. (15) is

$$r^2 + r + 9.25 = 0$$

so its roots are

$$r_1 = -\tfrac{1}{2} + 3i, \qquad r_2 = -\tfrac{1}{2} - 3i.$$

Therefore, two solutions of Eq. (15) are

$$y_1(t) = \exp\left[\left(-\tfrac{1}{2} + 3i\right)t\right] = e^{-t/2}(\cos 3t + i\sin 3t) \tag{17}$$

and

$$y_2(t) = \exp\left[\left(-\tfrac{1}{2} - 3i\right)t\right] = e^{-t/2}(\cos 3t - i\sin 3t). \tag{18}$$

You can verify that the Wronskian $W(y_1, y_2)(t) = -6ie^{-t}$, which is not zero, so the general solution of Eq. (15) can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$ with arbitrary coefficients.

However, the initial value problem (15), (16) has only real coefficients, and it is often desirable to express the solution of such a problem in terms of real-valued functions. To do this we can make use of Theorem 3.2.6, which states that the real and imaginary parts of a complex-valued solution of Eq. (15) are also solutions of Eq. (15). Thus, starting from either $y_1(t)$ or $y_2(t)$, we obtain

$$u(t) = e^{-t/2}\cos 3t, \qquad v(t) = e^{-t/2}\sin 3t \tag{19}$$

as real-valued solutions[5] of Eq. (15). On calculating the Wronskian of $u(t)$ and $v(t)$, we find that $W(u, v)(t) = 3e^{-t}$, which is not zero; thus $u(t)$ and $v(t)$ form a fundamental set of solutions, and the general solution of Eq. (15) can be written as

$$y = c_1u(t) + c_2v(t) = e^{-t/2}(c_1\cos 3t + c_2\sin 3t), \tag{20}$$

where c_1 and c_2 are arbitrary constants.

To satisfy the initial conditions (16), we first substitute $t = 0$ and $y = 2$ in Eq. (20) with the result that $c_1 = 2$. Then, by differentiating Eq. (20), setting $t = 0$, and setting $y' = 8$, we obtain $-\frac{1}{2}c_1 + 3c_2 = 8$, so that $c_2 = 3$. Thus the solution of the initial value problem (15), (16) is

$$y = e^{-t/2}(2\cos 3t + 3\sin 3t). \tag{21}$$

The graph of this solution is shown in Figure 3.3.1.

From the graph we see that the solution of this problem is a decaying oscillation. The sine and cosine factors control the oscillatory nature of the solution, and the negative exponential factor in each term causes the magnitude of the oscillations to diminish as time increases.

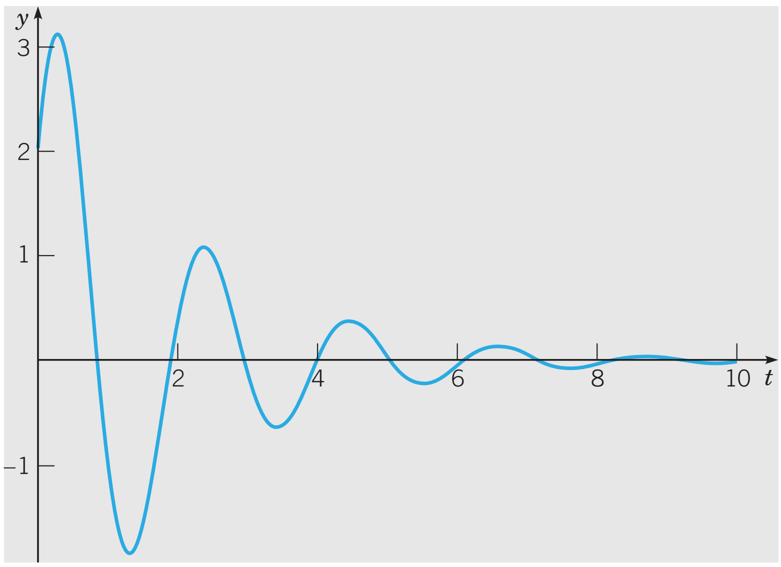

FIGURE 3.3.1 Solution of the initial value problem (15), (16): $y'' + y' + 9.25y = 0,\ \ y(0) = 2,\ \ y'(0) = 8.$

Complex Roots; The General Case. The functions $y_1(t)$ and $y_2(t)$, given by Eqs. (5) and with the meaning expressed by Eq. (13), are solutions of Eq. (1) when the roots of the characteristic equation (2) are complex numbers $\lambda \pm i\mu$. However, the solutions y_1 and y_2 are complex-valued functions, whereas in general we would prefer to have

[5] If you are not completely sure that $u(t)$ and $v(t)$ are solutions of the given differential equation, you should substitute these functions into Eq. (15) and confirm that they satisfy it.

real-valued solutions because the differential equation itself has real coefficients. Just as in Example 1, we can use Theorem 3.2.6 to find a fundamental set of real-valued solutions by choosing the real and imaginary parts of either $y_1(t)$ or $y_2(t)$. In this way we obtain the solutions

$$u(t) = e^{\lambda t}\cos\mu t, \qquad v(t) = e^{\lambda t}\sin\mu t. \tag{22}$$

By direct computation you can show that the Wronskian of u and v is

$$W(u, v)(t) = \mu e^{2\lambda t}. \tag{23}$$

Thus, as long as $\mu \neq 0$, the Wronskian W is not zero, so u and v form a fundamental set of solutions. (Of course, if $\mu = 0$, then the roots are real and the discussion in this section is not applicable.) Consequently, if the roots of the characteristic equation are complex numbers $\lambda \pm i\mu$, with $\mu \neq 0$, then the general solution of Eq. (1) is

$$y = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t, \tag{24}$$

where c_1 and c_2 are arbitrary constants. Note that the solution (24) can be written down as soon as the values of λ and μ are known. Let us now consider some further examples.

EXAMPLE 2

Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1. \tag{25}$$

The characteristic equation is $16r^2 - 8r + 145 = 0$ and its roots are $r = 1/4 \pm 3i$. Thus the general solution of the differential equation is

$$y = c_1 e^{t/4}\cos 3t + c_2 e^{t/4}\sin 3t. \tag{26}$$

To apply the first initial condition, we set $t = 0$ in Eq. (26); this gives

$$y(0) = c_1 = -2.$$

For the second initial condition, we must differentiate Eq. (26) and then set $t = 0$. In this way we find that

$$y'(0) = \tfrac{1}{4}c_1 + 3c_2 = 1,$$

from which $c_2 = 1/2$. Using these values of c_1 and c_2 in Eq. (26), we obtain

$$y = -2e^{t/4}\cos 3t + \tfrac{1}{2}e^{t/4}\sin 3t \tag{27}$$

as the solution of the initial value problem (25). The graph of this solution is shown in Figure 3.3.2.

In this case we observe that the solution is a growing oscillation. Again the trigonometric factors in Eq. (27) determine the oscillatory part of the solution, while the exponential factor (with a positive exponent this time) causes the magnitude of the oscillation to increase with time.

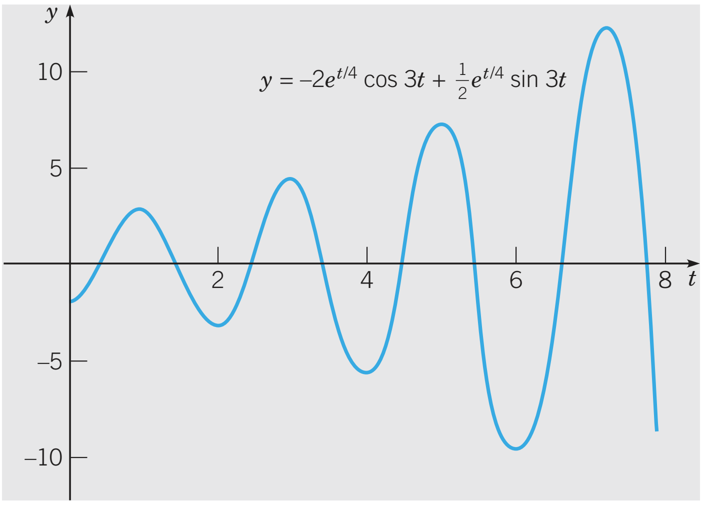

FIGURE 3.3.2 Solution of the initial value problem (25): $16y'' - 8y' + 145y = 0,\ \ y(0) = -2,\ \ y'(0) = 1.$

EXAMPLE 3

Find the general solution of

$$y'' + 9y = 0. \tag{28}$$

The characteristic equation is $r^2 + 9 = 0$ with the roots $r = \pm 3i$; thus $\lambda = 0$ and $\mu = 3$. The general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t; \tag{29}$$

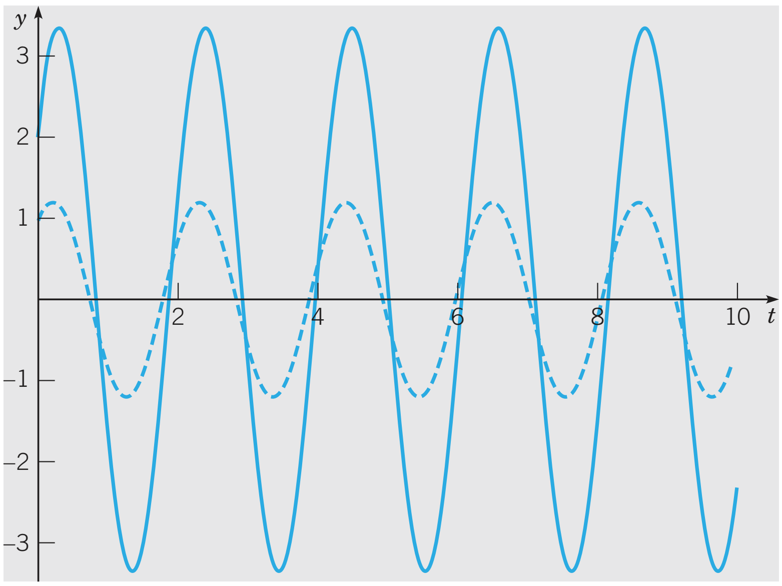

FIGURE 3.3.3 Two typical solutions of Eq. (28): $y'' + 9y = 0$.

note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution. Figure 3.3.3 shows the graph of two typical solutions of Eq. (28). In each case the solution is a pure oscillation whose amplitude is determined by the initial conditions. Since there is no exponential factor in the solution (29), the amplitude of each oscillation remains constant in time.

PROBLEMS

In each of Problems 1 through 6, use Euler's formula to write the given expression in the form $a + ib$.

1. $\exp(1 + 2i)$
2. $\exp(2 - 3i)$
3. $e^{i\pi}$
4. $e^{2-(\pi/2)i}$
5. 2^{1-i}
6. π^{-1+2i}

In each of Problems 7 through 16, find the general solution of the given differential equation.

7. $y'' - 2y' + 2y = 0$
8. $y'' - 2y' + 6y = 0$
9. $y'' + 2y' - 8y = 0$
10. $y'' + 2y' + 2y = 0$
11. $y'' + 6y' + 13y = 0$
12. $4y'' + 9y = 0$
13. $y'' + 2y' + 1.25y = 0$
14. $9y'' + 9y' - 4y = 0$
15. $y'' + y' + 1.25y = 0$
16. $y'' + 4y' + 6.25y = 0$

In each of Problems 17 through 22, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

17. $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
18. $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$
19. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$
20. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$
21. $y'' + y' + 1.25y = 0, \quad y(0) = 3, \quad y'(0) = 1$
22. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$
23. Consider the initial value problem

$$3u'' - u' + 2u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

(a) Find the solution $u(t)$ of this problem.
(b) For $t > 0$, find the first time at which $|u(t)| = 10$.

24. Consider the initial value problem

$$5u'' + 2u' + 7u = 0, \qquad u(0) = 2, \quad u'(0) = 1.$$

(a) Find the solution $u(t)$ of this problem.
(b) Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.

25. Consider the initial value problem

$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$

(a) Find the solution $y(t)$ of this problem.
(b) Find α such that $y = 0$ when $t = 1$.
(c) Find, as a function of α, the smallest positive value of t for which $y = 0$.
(d) Determine the limit of the expression found in part (c) as $\alpha \to \infty$.

26. Consider the initial value problem

$$y'' + 2ay' + (a^2 + 1)y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Find the solution $y(t)$ of this problem.
(b) For $a = 1$ find the smallest T such that $|y(t)| < 0.1$ for $t > T$.
(c) Repeat part (b) for $a = 1/4, 1/2$, and 2.
(d) Using the results of parts (b) and (c), plot T versus a and describe the relation between T and a.

27. Show that $W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}$.

28. In this problem we outline a different derivation of Euler's formula.
(a) Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.
(b) Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore,

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{i}$$

for some constants c_1 and c_2. Why is this so?
(c) Set $t = 0$ in Eq. (i) to show that $c_1 = 1$.
(d) Assuming that Eq. (14) is true, differentiate Eq. (i) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in Eq. (i) to arrive at Euler's formula.

29. Using Euler's formula, show that

$$\cos t = (e^{it} + e^{-it})/2, \qquad \sin t = (e^{it} - e^{-it})/2i.$$

30. If e^{rt} is given by Eq. (13), show that $e^{(r_1+r_2)t} = e^{r_1t}e^{r_2t}$ for any complex numbers r_1 and r_2.

31. If e^{rt} is given by Eq. (13), show that

$$\frac{d}{dt}e^{rt} = re^{rt}$$

for any complex number r.

32. Consider the differential equation

$$ay'' + by' + cy = 0,$$

where $b^2 - 4ac < 0$ and the characteristic equation has complex roots $\lambda \pm i\mu$. Substitute the functions

$$u(t) = e^{\lambda t}\cos\mu t \quad \text{and} \quad v(t) = e^{\lambda t}\sin\mu t$$

for y in the differential equation and thereby confirm that they are solutions.

33. If the functions y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$, show that between consecutive zeros of y_1 there is one and only one zero of y_2. Note that this result is illustrated by the solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of the equation $y'' + y = 0$.
Hint: Suppose that t_1 and t_2 are two zeros of y_1 between which there are no zeros of y_2. Apply Rolle's theorem to y_1/y_2 to reach a contradiction.

Change of Variables. Sometimes a differential equation with variable coefficients,

$$y'' + p(t)y' + q(t)y = 0, \tag{i}$$

can be put in a more suitable form for finding a solution by making a change of the independent variable. We explore these ideas in Problems 34 through 46. In particular, in Problem 34 we show that a class of equations known as Euler equations can be transformed into equations with constant coefficients by a simple change of the independent variable. Problems 35 through

42 are examples of this type of equation. Problem 43 determines conditions under which the more general Eq. (i) can be transformed into a differential equation with constant coefficients. Problems 44 through 46 give specific applications of this procedure.

34. **Euler Equations.** An equation of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \qquad t > 0, \tag{ii}$$

where α and β are real constants, is called an Euler equation.

(a) Let $x = \ln t$ and calculate dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2.

(b) Use the results of part (a) to transform Eq. (ii) into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{iii}$$

Observe that Eq. (iii) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of Eq. (iii), then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of Eq. (ii).

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

35. $t^2y'' + ty' + y = 0$
36. $t^2y'' + 4ty' + 2y = 0$
37. $t^2y'' + 3ty' + 1.25y = 0$
38. $t^2y'' - 4ty' - 6y = 0$
39. $t^2y'' - 4ty' + 6y = 0$
40. $t^2y'' - ty' + 5y = 0$
41. $t^2y'' + 3ty' - 3y = 0$
42. $t^2y'' + 7ty' + 10y = 0$

43. In this problem we determine conditions on p and q that enable Eq. (i) to be transformed into an equation with constant coefficients by a change of the independent variable. Let $x = u(t)$ be the new independent variable, with the relation between x and t to be specified later.

(a) Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \qquad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Show that the differential equation (i) becomes

$$\left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \tag{iv}$$

(c) In order for Eq. (iv) to have constant coefficients, the coefficients of d^2y/dx^2 and of y must be proportional. If $q(t) > 0$, then we can choose the constant of proportionality to be 1; hence

$$x = u(t) = \int [q(t)]^{1/2}\,dt. \tag{v}$$

(d) With x chosen as in part (c), show that the coefficient of dy/dx in Eq. (iv) is also a constant, provided that the expression

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} \tag{vi}$$

is a constant. Thus Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function $(q' + 2pq)/q^{3/2}$ is a constant. How must this result be modified if $q(t) < 0$?

In each of Problems 44 through 46, try to transform the given equation into one with constant coefficients by the method of Problem 43. If this is possible, find the general solution of the given equation.

44. $y'' + ty' + e^{-t^2}y = 0, \qquad -\infty < t < \infty$
45. $y'' + 3ty' + t^2y = 0, \qquad -\infty < t < \infty$
46. $ty'' + (t^2 - 1)y' + t^3y = 0, \qquad 0 < t < \infty$

3.4 Repeated Roots; Reduction of Order

In earlier sections we showed how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

either are real and different or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case is transitional between the other two and occurs when the discriminant $b^2 - 4ac$ is zero. Then it follows from the quadratic formula that

$$r_1 = r_2 = -b/2a. \tag{3}$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/2a} \tag{4}$$

of the differential equation (1), and it is not obvious how to find a second solution.

EXAMPLE 1

Solve the differential equation

$$y'' + 4y' + 4y = 0. \tag{5}$$

The characteristic equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0,$$

so $r_1 = r_2 = -2$. Therefore, one solution of Eq. (5) is $y_1(t) = e^{-2t}$. To find the general solution of Eq. (5), we need a second solution that is not a constant multiple of y_1. This second solution can be found in several ways (see Problems 20 through 22); here we use a method originated by D'Alembert[6] in the eighteenth century. Recall that since $y_1(t)$ is a solution of Eq. (1), so is $cy_1(t)$ for any constant c. The basic idea is to generalize this observation by replacing c by a

[6]Jean d'Alembert (1717–1783), a French mathematician, was a contemporary of Euler and Daniel Bernoulli and is known primarily for his work in mechanics and differential equations. D'Alembert's principle in mechanics and d'Alembert's paradox in hydrodynamics are named for him, and the wave equation first appeared in his paper on vibrating strings in 1747. In his later years he devoted himself primarily to philosophy and to his duties as science editor of Diderot's *Encyclopédie*.