25. (a) By the method of variation of parameters, show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t-\tau)\, d\tau.$$

(b) Show that if $f(t) = \delta(t-\pi)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t-\pi).$$

(c) Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t-\pi)$, and confirm that the solution agrees with the result of part (b).

6.6 The Convolution Integral

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other transforms $F(s)$ and $G(s)$, the latter transforms corresponding to known functions f and g, respectively. In this event, we might anticipate that $H(s)$ would be the transform of the product of f and g. However, this is not the case; in other words, the Laplace transform cannot be commuted with ordinary multiplication. On the other hand, if an appropriately defined "generalized product" is introduced, then the situation changes, as stated in the following theorem.

Theorem 6.6.1 If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a, \tag{1}$$

where

$$h(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau = \int_0^t f(\tau) g(t-\tau)\, d\tau. \tag{2}$$

The function h is known as the convolution of f and g; the integrals in Eq. (2) are called convolution integrals.

The equality of the two integrals in Eq. (2) follows by making the change of variable $t - \tau = \xi$ in the first integral. Before giving the proof of this theorem, let us make some observations about the convolution integral. According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f * g)(t). \tag{3}$$

In particular, the notation $(f * g)(t)$ serves to indicate the first integral appearing in Eq. (2).

The convolution $f * g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f * g = g * f \qquad \text{(commutative law)} \tag{4}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)} \tag{5}$$

$$(f * g) * h = f * (g * h) \qquad \text{(associative law)} \tag{6}$$

$$f * 0 = 0 * f = 0. \tag{7}$$

In Eq. (7) the zeros denote not the number 0 but the function that has the value 0 for each value of t. The proofs of these properties are left to you as exercises.

However, there are other properties of ordinary multiplication that the convolution integral does not have. For example, it is not true in general that $f * 1$ is equal to f. To see this, note that

$$(f * 1)(t) = \int_0^t f(t - \tau) \cdot 1 \, d\tau = \int_0^t f(t - \tau) \, d\tau.$$

If, for example, $f(t) = \cos t$, then

$$\begin{aligned} (f * 1)(t) &= \int_0^t \cos(t - \tau) \, d\tau = -\sin(t - \tau)\Big|_{\tau=0}^{\tau=t} \\ &= -\sin 0 + \sin t \\ &= \sin t. \end{aligned}$$

Clearly, $(f * 1)(t) \neq f(t)$ in this case. Similarly, it may not be true that $f * f$ is nonnegative. See Problem 3 for an example.

Convolution integrals arise in various applications in which the behavior of the system at time t depends not only on its state at time t but also on its past history. Systems of this kind are sometimes called hereditary systems and occur in such diverse fields as neutron transport, viscoelasticity, and population dynamics, among others.

Turning now to the proof of Theorem 6.6.1, we note first that if

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi) \, d\xi$$

and

$$G(s) = \int_0^\infty e^{-s\tau} g(\tau) \, d\tau,$$

then

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi) \, d\xi \int_0^\infty e^{-s\tau} g(\tau) \, d\tau. \tag{8}$$

Since the integrand of the first integral does not depend on the integration variable of the second, we can write $F(s)G(s)$ as an iterated integral

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\tau} g(\tau) \left[\int_0^\infty e^{-s\xi} f(\xi) \, d\xi \right] d\tau \\ &= \int_0^\infty g(\tau) \left[\int_0^\infty e^{-s(\xi+\tau)} f(\xi) \, d\xi \right] d\tau. \end{aligned} \tag{9}$$

The latter integral can be put into a more convenient form by introducing a change of variable. Let $\xi = t - \tau$, for fixed τ, so that $d\xi = dt$. Further, $\xi = 0$ corresponds to $t = \tau$, and $\xi = \infty$ corresponds to $t = \infty$; then the integral with respect to ξ in Eq. (9) is transformed into one with respect to t:

$$F(s)G(s) = \int_0^\infty g(\tau)\left[\int_\tau^\infty e^{-st} f(t-\tau)\, dt\right] d\tau. \tag{10}$$

The iterated integral on the right side of Eq. (10) is carried out over the shaded wedge-shaped region extending to infinity in the $t\tau$-plane shown in Figure 6.6.1. Assuming that the order of integration can be reversed, we rewrite Eq. (10) so that the integration with respect to τ is executed first. In this way we obtain

$$F(s)G(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)\, d\tau\right] dt \tag{11}$$

or

$$F(s)G(s) = \int_0^\infty e^{-st} h(t)\, dt = \mathcal{L}\{h(t)\}, \tag{12}$$

where $h(t)$ is defined by Eq. (2). This completes the proof of Theorem 6.6.1.

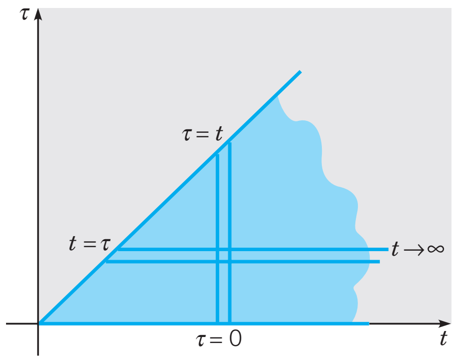

FIGURE 6.6.1 Region of integration in $F(s)G(s)$.

EXAMPLE 1

Find the inverse transform of

$$H(s) = \frac{a}{s^2(s^2 + a^2)}. \tag{13}$$

It is convenient to think of $H(s)$ as the product of s^{-2} and $a/(s^2 + a^2)$, which, according to lines 3 and 5 of Table 6.2.1, are the transforms of t and $\sin at$, respectively. Hence, by Theorem 6.6.1, the inverse transform of $H(s)$ is

$$h(t) = \int_0^t (t-\tau)\sin a\tau\, d\tau = \frac{at - \sin at}{a^2}. \tag{14}$$

You can verify that the same result is obtained if $h(t)$ is written in the alternative form

$$h(t) = \int_0^t \tau \sin a(t-\tau)\, d\tau,$$

which confirms Eq. (2) in this case. Of course, $h(t)$ can also be found by expanding $H(s)$ in partial fractions.

EXAMPLE 2

Find the solution of the initial value problem

$$y'' + 4y = g(t), \tag{15}$$
$$y(0) = 3, \qquad y'(0) = -1. \tag{16}$$

By taking the Laplace transform of the differential equation and using the initial conditions, we obtain

$$s^2 Y(s) - 3s + 1 + 4Y(s) = G(s)$$

or

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}. \tag{17}$$

Observe that the first and second terms on the right side of Eq. (17) contain the dependence of $Y(s)$ on the initial conditions and forcing function, respectively. It is convenient to write $Y(s)$ in the form

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s). \tag{18}$$

Then, using lines 5 and 6 of Table 6.2.1 and Theorem 6.6.1, we obtain

$$y = 3\cos 2t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau. \tag{19}$$

If a specific forcing function g is given, then the integral in Eq. (19) can be evaluated (by numerical means, if necessary).

Example 2 illustrates the power of the convolution integral as a tool for writing the solution of an initial value problem in terms of an integral. In fact, it is possible to proceed in much the same way in more general problems. Consider the problem consisting of the differential equation

$$ay'' + by' + cy = g(t), \tag{20}$$

where $a, b,$ and c are real constants and g is a given function, together with the initial conditions

$$y(0) = y_0, \qquad y'(0) = y_0'. \tag{21}$$

The transform approach yields some important insights concerning the structure of the solution of any problem of this type.

The initial value problem (20), (21) is often referred to as an input–output problem. The coefficients $a, b,$ and c describe the properties of some physical system, and $g(t)$ is the input to the system. The values y_0 and y_0' describe the initial state, and the solution y is the output at time t.

By taking the Laplace transform of Eq. (20) and using the initial conditions (21), we obtain

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_0' = G(s).$$

If we let

$$\Phi(s) = \frac{(as+b)y_0 + ay_0'}{as^2+bs+c}, \qquad \Psi(s) = \frac{G(s)}{as^2+bs+c}, \tag{22}$$

then we can write

$$Y(s) = \Phi(s) + \Psi(s). \tag{23}$$

Consequently,

$$y = \phi(t) + \psi(t), \tag{24}$$

where $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ and $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$. Observe that $\phi(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{25}$$

obtained from Eqs. (20) and (21) by setting $g(t)$ equal to zero. Similarly, $\psi(t)$ is the solution of

$$ay'' + by' + cy = g(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{26}$$

in which the initial values y_0 and y_0' are each replaced by zero.

Once specific values of $a, b,$ and c are given, we can find $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ by using Table 6.2.1, possibly in conjunction with a translation or a partial fraction expansion. To find $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$, it is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s), \tag{27}$$

where $H(s) = (as^2 + bs + c)^{-1}$. The function H is known as the **transfer function**[5] and depends only on the properties of the system under consideration; that is, $H(s)$ is determined entirely by the coefficients $a, b,$ and c. On the other hand, $G(s)$ depends only on the external excitation $g(t)$ that is applied to the system. By the convolution theorem we can write

$$\psi(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)\,d\tau, \tag{28}$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and $g(t)$ is the given forcing function.

To obtain a better understanding of the significance of $h(t)$, we consider the case in which $G(s) = 1$; consequently, $g(t) = \delta(t)$ and $\Psi(s) = H(s)$. This means that $y = h(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = \delta(t), \qquad y(0) = 0, \quad y'(0) = 0, \tag{29}$$

obtained from Eq. (26) by replacing $g(t)$ by $\delta(t)$. Thus $h(t)$ is the response of the system to a unit impulse applied at $t = 0$, and it is natural to call $h(t)$ the **impulse response** of the system. Equation (28) then says that $\psi(t)$ is the convolution of the impulse response and the forcing function.

Referring to Example 2, we note that in that case, the transfer function is $H(s) = 1/(s^2 + 4)$ and the impulse response is $h(t) = (\sin 2t)/2$. Also, the first two terms on the right side of Eq. (19) constitute the function $\phi(t)$, the solution of the corresponding homogeneous equation that satisfies the given initial conditions.

PROBLEMS

1. Establish the commutative, distributive, and associative properties of the convolution integral.
   (a) $f * g = g * f$
   (b) $f * (g_1 + g_2) = f * g_1 + f * g_2$
   (c) $f * (g * h) = (f * g) * h$

[5]This terminology arises from the fact that $H(s)$ is the ratio of the transforms of the output and the input of the problem (26).

2. Find an example different from the one in the text showing that $(f * 1)(t)$ need not be equal to $f(t)$.
3. Show, by means of the example $f(t) = \sin t$, that $f * f$ is not necessarily nonnegative.

In each of Problems 4 through 7, find the Laplace transform of the given function.

4. $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau$
5. $f(t) = \int_0^t e^{-(t-\tau)} \sin \tau \, d\tau$
6. $f(t) = \int_0^t (t-\tau)e^{\tau} \, d\tau$
7. $f(t) = \int_0^t \sin(t-\tau) \cos \tau \, d\tau$

In each of Problems 8 through 11, find the inverse Laplace transform of the given function by using the convolution theorem.

8. $F(s) = \dfrac{1}{s^4(s^2+1)}$
9. $F(s) = \dfrac{s}{(s+1)(s^2+4)}$
10. $F(s) = \dfrac{1}{(s+1)^2(s^2+4)}$
11. $F(s) = \dfrac{G(s)}{s^2+1}$
12. (a) If $f(t) = t^m$ and $g(t) = t^n$, where m and n are positive integers, show that

$$f * g = t^{m+n+1} \int_0^1 u^m (1-u)^n \, du.$$

(b) Use the convolution theorem to show that

$$\int_0^1 u^m (1-u)^n \, du = \frac{m!\,n!}{(m+n+1)!}.$$

(c) Extend the result of part (b) to the case where m and n are positive numbers but not necessarily integers.

In each of Problems 13 through 20, express the solution of the given initial value problem in terms of a convolution integral.

13. $y'' + \omega^2 y = g(t); \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' + 2y' + 2y = \sin \alpha t; \quad y(0) = 0, \quad y'(0) = 0$
15. $4y'' + 4y' + 17y = g(t); \quad y(0) = 0, \quad y'(0) = 0$
16. $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \quad y(0) = 1, \quad y'(0) = -1$
17. $y'' + 4y' + 4y = g(t); \quad y(0) = 2, \quad y'(0) = -3$
18. $y'' + 3y' + 2y = \cos \alpha t; \quad y(0) = 1, \quad y'(0) = 0$
19. $y^{(4)} - y = g(t); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
20. $y^{(4)} + 5y'' + 4y = g(t); \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
21. Consider the equation

$$\phi(t) + \int_0^t k(t-\xi)\phi(\xi) \, d\xi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as Volterra integral equations. Take the Laplace transform of the given integral equation and obtain an expression for $\mathcal{L}\{\phi(t)\}$ in terms of the transforms $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{k(t)\}$ of the given functions f and k. The inverse transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

22. Consider the Volterra integral equation (see Problem 21)

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin 2t. \tag{i}$$

(a) Solve the integral equation (i) by using the Laplace transform.

(b) By differentiating Eq. (i) twice, show that $\phi(t)$ satisfies the differential equation

$$\phi''(t) + \phi(t) = -4\sin 2t.$$

Show also that the initial conditions are

$$\phi(0) = 0, \qquad \phi'(0) = 2.$$

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

In each of Problems 23 through 25:

(a) Solve the given Volterra integral equation by using the Laplace transform.

(b) Convert the integral equation into an initial value problem, as in Problem 22(b).

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

23. $\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = 1$

24. $\phi(t) - \int_0^t (t-\xi)\phi(\xi)\,d\xi = 1$

25. $\phi(t) + 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = e^{-t}$

There are also equations, known as **integro-differential equations**, in which both derivatives and integrals of the unknown function appear. In each of Problems 26 through 28:

(a) Solve the given integro-differential equation by using the Laplace transform.

(b) By differentiating the integro-differential equation a sufficient number of times, convert it into an initial value problem.

(c) Solve the initial value problem in part (b), and verify that the solution is the same as the one in part (a).

26. $\phi'(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = t, \qquad \phi(0) = 0$

27. $\phi'(t) - \frac{1}{2}\int_0^t (t-\xi)^2\phi(\xi)\,d\xi = -t, \qquad \phi(0) = 1$

28. $\phi'(t) + \phi(t) = \int_0^t \sin(t-\xi)\phi(\xi)\,d\xi, \qquad \phi(0) = 1$

29. **The Tautochrone.** A problem of interest in the history of mathematics is that of finding the tautochrone[6]—the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its starting point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens (1629–1695) in 1673 by geometrical methods, and later by Leibniz and Jakob Bernoulli using analytical arguments. Bernoulli's solution (in 1690) was one of the first occasions on which

[6]The word "tautochrone" comes from the Greek words *tauto*, which means "same," and *chronos*, which means "time."

a differential equation was explicitly solved. The geometric configuration is shown in Figure 6.6.2. The starting point $P(a, b)$ is joined to the terminal point $(0, 0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}. \tag{i}$$

Then it follows from the principle of conservation of energy that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}} \int_0^b \frac{f(y)}{\sqrt{b-y}}\, dy. \tag{ii}$$

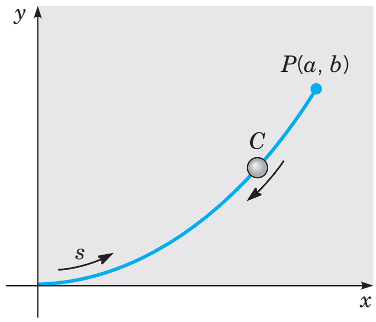

FIGURE 6.6.2 The tautochrone.

(a) Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of Eq. (ii) in this case, and using the convolution theorem, show that

$$F(s) = \sqrt{\frac{2g}{\pi}} \frac{T_0}{\sqrt{s}}; \tag{iii}$$

then show that

$$f(y) = \frac{\sqrt{2g}}{\pi} \frac{T_0}{\sqrt{y}}. \tag{iv}$$

Hint: See Problem 31 of Section 6.1.

(b) Combining Eqs. (i) and (iv), show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}, \tag{v}$$

where $\alpha = gT_0^2/\pi^2$.

(c) Use the substitution $y = 2\alpha \sin^2(\theta/2)$ to solve Eq. (v), and show that

$$x = \alpha(\theta + \sin\theta), \qquad y = \alpha(1 - \cos\theta). \tag{vi}$$

Equations (vi) can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

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The books listed below contain additional information on the Laplace transform and its applications.

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A further discussion of generalized functions can be found in

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