Introduction to Differential Equations
Assignment # 5

Date Given: May 9, 2022
Date Due: May 16, 2022

P1. (1 point) Find a differential equation whose general solution is $y = c_1 e^{-3t}\cos 4t + c_2 e^{-3t}\sin 4t$.

Solution: An algebraic equation with roots $-3+4\imath$ and $-3-4\imath$ is $\{r-(-3+4\imath)\}\{r-(-3-4\imath)\} = r^2 - 6r + 25 = 0$. This is the characteristic equation for the differential equation

$$y'' + 6y' + 25y = 0.$$

P2. (3 points) Find the general solution of the following differential equations

(a) $9y'' + 4y = 0$

(b) $y'' - 4y' + 5y = 0$

(c) $2y'' + 2y' + y = 0$

Solution:

(a) The characteristic equation is $9r^2 + 4 = 0$. Therefore the roots are $r = \pm\frac{2}{3}\imath$. Thus we arrive at the general solution

$$y(t) = (c_1 \cos \frac{2}{3}t + c_2 \sin \frac{2}{3}t).$$

(b) The characteristic equation is $r^2 - 4r + 5 = 0$. Therefore the roots are $r = 2 \pm \imath$. Thus we arrive at the general solution

$$y(t) = e^{2t}(c_1 \cos t + c_2 \sin t).$$

(c) The characteristic equation is $2r^2 + 2r + 1 = 0$. Therefore the roots are $r = \frac{1}{2}(-1 \pm \imath)$. Thus we arrive at the general solution

$$y(t) = e^{-t/2}(c_1 \cos \frac{1}{2}t + c_2 \sin \frac{1}{2}t).$$

P3. (2 points) Find the solution of the initial value problem $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$. Sketch the graph of the solution and describe its behavior for increasing t.

Solution: The characteristic equation is $r^2 - 2r + 5 = 0$, with the roots $r = 1 \pm 2\imath$. Hence the general solution is $y(t) = e^t(c_1 \cos 2t + c_2 \sin 2t)$.

Based on the initial condition $y(\pi/2) = 0$, we require $c_1 = 0$. It follows that $y(t) = c_2 e^t \sin 2t$, and so the first derivative is $y'(t) = c_2 e^t \sin 2t + 2c_2 e^t \cos 2t$. In order to satisfy the condition $y'(\pi/2) = 2$, we find that $-2c_2 e^{\pi/2} = 2$. Hence we have $c_2 = -e^{-\pi/2}$. Therefore the specific solution is

$$y(t) = -e^{(t-\pi/2)} \sin 2t.$$

The solution oscillates with an exponentially growing amplitude(see Figure 1).

P4. (2 points) Use the method of Problem 34 in Section 3.3 (Euler's equations) to solve the following differential equations

(a) $t^2 y'' + 4ty' + 2y = 0$ for $t > 0$

(b) $t^2 y'' + 2ty' + 0.25y = 0$ for $t > 0$

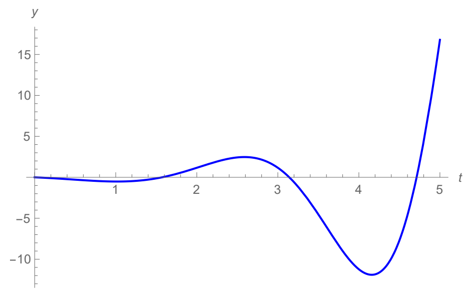

Figure 1: Illustration to problem P3.

Solution:

(a) Upon substitution $x = \ln t$ the equation transforms to $\mathrm{d}^2y/\mathrm{d}x + 3\mathrm{d}y/\mathrm{d}x + 2y = 0$. The characteristic roots are $r = -1, -2$. The solution is

$$y(x) = c_1e^{-x} + c_2e^{-2x} \quad \Longrightarrow \quad y(t) = c_1e^{-\ln t} + c_2e^{-2\ln t} = \frac{c_1}{t} + \frac{c_2}{t^2}.$$

(b) Upon substitution $x = \ln t$ the equation transforms to $\mathrm{d}^2y/\mathrm{d}x + \mathrm{d}y/\mathrm{d}x + (1/4)y = 0$. We get a repeated root $r = -1/2$. The solution is

$$y(x) = c_1e^{-x/2} + c_2xe^{-x/2} \quad \Longrightarrow \quad y(t) = c_1e^{-(\ln t)/2} + c_2 \ln te^{-(\ln t)/2} = c_1t^{-1/2} + c_2t^{-1/2}\ln t.$$

P5. (1 point) Find the general solution of the differential equation $25y'' - 20y' + 4y = 0$.

Solution: The characteristic equation is given by $25r^2 - 20r + 4 = 0$. Therefore we have one repeated root $r = 2/5$, and the general solution is given by

$$y(t) = c_1e^{2t/5} + c_2te^{2t/5}.$$

P6. (2 points) Solve the initial value problem $y'' - 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 2$. Sketch the graph of the solution and describe its behavior for increasing t.

Solution: The characteristic equation is given by $r^2 - 6r + 9 = 0$ Therefore we have one repeated root $r = 3$, and the general solution is given by $y(t) = c_1e^{3t} + c_2te^{3t}$. After differentiation $y'(t) = 3c_1e^{3t} + c_2(1 + 3t)e^{3t}$. From the initial conditions we get $c_1 = 0$ and $3c_1 + c_2 = 2$. Therefore $c_2 = 2$, and the specific solution is

$$y(t) = 2te^{3t}.$$

The solution $y \to \infty$ as $t \to \infty$ (see Figure 2).

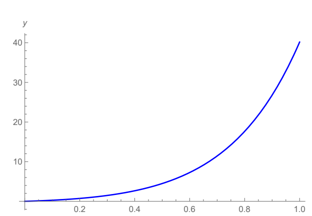

Figure 2: Illustration to problem P6.