Ch 7.4: Basic Theory of Systems of First Order Linear Equations

- The general theory of a system of n first order linear equations

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$

  parallels that of a single nth order linear equation.
- This system can be written as $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}$$

Vector Solutions of an ODE System

- A vector $\mathbf{x} = \phi(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ if the components of $\mathbf{x}$,
$$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t),$$
satisfy the system of equations on $I : \alpha < t < \beta$.
- For comparison, recall that $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ represents our system of equations
$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$
- Assuming $\mathbf{P}$ and $\mathbf{g}$ continuous on I, such a solution exists by Theorem 7.1.2.

Homogeneous Case; Vector Function Notation

- As in Chapters 3 and 4, we first examine the general homogeneous equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.
- Also, the following notation for the vector functions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)},\ldots, \mathbf{x}^{(k)},\ldots$ will be used:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \ldots, \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}, \ldots$$

Theorem 7.4.1

- If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

- Note: By repeatedly applying the result of this theorem, it can be seen that every finite linear combination

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_k\mathbf{x}^{(k)}(t)$$

  of solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ is itself a solution to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Theorem 7.4.2

- If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ for each point in $I: \alpha < t < \beta$, then each solution $\mathbf{x} = \phi(t)$ can be expressed uniquely in the form

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t)$$

- If solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent for each point in $I: \alpha < t < \beta$, then they are **fundamental solutions on *I***, and the **general solution** is given by

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t)$$

The Wronskian and Linear Independence

- The proof of Thm 7.4.2 uses the fact that if $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$,…, $\mathbf{x}^{(n)}$ are linearly independent on I, then $\det\mathbf{X}(t) \neq 0$ on I, where

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix},$$

- The Wronskian of $\mathbf{x}^{(1)}$,…, $\mathbf{x}^{(n)}$ is defined as

$$W[\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}](t) = \det\mathbf{X}(t).$$

- It follows that $W[\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}](t) \neq 0$ on I iff $\mathbf{x}^{(1)}$,…, $\mathbf{x}^{(n)}$ are linearly independent for each point in I.

Theorem 7.4.3

- If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $I: \alpha < t < \beta$, then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t)$ is either identically zero on I or else is never zero on I.
- This result relies on Abel's formula for the Wronskian

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn}) \Rightarrow W(t) = ce^{\int [p_{11}(t) + p_{22}(t) + \cdots + p_{nn}(t)]dt}$$

where c is an arbitrary constant (Refer to Section 3.2)

- This result enables us to determine whether a given set of solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are fundamental solutions by evaluating $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t)$ at *any* point t in $\alpha < t < \beta$.

Theorem 7.4.4

- Let

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots,\ \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ be solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, $\alpha < t < \beta$, that satisfy the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)},\ \ldots,\ \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)},$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are form a fundamental set of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Theorem 7.4.5

- Consider the system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

where each element of **P** is a real-valued continuous function. If **x =** **u**(t) + iv(t) is a complex-valued solution of Eq. (3), then its real part **u**(t) and its imaginary part v(t) are also solutions of this equation.