Introduction to Differential Equations
Assignment # 3

Date Given: April 25, 2022
Date Due: May 2, 2022

P1. (1 point) Determine (without solving the problem) an interval in which the solution of the initial value problem $(t-3)y' + (\ln t)y = 2t$, $y(1) = 2$ is certain to exist.

Solution: If this equation is written in the standard form $y' + p(t)y = g(t)$, then $p(t) = (\ln t)/(t-3)$ and $g(t) = 2t/(t-3)$. These are defined and continuous on the intervals $(0, 3)$ and $(3, \infty)$, but since the initial point is $t = 1$, the solution will be continuous on $0 < t < 3$.

P2. (1 point) Determine (without solving the problem) an interval in which the solution of the initial value problem $(4-t^2)y' + 2ty = 3t^2$, $y(-3) = 1$ is certain to exist.

Solution: If this equation is written in the standard form $y' + p(t)y = g(t)$, then $p(t) = 2t/(4-t^2)$ and $g(t) = 3t^2/(4-t^2)$, which have discontinuities at $t = \pm 2$. Since the initial point $t_0 = -3$, the solution will be continuous on $-\infty < t < -2$.

P3. (2 points) Determine whether the equation $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$ is exact or not. If it is exact, find the solution.

Solution: Here $M(x,y) = 3x^2 - 2xy + 2$ and $N(x,y) = 6y^2 - x^2 + 3$. Since $\partial M/\partial y = -2x = \partial N/\partial x$, the equation is exact.

- Since $\partial\psi/\partial x = M = 3x^2 - 2xy + 2$, to solve for ψ, we integrate M with respect to x and obtain $\psi = x^3 - x^2 y + 2x + h(y)$.
- Then $\partial\psi/\partial y = -x^2 + h'(y) = N = 6y^2 - x^2 + 3$ implies that $h'(y) = 6y^2 + 3$. Therefore $h(y) = 2y^3 + 3y$ and $\psi(x,y) = x^3 - x^2y + 2x + 2y^3 + 3y$. Thus, the solution of the equation, written in the implicit form, can be represented as

$$x^3 - x^2y + 2x + 2y^3 + 3y = C.$$

P4. (2 points) Determine whether the equation

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x) + (xe^{xy}\cos 2x - 3)\,y' = 0$$

is exact or not. If it is exact, find the solution.

Solution: Here $M(x,y) = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$ and $N(x,y) = xe^{xy}\cos 2x - 3$. Since $\partial M/\partial y = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x = \partial N/\partial x$, the equation is exact.

- If we try to find $\psi(x,y)$ by integrating $M(x,y)$ with respect to x, as in the conventional scheme, we must integrate by parts. It leads to longer (but still correct) computations. To reduce the amount of computations, we employ an alternative scheme in which the roles of x and y are interchanged. Specifically, we first find $\psi(x,y)$ by integrating $N(x,y)$ with respect to y.
- Since $\partial\psi/\partial y = N = xe^{xy}\cos 2x - 3$, to solve for ψ, we integrate N with respect to y and obtain $\psi = e^{xy}\cos 2x - 3y + h(x)$. Then we find $h(x)$ by differentiating $\psi(x,y)$ with respect to x and setting it equal to $M(x,y)$.
- Since $\partial\psi/\partial x = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + h'(x) = M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$ implies that $h'(x) = 2x$. Therefore $h(x) = x^2$ and $\psi(x,y) = e^{xy}\cos 2x - 3y + x^2$. Thus, the solution of the differential equation, written in the implicit form, can be represented as

$$e^{xy}\cos 2x - 3y + x^2 = C.$$

P5. (2 points) Solve the initial value problem

$$(2x - y) + (2y - x)\, y' = 0, \quad y(1) = 3$$

and determine the interval where the solution is valid.

Solution: Here $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Since $\partial M/\partial y = -1 = \partial N/\partial x$, the equation is exact.

- Since $\partial\psi/\partial x = M = 2x - y$, to solve for ψ, we integrate M with respect to x and obtain $\psi = x^2 - yx + h(y)$.
- Then $\partial\psi/\partial y = -x + h'(y) = N = 2y - x$ implies that $h'(y) = 2y$. Therefore $h(y) = y^2$ and $\psi(x,y) = x^2 - yx + y^2$. Thus, the solution of the equation, written in the implicit form, can be represented as $x^2 - yx + y^2 = C$.
- Since $y(1) = 3$, we have $C = 7$. Therefore

$$x^2 - yx + y^2 = 7.$$

- Resolving the implicit equation with respect to y, we obtain two explicit solutions (two curves), $y(x) = \left(x \pm \sqrt{28 - 3x^2}\right)/2$, but only one of them satisfies the given initial condition ($y(1) = 3$).
- Thus, the explicit form of the solution is $y(x) = \left(x + \sqrt{28 - 3x^2}\right)/2$. The solution is valid as long as $3x^2 \le 28$, that is
$$-2\sqrt{7/3} \le x \le 2\sqrt{7/3}.$$

P6. (2 points) Show that the equation $(x+2)\sin y + (x\cos y)\, y' = 0$ is not exact but becomes exact when multiplied by the integrating factor $\mu(x,y) = xe^x$. Then solve this equation.

Solution: Here $M(x,y) = (x+2)\sin y$ and $N(x,y) = x\cos y$. Since $\partial M/\partial y = (x+2)\cos y \neq \partial N/\partial x = \cos x$, <u>the equation is not exact</u>. Now, multiplying the equation by $\mu(x,y) = xe^x$, the equation becomes $\tilde{M}(x,y)\mathrm{d}x + \tilde{N}(x,y)\mathrm{d}y$, where $\tilde{M}(x,y) = (x^2+2x)e^x \sin y$ and $\tilde{N}(x,y) = x^2 e^x \cos y$. Now we see that for this equation $\partial\tilde{M}/\partial y = (x^2+2x)e^x\cos y = \partial\tilde{N}/\partial x$, so <u>the transformed equation is exact</u>.

- To solve this exact equation, it is easiest to integrate $\tilde{N}$ with respect to y. Since $\partial\psi/\partial y = \tilde{N} = x^2 e^x \cos y$, to solve for ψ, we integrate $\tilde{N}$ with respect to y and obtain $\psi(x,y) = x^2 e^x \sin y + g(x)$.
- Then $\partial\psi/\partial x = (x^2+2x)e^x \sin y + g'(x) = \tilde{M} = (x^2+2x)e^x \sin y$. Therefore $g'(x) = 0$. Therefore $g(x) = \text{const}$ and $\psi(x,y) = x^2 e^x \sin y + \text{const}$. Thus, the solution of the equation, written in the implicit form, can be represented as

$$x^2 e^x \sin y = C.$$