Ch 3.5. Nonhomogeneous Equations: Method of Undetermined Coefficients

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, g are continuous functions on an open interval I.

- The associated homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0$$

- In this section we will learn the method of undetermined coefficients to solve the nonhomogeneous equation, which relies on knowing solutions to the homogeneous equation.

Theorem 3.5.1

- If Y_1 and Y_2 are solutions of the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

then $Y_1 - Y_2$ is a solution of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

- If, in addition, $\{y_1, y_2\}$ forms a fundamental solution set of the homogeneous equation, then there exist constants c_1 and c_2 such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem 3.5.2 (General Solution)

- To solve the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

  we need to do three things:

1. Find the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous equation. This is called the **complementary solution** and may be denoted by $y_c(t)$.
2. Find any solution $Y(t)$ of the nonhomogeneous equation. This is often referred to as a **particular solution**.
3. Form the sum of the functions found in steps 1 and 2.

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

Method of Undetermined Coefficients

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

- In this section we use the method of **undetermined coefficients** to find a particular solution Y to the nonhomogeneous equation, assuming we can find solutions y_1, y_2 for the homogeneous case.
- The method of undetermined coefficients is usually limited to when p and q are constant, and $g(t)$ is a polynomial, exponential, sine or cosine function.

Example 1: Exponential *g*(*t*)

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}$$

- We seek Y satisfying this equation. Since exponentials replicate through differentiation, a good start for Y is:

$$Y(t) = Ae^{2t} \Rightarrow Y'(t) = 2Ae^{2t},\ Y''(t) = 4Ae^{2t}$$

- Substituting these derivatives into the differential equation,

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$\Leftrightarrow \quad -6Ae^{2t} = 3e^{2t} \quad \Leftrightarrow \quad A = -1/2$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = -\frac{1}{2}e^{2t}$$

Example 2: Sine $g(t)$, First Attempt (1 of 2)

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin t$$

- We seek Y satisfying this equation. Since sines replicate through differentiation, a good start for Y is:

$$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\; Y''(t) = -A\sin t$$

- Substituting these derivatives into the differential equation,

$$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t$$

$$\Leftrightarrow\; (2+5A)\sin t + 3A\cos t = 0$$

$$\Leftrightarrow\; c_1\sin t + c_2\cos t = 0$$

- Since sin(x) and cos(x) are not multiples of each other, we must have $c_1 = c_2 = 0$, and hence $2 + 5A = 3A = 0$, which is impossible.

$$y'' - 3y' - 4y = 2\sin t$$

Example 2: Sine $g(t)$, Particular Solution (2 of 2)

- Our next attempt at finding a Y is

$$Y(t) = A\sin t + B\cos t$$

$$\Rightarrow Y'(t) = A\cos t - B\sin t,\ Y''(t) = -A\sin t - B\cos t$$

- Substituting these derivatives into ODE, we obtain

$$\left(-A\sin t - B\cos t\right) - 3\left(A\cos t - B\sin t\right) - 4\left(A\sin t + B\cos t\right) = 2\sin t$$

$$\Leftrightarrow \left(-5A + 3B\right)\sin t + \left(-3A - 5B\right)\cos t = 2\sin t$$

$$\Leftrightarrow -5A + 3B = 2,\ -3A - 5B = 0$$

$$\Leftrightarrow A = -\frac{5}{17},\ B = \frac{3}{17}$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$$

Example 3: Product $g(t)$

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = -8e^t \cos(2t)$$

- We seek Y satisfying this equation, as follows:

$$Y(t) = Ae^t \cos(2t) + Be^t \sin(2t)$$

$$Y'(t) = Ae^t \cos(2t) - 2Ae^t \sin(2t) + Be^t \sin(2t) + 2Be^t \cos(2t)$$

$$= (A + 2B)e^t \cos(2t) + (-2A + B)e^t \sin(2t)$$

$$Y''(t) = (A + 2B)e^t \cos(2t) - 2(A + 2B)e^t \sin(2t) + (-2A + B)e^t \sin(2t)$$

$$+ 2(-2A + B)e^t \cos(2t)$$

$$= (-3A + 4B)e^t \cos(2t) + (-4A - 3B)e^t \sin(2t)$$

- Substituting these into the ODE and solving for A and B:

$$A = \frac{10}{13},\; B = \frac{2}{13} \quad \Rightarrow Y(t) = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t)$$

Discussion: Sum *g*(*t*)

- Consider again our general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

- Suppose that $g(t)$ is sum of functions:

$$g(t) = g_1(t) + g_2(t)$$

- If Y_1, Y_2 are solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$
$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

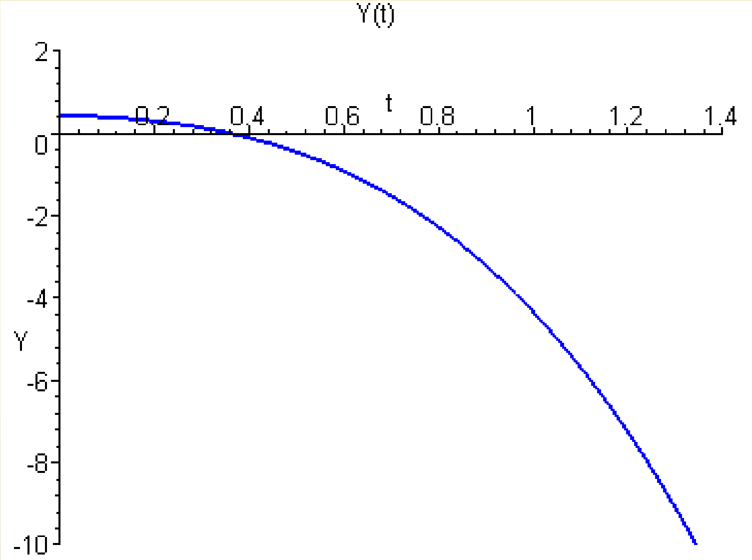
Example 4: Sum $g(t)$

- Consider the equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

- Our equations to solve individually are

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$



- Our particular solution is then

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

Example 5: First Attempt (1 of 3)

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 2e^{-t}$$

- We seek Y satisfying this equation. We begin with

$$Y(t) = Ae^{-t} \Rightarrow Y'(t) = -Ae^{-t},\ Y''(t) = Ae^{-t}$$

- Substituting these derivatives into differential equation,

$$(A + 3A - 4A)e^{-t} = 2e^{-t}$$

- Since the left side of the above equation is always 0, no value of A can be found to make $Y(t) = Ae^{-t}$ a solution to the nonhomogeneous equation.
- To understand why this happens, we will look at the solution of the corresponding homogeneous differential equation

Example 5: Homogeneous Solution (2 of 3)

- To solve the corresponding homogeneous equation:

$$y'' - 3y' - 4y = 0$$

- We use the techniques from Section 3.1 and get

$$y_1(t) = e^{-t} \text{and } y_2(t) = e^{4t}$$

- Thus our assumed particular solution $Y(t) = Ae^{-t}$ solves the homogeneous equation instead of the nonhomogeneous equation.
- So we need another form for $Y(t)$ to arrive at the general solution of the form:

$$y(t) = c_1 e^{-t} + c_2 e^{4t} + Y(t)$$

$$y'' - 3y' - 4y = 2e^{-t}$$

Example 5: Particular Solution

- Our next attempt at finding a $Y(t)$ is:

$$Y(t) = Ate^{-t}$$

$$Y'(t) = Ae^{-t} - Ate^{-t}$$

$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = Ate^{-t} - 2Ae^{-t}$$

- Substituting these into the ODE,

e^{-t}

$$Ate^{-t} - 2Ae^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$$

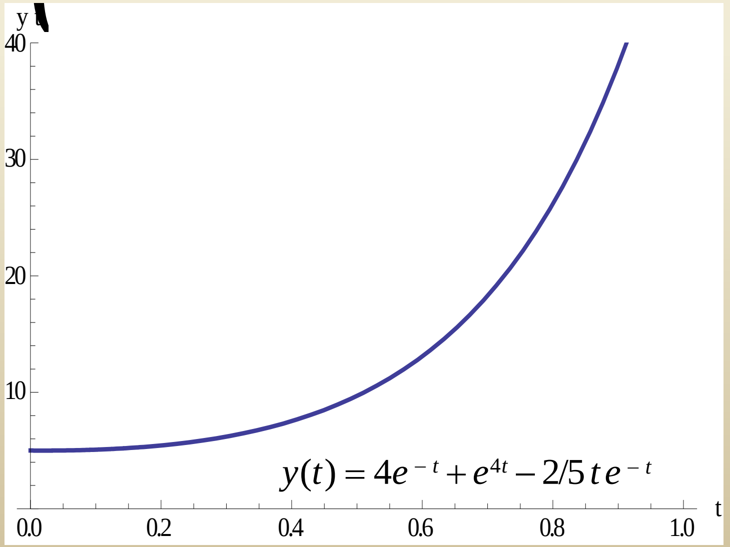
$$0 \cdot Ate^{-t} - 5Ate^{-t} = -5Ate^{-t} = 2e^{-t}$$

$$\Rightarrow A = -2/5$$

$$\Rightarrow Y(t) = -\frac{2}{5}te^{-t}$$

- So the general solution to the IVP is

$$y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{2}{5}te^{-t}$$



Summary – Undetermined Coefficients (1 of 3)

- For the differential equation
$$ay'' + by' + cy = g(t)$$
where a, b, and c are constants, if $g(t)$ belongs to the class of functions discussed in this section (involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of these), the method of undetermined coefficients may be used to find a particular solution to the nonhomogeneous equation.
- The first step is to find the general solution for the corresponding homogeneous equation with $g(t) = 0$.
$$y_C(t) = c_1 y_1(t) + c_2 y_2(t)$$

Summary – Undetermined Coefficients (2 of 3)

- The second step is to select an appropriate form for the particular solution, $Y(t)$, to the nonhomogeneous equation and determine the derivatives of that function.
- After substituting $Y(t)$, $Y'(t)$, and $Y''(t)$ into the nonhomogeneous differential equation, if the form for $Y(t)$ is correct, all the coefficients in $Y(t)$ can be determined.
- Finally, the general solution to the nonhomogeneous differential equation can be written as

$$y_{gen}(t) = y_C(t) + Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

Summary – Undetermined Coefficients (2 of 3)

- The particular solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$t^s\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_n\right)$
$P_n(t)e^{\alpha t}$	$t^s\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_n\right)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\cos\beta t\\ \sin\beta t\end{cases}$	$t^s\left\{\left(A_0 t^n + A_1 t^{n-1} + \cdots + A_n\right)e^{\alpha t}\cos\beta t + \right.$ $\left. + \left(B_0 t^n + B_1 t^{n-1} + \cdots + B_n\right)e^{\alpha t}\sin\beta t\right\}$

- 1st row: s is the number of times when 0 is a root of the characteristic equation
- 2nd row: s is the number of times when α is a root of the characteristic equation
- 3rd row: s is the number of times when $\alpha + i\beta$ is a root of the characteristic equation

Ch 3.6. Nonhomogeneous Equations: Method of Variation of Parameters

- Recall the nonhomogeneous equation
$$y'' + p(t)y' + q(t)y = g(t)$$
where p, q, g are continuous functions on an open interval I.
- The associated homogeneous equation is
$$y'' + p(t)y' + q(t)y = 0$$
- In this section we will learn the **variation of parameters** method to solve the nonhomogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to the homogeneous equation.
- Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals need to be evaluated, and this can present difficulties.

Example 1: Variation of Parameters (1 of 6)

- We seek a particular solution to the equation below.

$$y'' + 4y = 8\tan t, \quad -\pi/2 < t < \pi/2$$

- We cannot use the undetermined coefficients method since $g(t)$ is a quotient of $\sin t$ or $\cos t$, instead of a sum or product.
- Recall that the solution to the homogeneous equation is

$$y_C(t) = c_1\cos(2t) + c_2\sin(2t)$$

- To find a particular solution to the nonhomogeneous equation, we begin with the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$$

- Then

$$y'(t) = u_1'(t)\cos(2t) - 2u_1(t)\sin(2t) + u_2'(t)\sin(2t) + 2u_2(t)\cos(2t)$$

- or

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$$

Example 1: Derivatives, 2nd Equation (2 of 6)

- From the previous slide,

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$$

- Note that we need two equations to solve for u_1 and u_2. The first equation is the differential equation. To get a second equation, we will require

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

- Then

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t)$$

- Next,

$$y''(t) = -2u_1'(t)\sin(2t) - 4u_1(t)\cos(2t) + 2u_2'(t)\cos(2t) - 4u_2(t)\sin(2t)$$

Example 1: Two Equations (3 of 6)

- Recall that our differential equation is

$$y'' + 4y = 8\tan t$$

- Substituting y'' and y into this equation, we obtain

$$-2u_1'(t)\sin(2t) - 4u_1(t)\cos(2t) + 2u_2'(t)\cos(2t) - 4u_2(t)\sin(2t)$$
$$+ 4\big(u_1(t)\cos(2t) + u_2(t)\sin(2t)\big) = 8\tan t$$

- This equation simplifies to

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t$$

- Thus, to solve for u_1 and u_2, we have the two equations:

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t$$
$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

Example 1: Solve for u_1' (4 of 6)

- To find u_1 and u_2 , we first need to solve for u_1' and u_2'

$$-2u_1'(t)\sin(2t)+2u_2'(t)\cos(2t)=8\tan t$$
$$u_1'(t)\cos(2t)+u_2'(t)\sin(2t)=0$$

- From second equation,

$$u_2'(t)=-u_1'(t)\frac{\cos 2t}{\sin 2t}$$

- Substituting this into the first equation,

$$-2u_1'(t)\sin(2t)+2\left[-u_1'(t)\frac{\cos(2t)}{\sin(2t)}\right]\cos(2t)=8\tan t$$
$$-2u_1'(t)\sin^2\left(2t\right)-2u_1'(t)\cos^2\left(2t\right)=8\tan t\sin(2t)$$
$$-2u_1'(t)\left[\sin^2\left(2t\right)+\cos^2\left(2t\right)\right]=8\left[\frac{2\sin^2 t\cos t}{\cos t}\right]$$
$$u_1'(t)=-8\sin^2 t$$

Example 1: Solve for u_1 and u_2 (5 of 6)

- From the previous slide,

$$u_1'(t) = -8\sin^2 t, \quad u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$$

- Then

$$u_2'(t) = 8\sin^2 t\,\frac{\cos(2t)}{\sin(2t)} = 4\,\frac{\sin t(2\cos^2 t - 1)}{\cos t} = 4\sin t\left(2\cos t - \frac{1}{\cos t}\right)$$

- Thus

$$u_1(t) = \int u_1'(t)\,dt = 4\sin t\cos t - 4t + c_1$$

$$u_2(t) = \int u_2'(t)\,dt = 4\ln(\cos t) - 4\cos^2 t + c_2$$

Example 1: General Solution (6 of 6)

- Recall our equation and homogeneous solution y_C:

$$y'' + 4y = 8\tan t, \quad y_C(t) = c_1\cos(2t) + c_2\sin(2t)$$

- Using the expressions for u_1 and u_2 on the previous slide, the general solution to the differential equation is

$$\begin{aligned} y(t) &= u_1(t)\cos 2t + u_2(t)\sin 2t + y_C(t) \\ &= (4\sin t\cos t)\cos(2t) + (4\ln(\cos t) - 4\cos^2 t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \\ &= -2\sin(2t) - 4t\cos(2t) + 4\ln(\cos t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \end{aligned}$$

$$y'' + p(t)y' + q(t)y = g(t)$$

Summary

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

- Suppose y_1, y_2 are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation above, where we note that the coefficient on y'' is 1.
- To find u_1 and u_2, we need to solve the equations

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$
$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

- Doing so, and using the Wronskian, we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$

- Thus

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt + c_1, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt + c_2$$

Theorem 3.6.1

- Consider the equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

- If the functions *p*, *q* and *g* are continuous on an open interval *I*, and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt$$

and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$