CHAPTER

4

Higher Order Linear Equations

The theoretical structure and methods of solution developed in the preceding chapter for second order linear equations extend directly to linear equations of third and higher order. In this chapter we briefly review this generalization, taking particular note of those instances where new phenomena may appear, because of the greater variety of situations that can occur for equations of higher order.

4.1 General Theory of *n*th Order Linear Equations

An nth order linear differential equation is an equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t). \tag{1}$$

We assume that the functions $P_0, \dots, P_n$, and G are continuous real-valued functions on some interval $I\colon \alpha < t < \beta$, and that P_0 is nowhere zero in this interval. Then, dividing Eq. (1) by $P_0(t)$, we obtain

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t). \tag{2}$$

The linear differential operator L of order n defined by Eq. (2) is similar to the second order operator introduced in Chapter 3. The mathematical theory associated with Eq. (2) is completely analogous to that for the second order linear equation; for this reason we simply state the results for the nth order problem. The proofs of most of the results are also similar to those for the second order equation and are usually left as exercises.

Since Eq. (2) involves the nth derivative of y with respect to t, it will, so to speak, require n integrations to solve Eq. (2). Each of these integrations introduces an arbitrary constant. Hence we expect that to obtain a unique solution it is necessary to specify n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}, \tag{3}$$

where t_0 may be any point in the interval I and $y_0, y'_0, \ldots, y_0^{(n-1)}$ is any set of prescribed real constants. The following theorem, which is similar to Theorem 3.2.1, guarantees that the initial value problem (2), (3) has a solution and that it is unique.

Theorem 4.1.1 If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (2) that also satisfies the initial conditions (3), where t_0 is any point in I. This solution exists throughout the interval I.

We will not give a proof of this theorem here. However, if the coefficients $p_1, \ldots, p_n$ are constants, then we can construct the solution of the initial value problem (2), (3) much as in Chapter 3; see Sections 4.2 through 4.4. Even though we may find a solution in this case, we do not know that it is unique without the use of Theorem 4.1.1. A proof of the theorem can be found in Ince (Section 3.32) or Coddington (Chapter 6).

The Homogeneous Equation. As in the corresponding second order problem, we first discuss the homogeneous equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0. \tag{4}$$

If the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. (4), then it follows by direct computation that the linear combination

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t), \tag{5}$$

where $c_1, \ldots, c_n$ are arbitrary constants, is also a solution of Eq. (4). It is then natural to ask whether every solution of Eq. (4) can be expressed as a linear combination of $y_1, \ldots, y_n$. This will be true if, regardless of the initial conditions (3) that are prescribed, it is possible to choose the constants $c_1, \ldots, c_n$ so that the linear combination (5) satisfies the initial conditions. That is, for any choice of the point t_0 in I, and for any choice of $y_0, y'_0, \ldots, y_0^{(n-1)}$, we must be able to determine $c_1, \ldots, c_n$ so that the equations

$$\begin{aligned} c_1y_1(t_0) + \cdots + c_ny_n(t_0) &= y_0 \\ c_1y'_1(t_0) + \cdots + c_ny'_n(t_0) &= y'_0 \\ &\vdots \\ c_1y_1^{(n-1)}(t_0) + \cdots + c_ny_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned} \tag{6}$$

are satisfied. Equations (6) can be solved uniquely for the constants $c_1, \ldots, c_n$, provided that the determinant of coefficients is not zero. On the other hand, if the determinant of coefficients is zero, then it is always possible to choose values of

$y_0, y_0', \ldots, y_0^{(n-1)}$ so that Eqs. (6) do not have a solution. Therefore a necessary and sufficient condition for the existence of a solution of Eqs. (6) for arbitrary values of $y_0, y_0', \ldots, y_0^{(n-1)}$ is that the Wronskian

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{7}$$

is not zero at $t = t_0$. Since t_0 can be any point in the interval I, it is necessary and sufficient that $W(y_1, y_2, \ldots, y_n)$ be nonzero at every point in the interval. Just as for the second order linear equation, it can be shown that if $y_1, y_2, \ldots, y_n$ are solutions of Eq. (4), then $W(y_1, y_2, \ldots, y_n)$ either is zero for every t in the interval I or else is never zero there; see Problem 20. Hence we have the following theorem.

Theorem 4.1.2 If the functions $p_1, p_2, \ldots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. (4), and if $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for at least one point in I, then every solution of Eq. (4) can be expressed as a linear combination of the solutions $y_1, y_2, \ldots, y_n$.

A set of solutions $y_1, \ldots, y_n$ of Eq. (4) whose Wronskian is nonzero is referred to as a **fundamental set of solutions**. The existence of a fundamental set of solutions can be demonstrated in precisely the same way as for the second order linear equation (see Theorem 3.2.5). Since all solutions of Eq. (4) are of the form (5), we use the term **general solution** to refer to an arbitrary linear combination of any fundamental set of solutions of Eq. (4).

Linear Dependence and Independence. We now explore the relationship between fundamental sets of solutions and the concept of linear independence, a central idea in the study of linear algebra. The functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on an interval I if there exists a set of constants $k_1, k_2, \ldots, k_n$, not all zero, such that

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0 \tag{8}$$

for all t in I. The functions $f_1, \ldots, f_n$ are said to be **linearly independent** on I if they are not linearly dependent there.

EXAMPLE 1

Determine whether the functions $f_1(t) = 1$, $f_2(t) = t$, and $f_3(t) = t^2$ are linearly independent or dependent on the interval $I: -\infty < t < \infty$.

Form the linear combination

$$k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = k_1 + k_2 t + k_3 t^2,$$

and set it equal to zero to obtain

$$k_1 + k_2 t + k_3 t^2 = 0. \tag{9}$$

If Eq. (9) is to hold for all t in I, then it must certainly be true at any three distinct points in I. Any three points will serve our purpose, but it is convenient to choose $t = 0, t = 1$, and $t = -1$. Evaluating Eq. (9) at each of these points, we obtain the system of equations

$$\begin{aligned} k_1 \qquad\qquad &= 0, \\ k_1 + k_2 + k_3 &= 0, \\ k_1 - k_2 + k_3 &= 0. \end{aligned} \tag{10}$$

From the first of Eqs. (10) we note that $k_1 = 0$; then from the other two equations it follows that $k_2 = k_3 = 0$ as well. Therefore, there is no set of constants k_1, k_2, k_3, not all zero, for which Eq. (9) holds even at the three chosen points, much less throughout I. Thus the given functions are not linearly dependent on I, so they must be linearly independent. Indeed, they are linearly independent on any interval. This can be established just as in this example, possibly using a different set of three points.

EXAMPLE 2

Determine whether the functions

$$f_1(t) = 1, \quad f_2(t) = 2 + t, \quad f_3(t) = 3 - t^2, \quad \text{and} \quad f_4(t) = 4t + t^2$$

are linearly independent or dependent on any interval I.

Form the linear combination

$$\begin{aligned} k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) + k_4 f_4(t) &= k_1 + k_2(2 + t) + k_3(3 - t^2) + k_4(4t + t^2) \\ &= (k_1 + 2k_2 + 3k_3) + (k_2 + 4k_4)t + (-k_3 + k_4)t^2. \end{aligned} \tag{11}$$

For this expression to be zero throughout an interval, it is certainly sufficient to require that

$$k_1 + 2k_2 + 3k_3 = 0, \quad k_2 + 4k_4 = 0, \quad -k_3 + k_4 = 0.$$

These three equations, with four unknowns, have many solutions. For instance, if $k_4 = 1$, then $k_3 = 1$, $k_2 = -4$, and $k_1 = 5$. If we use these values for the coefficients in Eq. (11), then we have

$$5f_1(t) - 4f_2(t) + f_3(t) + f_4(t) = 0$$

for each value of t. Thus the given functions are linearly dependent on every interval.

The concept of linear independence provides an alternative characterization of fundamental sets of solutions of the homogeneous equation (4). Suppose that the functions $y_1, \ldots, y_n$ are solutions of Eq. (4) on an interval I, and consider the equation

$$k_1 y_1(t) + \cdots + k_n y_n(t) = 0. \tag{12}$$

By differentiating Eq. (12) repeatedly, we obtain the additional $n - 1$ equations

$$\begin{aligned} k_1 y_1'(t) + \cdots + k_n y_n'(t) &= 0, \\ &\vdots \\ k_1 y_1^{(n-1)}(t) + \cdots + k_n y_n^{(n-1)}(t) &= 0. \end{aligned} \tag{13}$$

The system consisting of Eqs. (12) and (13) is a system of n linear algebraic equations for the n unknowns $k_1, \ldots, k_n$. The determinant of coefficients for this system is the Wronskian $W(y_1, \ldots, y_n)(t)$ of $y_1, \ldots, y_n$. This leads to the following theorem.

Theorem 4.1.3 If $y_1(t), \ldots, y_n(t)$ is a fundamental set of solutions of Eq. (4)

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

on an interval I, then $y_1(t), \ldots, y_n(t)$ are linearly independent on I. Conversely, if $y_1(t), \ldots, y_n(t)$ are linearly independent solutions of Eq. (4) on I, then they form a fundamental set of solutions on I.

To prove this theorem, first suppose that $y_1(t), \ldots, y_n(t)$ is a fundamental set of solutions of Eq. (4) on I. Then the Wronskian $W(y_1, \ldots, y_n)(t) \neq 0$ for every t in I. Hence the system (12), (13) has only the solution $k_1 = \cdots = k_n = 0$ for every t in I. Thus $y_1(t), \ldots, y_n(t)$ cannot be linearly dependent on I and must therefore be linearly independent there.

To demonstrate the converse, let $y_1(t), \ldots, y_n(t)$ be linearly independent on I. To show that they form a fundamental set of solutions, we need to show that their Wronskian is never zero in I. Suppose that this is not true; then there is at least one point t_0 where the Wronskian is zero. At this point the system (12), (13) has a nonzero solution; let us denote it by $k_1^*, \ldots, k_n^*$. Now form the linear combination

$$\phi(t) = k_1^* y_1(t) + \cdots + k_n^* y_n(t). \tag{14}$$

Then $\phi(t)$ satisfies the initial value problem

$$L[y] = 0, \qquad y(t_0) = 0, \quad y'(t_0) = 0, \quad \ldots, \quad y^{(n-1)}(t_0) = 0. \tag{15}$$

The function ϕ satisfies the differential equation because it is a linear combination of solutions; it satisfies the initial conditions because these are just the equations in the system (12), (13) evaluated at t_0. However, the function $y(t) = 0$ for all t in I is also a solution of this initial value problem, and by Theorem 4.1.1, the solution is unique. Thus $\phi(t) = 0$ for all t in I. Consequently, $y_1(t), \ldots, y_n(t)$ are linearly dependent on I, which is a contradiction. Hence the assumption that there is a point where the Wronskian is zero is untenable. Therefore, the Wronskian is never zero on I, as was to be proved.

Note that for a set of functions $f_1, \ldots, f_n$ that are not solutions of Eq. (4), the converse part of Theorem 4.1.3 is not necessarily true. They may be linearly independent on I even though the Wronskian is zero at some points, or even every point, but with different sets of constants $k_1, \ldots, k_n$ at different points. See Problem 25 for an example.

The Nonhomogeneous Equation. Now consider the nonhomogeneous equation (2)

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t).$$

If Y_1 and Y_2 are any two solutions of Eq. (2), then it follows immediately from the linearity of the operator L that

$$L[Y_1 - Y_2](t) = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0.$$

Hence the difference of any two solutions of the nonhomogeneous equation (2) is a solution of the homogeneous equation (4). Since any solution of the homogeneous equation can be expressed as a linear combination of a fundamental set of solutions $y_1, \ldots, y_n$, it follows that any solution of Eq. (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \tag{16}$$

where Y is some particular solution of the nonhomogeneous equation (2). The linear combination (16) is called the general solution of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions $y_1, \ldots, y_n$ of the homogeneous equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

To find a particular solution $Y(t)$ in Eq. (16), the methods of undetermined coefficients and variation of parameters are again available. They are discussed and illustrated in Sections 4.3 and 4.4, respectively.

The method of reduction of order (Section 3.4) also applies to nth order linear equations. If y_1 is one solution of Eq. (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n - 1$ for v' (see Problem 26 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

PROBLEMS

In each of Problems 1 through 6, determine intervals in which solutions are sure to exist.

1. $y^{(4)} + 4y''' + 3y = t$
2. $ty''' + (\sin t)y'' + 3y = \cos t$
3. $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$
4. $y''' + ty'' + t^2 y' + t^3 y = \ln t$
5. $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$
6. $(x^2 - 4)y^{(6)} + x^2 y''' + 9y = 0$

In each of Problems 7 through 10, determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

7. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t$
8. $f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t$
9. $f_1(t) = 2t - 3, \quad f_2(t) = t^2 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$
10. $f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1$

In each of Problems 11 through 16, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

11. $y''' + y' = 0; \qquad 1, \quad \cos t, \quad \sin t$
12. $y^{(4)} + y'' = 0; \qquad 1, \quad t, \quad \cos t, \quad \sin t$
13. $y''' + 2y'' - y' - 2y = 0; \qquad e^t, \quad e^{-t}, \quad e^{-2t}$
14. $y^{(4)} + 2y''' + y'' = 0; \qquad 1, \quad t, \quad e^{-t}, \quad te^{-t}$
15. $xy''' - y'' = 0; \qquad 1, \quad x, \quad x^3$
16. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \qquad x, \quad x^2, \quad 1/x$
17. Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?
18. Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2],$$

where y_1 and y_2 are n-times-differentiable functions and c_1 and c_2 are arbitrary constants. Hence, show that if $y_1, y_2, \ldots, y_n$ are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \cdots + c_ny_n$ is also a solution of $L[y] = 0$.

19. Let the linear differential operator L be defined by

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny,$$

where $a_0, a_1, \ldots, a_n$ are real constants.
(a) Find $L[t^n]$.
(b) Find $L[e^{rt}]$.
(c) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Why?

20. In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2, and y_3 be solutions of this equation on an interval I.
(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.
(b) Substitute for y_1''', y_2''', and y_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c \exp\left[-\int p_1(t)\,dt\right].$$

It follows that W is either always zero or nowhere zero on I.
(d) Generalize this argument to the nth order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions $y_1, \ldots, y_n$. That is, establish Abel's formula

$$W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\,dt\right]$$

for this case.

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

21. $y''' + 2y'' - y' - 3y = 0$
22. $y^{(4)} + y = 0$
23. $ty''' + 2y'' - y' + ty = 0$
24. $t^2y^{(4)} + ty''' + y'' - 4y = 0$

25. (a) Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$.
(b) Show that $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$.
(c) Show that $W(f, g)(t)$ is zero for all t in $-1 < t < 1$.

26. Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second order equation for v':

$$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0.$$

In each of Problems 27 and 28, use the method of reduction of order (Problem 26) to solve the given differential equation.

27. $(2 - t)y''' + (2t - 3)y'' - ty' + y = 0, \quad t < 2; \qquad y_1(t) = e^t$
28. $t^2(t + 3)y''' - 3t(t + 2)y'' + 6(1 + t)y' - 6y = 0, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^3$

4.2 Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0, \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$. From our knowledge of second order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (1) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt}Z(r) \tag{2}$$

for all r, where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n. \tag{3}$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution of Eq. (1). The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (1). Since $a_0 \neq 0$, we know that $Z(r)$ is a polynomial of degree n and therefore has n zeros,[1] say, $r_1, r_2, \ldots, r_n$, some of which may be equal. Hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2)\cdots(r - r_n). \tag{4}$$

[1]An important question in mathematics for more than 200 years was whether every polynomial equation has at least one root. The affirmative answer to this question, the fundamental theorem of algebra, was given by Carl Friedrich Gauss (1777–1855) in his doctoral dissertation in 1799, although his proof does not meet modern standards of rigor. Several other proofs have been discovered since, including three by Gauss himself. Today, students often meet the fundamental theorem of algebra in a first course on complex variables, where it can be established as a consequence of some of the basic properties of complex analytic functions.

Real and Unequal Roots. If the roots of the characteristic equation are real and no two are equal, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ of Eq. (1). If these functions are linearly independent, then the general solution of Eq. (1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}. \tag{5}$$

One way to establish the linear independence of $e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$ is to evaluate their Wronskian determinant. Another way is outlined in Problem 40.

EXAMPLE 1

Find the general solution of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0. \tag{6}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = -2, \qquad y'''(0) = -1 \tag{7}$$

and plot its graph.

Assuming that $y = e^{rt}$, we must determine r by solving the polynomial equation

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \tag{8}$$

The roots of this equation are $r_1 = 1$, $r_2 = -1$, $r_3 = 2$, and $r_4 = -3$. Therefore, the general solution of Eq. (6) is

$$y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}. \tag{9}$$

The initial conditions (7) require that $c_1, \dots, c_4$ satisfy the four equations

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1, \\ c_1 - c_2 + 2c_3 - 3c_4 &= 0, \\ c_1 + c_2 + 4c_3 + 9c_4 &= -2, \\ c_1 - c_2 + 8c_3 - 27c_4 &= -1. \end{aligned} \tag{10}$$

By solving this system of four linear algebraic equations, we find that

$$c_1 = \tfrac{11}{8}, \qquad c_2 = \tfrac{5}{12}, \qquad c_3 = -\tfrac{2}{3}, \qquad c_4 = -\tfrac{1}{8}.$$

Thus the solution of the initial value problem is

$$y = \tfrac{11}{8} e^{t} + \tfrac{5}{12} e^{-t} - \tfrac{2}{3} e^{2t} - \tfrac{1}{8} e^{-3t}. \tag{11}$$

The graph of the solution is shown in Figure 4.2.1.

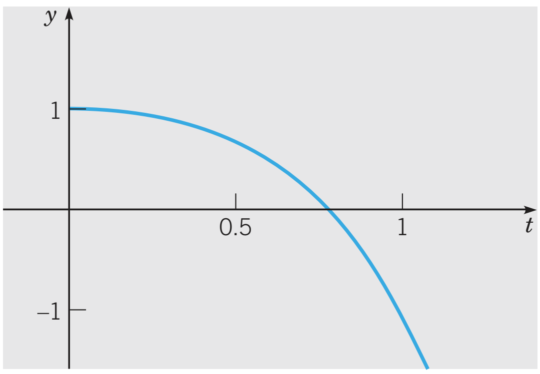

FIGURE 4.2.1 Solution of the initial value problem (6), (7): $y^{(4)} + y''' - 7y'' - y' + 6y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = -1$.

As Example 1 illustrates, the procedure for solving an nth order linear differential equation with constant coefficients depends on finding the roots of a corresponding nth degree polynomial equation. If initial conditions are prescribed, then a system of n linear algebraic equations must be solved to determine the proper values of the constants $c_1, \dots, c_n$. Each of these tasks becomes much more complicated as n increases, and we have omitted the detailed calculations in Example 1. Computer assistance can be very helpful in such problems.

For third and fourth degree polynomials there are formulas,[2] analogous to the formula for quadratic equations but more complicated, that give exact expressions for the roots. Root-finding algorithms are readily available on calculators and computers. Sometimes they are included in the differential equation solver, so that the process of factoring the characteristic polynomial is hidden and the solution of the differential equation is produced automatically.

If you are faced with the need to factor the characteristic polynomial by hand, here is one result that is sometimes helpful. Suppose that the polynomial

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0 \tag{12}$$

has integer coefficients. If $r = p/q$ is a rational root, where p and q have no common factors, then p must be a factor of a_n, and q must be a factor of a_0. For example, in Eq. (8) the factors of a_0 are ± 1 and the factors of a_n are ± 1, ± 2, ± 3, and ± 6. Thus the only possible rational roots of this equation are ± 1, ± 2, ± 3, and ± 6. By testing these possible roots, we find that $1, -1, 2$, and -3 are actual roots. In this case there are no other roots, since the polynomial is of fourth degree. If some of the roots are irrational or complex, as is usually the case, then this process will not find them, but at least the degree of the polynomial can be reduced by dividing the polynomial by the factors corresponding to the rational roots.

If the roots of the characteristic equation are real and different, we have seen that the general solution (5) is simply a sum of exponential functions. For large values of t the solution is dominated by the term corresponding to the algebraically largest root. If this root is positive, then solutions become exponentially unbounded, whereas if it is negative, then solutions tend exponentially to zero. Finally, if the largest root is zero, then solutions approach a nonzero constant as t becomes large. Of course, for certain initial conditions, the coefficient of the otherwise dominant term may be zero; then the nature of the solution for large t is determined by the next largest root.

Complex Roots. If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficients $a_0, a_1, a_2, \dots, a_n$ are real numbers. Provided that none of the roots is repeated, the general solution of Eq. (1) is still of the

[2]The method for solving the cubic equation was apparently discovered by Scipione dal Ferro (1465–1526) about 1500, although it was first published in 1545 by Girolamo Cardano (1501–1576) in his *Ars Magna*. This book also contains a method for solving quartic equations that Cardano attributes to his pupil Ludovico Ferrari (1522–1565). The question of whether analogous formulas exist for the roots of higher degree equations remained open for more than two centuries, until 1826, when Niels Abel showed that no general solution formulas can exist for polynomial equations of degree five or higher. A more general theory was developed by Evariste Galois (1811–1832) in 1831, but unfortunately it did not become widely known for several decades.

form of Eq. (5). However, just as for the second order equation (Section 3.3), we can replace the complex-valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions

$$e^{\lambda t}\cos \mu t, \qquad e^{\lambda t}\sin \mu t \tag{13}$$

obtained as the real and imaginary parts of $e^{(\lambda+i\mu)t}$. Thus, even though some of the roots of the characteristic equation are complex, it is still possible to express the general solution of Eq. (1) as a linear combination of real-valued solutions.

EXAMPLE 2

Find the general solution of

$$y^{(4)} - y = 0. \tag{14}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 7/2, \qquad y'(0) = -4, \qquad y''(0) = 5/2, \qquad y'''(0) = -2 \tag{15}$$

and draw its graph.

Substituting e^{rt} for y, we find that the characteristic equation is

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0.$$

Therefore, the roots are $r = 1, -1, i, -i$, and the general solution of Eq. (14) is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

If we impose the initial conditions (15), we obtain

$$c_1 = 0, \qquad c_2 = 3, \qquad c_3 = 1/2, \qquad c_4 = -1;$$

thus the solution of the given initial value problem is

$$y = 3e^{-t} + \tfrac{1}{2}\cos t - \sin t. \tag{16}$$

The graph of this solution is shown in Figure 4.2.2.

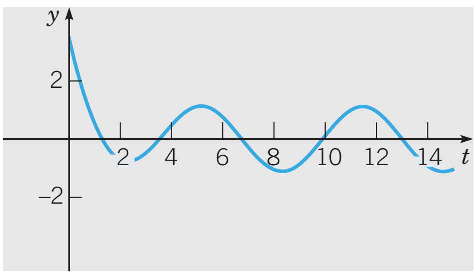

FIGURE 4.2.2 Solution of the initial value problem (14), (15): $y^{(4)} - y = 0$, $y(0) = 7/2$, $y'(0) = -4$, $y''(0) = 5/2$, $y'''(0) = -2$.

Observe that the initial conditions (15) cause the coefficient c_1 of the exponentially growing term in the general solution to be zero. Therefore, this term is absent in the solution (16), which describes an exponential decay to a steady oscillation, as Figure 4.2.2 shows. However, if the initial conditions are changed slightly, then c_1 is likely to be nonzero, and the nature of the solution changes enormously. For example, if the first three initial conditions remain the same, but the value of $y'''(0)$ is changed from -2 to $-15/8$, then the solution of the initial value problem becomes

$$y = \tfrac{1}{32}e^t + \tfrac{95}{32}e^{-t} + \tfrac{1}{2}\cos t - \tfrac{17}{16}\sin t. \tag{17}$$

The coefficients in Eq. (17) differ only slightly from those in Eq. (16), but the exponentially growing term, even with the relatively small coefficient of $1/32$, completely dominates the solution by the time t is larger than about 4 or 5. This is clearly seen in Figure 4.2.3, which shows the graphs of the two solutions (16) and (17).

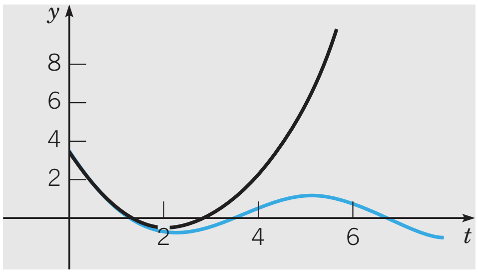

FIGURE 4.2.3 The blue curve is the solution of the initial value problem (14), (15) and is the same as the curve in Figure 4.2.2. The black curve is the solution of the modified problem in which the last initial condition is changed to $y'''(0) = -15/8$.

Repeated Roots. If the roots of the characteristic equation are not distinct—that is, if some of the roots are repeated—then the solution (5) is clearly not the general solution of Eq. (1). Recall that if r_1 is a repeated root for the second order linear equation $a_0y'' + a_1y' + a_2y = 0$, then two linearly independent solutions are e^{r_1t} and te^{r_1t}. For an equation of order n, if a root of $Z(r) = 0$, say $r = r_1$, has multiplicity s (where $s \leq n$), then

$$e^{r_1t}, \quad te^{r_1t}, \quad t^2e^{r_1t}, \quad \ldots, \quad t^{s-1}e^{r_1t} \tag{18}$$

are corresponding solutions of Eq. (1). See Problem 41 for a proof of this statement, which is valid whether the repeated root is real or complex.

Note that a complex root can be repeated only if the differential equation (1) is of order four or higher. If a complex root $\lambda + i\mu$ is repeated s times, the complex conjugate $\lambda - i\mu$ is also repeated s times. Corresponding to these $2s$ complex-valued solutions, we can find $2s$ real-valued solutions by noting that the real and imaginary parts of $e^{(\lambda+i\mu)t}, te^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ are also linearly independent solutions:

$$\begin{gathered} e^{\lambda t}\cos\mu t, \quad e^{\lambda t}\sin\mu t, \quad te^{\lambda t}\cos\mu t, \quad te^{\lambda t}\sin\mu t, \\ \ldots, \quad t^{s-1}e^{\lambda t}\cos\mu t, \quad t^{s-1}e^{\lambda t}\sin\mu t. \end{gathered}$$

Hence the general solution of Eq. (1) can always be expressed as a linear combination of n real-valued solutions. Consider the following example.

EXAMPLE 3

Find the general solution of

$$y^{(4)} + 2y'' + y = 0. \tag{19}$$

The characteristic equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0.$$

The roots are $r = i, i, -i, -i$, and the general solution of Eq. (19) is

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

In determining the roots of the characteristic equation, it may be necessary to compute the cube roots, the fourth roots, or even higher roots of a (possibly complex) number. This can usually be done most conveniently by using Euler's formula $e^{it} = \cos t + i \sin t$ and the algebraic laws given in Section 3.3. This is illustrated in the following example.

EXAMPLE 4

Find the general solution of

$$y^{(4)} + y = 0. \tag{20}$$

The characteristic equation is

$$r^4 + 1 = 0.$$

To solve the equation, we must compute the fourth roots of -1. Now -1, thought of as a complex number, is $-1 + 0i$. It has magnitude 1 and polar angle π. Thus

$$-1 = \cos \pi + i \sin \pi = e^{i\pi}.$$

Moreover, the angle is determined only up to a multiple of 2π. Thus

$$-1 = \cos(\pi + 2m\pi) + i \sin(\pi + 2m\pi) = e^{i(\pi+2m\pi)},$$

where m is zero or any positive or negative integer. Thus

$$(-1)^{1/4} = e^{i(\pi/4+m\pi/2)} = \cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right).$$

The four fourth roots of -1 are obtained by setting $m = 0, 1, 2,$ and 3; they are

$$\frac{1+i}{\sqrt{2}}, \quad \frac{-1+i}{\sqrt{2}}, \quad \frac{-1-i}{\sqrt{2}}, \quad \frac{1-i}{\sqrt{2}}.$$

It is easy to verify that, for any other value of m, we obtain one of these four roots. For example, corresponding to $m = 4$, we obtain $(1 + i)/\sqrt{2}$. The general solution of Eq. (20) is

$$y = e^{t/\sqrt{2}}\left(c_1 \cos\frac{t}{\sqrt{2}} + c_2 \sin\frac{t}{\sqrt{2}}\right) + e^{-t/\sqrt{2}}\left(c_3 \cos\frac{t}{\sqrt{2}} + c_4 \sin\frac{t}{\sqrt{2}}\right). \tag{21}$$

In conclusion, we note that the problem of finding all the roots of a polynomial equation may not be entirely straightforward, even with computer assistance. For instance, it may be difficult to determine whether two roots are equal or merely very close together. Recall that the form of the general solution is different in these two cases.

If the constants $a_0, a_1, \dots, a_n$ in Eq. (1) are complex numbers, the solution of Eq. (1) is still of the form (4). In this case, however, the roots of the characteristic equation are, in general, complex numbers, and it is no longer true that the complex conjugate of a root is also a root. The corresponding solutions are complex-valued.

PROBLEMS

In each of Problems 1 through 6, express the given complex number in the form $R(\cos\theta + i \sin\theta) = Re^{i\theta}$.

1. $1 + i$
2. $-1 + \sqrt{3}i$
3. -3
4. $-i$
5. $\sqrt{3} - i$
6. $-1 - i$

In each of Problems 7 through 10, follow the procedure illustrated in Example 4 to determine the indicated roots of the given complex number.

7. $1^{1/3}$
8. $(1-i)^{1/2}$
9. $1^{1/4}$
10. $[2(\cos\pi/3 + i\sin\pi/3)]^{1/2}$

In each of Problems 11 through 28, find the general solution of the given differential equation.

11. $y''' - y'' - y' + y = 0$
12. $y''' - 3y'' + 3y' - y = 0$
13. $2y''' - 4y'' - 2y' + 4y = 0$
14. $y^{(4)} - 4y''' + 4y'' = 0$
15. $y^{(6)} + y = 0$
16. $y^{(4)} - 5y'' + 4y = 0$
17. $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$
18. $y^{(6)} - y'' = 0$
19. $y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0$
20. $y^{(4)} - 8y' = 0$
21. $y^{(8)} + 8y^{(4)} + 16y = 0$
22. $y^{(4)} + 2y'' + y = 0$
23. $y''' - 5y'' + 3y' + y = 0$
24. $y''' + 5y'' + 6y' + 2y = 0$
25. $18y''' + 21y'' + 14y' + 4y = 0$
26. $y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0$
27. $12y^{(4)} + 31y''' + 75y'' + 37y' + 5y = 0$
28. $y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0$

In each of Problems 29 through 36, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

29. $y''' + y' = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2$
30. $y^{(4)} + y = 0; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0$
31. $y^{(4)} - 4y''' + 4y'' = 0; \quad y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0$
32. $y''' - y'' + y' - y = 0; \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = -2$
33. $2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0; \quad y(0) = -2, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
34. $4y''' + y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$
35. $6y''' + 5y'' + y' = 0; \quad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0$
36. $y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0; \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 0,$ $y'''(0) = 3$
37. Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1\cos t + c_2\sin t + c_3\cosh t + c_4\sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?

38. Consider the equation $y^{(4)} - y = 0$.

(a) Use Abel's formula [Problem 20(d) of Section 4.1] to find the Wronskian of a fundamental set of solutions of the given equation.

(b) Determine the Wronskian of the solutions e^t, e^{-t}, $\cos t$, and $\sin t$.

(c) Determine the Wronskian of the solutions $\cosh t$, $\sinh t$, $\cos t$, and $\sin t$.

39. Consider the spring–mass system, shown in Figure 4.2.4, consisting of two unit masses suspended from springs with spring constants 3 and 2, respectively. Assume that there is no damping in the system.

(a) Show that the displacements u_1 and u_2 of the masses from their respective equilibrium positions satisfy the equations

$$u_1'' + 5u_1 = 2u_2, \qquad u_2'' + 2u_2 = 2u_1. \tag{i}$$

(b) Solve the first of Eqs. (i) for u_2 and substitute into the second equation, thereby obtaining the following fourth order equation for u_1:

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0. \tag{ii}$$

Find the general solution of Eq. (ii).

(c) Suppose that the initial conditions are

$$u_1(0) = 1, \qquad u_1'(0) = 0, \qquad u_2(0) = 2, \qquad u_2'(0) = 0. \tag{iii}$$

Use the first of Eqs. (i) and the initial conditions (iii) to obtain values for $u_1''(0)$ and $u_1'''(0)$. Then show that the solution of Eq. (ii) that satisfies the four initial conditions on u_1 is $u_1(t) = \cos t$. Show that the corresponding solution u_2 is $u_2(t) = 2\cos t$.

(d) Now suppose that the initial conditions are

$$u_1(0) = -2, \qquad u_1'(0) = 0, \qquad u_2(0) = 1, \qquad u_2'(0) = 0. \tag{iv}$$

Proceed as in part (c) to show that the corresponding solutions are $u_1(t) = -2\cos\sqrt{6}\,t$ and $u_2(t) = \cos\sqrt{6}\,t$.

(e) Observe that the solutions obtained in parts (c) and (d) describe two distinct modes of vibration. In the first, the frequency of the motion is 1, and the two masses move in phase, both moving up or down together; the second mass moves twice as far as the first. The second motion has frequency $\sqrt{6}$, and the masses move out of phase with each other, one moving down while the other is moving up, and vice versa. In this mode the first mass moves twice as far as the second. For other initial conditions, not proportional to either of Eqs. (iii) or (iv), the motion of the masses is a combination of these two modes.

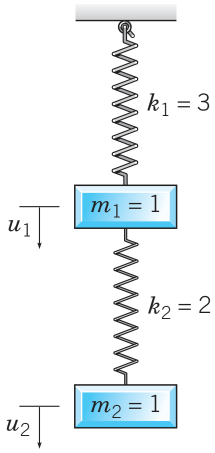

FIGURE 4.2.4 A two-spring, two-mass system.

40. In this problem we outline one way to show that if $r_1, \dots, r_n$ are all real and different, then $e^{r_1 t}, \dots, e^{r_n t}$ are linearly independent on $-\infty < t < \infty$. To do this, we consider the linear relation

$$c_1 e^{r_1 t} + \cdots + c_n e^{r_n t} = 0, \qquad -\infty < t < \infty \tag{i}$$

and show that all the constants are zero.

(a) Multiply Eq. (i) by $e^{-r_1 t}$ and differentiate with respect to t, thereby obtaining

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0.$$

(b) Multiply the result of part (a) by $e^{-(r_2-r_1)t}$ and differentiate with respect to t to obtain

$$c_3(r_3-r_2)(r_3-r_1)e^{(r_3-r_2)t}+\cdots+c_n(r_n-r_2)(r_n-r_1)e^{(r_n-r_2)t}=0.$$

(c) Continue the procedure from parts (a) and (b), eventually obtaining

$$c_n(r_n-r_{n-1})\cdots(r_n-r_1)e^{(r_n-r_{n-1})t}=0.$$

Hence $c_n=0$, and therefore,

$$c_1e^{r_1t}+\cdots+c_{n-1}e^{r_{n-1}t}=0.$$

(d) Repeat the preceding argument to show that $c_{n-1}=0$. In a similar way it follows that $c_{n-2}=\cdots=c_1=0$. Thus the functions $e^{r_1t},\ldots,e^{r_nt}$ are linearly independent.

41. In this problem we indicate one way to show that if $r=r_1$ is a root of multiplicity s of the characteristic polynomial $Z(r)$, then e^{r_1t}, $te^{r_1t},\ldots,$ $t^{s-1}e^{r_1t}$ are solutions of Eq. (1). This problem extends to nth order equations the method for second order equations given in Problem 22 of Section 3.4. We start from Eq. (2) in the text

$$L[e^{rt}]=e^{rt}Z(r) \tag{i}$$

and differentiate repeatedly with respect to r, setting $r=r_1$ after each differentiation.

(a) Observe that if r_1 is a root of multiplicity s, then $Z(r)=(r-r_1)^sq(r)$, where $q(r)$ is a polynomial of degree $n-s$ and $q(r_1)\neq 0$. Show that $Z(r_1), Z'(r_1),\ldots,Z^{(s-1)}(r_1)$ are all zero, but $Z^{(s)}(r_1)\neq 0$.

(b) By differentiating Eq. (i) repeatedly with respect to r, show that

$$\frac{\partial}{\partial r}L[e^{rt}]=L\left[\frac{\partial}{\partial r}e^{rt}\right]=L[te^{rt}],$$
$$\vdots$$
$$\frac{\partial^{s-1}}{\partial r^{s-1}}L[e^{rt}]=L[t^{s-1}e^{rt}].$$

(c) Show that $e^{r_1t},te^{r_1t},\ldots,t^{s-1}e^{r_1t}$ are solutions of Eq. (1).

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y]=a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=g(t) \tag{1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. Although the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when it is applicable.

Just as for the second order linear equation, when the constant coefficient linear differential operator L is applied to a polynomial $A_0t^m+A_1t^{m-1}+\cdots+A_m$, an