CHAPTER

6

The Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in Chapter 3 are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform. In this chapter we describe how this important method works, emphasizing problems typical of those that arise in engineering applications.

6.1 Definition of the Laplace Transform

Improper Integrals. Since the Laplace transform involves an integral from zero to infinity, a knowledge of improper integrals of this type is necessary to appreciate the subsequent development of the properties of the transform. We provide a brief review of such improper integrals here. If you are already familiar with improper integrals, you may wish to skip over this review. On the other hand, if improper integrals are new to you, then you should probably consult a calculus book, where you will find many more details and examples.

An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt, \tag{1}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or to fail to exist. The following examples illustrate both possibilities.

EXAMPLE 1

Let $f(t) = e^{ct}$, $t \geq 0$, where c is a real nonzero constant. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty}\int_0^A e^{ct}\,dt = \lim_{A\to\infty}\left.\frac{e^{ct}}{c}\right|_0^A$$

$$= \lim_{A\to\infty}\frac{1}{c}(e^{cA}-1).$$

It follows that the improper integral converges to the value $-1/c$ if $c < 0$ and diverges if $c > 0$. If $c = 0$, the integrand $f(t)$ is the constant function with value 1. In this case

$$\lim_{A\to\infty}\int_0^A 1\,dt = \lim_{A\to\infty}(A-0) = \infty,$$

so the integral again diverges.

EXAMPLE 2

Let $f(t) = 1/t$, $t \geq 1$. Then

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty}\int_1^A \frac{dt}{t} = \lim_{A\to\infty}\ln A.$$

Since $\lim\limits_{A\to\infty}\ln A = \infty$, the improper integral diverges.

EXAMPLE 3

Let $f(t) = t^{-p}$, $t \geq 1$, where p is a real constant and $p \neq 1$; the case $p = 1$ was considered in Example 2. Then

$$\int_1^\infty t^{-p}\,dt = \lim_{A\to\infty}\int_1^A t^{-p}\,dt = \lim_{A\to\infty}\frac{1}{1-p}(A^{1-p}-1).$$

As $A \to \infty$, $A^{1-p} \to 0$ if $p > 1$, but $A^{1-p} \to \infty$ if $p < 1$. Hence $\int_1^\infty t^{-p}\,dt$ converges to the value $1/(p-1)$ for $p > 1$ but (incorporating the result of Example 2) diverges for $p \leq 1$. These results are analogous to those for the infinite series $\sum\limits_{n=1}^{\infty} n^{-p}$.

Before discussing the possible existence of $\int_a^\infty f(t)\,dt$, it is helpful to define certain terms. A function f is said to be **piecewise continuous** on an interval $\alpha \leq t \leq \beta$ if the interval[1] can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. f is continuous on each open subinterval $t_{i-1} < t < t_i$.
2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is piecewise continuous on $\alpha \leq t \leq \beta$ if it is continuous there except for a finite number of jump discontinuities. If f is piecewise continuous on $\alpha \leq t \leq \beta$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \geq \alpha$. An example of a piecewise continuous function is shown in Figure 6.1.1.

[1] It is not essential that the interval be closed; the same definition applies if the interval is open at one or both ends.

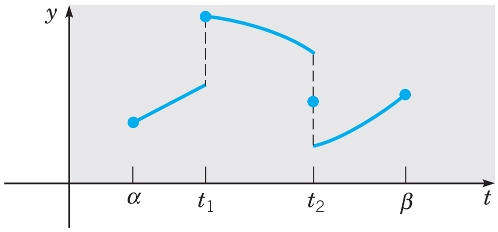

FIGURE 6.1.1 A piecewise continuous function $y = f(t)$.

The integral of a piecewise continuous function on a finite interval is just the sum of the integrals on the subintervals created by the partition points. For instance, for the function $f(t)$ shown in Figure 6.1.1, we have

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{\beta} f(t)\,dt. \tag{2}$$

For the function shown in Figure 6.1.1, we have assigned values to the function at the endpoints α and β and at the partition points t_1 and t_2. However, as far as the integrals in Eq. (2) are concerned, it does not matter whether $f(t)$ is defined at these points, or what values may be assigned to $f(t)$ at them. The values of the integrals in Eq. (2) remain the same regardless.

Thus, if f is piecewise continuous on the interval $a \le t \le A$, then $\int_a^A f(t)\,dt$ exists. Hence, if f is piecewise continuous for $t \ge a$, then $\int_a^A f(t)\,dt$ exists for each $A > a$. However, piecewise continuity is not enough to ensure convergence of the improper integral $\int_a^\infty f(t)\,dt$, as the preceding examples show.

If f cannot be integrated easily in terms of elementary functions, the definition of convergence of $\int_a^\infty f(t)\,dt$ may be difficult to apply. Frequently, the most convenient way to test the convergence or divergence of an improper integral is by the following comparison theorem, which is analogous to a similar theorem for infinite series.

Theorem 6.1.1 If f is piecewise continuous for $t \ge a$, if $|f(t)| \le g(t)$ when $t \ge M$ for some positive constant M, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges. On the other hand, if $f(t) \ge g(t) \ge 0$ for $t \ge M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

The proof of this result from calculus will not be given here. It is made plausible, however, by comparing the areas represented by $\int_M^\infty g(t)\,dt$ and $\int_M^\infty |f(t)|\,dt$. The functions most useful for comparison purposes are e^{ct} and t^{-p}, which we considered in Examples 1, 2, and 3.

The Laplace Transform. Among the tools that are very useful for solving linear differential equations are **integral transforms**. An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t)f(t)\,dt, \tag{3}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration α and β are also given. It is possible that $\alpha = -\infty$, or $\beta = \infty$, or both. The relation (3) transforms the function f into another function F, which is called the **transform** of f.

There are several integral transforms that are useful in applied mathematics, but in this chapter we consider only the Laplace[2] transform. This transform is defined in the following way. Let $f(t)$ be given for $t \geq 0$, and suppose that f satisfies certain conditions to be stated a little later. Then the Laplace transform of f, which we will denote by $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st}f(t)\,dt, \tag{4}$$

whenever this improper integral converges. The Laplace transform makes use of the kernel $K(s,t) = e^{-st}$. Since the solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations. The general idea in using the Laplace transform to solve a differential equation is as follows:

1. Use the relation (4) to transform an initial value problem for an unknown function f in the t-domain into a simpler problem (indeed, an algebraic problem) for F in the s-domain.
2. Solve this algebraic problem to find F.
3. Recover the desired function f from its transform F. This last step is known as "inverting the transform."

In general, the parameter s may be complex, and the full power of the Laplace transform becomes available only when we regard $F(s)$ as a function of a complex variable. However, for the problems discussed here, it is sufficient to consider only real values of s. The Laplace transform F of a function f exists if f satisfies certain conditions, such as those stated in the following theorem.

Theorem 6.1.2 Suppose that

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality, K, a, and M are real constants, K and M necessarily positive.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by Eq. (4), exists for $s > a$.

[2]The Laplace transform is named for the eminent French mathematician P. S. Laplace, who studied the relation (3) in 1782. However, the techniques described in this chapter were not developed until a century or more later. We owe them mainly to Oliver Heaviside (1850–1925), an innovative self-taught English electrical engineer, who made significant contributions to the development and application of electromagnetic theory. He was also one of the developers of vector calculus.

To establish this theorem, we must show that the integral in Eq. (4) converges for $s > a$. Splitting the improper integral into two parts, we have

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^M e^{-st} f(t)\, dt + \int_M^\infty e^{-st} f(t)\, dt. \tag{5}$$

The first integral on the right side of Eq. (5) exists by hypothesis (1) of the theorem; hence the existence of $F(s)$ depends on the convergence of the second integral. By hypothesis (2) we have, for $t \geq M$,

$$|e^{-st} f(t)| \leq Ke^{-st}e^{at} = Ke^{(a-s)t},$$

and thus, by Theorem 6.1.1, $F(s)$ exists provided that $\int_M^\infty e^{(a-s)t}\, dt$ converges. Referring to Example 1 with c replaced by $a - s$, we see that this latter integral converges when $a - s < 0$, which establishes Theorem 6.1.2.

In this chapter (except in Section 6.5), we deal almost exclusively with functions that satisfy the conditions of Theorem 6.1.2. Such functions are described as piecewise continuous and of **exponential order** as $t \to \infty$. Note that there are functions that are not of exponential order as $t \to \infty$. One such function is $f(t) = e^{t^2}$. As $t \to \infty$, this function increases faster than Ke^{at} regardless of how large the constants K and a may be.

The Laplace transforms of some important elementary functions are given in the following examples.

EXAMPLE 4

Let $f(t) = 1,\ t \geq 0$. Then, as in Example 1,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = -\lim_{A\to\infty} \left.\frac{e^{-st}}{s}\right|_0^A = \frac{1}{s}, \qquad s > 0.$$

EXAMPLE 5

Let $f(t) = e^{at},\ t \geq 0$. Then, again referring to Example 1,

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \frac{1}{s-a}, \qquad s > a.$$

EXAMPLE 6

Let

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ k, & t = 1, \\ 0, & t > 1, \end{cases}$$

where k is a constant. In engineering contexts $f(t)$ often represents a unit pulse, perhaps of force or voltage.

Note that f is a piecewise continuous function. Then

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt = -\left.\frac{e^{-st}}{s}\right|_0^1 = \frac{1 - e^{-s}}{s}, \qquad s > 0.$$

Observe that $\mathcal{L}\{f(t)\}$ does not depend on k, the function value at the point of discontinuity. Even if $f(t)$ is not defined at this point, the Laplace transform of f remains the same. Thus there are many functions, differing only in their value at a single point, that have the same Laplace transform.

EXAMPLE 7

Let $f(t) = \sin at,\ t \geq 0$. Then

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at\, dt, \qquad s > 0.$$

Since

$$F(s) = \lim_{A\to\infty} \int_0^A e^{-st} \sin at\, dt,$$

upon integrating by parts, we obtain

$$\begin{aligned} F(s) &= \lim_{A\to\infty} \left[-\frac{e^{-st}\cos at}{a}\bigg|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos at\, dt \right] \\ &= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\, dt. \end{aligned}$$

A second integration by parts then yields

$$\begin{aligned} F(s) &= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\, dt \\ &= \frac{1}{a} - \frac{s^2}{a^2}F(s). \end{aligned}$$

Hence, solving for $F(s)$, we have

$$F(s) = \frac{a}{s^2 + a^2}, \qquad s > 0.$$

Now let us suppose that f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for s greater than the maximum of a_1 and a_2,

$$\begin{aligned} \mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\, dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt; \end{aligned}$$

hence

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}. \tag{6}$$

Equation (6) states that the Laplace transform is a **linear operator**, and we make frequent use of this property later. The sum in Eq. (6) can be readily extended to an arbitrary number of terms.

EXAMPLE 8

Find the Laplace transform of $f(t) = 5e^{-2t} - 3\sin 4t,\ t \geq 0$.

Using Eq. (6), we write

$$\mathcal{L}\{f(t)\} = 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\}.$$

Then, from Examples 5 and 7, we obtain

$$\mathcal{L}\{f(t)\} = \frac{5}{s+2} - \frac{12}{s^2+16}, \qquad s > 0.$$

PROBLEMS

In each of Problems 1 through 4, sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \leq t \leq 3$.

1. $f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 2+t, & 1 < t \leq 2 \\ 6-t, & 2 < t \leq 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ (t-1)^{-1}, & 1 < t \leq 2 \\ 1, & 2 < t \leq 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 1, & 1 < t \leq 2 \\ 3-t, & 2 < t \leq 3 \end{cases}$

4. $f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 3-t, & 1 < t \leq 2 \\ 1, & 2 < t \leq 3 \end{cases}$

5. Find the Laplace transform of each of the following functions:
   (a) $f(t) = t$
   (b) $f(t) = t^2$
   (c) $f(t) = t^n$, where n is a positive integer
6. Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

7. $f(t) = \cosh bt$
8. $f(t) = \sinh bt$
9. $f(t) = e^{at} \cosh bt$
10. $f(t) = e^{at} \sinh bt$

Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. In each of Problems 11 through 14, find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

11. $f(t) = \sin bt$
12. $f(t) = \cos bt$
13. $f(t) = e^{at} \sin bt$
14. $f(t) = e^{at} \cos bt$

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

15. $f(t) = te^{at}$
16. $f(t) = t \sin at$
17. $f(t) = t \cosh at$
18. $f(t) = t^n e^{at}$
19. $f(t) = t^2 \sin at$
20. $f(t) = t^2 \sinh at$

In each of Problems 21 through 24, find the Laplace transform of the given function.

21. $f(t) = \begin{cases} 1, & 0 \leq t < \pi \\ 0, & \pi \leq t < \infty \end{cases}$

22. $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 0, & 1 \leq t < \infty \end{cases}$

23. $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t < \infty \end{cases}$

24. $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2-t, & 1 \leq t < 2 \\ 0, & 2 \leq t < \infty \end{cases}$

In each of Problems 25 through 28, determine whether the given integral converges or diverges.

25. $\displaystyle\int_0^\infty (t^2 + 1)^{-1}\, dt$
26. $\displaystyle\int_0^\infty te^{-t}\, dt$
27. $\displaystyle\int_1^\infty t^{-2} e^t\, dt$
28. $\displaystyle\int_0^\infty e^{-t} \cos t\, dt$

29. Suppose that f and f' are continuous for $t \geq 0$ and of exponential order as $t \to \infty$. Use integration by parts to show that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.
30. **The Gamma Function.** The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx. \tag{i}$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(a) Show that, for $p > 0$,

$$\Gamma(p+1) = p\Gamma(p).$$

(b) Show that $\Gamma(1) = 1$.

(c) If p is a positive integer n, show that

$$\Gamma(n+1) = n!.$$

Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

(d) Show that, for $p > 0$,

$$p(p+1)(p+2)\cdots(p+n-1) = \Gamma(p+n)/\Gamma(p).$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length—say, $0 < p \leq 1$. It is possible to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Find $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{11}{2}\right)$.

31. Consider the Laplace transform of t^p, where $p > -1$.

(a) Referring to Problem 30, show that

$$\begin{aligned}\mathcal{L}\{t^p\} &= \int_0^\infty e^{-st}t^p\,dt = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\,dx \\ &= \Gamma(p+1)/s^{p+1}, \qquad s > 0.\end{aligned}$$

(b) Let p be a positive integer n in part (a); show that

$$\mathcal{L}\{t^n\} = n!/s^{n+1}, \qquad s > 0.$$

(c) Show that

$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx, \qquad s > 0.$$

It is possible to show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2};$$

hence

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}, \qquad s > 0.$$

(d) Show that

$$\mathcal{L}\{t^{1/2}\} = \sqrt{\pi}/(2s^{3/2}), \qquad s > 0.$$

6.2 Solution of Initial Value Problems

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. The usefulness of the Laplace transform for this purpose rests primarily on the fact that the transform of f' is related in a simple way to the transform of f. The relationship is expressed in the following theorem.

Theorem 6.2.1 Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{1}$$

To prove this theorem, we consider the integral

$$\int_0^A e^{-st} f'(t)\, dt,$$

whose limit as $A \to \infty$, if it exists, is the Laplace transform of f'. To calculate this limit we first need to write the integral in a suitable form. If f' has points of discontinuity in the interval $0 \le t \le A$, let them be denoted by $t_1, t_2, \dots, t_k$. Then we can write the integral as

$$\int_0^A e^{-st} f'(t)\, dt = \int_0^{t_1} e^{-st} f'(t)\, dt + \int_{t_1}^{t_2} e^{-st} f'(t)\, dt + \cdots + \int_{t_k}^{A} e^{-st} f'(t)\, dt.$$

Integrating each term on the right by parts yields

$$\begin{aligned}\int_0^A e^{-st} f'(t)\, dt = {}& e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\Big|_{t_k}^{A} \\ &+ s\left[\int_0^{t_1} e^{-st} f(t)\, dt + \int_{t_1}^{t_2} e^{-st} f(t)\, dt + \cdots + \int_{t_k}^{A} e^{-st} f(t)\, dt\right].\end{aligned}$$

Since f is continuous, the contributions of the integrated terms at $t_1, t_2, \dots, t_k$ cancel. Further, the integrals on the right side can be combined into a single integral, so that we obtain

$$\int_0^A e^{-st} f'(t)\, dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)\, dt. \tag{2}$$

Now we let $A \to \infty$ in Eq. (2). The integral on the right side of this equation approaches $\mathcal{L}\{f(t)\}$. Further, for $A \ge M$, we have $|f(A)| \le Ke^{aA}$; consequently, $|e^{-sA} f(A)| \le Ke^{-(s-a)A}$. Hence $e^{-sA} f(A) \to 0$ as $A \to \infty$ whenever $s > a$. Thus the right side of Eq. (2) has the limit $s\mathcal{L}\{f(t)\} - f(0)$. Consequently, the left side of Eq. (2) also has a limit, and as noted above, this limit is $\mathcal{L}\{f'(t)\}$. Therefore, for $s > a$, we conclude that

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$

which establishes the theorem.

If f' and f'' satisfy the same conditions that are imposed on f and f', respectively, in Theorem 6.2.1, then it follows that the Laplace transform of f'' also exists for $s > a$ and is given by

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)\\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \end{aligned} \tag{3}$$

Indeed, provided the function f and its derivatives satisfy suitable conditions, an expression for the transform of the nth derivative $f^{(n)}$ can be derived by n successive applications of this theorem. The result is given in the following corollary.

Corollary 6.2.2 Suppose that the functions $f, f', \dots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}, \dots, |f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{4}$$

We now show how the Laplace transform can be used to solve initial value problems. It is most useful for problems involving nonhomogeneous differential equations, as we will demonstrate in later sections of this chapter. However, we begin by looking at some homogeneous equations, which are a bit simpler.

EXAMPLE 1

Consider the differential equation

$$y'' - y' - 2y = 0 \tag{5}$$

and the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{6}$$

This problem is easily solved by the methods of Section 3.1. The characteristic equation is

$$r^2 - r - 2 = (r-2)(r+1) = 0,$$

and consequently, the general solution of Eq. (5) is

$$y = c_1e^{-t} + c_2e^{2t}. \tag{7}$$

To satisfy the initial conditions (6), we must have $c_1 + c_2 = 1$ and $-c_1 + 2c_2 = 0$; hence $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$, so the solution of the initial value problem (5) and (6) is

$$y = \phi(t) = \tfrac{2}{3}e^{-t} + \tfrac{1}{3}e^{2t}. \tag{8}$$

Now let us solve the same problem by using the Laplace transform. To do this, we must assume that the problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation (5), we obtain

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0, \tag{9}$$

where we have used the linearity of the transform to write the transform of a sum as the sum of the separate transforms. Upon using the corollary to express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$, we find that Eq. (9) becomes

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - [s\mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = 0,$$

or

$$(s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) = 0, \tag{10}$$

where $Y(s) = \mathcal{L}\{y\}$. Substituting for $y(0)$ and $y'(0)$ in Eq. (10) from the initial conditions (6), and then solving for $Y(s)$, we obtain

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}. \tag{11}$$

We have thus obtained an expression for the Laplace transform $Y(s)$ of the solution $y = \phi(t)$ of the given initial value problem. To determine the function ϕ, we must find the function whose Laplace transform is $Y(s)$, as given by Eq. (11).

This can be done most easily by expanding the right side of Eq. (11) in partial fractions. Thus we write

$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}, \tag{12}$$

where the coefficients a and b are to be determined. By equating numerators of the second and fourth members of Eq. (12), we obtain

$$s - 1 = a(s+1) + b(s-2),$$

an equation that must hold for all s. In particular, if we set $s = 2$, then it follows that $a = \frac{1}{3}$. Similarly, if we set $s = -1$, then we find that $b = \frac{2}{3}$. By substituting these values for a and b, respectively, we have

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}. \tag{13}$$

Finally, if we use the result of Example 5 of Section 6.1, it follows that $\frac{1}{3}e^{2t}$ has the transform $\frac{1}{3}(s-2)^{-1}$; similarly, $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}(s+1)^{-1}$. Hence, by the linearity of the Laplace transform,

$$y = \phi(t) = \tfrac{1}{3}e^{2t} + \tfrac{2}{3}e^{-t}$$

has the transform (13) and is therefore the solution of the initial value problem (5), (6). Observe that it does satisfy the conditions of Corollary 6.2.2, as we assumed initially. Of course, this is the same solution that we obtained earlier.

The same procedure can be applied to the general second order linear equation with constant coefficients

$$ay'' + by' + cy = f(t). \tag{14}$$

Assuming that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 2$, we can take the transform of Eq. (14) and thereby obtain

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s), \tag{15}$$

where $F(s)$ is the transform of $f(t)$. By solving Eq. (15) for $Y(s)$, we find that

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{16}$$

The problem is then solved, provided that we can find the function $y = \phi(t)$ whose transform is $Y(s)$.

Even at this early stage of our discussion we can point out some of the essential features of the transform method. In the first place, the transform $Y(s)$ of the unknown function $y = \phi(t)$ is found by solving an *algebraic equation* rather than a *differential equation*, Eq. (10) rather than Eq. (5) in Example 1, or in general Eq. (15) rather than Eq. (14). This is the key to the usefulness of Laplace transforms for solving linear, constant coefficient, ordinary differential equations—the problem is reduced from a differential equation to an algebraic one. Next, the solution satisfying given initial conditions is automatically found, so that the task of determining appropriate values for the arbitrary constants in the general solution does not arise. Further, as indicated in Eq. (15), nonhomogeneous equations are handled in exactly the same way as homogeneous ones; it is not necessary to solve the corresponding homogeneous equation first. Finally, the method can be applied in the same way to higher order equations, as long as we assume that the solution satisfies the conditions of Corollary 6.2.2 for the appropriate value of n.

Observe that the polynomial $as^2 + bs + c$ in the denominator on the right side of Eq. (16) is precisely the characteristic polynomial associated with Eq. (14). Since the use of a partial fraction expansion of $Y(s)$ to determine $\phi(t)$ requires us to factor this polynomial, the use of Laplace transforms does not avoid the necessity of finding roots of the characteristic equation. For equations of higher than second order, this may require a numerical approximation, particularly if the roots are irrational or complex.

The main difficulty that occurs in solving initial value problems by the transform method lies in the problem of determining the function $y = \phi(t)$ corresponding to the transform $Y(s)$. This problem is known as the inversion problem for the Laplace transform; $\phi(t)$ is called the inverse transform corresponding to $Y(s)$, and the process of finding $\phi(t)$ from $Y(s)$ is known as inverting the transform. We also use the notation $\mathcal{L}^{-1}\{Y(s)\}$ to denote the inverse transform of $Y(s)$. There is a general formula for the inverse Laplace transform, but its use requires a familiarity with functions of a complex variable, and we do not consider it in this book. However, it is still possible to develop many important properties of the Laplace transform, and to solve many interesting problems, without the use of complex variables.

In solving the initial value problem (5), (6), we did not consider the question of whether there may be functions other than the one given by Eq. (8) that also have the transform (13). By Theorem 3.2.1 we know that the initial value problem has no other solutions. We also know that the unique solution (8) of the initial value problem is continuous. Consistent with this fact, it can be shown that if f and g are continuous functions with the same Laplace transform, then f and g must be identical. On the other hand, if f and g are only piecewise continuous, then they may differ at one or more points of discontinuity and yet have the same Laplace transform; see Example 6 in Section 6.1. This lack of uniqueness of the inverse Laplace transform for piecewise continuous functions is of no practical significance in applications.

Thus there is essentially a one-to-one correspondence between functions and their Laplace transforms. This fact suggests the compilation of a table, such as Table 6.2.1, giving the transforms of functions frequently encountered, and vice versa. The entries in the second column of Table 6.2.1 are the transforms of those in the first column. Perhaps more important, the functions in the first column are the inverse transforms

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

of those in the second column. Thus, for example, if the transform of the solution of a differential equation is known, the solution itself can often be found merely by looking it up in the table. Some of the entries in Table 6.2.1 have been used as examples, or appear as problems in Section 6.1, while others will be developed later in the chapter. The third column of the table indicates where the derivation of the given transforms may be found. Although Table 6.2.1 is sufficient for the examples and problems in this book, much larger tables are also available (see the list of references at the end of the chapter). Transforms and inverse transforms can also be readily obtained electronically by using a computer algebra system.

Frequently, a Laplace transform $F(s)$ is expressible as a sum of several terms

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s). \tag{17}$$

Suppose that $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \ldots, f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$. Then the function

$$f(t) = f_1(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$. By the uniqueness property stated previously, no other continuous function f has the same transform. Thus

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}; \tag{18}$$

that is, the inverse Laplace transform is also a linear operator.

In many problems it is convenient to make use of this property by decomposing a given transform into a sum of functions whose inverse transforms are already known or can be found in the table. Partial fraction expansions are particularly useful for this purpose, and a general result covering many cases is given in Problem 39. Other useful properties of Laplace transforms are derived later in this chapter.

As further illustrations of the technique of solving initial value problems by means of the Laplace transform and partial fraction expansions, consider the following examples.

EXAMPLE 2

Find the solution of the differential equation

$$y'' + y = \sin 2t \tag{19}$$

satisfying the initial conditions

$$y(0) = 2, \qquad y'(0) = 1. \tag{20}$$

We assume that this initial value problem has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Corollary 6.2.2. Then, taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4),$$

where the transform of $\sin 2t$ has been obtained from line 5 of Table 6.2.1. Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{21}$$

Using partial fractions, we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{22}$$

By expanding the numerator on the right side of Eq. (22) and equating it to the numerator in Eq. (21), we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d)$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \qquad b + d = 1,$$
$$4a + c = 8, \qquad 4b + d = 6.$$

Consequently, $a = 2$, $c = 0$, $b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}. \tag{23}$$

From lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \tfrac{5}{3}\sin t - \tfrac{1}{3}\sin 2t. \tag{24}$$

EXAMPLE 3

Find the solution of the initial value problem

$$y^{(4)} - y = 0, \tag{25}$$

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y'''(0) = 0. \tag{26}$$

In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (25) is

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (26) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1}. \tag{27}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as + b}{s^2 - 1} + \frac{cs + d}{s^2 + 1},$$

and it follows that

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2 \tag{28}$$

for all s. By setting $s = 1$ and $s = -1$, respectively, in Eq. (28), we obtain the pair of equations

$$2(a + b) = 1, \qquad 2(-a + b) = 1,$$

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq. (28), then $b - d = 0$, so $d = \frac{1}{2}$. Finally, equating the coefficients of the cubic terms on each side of Eq. (28), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}, \tag{29}$$

and from lines 7 and 5 of Table 6.2.1, the solution of the initial value problem (25), (26) is

$$y = \phi(t) = \frac{\sinh t + \sin t}{2}. \tag{30}$$

The most important elementary applications of the Laplace transform are in the study of mechanical vibrations and in the analysis of electric circuits; the governing equations were derived in Section 3.7. A vibrating spring–mass system has the equation of motion

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F(t), \tag{31}$$

where m is the mass, γ the damping coefficient, k the spring constant, and $F(t)$ the applied external force. The equation that describes an electric circuit containing an inductance L, a resistance R, and a capacitance C (an LRC circuit) is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \tag{32}$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is the applied voltage. In terms of the current $I(t) = dQ(t)/dt$, we can differentiate Eq. (32) and write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t). \tag{33}$$

Suitable initial conditions on u, Q, or I must also be prescribed.

We have noted previously, in Section 3.7, that Eq. (31) for the spring–mass system and Eqs. (32) or (33) for the electric circuit are identical mathematically, differing only in the interpretation of the constants and variables appearing in them. There are other physical problems that also lead to the same differential equation. Thus, once the mathematical problem is solved, its solution can be interpreted in terms of whichever corresponding physical problem is of immediate interest.

In the problem lists following this and other sections in this chapter are numerous initial value problems for second order linear differential equations with constant coefficients. Many can be interpreted as models of particular physical systems, but usually we do not point this out explicitly.

PROBLEMS

In each of Problems 1 through 10, find the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{3}{s^2 + 4}$
2. $F(s) = \dfrac{4}{(s-1)^3}$
3. $F(s) = \dfrac{2}{s^2 + 3s - 4}$
4. $F(s) = \dfrac{3s}{s^2 - s - 6}$
5. $F(s) = \dfrac{2s + 2}{s^2 + 2s + 5}$
6. $F(s) = \dfrac{2s - 3}{s^2 - 4}$
7. $F(s) = \dfrac{2s + 1}{s^2 - 2s + 2}$
8. $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2 + 4)}$
9. $F(s) = \dfrac{1 - 2s}{s^2 + 4s + 5}$
10. $F(s) = \dfrac{2s - 3}{s^2 + 2s + 10}$

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

11. $y'' - y' - 6y = 0; \qquad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = 0; \qquad y(0) = 1, \quad y'(0) = 0$

13. $y'' - 2y' + 2y = 0; \qquad y(0) = 0, \quad y'(0) = 1$
14. $y'' - 4y' + 4y = 0; \qquad y(0) = 1, \quad y'(0) = 1$
15. $y'' - 2y' + 4y = 0; \qquad y(0) = 2, \quad y'(0) = 0$
16. $y'' + 2y' + 5y = 0; \qquad y(0) = 2, \quad y'(0) = -1$
17. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0; \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$
18. $y^{(4)} - y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$
19. $y^{(4)} - 4y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
20. $y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4; \qquad y(0) = 1, \quad y'(0) = 0$
21. $y'' - 2y' + 2y = \cos t; \qquad y(0) = 1, \quad y'(0) = 0$
22. $y'' - 2y' + 2y = e^{-t}; \qquad y(0) = 0, \quad y'(0) = 1$
23. $y'' + 2y' + y = 4e^{-t}; \qquad y(0) = 2, \quad y'(0) = -1$

In each of Problems 24 through 27, find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3. You may wish to refer to Problems 21 through 24 in Section 6.1.

24. $y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases} \qquad y(0) = 1, \quad y'(0) = 0$

25. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 0, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

26. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

27. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2, \\ 0, & 2 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$

28. The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.
    (a) Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \qquad s > 1.$$

    (b) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.$$

(c) The Bessel function of the first kind of order zero, J_0, has the Taylor series (see Section 5.7)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Assuming that the following Laplace transforms can be computed term by term, verify that

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}, \qquad s > 1$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/(4s)}, \qquad s > 0.$$

Problems 29 through 37 are concerned with differentiation of the Laplace transform.

29. Let

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s > a$.

(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

30. $f(t) = te^{at}$
31. $f(t) = t^2 \sin bt$
32. $f(t) = t^n$
33. $f(t) = t^n e^{at}$
34. $f(t) = te^{at} \sin bt$
35. $f(t) = te^{at} \cos bt$
36. Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$$(1+s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Writing $(1+s^2)^{-1/2} = s^{-1}(1+s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.

37. For each of the following initial value problems, use the results of Problem 29 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{\phi(t)\}$, where $y = \phi(t)$ is the solution of the given initial value problem.

(a) $y'' - ty = 0; \quad y(0) = 1, \quad y'(0) = 0$ (Airy's equation)

(b) $(1 - t^2)y'' - 2ty' + \alpha(\alpha + 1)y = 0; \quad y(0) = 0, \quad y'(0) = 1$ (Legendre's equation)

Note that the differential equation for $Y(s)$ is of first order in part (a), but of second order in part (b). This is due to the fact that t appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

38. Suppose that

$$g(t) = \int_0^t f(\tau)\, d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = F(s)/s.$$

39. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

$$F(s) = P(s)/Q(s),$$

where $Q(s)$ is a polynomial of degree n with distinct zeros $r_1, \ldots, r_n$, and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{i}$$

where the coefficients $A_1, \ldots, A_n$ must be determined.

(a) Show that

$$A_k = P(r_k)/Q'(r_k), \qquad k = 1, \ldots, n. \tag{ii}$$

Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.

(b) Show that

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii}$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the following ones, we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing