# Ch 7.6: Complex Eigenvalues

- We consider again a homogeneous system of $n$ first order linear equations with constant, real coefficients,

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{aligned}$$

and thus the system can be written as **x**' = **Ax**, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

# Conjugate Eigenvalues and Eigenvectors

- We know that $\mathbf{x} = \xi e^{rt}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, provided $r$ is an eigenvalue and $\xi$ is an eigenvector of $\mathbf{A}$.
- The eigenvalues $r_1, \ldots, r_n$ are the roots of $\det(\mathbf{A} - r\mathbf{I}) = 0$, and the corresponding eigenvectors satisfy $(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$.
- If $\mathbf{A}$ is real, then the coefficients in the polynomial equation $\det(\mathbf{A} - r\mathbf{I}) = 0$ are real, and hence any complex eigenvalues must occur in conjugate pairs. Thus if $r_1 = \lambda + i\mu$ is an eigenvalue, then so is $r_2 = \lambda - i\mu$.
- The corresponding eigenvectors $\xi^{(1)}$, $\xi^{(2)}$ are conjugates also.

To see this, recall $\mathbf{A}$ and $\mathbf{I}$ have real entries, and hence

$$\left(\mathbf{A} - r_1\mathbf{I}\right)\boldsymbol{\xi}^{(1)} = \mathbf{0} \Rightarrow \left(\mathbf{A} - \bar{r}_1\mathbf{I}\right)\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{0} \Rightarrow \left(\mathbf{A} - r_2\mathbf{I}\right)\boldsymbol{\xi}^{(2)} = \mathbf{0}$$

# Conjugate Solutions

- It follows from the previous slide that the solutions

$$\mathbf{x}^{(1)} = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \quad \mathbf{x}^{(2)} = \boldsymbol{\xi}^{(2)} e^{r_2 t}$$

corresponding to these eigenvalues and eigenvectors are conjugates conjugates as well, since

$$\mathbf{x}^{(2)} = \boldsymbol{\xi}^{(2)} e^{r_2 t} = \overline{\boldsymbol{\xi}}^{(1)} e^{\bar{r}_2 t} = \overline{\mathbf{x}}^{(1)}$$

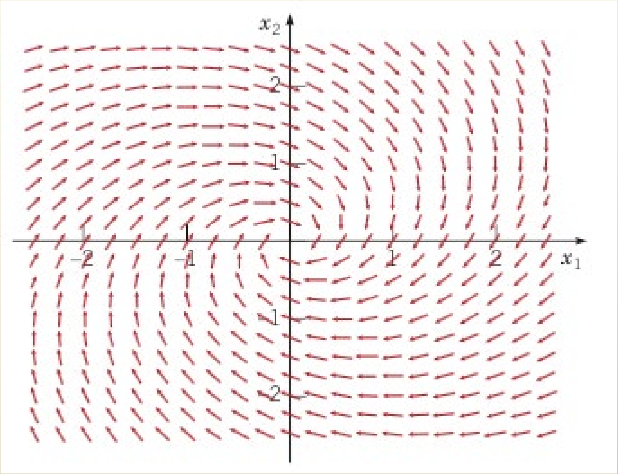
# Example 1: Direction Field (1 of 7)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \xi e^{rt}$ in for $\mathbf{x}$, and rewriting system as $(\mathbf{A} - r\mathbf{I})\,\xi = \mathbf{0}$, we obtain

$$\begin{pmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



# Example 1: Complex Eigenvalues (2 of 7)

- We determine $r$ by solving $\det(\mathbf{A} - r\mathbf{I}) = 0$. Now

$$\begin{vmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{vmatrix} = (r + 1/2)^2 + 1 = r^2 + r + \frac{5}{4}$$

- Thus

$$r = \frac{-1 \pm \sqrt{1^2 - 4(5/4)}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

- Therefore the eigenvalues are $r_1 = -1/2 + i$ and $r_2 = -1/2 - i$.

# Example 1: First Eigenvector (3 of 7)

- Eigenvector for $r_1 = -1/2 + i$: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & i & 0 \\ -1 & -i & 0 \end{pmatrix} \to \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -i\xi_2 \\ \xi_2 \end{pmatrix} \to \text{choose } \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

- Thus

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

# Example 1: Second Eigenvector (4 of 7)

- Eigenvector for $r_1 = -1/2 - i$: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \;\Leftrightarrow\; \begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Leftrightarrow\; \begin{pmatrix} 1 & -i \\ -1 & i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & -i & 0 \\ -1 & i & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \boldsymbol{\xi}^{(2)} = \begin{pmatrix} i\xi_2 \\ \xi_2 \end{pmatrix} \to \text{choose } \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

- Thus

$$\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

# Example 1: General Solution (5 of 7)

- The corresponding solutions $\mathbf{x} = \xi e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{u}(t) = e^{-t/2}\left[\begin{pmatrix}1\\0\end{pmatrix}\cos t - \begin{pmatrix}0\\1\end{pmatrix}\sin t\right] = e^{-t/2}\begin{pmatrix}\cos t\\-\sin t\end{pmatrix}$$

$$\mathbf{v}(t) = e^{-t/2}\left[\begin{pmatrix}1\\0\end{pmatrix}\sin t + \begin{pmatrix}0\\1\end{pmatrix}\cos t\right] = e^{-t/2}\begin{pmatrix}\sin t\\\cos t\end{pmatrix}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} = e^{-t} \neq 0$$

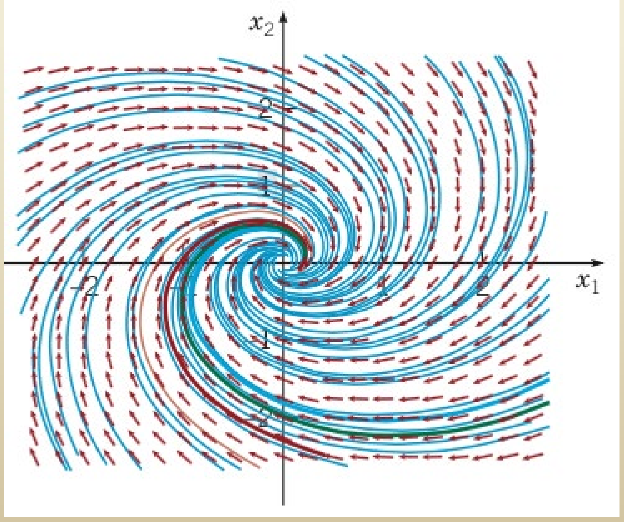
- Thus $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are real-valued fundamental solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with general solution $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$.

# Example 1: Phase Plane (6 of 7)

- Given below is the phase plane plot for solutions **x**, with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}$$

- Each solution trajectory approaches origin along a spiral path as $t \to \infty$, since coordinates are products of decaying exponential and sine or cosine factors.
- The graph of **u** passes through (1,0), since u(0) = (1,0). Similarly, the graph of **v** passes through (0,1).
- The origin is a **spiral point**, and is asymptotically stable.

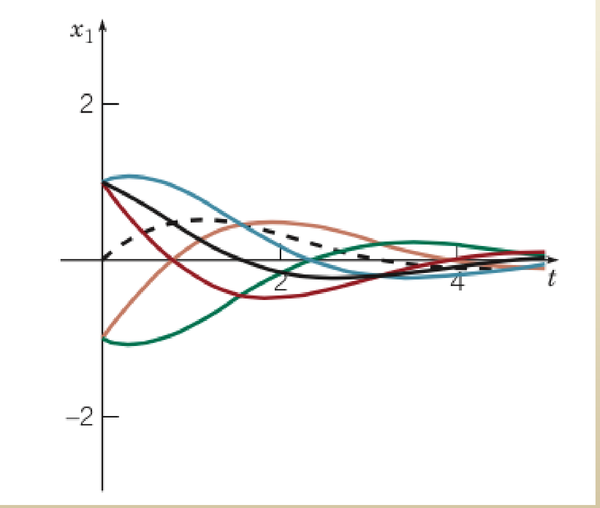


# Example 1: Time Plots (7 of 7)

- The general solution is $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$:

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t \\ -c_1 e^{-t/2}\sin t + c_2 e^{-t/2}\cos t \end{pmatrix}$$

- As an alternative to phase plane plots, we can graph $x_1$ or $x_2$ as a function of $t$. A few plots of $x_1$ are given below, each one a decaying oscillation as $t \to \infty$.



# General Solution

- To summarize, suppose $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$ and that $r_3, \ldots, r_n$ are all real and distinct eigenvalues of $\mathbf{A}$. Let the corresponding eigenvectors be

$$\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b},\ \boldsymbol{\xi}^{(2)} = \mathbf{a} - i\mathbf{b},\ \boldsymbol{\xi}^{(3)}, \boldsymbol{\xi}^{(4)}, \ldots, \boldsymbol{\xi}^{(n)}$$

- Then the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\boldsymbol{\xi}^{(3)}e^{r_3 t} + \ldots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}$$

where

$$\mathbf{u}(t) = e^{\lambda t}\left(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t\right),\ \mathbf{v}(t) = e^{\lambda t}\left(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t\right)$$

# Real-Valued Solutions

- Thus for complex conjugate eigenvalues $r_1$ and $r_2$, the corresponding solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are conjugates also.
- To obtain real-valued solutions, use real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. To see this, let $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$. Then

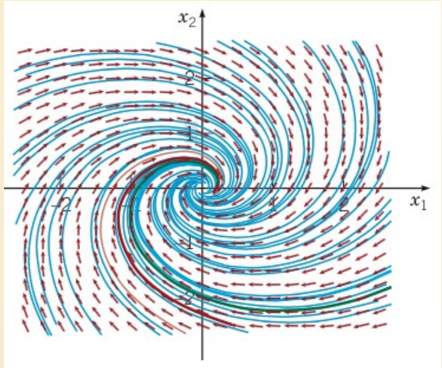
$$\begin{aligned}\mathbf{x}^{(1)} &= \boldsymbol{\xi}^{(1)} e^{(\lambda + i\mu)t} = (\mathbf{a} + i\mathbf{b}) e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ &= e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t) + i e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t) \\ &= \mathbf{u}(t) + i\,\mathbf{v}(t)\end{aligned}$$

where

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a} \cos \mu t - \mathbf{b} \sin \mu t),\ \mathbf{v}(t) = e^{\lambda t}(\mathbf{a} \sin \mu t + \mathbf{b} \cos \mu t),$$

are real valued solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and can be shown to be linearly independent.

# Spiral Points, Centers, Eigenvalues, and Trajectories



- In previous example, general solution was

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}$$

- The origin was a **spiral point**, and was asymptotically stable.
- If real part of complex eigenvalues is positive, then trajectories spiral away, unbounded, from origin, and hence origin would be an unstable spiral point.
- If real part of complex eigenvalues is zero, then trajectories circle origin, neither approaching nor departing. Then origin is called a **center** and is stable, but not asymptotically stable. Trajectories periodic in time.
- The direction of trajectory motion depends on entries in **A**.

# Example 2:
# Second Order System with Parameter (1 of 2)

- The system **x'** = **Ax** below contains a parameter $\alpha$.

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}$$

- Substituting **x** = $\xi e^{rt}$ in for **x** and rewriting system as (**A** – $r$**I**) $\xi$ = **0**, we obtain

$$\begin{pmatrix} \alpha - r & 2 \\ -2 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Next, solve for $r$ in terms of $\alpha$:

$$\begin{vmatrix} \alpha - r & 2 \\ -2 & -r \end{vmatrix} = r(r-\alpha) + 4 = r^2 - \alpha r + 4 \Rightarrow r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

# Example 2: Eigenvalue Analysis (2 of 2)

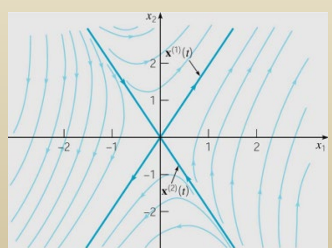
$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

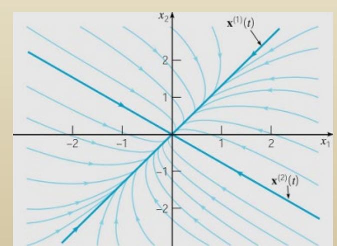
- The eigenvalues are given by the quadratic formula above.
- For $\alpha < -4$, both eigenvalues are real and negative, and hence origin is asymptotically stable node.
- For $\alpha > 4$, both eigenvalues are real and positive, and hence the origin is an unstable node.
- For $-4 < \alpha < 0$, eigenvalues are complex with a negative real part, and hence origin is asymptotically stable spiral point.
- For $0 < \alpha < 4$, eigenvalues are complex with a positive real part, and the origin is an unstable spiral point.
- For $\alpha = 0$, eigenvalues are purely imaginary, origin is a center. Trajectories closed curves about origin & periodic.
- For $\alpha = \pm\ 4$, eigenvalues real & equal, origin is a node (Ch 7.8)

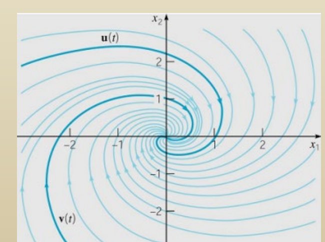
# Second Order Solution Behavior and Eigenvalues: Three Main Cases

- For second order systems, the three main cases are:
  - Eigenvalues are real and have opposite signs; $\mathbf{x} = \mathbf{0}$ is a saddle point.
  - Eigenvalues are real, distinct and have same sign; $\mathbf{x} = \mathbf{0}$ is a node.
  - Eigenvalues are complex with nonzero real part; $\mathbf{x} = \mathbf{0}$ a spiral point.
- Other possibilities exist and occur as transitions between two of the cases listed above:
  - A zero eigenvalue occurs during transition between saddle point and node. Real and equal eigenvalues occur during transition between nodes and spiral points. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$







# Example 3: Multiple Spring-Mass System (1 of 6)

- The equations for the system of two masses and three springs discussed in Section 7.1, assuming no external forces, can be expressed as:

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2 x_2 \quad \text{and} \quad m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - (k_2 + k_3)x_2$$

or $m_1 y_3' = -(k_1 + k_2) y_1 + k_2 y_2$ and $m_2 y_4' = k_2 y_1 - (k_2 + k_3) y_2$

where $y_1 = x_1,\ y_2 = x_2,\ y_3 = x_1'$, and $y_4 = x_2'$

- Given $m_1 = 2,\ m_2 = 9/4,\ k_1 = 1,\ k_2 = 3$, and $k_3,= 15/4$, the equations become

$$y_1' = y_3,\ y_2' = y_4,\ y_3' = -2y_1 + 3/2\ y_2, \text{ and } y_4' = 4/3\ y_1 - 3y_2$$

$y_1' = y_3,\ y_2' = y_4,\ y_3' = -2y_1 + 3/2\ y_2$, and $y_4' = 4/3\ y_1 - 3y_2$

# Example 3: Multiple Spring-Mass System (2 of 6)

- Writing the system of equations in matrix form:

$$y' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix} y = \mathrm{Ay}$$

- Assuming a solution of the form $\boldsymbol{y} = \xi e^{rt}$, where $r$ must be an eigenvalue of the matrix **A** and $\xi$ is the corresponding eigenvector, the characteristic polynomial of **A** is

$$r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4)$$

$$r_1 = i,\ r_2 = -i,\ r_3 = 2i, \text{ and } r_4 = -2i$$

yielding the eigenvalues:

$$y' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3/2 & 0 & 0 \\ 4/3 & -3 & 0 & 0 \end{pmatrix} y = Ay$$

# Example 3: Multiple Spring-Mass System (3 of 6)

- For the eigenvalues $r_1 = i,\ r_2 = -i,\ r_3 = 2i$, and $r_4 = -2i$ the corresponding eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix},\ \xi^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix},\ \xi^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \text{ and } \xi^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}$$

- The products $\boldsymbol{\xi}^{(1)} e^{it}$ and $\boldsymbol{\xi}^{(3)} e^{2it}$ yield the complex-valued solutions:

$$\xi^{(1)} e^{it} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + i \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} = \mathbf{u}^{(1)}(t) + i\,\mathbf{v}^{(1)}(t)$$

$$\xi^{(3)} e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix} (\cos 2t + i \sin 2t) = \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + i \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} = \mathbf{u}^{(2)}(t) + i\,\mathbf{v}^{(2)}(t)$$

$y_1' = y_3,\ y_2' = y_4,\ y_3' = -2y_1 + 3/2\ y_2$, and $y_4' = 4/3\ y_1 - 3y_2$

# Example 3: Multiple Spring-Mass System (4 of 6)

- After validating that $\mathbf{u}^{(1)}(t),\ \mathbf{v}^{(1)}(t),\ \mathbf{u}^{(2)}(t),\ \mathbf{v}^{(2)}(t)$ are linearly independent, the general solution of the system of equations can be written as

$$y = c_1\begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + c_2\begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} + c_3\begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + c_4\begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix}$$

- where $c_1, c_2, c_3, c_4$ are arbitrary constants.
- Each solution will be periodic with period 2π, so each trajectory is a closed curve. The first two terms of the solution describe motions with frequency 1 and period 2π while the second two terms describe motions with frequency 2 and period π. The motions of the two masses will be different relative to one another for solutions involving only the first two terms or the second two terms.

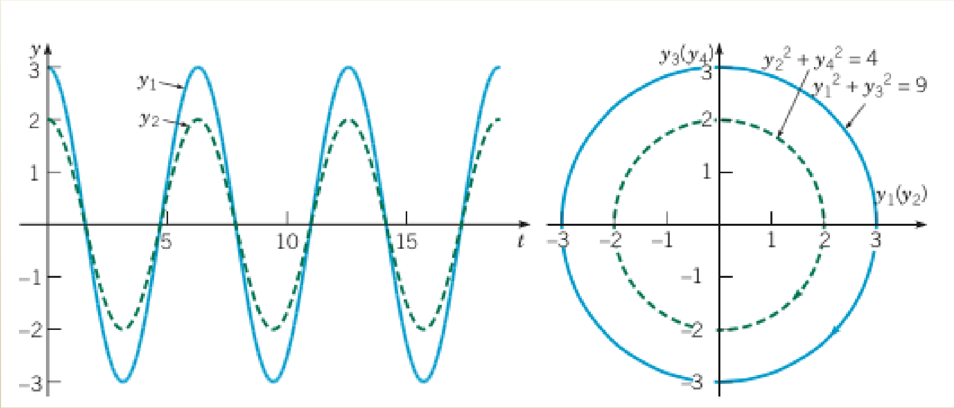
$y_1$ and $y_2$ represent the motion of the masses and $y_3 = y_1', y_4 = y_1'$

# Example 3: Multiple Spring-Mass System (5 of 6)

- To obtain the fundamental mode of vibration with frequency 1

  $c_3 = c_4 = 0 \rightarrow$ occurs when $3y_2(0) = 2\,y_1(0)$ and $3y_4(0) = 2\,y_3(0)$
- To obtain the fundamental mode of vibration with frequency 2

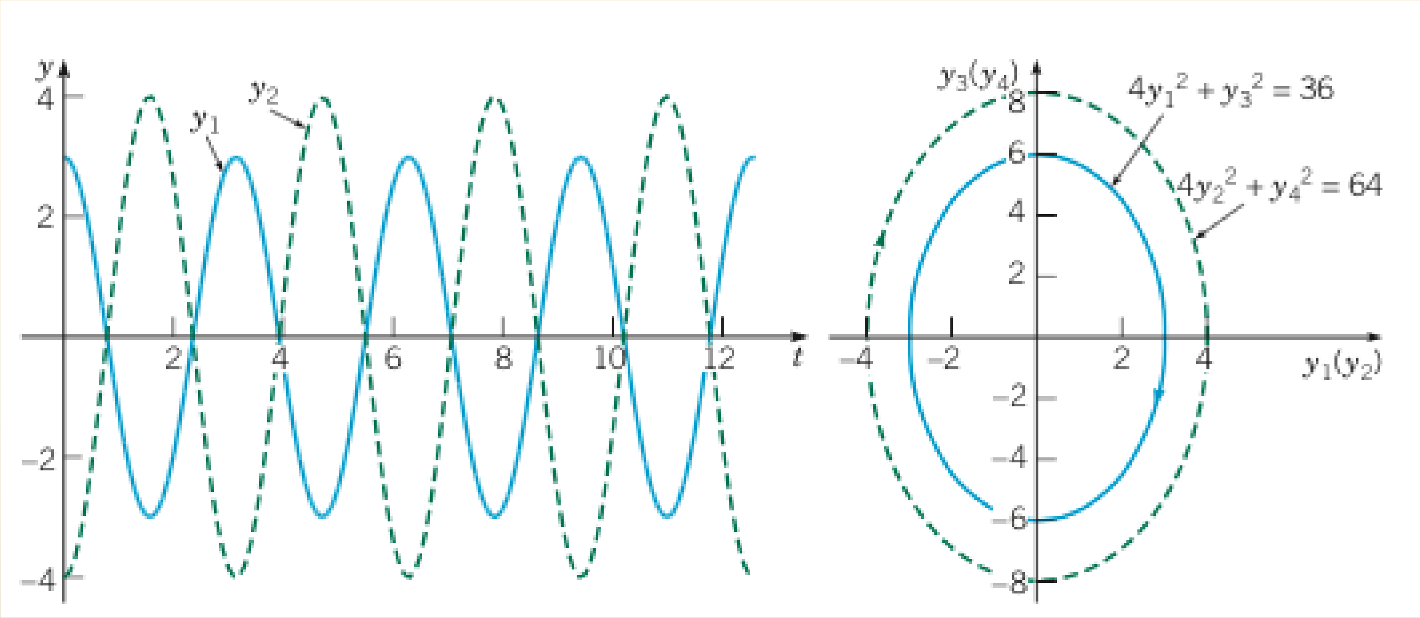
  $c_1 = c_2 = 0 \rightarrow$ occurs when $3y_2(0) = -4\,y_1(0)$ and $3y_4(0) = -4\,y_3(0)$
- Plots of $y_1$ and $y_2$ and parametric plots $(y, y')$ are shown for a selected solution with frequency 1



$y_1$ and $y_2$ represent the motion of the masses and $y_3 = y_1', y_4 = y_1'$

# Example 3: Multiple Spring-Mass System (6 of 6)

- Plots of $y_1$ and $y_2$ and parametric plots $(y, y')$ are shown for a selected solution with frequency 2



# Ch 7.7: Fundamental Matrices

- Suppose that $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on $\alpha < t < \beta$.
- The matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix},$$

whose columns are $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$, is a fundamental matrix for the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. This matrix is nonsingular since its columns are linearly independent, and hence det $\mathbf{\Psi} \neq 0$.

- Note also that since $\mathbf{x}^{(1)}(t),\ldots, \mathbf{x}^{(n)}(t)$ are solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, $\mathbf{\Psi}$ satisfies the matrix differential equation $\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}$.

# Example 1:

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

- In Example 2 of Chapter 7.5, we found the following fundamental solutions for this system:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- Thus a fundamental matrix for this system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

# Fundamental Matrices and General Solution

- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}$$

can be expressed $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector with components $c_1, \ldots, c_n$:

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

# Fundamental Matrix & Initial Value Problem

- Consider an initial value problem
$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \ \ \mathbf{x}(t_0) = \mathbf{x}^0$$
where $\alpha < t_0 < \beta$ and $\mathbf{x}^0$ is a given initial vector.
- Now the solution has the form $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$, hence we choose $\mathbf{c}$ so as to satisfy $\mathbf{x}(t_0) = \mathbf{x}^0$.
- Recalling $\mathbf{\Psi}(t_0)$ is nonsingular, it follows that
$$\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x}^0 \quad \Rightarrow \quad \mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$$
- Thus our solution $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$ can be expressed as

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$$

# Recall: Theorem 7.4.4

- Let

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on $I$: $\alpha < t < \beta$ that satisfy the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \ \ldots, \ \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)}, \ \ \alpha < t_0 < \beta$$

  Then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are fundamental solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

# Fundamental Matrix & Theorem 7.4.4

- Suppose $\mathbf{x}^{(1)}(t),\ldots,\ \mathbf{x}^{(n)}(t)$ form the fundamental solutions given by Thm 7.4.4. Denote the corresponding fundamental matrix by $\mathbf{\Phi}(t)$. Then columns of $\mathbf{\Phi}(t)$ are $\mathbf{x}^{(1)}(t),\ldots,\ \mathbf{x}^{(n)}(t)$, and hence

$$\Phi(t_0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{I}$$

- Thus $\mathbf{\Phi}^{-1}(t_0) = \mathbf{I}$, and the hence general solution to the corresponding initial value problem is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}^0 = \mathbf{\Phi}(t)\mathbf{x}^0$$

- It follows that for any fundamental matrix $\mathbf{\Phi}(t)$,

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0 = \mathbf{\Phi}(t)\mathbf{x}^0 \;\Rightarrow\; \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$$

# The Fundamental Matrix $\Phi$ and Varying Initial Conditions

- Thus when using the fundamental matrix $\Phi(t)$, the general solution to an IVP is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}^0 = \mathbf{\Phi}(t)\mathbf{x}^0$$

- This representation is useful if same system is to be solved for many different initial conditions, such as a physical system that can be started from many different initial states.
- Also, once $\Phi(t)$ has been determined, the solution to each set of initial conditions can be found by matrix multiplication, as indicated by the equation above.
- Thus $\Phi(t)$ represents a linear transformation of the initial conditions $\mathbf{x}^0$ into the solution $\mathbf{x}(t)$ at time $t$.

# Example 2: Find $\Phi(t)$ for 2 x 2 System (1 of 5)

- Find $\Phi(t)$ such that $\Phi(0) = \mathbf{I}$ for the system below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$$

- Solution: First, we must obtain $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ such that

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- We know from previous results that the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}$$

- Every solution can be expressed in terms of the general solution, and we use this fact to find $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$.

# Example 2: Use General Solution (2 of 5)

- Thus, to find $\mathbf{x}^{(1)}(t)$, express it terms of the general solution

$$\mathbf{x}^{(1)}(t) = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-t}$$

and then find the coefficients $c_1$ and $c_2$.

- To do so, use the initial conditions to obtain

$$\mathbf{x}^{(1)}(0) = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

or equivalently,

$$\begin{pmatrix}1 & 1\\2 & -2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

# Example 2: Solve for $\mathbf{x}^{(1)}(t)$ (3 of 5)

- To find $\mathbf{x}^{(1)}(t)$, we therefore solve

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{pmatrix}$$

$$\to \begin{array}{ll} c_1 & = 1/2 \\ \quad c_2 & = 1/2 \end{array}$$

- Thus

$$\mathbf{x}^{(1)}(t) = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \frac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

# Example 2: Solve for $\mathbf{x}^{(2)}(t)$ (4 of 5)

- To find $\mathbf{x}^{(2)}(t)$, we similarly solve

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1/4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -1/4 \end{pmatrix}$$

$$\rightarrow \begin{array}{rl} c_1 & = 1/4 \\ c_2 & = -1/4 \end{array}$$

- Thus

$$\mathbf{x}^{(2)}(t) = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} - \frac{1}{4}\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

# Example 2: Obtain $\mathbf{\Phi}(t)$ (5 of 5)

- The columns of $\mathbf{\Phi}(t)$ are given by $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$, and thus from the previous slide we have

$$\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

- Note $\mathbf{\Phi}(t)$ is more complicated than $\mathbf{\Psi}(t)$ found in Ex 1. However, it is now much easier to determine the solution to any set of initial conditions.

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

# Matrix Exponential Functions

- Consider the following two cases:
  - The solution to $x' = ax$, $x(0) = x_0$, is $x = x_0 e^{at}$, where $e^0 = 1$.
  - The solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$, is $\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0$, where $\mathbf{\Phi}(0) = \mathbf{I}$.
- Comparing the form and solution for both of these cases, we might expect $\mathbf{\Phi}(t)$ to have an exponential character.
- Indeed, it can be shown that $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$, where

$$e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$$

  is a well defined matrix function that has all the usual properties of an exponential function. See text for details.
- Thus the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$, is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}^0$.

# Coupled Systems of Equations

- Recall that our constant coefficient homogeneous system

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n,
\end{aligned}
$$

written as $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with

$$
\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},
$$

is a system of *coupled* equations that must be solved *simultaneously* to find all the unknown variables.

# Uncoupled Systems & Diagonal Matrices

- In contrast, if each equation had only one variable, solved for independently of other equations, then task would be easier.
- In this case our system would have the form

$$
\begin{aligned}
x_1' &= d_{11}x_1 + 0x_2 + \ldots + 0x_n \\
x_2' &= 0x_1 + d_{11}x_2 + \ldots + 0x_n \\
&\vdots \\
x_n' &= 0x_1 + 0x_2 + \ldots + d_{nn}x_n,
\end{aligned}
$$

or $\mathbf{x}' = \mathbf{D}\mathbf{x}$, where $\mathbf{D}$ is a diagonal matrix:

$$
\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}
$$

# Uncoupling: Transform Matrix T

- In order to explore transforming our given system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ of coupled equations into an uncoupled system $\mathbf{x}' = \mathbf{D}\mathbf{x}$, where **D** is a diagonal matrix, we will use the eigenvectors of **A**.
- Suppose **A** is *n* x *n* with *n* linearly independent eigenvectors $\xi^{(1)}, \ldots, \xi^{(n)}$, and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.
- Define *n* x *n* matrices **T** and **D** using the eigenvalues & eigenvectors of **A**:

$$\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

- Note that **T** is nonsingular, and hence $\mathbf{T}^{-1}$ exists.

# Uncoupling: $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$

- Recall here the definitions of **A**, **T** and **D**:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

- Then the columns of **AT** are $\mathbf{A}\,\xi^{(1)},\ldots,\ \mathbf{A}\ \xi^{(n)}$, and hence

$$\mathbf{AT} = \begin{pmatrix} \lambda_1\xi_1^{(1)} & \cdots & \lambda_n\xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \lambda_1\xi_n^{(1)} & \cdots & \lambda_n\xi_n^{(n)} \end{pmatrix} = \mathbf{TD}$$

- It follows that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$.

# Similarity Transformations

- Thus, if the eigenvalues and eigenvectors of **A** are known, then **A** can be transformed into a diagonal matrix **D**, with

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

- This process is known as a **similarity transformation**, and **A** is said to be **similar** to **D**. Alternatively, we could say that **A** is **diagonalizable**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

# Similarity Transformations: Hermitian Case

- Recall: Our similarity transformation of **A** has the form

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

  where **D** is diagonal and columns of **T** are eigenvectors of **A**.
- If **A** is Hermitian, then **A** has $n$ linearly independent orthogonal eigenvectors $\xi^{(1)},\ldots,\xi^{(n)}$, normalized so that $(\xi^{(i)},\xi^{(i)}) = 1$ for $i = 1,\ldots, n$, and $(\xi^{(i)},\xi^{(k)}) = 0$ for $i \neq k$.
- With this selection of eigenvectors, it can be shown that $\mathbf{T}^{-1} = \mathbf{T}^*$. In this case we can write our similarity transform as

$$\mathbf{T}^*\mathbf{A}\mathbf{T} = \mathbf{D}$$

# Nondiagonalizable A

- Finally, if **A** is $n$ x $n$ with fewer than $n$ linearly independent eigenvectors, then there is no matrix **T** such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$.
- In this case, **A** is not similar to a diagonal matrix and **A** is not diagonlizable.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

# Example 3: Find Transformation Matrix T (1 of 2)

- For the matrix **A** below, find the similarity transformation matrix **T** and show that **A** can be diagonalized.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

- We already know that the eigenvalues are $\lambda_1 = 3, \ \lambda_2 = -1$ with corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ \boldsymbol{\xi}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- Thus

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

# Example 3: Similarity Transformation (2 of 2)

- To find $\mathbf{T}^{-1}$, augment the identity to $\mathbf{T}$ and row reduce:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/2 & -1/4 \end{pmatrix} \to \mathbf{T}^{-1} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix}$$

- Then

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \left[ \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{D}$$

- Thus **A** is similar to **D**, and hence **A** is diagonalizable.

# Fundamental Matrices for Similar Systems (1 of 3)

- Recall our original system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- If $\mathbf{A}$ is $n$ x $n$ with $n$ linearly independent eigenvectors, then $\mathbf{A}$ is diagonalizable. The eigenvectors form the columns of the nonsingular transform matrix $\mathbf{T}$, and the eigenvalues are the corresponding nonzero entries in the diagonal matrix $\mathbf{D}$.
- Suppose $\mathbf{x}$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$, let $\mathbf{y}$ be the $n$ x 1 vector such that $\mathbf{x} = \mathbf{T}\mathbf{y}$. That is, let $\mathbf{y}$ be defined by $\mathbf{y} = \mathbf{T}^{-1}\mathbf{x}$.
- Since $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{T}$ is a constant matrix, we have $\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y}$, and hence $\mathbf{y}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} = \mathbf{D}\mathbf{y}$.
- Therefore $\mathbf{y}$ satisfies $\mathbf{y}' = \mathbf{D}\mathbf{y}$, the system similar to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- Both of these systems have fundamental matrices, which we examine next.

# Fundamental Matrix for Diagonal System (2 of 3)

- A fundamental matrix for $\mathbf{y}' = \mathbf{D}\mathbf{y}$ is given by $\mathbf{Q}(t) = e^{\mathbf{D}t}$.
- Recalling the definition of $e^{\mathbf{D}t}$, we have

$$\mathbf{Q}(t) = \sum_{n=0}^{\infty} \frac{\mathbf{D}^n t^n}{n!} = \sum_{n=0}^{\infty} \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^n \end{pmatrix} \frac{t^n}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{n!} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sum_{n=0}^{\infty} \frac{(\lambda_n t)^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{pmatrix}$$

# Fundamental Matrix for Original System (3 of 3)

- To obtain a fundamental matrix $\Psi(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$, recall that the columns of $\Psi(t)$ consist of fundamental solutions $\mathbf{x}$ satisfying $\mathbf{x}' = \mathbf{A}\mathbf{x}$. We also know $\mathbf{x} = \mathbf{T}\mathbf{y}$, and hence it follows that

$$\mathbf{\Psi} = \mathbf{TQ} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{pmatrix} = \begin{pmatrix} \xi_1^{(1)} e^{\lambda_1 t} & \cdots & \xi_1^{(n)} e^{\lambda_n t} \\ \vdots & \ddots & \vdots \\ \xi_n^{(1)} e^{\lambda_1 t} & \cdots & \xi_n^{(n)} e^{\lambda_n t} \end{pmatrix}$$

- The columns of $\Psi(t)$ given the expected fundamental solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

# Example 4:
# Fundamental Matrices for Similar Systems

- We now use the analysis and results of the last few slides.
- Applying the transformation $\mathbf{x} = \mathbf{Ty}$ to $\mathbf{x}' = \mathbf{Ax}$ below, this system becomes $\mathbf{y}' = \mathbf{T}^{-1}\mathbf{ATy} = \mathbf{Dy}$:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} \quad \Rightarrow \quad \mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y}$$

- A fundamental matrix for $\mathbf{y}' = \mathbf{Dy}$ is given by $\mathbf{Q}(t) = e^{\mathbf{D}t}$:

$$\mathbf{Q}(t) = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

- Thus a fundamental matrix $\Psi(t)$ for $\mathbf{x}' = \mathbf{Ax}$ is

$$\Psi(t) = \mathbf{TQ} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$