23. Consider two interconnected tanks similar to those in Figure 7.1.6. Initially, Tank 1 contains 60 gal of water and Q_1^0 oz of salt, and Tank 2 contains 100 gal of water and Q_2^0 oz of salt. Water containing q_1 oz/gal of salt flows into Tank 1 at a rate of 3 gal/min. The mixture in Tank 1 flows out at a rate of 4 gal/min, of which half flows into Tank 2, while the remainder leaves the system. Water containing q_2 oz/gal of salt also flows into Tank 2 from the outside at the rate of 1 gal/min. The mixture in Tank 2 leaves it at a rate of 3 gal/min, of which some flows back into Tank 1 at a rate of 1 gal/min, while the rest leaves the system.

(a) Draw a diagram that depicts the flow process described above. Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions for Q_1 and Q_2 that model the flow process.

(b) Find the equilibrium values Q_1^E and Q_2^E in terms of the concentrations q_1 and q_2.

(c) Is it possible (by adjusting q_1 and q_2) to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state?

(d) Describe which equilibrium states are possible for this system for various values of q_1 and q_2.

7.2 Review of Matrices

For both theoretical and computational reasons, it is advisable to bring some of the results of matrix algebra[2] to bear on the initial value problem for a system of linear differential equations. For reference purposes, this section and the next are devoted to a brief summary of the facts that will be needed later. More details can be found in any elementary book on linear algebra. We assume, however, that you are familiar with determinants and how to evaluate them.

We designate matrices by boldfaced capitals $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$, occasionally using boldfaced Greek capitals $\boldsymbol{\Phi}, \boldsymbol{\Psi}, \dots$. A matrix $\mathbf{A}$ consists of a rectangular array of numbers, or elements, arranged in m rows and n columns—that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{1}$$

We speak of $\mathbf{A}$ as an $m \times n$ matrix. Although later in the chapter we will often assume that the elements of certain matrices are real numbers, in this section we assume that

[2]The properties of matrices were first extensively explored in 1858 in a paper by the English algebraist Arthur Cayley (1821–1895), although the word "matrix" was introduced by his good friend James Sylvester (1814–1897) in 1850. Cayley did some of his best mathematical work while practicing law from 1849 to 1863; he then became professor of mathematics at Cambridge, a position he held for the rest of his life. After Cayley's groundbreaking work, the development of matrix theory proceeded rapidly, with significant contributions by Charles Hermite, Georg Frobenius, and Camille Jordan, among others.

the elements of matrices may be complex numbers. The element lying in the ith row and jth column is designated by a_{ij}, the first subscript identifying its row and the second its column. Sometimes the notation (a_{ij}) is used to denote the matrix whose generic element is a_{ij}.

Associated with each matrix $\mathbf{A}$ is the matrix $\mathbf{A}^T$, which is known as the **transpose** of $\mathbf{A}$ and is obtained from $\mathbf{A}$ by interchanging the rows and columns of $\mathbf{A}$. Thus, if $\mathbf{A} = (a_{ij})$, then $\mathbf{A}^T = (a_{ji})$. Also, we will denote by $\overline{a}_{ij}$ the complex conjugate of a_{ij}, and by $\overline{\mathbf{A}}$ the matrix obtained from $\mathbf{A}$ by replacing each element a_{ij} by its conjugate $\overline{a}_{ij}$. The matrix $\overline{\mathbf{A}}$ is called the **conjugate** of $\mathbf{A}$. It will also be necessary to consider the transpose of the conjugate matrix $\overline{\mathbf{A}}^T$. This matrix is called the **adjoint** of $\mathbf{A}$ and will be denoted by $\mathbf{A}^*$.

For example, let

$$\mathbf{A} = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}.$$

Then

$$\mathbf{A}^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix}, \qquad \overline{\mathbf{A}} = \begin{pmatrix} 3 & 2+i \\ 4-3i & -5-2i \end{pmatrix},$$

$$\mathbf{A}^* = \begin{pmatrix} 3 & 4-3i \\ 2+i & -5-2i \end{pmatrix}.$$

We are particularly interested in two somewhat special kinds of matrices: square matrices, which have the same number of rows and columns—that is, $m = n$; and vectors (or column vectors), which can be thought of as $n \times 1$ matrices, or matrices having only one column. Square matrices having n rows and n columns are said to be of order n. We denote (column) vectors by boldfaced lowercase letters: $\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$. The transpose $\mathbf{x}^T$ of an $n \times 1$ column vector is a $1 \times n$ row vector—that is, the matrix consisting of one row whose elements are the same as the elements in the corresponding positions of $\mathbf{x}$.

Properties of Matrices.

1. **Equality.** Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be equal if all corresponding elements are equal—that is, if $a_{ij} = b_{ij}$ for each i and j.

2. **Zero.** The symbol $\mathbf{0}$ will be used to denote the matrix (or vector) each of whose elements is zero.

3. **Addition.** The sum of two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as the matrix obtained by adding corresponding elements:

$$\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}). \tag{2}$$

With this definition, it follows that matrix addition is commutative and associative, so that

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}, \qquad \mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}. \tag{3}$$

4. **Multiplication by a Number.** The product of a matrix $\mathbf{A}$ by a real or complex number α is defined as follows:

$$\alpha\mathbf{A}=\alpha(a_{ij})=(\alpha a_{ij}); \tag{4}$$

that is, each element of $\mathbf{A}$ is multiplied by α. The distributive laws

$$\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}, \qquad (\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A} \tag{5}$$

are satisfied for this type of multiplication. In particular, the negative of $\mathbf{A}$, denoted by $-\mathbf{A}$, is defined by

$$-\mathbf{A}=(-1)\mathbf{A}. \tag{6}$$

5. **Subtraction.** The difference $\mathbf{A}-\mathbf{B}$ of two $m \times n$ matrices is defined by

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B}). \tag{7}$$

Thus

$$\mathbf{A}-\mathbf{B}=(a_{ij})-(b_{ij})=(a_{ij}-b_{ij}), \tag{8}$$

which is similar to Eq. (2).

6. **Multiplication.** The product $\mathbf{AB}$ of two matrices is defined whenever the number of columns in the first factor is the same as the number of rows in the second. If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ and $n \times r$ matrices, respectively, then the product $\mathbf{C}=\mathbf{AB}$ is an $m \times r$ matrix. The element in the ith row and jth column of $\mathbf{C}$ is found by multiplying each element of the ith row of $\mathbf{A}$ by the corresponding element of the jth column of $\mathbf{B}$ and then adding the resulting products. In symbols,

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}. \tag{9}$$

By direct calculation, it can be shown that matrix multiplication satisfies the associative law

$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC}) \tag{10}$$

and the distributive law

$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}. \tag{11}$$

However, in general, matrix multiplication is not commutative. For both products $\mathbf{AB}$ and $\mathbf{BA}$ to exist and to be of the same size, it is necessary that $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same order. Even in that case the two products are usually unequal, so that, in general,

$$\mathbf{AB} \neq \mathbf{BA}. \tag{12}$$

EXAMPLE 1

To illustrate the multiplication of matrices, and also the fact that matrix multiplication is not necessarily commutative, consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}.$$

From the definition of multiplication given in Eq. (9), we have

$$\mathbf{AB} = \begin{pmatrix} 2-2+2 & 1+2-1 & -1+0+1 \\ 0+2-2 & 0-2+1 & 0+0-1 \\ 4+1+2 & 2-1-1 & -2+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{pmatrix}.$$

Similarly, we find that

$$\mathbf{BA} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{pmatrix}.$$

Clearly, $\mathbf{AB} \neq \mathbf{BA}$.

7. **Multiplication of Vectors.** There are several ways of forming a product of two vectors $\mathbf{x}$ and $\mathbf{y}$, each with n components. One is a direct extension to n dimensions of the familiar dot product from physics and calculus; we denote it by $\mathbf{x}^T\mathbf{y}$ and write

$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i. \tag{13}$$

The result of Eq. (13) is a real or complex number, and it follows directly from Eq. (13) that

$$\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}, \qquad \mathbf{x}^T(\mathbf{y}+\mathbf{z}) = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z}, \qquad (\alpha\mathbf{x})^T\mathbf{y} = \alpha(\mathbf{x}^T\mathbf{y}) = \mathbf{x}^T(\alpha\mathbf{y}). \tag{14}$$

There is another vector product that is also defined for any two vectors having the same number of components. This product, denoted by $(\mathbf{x}, \mathbf{y})$, is called the **scalar** or **inner product** and is defined by

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \overline{y}_i. \tag{15}$$

The scalar product is also a real or complex number, and by comparing Eqs. (13) and (15), we see that

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\overline{\mathbf{y}}. \tag{16}$$

Thus, if all the elements of $\mathbf{y}$ are real, then the two products (13) and (15) are identical. From Eq. (15) it follows that

$$\begin{aligned} (\mathbf{x}, \mathbf{y}) &= \overline{(\mathbf{y}, \mathbf{x})}, & (\mathbf{x}, \mathbf{y}+\mathbf{z}) &= (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z}), \\ (\alpha\mathbf{x}, \mathbf{y}) &= \alpha(\mathbf{x}, \mathbf{y}), & (\mathbf{x}, \alpha\mathbf{y}) &= \overline{\alpha}(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{17}$$

Note that even if the vector **x** has elements with nonzero imaginary parts, the scalar product of **x** with itself yields a nonnegative real number

$$(\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2. \tag{18}$$

The nonnegative quantity $(\mathbf{x}, \mathbf{x})^{1/2}$, often denoted by $\|\mathbf{x}\|$, is called the **length**, or **magnitude**, of **x**. If $(\mathbf{x}, \mathbf{y}) = 0$, then the two vectors **x** and **y** are said to be **orthogonal**. For example, the unit vectors **i**, **j**, **k** of three-dimensional vector geometry form an orthogonal set. On the other hand, if some of the elements of **x** are not real, then the product

$$\mathbf{x}^T\mathbf{x} = \sum_{i=1}^{n} x_i^2 \tag{19}$$

may not be a real number.

For example, let

$$\mathbf{x} = \begin{pmatrix} i \\ -2 \\ 1+i \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} 2-i \\ i \\ 3 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{x}^T\mathbf{y} &= (i)(2-i) + (-2)(i) + (1+i)(3) = 4+3i, \\ (\mathbf{x}, \mathbf{y}) &= (i)(2+i) + (-2)(-i) + (1+i)(3) = 2+7i, \\ \mathbf{x}^T\mathbf{x} &= (i)^2 + (-2)^2 + (1+i)^2 = 3+2i, \\ (\mathbf{x}, \mathbf{x}) &= (i)(-i) + (-2)(-2) + (1+i)(1-i) = 7. \end{aligned}$$

8. **Identity.** The multiplicative identity, or simply the identity matrix **I**, is given by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{20}$$

From the definition of matrix multiplication, we have

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A} \tag{21}$$

for any (square) matrix **A**. Hence the commutative law does hold for square matrices if one of the matrices is the identity.

9. **Inverse.** The square matrix **A** is said to be **nonsingular** or **invertible** if there is another matrix **B** such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$, where **I** is the identity. If there is such a **B**, it can be shown that there is only one. It is called the multiplicative inverse, or simply the inverse, of **A**, and we write $\mathbf{B} = \mathbf{A}^{-1}$. Then

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{22}$$

Matrices that do not have an inverse are called **singular** or **noninvertible**.

There are various ways to compute $\mathbf{A}^{-1}$ from **A**, assuming that it exists. One way involves the use of determinants. Associated with each element a_{ij} of a given matrix is the minor M_{ij}, which is the determinant of the matrix obtained by deleting the ith row and jth column of the original matrix—that is, the row and column containing a_{ij}. Also associated with each element a_{ij} is the cofactor C_{ij} defined by the equation

$$C_{ij} = (-1)^{i+j} M_{ij}. \tag{23}$$

If $\mathbf{B} = \mathbf{A}^{-1}$, then it can be shown that the general element b_{ij} is given by

$$b_{ij} = \frac{C_{ji}}{\det \mathbf{A}}. \tag{24}$$

Although Eq. (24) is not an efficient way[3] to calculate $\mathbf{A}^{-1}$, it does suggest a condition that $\mathbf{A}$ must satisfy for it to have an inverse. In fact, the condition is both necessary and sufficient: $\mathbf{A}$ is nonsingular if and only if $\det \mathbf{A} \neq 0$. If $\det \mathbf{A} = 0$, then $\mathbf{A}$ is singular.

Another (and usually better) way to compute $\mathbf{A}^{-1}$ is by means of elementary row operations. There are three such operations:

1. Interchange of two rows.
2. Multiplication of a row by a nonzero scalar.
3. Addition of any multiple of one row to another row.

The transformation of a matrix by a sequence of elementary row operations is referred to as **row reduction** or **Gaussian[4] elimination**. Any nonsingular matrix $\mathbf{A}$ can be transformed into the identity $\mathbf{I}$ by a systematic sequence of these operations. It is possible to show that if the same sequence of operations is then performed on $\mathbf{I}$, it is transformed into $\mathbf{A}^{-1}$. It is most efficient to perform the sequence of operations on both matrices at the same time by forming the augmented matrix $\mathbf{A} \mid \mathbf{I}$. The following example illustrates the calculation of an inverse matrix in this way.

EXAMPLE 2

Find the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

---

[3] For large n the number of multiplications required to evaluate $\mathbf{A}^{-1}$ by Eq. (24) is proportional to $n!$. If we use a more efficient method, such as the row reduction procedure described in this section, the number of multiplications is proportional only to n^3. Even for small values of n (such as $n = 4$), determinants are not an economical tool in calculating inverses, and row reduction methods are preferred.

[4] Carl Friedrich Gauss (1777–1855) was born in Brunswick (Germany) and spent most of his life as professor of astronomy and director of the Observatory at the University of Göttingen. Gauss made major contributions to many areas of mathematics, including number theory, algebra, non-Euclidean and differential geometry, and analysis, as well as to more applied fields such as geodesy, statistics, and celestial mechanics. He is generally considered to be among the half-dozen best mathematicians of all time.

We begin by forming the augmented matrix $\mathbf{A}\,|\,\mathbf{I}$:

$$\mathbf{A}\,|\,\mathbf{I} = \left(\begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right).$$

The matrix $\mathbf{A}$ can be transformed into $\mathbf{I}$ by the following sequence of operations, and at the same time, $\mathbf{I}$ is transformed into $\mathbf{A}^{-1}$. The result of each step appears below the statement.

(a) Obtain zeros in the off-diagonal positions in the first column by adding (−3) times the first row to the second row and adding (−2) times the first row to the third row.

$$\left(\begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

(b) Obtain a 1 in the diagonal position in the second column by multiplying the second row by $\frac{1}{2}$.

$$\left(\begin{array}{rrr|rrr} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

(c) Obtain zeros in the off-diagonal positions in the second column by adding the second row to the first row and adding (−4) times the second row to the third row.

$$\left(\begin{array}{rrr|rrr} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right)$$

(d) Obtain a 1 in the diagonal position in the third column by multiplying the third row by $(-\frac{1}{5})$.

$$\left(\begin{array}{rrr|rrr} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right)$$

(e) Obtain zeros in the off-diagonal positions in the third column by adding $(-\frac{3}{2})$ times the third row to the first row and adding $(-\frac{5}{2})$ times the third row to the second row.

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right)$$

The last of these matrices is $\mathbf{I}\,|\,\mathbf{A}^{-1}$, a fact that can be verified by direct multiplication with the original matrix $\mathbf{A}$.

This example was made slightly simpler by the fact that the given matrix $\mathbf{A}$ had a 1 in the upper left corner ($a_{11} = 1$). If this is not the case, then the first step is to produce a 1 there by multiplying the first row by $1/a_{11}$, as long as $a_{11} \neq 0$. If $a_{11} = 0$, then the first row must be interchanged with some other row to bring a nonzero element into the upper left position before proceeding. If this cannot be done, because every element in the first column is zero, then the matrix has no inverse and is singular.

A similar situation may occur at later stages of the process as well, and the remedy is the same: interchange the given row with a lower row so as to bring a nonzero element to the desired diagonal location. If this cannot be done, then the original matrix is singular.

Matrix Functions. We sometimes need to consider vectors or matrices whose elements are functions of a real variable t. We write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix}, \tag{25}$$

respectively.

The matrix $\mathbf{A}(t)$ is said to be continuous at $t = t_0$ or on an interval $\alpha < t < \beta$ if each element of $\mathbf{A}$ is a continuous function at the given point or on the given interval. Similarly, $\mathbf{A}(t)$ is said to be differentiable if each of its elements is differentiable, and its derivative $d\mathbf{A}/dt$ is defined by

$$\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt}\right); \tag{26}$$

that is, each element of $d\mathbf{A}/dt$ is the derivative of the corresponding element of $\mathbf{A}$. In the same way, the integral of a matrix function is defined as

$$\int_a^b \mathbf{A}(t)\,dt = \left(\int_a^b a_{ij}(t)\,dt\right). \tag{27}$$

For example, if

$$\mathbf{A}(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix},$$

then

$$\mathbf{A}'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix}, \qquad \int_0^\pi \mathbf{A}(t)\,dt = \begin{pmatrix} 2 & \pi^2/2 \\ \pi & 0 \end{pmatrix}.$$

Many of the rules of elementary calculus extend easily to matrix functions; in particular,

$$\frac{d}{dt}(\mathbf{CA}) = \mathbf{C}\frac{d\mathbf{A}}{dt}, \qquad \text{where } \mathbf{C} \text{ is a constant matrix;} \tag{28}$$

$$\frac{d}{dt}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}; \tag{29}$$

$$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}. \tag{30}$$

In Eqs. (28) and (30), care must be taken in each term to avoid interchanging the order of multiplication. The definitions expressed by Eqs. (26) and (27) also apply as special cases to vectors.

To conclude this section: some important operations on matrices are accomplished by applying the operation separately to each element of the matrix. Examples include multiplication by a number, differentiation, and integration. However, this is not true of many other operations. For instance, the square of a matrix is not calculated by squaring each of its elements.

PROBLEMS

1. If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$, find
   (a) $2\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - 4\mathbf{B}$
   (c) $\mathbf{AB}$ (d) $\mathbf{BA}$
2. If $\mathbf{A} = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$, find
   (a) $\mathbf{A} - 2\mathbf{B}$ (b) $3\mathbf{A} + \mathbf{B}$
   (c) $\mathbf{AB}$ (d) $\mathbf{BA}$
3. If $\mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$, find
   (a) $\mathbf{A}^T$ (b) $\mathbf{B}^T$
   (c) $\mathbf{A}^T + \mathbf{B}^T$ (d) $(\mathbf{A} + \mathbf{B})^T$
4. If $\mathbf{A} = \begin{pmatrix} 3-2i & 1+i \\ 2-i & -2+3i \end{pmatrix}$, find
   (a) $\mathbf{A}^T$ (b) $\overline{\mathbf{A}}$ (c) $\mathbf{A}^*$
5. If $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, verify that $2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$.
6. If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, verify that
   (a) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
   (c) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
7. Prove each of the following laws of matrix algebra:
   (a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (b) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
   (c) $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$ (d) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$
   (e) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (f) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
8. If $\mathbf{x} = \begin{pmatrix} 2 \\ 3i \\ 1-i \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix}$, find
   (a) $\mathbf{x}^T\mathbf{y}$ (b) $\mathbf{y}^T\mathbf{y}$
   (c) $(\mathbf{x}, \mathbf{y})$ (d) $(\mathbf{y}, \mathbf{y})$

9. If $\mathbf{x} = \begin{pmatrix} 1-2i \\ i \\ 2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 2 \\ 3-i \\ 1+2i \end{pmatrix}$, show that

(a) $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$

(b) $(\mathbf{x},\mathbf{y}) = \overline{(\mathbf{y},\mathbf{x})}$

In each of Problems 10 through 19, either compute the inverse of the given matrix, or else show that it is singular.

10. $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$

11. $\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

13. $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

14. $\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}$

15. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

16. $\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 2 & 3 & 1 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{pmatrix}$

18. $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$

19. $\begin{pmatrix} 1 & -1 & 2 & 0 \\ -1 & 2 & -4 & 2 \\ 1 & 0 & 1 & 3 \\ -2 & 2 & 0 & -1 \end{pmatrix}$

20. If $\mathbf{A}$ is a square matrix, and if there are two matrices $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$, show that $\mathbf{B} = \mathbf{C}$. Thus, if a matrix has an inverse, it can have only one.

21. If $\mathbf{A}(t) = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix}$ and $\mathbf{B}(t) = \begin{pmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{pmatrix}$, find

(a) $\mathbf{A} + 3\mathbf{B}$

(b) $\mathbf{AB}$

(c) $d\mathbf{A}/dt$

(d) $\int_0^1 \mathbf{A}(t)\,dt$

In each of Problems 22 through 24, verify that the given vector satisfies the given differential equation.

22. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$

23. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t, \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t$

24. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x} = \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$

In each of Problems 25 and 26, verify that the given matrix satisfies the given differential equation.

25. $\boldsymbol{\Psi}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\boldsymbol{\Psi}, \qquad \boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$

26. $\boldsymbol{\Psi}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\boldsymbol{\Psi}, \qquad \boldsymbol{\Psi}(t) = \begin{pmatrix} e^{t} & e^{-2t} & e^{3t} \\ -4e^{t} & -e^{-2t} & 2e^{3t} \\ -e^{t} & -e^{-2t} & e^{3t} \end{pmatrix}$

7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors

In this section we review some results from linear algebra that are important for the solution of systems of linear differential equations. Some of these results are easily proved and others are not; since we are interested simply in summarizing some useful information in compact form, we give no indication of proofs in either case. All the results in this section depend on some basic facts about the solution of systems of linear algebraic equations.

Systems of Linear Algebraic Equations. A set of n simultaneous linear algebraic equations in n variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ &\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

can be written as

$$\mathbf{Ax} = \mathbf{b}, \tag{2}$$

where the $n \times n$ matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are given, and the components of $\mathbf{x}$ are to be determined. If $\mathbf{b} = \mathbf{0}$, the system is said to be **homogeneous**; otherwise, it is **nonhomogeneous**.

If the coefficient matrix $\mathbf{A}$ is nonsingular—that is, if $\det\mathbf{A}$ is not zero—then there is a unique solution of the system (2). Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and the solution can be found by multiplying each side of Eq. (2) on the left by $\mathbf{A}^{-1}$; thus

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3}$$

In particular, the homogeneous problem $\mathbf{Ax} = \mathbf{0}$, corresponding to $\mathbf{b} = \mathbf{0}$ in Eq. (2), has only the trivial solution $\mathbf{x} = \mathbf{0}$.

On the other hand, if $\mathbf{A}$ is singular—that is, if $\det \mathbf{A}$ is zero—then solutions of Eq. (2) either do not exist, or do exist but are not unique. Since $\mathbf{A}$ is singular, $\mathbf{A}^{-1}$ does not exist, so Eq. (3) is no longer valid. The homogeneous system

$$\mathbf{A}\mathbf{x} = \mathbf{0} \tag{4}$$

has (infinitely many) nonzero solutions in addition to the trivial solution. The situation for the nonhomogeneous system (2) is more complicated. This system has no solution unless the vector $\mathbf{b}$ satisfies a certain further condition. This condition is that

$$(\mathbf{b}, \mathbf{y}) = 0, \tag{5}$$

for all vectors $\mathbf{y}$ satisfying $\mathbf{A}^*\mathbf{y} = \mathbf{0}$, where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$. If condition (5) is met, then the system (2) has (infinitely many) solutions. These solutions are of the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \boldsymbol{\xi}, \tag{6}$$

where $\mathbf{x}^{(0)}$ is a particular solution of Eq. (2), and $\boldsymbol{\xi}$ is the most general solution of the homogeneous system (4). Note the resemblance between Eq. (6) and the solution of a nonhomogeneous linear differential equation. The proofs of some of the preceding statements are outlined in Problems 26 through 30.

The results in the preceding paragraph are important as a means of classifying the solutions of linear systems. However, for solving particular systems, it is generally best to use row reduction to transform the system into a much simpler one from which the solution(s), if there are any, can be written down easily. To do this efficiently, we can form the augmented matrix

$$\mathbf{A} \,|\, \mathbf{b} = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right) \tag{7}$$

by adjoining the vector $\mathbf{b}$ to the coefficient matrix $\mathbf{A}$ as an additional column. The vertical line replaces the equals sign and is said to partition the augmented matrix. We now perform row operations on the augmented matrix so as to transform $\mathbf{A}$ into an upper triangular matrix—that is, a matrix whose elements below the main diagonal are all zero. Once this is done, it is easy to see whether the system has solutions, and to find them if it does. Observe that elementary row operations on the augmented matrix (7) correspond to legitimate operations on the equations in the system (1). The following examples illustrate the process.

EXAMPLE 1

Solve the system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7, \\ -x_1 + x_2 - 2x_3 &= -5, \\ 2x_1 - x_2 - x_3 &= 4. \end{aligned} \tag{8}$$

The augmented matrix for the system (8) is

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right). \tag{9}$$

We now perform row operations on the matrix (9) with a view to introducing zeros in the lower left part of the matrix. Each step is described and the result recorded below.

(a) Add the first row to the second row, and add (-2) times the first row to the third row.

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{array}\right)$$

(b) Multiply the second row by -1.

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & -7 & -10 \end{array}\right)$$

(c) Add (-3) times the second row to the third row.

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -4 \end{array}\right)$$

(d) Divide the third row by -4.

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The matrix obtained in this manner corresponds to the system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7, \\ x_2 - x_3 &= -2, \\ x_3 &= 1, \end{aligned} \tag{10}$$

which is equivalent to the original system (8). Note that the coefficients in Eqs. (10) form a triangular matrix. From the last of Eqs. (10) we have $x_3 = 1$, from the second equation $x_2 = -2 + x_3 = -1$, and from the first equation $x_1 = 7 + 2x_2 - 3x_3 = 2$. Thus we obtain

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

which is the solution of the given system (8). Incidentally, since the solution is unique, we conclude that the coefficient matrix is nonsingular.

EXAMPLE 2

Discuss solutions of the system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1, \\ -x_1 + x_2 - 2x_3 &= b_2, \\ 2x_1 - x_2 + 3x_3 &= b_3 \end{aligned} \tag{11}$$

for various values of b_1, b_2, and b_3.

Observe that the coefficients in the system (11) are the same as those in the system (8) except for the coefficient of x_3 in the third equation. The augmented matrix for the system (11) is

$$\left(\begin{array}{rrr|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right). \tag{12}$$

By performing steps (a), (b), and (c) as in Example 1, we transform the matrix (12) into

$$\left(\begin{array}{rrr|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{array}\right). \tag{13}$$

The equation corresponding to the third row of the matrix (13) is

$$b_1 + 3b_2 + b_3 = 0; \tag{14}$$

thus the system (11) has no solution unless the condition (14) is satisfied by b_1, b_2, and b_3. It is possible to show that this condition is just Eq. (5) for the system (11).

Let us now assume that $b_1 = 2, b_2 = 1$, and $b_3 = -5$, in which case Eq. (14) is satisfied. Then the first two rows of the matrix (13) correspond to the equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 2, \\ x_2 - x_3 &= -3. \end{aligned} \tag{15}$$

To solve the system (15), we can choose one of the unknowns arbitrarily and then solve for the other two. If we let $x_3 = \alpha$, where α is arbitrary, it then follows that

$$\begin{aligned} x_2 &= \alpha - 3, \\ x_1 &= 2(\alpha - 3) - 3\alpha + 2 = -\alpha - 4. \end{aligned}$$

If we write the solution in vector notation, we have

$$\mathbf{x} = \begin{pmatrix} -\alpha - 4 \\ \alpha - 3 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix}. \tag{16}$$

It is easy to verify that the second term on the right side of Eq. (16) is a solution of the nonhomogeneous system (11) and that the first term is the most general solution of the homogeneous system corresponding to (11).

Row reduction is also useful in solving homogeneous systems and systems in which the number of equations is different from the number of unknowns.

Linear Dependence and Independence. A set of k vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ is said to be **linearly dependent** if there exists a set of real or complex numbers $c_1, \dots, c_k$, at least one of which is nonzero, such that

$$c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0}. \tag{17}$$

In other words, $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ are linearly dependent if there is a linear relation among them. On the other hand, if the only set $c_1, \dots, c_k$ for which Eq. (17) is satisfied is $c_1 = c_2 = \cdots = c_k = 0$, then $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ are said to be **linearly independent**.

Consider now a set of n vectors, each of which has n components. Let $x_{ij} = x_i^{(j)}$ be the ith component of the vector $\mathbf{x}^{(j)}$, and let $\mathbf{X} = (x_{ij})$. Then Eq. (17) can be written as

$$\begin{pmatrix} x_1^{(1)}c_1 + \cdots + x_1^{(n)}c_n \\ \vdots \qquad\qquad \vdots \\ x_n^{(1)}c_1 + \cdots + x_n^{(n)}c_n \end{pmatrix} = \begin{pmatrix} x_{11}c_1 + \cdots + x_{1n}c_n \\ \vdots \qquad\qquad \vdots \\ x_{n1}c_1 + \cdots + x_{nn}c_n \end{pmatrix} = \mathbf{0},$$

or, equivalently,

$$\mathbf{Xc} = \mathbf{0}. \tag{18}$$

If $\det \mathbf{X} \neq 0$, then the only solution of Eq. (18) is $\mathbf{c} = \mathbf{0}$, but if $\det \mathbf{X} = 0$, there are nonzero solutions. Thus the set of vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ is linearly independent if and only if $\det \mathbf{X} \neq 0$.

EXAMPLE 3

Determine whether the vectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix} \tag{19}$$

are linearly independent or linearly dependent. If they are linearly dependent, find a linear relation among them.

To determine whether $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ are linearly dependent, we seek constants c_1, c_2, and c_3 such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}. \tag{20}$$

Equation (20) can also be written in the form

$$\begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

and solved by means of elementary row operations starting from the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{array}\right). \tag{22}$$

We proceed as in Examples 1 and 2.

(a) Add (-2) times the first row to the second row, and add the first row to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right)$$

(b) Divide the second row by -3; then add (-5) times the second row to the third row.

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus we obtain the equivalent system

$$\begin{aligned} c_1 + 2c_2 - 4c_3 &= 0, \\ c_2 - 3c_3 &= 0. \end{aligned} \tag{23}$$

From the second of Eqs. (23) we have $c_2 = 3c_3$, and then from the first we obtain $c_1 = 4c_3 - 2c_2 = -2c_3$. Thus we have solved for c_1 and c_2 in terms of c_3, with the latter remaining arbitrary. If we choose $c_3 = -1$ for convenience, then $c_1 = 2$ and $c_2 = -3$. In this case the relation (20) becomes

$$2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} - \mathbf{x}^{(3)} = \mathbf{0},$$

and the given vectors are linearly dependent.

Alternatively, we can compute $\det(x_{ij})$, whose columns are the components of $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$, respectively. Thus

$$\det(x_{ij}) = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix}$$

and direct calculation shows that it is zero. Hence $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, and $\mathbf{x}^{(3)}$ are linearly dependent. However, if the coefficients c_1, c_2, and c_3 are required, we still need to solve Eq. (20) to find them.

Frequently, it is useful to think of the columns (or rows) of a matrix $\mathbf{A}$ as vectors. These column (or row) vectors are linearly independent if and only if $\det \mathbf{A} \neq 0$. Further, if $\mathbf{C} = \mathbf{AB}$, then it can be shown that $\det \mathbf{C} = (\det \mathbf{A})(\det \mathbf{B})$. Therefore, if the columns (or rows) of both $\mathbf{A}$ and $\mathbf{B}$ are linearly independent, then the columns (or rows) of $\mathbf{C}$ are also linearly independent.

Now let us extend the concepts of linear dependence and independence to a set of vector functions $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(k)}(t)$ defined on an interval $\alpha < t < \beta$. The vectors $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(k)}(t)$ are said to be linearly dependent on $\alpha < t < \beta$ if there exists a set of constants $c_1, \dots, c_k$, not all of which are zero, such that

$$c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0} \quad \text{for all } t \text{ in the interval.}$$

Otherwise, $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(k)}(t)$ are said to be linearly independent. Note that if $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(k)}(t)$ are linearly dependent on an interval, they are linearly dependent at each point in the interval. However, if $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(k)}(t)$ are linearly independent on an interval, they may or may not be linearly independent at each point; they may, in fact, be linearly dependent at each point, but with different sets of constants at different points. See Problem 15 for an example.

Eigenvalues and Eigenvectors. The equation

$$\mathbf{A}\mathbf{x} = \mathbf{y} \tag{24}$$

can be viewed as a linear transformation that maps (or transforms) a given vector $\mathbf{x}$ into a new vector $\mathbf{y}$. Vectors that are transformed into multiples of themselves are important in many applications.[5] To find such vectors, we set $\mathbf{y} = \lambda\mathbf{x}$, where λ is a scalar proportionality factor, and seek solutions of the equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \tag{25}$$

or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{26}$$

The latter equation has nonzero solutions if and only if λ is chosen so that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{27}$$

Equation (27) is a polynomial equation of degree n in λ and is called the **characteristic equation** of the matrix $\mathbf{A}$. Values of λ that satisfy Eq. (27) may be either real- or complex-valued and are called **eigenvalues** of $\mathbf{A}$. The nonzero solutions of Eq. (25) or (26) that are obtained by using such a value of λ are called the **eigenvectors** corresponding to that eigenvalue.

If $\mathbf{A}$ is a 2×2 matrix, then Eq. (26) is

$$\begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{28}$$

and Eq. (27) becomes

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0,$$

or

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{29}$$

The following example illustrates how eigenvalues and eigenvectors are found.

EXAMPLE 4

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}. \tag{30}$$

The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{31}$$

[5]For example, this problem is encountered in finding the principal axes of stress or strain in an elastic body, and in finding the modes of free vibration in a conservative system with a finite number of degrees of freedom.

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0. \tag{32}$$

Thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$.

To find the eigenvectors, we return to Eq. (31) and replace λ by each of the eigenvalues in turn. For $\lambda = 2$ we have

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{33}$$

Hence each row of this vector equation leads to the condition $x_1 - x_2 = 0$, so x_1 and x_2 are equal but their value is not determined. If $x_1 = c$, then $x_2 = c$ also, and the eigenvector $\mathbf{x}^{(1)}$ is

$$\mathbf{x}^{(1)} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad c \neq 0. \tag{34}$$

Thus there is an infinite family of eigenvectors, indexed by the arbitrary constant c, corresponding to the eigenvalue λ_1. We will choose a single member of this family as a representative of the rest; in this example it seems simplest to let $c = 1$. Then, instead of Eq. (34), we write

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{35}$$

and remember that any nonzero multiple of this vector is also an eigenvector. We say that $\mathbf{x}^{(1)}$ is the eigenvector corresponding to the eigenvalue $\lambda_1 = 2$.

Now, setting $\lambda = -1$ in Eq. (31), we obtain

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{36}$$

Again we obtain a single condition on x_1 and x_2, namely, $4x_1 - x_2 = 0$. Thus the eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{37}$$

or any nonzero multiple of this vector.

As Example 4 illustrates, eigenvectors are determined only up to an arbitrary nonzero multiplicative constant; if this constant is specified in some way, then the eigenvectors are said to be **normalized**. In Example 4, we chose the constant c so that the components of the eigenvectors would be small integers. However, any other choice of c is equally valid, although perhaps less convenient. Sometimes it is useful to normalize an eigenvector $\mathbf{x}$ by choosing the constant so that its length $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$.

Since the characteristic equation (27) for an $n \times n$ matrix $\mathbf{A}$ is a polynomial equation of degree n in λ, each such matrix has n eigenvalues $\lambda_1, \ldots, \lambda_n$, some of which may be repeated. If a given eigenvalue appears m times as a root of Eq. (27), then that eigenvalue is said to have **algebraic multiplicity** m. Each eigenvalue has at least one associated eigenvector, and an eigenvalue of algebraic multiplicity m may have q

linearly independent eigenvectors. The integer q is called the **geometric multiplicity** of the eigenvalue, and it is possible to show that

$$1 \le q \le m. \tag{38}$$

Further, examples demonstrate that q may be any integer in this interval. If each eigenvalue of $\mathbf{A}$ is **simple** (has algebraic multiplicity 1), then each eigenvalue also has geometric multiplicity 1.

It is possible to show that if λ_1 and λ_2 are two eigenvalues of $\mathbf{A}$ and if $\lambda_1 \neq \lambda_2$, then their corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent (Problem 34). This result extends to any set $\lambda_1, \dots, \lambda_k$ of distinct eigenvalues: their eigenvectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ are linearly independent. Thus, if each eigenvalue of an $n \times n$ matrix is simple, then the n eigenvectors of $\mathbf{A}$, one for each eigenvalue, are linearly independent. On the other hand, if $\mathbf{A}$ has one or more repeated eigenvalues, then there may be fewer than n linearly independent eigenvectors associated with $\mathbf{A}$, since for a repeated eigenvalue we may have $q < m$. As we will see in Section 7.8, this fact may lead to complications later on in the solution of systems of differential equations.

EXAMPLE 5

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{39}$$

The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{40}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = 0. \tag{41}$$

The roots of Eq. (41) are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = -1$. Thus 2 is a simple eigenvalue, and -1 is an eigenvalue of algebraic multiplicity 2, or a double eigenvalue.

To find the eigenvector $\mathbf{x}^{(1)}$ corresponding to the eigenvalue λ_1, we substitute $\lambda = 2$ in Eq. (40); this gives the system

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{42}$$

We can reduce this to the equivalent system

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{43}$$

by elementary row operations. Solving this system yields the eigenvector

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{44}$$

For $\lambda = -1$, Eqs. (40) reduce immediately to the single equation

$$x_1 + x_2 + x_3 = 0. \tag{45}$$

Thus values for two of the quantities x_1, x_2, x_3 can be chosen arbitrarily, and the third is determined from Eq. (45). For example, if $x_1 = c_1$ and $x_2 = c_2$, then $x_3 = -c_1 - c_2$. In vector notation we have

$$\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{46}$$

For example, by choosing $c_1 = 1$ and $c_2 = 0$, we obtain the eigenvector

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{47}$$

Any nonzero multiple of $\mathbf{x}^{(2)}$ is also an eigenvector, but a second independent eigenvector can be found by making another choice of c_1 and c_2—for instance, $c_1 = 0$ and $c_2 = 1$. In this case we obtain

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tag{48}$$

which is linearly independent of $\mathbf{x}^{(2)}$. Therefore, in this example, two linearly independent eigenvectors are associated with the double eigenvalue.

An important special class of matrices, called **self-adjoint** or **Hermitian** matrices, are those for which $\mathbf{A}^* = \mathbf{A}$; that is, $\overline{a}_{ji} = a_{ij}$. Hermitian matrices include as a subclass real symmetric matrices—that is, matrices that have real elements and for which $\mathbf{A}^T = \mathbf{A}$. The eigenvalues and eigenvectors of Hermitian matrices always have the following useful properties:

1. All eigenvalues are real.
2. There always exists a full set of n linearly independent eigenvectors, regardless of the algebraic multiplicities of the eigenvalues.
3. If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues, then $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$. Thus, if all eigenvalues are simple, then the associated eigenvectors form an orthogonal set of vectors.
4. Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m eigenvectors that are mutually orthogonal. Thus the full set of n eigenvectors can always be chosen to be orthogonal as well as linearly independent.

The proofs of statements 1 and 3 above are outlined in Problems 32 and 33. Example 5 involves a real symmetric matrix and illustrates properties 1, 2, and 3, but the choice we have made for $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ does not illustrate property 4. However, it is always possible to choose an $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ so that $(\mathbf{x}^{(2)}, \mathbf{x}^{(3)}) = 0$. For instance, in Example 5 we could have chosen $\mathbf{x}^{(2)}$ as before and $\mathbf{x}^{(3)}$ by using $c_1 = 1$ and $c_2 = -2$ in Eq. (46). In this way we obtain

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

as the eigenvectors associated with the eigenvalue $\lambda = -1$. These eigenvectors are orthogonal to each other as well as to the eigenvector $\mathbf{x}^{(1)}$ that corresponds to the eigenvalue $\lambda = 2$.

PROBLEMS

In each of Problems 1 through 6, either solve the given system of equations, or else show that there is no solution.

1. $$\begin{aligned} x_1 \qquad - x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 1 \\ -x_1 + x_2 + 2x_3 &= 2 \end{aligned}$$

2. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 1 \end{aligned}$$

3. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

4. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

5. $$\begin{aligned} x_1 \qquad - x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x_1 + 2x_2 - x_3 &= -2 \\ -2x_1 - 4x_2 + 2x_3 &= 4 \\ 2x_1 + 4x_2 - 2x_3 &= -4 \end{aligned}$$

In each of Problems 7 through 11, determine whether the members of the given set of vectors are linearly independent. If they are linearly dependent, find a linear relation among them. The vectors are written as row vectors to save space but may be considered as column vectors; that is, the transposes of the given vectors may be used instead of the vectors themselves.

7. $\mathbf{x}^{(1)} = (1, 1, 0)$, $\quad \mathbf{x}^{(2)} = (0, 1, 1)$, $\quad \mathbf{x}^{(3)} = (1, 0, 1)$
8. $\mathbf{x}^{(1)} = (2, 1, 0)$, $\quad \mathbf{x}^{(2)} = (0, 1, 0)$, $\quad \mathbf{x}^{(3)} = (-1, 2, 0)$
9. $\mathbf{x}^{(1)} = (1, 2, 2, 3)$, $\quad \mathbf{x}^{(2)} = (-1, 0, 3, 1)$, $\quad \mathbf{x}^{(3)} = (-2, -1, 1, 0)$, $\quad \mathbf{x}^{(4)} = (-3, 0, -1, 3)$
10. $\mathbf{x}^{(1)} = (1, 2, -1, 0)$, $\quad \mathbf{x}^{(2)} = (2, 3, 1, -1)$, $\quad \mathbf{x}^{(3)} = (-1, 0, 2, 2)$, $\quad \mathbf{x}^{(4)} = (3, -1, 1, 3)$
11. $\mathbf{x}^{(1)} = (1, 2, -2)$, $\quad \mathbf{x}^{(2)} = (3, 1, 0)$, $\quad \mathbf{x}^{(3)} = (2, -1, 1)$, $\quad \mathbf{x}^{(4)} = (4, 3, -2)$
12. Suppose that each of the vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ has n components, where $n < m$. Show that $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent.

In each of Problems 13 and 14, determine whether the members of the given set of vectors are linearly independent for $-\infty < t < \infty$. If they are linearly dependent, find the linear relation among them. As in Problems 7 through 11, the vectors are written as row vectors to save space.

13. $\mathbf{x}^{(1)}(t) = (e^{-t}, 2e^{-t})$, $\quad \mathbf{x}^{(2)}(t) = (e^{-t}, e^{-t})$, $\quad \mathbf{x}^{(3)}(t) = (3e^{-t}, 0)$
14. $\mathbf{x}^{(1)}(t) = (2\sin t, \sin t)$, $\quad \mathbf{x}^{(2)}(t) = (\sin t, 2\sin t)$

15. Let

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$. Nevertheless, show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $0 \le t \le 1$.

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

16. $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

18. $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

19. $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

20. $\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

21. $\begin{pmatrix} -3 & 3/4 \\ -5 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

23. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

24. $\begin{pmatrix} 11/9 & -2/9 & 8/9 \\ -2/9 & 2/9 & 10/9 \\ 8/9 & 10/9 & 5/9 \end{pmatrix}$

25. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Problems 26 through 30 deal with the problem of solving $\mathbf{Ax} = \mathbf{b}$ when $\det \mathbf{A} = 0$.

26. (a) Suppose that $\mathbf{A}$ is a real-valued $n \times n$ matrix. Show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^T\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.
Hint: You may find it simpler to consider first the case $n = 2$; then extend the result to an arbitrary value of n.
(b) If $\mathbf{A}$ is not necessarily real, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.
(c) If $\mathbf{A}$ is Hermitian, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{Ay})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.

27. Suppose that, for a given matrix $\mathbf{A}$, there is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. Show that there is also a nonzero vector $\mathbf{y}$ such that $\mathbf{A}^*\mathbf{y} = \mathbf{0}$.

28. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{Ax} = \mathbf{b}$ has solutions. Show that $(\mathbf{b}, \mathbf{y}) = 0$, where $\mathbf{y}$ is any solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Verify that this statement is true for the set of equations in Example 2.
Hint: Use the result of Problem 26(b).

29. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{x} = \mathbf{x}^{(0)}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Show that if $\boldsymbol{\xi}$ is a solution of $\mathbf{A}\boldsymbol{\xi} = \mathbf{0}$ and α is any constant, then $\mathbf{x} = \mathbf{x}^{(0)} + \alpha\boldsymbol{\xi}$ is also a solution of $\mathbf{Ax} = \mathbf{b}$.

30. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{y}$ is a solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Show that if $(\mathbf{b}, \mathbf{y}) = 0$ for every such $\mathbf{y}$, then $\mathbf{Ax} = \mathbf{b}$ has solutions. Note that this is the converse of Problem 28; the form of the solution is given by Problem 29.
Hint: What does the relation $\mathbf{A}^*\mathbf{y} = \mathbf{0}$ say about the rows of $\mathbf{A}$? Again, it may be helpful to consider the case $n = 2$ first.

31. Prove that $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.

32. In this problem we show that the eigenvalues of a Hermitian matrix $\mathbf{A}$ are real. Let $\mathbf{x}$ be an eigenvector corresponding to the eigenvalue λ.
    (a) Show that $(\mathbf{Ax}, \mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$. *Hint:* See Problem 26(c).
    (b) Show that $\lambda(\mathbf{x}, \mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x})$. *Hint:* Recall that $\mathbf{Ax} = \lambda\mathbf{x}$.
    (c) Show that $\lambda = \overline{\lambda}$; that is, the eigenvalue λ is real.

33. Show that if λ_1 and λ_2 are eigenvalues of a Hermitian matrix $\mathbf{A}$, and if $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
    Hint: Use the results of Problems 26(c) and 32 to show that $(\lambda_1 - \lambda_2)(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$.

34. Show that if λ_1 and λ_2 are eigenvalues of any matrix $\mathbf{A}$, and if $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent.
    Hint: Start from $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \mathbf{0}$; multiply by $\mathbf{A}$ to obtain $c_1\lambda_1\mathbf{x}^{(1)} + c_2\lambda_2\mathbf{x}^{(2)} = \mathbf{0}$. Then show that $c_1 = c_2 = 0$.

7.4 Basic Theory of Systems of First Order Linear Equations

The general theory of a system of n first order linear equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\ \ \vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{1}$$

closely parallels that of a single linear equation of nth order. The discussion in this section therefore follows the same general lines as that in Sections 3.2 and 4.1. To discuss the system (1) most effectively, we write it in matrix notation. That is, we consider $x_1 = \phi_1(t), \dots, x_n = \phi_n(t)$ to be components of a vector $\mathbf{x} = \boldsymbol{\phi}(t)$; similarly, $g_1(t), \dots, g_n(t)$ are components of a vector $\mathbf{g}(t)$, and $p_{11}(t), \dots, p_{nn}(t)$ are elements of an $n \times n$ matrix $\mathbf{P}(t)$. Equation (1) then takes the form

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{2}$$

The use of vectors and matrices not only saves a great deal of space and facilitates calculations but also emphasizes the similarity between systems of equations and single (scalar) equations.

A vector $\mathbf{x} = \boldsymbol{\phi}(t)$ is said to be a solution of Eq. (2) if its components satisfy the system of equations (1). Throughout this section we assume that $\mathbf{P}$ and $\mathbf{g}$ are continuous on some interval $\alpha < t < \beta$; that is, each of the scalar functions $p_{11}, \dots, p_{nn}, g_1, \dots, g_n$ is continuous there. According to Theorem 7.1.2, this is sufficient to guarantee the existence of solutions of Eq. (2) on the interval $\alpha < t < \beta$.

It is convenient to consider first the homogeneous equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{3}$$

obtained from Eq. (2) by setting $\mathbf{g}(t) = \mathbf{0}$. Once the homogeneous equation has been solved, there are several methods that can be used to solve the nonhomogeneous