35. In this problem we indicate an alternative procedure[7] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \tag{i}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

(a) Verify that Eq. (i) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

(b) Let $u = (D - r_2)y$. Then show that the solution of Eq (i) can be found by solving the following two first order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

In each of Problems 36 through 39, use the method of Problem 35 to solve the given differential equation.

36. $y'' - 3y' - 4y = 3e^{2t}$ (see Example 1)
37. $2y'' + 3y' + y = t^2 + 3\sin t$ (see Problem 9)
38. $y'' + 2y' + y = 2e^{-t}$ (see Problem 8)
39. $y'' + 2y' = 3 + 4\sin 2t$ (see Problem 6)

3.6 Variation of Parameters

In this section we describe another method of finding a particular solution of a nonhomogeneous equation. This method, **variation of parameters**, is due to Lagrange and complements the method of undetermined coefficients rather well. The main advantage of variation of parameters is that it is a *general method*; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second order linear nonhomogeneous differential equation. On the other hand, the method of variation of parameters eventually requires us to evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties. Before looking at this method in the general case, we illustrate its use in an example.

EXAMPLE 1

Find a particular solution of

$$y'' + 4y = 3\csc t. \tag{1}$$

Observe that this problem is not a good candidate for the method of undetermined coefficients, as described in Section 3.5, because the nonhomogeneous term $g(t) = 3\csc t$ involves

[7] R. S. Luthar, "Another Approach to a Standard Differential Equation," *Two Year College Mathematics Journal 10* (1979), pp. 200–201. Also see D. C. Sandell and F. M. Stein, "Factorization of Operators of Second Order Linear Homogeneous Ordinary Differential Equations," *Two Year College Mathematics Journal 8* (1977), pp. 132–141, for a more general discussion of factoring operators.

a quotient (rather than a sum or a product) of $\sin t$ or $\cos t$. Therefore, we need a different approach. Observe also that the homogeneous equation corresponding to Eq. (1) is

$$y'' + 4y = 0, \tag{2}$$

and that the general solution of Eq. (2) is

$$y_c(t) = c_1 \cos 2t + c_2 \sin 2t. \tag{3}$$

The basic idea in the method of variation of parameters is to replace the constants c_1 and c_2 in Eq. (3) by functions $u_1(t)$ and $u_2(t)$, respectively, and then to determine these functions so that the resulting expression

$$y = u_1(t) \cos 2t + u_2(t) \sin 2t \tag{4}$$

is a solution of the nonhomogeneous equation (1).

To determine u_1 and u_2, we need to substitute for y from Eq. (4) in Eq. (1). However, even without carrying out this substitution, we can anticipate that the result will be a single equation involving some combination of u_1, u_2, and their first two derivatives. Since there is only one equation and two unknown functions, we can expect that there are many possible choices of u_1 and u_2 that will meet our needs. Alternatively, we may be able to impose a second condition of our own choosing, thereby obtaining two equations for the two unknown functions u_1 and u_2. We will soon show (following Lagrange) that it is possible to choose this second condition in a way that makes the computation markedly more efficient.

Returning now to Eq. (4), we differentiate it and rearrange the terms, thereby obtaining

$$y' = -2u_1(t) \sin 2t + 2u_2(t) \cos 2t + u_1'(t) \cos 2t + u_2'(t) \sin 2t. \tag{5}$$

Keeping in mind the possibility of choosing a second condition on u_1 and u_2, let us require the sum of the last two terms on the right side of Eq. (5) to be zero; that is, we require that

$$u_1'(t) \cos 2t + u_2'(t) \sin 2t = 0. \tag{6}$$

It then follows from Eq. (5) that

$$y' = -2u_1(t) \sin 2t + 2u_2(t) \cos 2t. \tag{7}$$

Although the ultimate effect of the condition (6) is not yet clear, at the very least it has simplified the expression for y'. Further, by differentiating Eq. (7) we obtain

$$y'' = -4u_1(t) \cos 2t - 4u_2(t) \sin 2t - 2u_1'(t) \sin 2t + 2u_2'(t) \cos 2t. \tag{8}$$

Then, substituting for y and y'' in Eq. (1) from Eqs. (4) and (8), respectively, we find that

$$\begin{aligned} y'' + 4y = &- 4u_1(t) \cos 2t - 4u_2(t) \sin 2t - 2u_1'(t) \sin 2t + 2u_2'(t) \cos 2t \\ &+ 4u_1(t) \cos 2t + 4u_2(t) \sin 2t = 3 \csc t. \end{aligned}$$

Hence u_1 and u_2 must satisfy

$$-2u_1'(t) \sin 2t + 2u_2'(t) \cos 2t = 3 \csc t. \tag{9}$$

Summarizing our results to this point, we want to choose u_1 and u_2 so as to satisfy Eqs. (6) and (9). These equations can be viewed as a pair of linear *algebraic* equations for the two unknown quantities $u_1'(t)$ and $u_2'(t)$. Equations (6) and (9) can be solved in various ways. For example, solving Eq. (6) for $u_2'(t)$, we have

$$u_2'(t) = -u_1'(t) \frac{\cos 2t}{\sin 2t}. \tag{10}$$

Then, substituting for $u_2'(t)$ in Eq. (9) and simplifying, we obtain

$$u_1'(t) = -\frac{3\csc t \sin 2t}{2} = -3\cos t. \tag{11}$$

Further, putting this expression for $u_1'(t)$ back in Eq. (10) and using the double-angle formulas, we find that

$$u_2'(t) = \frac{3\cos t \cos 2t}{\sin 2t} = \frac{3(1 - 2\sin^2 t)}{2\sin t} = \frac{3}{2}\csc t - 3\sin t. \tag{12}$$

Having obtained $u_1'(t)$ and $u_2'(t)$, we next integrate so as to find $u_1(t)$ and $u_2(t)$. The result is

$$u_1(t) = -3\sin t + c_1 \tag{13}$$

and

$$u_2(t) = \tfrac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2. \tag{14}$$

On substituting these expressions in Eq. (4), we have

$$y = -3\sin t \cos 2t + \tfrac{3}{2}\ln|\csc t - \cot t| \sin 2t + 3\cos t \sin 2t$$
$$+ c_1 \cos 2t + c_2 \sin 2t.$$

Finally, by using the double-angle formulas once more, we obtain

$$y = 3\sin t + \tfrac{3}{2}\ln|\csc t - \cot t| \sin 2t + c_1\cos 2t + c_2 \sin 2t. \tag{15}$$

The terms in Eq. (15) involving the arbitrary constants c_1 and c_2 are the general solution of the corresponding homogeneous equation, while the remaining terms are a particular solution of the nonhomogeneous equation (1). Thus Eq. (15) is the general solution of Eq. (1).

In the preceding example the method of variation of parameters worked well in determining a particular solution, and hence the general solution, of Eq. (1). The next question is whether this method can be applied effectively to an arbitrary equation. Therefore, we consider

$$y'' + p(t)y' + q(t)y = g(t), \tag{16}$$

where p, q, and g are given continuous functions. As a starting point, we assume that we know the general solution

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) \tag{17}$$

of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{18}$$

This is a major assumption. So far we have shown how to solve Eq. (18) only if it has constant coefficients. If Eq. (18) has coefficients that depend on t, then usually the methods described in Chapter 5 must be used to obtain $y_c(t)$.

The crucial idea, as illustrated in Example 1, is to replace the constants c_1 and c_2 in Eq. (17) by functions $u_1(t)$ and $u_2(t)$, respectively; thus we have

$$y = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{19}$$

Then we try to determine $u_1(t)$ and $u_2(t)$ so that the expression in Eq. (19) is a solution of the nonhomogeneous equation (16) rather than the homogeneous equation (18). Thus we differentiate Eq. (19), obtaining

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t). \tag{20}$$

As in Example 1, we now set the terms involving $u_1'(t)$ and $u_2'(t)$ in Eq. (20) equal to zero; that is, we require that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{21}$$

Then, from Eq. (20), we have

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t). \tag{22}$$

Further, by differentiating again, we obtain

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t). \tag{23}$$

Now we substitute for y, y', and y'' in Eq. (16) from Eqs. (19), (22), and (23), respectively. After rearranging the terms in the resulting equation, we find that

$$\begin{aligned} u_1(t)\big[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\big] \\ + u_2(t)\big[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)\big] \\ + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \end{aligned} \tag{24}$$

Each of the expressions in square brackets in Eq. (24) is zero because both y_1 and y_2 are solutions of the homogeneous equation (18). Therefore, Eq. (24) reduces to

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{25}$$

Equations (21) and (25) form a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions. They correspond exactly to Eqs. (6) and (9) in Example 1.

By solving the system (21), (25) we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}, \tag{26}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Note that division by W is permissible since y_1 and y_2 are a fundamental set of solutions, and therefore their Wronskian is nonzero. By integrating Eqs. (26), we find the desired functions $u_1(t)$ and $u_2(t)$, namely,

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_2. \tag{27}$$

If the integrals in Eqs. (27) can be evaluated in terms of elementary functions, then we substitute the results in Eq. (19), thereby obtaining the general solution of Eq. (16). More generally, the solution can always be expressed in terms of integrals, as stated in the following theorem.

Theorem 3.6.1 If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are a fundamental set of solutions of the homogeneous equation (18) corresponding to the nonhomogeneous equation (16)

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of Eq. (16) is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}\,ds, \tag{28}$$

where t_0 is any conveniently chosen point in I. The general solution is

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t), \tag{29}$$

as prescribed by Theorem 3.5.2.

By examining the expression (28) and reviewing the process by which we derived it, we can see that there may be two major difficulties in using the method of variation of parameters. As we have mentioned earlier, one is the determination of $y_1(t)$ and $y_2(t)$, a fundamental set of solutions of the homogeneous equation (18), when the coefficients in that equation are not constants. The other possible difficulty lies in the evaluation of the integrals appearing in Eq. (28). This depends entirely on the nature of the functions y_1, y_2, and g. In using Eq. (28), be sure that the differential equation is exactly in the form (16); otherwise, the nonhomogeneous term $g(t)$ will not be correctly identified.

A major advantage of the method of variation of parameters is that Eq. (28) provides an expression for the particular solution $Y(t)$ in terms of an arbitrary forcing function $g(t)$. This expression is a good starting point if you wish to investigate the effect of variations in the forcing function, or if you wish to analyze the response of a system to a number of different forcing functions.

PROBLEMS

In each of Problems 1 through 4, use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' - y' - 2y = 2e^{-t}$
3. $y'' + 2y' + y = 3e^{-t}$
4. $4y'' - 4y' + y = 16e^{t/2}$

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

5. $y'' + y = \tan t, \quad 0 < t < \pi/2$
6. $y'' + 9y = 9\sec^2 3t, \quad 0 < t < \pi/6$
7. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$
8. $y'' + 4y = 3\csc 2t, \quad 0 < t < \pi/2$
9. $4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$
10. $y'' - 2y' + y = e^t/(1 + t^2)$
11. $y'' - 5y' + 6y = g(t)$
12. $y'' + 4y = g(t)$

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

13. $t^2y'' - 2y = 3t^2 - 1, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^{-1}$
14. $t^2y'' - t(t + 2)y' + (t + 2)y = 2t^3, \quad t > 0; \qquad y_1(t) = t, \quad y_2(t) = te^t$
15. $ty'' - (1 + t)y' + y = t^2e^{2t}, \quad t > 0; \qquad y_1(t) = 1 + t, \quad y_2(t) = e^t$
16. $(1 - t)y'' + ty' - y = 2(t - 1)^2e^{-t}, \quad 0 < t < 1; \qquad y_1(t) = e^t, \quad y_2(t) = t$
17. $x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0; \qquad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x$

18. $x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x, \quad x > 0;$
$y_1(x) = x^{-1/2}\sin x, \quad y_2(x) = x^{-1/2}\cos x$
19. $(1-x)y'' + xy' - y = g(x), \quad 0 < x < 1; \qquad y_1(x) = e^x, \quad y_2(x) = x$
20. $x^2y'' + xy' + (x^2 - 0.25)y = g(x), \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x, \quad y_2(x) = x^{-1/2}\cos x$
21. Show that the solution of the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{i}$$

can be written as $y = u(t) + v(t)$, where u and v are solutions of the two initial value problems

$$L[u] = 0, \qquad u(t_0) = y_0, \quad u'(t_0) = y_0', \tag{ii}$$

$$L[v] = g(t), \qquad v(t_0) = 0, \quad v'(t_0) = 0, \tag{iii}$$

respectively. In other words, the nonhomogeneities in the differential equation and in the initial conditions can be dealt with separately. Observe that u is easy to find if a fundamental set of solutions of $L[u] = 0$ is known.
22. By choosing the lower limit of integration in Eq. (28) in the text as the initial point t_0, show that $Y(t)$ becomes

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}\, g(s)\, ds.$$

Show that $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Thus Y can be identified with v in Problem 21.
23. (a) Use the result of Problem 22 to show that the solution of the initial value problem

$$y'' + y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0 \tag{i}$$

is

$$y = \int_{t_0}^{t} \sin(t-s)g(s)\, ds. \tag{ii}$$

(b) Use the result of Problem 21 to find the solution of the initial value problem

$$y'' + y = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'.$$

24. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D-a)(D-b)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a and b are real numbers with $a \neq b$.
25. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = [D^2 - 2\lambda D + (\lambda^2 + \mu^2)]y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Note that the roots of the characteristic equation are $\lambda \pm i\mu$.
26. Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D-a)^2 y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a is any real number.

27. By combining the results of Problems 24 through 26, show that the solution of the initial value problem

$$L[y] = (D^2 + bD + c)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where b and c are constants, has the form

$$y = \phi(t) = \int_{t_0}^{t} K(t-s)g(s)\,ds. \tag{i}$$

The function K depends only on the solutions y_1 and y_2 of the corresponding homogeneous equation and is independent of the nonhomogeneous term. Once K is determined, all nonhomogeneous problems involving the same differential operator L are reduced to the evaluation of an integral. Note also that although K depends on both t and s, only the combination $t - s$ appears, so K is actually a function of a single variable. When we think of $g(t)$ as the input to the problem and of $\phi(t)$ as the output, it follows from Eq. (i) that the output depends on the input over the entire interval from the initial point t_0 to the current value t. The integral in Eq. (i) is called the **convolution** of K and g, and K is referred to as the **kernel**.

28. The method of reduction of order (Section 3.4) can also be used for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{i}$$

provided one solution y_1 of the corresponding homogeneous equation is known. Let $y = v(t)y_1(t)$ and show that y satisfies Eq. (i) if v is a solution of

$$y_1(t)v'' + [2y_1'(t) + p(t)y_1(t)]v' = g(t). \tag{ii}$$

Equation (ii) is a first order linear equation for v'. By solving this equation, integrating the result, and then multiplying by $y_1(t)$, you can find the general solution of Eq. (i).

In each of Problems 29 through 32, use the method outlined in Problem 28 to solve the given differential equation.

29. $t^2y'' - 2ty' + 2y = 4t^2, \quad t > 0; \qquad y_1(t) = t$
30. $t^2y'' + 7ty' + 5y = t, \quad t > 0; \qquad y_1(t) = t^{-1}$
31. $ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \qquad y_1(t) = 1 + t \qquad$ (see Problem 15)
32. $(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1; \qquad y_1(t) = e^t \qquad$ (see Problem 16)

3.7 Mechanical and Electrical Vibrations

One of the reasons why second order linear equations with constant coefficients are worth studying is that they serve as mathematical models of some important physical processes. Two important areas of application are the fields of mechanical and electrical oscillations. For example, the motion of a mass on a vibrating spring, the torsional oscillations of a shaft with a flywheel, the flow of electric current in a simple series circuit, and many other physical problems are all described by the solution of an initial value problem of the form

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'. \tag{1}$$

This illustrates a fundamental relationship between mathematics and physics: *many physical problems may have the same mathematical model*. Thus, once we know