Ch 4.1: Higher Order Linear ODEs: General Theory

- An ***n*th order ODE** has the general form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

- We assume that $P_0, \ldots, P_n$, and G are continuous real-valued functions on some interval $I = (\alpha, \beta)$, and that P_0 is nowhere zero on I.
- Dividing by P_0, the ODE becomes

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

- For an nth order ODE, there are typically n initial conditions:

$$y(t_0) = y_0,\ \ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Theorem 4.1.1

- Consider the *n*th order initial value problem

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

$$y(t_0) = y_0,\ \ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

- If the functions $p_1, \ldots, p_n$, and g are continuous on an open interval I, then there exists exactly one solution $y = \phi(t)$ that satisfies the initial value problem. This solution exists throughout the interval I.

Homogeneous Equations

- As with 2nd order case, we begin with homogeneous ODE:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0$$

- If $y_1,\ldots, y_n$ are solns to ODE, then so is linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Every soln can be expressed in this form, with coefficients determined by initial conditions, iff we can solve:

$$\begin{aligned}
c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\
c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\
&\vdots \\
c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)}
\end{aligned}$$

Homogeneous Equations & Wronskian

- The system of equations on the previous slide has a unique solution iff its determinant, or Wronskian, is nonzero at t_0:

$$W(y_1, y_2, \ldots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix}$$

- Since t_0 can be any point in the interval I, the Wronskian determinant needs to be nonzero at every point in I.
- As before, it turns out that the Wronskian is either zero for every point in I, or it is never zero on I.

Theorem 4.1.2

- Consider the *n*th order initial value problem

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0$$
$$y(t_0) = y_0,\ \ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y^{(n-1)}$$

- If the functions $p_1, \ldots, p_n$ are continuous on an open interval I, and if $y_1, \ldots, y_n$ are solutions with $W(y_1, \ldots, y_n)(t) \neq 0$ for at least one t in I, then every solution y of the ODE can be expressed as a linear combination of $y_1, \ldots, y_n$:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Linear Dependence and Independence

- Two functions f and g are **linearly dependent** if there exist constants c_1 and c_2, not both zero, such that
$$c_1 f(t) + c_2 g(t) = 0$$
for all t in I. Note that this reduces to determining whether f and g are multiples of each other.
- If the only solution to this equation is $c_1 = c_2 = 0$, then f and g are **linearly independent**.
- For example, let $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$, and consider the linear combination
$$c_1 \sin 2x + c_2 \sin x \cos x = 0$$
This equation is satisfied if we choose $c_1 = 1$, $c_2 = -2$, and hence f and g are linearly dependent.

Example 1

- Are the following functions linearly independent or dependent on the interval I: $0 < t < \infty$

$$f_1(t) = 1,\ f_2(t) = t,\ f_3(t) = t^2$$

- Form the linear combination and set it equal to zero

$$k_1 + k_2 t + k_3 t^2 = 0$$

- Evaluating this at t = 0, t = 1, and t = =1, we get

$$\begin{aligned} k_1 &= 0 \\ k_1 + k_2 + k_3 &= 0 \\ k_1 - k_2 + k_3 &= 0 \end{aligned}$$

- The only solution to this system is $k_1 = k_2 = k_3 = 0$
- Therefore, the given functions are linearly independent

Example 2

- Are the following functions linearly independent or dependent on any interval I:

$$f_1(t) = 1,\ f_2(t) = 2 + t,\ f_3(t) = 3 - t^2,\ f_4(t) = 4t + t^2$$

- Form the linear combination and set it equal to zero

$$k_1 + k_2(2 + t) + k_3(3 - t^2) + k_4(4t + t^2) = 0$$

- Evaluating this at t = 0, t = 1, and t = =1, we get

$$\begin{aligned} &k_1 + 2k_2 + k_3 = 0 \\ &k_2 + 4k_4 = 0 \\ &-k_3 + k_4 = 0 \end{aligned}$$

- There are many nonzero solutions to this system of equations
- Therefore, the given functions are linearly dependent

Theorem 4.1.3

- If $\{y_1, \ldots, y_n\}$ is a fundamental set of solutions of
$$L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$
on an interval I, then $\{y_1, \ldots, y_n\}$ are linearly independent on that interval.

- Conversely, if $\{y_1, \ldots, y_n\}$ are linearly independent solutions to the above differential equation, then they form a fundamental set of solutions on the interval I

Fundamental Solutions & Linear Independence

- Consider the nth order ODE:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

- A set $\{y_1, \ldots, y_n\}$ of solutions with $W(y_1, \ldots, y_n) \neq 0$ on I is called a **fundamental set of solutions**.
- Since all solutions can be expressed as a linear combination of the fundamental set of solutions, the **general solution** is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- If $y_1, \ldots, y_n$ are fundamental solutions, then $W(y_1, \ldots, y_n) \neq 0$ on I. It can be shown that this is equivalent to saying that $y_1, \ldots, y_n$ are **linearly independent**:

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0 \text{ iff } c_1 = c_2 = \cdots = c_n = 0$$

Nonhomogeneous Equations

- Consider the nonhomogeneous equation:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

- If Y_1, Y_2 are solns to nonhomogeneous equation, then Y_1 - Y_2 is a solution to the homogeneous equation:

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$$

- Then there exist coefficients $c_1, \ldots, c_n$ such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Thus the general solution to the nonhomogeneous ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)$$

where Y is any particular solution to nonhomogeneous ODE.

Ch 4.2: Homogeneous Differential Equations with Constant Coefficients

- Consider the *n*th order linear homogeneous differential equation with constant, real coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- As with second order linear equations with constant coefficients, $y = e^{rt}$ is a solution for values of r that make characteristic polynomial $Z(r)$ zero:

$$L\left[e^{rt}\right] = e^{rt} \underbrace{\left[a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n\right]}_{\text{characteristic polynomial } Z(r)} = 0$$

- By the fundamental theorem of algebra, a polynomial of degree n has n roots $r_1, r_2, \ldots, r_n$, and hence

$$Z(r) = a_0 (r - r_1)(r - r_2) \cdots (r - r_n)$$

Real and Unequal Roots

- If roots of characteristic polynomial $Z(r)$ are real and unequal, then there are n distinct solutions of the differential equation:

$$e^{r_1 t},\ e^{r_2 t},\ \ldots,\ e^{r_n t}$$

- If these functions are linearly independent, then general solution of differential equation is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$$

- The Wronskian can be used to determine linear independence of solutions.

Example 1: Distinct Real Roots (1 of 3)

- Consider the initial value problem

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$
$$y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = -1$$

- Assuming exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 + r^3 - 7r^2 - r + 6 = 0$$
$$\Leftrightarrow \quad (r-1)(r+1)(r-2)(r+3) = 0$$

- Thus the general solution is

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}$$

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + c_3 e^{-3t}$$

Example 1: Solution (2 of 3)

- The initial conditions

$$y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = -1$$

yield

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ c_1 - c_2 + 2c_3 - 3c_4 &= 0 \\ c_1 + c_2 + 4c_3 + 9c_4 &= -2 \\ c_1 - c_2 + 8c_3 - 27c_4 &= -1 \end{aligned}$$

- Solving,

$$c_1 = \frac{11}{8},\ c_2 = \frac{5}{12},\ c_3 = -\frac{2}{3},\ c_4 = -\frac{1}{8}$$

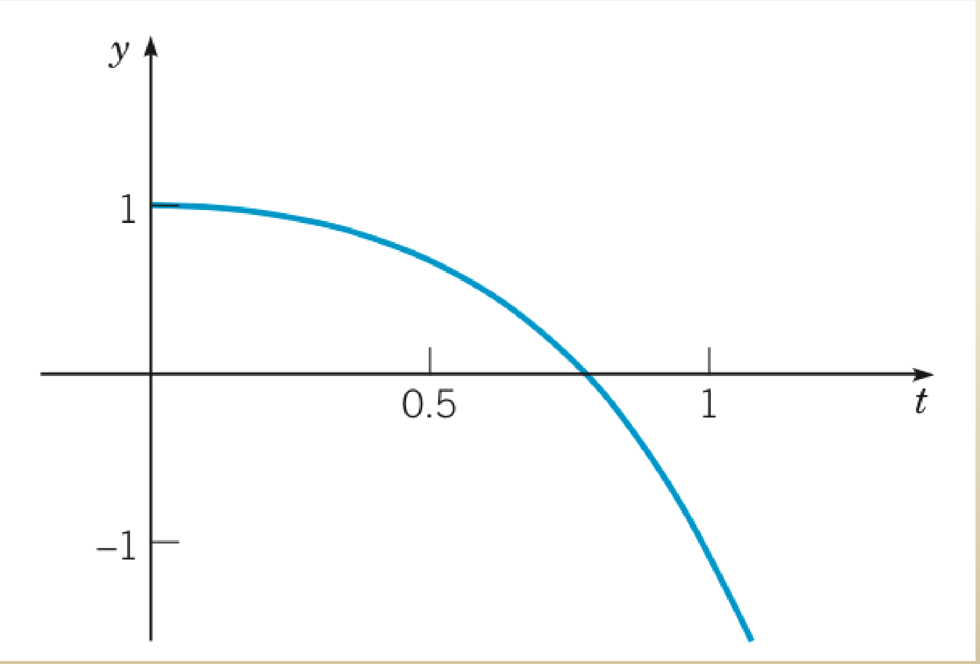
- Hence

$$y(t) = \frac{11}{8} e^{t} + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t}$$

Example 1: Graph of Solution (3 of 3)

- The graph of the solution is given below. Note the effect of the largest root of the characteristic equation.

$$y(t) = \frac{11}{8}e^{t} + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}$$



Complex Roots

- If the characteristic polynomial $Z(r)$ has complex roots, then they must occur in conjugate pairs, $\lambda \pm i\mu$
- Note that not all the roots need be complex.
- Solutions corresponding to complex roots have the form

$$e^{(\lambda+i\mu)t} = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t$$
$$e^{(\lambda-i\mu)t} = e^{\lambda t}\cos\mu t - ie^{\lambda t}\sin\mu t$$

- As in Chapter 3.4, we use the real-valued solutions

$$e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t$$

Example 2: Complex Roots

- Consider the initial value problem

$$y^{(4)} - y = 0, \quad y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2$$

- Then

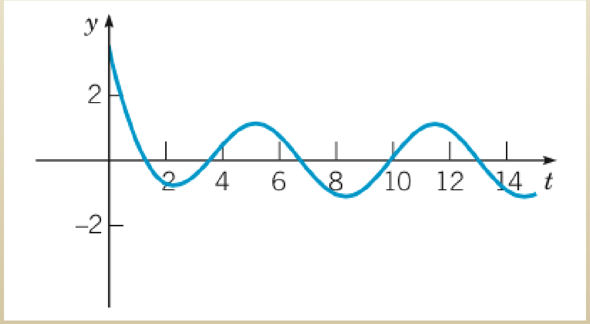
$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 - 1 = 0 \Leftrightarrow \left(r^2 - 1\right)\left(r^2 + 1\right) = 0$$

- The roots are 1, -1, *i*, -*i*. Thus the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$$

- Using the initial conditions, we obtain

$$y(t) = 0e^t + 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$$



- The graph of solution is given on right.

$$y(t) = 0e^{t} + 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$$

Example 2: Small Change in an Initial Condition (2 of 2)

- Note that if one initial condition is slightly modified, then the solution can change significantly. For example, replace

$$y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2$$

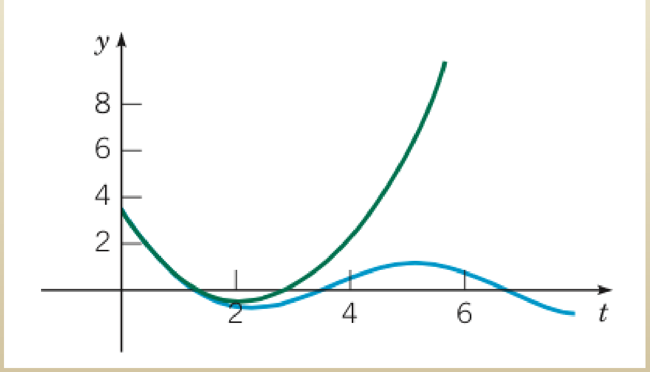
with

$$y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -15/8$$

then

$$y(t) = \frac{1}{32}e^{t} + \frac{95}{32}e^{-t} + \frac{1}{2}\cos(t) - \frac{17}{16}\sin(t)$$

- The graph of this solution and original are given below.



Repeated Roots

- Suppose a root r_k of characteristic polynomial $Z(r)$ is a repeated root with multiplicty s. Then linearly independent solutions corresponding to this repeated root have the form

$$e^{r_k t},\ te^{r_k t},\ t^2 e^{r_k t},\ \ldots,\ t^{s-1} e^{r_k t}$$

- If a complex root $\lambda + i\mu$ is repeated s times, then so is its conjugate $\lambda - i\mu$. There are $2s$ corresponding linearly independent solns, derived from real and imaginary parts of

$$e^{(\lambda+iu)t},\ te^{(\lambda+iu)t},\ t^2 e^{(\lambda+iu)t},\ \ldots,\ t^{s-1} e^{(\lambda+iu)t}$$

or

$$e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t,\ te^{\lambda t}\cos\mu t,\ te^{\lambda t}\sin\mu t, \ldots,$$

$$t^{s-1} e^{r_k t}\cos\mu t,\quad t^{s-1} e^{r_k t} e^{\lambda t}\sin\mu t,$$

Example 3: Repeated Roots

- Consider the equation

$$y^{(4)} + 2y'' + y = 0$$

- Then

$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 + 2r + 1 = 0 \Leftrightarrow \left(r^2 + 1\right)\left(r^2 + 1\right) = 0$$

- The roots are *i*, *i*, -*i*, -*i*. Thus the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos(t) + c_4 t \sin(t)$$

Example 4: Complex Roots of –1 (1 of 2)

- For the general solution of $y^{(4)} + y = 0$, the characteristic equation is $r^4 + 1 = 0$.
- To solve this equation, we need to use Euler's equation to find the four 4th roots of –1:

$$-1 = \cos\pi + i\sin\pi = e^{i\pi} \text{ or}$$

$$-1 = \cos(\pi + 2m\pi) + i\sin(\pi + 2m\pi) = e^{i(\pi+2m\pi)} \text{ for any integer } m$$

$$(-1)^{1/4} = e^{i(\pi+2m\pi)/4} = \cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right)$$

- Letting m = 0, 1, 2, and 3, we get the roots:

$$\frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \text{ respectively.}$$

$$r = \left\{ \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}} \right\}$$

Example 4: Complex Roots of –1 (2 of 2)

- Given the four complex roots, extending the ideas from Chapter 4, we can form four linearly independent real solutions.
- For the complex conjugate pair $\frac{1 \pm i}{\sqrt{2}}$, we get the solutions

$$y_1 = e^{t/\sqrt{2}} \cos(t/\sqrt{2}),\ y_2 = e^{t/\sqrt{2}} \sin(t/\sqrt{2})$$

- For the complex conjugate pair $\frac{-1 \pm i}{\sqrt{2}}$, we get the solutions

$$y_3 = e^{-t/\sqrt{2}} \cos(t/\sqrt{2}),\ y_4 = e^{-t/\sqrt{2}} \sin(t/\sqrt{2})$$

- So the general solution can be written as

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$$