Introduction to Differential Equations
Sample problems # 10

Date Given: June 13, 2022

P1. Find the Laplace transform of

$$f(t) = \begin{cases} 0, & \text{if } t < 2 \\ (t-2)^2, & \text{if } t \geq 2 \end{cases}$$

Solution: Using the Heaviside function, we can write $f(t) = \left((t-2)^2\right) u_2(t)$. The Laplace transform has the property that $\mathcal{L}[u_c(t)\, f(t-c)] = e^{-cs}\mathcal{L}[f(t)]$. Hence

$$\mathcal{L}\left[(t-2)^2\right] u_2(t) = \frac{2e^{-2s}}{s^3}$$

P2. Find the Laplace transform of

$$f(t) = \begin{cases} 0, & \text{if } t < \pi \\ t-\pi, & \text{if } \pi \leq t < 2\pi \\ 0, & \text{if } t \geq 2\pi \end{cases}$$

P3. Find the Laplace transform of $f(t) = (t-3)\, u_2(t) - (t-2)\, u_3(t)$.

Solution: Before invoking the translation property of the transform, write the function as

$$f(t) = (t-2)\, u_2(t) - u_2(t) - (t-3)\, u_3(t) - u_3(t)$$

It follows directly from the translation property of the transform that

$$\mathcal{L}\left[f(t)\right] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}$$

P4. Find the inverse Laplace transform of $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2}$.

Solution: First, consider the function

$$G(s) = \frac{2(s-1)}{s^2+s-2}$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2+1}$$

It follows that

$$\mathcal{L}^{-1}\left[G(s)\right] = 2e^t \cos t$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s}G(s)\right] = 2e^{t-2}\cos(t-2)\, u_2(t)$$

P5. Find the inverse Laplace transform of $F(s) = \dfrac{(s-2)e^{-s}}{s^2-4s+3}$.

Solution: First, consider the function

$$G(s) = \frac{(s-2)}{s^2 - 4s + 3}$$

Completing the square in the denominator,

$$G(s) = \frac{(s-2)}{(s-2)^2 - 1}$$

It follows that

$$\mathcal{L}^{-1}\left[G(s)\right] = e^{2t}\cosh t$$

Hence

$$\mathcal{L}^{-1}\left[e^{-s}G(s)\right] = e^{2(t-1)}\cosh(t-1)\,u_1(t)$$

P6. Find the inverse Laplace transform of $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$.

Solution: Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}$$

It follows from the translation property of the transform that

$$\mathcal{L}^{-1}\left[\frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) - u_4(t)$$

P7. Suppose that $F(s) = \mathcal{L}\left[f(t)\right]$ exists for $s > a \geq 0$.

(a) Show that if c is a positive constant, then

$$\mathcal{L}\left[f(ct)\right] = \frac{1}{c}\,F\left(\frac{s}{c}\right), \quad s > ca$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\left[F(ks)\right] = \frac{1}{k}\,f\left(\frac{t}{k}\right)$$

(c) Show that if a and b are positive constants with $a > 0$, then

$$\mathcal{L}^{-1}\left[F(as+b)\right] = \frac{1}{a}\,e^{-bt/a} f\left(\frac{t}{a}\right)$$

Solution:

(a) By definition of the Laplace transform,

$$\mathcal{L}\left[f(ct)\right] = \int_0^\infty e^{-st} f(ct)\,\mathrm{d}t$$

Making a change of variable, $\tau = ct$, we have

$$\mathcal{L}\left[f(ct)\right] = \frac{1}{c}\int_0^\infty e^{-(s/c)\tau} f(\tau)\,\mathrm{d}\tau$$

Hence $\mathcal{L}\left[f(ct)\right] = \dfrac{1}{c}\,F\left(\dfrac{s}{c}\right)$, where $s > ca$.

(b) Using the result in part (a),

$$\mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = kF(ks)$$

Hence

$$\mathcal{L}^{-1}\left[F(ks)\right] = \frac{1}{k}\,f\left(\frac{t}{k}\right)$$

(c) From part (b), $\mathcal{L}^{-1}\left[F(as)\right] = (1/a)f\left(t/a\right)$. Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}\left[e^{ct}f(t)\right] = \mathcal{L}\left[f(t)\right]_{s\to(s-c)}$, we get

$$\mathcal{L}^{-1}\left[F(as+b)\right] = \frac{1}{a}\,e^{-bt/a}f\left(\frac{t}{a}\right)$$

P8. Find the solution of the following initial value problem

$$y'' + 2y' + 2y = h(t), \quad y(0) = 0, y'(0) = 1, \quad h(t) = \begin{cases} 1, & \text{if } \pi \le t < 2\pi \\ 0, & \text{if } 0 \le t < \pi \text{ and } t \ge 2\pi \end{cases}$$

Draw the graphs of the forcing function and of the solution.

Solution: Let $h(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] + 2Y(s) = \mathcal{L}\left[h(t)\right].$$

Applying the initial conditions, we have

$$s^2Y(s) + 2sY(s) + 2Y(s) - 1 = \mathcal{L}\left[h(t)\right].$$

The forcing function can be written as $h(t) = u_\pi(t) - u_{2\pi}(t)$. Its transform

$$\mathcal{L}\left[h(t)\right] = \frac{e^{-\pi s} - e^{-2\pi s}}{s}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2+2s+2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}$$

First note that

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}.$$

Using partial fractions,

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{(s+1)+1}{(s+1)^2+1}$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t$$

Now let

$$G(s) = \frac{1}{s(s^2+2s+2)}.$$

Then

$$\mathcal{L}^{-1}\left[G(s)\right] = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$$

Applying Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[e^{-cs}G(s)\right] = \frac{1}{2}u_c(t) - \frac{1}{2}e^{-(t-c)}\cos(t-c)\,u_c(t) - \frac{1}{2}e^{-(t-c)}\sin(t-c)\,u_c(t)$$

Hence the solution of the initial value problem is

$$\begin{aligned} y(t) = e^{-t}\sin t + \frac{1}{2}u_\pi(t) - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi)\,u_\pi(t) - \frac{1}{2}e^{-(t-\pi)}\sin(t-\pi)\,u_\pi(t) \\ -\frac{1}{2}u_{2\pi}(t) + \frac{1}{2}e^{-(t-2\pi)}\cos(t-2\pi)\,u_{2\pi}(t) + \frac{1}{2}e^{-(t-2\pi)}\sin(t-2\pi)\,u_{2\pi}(t) \end{aligned}$$

That is

$$y(t) = e^{-t}\sin t + \frac{1}{2}\left\{u_\pi(t) - u_{2\pi}(t)\right\} + \frac{1}{2}(\cos t + \sin t)\left\{e^{-(t-\pi)}u_\pi(t) + e^{-(t-2\pi)}u_{2\pi}(t)\right\}$$

Graphs of the solution and the forcing function are shown in Figure 1.

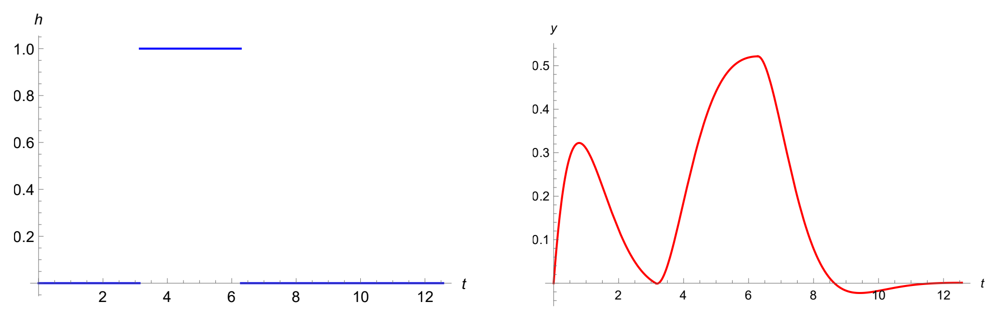

Figure 1: Illustration to problem P8. The solution starts out as free oscillation, due to the initial conditions. The amplitude increases, as long as the forcing is present. Thereafter, the solution rapidly decays.

P9. Find the solution of the following initial value problem

$$y'' + 3y' + 2y = f(t), \quad y(0) = 0, y'(0) = 0, \quad h(t) = \begin{cases} 1, & \text{if } 0 \le t < 10 \\ 0, & \text{if } t \ge 10 \end{cases}$$

Draw the graphs of the forcing function and of the solution.

Solution: Let $f(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = \mathcal{L}\left[f(t)\right].$$

Applying the initial conditions, we have

$$s^2Y(s) + 3sY(s) + 2Y(s) = \mathcal{L}\left[h(t)\right].$$

The forcing function can be written as $f(t) = 1 - u_{10}(t)$. Its transform

$$\mathcal{L}\left[f(t)\right] = \frac{1}{s} - \frac{e^{-10s}}{s}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}$$

Using partial fractions,

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{2}\,\frac{1}{s} + \frac{1}{2}\,\frac{1}{(s+2)} - \frac{1}{2}\,\frac{2}{(s+1)}$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = \frac{1}{2} + \frac{1}{2}\,e^{-2t} - e^{-t}$$

Based on Theorem 6.3.1,

$$\mathcal{L}^{-1}\left[\frac{e^{-10s}}{s(s^2+3s+2)}\right] = \frac{1}{2}\left\{1 + e^{-2(t-10)} - 2e^{-(t-10)}\right\}u_{10}(t).$$

Hence the solution of the initial value problem is

$$y(t) = \frac{1}{2}\{1 - u_{10}(t)\} + \frac{1}{2}\,e^{-2t} - e^{-t} - \frac{1}{2}\left\{e^{-2(t-10)} - 2e^{-(t-10)}\right\}u_{10}(t)$$

Graphs of the solution and the forcing function are shown in Figure 2.

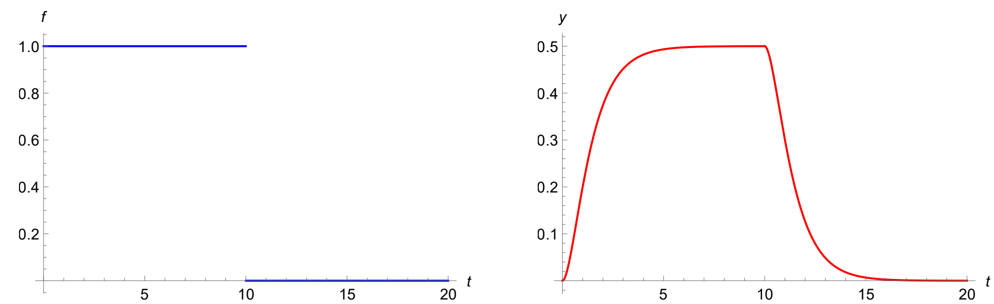

Figure 2: Illustration to problem P9. The solution increases to a temporary steady value of $y = 1/2$. After the forcing ceases, the response decays exponentially to $y = 0$.

P10. Find the solution of the following initial value problem

$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, y'(0) = 1.$$

Draw the graphs of the forcing function and of the solution.

P11. Find the solution of the following initial value problem $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$; $y(0) = 1, y'(0) = 0$. Draw the graph of the solution.

Solution: Let $Y(s) = \mathcal{L}[y]$ and take the Laplace transform of the differential equation. We arrive at

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Applying the initial conditions, we have

$$s^2Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s},$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4} = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}.$$

Applying Theorem 6.3.1, the solution of the initial value problem is

$$y(t) = \frac{1}{2}\sin(2t-2\pi)u_\pi(t) - \frac{1}{2}\sin(2t-4\pi)u_{2\pi}(t) = \frac{1}{2}\sin(2t)\left(u_\pi(t) - u_{2\pi}(t)\right).$$

A graph of the solution is shown in Figure 3.

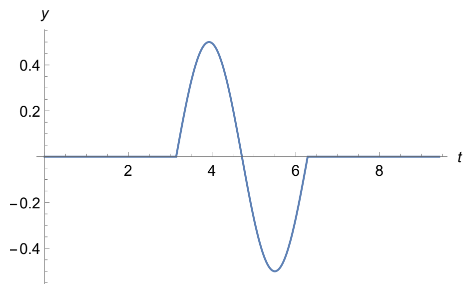

Figure 3: Illustration to problem P11.

P12. The problem deals with the effect of a sequence of impulses on an undamped oscillator. Find the solution of the following initial value problem $y'' + y = \sum_{k=1}^{20} \delta(t - k\pi)$; $y(0) = 0, y'(0) = 0$. Determine what happens after the sequence of impulses ends. Draw a graph of the solution.

Solution: Applying the Laplace transform to the equation and using the initial conditions, we have We arrive at

$$(s^2 + 1)Y(s) = \sum_{k=1}^{20} e^{-k\pi s}$$

Therefore,

$$Y(s) = \frac{1}{s^2 + 1} \sum_{k=1}^{20} e^{-k\pi s},$$

which implies

$$y(t) = \sum_{k=1}^{20} \sin(t - k\pi)\, u_{k\pi}(t).$$

After after the sequence of impulses ends, the oscillator returns to equilibrium[1]. A graph of the solution is shown in Figure 4.

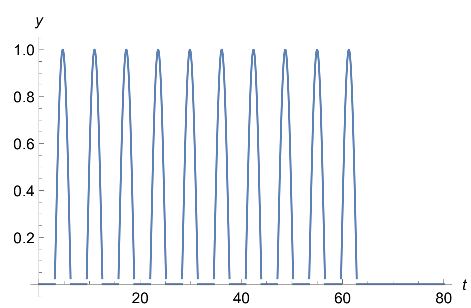

Figure 4: Illustration to problem P12.

P13. The problem deals with the effect of a sequence of impulses on an undamped oscillator. Find the solution of the following initial value problem $y'' + y = \sum_{k=1}^{20} (-1)^{k+1} \delta(t - k\pi)$; $y(0) = 0, y'(0) = 0$. Determine what happens after the sequence of impulses ends. Draw a graph of the solution.

[1]For $t > 20\pi$, we will have the motion of a free (unforced) oscillator with the initial conditions $y(20\pi) = 0$ and $y'(20\pi) = 0$. Therefore, $y(t) = 0$.

Solution: Applying the Laplace transform to the equation and using the initial conditions, we have We arrive at

$$(s^2+1)Y(s) = \sum_{k=1}^{20}(-1)^{k+1}e^{-k\pi s}$$

Therefore,

$$Y(s) = \frac{1}{s^2+1}\sum_{k=1}^{20}(-1)^{k+1}e^{-k\pi s}.$$

Applying Theorem 6.3.1 term-by-term, we get

$$y(t) = \sum_{k=1}^{20}(-1)^{k+1}\sin(t-k\pi)\,u_{k\pi}(t) = -\sin t\sum_{k=1}^{20}u_{k\pi}(t).$$

After after the sequence of impulses ends, for $t > 20\pi$, we will have the motion of a free (unforced) oscillator with the initial conditions $y(20\pi) = 0$ and $y'(20\pi) = -20$. Therefore, $y(t) = -20\sin t$. A graph of the solution is shown in Figure 5.

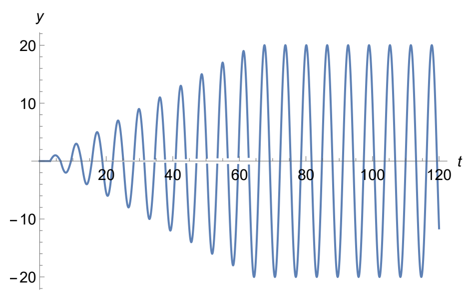

Figure 5: Illustration to problem P13.

P14. The problem deals with the effect of a sequence of impulses on an undamped oscillator. Find the solution of the following initial value problem $y'' + y = \sum_{k=1}^{40}(-1)^{k+1}\delta(t-3k);\ y(0) = 0, y'(0) = 0.$ Determine what happens after the sequence of impulses ends. Draw a graph of the solution.

Solution: Applying the Laplace transform to the equation and using the initial conditions, we have We arrive at

$$(s^2+1)Y(s) = \sum_{k=1}^{40}(-1)^{k+1}e^{-3ks}$$

Therefore,

$$Y(s) = \frac{1}{s^2+1}\sum_{k=1}^{40}(-1)^{k+1}e^{-3ks}.$$

Applying Theorem 6.3.1 term-by-term, we get

$$y(t) = \sum_{k=1}^{40}(-1)^{k+1}\sin(t-3k)\,u_{3k}(t)$$

After after the sequence of impulses ends, for $t > 120$, we will have the motion of a free (unforced) oscillator with the initial conditions $y(120) \approx 1.60086$ and $y'(120) \approx 4.00065$. Therefore,

$$y(t) = \sum_{k=1}^{40}(-1)^{k+1}\sin(t-3k)\,u_{3k}(t) \approx 4.30906\cos(t-1.80974)$$

A graph of the solution is shown in Figure 6.

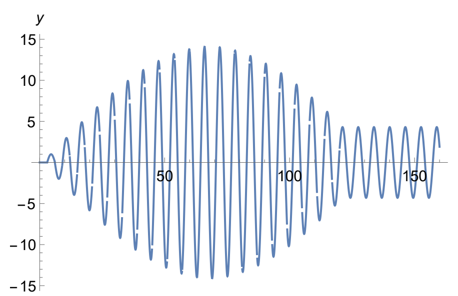

Figure 6: Illustration to problem P14.