CHAPTER

3

Second Order Linear Equations

Linear equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and of these methods is understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations. Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations. As an example, we discuss the oscillations of some basic mechanical and electrical systems at the end of the chapter.

3.1 Homogeneous Equations with Constant Coefficients

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \tag{1}$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we will use x instead. We will use y, or occasionally some other letter, to designate the dependent variable. Equation (1) is said to be **linear** if the function f

has the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{2}$$

that is, if f is linear in y and dy/dt. In Eq. (2) g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite Eq. (1) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{3}$$

where the primes denote differentiation with respect to t. Instead of Eq. (3), we often see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4}$$

Of course, if $P(t) \neq 0$, we can divide Eq. (4) by $P(t)$ and thereby obtain Eq. (3) with

$$p(t) = \frac{Q(t)}{P(t)}, \qquad q(t) = \frac{R(t)}{P(t)}, \qquad g(t) = \frac{G(t)}{P(t)}. \tag{5}$$

In discussing Eq. (3) and in trying to solve it, we will restrict ourselves to intervals in which p, q, and g are continuous functions.[1]

If Eq. (1) is not of the form (3) or (4), then it is called **nonlinear**. Analytical investigations of nonlinear equations are relatively difficult, so we will have little to say about them in this book. Numerical or geometical approaches are often more appropriate, and these are discussed in Chapters 8 and 9.

An initial value problem consists of a differential equation such as Eq. (1), (3), or (4) together with a pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y'_0, \tag{6}$$

where y_0 and y'_0 are given numbers prescribing values for y and y' at the initial point t_0. Observe that the initial conditions for a second order equation identify not only a particular point (t_0, y_0) through which the graph of the solution must pass, but also the slope y'_0 of the graph at that point. It is reasonable to expect that two initial conditions are needed for a second order equation because, roughly speaking, two integrations are required to find a solution and each integration introduces an arbitrary constant. Presumably, two initial conditions will suffice to determine values for these two constants.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ in Eq. (3), or the term $G(t)$ in Eq. (4), is zero for all t. Otherwise, the equation is called **nonhomogeneous**. As a result, the term $g(t)$, or $G(t)$, is sometimes called the nonhomogeneous term. We begin our discussion with homogeneous equations, which we will write in the form

$$P(t)y'' + Q(t)y' + R(t)y = 0. \tag{7}$$

Later, in Sections 3.5 and 3.6, we will show that once the homogeneous equation has been solved, it is always possible to solve the corresponding nonhomogeneous

[1]There is a corresponding treatment of higher order linear equations in Chapter 4. If you wish, you may read the appropriate parts of Chapter 4 in parallel with Chapter 3.

equation (4), or at least to express the solution in terms of an integral. Thus the problem of solving the homogeneous equation is the more fundamental one.

In this chapter we will concentrate our attention on equations in which the functions P, Q, and R are constants. In this case, Eq. (7) becomes

$$ay'' + by' + cy = 0, \tag{8}$$

where a, b, and c are given constants. It turns out that Eq. (8) can always be solved easily in terms of the elementary functions of calculus. On the other hand, it is usually much more difficult to solve Eq. (7) if the coefficients are not constants, and a treatment of that case is deferred until Chapter 5. Before taking up Eq. (8), let us first gain some experience by looking at a simple example that in many ways is typical.

EXAMPLE 1

Solve the equation

$$y'' - y = 0, \tag{9}$$

and also find the solution that satisfies the initial conditions

$$y(0) = 2, \qquad y'(0) = -1. \tag{10}$$

Observe that Eq. (9) is just Eq. (8) with $a = 1, b = 0$, and $c = -1$. In words, Eq. (9) says that we seek a function with the property that the second derivative of the function is the same as the function itself. Do any of the functions that you studied in calculus have this property? A little thought will probably produce at least one such function, namely, $y_1(t) = e^t$, the exponential function. A little more thought may also produce a second function, $y_2(t) = e^{-t}$. Some further experimentation reveals that constant multiples of these two solutions are also solutions. For example, the functions $2e^t$ and $5e^{-t}$ also satisfy Eq. (9), as you can verify by calculating their second derivatives. In the same way, the functions $c_1y_1(t) = c_1e^t$ and $c_2y_2(t) = c_2e^{-t}$ satisfy the differential equation (9) for all values of the constants c_1 and c_2.

Next, it is vital to notice that the sum of any two solutions of Eq. (9) is also a solution. In particular, since $c_1y_1(t)$ and $c_2y_2(t)$ are solutions of Eq. (9) for any values of c_1 and c_2, so is the function

$$y = c_1y_1(t) + c_2y_2(t) = c_1e^t + c_2e^{-t}. \tag{11}$$

Again, this can be verified by calculating the second derivative y'' from Eq. (11). We have $y' = c_1e^t - c_2e^{-t}$ and $y'' = c_1e^t + c_2e^{-t}$; thus y'' is the same as y, and Eq. (9) is satisfied.

Let us summarize what we have done so far in this example. Once we notice that the functions $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions of Eq. (9), it follows that the general linear combination (11) of these functions is also a solution. Since the coefficients c_1 and c_2 in Eq. (11) are arbitrary, this expression represents an infinite family of solutions of the differential equation (9).

It is now possible to consider how to pick out a particular member of this infinite family of solutions that also satisfies a given set of initial conditions (10). In other words, we seek the solution that passes through the point $(0, 2)$ and at that point has the slope -1. First, we set $t = 0$ and $y = 2$ in Eq. (11); this gives the equation

$$c_1 + c_2 = 2. \tag{12}$$

Next, we differentiate Eq. (11) with the result that

$$y' = c_1e^t - c_2e^{-t}.$$

Then, setting $t = 0$ and $y' = -1$, we obtain

$$c_1 - c_2 = -1. \tag{13}$$

By solving Eqs. (12) and (13) simultaneously for c_1 and c_2, we find that

$$c_1 = \tfrac{1}{2}, \qquad c_2 = \tfrac{3}{2}. \tag{14}$$

Finally, inserting these values in Eq. (11), we obtain

$$y = \tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}, \tag{15}$$

the solution of the initial value problem consisting of the differential equation (9) and the initial conditions (10).

What conclusions can we draw from the preceding example that will help us to deal with the more general equation (8),

$$ay'' + by' + cy = 0,$$

whose coefficients a, b, and c are arbitrary (real) constants? In the first place, in the example the solutions were exponential functions. Further, once we had identified two solutions, we were able to use a linear combination of them to satisfy the given initial conditions as well as the differential equation itself.

It turns out that by exploiting these two ideas, we can solve Eq. (8) for any values of its coefficients and also satisfy any given set of initial conditions for y and y'. We start by seeking exponential solutions of the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2e^{rt}$. By substituting these expressions for y, y', and y'' in Eq. (8), we obtain

$$(ar^2 + br + c)e^{rt} = 0,$$

or, since $e^{rt} \neq 0$,

$$ar^2 + br + c = 0. \tag{16}$$

Equation (16) is called the **characteristic equation** for the differential equation (8). Its significance lies in the fact that if r is a root of the polynomial equation (16), then $y = e^{rt}$ is a solution of the differential equation (8). Since Eq. (16) is a quadratic equation with real coefficients, it has two roots, which may be real and different, real but repeated, or complex conjugates. We consider the first case here and the latter two cases in Sections 3.3 and 3.4.

Assuming that the roots of the characteristic equation (16) are real and different, let them be denoted by r_1 and r_2, where $r_1 \neq r_2$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of Eq. (8). Just as in Example 1, it now follows that

$$y = c_1y_1(t) + c_2y_2(t) = c_1e^{r_1 t} + c_2e^{r_2 t} \tag{17}$$

is also a solution of Eq. (8). To verify that this is so, we can differentiate the expression in Eq. (17); hence

$$y' = c_1r_1e^{r_1 t} + c_2r_2e^{r_2 t} \tag{18}$$

and

$$y'' = c_1r_1^2e^{r_1 t} + c_2r_2^2e^{r_2 t}. \tag{19}$$

Substituting these expressions for y, y', and y'' in Eq. (8) and rearranging terms, we obtain

$$ay'' + by' + cy = c_1(ar_1^2 + br_1 + c)e^{r_1 t} + c_2(ar_2^2 + br_2 + c)e^{r_2 t}. \tag{20}$$

The quantities in the two sets of parentheses on the right-hand side of Eq. (20) are zero because r_1 and r_2 are roots of Eq. (16); therefore, y as given by Eq. (17) is indeed a solution of Eq. (8), as we wished to verify.

Now suppose that we want to find the particular member of the family of solutions (17) that satisfies the initial conditions (6)

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

By substituting $t = t_0$ and $y = y_0$ in Eq. (17), we obtain

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0. \tag{21}$$

Similarly, setting $t = t_0$ and $y' = y_0'$ in Eq. (18) gives

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'. \tag{22}$$

On solving Eqs. (21) and (22) simultaneously for c_1 and c_2, we find that

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \qquad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}. \tag{23}$$

Recall that $r_1 - r_2 \neq 0$ so that the expressions in Eq. (23) always make sense. Thus, no matter what initial conditions are assigned—that is, regardless of the values of t_0, y_0, and y_0' in Eqs. (6)—it is always possible to determine c_1 and c_2 so that the initial conditions are satisfied. Moreover, there is only one possible choice of c_1 and c_2 for each set of initial conditions. With the values of c_1 and c_2 given by Eq. (23), the expression (17) is the solution of the initial value problem

$$ay'' + by' + cy = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{24}$$

It is possible to show, on the basis of the fundamental theorem cited in the next section, that all solutions of Eq. (8) are included in the expression (17), at least for the case in which the roots of Eq. (16) are real and different. Therefore, we call Eq. (17) the general solution of Eq. (8). The fact that any possible initial conditions can be satisfied by the proper choice of the constants in Eq. (17) makes more plausible the idea that this expression does include all solutions of Eq. (8).

Let us now look at some further examples.

EXAMPLE 2

Find the general solution of

$$y'' + 5y' + 6y = 0. \tag{25}$$

We assume that $y = e^{rt}$, and it then follows that r must be a root of the characteristic equation

$$r^2 + 5r + 6 = (r + 2)(r + 3) = 0.$$

Thus the possible values of r are $r_1 = -2$ and $r_2 = -3$; the general solution of Eq. (25) is

$$y = c_1 e^{-2t} + c_2 e^{-3t}. \tag{26}$$

EXAMPLE 3

Find the solution of the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = 3. \tag{27}$$

The general solution of the differential equation was found in Example 2 and is given by Eq. (26). To satisfy the first initial condition, we set $t = 0$ and $y = 2$ in Eq. (26); thus c_1 and c_2 must satisfy

$$c_1 + c_2 = 2. \tag{28}$$

To use the second initial condition, we must first differentiate Eq. (26). This gives $y' = -2c_1e^{-2t} - 3c_2e^{-3t}$. Then, setting $t = 0$ and $y' = 3$, we obtain

$$-2c_1 - 3c_2 = 3. \tag{29}$$

By solving Eqs. (28) and (29), we find that $c_1 = 9$ and $c_2 = -7$. Using these values in the expression (26), we obtain the solution

$$y = 9e^{-2t} - 7e^{-3t} \tag{30}$$

of the initial value problem (27). The graph of the solution is shown in Figure 3.1.1.

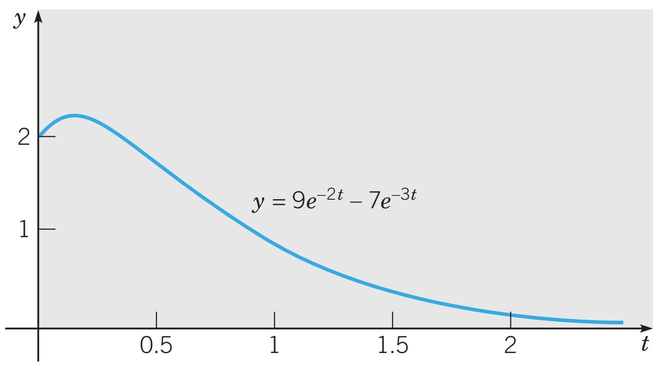

FIGURE 3.1.1 Solution of the initial value problem (27): $y'' + 5y' + 6y = 0,\ y(0) = 2,\ y'(0) = 3$.

EXAMPLE 4

Find the solution of the initial value problem

$$4y'' - 8y' + 3y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}. \tag{31}$$

If $y = e^{rt}$, then we obtain the characteristic equation

$$4r^2 - 8r + 3 = 0$$

whose roots are $r = 3/2$ and $r = 1/2$. Therefore, the general solution of the differential equation is

$$y = c_1e^{3t/2} + c_2e^{t/2}. \tag{32}$$

Applying the initial conditions, we obtain the following two equations for c_1 and c_2:

$$c_1 + c_2 = 2, \qquad \tfrac{3}{2}c_1 + \tfrac{1}{2}c_2 = \tfrac{1}{2}.$$

The solution of these equations is $c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$, so the solution of the initial value problem (31) is

$$y = -\tfrac{1}{2}e^{3t/2} + \tfrac{5}{2}e^{t/2}. \tag{33}$$

Figure 3.1.2 shows the graph of the solution.

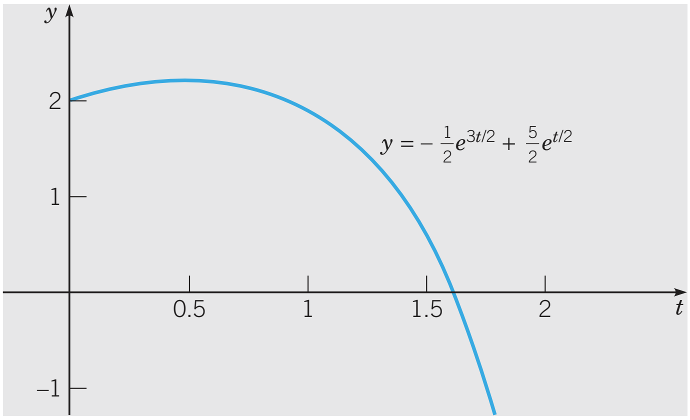

FIGURE 3.1.2 Solution of the initial value problem (31): $4y'' - 8y' + 3y = 0,\ \ y(0) = 2,\ \ y'(0) = 0.5.$

EXAMPLE 5

The solution (30) of the initial value problem (27) initially increases (because its initial slope is positive) but eventually approaches zero (because both terms involve negative exponential functions). Therefore, the solution must have a maximum point, and the graph in Figure 3.1.1 confirms this. Determine the location of this maximum point.

The coordinates of the maximum point can be estimated from the graph, but to find them more precisely, we seek the point where the solution has a horizontal tangent line. By differentiating the solution (30), $y = 9e^{-2t} - 7e^{-3t}$, with respect to t, we obtain

$$y' = -18e^{-2t} + 21e^{-3t}. \tag{34}$$

Setting y' equal to zero and multiplying by e^{3t}, we find that the critical value t_m satisfies $e^t = 7/6$; hence

$$t_m = \ln(7/6) \cong 0.15415. \tag{35}$$

The corresponding maximum value y_m is given by

$$y_m = 9e^{-2t_m} - 7e^{-3t_m} = \frac{108}{49} \cong 2.20408. \tag{36}$$

In this example the initial slope is 3, but the solution of the given differential equation behaves in a similar way for any other positive initial slope. In Problem 26 you are asked to determine how the coordinates of the maximum point depend on the initial slope.

Returning to the equation $ay'' + by' + cy = 0$ with arbitrary coefficients, recall that when $r_1 \neq r_2$, its general solution (17) is the sum of two exponential functions. Therefore, the solution has a relatively simple geometrical behavior: as t increases, the magnitude of the solution either tends to zero (when both exponents are negative) or else grows rapidly (when at least one exponent is positive). These two cases are illustrated by the solutions of Examples 3 and 4, which are shown in Figures 3.1.1 and 3.1.2, respectively. There is also a third case that occurs less often: the solution approaches a constant when one exponent is zero and the other is negative.

In Sections 3.3 and 3.4, respectively, we return to the problem of solving the equation $ay'' + by' + cy = 0$ when the roots of the characteristic equation either are complex conjugates or are real and equal. In the meantime, in Section 3.2, we provide a systematic discussion of the mathematical structure of the solutions of all second order linear homogeneous equations.

PROBLEMS

In each of Problems 1 through 8, find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $y'' + 3y' + 2y = 0$
3. $6y'' - y' - y = 0$
4. $2y'' - 3y' + y = 0$
5. $y'' + 5y' = 0$
6. $4y'' - 9y = 0$
7. $y'' - 9y' + 9y = 0$
8. $y'' - 2y' - 2y = 0$

In each of Problems 9 through 16, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

9. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$
10. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
11. $6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0$
12. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$
13. $y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$
14. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
15. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$
16. $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$
17. Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.
18. Find a differential equation whose general solution is $y = c_1e^{-t/2} + c_2e^{-2t}$.
19. Find the solution of the initial value problem

$$y'' - y = 0, \qquad y(0) = \tfrac{5}{4}, \quad y'(0) = -\tfrac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

20. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

21. Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.
22. Solve the initial value problem $4y'' - y = 0$, $y(0) = 2$, $y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

In each of Problems 23 and 24, determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

23. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
24. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$
25. Consider the initial value problem

$$2y'' + 3y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = -\beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Plot the solution when $\beta = 1$. Find the coordinates (t_0, y_0) of the minimum point of the solution in this case.

(c) Find the smallest value of β for which the solution has no minimum point.

26. Consider the initial value problem (see Example 5)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.
(a) Solve the initial value problem.
(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.
(c) Determine the smallest value of β for which $y_m \geq 4$.
(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

27. Consider the equation $ay'' + by' + cy = d$, where a, b, c, and d are constants.
(a) Find all equilibrium, or constant, solutions of this differential equation.
(b) Let y_e denote an equilibrium solution, and let $Y = y - y_e$. Thus Y is the deviation of a solution y from an equilibrium solution. Find the differential equation satisfied by Y.

28. Consider the equation $ay'' + by' + cy = 0$, where a, b, and c are constants with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic equation are:
(a) real, different, and negative.
(b) real with opposite signs.
(c) real, different, and positive.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In turn, this understanding will assist us in finding the solutions of other problems that we will encounter later.

To discuss general properties of linear differential equations, it is helpful to introduce a differential operator notation. Let p and q be continuous functions on an open interval I—that is, for $\alpha < t < \beta$. The cases for $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

Note that $L[\phi]$ is a function on I. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin 3t$, then

$$\begin{aligned} L[\phi](t) &= (\sin 3t)'' + t^2(\sin 3t)' + (1+t)\sin 3t \\ &= -9\sin 3t + 3t^2\cos 3t + (1+t)\sin 3t. \end{aligned}$$