Introduction to Differential Equations
Assignment # 2

Date Given: April 18, 2022
Date Due: April 25, 2022

P1. (2 points) Find the solution of the initial value problem $ty' + 2y = t^2 - t + 1$, $y(1) = 1/2$, $t > 0$.

Solution: First, convert the equation to the standard form: $y' + 2y/t = t - 1 + 1/t$. The integrating factor $\mu(t) = e^{\int 2/t \mathrm{d}t} = e^{2\ln|t|} = e^{\ln|t|^2} = |t|^2 = t^2$, so $(t^2y)' = t^3 - t^2 + t$. Therefore

$$t^2y = t^4/4 - t^3/3 + t^2/2 + c \quad \Longrightarrow \quad y(t) = t^2/4 - t/3 + 1/2 + c/t^2$$

Setting $t = 1$ and $y = 1/2$, we have $c = 1/12$. Hence

$$y(t) = t^2/4 - t/3 + 1/2 + 1/(12t^2).$$

P2. (2 points) Find the solution of the initial value problem $ty' + 2y = 2\sin t$, $y(\pi/2) = 1$, $t > 0$.

Solution: First, convert the equation to the standard form: $y' + 2y/t = (2\sin t)/t$. The integrating factor $\mu(t) = e^{\int (2/t)\mathrm{d}t} = e^{2\ln|t|} = e^{\ln|t|^2} = t^2$, so $(t^2y)' = 2t\sin t$. Therefore,

$$t^2y = 2\int t\sin t\mathrm{d}t + c = -2\int t\mathrm{d}(\cos t) + c = -2t\cos t + 2\int \cos t\mathrm{d}t + c = -2t\cos t + 2\sin t + c,$$

and

$$y(t) = (-2t\cos t + 2\sin t + c)/t^2.$$

Setting $t = \pi/2$ and $y = 1$, we have $c = \pi^2/4 - 2$. Hence

$$y(t) = (-2t\cos t + 2\sin t + \pi^2/4 - 2)/t^2.$$

P3. (1 point) Use the method of variation of parameter to solve the differential equation $y' + (1/t)y = 3\cos 2t$, $t > 0$.

Solution: The solution of the homogeneous equation (with zero right-hand side), $y' + (1/t)y = 0$, is $y(t) = Ce^{-\int (1/t)\mathrm{d}t} = Ce^{-\ln t} = C/t$, where C is a constant.

Assume now that $C = C(t)$ is a function of t. Differentiating $y(t) = C(t)/t$ gives

$$y'(t) = C'(t)\frac{1}{t} - C(t)\frac{1}{t^2}.$$

Substituting $y(t)$ and $y'(t)$ into the original non-homogeneous differential equation yields

$$C'(t)\frac{1}{t} - C(t)\frac{1}{t^2} + \frac{1}{t}C(t)\frac{1}{t} = 3\cos 2t,$$

and therefore $C'(t) = 3\cos 2t$. This implies that

$$C(t) = \frac{3\cos 2t}{4} + \frac{3t\sin 2t}{2} + c,$$

where c is a constant. Thus, the solution is

$$y(t) = \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + c/t.$$

P4. (1 point) Solve the differential equation $y' + y^2\sin x = 0$.

Solution: The differential equation may be written as $y^{-2}\mathrm{d}y = -\sin x \mathrm{d}x$ Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos x + c$, where c is an arbitrary constant. Therefore, $(c + \cos x)y = -1$. Solving for the dependent variable, explicitly,

$$y(x) = -\frac{1}{(c + \cos x)}.$$

P5. (1 point) Solve the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2}{1 + y^2}$.

Solution: The differential equation may be written as $(1 + y^2)\mathrm{d}y = x^2\mathrm{d}x$ Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation

$$y + y^3/3 = x^3/3 + c,$$

where c is an arbitrary constant.

P6. (3 points) In this problem:

(a) Find the solution of the initial value problem $y' = 2x/(1 + 2y)$, $y(2) = 0$ in explicit form.

(b) Plot the graph of the solution.

(c) Determine the interval in which the solution is defined.

Solution:

(a) Separating variables gives $(1 + 2y)\mathrm{d}y = 2x\mathrm{d}x$ and integrating yields $y + y^2 = x^2 + c$. Setting $y = 0$ when $x = 2$ yields $c = -4$ or $y + y^2 = x^2 - 4$. To solve for y, complete the square on the left side by adding $1/4$ to both sides. This yields $y + y^2 + 1/4 = x^2 - 4 + 1/4$ or $(y + 1/2)^2 = x^2 - 15/4$. Taking the square root of both sides gives $y + 1/2 = \pm\sqrt{x^2 - 15/4}$, where positive square root must be taken in order satisfy the given initial condition. Thus,

$$y(x) = -1/2 + \sqrt{x^2 - 15/4}.$$

(b) The graph of the solution is shown in Figure 1.

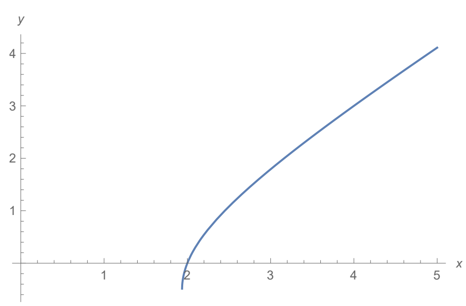

Figure 1: Illustration to problem P6.

(c) The solution $y(x) = -1/2 + \sqrt{x^2 - 15/4}$ is defined for $x^2 \geq 15/4$. Hence $x \geq \sqrt{15}/2$.