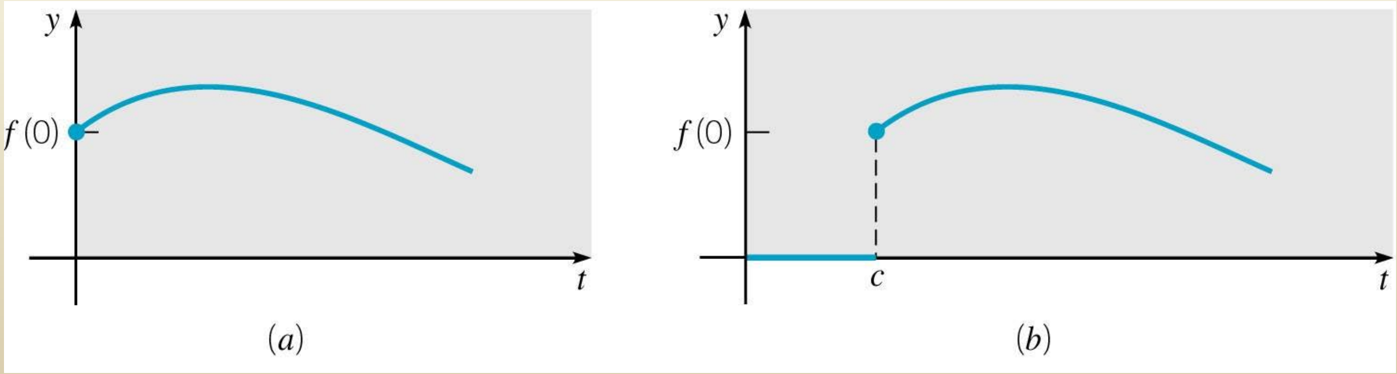
# Ch 6.3: Step Functions

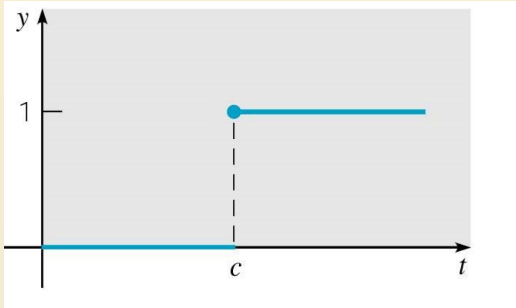
- Some of the most interesting elementary applications of the Laplace Transform method occur in the solution of linear equations with discontinuous or impulsive forcing functions.
- In this section, we will assume that all functions considered are piecewise continuous and of exponential order, so that their Laplace Transforms all exist, for $s$ large enough.



# Step Function definition

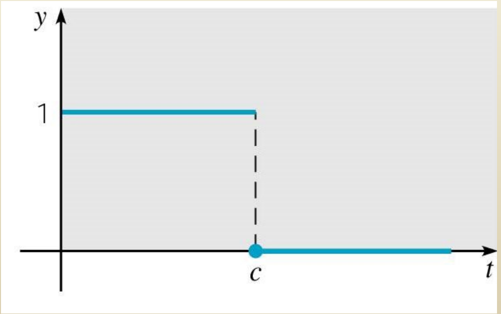
- Let $c \geq 0$. The **unit step function**, or Heaviside function, is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$




- A negative step can be represented by

$$y(t) = 1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$



# Example 1

- Sketch the graph of

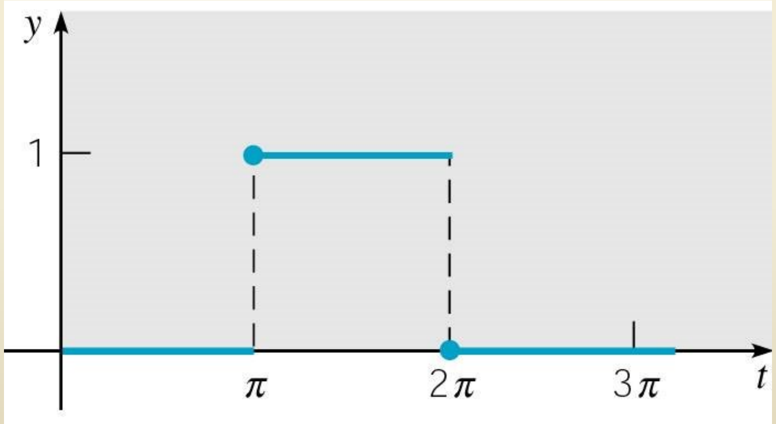
$$h(t) = u_\pi(t) - u_{2\pi}(t), \quad t \geq 0$$

- Solution: Recall that $u_c(t)$ is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- Thus

$$h(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t < \infty \end{cases}$$



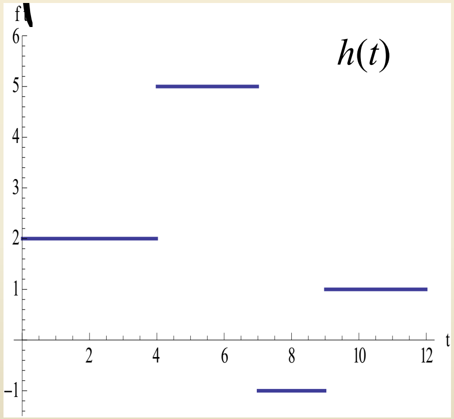

and hence the graph of $h(t)$ is a rectangular pulse.

# Example 2

- For the function

$$h(t) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & 7 \le t < 9 \\ 1, & t \ge 9 \end{cases}$$

  whose graph is shown




- To write $h(t)$ in terms of $u_c(t)$, we will need $u_4(t)$, $u_7(t)$, and $u_9(t)$. We begin with the 2, then add 3 to get 5, then subtract 6 to get -1, and finally add 2 to get 1 – each quantity is multiplied by the appropriate $u_c(t)$

$$h(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t), \quad t \ge 0$$

# Laplace Transform of Step Function

- The Laplace Transform of $u_c(t)$ is

$$L\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st}\,dt = \lim_{b\to\infty} \left[ -\frac{1}{s} e^{-st} \Big|_c^b \right]$$

$$= \lim_{b\to\infty} \left[ -\frac{e^{-bs}}{s} + \frac{e^{-cs}}{s} \right]$$

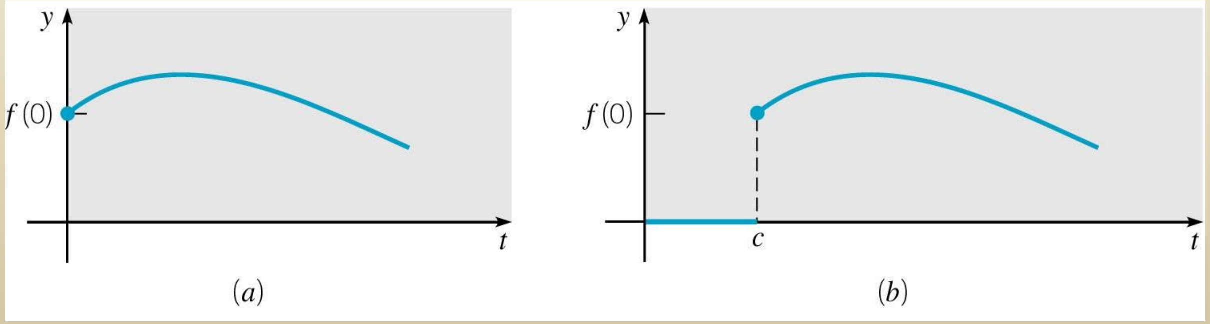
$$= \frac{e^{-cs}}{s}$$

# Translated Functions

- Given a function $f(t)$ defined for $t \geq 0$, we will often want to consider the related function $g(t) = u_c(t) f(t - c)$:

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

- Thus $g$ represents a translation of $f$ a distance $c$ in the positive $t$ direction.
- In the figure below, the graph of $f$ is given on the left, and the graph of $g$ on the right.



# Theorem 6.3.1

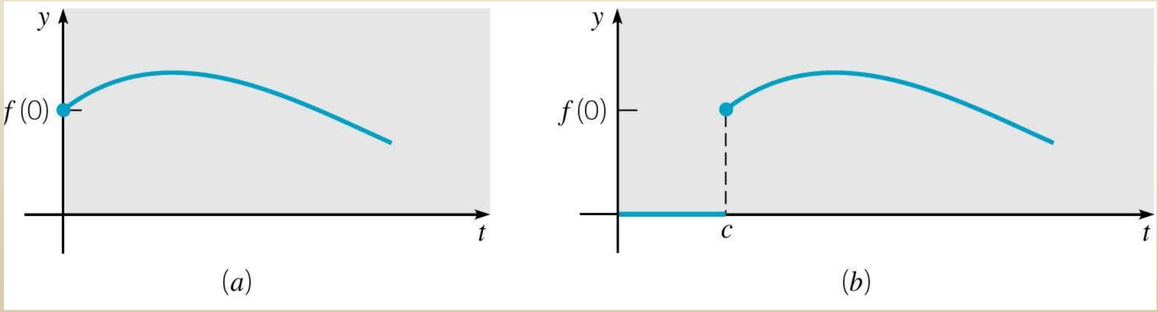
- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if $c > 0$, then

$$L\left\{u_c(t)f(t-c)\right\} = e^{-cs}L\left\{f(t)\right\} = e^{-cs}F(s)$$

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = L^{-1}\left\{e^{-cs}F(s)\right\}$$

- Thus the translation of $f(t)$ a distance $c$ in the positive $t$ direction corresponds to a multiplication of $F(s)$ by $e^{-cs}$.



# Theorem 6.3.1: Proof Outline

- We need to show

$$L\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

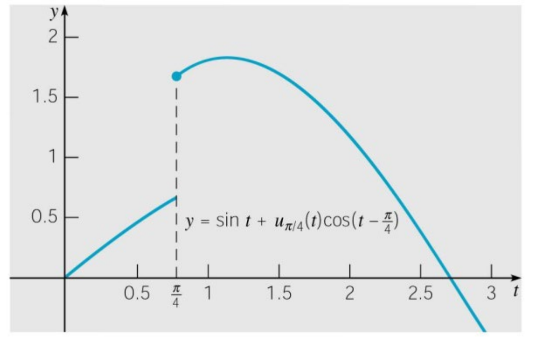
- Using the definition of the Laplace Transform, we have

$$\begin{aligned}
L\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\
&= \int_c^\infty e^{-st}f(t-c)dt \\
&\overset{u=t-c}{=} \int_0^\infty e^{-s(u+c)}f(u)du \\
&= e^{-cs}\int_0^\infty e^{-su}f(u)du \\
&= e^{-cs}F(s)
\end{aligned}$$

# Example 3

- Find $L\{f(t)\}$, where $f$ is defined by

$$f(t)=\begin{cases}\sin t, & 0\le t<\pi/4\\ \sin t+\cos(t-\pi/4), & t\ge \pi/4\end{cases}$$




- Note that $f(t) = \sin(t) + u_{\pi/4}(t)\cos(t - \pi/4)$, and

$$\begin{aligned}L\{f(t)\}&=L\{\sin t\}+L\{u_{\pi/4}(t)\cos(t-\pi/4)\}\\ &=L\{\sin t\}+e^{-\pi s/4}L\{\cos t\}\\ &=\frac{1}{s^2+1}+e^{-\pi s/4}\frac{s}{s^2+1}\\ &=\frac{1+se^{-\pi s/4}}{s^2+1}\end{aligned}$$

# Example 4

- Find $L^{-1}\{F(s)\}$, where

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

- Solution:

$$f(t) = L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

$$= t - u_2(t)\left(t - 2\right)$$

- The function may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$$

# Theorem 6.3.2

- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if $c$ is a constant, then

$$L\left\{e^{ct} f(t)\right\} = F(s-c), \quad s > a+c$$

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = L^{-1}\left\{F(s-c)\right\}$$

- Thus multiplication $f(t)$ by $e^{ct}$ results in translating $F(s)$ a distance $c$ in the positive $t$ direction, and conversely.
- Proof Outline:

$$L\left\{e^{ct} f(t)\right\} = \int_0^\infty e^{-st} e^{ct} f(t) dt = \int_0^\infty e^{-(s-c)t} f(t) dt = F(s-c)$$

# Example 5

- To find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}$$

- We first complete the square:

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{\left(s^2 - 4s + 4\right) + 1} = \frac{1}{\left(s - 2\right)^2 + 1} = F(s-2)$$

- Since

$$L^{-1}\left\{F(s)\right\} = L^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t \text{ and } L^{-1}\left\{F(s-2)\right\} = e^{2t} f(t)$$

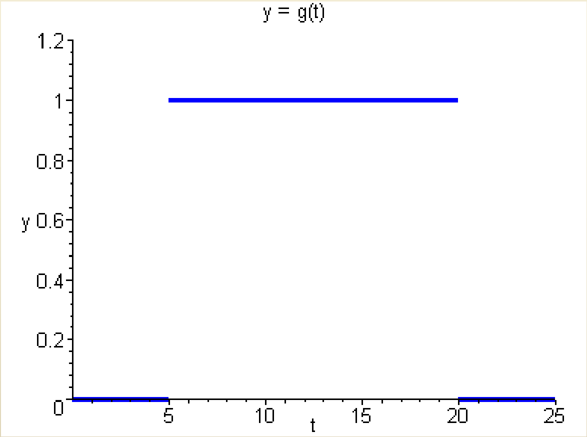
it follows that

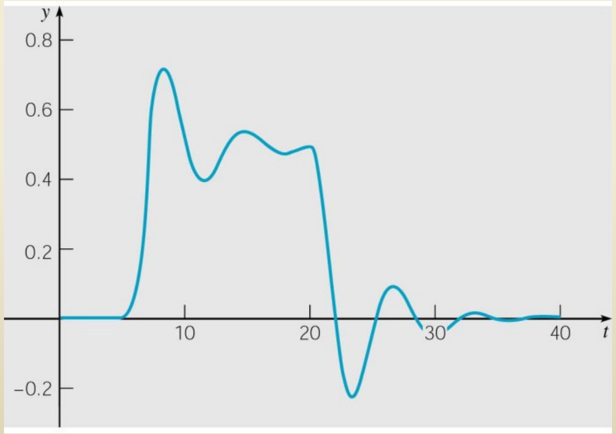
$$g(t) = L^{-1}\left\{G(s)\right\} = e^{2t} \sin t$$

# Ch 6.4: Differential Equations with Discontinuous Forcing Functions

- In this section focus on examples of nonhomogeneous initial value problems in which the forcing function is discontinuous.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \;\; y'(0) = y_0'$$



# Example 1: Initial Value Problem (1 of 12)

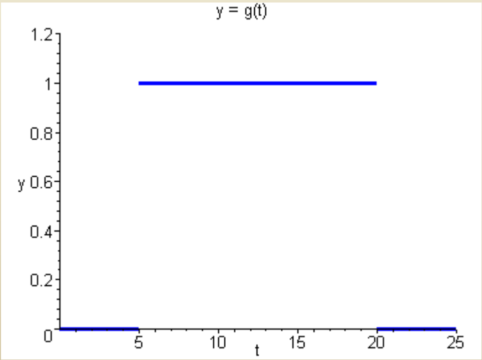
- Find the solution to the initial value problem

$$2y'' + y' + 2y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & 0 \le t < 5 \text{ and } t \ge 20 \end{cases}$$

- Such an initial value problem might model the response of a damped oscillator subject to $g(t)$, or current in a circuit for a unit voltage pulse.



$$2y'' + y' + 2y = u_5(t) - u_{20}(t), \quad y(0) = 0, \; y'(0) = 0$$

# Example 1: Laplace Transform (2 of 12)

- Assume the conditions of Corollary 6.2.2 are met. Then

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{u_5(t)\} - L\{u_{20}(t)\}$$

or

$$\left[2s^2L\{y\} - 2sy(0) - 2y'(0)\right] + \left[sL\{y\} - y(0)\right] + 2L\{y\} = \frac{e^{-5s} - e^{-20s}}{s}$$

- Letting $Y(s) = L\{y\}$,

$$\left(2s^2 + s + 2\right)Y(s) - \left(2s + 1\right)y(0) - 2y'(0) = \left(e^{-5s} - e^{-20s}\right)/s$$

- Substituting in the initial conditions, we obtain

$$\left(2s^2 + s + 2\right)Y(s) = \left(e^{-5s} - e^{-20s}\right)/s$$

- Thus

$$Y(s) = \frac{\left(e^{-5s} - e^{-20s}\right)}{s\left(2s^2 + s + 2\right)}$$

# Example 1: Factoring $Y(s)$ (3 of 12)

- We have

$$Y(s) = \frac{\left(e^{-5s} - e^{-20s}\right)}{s\left(2s^2 + s + 2\right)} = \left(e^{-5s} - e^{-20s}\right)H(s)$$

where

$$H(s) = \frac{1}{s\left(2s^2 + s + 2\right)}$$

- If we let $h(t) = L^{-1}\{H(s)\}$, then

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

by Theorem 6.3.1.

# Example 1: Partial Fractions 

- Thus we examine $H(s)$, as follows.

$$H(s) = \frac{1}{s\left(2s^2 + s + 2\right)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2}$$

- This partial fraction expansion yields the equations

$$(2A + B)s^2 + (A + C)s + 2A = 1$$

$$\Rightarrow A = 1/2,\ B = -1,\ C = -1/2$$

- Thus

$$H(s) = \frac{1/2}{s} - \frac{s + 1/2}{2s^2 + s + 2}$$

# Example 1: Completing the Square (5 of 12)

- Completing the square,

$$
\begin{aligned}
H(s) &= \frac{1/2}{s} - \frac{s+1/2}{2s^2+s+2} \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{s^2+s/2+1}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{s^2+s/2+1/16+15/16}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{(s+1/4)^2+15/16}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{(s+1/4)+1/4}{(s+1/4)^2+15/16}\right]
\end{aligned}
$$

# Example 1: Solution (6 of 12)

- Thus

$$H(s)=\frac{1/2}{s}-\frac{1}{2}\left[\frac{(s+1/4)+1/4}{(s+1/4)^2+15/16}\right]$$

$$=\frac{1/2}{s}-\frac{1}{2}\left[\frac{(s+1/4)}{(s+1/4)^2+15/16}\right]-\frac{1}{2\sqrt{15}}\left[\frac{\sqrt{15}/4}{(s+1/4)^2+15/16}\right]$$

and hence

$$h(t)=L^{-1}\{H(s)\}=\frac{1}{2}-\frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right)-\frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

- For $h(t)$ as given above, and recalling our previous results, the solution to the initial value problem is then

$$\phi(t)=u_5(t)h(t-5)-u_{20}(t)h(t-20)$$

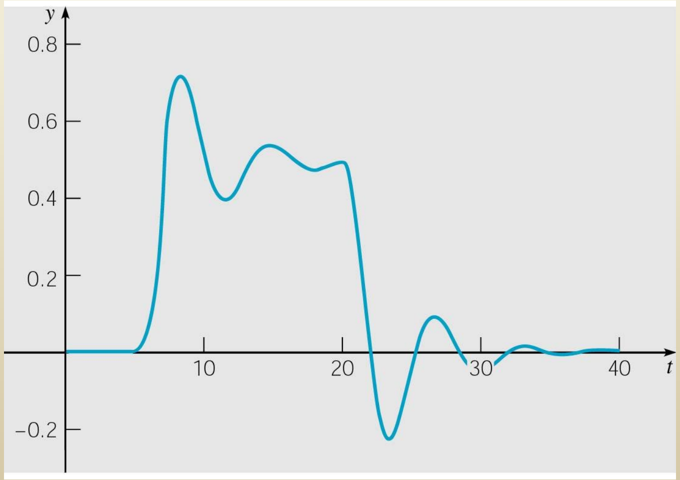
# Example 1: Solution Graph (7 of 12)

- Thus the solution to the initial value problem is

$$\phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20), \quad \text{where}$$

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\sqrt{15}t/4\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\sqrt{15}t/4\right)$$

- The graph of this solution is given below.



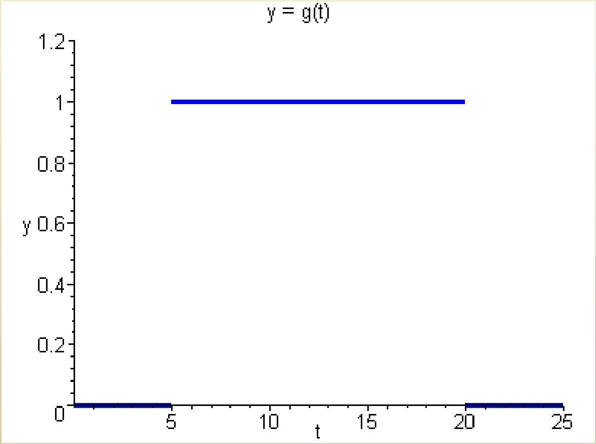
# Example 1: Composite IVPs (8 of 12)

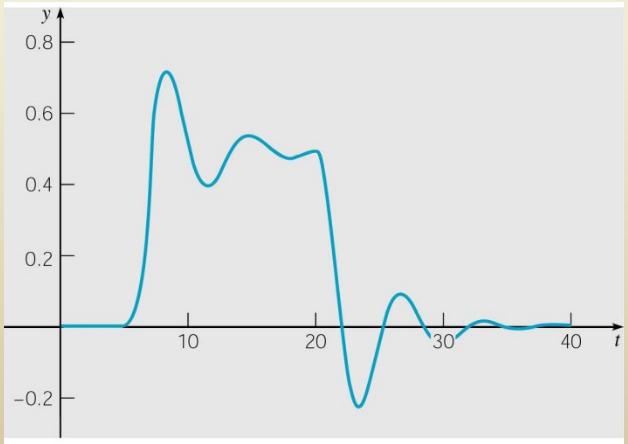
- The solution to original IVP can be viewed as a composite of three separate solutions to three separate IVPs:

$$0 \le t < 5: \quad 2y_1'' + y_1' + 2y_1 = 0, \quad y_1(0) = 0,\ y_1'(0) = 0$$

$$5 < t < 20: \quad 2y_2'' + y_2' + 2y_2 = 1, \quad y_2(5) = 0,\ y_2'(5) = 0$$

$$t > 20: \quad 2y_3'' + y_3' + 2y_3 = 0, \quad y_3(20) = y_2(20),\ y_3'(20) = y_2'(20)$$



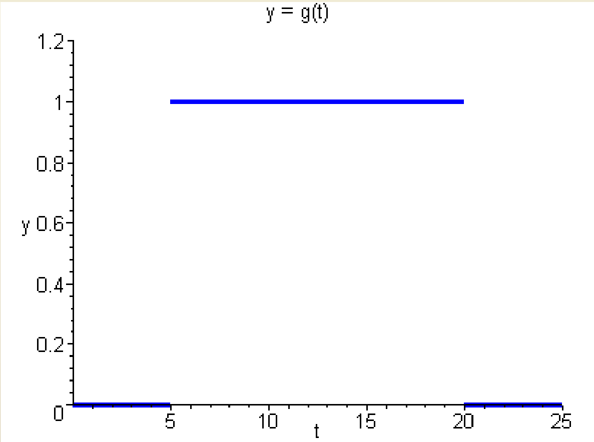



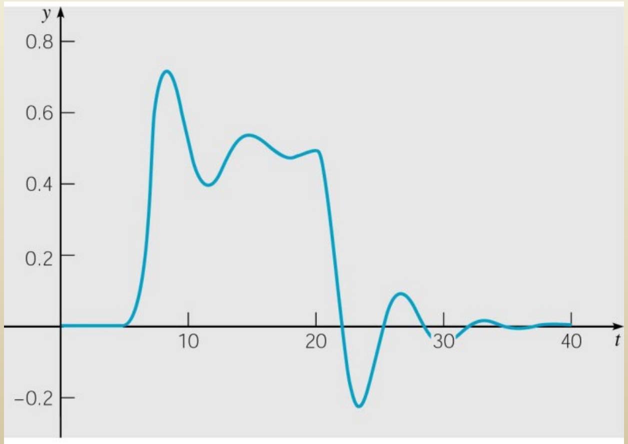
# Example 1: First IVP (9 of 12)

- Consider the first initial value problem

$$2y_1'' + y_1' + 2y_1 = 0, \qquad y_1(0) = 0,\ y_1'(0) = 0; \quad 0 \le t < 5$$

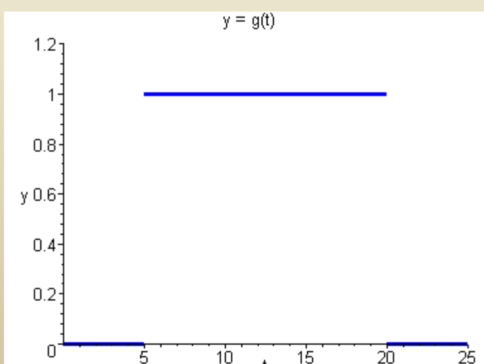
- From a physical point of view, the system is initially at rest, and since there is no external forcing, it remains at rest.
- Thus the solution over [0, 5) is $y_1 = 0$, and this can be verified analytically as well. See graphs below.

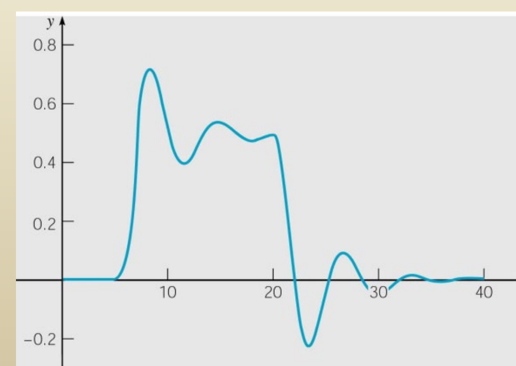


# Example 1: Second IVP 

- Consider the second initial value problem
  $$2y_2'' + y_2' + 2y_2 = 1, \qquad y_2(5) = 0,\ y_2'(5) = 0; \quad 5 < t < 20$$
- Using methods of Chapter 3, the solution has the form
  $$y_2 = c_1 e^{-t/4} \cos\left(\sqrt{15}t/4\right) + c_2 e^{-t/4} \sin\left(\sqrt{15}t/4\right) + 1/2$$
- Physically, the system responds with the sum of a constant (the response to the constant forcing function) and a damped oscillation, over the time interval (5, 20). See graphs below.



# Example 1: Third IVP 

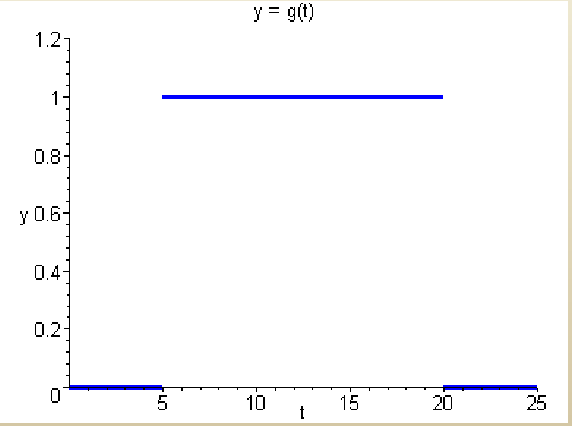
- Consider the third initial value problem

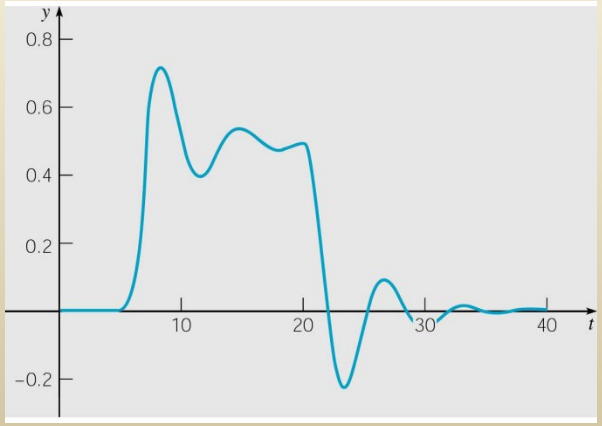
$$2y_3'' + y_3' + 2y_3 = 0, \qquad y_3(20) = y_2(20),\ y_3'(20) = y_2'(20); \quad t > 20$$

- Using methods of Chapter 3, the solution has the form

$$y_3 = c_1 e^{-t/4}\cos\left(\sqrt{15}t/4\right) + c_2 e^{-t/4}\sin\left(\sqrt{15}t/4\right)$$

- Physically, since there is no external forcing, the response is a damped oscillation about $y = 0$, for $t > 20$. See graphs below.



# Example 1: Solution Smoothness (12 of 12)

- Our solution is

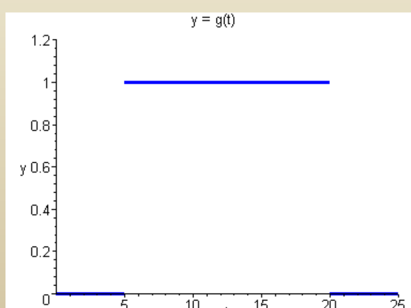
$$\phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

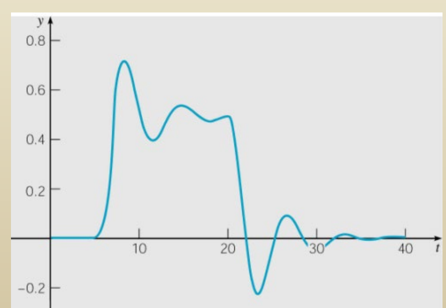
- It can be shown that $\phi$ and $\phi'$ are continuous at $t = 5$ and $t = 20$, and $\phi''$ has a jump of 1/2 at $t = 5$ and a jump of –1/2 at $t = 20$:

$$\lim_{t \to 5^-} \phi''(t) = 0, \quad \lim_{t \to 5^+} \phi''(t) = 1/2$$

$$\lim_{t \to 20^-} \phi''(t) \cong -.0072, \quad \lim_{t \to 20^+} \phi''(t) \cong -.5072$$

- Thus jump in forcing term $g(t)$ at these points is balanced by a corresponding jump in highest order term $2y''$ in ODE.



# Smoothness of Solution in General

- Consider a general second order linear equation
$$y'' + p(t)y' + q(t)y = g(t)$$
where $p$ and $q$ are continuous on some interval $(a, b)$ but $g$ is only piecewise continuous there.
- If $y = \psi(t)$ is a solution, then $\psi$ and $\psi'$ are continuous on $(a, b)$ but $\psi''$ has jump discontinuities at the same points as $g$.
- Similarly for higher order equations, where the highest derivative of the solution has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous over $(a, b)$.

# Example 2: Initial Value Problem 

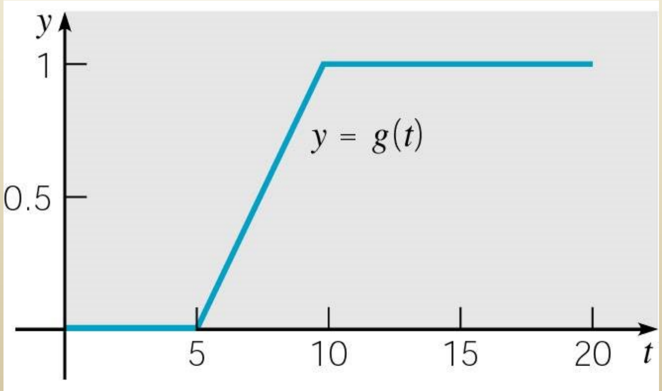
- Find the solution to the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

where

$$g(t) = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5} = \begin{cases} 0, & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$

- The graph of forcing function $g(t)$ is given on right, and is known as ramp loading.



$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0)=0,\ y'(0)=0$$

# Example 2: Laplace Transform 

- Assume that this ODE has a solution $y = \phi(t)$ and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{y''\} + 4L\{y\} = \left[L\{u_5(t)(t-5)\}\right]/5 - \left[L\{u_{10}(t)(t-10)\}\right]/5$$

or

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 4L\{y\} = \frac{e^{-5s} - e^{-10s}}{5s^2}$$

- Letting $Y(s) = L\{y\}$, and substituting in initial conditions,

$$\left(s^2+4\right)Y(s) = \left(e^{-5s} - e^{-10s}\right)/5s^2$$

- Thus

$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2+4\right)}$$

# Example 2: Factoring $Y(s)$ (3 of 12)

- We have

$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2+4\right)} = \frac{e^{-5s} - e^{-10s}}{5} H(s)$$

where

$$H(s) = \frac{1}{s^2\left(s^2+4\right)}$$

- If we let $h(t) = L^{-1}\{H(s)\}$, then

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

by Theorem 6.3.1.

# Example 2: Partial Fractions (4 of 12)

- Thus we examine $H(s)$, as follows.

$$H(s) = \frac{1}{s^2\left(s^2+4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

- This partial fraction expansion yields the equations

$$(A+C)s^3 + (B+D)s^2 + 4As + 4B = 1$$
$$\Rightarrow A = 0,\ B = 1/4,\ C = 0,\ D = -1/4$$

- Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

# Example 2: Solution 

- Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

$$= \frac{1}{4}\left[\frac{1}{s^2}\right] - \frac{1}{8}\left[\frac{2}{s^2+4}\right]$$

and hence

$$h(t) = L^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- For $h(t)$ as given above, and recalling our previous results, the solution to the initial value problem is then

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

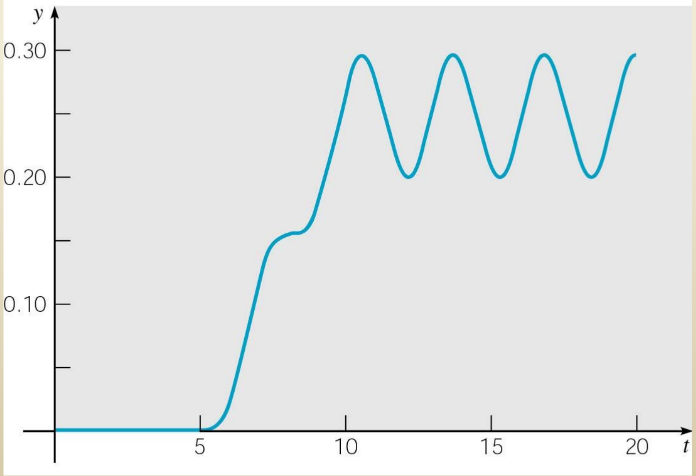
# Example 2: Graph of Solution (6 of 12)

- Thus the solution to the initial value problem is

$$\phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \quad \text{where}$$

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- The graph of this solution is given below.



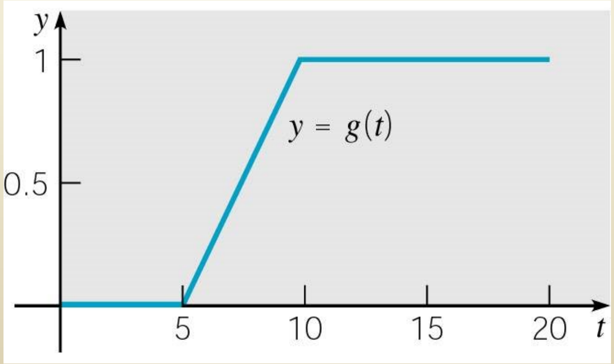
# Example 2: Composite IVPs (7 of 12)

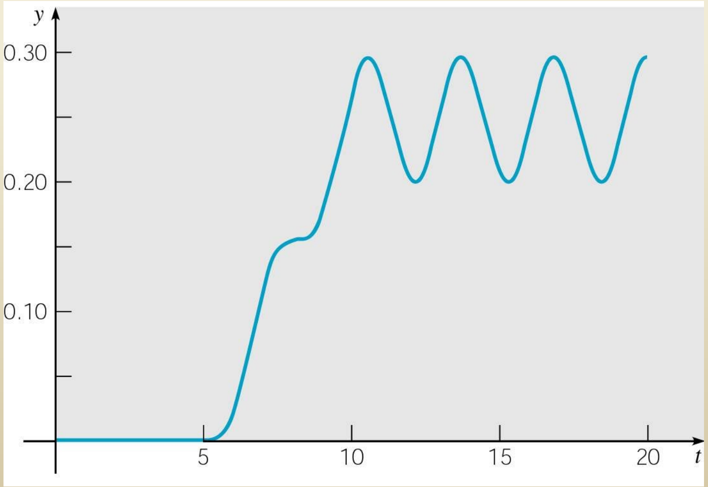
- The solution to original IVP can be viewed as a composite of three separate solutions to three separate IVPs (discuss):

$$0 \le t < 5: \quad y_1'' + 4y_1 = 0, \qquad y_1(0) = 0,\ y_1'(0) = 0$$

$$5 < t < 10: \quad y_2'' + 4y_2 = (t-5)/5, \qquad y_2(5) = 0,\ y_2'(5) = 0$$

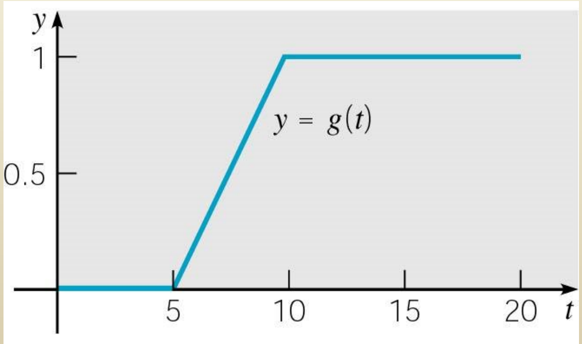
$$t > 10: \quad y_3'' + 4y_3 = 1, \qquad y_3(10) = y_2(10),\ y_3'(10) = y_2'(10)$$

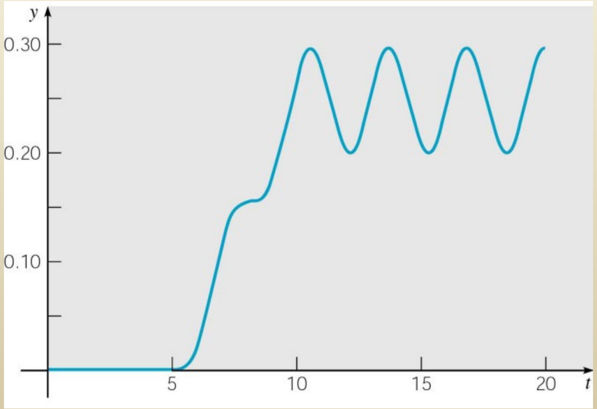


# Example 2: First IVP (8 of 12)

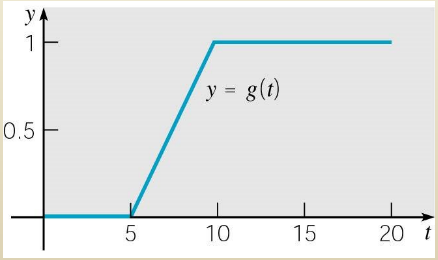
- Consider the first initial value problem
$$y_1'' + 4y_1 = 0, \qquad y_1(0) = 0,\ y_1'(0) = 0; \quad 0 \le t < 5$$
- From a physical point of view, the system is initially at rest, and since there is no external forcing, it remains at rest.
- Thus the solution over [0, 5) is $y_1 = 0$, and this can be verified analytically as well. See graphs below.

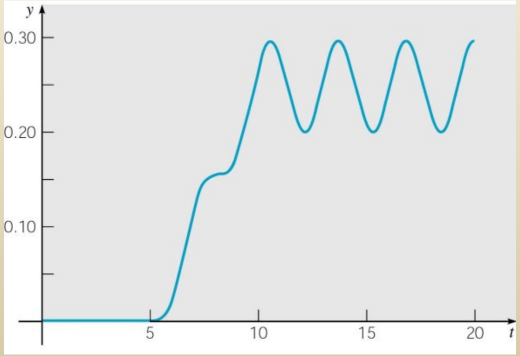


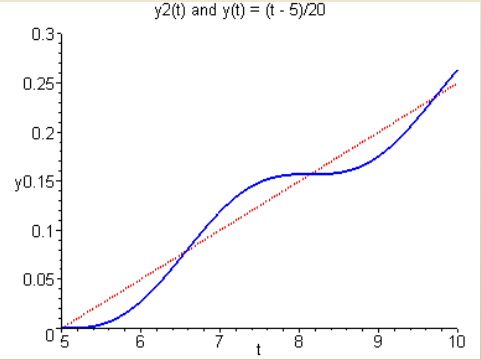



# Example 2: Second IVP 

- Consider the second initial value problem
  $$y_2'' + 4y_2 = (t-5)/5, \qquad y_2(5) = 0,\ y_2'(5) = 0; \quad 5 < t < 10$$
- Using methods of Chapter 3, the solution has the form
  $$y_2 = c_1 \cos(2t) + c_2 \sin(2t) + t/20 - 1/4$$
- Thus the solution is an oscillation about the line ($t$ – 5)/20, over the time interval (5, 10). See graphs below.



# Example 2: Third IVP 

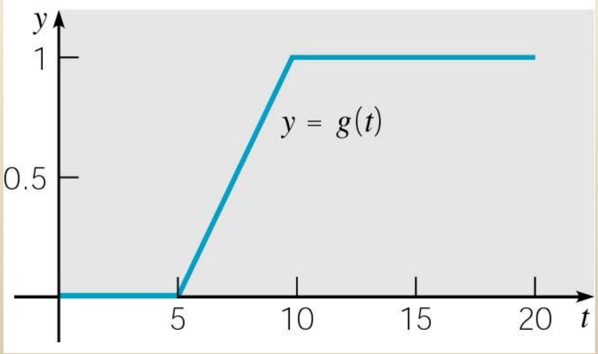
- Consider the third initial value problem

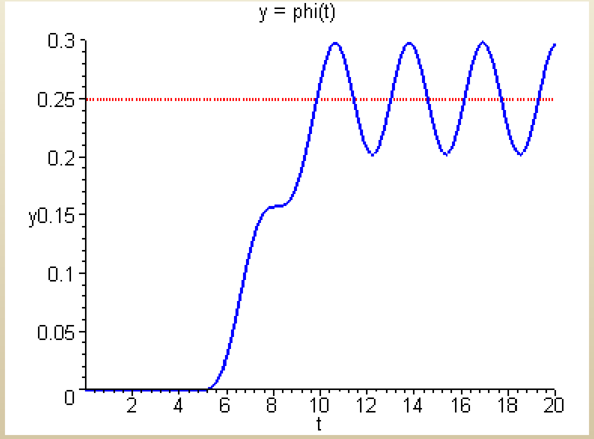
$$y_3'' + 4y_3 = 1, \qquad y_3(10) = y_2(10),\ y_3'(10) = y_2'(10); \quad t > 10$$

- Using methods of Chapter 3, the solution has the form

$$y_3 = c_1 \cos(2t) + c_2 \sin(2t) + 1/4$$

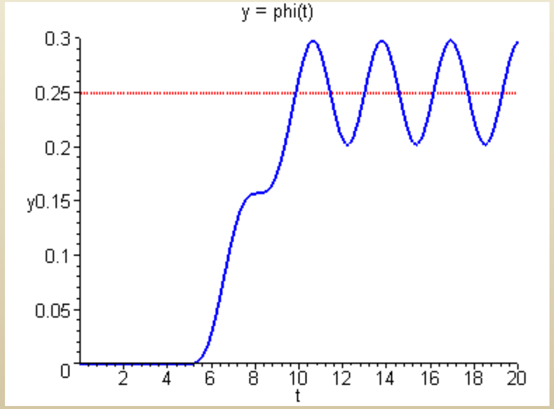
- Thus the solution is an oscillation about $y = 1/4$, for $t > 10$. See graphs below.

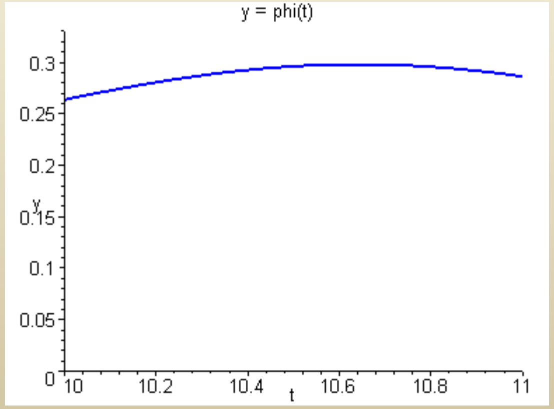


# Example 2: Amplitude 

- Recall that the solution to the initial value problem is
$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$
- To find the amplitude of the eventual steady oscillation, we locate one of the maximum or minimum points for $t > 10$.
- Solving $y' = 0$, the first maximum is (10.642, 0.2979).
- Thus the amplitude of the oscillation is about 0.0479.



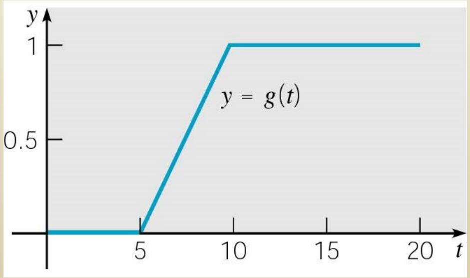



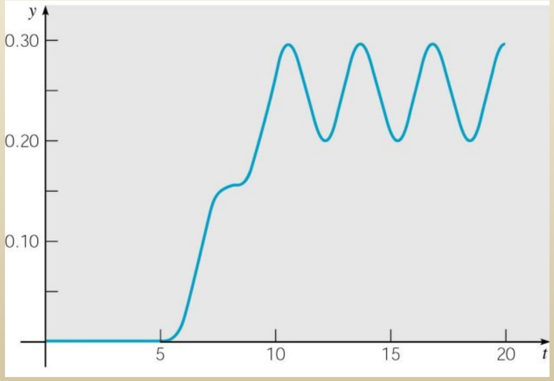
# Example 2: Solution Smoothness (12 of 12)

- Our solution is

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin\left(2t\right)$$

- In this example, the forcing function $g$ is continuous but $g'$ is discontinuous at $t = 5$ and $t = 10$.
- It follows that $\phi$ and its first two derivatives are continuous everywhere, but $\phi'''$ has discontinuities at $t = 5$ and $t = 10$ that match the discontinuities of $g'$ at $t = 5$ and $t = 10$.



# Ch 6.5: Impulse Functions

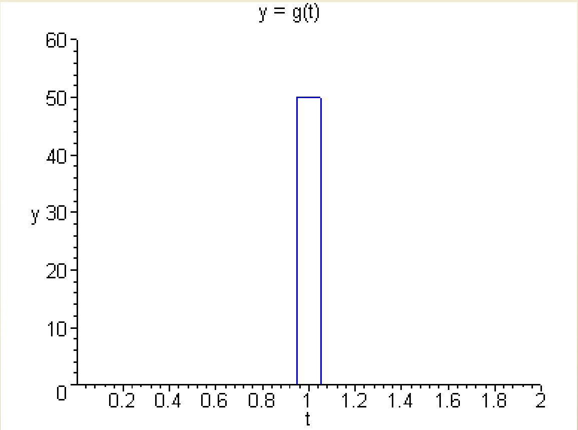
- In some applications, it is necessary to deal with phenomena of an impulsive nature.
- For example, an electrical circuit or mechanical system subject to a sudden voltage or force $g(t)$ of large magnitude that acts over a short time interval about $t_0$. The differential equation will then have the form

$$ay'' + by' + cy = g(t),$$

where

$$g(t) = \begin{cases} \text{big}, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

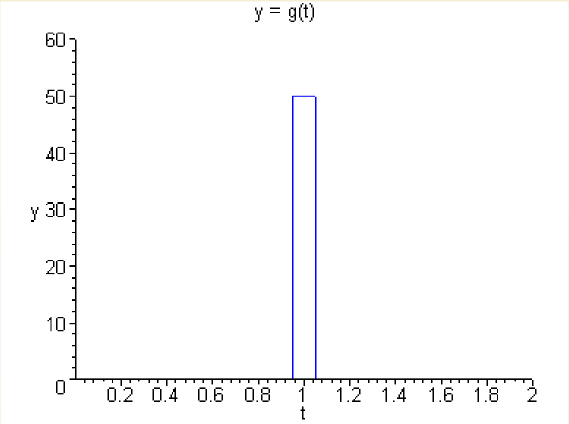
and $\tau > 0$ is small.



# Measuring Impulse

- In a mechanical system, where $g(t)$ is a force, the total **impulse** of this force is measured by the integral

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g(t)dt$$




- Note that if $g(t)$ has the form

$$g(t) = \begin{cases} c, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

then

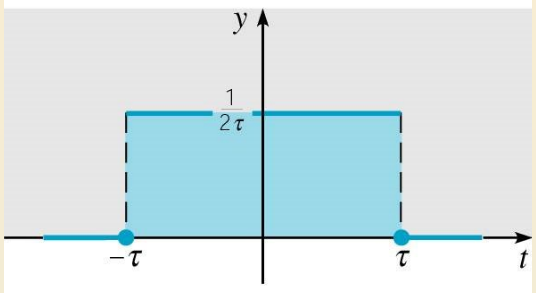
$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g(t)dt = 2\tau c, \quad \tau > 0$$

- In particular, if $c = 1/(2\tau)$, then $I(\tau) = 1$ (independent of $\tau$).

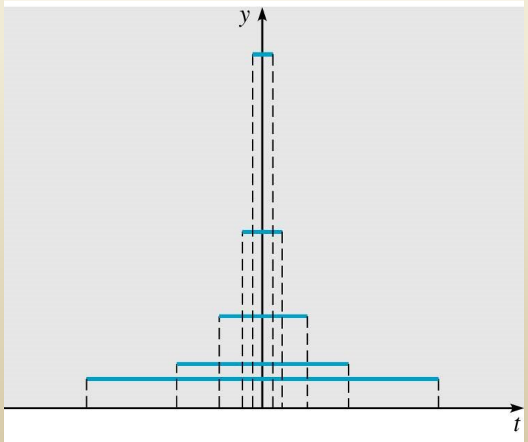
# Unit Impulse Function

- Suppose the forcing function $d_\tau(t)$ has the form

$$d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$




- Then as we have seen, $I(\tau) = 1$.
- We are interested $d_\tau(t)$ acting over shorter and shorter time intervals (i.e., $\tau \to 0$). See graph on right.




- Note that $d_\tau(t)$ gets taller and narrower as $\tau \to 0$. Thus for $t \neq 0$, we have

$$\lim_{\tau \to 0} d_\tau(t) = 0, \text{ and } \lim_{\tau \to 0} I(\tau) = 1$$

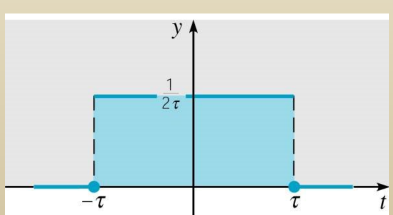
# Dirac Delta Function

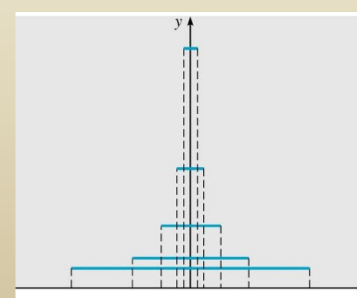
- Thus for $t \neq 0$, we have $\lim_{\tau \to 0} d_\tau(t) = 0$, and $\lim_{\tau \to 0} I(\tau) = 1$
- The **unit impulse function** $\delta$ is defined to have the properties

$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

- The unit impulse function is an example of a generalized function and is usually called the **Dirac delta function**.
- In general, for a unit impulse at an arbitrary point $t_0$,

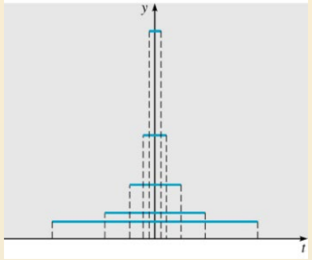
$$\delta(t - t_0) = 0 \text{ for } t \neq t_0, \text{ and } \int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1$$



# Laplace Transform of $\delta$ (1 of 2)

- The Laplace Transform of $\delta$ is defined by



$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0} L\{d_\tau(t-t_0)\},\ t_0 > 0$$

and thus

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0}\int_0^\infty e^{-st} d_\tau(t-t_0)\,dt = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\,dt$$

$$= \lim_{\tau\to 0}\left.\frac{-e^{-st}}{2s\tau}\right|_{t_0-\tau}^{t_0+\tau} = \lim_{\tau\to 0}\frac{1}{2s\tau}\left[-e^{-s(t_0+\tau)} + e^{-s(t_0-\tau)}\right]$$

$$= \lim_{\tau\to 0}\frac{e^{-st_0}}{s\tau}\left[\frac{e^{s\tau}-e^{-s\tau}}{2}\right] = e^{-st_0}\left[\lim_{\tau\to 0}\frac{\sinh(s\tau)}{s\tau}\right]$$

$$= e^{-st_0}\left[\lim_{\tau\to 0}\frac{s\cosh(s\tau)}{s}\right] = e^{-st_0}$$

# Laplace Transform of $\delta$ (2 of 2)

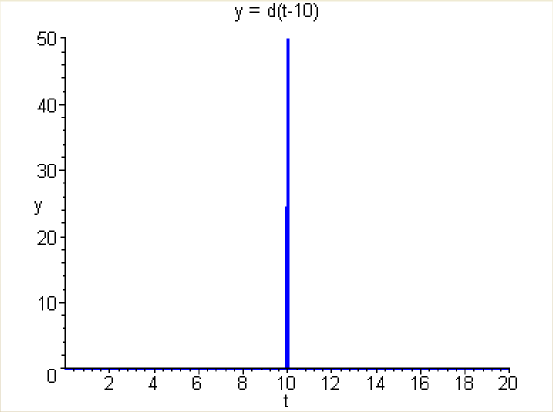
- Thus the Laplace Transform of $\delta$ is

$$L\{\delta(t-t_0)\}=e^{-st_0},\ t_0>0$$

- For Laplace Transform of $\delta$ at $t_0 = 0$, take limit as follows:

$$L\{\delta(t)\}=\lim_{t_0\to 0} L\{d_\tau(t-t_0)\}=\lim_{\tau_0\to 0} e^{-st_0}=1$$

- For example, when $t_0 = 10$, we have $L\{\delta(t\text{-}10)\} = e^{-10s}$.



# Product of Continuous Functions and $\delta$

- The product of the delta function and a continuous function $f$ can be integrated, using the mean value theorem for integrals:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t-t_0) f(t)dt &= \lim_{\tau\to 0}\int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)dt \\ &= \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)dt \\ &= \lim_{\tau\to 0}\frac{1}{2\tau}\left[2\tau\, f(t^*)\right] \quad (\text{where } t_0-\tau < t^* < t_0+\tau) \\ &= \lim_{\tau\to 0} f(t^*) \\ &= f(t_0)\end{aligned}$$

- Thus

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)dt = f(t_0)$$

# Example 1: Initial Value Problem (1 of 3)

- Consider the solution to the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \quad y(0) = 0, \; y'(0) = 0$$

- Then

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{\delta(t-5)\}$$

- Letting $Y(s) = L\{y\}$,

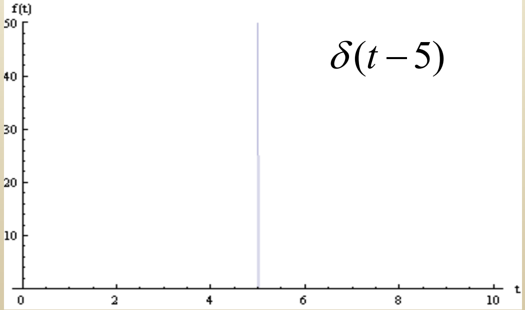
$$\left[2s^2 Y(s) - 2sy(0) - 2y'(0)\right] + \left[sY(s) - y(0)\right] + 2Y(s) = e^{-5s}$$

- Substituting in the initial conditions, we obtain

$$\left(2s^2 + s + 2\right) Y(s) = e^{-5s}$$

or

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$



# Example 1: Solution (2 of 3)

- We have

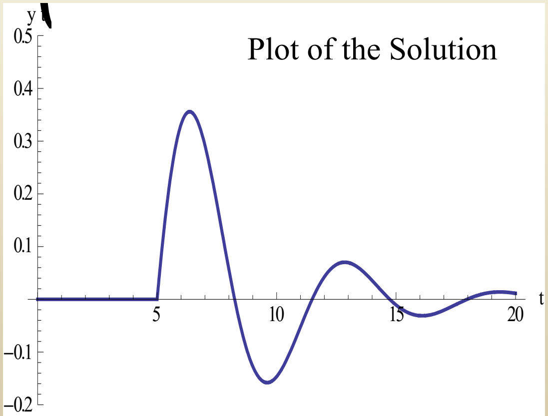
$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$

- The partial fraction expansion of $Y(s)$ yields

$$Y(s) = \frac{e^{-5s}}{2\sqrt{15}}\left[\frac{\sqrt{15}/4}{(s+1/4)^2 + 15/16}\right]$$

and hence

$$y(t) = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$



# Example 1: Solution Behavior (3 of 3)

- With homogeneous initial conditions at $t = 0$ and no external excitation until $t = 5$, there is no response on (0, 5).
- The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely.
- Response is continuous at $t = 5$ despite singularity in forcing function. Since $y'$ has a jump discontinuity at $t = 5$, $y''$ has an infinite discontinuity there. Thus a singularity in the forcing function is balanced by a corresponding singularity in $y''$.