# Ch 7.8: Repeated Eigenvalues

- We consider again a homogeneous system of $n$ first order linear equations with constant real coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- If the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ are real and different, then there are $n$ linearly independent eigenvectors $\xi^{(1)}, \ldots, \xi^{(n)}$, and $n$ linearly independent solutions of the form

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}$$

- If some of the eigenvalues $r_1, \ldots, r_n$ are repeated, then there may not be $n$ corresponding linearly independent solutions of the above form.
- In this case, we will seek additional solutions that are products of polynomials and exponential functions.

# Example 1: Eigenvalues (1 of 2)

- We need to find the eigenvectors for the matrix:

$$\mathrm{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$$

- The eigenvalues $r$ and eigenvectors $\xi$ satisfy the equation $(\mathbf{A} - r\mathbf{I})\,\xi = 0$ or

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- To determine $r$, solve $\det(\mathbf{A} - r\mathbf{I}) = 0$:

$$\begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-1)(r-3)+1 = r^2 - 4r + 4 = (r-2)^2$$

- Thus $r_1 = 2$ and $r_2 = 2$.

# Example 1: Eigenvectors (2 of 2)

- To find the eigenvectors, we solve

$$(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi}=\mathbf{0} \;\Leftrightarrow\; \begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{matrix} 1\xi_1 & +1\xi_2 & =0 \\ & 0\xi_2 & =0 \end{matrix}$$

$$\to \boldsymbol{\xi}^{(1)}=\begin{pmatrix} -\xi_2 \\ \xi_2 \end{pmatrix} \to \text{ choose } \boldsymbol{\xi}^{(1)}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Thus there is only one linearly independent eigenvector for the repeated eigenvalue $r = 2$.

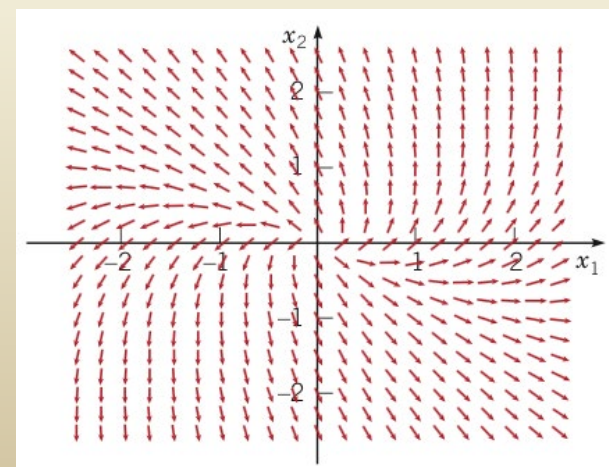
# Example 2: Direction Field 

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \xi e^{rt}$ in for $\mathbf{x}$, where $r$ is $\mathbf{A}$'s eigenvalue and $\xi$ is its corresponding eigenvector, the previous example showed the existence of only one eigenvalue, $r = 2$, with one eigenvector:

$$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



# Example 2: First Solution; and Second Solution, First Attempt 

- The corresponding solution $\mathbf{x} = \xi e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

- Since there is no second solution of the form $\mathbf{x} = \xi e^{rt}$, we need to try a different form. Based on methods for second order linear equations in Ch 3.5, we first try $\mathbf{x} = \xi te^{2t}$.
- Substituting $\mathbf{x} = \xi te^{2t}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} te^{2t} = \mathbf{A}\boldsymbol{\xi} te^{2t}$$

or

$$2\boldsymbol{\xi} te^{2t} + \boldsymbol{\xi} e^{2t} - \mathbf{A}\boldsymbol{\xi} te^{2t} = 0$$

# Example 2:
# Second Solution, Second Attempt (3 of 10)

- From the previous slide, we have

$$2\boldsymbol{\xi}te^{2t} + \boldsymbol{\xi}e^{2t} - \mathbf{A}\boldsymbol{\xi}te^{2t} = 0$$

- In order for this equation to be satisfied for all $t$, it is necessary for the coefficients of $te^{2t}$ and $e^{2t}$ to both be zero.
- From the $e^{2t}$ term, we see that $\boldsymbol{\xi} = \mathbf{0}$, and hence there is no nonzero solution of the form $\mathbf{x} = \boldsymbol{\xi}te^{2t}$.
- Since $te^{2t}$ and $e^{2t}$ appear in the above equation, we next consider a solution of the form

$$\mathbf{x} = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t}$$

# Example 2: Second Solution and its Defining Matrix Equations 

- Substituting $\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$\boldsymbol{\xi} e^{2t} + 2\boldsymbol{\xi} t e^{2t} + 2\boldsymbol{\eta} e^{2t} = \mathbf{A}\left(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}\right)$$

or

$$2\boldsymbol{\xi} t e^{2t} + \left(\boldsymbol{\xi} + 2\boldsymbol{\eta}\right)e^{2t} = \mathbf{A}\boldsymbol{\xi} t e^{2t} + \mathbf{A}\boldsymbol{\eta} e^{2t}$$

- Equating coefficients yields $\mathbf{A}\,\xi = 2\,\xi$ and $\mathbf{A}\,\eta = \xi + 2\,\eta$, or

$$\left(\mathbf{A} - 2\mathbf{I}\right)\boldsymbol{\xi} = \mathbf{0} \quad \text{and} \quad \left(\mathbf{A} - 2\mathbf{I}\right)\boldsymbol{\eta} = \boldsymbol{\xi}$$

- The first equation is satisfied if $\xi$ is an eigenvector of **A** corresponding to the eigenvalue $r = 2$. Thus

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

# Example 2: Solving for Second Solution (5 of 10)

- Recall that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Thus to solve $(\mathbf{A} - 2\mathbf{I})\,\eta = \xi$ for $\eta$, we row reduce the corresponding augmented matrix:

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \eta_2 = -1 - \eta_1$$

$$\to \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ -1 - \eta_1 \end{pmatrix} \to \boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

# Example 2: Second Solution (6 of 10)

- Our second solution $\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$ is now

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

- Recalling that the first solution was

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t},$$

we see that our second solution is simply

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t},$$

since the last term of third term of $\mathbf{x}$ is a multiple of $\mathbf{x}^{(1)}$.

# Example 2: General Solution (7 of 10)

- The two solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{2t} & te^{2t} \\ -e^{2t} & te^{2t} - e^{2t} \end{vmatrix} = -e^{4t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

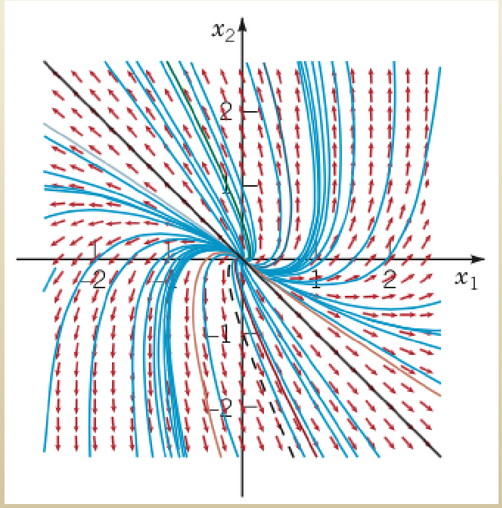
$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) \\ &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right] \end{aligned}$$

# Example 2: Phase Plane (8 of 10)

- The general solution is

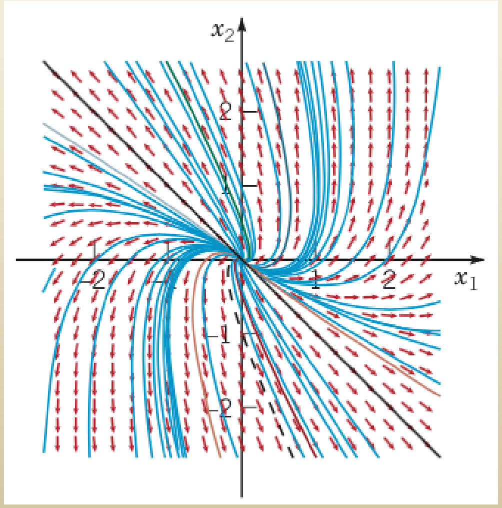
$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t}\right]$$

- Thus **x** is unbounded as $t \to \infty$, and **x** $\to$ **0** as $t \to -\infty$.
- Further, it can be shown that as $t \to -\infty$, **x** $\to$ **0** asymptotic to the line $x_2 = -x_1$ determined by the first eigenvector.
- Similarly, as $t \to \infty$, **x** is asymptotic to a line parallel to $x_2 = -x_1$.



# Example 2: Phase Plane (9 of 10)

- The origin is an **improper node**, and is unstable. See graph.
- The pattern of trajectories is typical for two repeated eigenvalues with only one eigenvector.
- If the eigenvalues are negative, then the trajectories are similar but are traversed in the inward direction. In this case the origin is an asymptotically stable improper node.

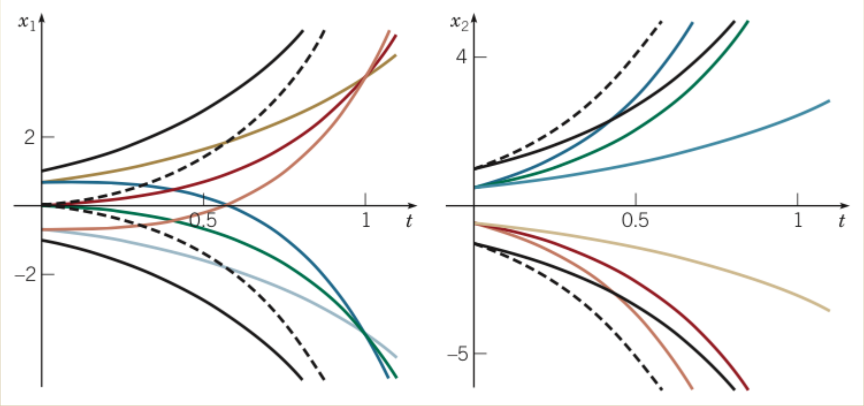


# Example 2:
# Time Plots for General Solution 

- Time plots for $x_1(t)$ are given below, where we note that the general solution $\mathbf{x}$ can be written as follows.

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t}\right]$$

$$\Leftrightarrow \quad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1e^{2t} + c_2te^{2t} \\ -(c_1+c_2)e^{2t} - c_2te^{2t} \end{pmatrix}$$



# General Case for Double Eigenvalues

- Suppose the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has a double eigenvalue $r = \rho$ and a single corresponding eigenvector $\xi$.
- The first solution is
$$\mathbf{x}^{(1)}(t) = \xi e^{\rho t},$$
where $\xi$ satisfies $(\mathbf{A} - \rho\mathbf{I})\,\xi = \mathbf{0}$.
- As in Example 1, the second solution has the form
$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{\rho t} + \boldsymbol{\eta} e^{\rho t}$$
where $\xi$ is as above and $\eta$ satisfies $(\mathrm{A} - \rho \mathrm{I})\eta = \xi$.
- Even though $\det(\mathbf{A} - \rho\mathbf{I}) = 0$, it can be shown that $(\mathrm{A} - \rho \mathrm{I})\eta = \xi$ can always be solved for $\eta$.
- The vector $\eta$ is called a **generalized eigenvector**.

# Example 2 Extension: Fundamental Matrix (1 of 2)

- Recall that a fundamental matrix $\Psi(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has linearly independent solution for its columns.
- In Example 1, our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ was

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$$

and the two solutions we found were

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

- Thus the corresponding fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix} = e^{2t}\begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix}$$

# Example 2 Extension: Fundamental Matrix (2 of 2)

- The fundamental matrix $\Phi(t)$ that satisfies $\Phi(\mathbf{0}) = \mathbf{I}$ can be found using $\Phi(t) = \Psi(t)\Psi^{-1}(0)$, where

$$\mathbf{\Psi}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{\Psi}^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

where $\Psi^{-1}(0)$ is found as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

- Thus

$$\mathbf{\Phi}(t) = e^{2t}\begin{pmatrix} 1 & t \\ -1 & -t-1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = e^{2t}\begin{pmatrix} 1-t & -t \\ t & t+1 \end{pmatrix}$$

# Jordan Forms

- If **A** is $n$ x $n$ with $n$ linearly independent eigenvectors, then **A** can be diagonalized using a similarity transform $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$. The transform matrix **T** consisted of eigenvectors of **A**, and the diagonal entries of D consisted of the eigenvalues of **A**.
- In the case of repeated eigenvalues and fewer than $n$ linearly independent eigenvectors, **A** can be transformed into a nearly diagonal matrix **J**, called the **Jordan form** of **A**, with

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{J}.$$

# Example 2 Extension: Transform Matrix (1 of 2)

- In Example 2, our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ was

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

with eigenvalues $r_1 = 2$ and $r_2 = 2$ and eigenvectors

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; \boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- Choosing $k = 0$, the transform matrix $\mathbf{T}$ formed from the two eigenvectors $\xi$ and $\eta$ is

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

# Example 2 Extension: Jordan Form (2 of 2)

- The Jordan form $\mathbf{J}$ of $\mathbf{A}$ is defined by $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{J}$.
- Now

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \ \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

and hence

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\left[\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\right] = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

- Note that the eigenvalues of $\mathbf{A}$, $r_1 = 2$ and $r_2 = 2$, are on the main diagonal of $\mathbf{J}$, and that there is a 1 directly above the second eigenvalue. This pattern is typical of Jordan forms.

# Ch 7.9: Nonhomogeneous Linear Systems

- The general theory of a nonhomogeneous system of equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

  parallels that of a single *n*th order linear equation.
- This system can be written as $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}$$

# General Solution

- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ on $I$: $\alpha < t < \beta$ has the form

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t)$$

where

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t)$$

is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$.

# Diagonalization

- Suppose $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, where $\mathbf{A}$ is an $n$ x $n$ diagonalizable constant matrix.
- Let $\mathbf{T}$ be the nonsingular transform matrix whose columns are the eigenvectors of $\mathbf{A}$, and $\mathbf{D}$ the diagonal matrix whose diagonal entries are the corresponding eigenvalues of $\mathbf{A}$.
- Suppose $\mathbf{x}$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$, let $\mathbf{y}$ be defined by $\mathbf{x} = \mathbf{T}\mathbf{y}$.
- Substituting $\mathbf{x} = \mathbf{T}\mathbf{y}$ into $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we obtain

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t).$$

or
$$\mathbf{y}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t)$$

or
$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t), \text{ where } \mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t).$$

- Note that if we can solve diagonal system $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ for $\mathbf{y}$, then $\mathbf{x} = \mathbf{T}\mathbf{y}$ is a solution to the original system.

# Solving Diagonal System

- Now $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ is a diagonal system of the form

$$\left.\begin{aligned} y_1' &= r_1 y_1 + 0y_2 + \ldots + 0y_n + h_1(t) \\ y_2' &= 0y_1 + r_2 y_2 + \ldots + 0y_n + h_2(t) \\ &\ \vdots \\ y_n' &= 0y_1 + 0y_2 + \ldots + r_1 y_n + h_n(t) \end{aligned}\right\} \Leftrightarrow \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

  where $r_1, \ldots, r_n$ are the eigenvalues of $\mathbf{A}$.

- Thus $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ is an uncoupled system of $n$ linear first order equations in the unknowns $y_k(t)$, which can be isolated

$$y_k' = r_1 y_k + h_k(t), \;\; k = 1, \ldots, n$$

  and solved separately, using methods of Section 2.1:

$$y_k = e^{r_k t} \int_{t_0}^{t} e^{-r_k s} h_k(s) ds + c_k e^{r_k t}, \;\; k = 1, \ldots, n$$

# Solving Original System

- The solution $\mathbf{y}$ to $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t)$ has components

$$y_k = e^{r_k t}\int_{t_0}^{t} e^{-r_k s} h_k(s)\,ds + c_k e^{r_k t}, \;\; k = 1, \ldots, n$$

- For this solution vector $\mathbf{y}$, the solution to the original system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ is then $\mathbf{x} = \mathbf{T}\mathbf{y}$.
- Recall that $\mathbf{T}$ is the nonsingular transform matrix whose columns are the eigenvectors of $\mathbf{A}$.
- Thus, when multiplied by $\mathbf{T}$, the second term on right side of $y_k$ produces general solution of homogeneous equation, while the integral term of $y_k$ produces a particular solution of nonhomogeneous system.

# Example 1: General Solution of Homogeneous Case (1 of 5)

- Consider the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$ below.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$$

- Note: $\mathbf{A}$ is a Hermitian matrix, since it is real and symmetric.
- The eigenvalues of $\mathbf{A}$ are $r_1 = -3$ and $r_2 = -1$, with corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- The general solution of the homogeneous system is then

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}$$

# Example 1: Transformation Matrix (2 of 5)

- Consider next the transformation matrix **T** of eigenvectors. Using a Section 7.7 comment, and **A** Hermitian, we have $\mathbf{T}^{-1} = \mathbf{T}^{*} = \mathbf{T}^{T}$, provided we normalize $\xi^{(1)}$ and $\xi^{(2)}$ so that $(\xi^{(1)}, \xi^{(1)}) = 1$ and $(\xi^{(2)}, \xi^{(2)}) = 1$. Thus normalize as follows:

$$\xi^{(1)} = \frac{1}{\sqrt{(1)(1)+(-1)(-1)}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\xi^{(2)} = \frac{1}{\sqrt{(1)(1)+(1)(1)}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Then for this choice of eigenvectors,

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

# Example 1: Diagonal System and its Solution (3 of 5)

- Under the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, we obtain the diagonal system $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t)$:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$= \begin{pmatrix} -3y_1 \\ -y_2 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}$$

- Then, using methods of Section 2.1,

$$y_1' + 3y_1 = \sqrt{2}e^{-t} - \frac{3}{\sqrt{2}}t \;\Rightarrow\; y_1 = \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}\left(\frac{t}{3} - \frac{1}{9}\right) + c_1 e^{-3t}$$

$$y_2' + y_2 = \sqrt{2}e^{-t} + \frac{3}{\sqrt{2}}t \;\Rightarrow\; y_2 = \sqrt{2}te^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2 e^{-t}$$

# Example 1:
# Transform Back to Original System (4 of 5)

- We next use the transformation **x** = **Ty** to obtain the solution to the original system **x**′ = **Ax** + **g**($t$):

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^{-t} - \left(\frac{t}{2} - \frac{1}{6}\right) + k_1 e^{-3t} \\ te^{-t} + \frac{3}{2}(t-1) + k_2 e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} k_1 e^{-3t} + \left(k_2 + \frac{1}{2}\right)e^{-t} + t - \frac{4}{3} + te^{-t} \\ -k_1 e^{-3t} + \left(k_2 - \frac{1}{2}\right)e^{-t} + 2t - \frac{5}{3} + te^{-t} \end{pmatrix}, \quad k_1 = \frac{c_1}{\sqrt{2}}, \; k_2 = \frac{c_2}{\sqrt{2}}$$

# Example 1: Solution of Original System (5 of 5)

- Simplifying further, the solution **x** can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k_1 e^{-3t} + \left(k_2 + \frac{1}{2}\right)e^{-t} + t - \frac{4}{3} + te^{-t} \\ -k_1 e^{-3t} + \left(k_2 - \frac{1}{2}\right)e^{-t} + 2t - \frac{5}{3} + te^{-t} \end{pmatrix}$$

$$= k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

- Note that the first two terms on right side form the general solution to homogeneous system, while the remaining terms are a particular solution to nonhomogeneous system.

# Nondiagonal Case

- If **A** cannot be diagonalized, (repeated eigenvalues and a shortage of eigenvectors), then it can be transformed to its Jordan form **J**, which is nearly diagonal.
- In this case the differential equations are not totally uncoupled, because some rows of **J** have two nonzero entries: an eigenvalue in diagonal position, and a 1 in adjacent position to the right of diagonal position.
- However, the equations for $y_1, \ldots, y_n$ can still be solved consecutively, starting with $y_n$. Then the solution **x** to original system can be found using $\mathbf{x} = \mathbf{T}\mathbf{y}$.

# Undetermined Coefficients

- A second way of solving $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the method of undetermined coefficients. Assume $\mathbf{P}$ is a constant matrix, and that the components of $\mathbf{g}$ are polynomial, exponential or sinusoidal functions, or sums or products of these.
- The procedure for choosing the form of solution is usually directly analogous to that given in Section 3.6.
- The main difference arises when $\mathbf{g}(t)$ has the form $\mathbf{u}e^{\lambda t}$, where $\lambda$ is a simple eigenvalue of $\mathbf{P}$. In this case, $\mathbf{g}(t)$ matches solution form of homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and as a result, it is necessary to take nonhomogeneous solution to be of the form $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$. This form differs from the Section 3.6 analog, $\mathbf{a}te^{\lambda t}$.

# Example 2: Undetermined Coefficients (1 of 5)

- Consider again the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}t$$

- Assume a particular solution of the form

$$\mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}$$

  where the vector coefficents **a**, **b**, **c**, **d** are to be determined.
- Since $r = -1$ is an eigenvalue of **A**, it is necessary to include both $\mathbf{a}te^{-t}$ and $\mathbf{b}e^{-t}$, as mentioned on the previous slide.

# Example 2:
# Matrix Equations for Coefficients (2 of 5)

- Substituting

$$\mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}$$

in for $\mathbf{x}$ in our nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$,

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}t,$$

we obtain

$$-\mathbf{a}te^{-t} + (\mathbf{a} - \mathbf{b})e^{-t} + \mathbf{c} = \mathbf{A}\mathbf{a}te^{-t} + \mathbf{A}\mathbf{b}e^{-t} + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}t$$

- Equating coefficients, we conclude that

$$\mathbf{A}\mathbf{a} = -\mathbf{a},\ \mathbf{A}\mathbf{b} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ \mathbf{A}\mathbf{c} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix},\ \mathbf{A}\mathbf{d} = \mathbf{c}$$

# Example 2: Solving Matrix Equation for (a) (3 of 5)

- Our matrix equations for the coefficients are:

$$\mathbf{Aa} = -\mathbf{a}, \;\; \mathbf{Ab} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \;\; \mathbf{Ac} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \;\; \mathbf{Ad} = \mathbf{c}$$

- From the first equation, we see that **a** is an eigenvector of **A** corresponding to eigenvalue $r$ = -1, and hence has the form

$$\mathbf{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

- We will see on the next slide that $\alpha = 1$, and hence

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

# Example 2: Solving Matrix Equation for (b) (4 of 5)

- Our matrix equations for the coefficients are:

$$\mathbf{Aa} = -\mathbf{a}, \quad \mathbf{Ab} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{Ac} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \mathbf{Ad} = \mathbf{c}$$

- Substituting $\mathbf{a} = (\alpha, \alpha)^T$ into the second equation,

$$\mathbf{Ab} = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Leftrightarrow \left[ \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha - 1 \end{pmatrix}$$

- Thus $\alpha = 1$, and solving for **b**, we obtain

$$\mathbf{b} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{choose } k = 0 \rightarrow \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

# Example 2: Particular Solution (5 of 5)

- Our matrix equations for the coefficients are:

$$\mathbf{Aa} = -\mathbf{a}, \ \ \mathbf{Ab} = \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ \ \mathbf{Ac} = -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \ \ \mathbf{Ad} = \mathbf{c}$$

- Solving third equation for **c**, and then fourth equation for **d**, it is straightforward to obtain $\mathbf{c}^T = (1, 2)$, $\mathbf{d}^T = (-4/3, -5/3)$.
- Thus our particular solution of $\mathbf{x}' = \mathbf{Ax} + \mathbf{g}$ is

$$\mathbf{v}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

- Comparing this to the result obtained in Example 1, we see that both particular solutions would be the same if we had chosen $k = ½$ for **b** on previous slide, instead of $k = 0$.

# Variation of Parameters: Preliminaries

- A more general way of solving $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the method of variation of parameters.
- Assume $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on $\alpha < t < \beta$, and let $\Psi(t)$ be a fundamental matrix for the homogeneous system.
- Recall that the columns of $\Psi$ are linearly independent solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and hence $\Psi(t)$ is invertible on the interval $\alpha < t < \beta$, and also $\Psi'(t) = P(t)\Psi(t)$.
- Next, recall that the solution of the homogeneous system can be expressed as $\mathbf{x} = \Psi(t)\mathbf{c}$.
- Analogous to Section 3.7, assume the particular solution of the nonhomogeneous system has the form $\mathbf{x} = \Psi(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ is a vector to be found.

# Variation of Parameters: Solution

- We assume a particular solution of the form $\mathbf{x} = \Psi(t)\mathbf{u}(t)$.
- Substituting this into $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, we obtain

$$\Psi'(t)\mathbf{u}(t) + \Psi(t)\mathbf{u}'(t) = \mathbf{P}(t)\Psi(t)\mathbf{u}(t) + \mathbf{g}(t)$$

- Since $\Psi'(t) = \mathbf{P}(t)\Psi(t)$, the above equation simplifies to

$$\mathbf{u}'(t) = \Psi^{-1}(t)\mathbf{g}(t)$$

- Thus

$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t)g(t)dt + \mathbf{c}$$

where the vector **c** is an arbitrary constant of integration.

- The general solution to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is therefore

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int_{t_1}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds, \quad t_1 \in (\alpha, \beta) \text{ arbitrary}$$

# Variation of Parameters: Initial Value Problem

- For an initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),\ \mathbf{x}(t_0) = \mathbf{x}^{(0)},$$

the general solution to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^{(0)} + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds$$

- Alternatively, recall that the fundamental matrix $\mathbf{\Phi}(t)$ satisfies $\mathbf{\Phi}(t_0) = \mathbf{I}$, and hence the general solution is

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^{(0)} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds$$

- In practice, it may be easier to row reduce matrices and solve necessary equations than to compute $\mathbf{\Psi}^{-1}(t)$ and substitute into equations. See next example.

# Example 3: Variation of Parameters (1 of 3)

- Consider again the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}t$$

- We have previously found general solution to homogeneous case, with corresponding fundamental matrix:

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

- Using variation of parameters method, our solution is given by $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ satisfies $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$, or

$$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

# Example 3: Solving for $\mathbf{u}(t)$ (2 of 3)

- Solving $\boldsymbol{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$ by row reduction,

$$\begin{pmatrix} e^{-3t} & e^{-t} & 2e^{-t} \\ -e^{-3t} & e^{-t} & 3t \end{pmatrix} \to \begin{pmatrix} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & 2e^{-t} & 2e^{-t}+3t \end{pmatrix}$$

$$\to \begin{pmatrix} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & e^{-t} & e^{-t}+3t/2 \end{pmatrix} \to \begin{pmatrix} e^{-3t} & 0 & e^{-t}-3t/2 \\ 0 & e^{-t} & e^{-t}+3t/2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & e^{2t}-3te^{3t}/2 \\ 0 & 1 & 1+3te^{t}/2 \end{pmatrix} \to \begin{aligned} u_1' &= e^{2t}-3te^{3t}/2 \\ u_2' &= 1+3te^{t}/2 \end{aligned}$$

- It follows that

$$\mathbf{u}(t) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{2t}/2 - te^{3t}/2 + e^{3t}/6 + c_1 \\ t + 3te^{t}/2 - 3e^{t}/2 + c_2 \end{pmatrix}$$

# Example 3: Solving for $\mathbf{x}(t)$ (3 of 3)

- Now $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{u}(t)$, and hence we multiply

$$\mathbf{x} = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} e^{2t}/2 - te^{3t}/2 + e^{3t}/6 + c_1 \\ t + 3te^{t}/2 - 3e^{t}/2 + c_2 \end{pmatrix}$$

to obtain, after collecting terms and simplifying,

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

- Note that this is the same solution as in Example 1.

# Laplace Transforms

- The Laplace transform can be used to solve systems of equations. Here, the transform of a vector is the vector of component transforms, denoted by $\mathbf{X}(s)$:

$$L\{\mathbf{x}(t)\} = L\left\{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\right\} = \begin{pmatrix} L\{x_1(t)\} \\ L\{x_2(t)\} \end{pmatrix} = \mathbf{X}(s)$$

- Then by extending Theorem 6.2.1, we obtain

$$L\{\mathbf{x}'(t)\} = s\mathbf{X}(s) - \mathbf{x}(0)$$

# Example 4: Laplace Transform 

- Consider again the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

- Taking the Laplace transform of each term, we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s)$$

where $\mathbf{G}(s)$ is the transform of $g(t)$, and is given by

$$\mathbf{G}(s) = \begin{pmatrix} \dfrac{2}{s+1} \\ \dfrac{3}{s^2} \end{pmatrix}$$

# Example 4: Transfer Matrix (2 of 5)

- Our transformed equation is

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s)$$

- If we take $\mathbf{x}(\mathbf{0}) = \mathbf{0}$, then the above equation becomes

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s)$$

or

$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{G}(s)$$

- Solving for $\mathbf{X}(s)$, we obtain

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{G}(s)$$

- The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is called the **transfer matrix**.

# Example 4: Finding Transfer Matrix (3 of 5)

- Then

$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \Rightarrow (s\mathbf{I} - \mathbf{A}) = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}$$

- Solving for $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)} \begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}$$

# Example 4: Transfer Matrix (4 of 5)

- Next, $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s)$, and hence

$$\mathbf{X}(s) = \frac{1}{(s+1)(s+3)}\begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}\begin{pmatrix} 2/(s+1) \\ 3/s^2 \end{pmatrix}$$

or

$$\mathbf{X}(s) = \begin{pmatrix} \dfrac{2(s+2)}{(s+1)^2(s+3)} + \dfrac{3}{s^2(s+1)(s+3)} \\ \dfrac{2}{(s+1)^2(s+3)} + \dfrac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}$$

# Example 4: Transfer Matrix (5 of 5)

- Thus

$$\mathbf{X}(s) = \begin{pmatrix} \dfrac{2(s+2)}{(s+1)^2(s+3)} + \dfrac{3}{s^2(s+1)(s+3)} \\ \dfrac{2}{(s+1)^2(s+3)} + \dfrac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}$$

- To solve for $\mathbf{x}(t) = L^{-1}\{\mathbf{X}(s)\}$, use partial fraction expansions of both components of $\mathbf{X}(s)$, and then Table 6.2.1 to obtain:

$$\mathbf{x} = -\frac{2}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

- Since we assumed $\mathbf{x}(\mathbf{0}) = \mathbf{0}$, this solution differs slightly from the previous particular solutions.

# Summary (1 of 2)

- The method of undetermined coefficients requires no integration but is limited in scope and may involve several sets of algebraic equations.
- Diagonalization requires finding inverse of transformation matrix and solving uncoupled first order linear equations. When coefficient matrix is Hermitian, the inverse of transformation matrix can be found without calculation, which is very helpful for large systems.
- The Laplace transform method involves matrix inversion, matrix multiplication, and inverse transforms. This method is particularly useful for problems with discontinuous or impulsive forcing functions.

# Summary (2 of 2)

- Variation of parameters is the most general method, but it involves solving linear algebraic equations with variable coefficients, integration, and matrix multiplication, and hence may be the most computationally complicated method.
- For many small systems with constant coefficients, all of these methods work well, and there may be little reason to select one over another.