Introduction to Ordinary Differential Equations (ODE)

- <u>Prerequisites:</u> calculus (differentiation & integration), linear algebra

- <u>Textbook:</u> "Elementary Differential Equations and Boundary Value Problems," by William E. Boyce, Richard C. DiPrima, and Doug Meade, John Wiley & Sons (any edition)

- <u>Extra-references</u> (see syllabus)

- <u>Evaluation:</u>
  - Attendance (10%)
  - Homework (20%)
  - End-term test (70%)

Chapter 1.1: Basic Mathematical Models; Direction Fields

- **Differential equations** are equations containing derivatives.
- The following are examples of physical phenomena involving rates of change:
    - Motion of fluids
    - Motion of mechanical systems
    - Flow of current in electrical circuits
    - Dissipation of heat in solid objects
    - Seismic waves
    - Population dynamics
- A differential equation that describes a physical process is often called a **mathematical model**.

Example 1: Free Fall

- Formulate a differential equation describing motion of an object falling in the atmosphere near sea level.

- Variables: time t, velocity v

- Newton's 2nd Law: $F = ma = m(dv/dt)$ (net force)

- Force of gravity: $F = mg$ (downward force)

- Force of air resistance: $F = \gamma\, v$ (upward force)

- Then

$$m\frac{dv}{dt} = mg - \gamma\, v$$

- Taking $g = 9.8$ m/sec^2, $m = 10$ kg, $\gamma = 2$ kg/sec, we obtain
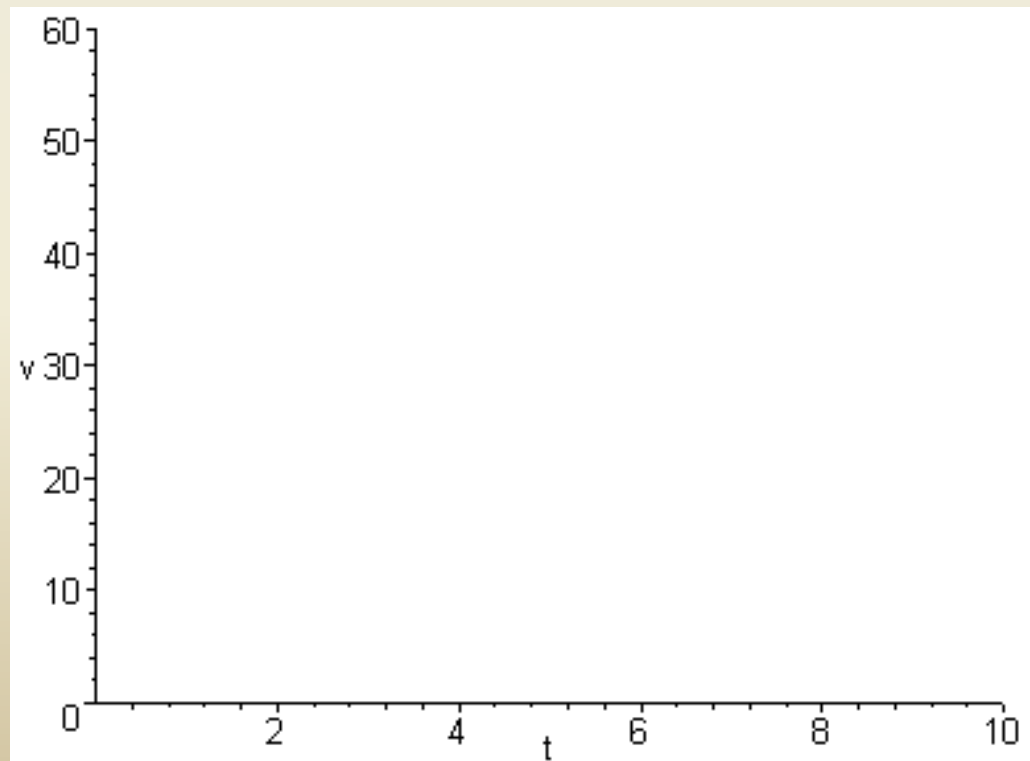
$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$
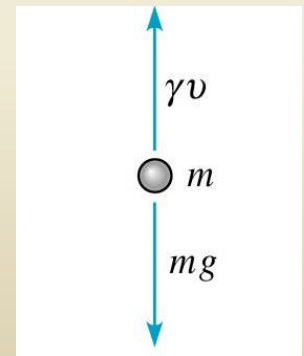
- Using differential equation and table, plot slopes (estimates) on axes below. The resulting graph is called a **direction field**. (Note that values of *v* do not depend on *t*.)

v	v'
0	9.8
5	8.8
10	7.8
15	6.8
20	5.8
25	4.8
30	3.8
35	2.8
40	1.8
45	0.8
50	-0.2
55	-1.2
60	-2.2

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

- When graphing direction fields, be sure to use an appropriate window, in order to display all equilibrium solutions and relevant solution behavior.
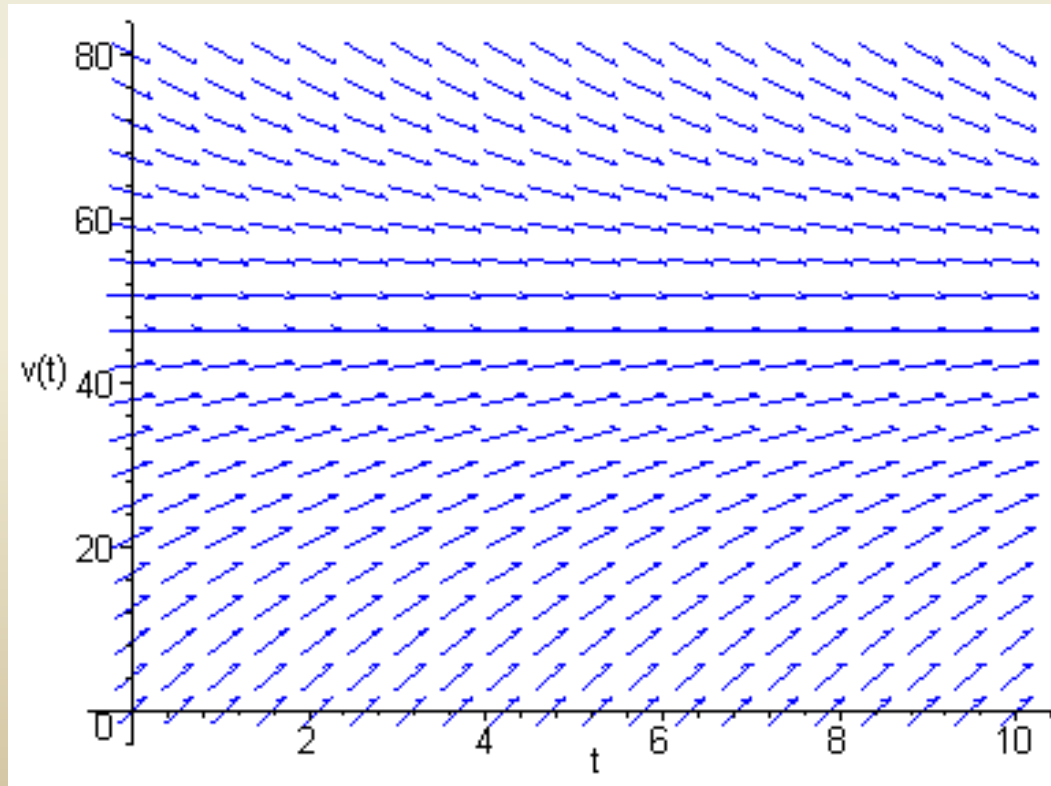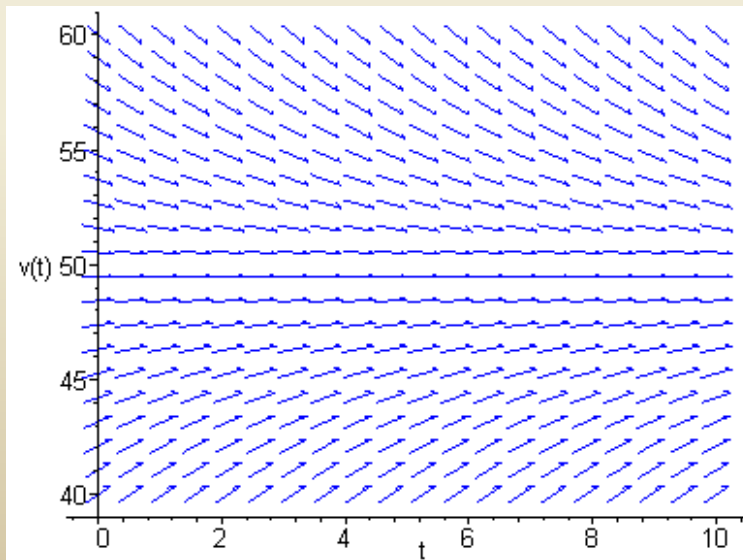
$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

- Arrows give tangent lines to solution curves, and indicate where soln is increasing & decreasing (and by how much).

- Horizontal solution curves are called **equilibrium solutions**.

- Use the graph below to solve for equilibrium solution, and then determine analytically by setting $v' = 0$.

Set $v' = 0$:

$$\Leftrightarrow 9.8 - 0.2v = 0$$

$$\Leftrightarrow v = (5)(9.8)$$

$$\Leftrightarrow v = 49$$

Mice and Owls

- Consider a mouse population that reproduces at a rate proportional to the current population, with a rate constant equal to 0.5 mice/month (assuming no owls present).

- When owls are present, they eat the mice. Suppose that the owls eat 15 per day (average). Write a differential equation describing mouse population in the presence of owls. (Assume that there are 30 days in a month.)

- Solution:

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

Example 3: Direction Field

- Discuss solution curve behavior, and find equilibrium soln.
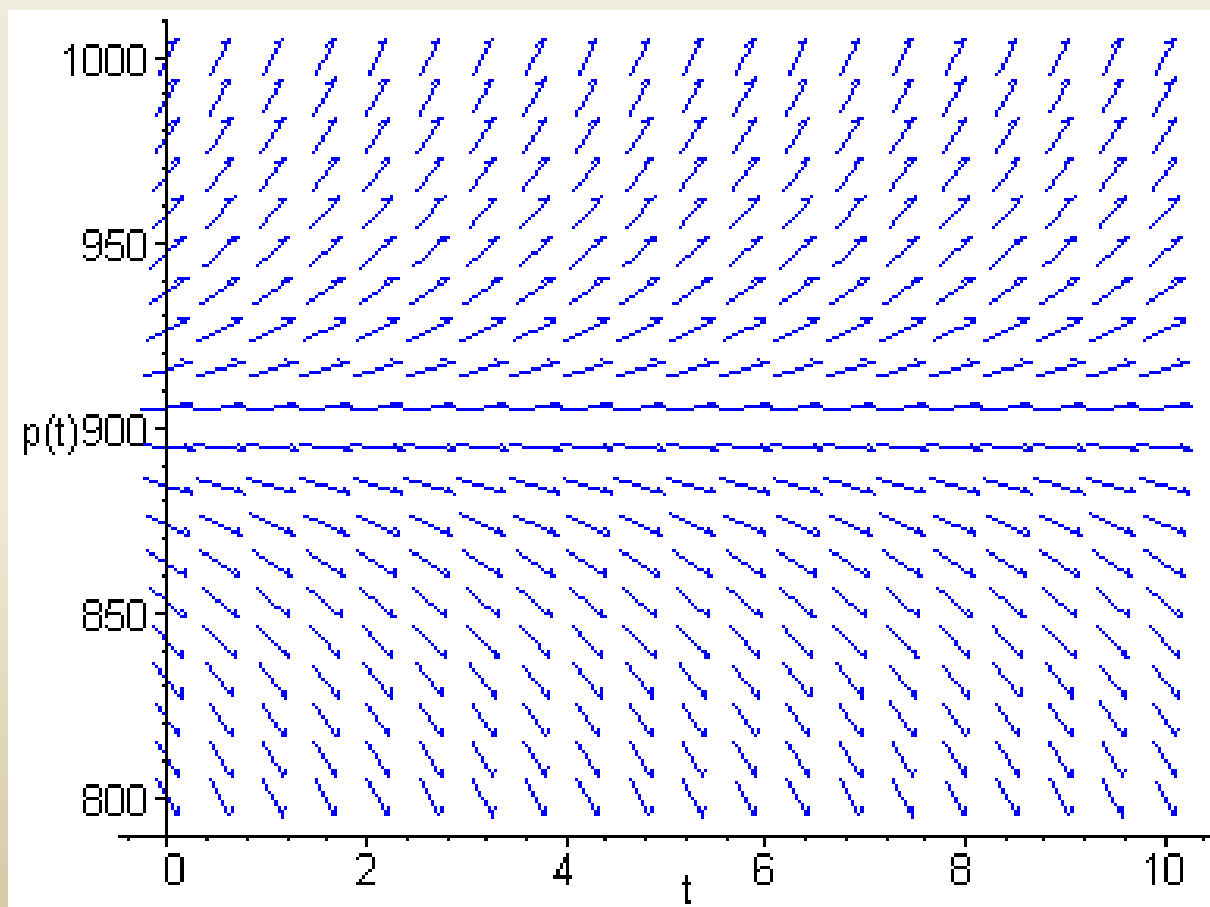
$$\frac{dp}{dt} = \frac{p}{2} - 450$$

Steps in Constructing Mathematical Models Using Differential Equations

- Identify independent and dependent variables and assign letters to represent them.

- Choose the units of measure for each variable.

- Articulate the basic principle that underlies or governs the problem you are investigating. This requires your being familiar with the field in which the problem originates.

- Express the principle or law in the previous step in terms of the variables identified at the start. This may involve the use of intermediate variables related to the primary variables.

- Make sure each term of your equation has the same physical units.

- The result may involve one or more differential equations.

Chapter 1.2: Solutions of Some Differential Equations

- Recall the differential equations:

$$m\frac{dv}{dt} = mg - \gamma v \qquad\qquad \frac{dp}{dt} = rp - k$$

- These equations have the general form $\frac{dy}{dt} = ay - b$

- We can use methods of calculus to solve differential equations of this form.

Example 1: Mice and Owls

- To solve the differential equation

$$\frac{dp}{dt} = 0.5p - 450$$

we use methods of calculus, as follows.

$$\frac{dp}{dt} = \frac{p-900}{2} \implies \frac{dp/dt}{p-900} = \frac{1}{2} \implies \int \frac{dp}{p-900} = \int \frac{1}{2} dt$$

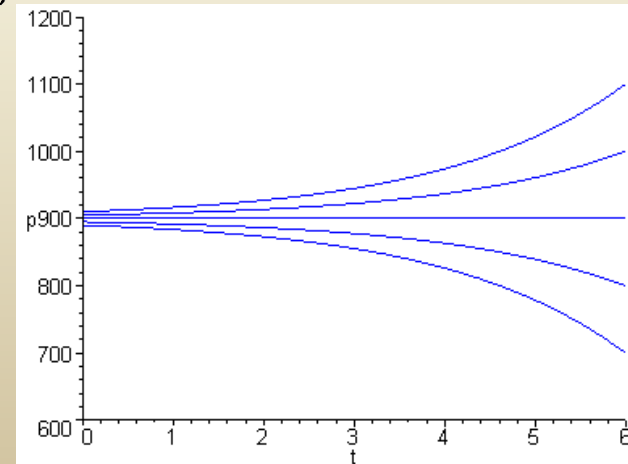$$\implies \ln|p-900| = \frac{1}{2}t + C \implies |p-900| = e^{\frac{1}{2}t+C}$$

$$\implies p-900 = \pm e^{\frac{1}{2}t}e^C \implies p = 900 + ce^{\frac{1}{2}t}, \ c = \pm e^C$$

- Thus the solution is

$$p = 900 + ce^{\frac{1}{2}t}$$

where c is a constant.

- Thus we have infinitely many solutions to our equation,

$$\frac{dp}{dt} = 0.5p - 450 \;\Rightarrow\; p = 900 + ce^{\frac{1}{2}t},$$

  since k is an arbitrary constant.

- Graphs of solutions (**integral curves**) for several values of c, and direction field for differential equation, are given below.

- Choosing $c = 0$, we obtain the equilibrium solution, while for $c > 0$, the solutions diverge from equilibrium solution.
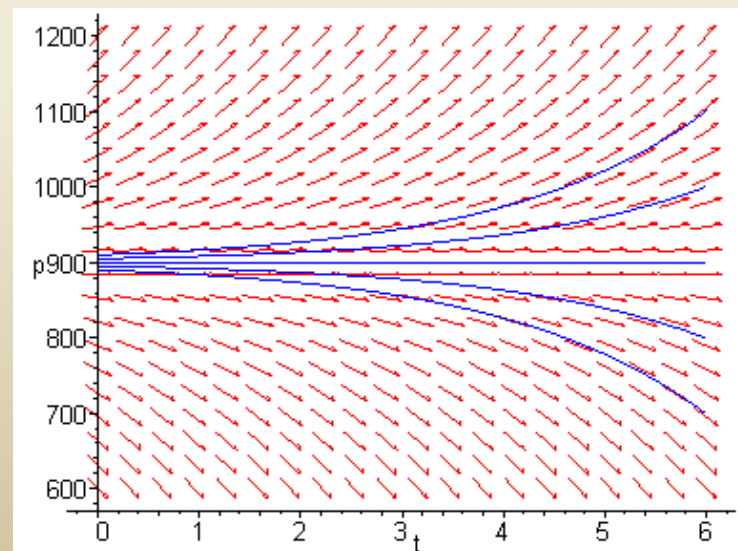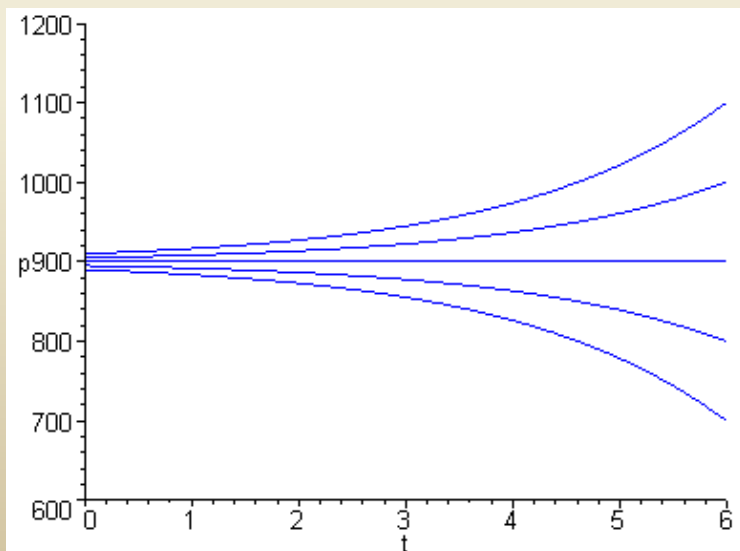
- A differential equation often has infinitely many solutions. If a point on the solution curve is known, such as an initial condition, then this determines a unique solution.

- In the mice/owl differential equation, suppose we know that the mice population starts out at 850. Then $p(0) = 850$, and

$$p(t) = 900 + ce^{0.5t}$$

$$p(0) = 850 = 900 + ce^0$$

$$c = -50$$

Solution:

$$p(t) = 900 - 50e^{\frac{1}{2}t}$$
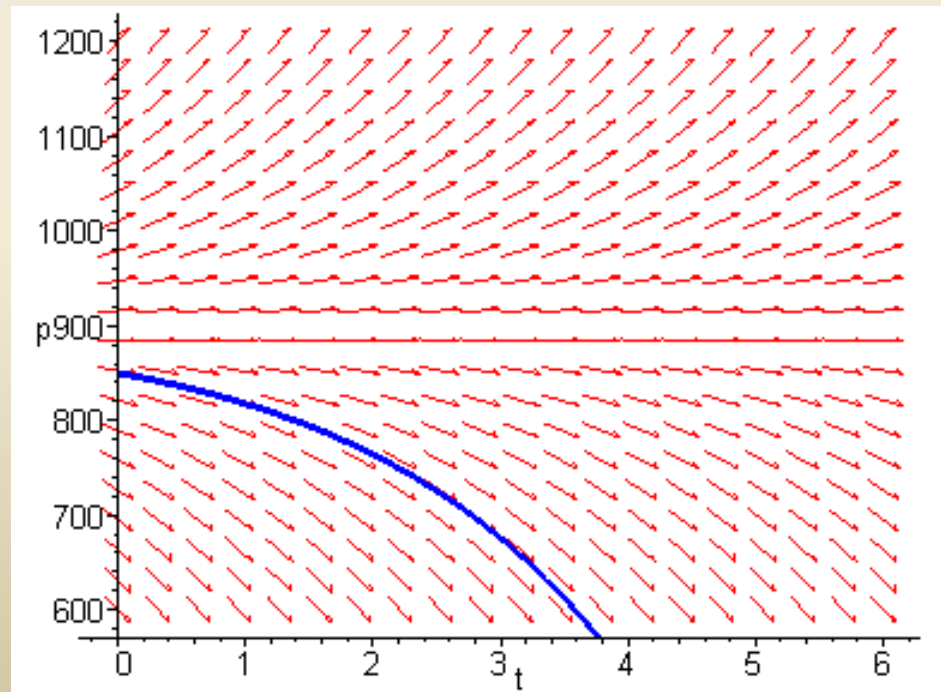
Solution to General Equation

- To solve the general equation $\dfrac{dy}{dt} = ay - b$

  we use methods of calculus, as follows.

$$\frac{dy}{dt} = a\left(y - \frac{b}{a} \right) \implies \frac{dy/dt}{y - b/a} = a \implies \int \frac{dy}{y - b/a} = \int a\,dt$$

$$\implies \ln|y - b/a| = at + C \implies |y - b/a| = e^{at+C}$$

$$\implies y - b/a = \pm e^{at}e^{C} \implies y = b/a + ce^{at}, \ c = \pm e^{C}$$

- Thus the general solution is

$$y = \frac{b}{a} + ce^{at},$$

  where c is a constant.

Initial Value Problem

- Next, we solve the initial value problem
$$y' = ay - b, \quad y(0) = y_0$$

- From previous slide, the solution to differential equation is
$$y = \frac{b}{a} + ce^{at}$$

- Using the initial condition to solve for c, we obtain
$$y(0) = y_0 = \frac{b}{a} + ce^0 \implies c = y_0 - \frac{b}{a}$$

  and hence the solution to the initial value problem is
$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}$$

Equilibrium Solution

- To find the equilibrium solution, set $y' = 0$ & solve for y:

$$y' = ay - b \overset{set}{=} 0 \implies y(t) = \frac{b}{a}$$

- From the previous slide, our solution to the initial value problem is:

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}$$

- Note the following solution behavior:
  - If $y_0 = b/a$, then y is constant, with $y(t) = b/a$
  - If $y_0 > b/a$ and $a > 0$, then y increases exponentially without bound
  - If $y_0 > b/a$ and $a < 0$, then y decays exponentially to b/a
  - If $y_0 < b/a$ and $a > 0$, then y decreases exponentially without bound
  - If $y_0 < b/a$ and $a < 0$, then y increases asymptotically to b/a

Example 2: Free Fall Equation

- Recall equation modeling free fall descent of 10 kg object, assuming an air resistance coefficient $\gamma = 2$ kg/sec:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

- Suppose object is dropped from 300 m. above ground.

  (a) Find velocity at any time t.

  (b) How long until it hits ground and how fast will it be moving then?

- For part (a), we need to solve the initial value problem

$$v' = 9.8 - 0.2v, \quad v(0) = 0$$

- Using result from previous slide, we have

$$\frac{dv/dt}{v-49} = -\frac{1}{5} \Rightarrow \ln|v(t) - 49| = -\frac{1}{5} + C \Rightarrow v(t) = 49 + ce^{-t/5}$$

Example 2: Free Fall Equation

- To determine the particular value of c that corresponds to the initial condition, we substitute $t = 0$ and $v = 0$.

- This gives us

$$v(t) = 49 + ce^{-t/5} \Rightarrow 0 = 49 + ce^{0} \Rightarrow c = -49$$

- With the result that $c = -49$. And the solution of the initial value problem is
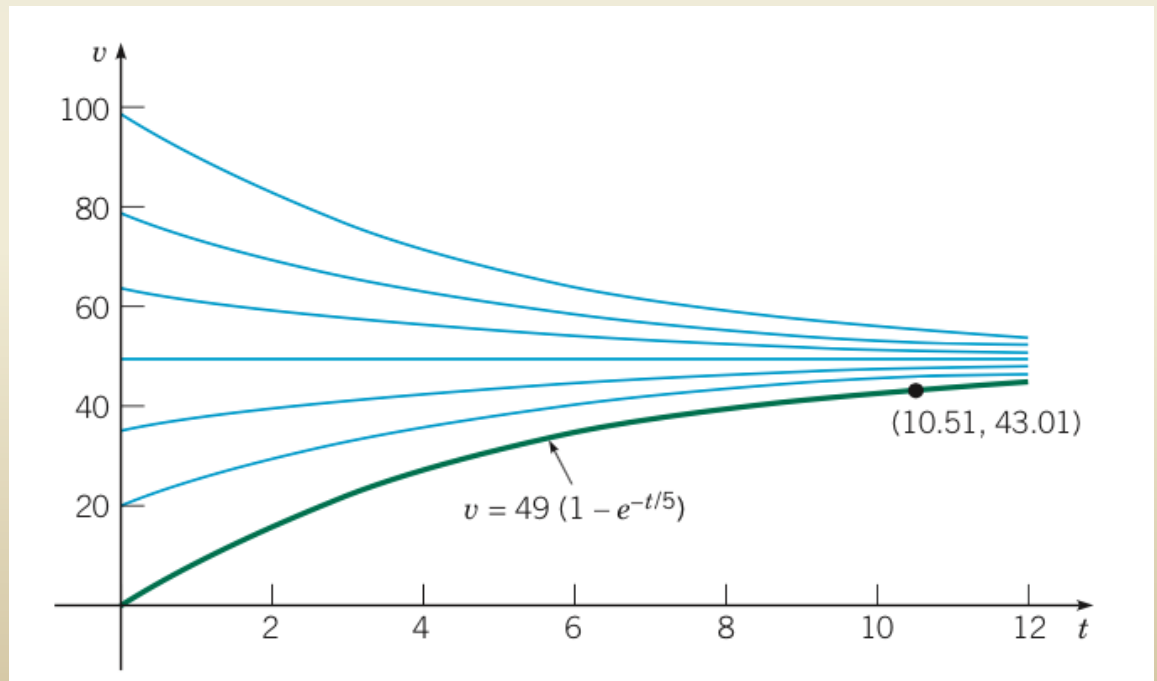
$$v(t) = 49(1 - e^{-t/5})$$

- The graph of the solution found in part (a), along with the direction field for the differential equation, is given below.

$$v' = 9.8 - 0.5v, \quad v(0) = 0$$

$$v = 49\left(1 - e^{-t/5}\right)$$

Example 2

Part (b): Time and Speed of Impact

- Next, given that the object is dropped from 300 m above ground, how long will it take to hit ground, and how fast will it be moving at impact?

- Solution: Let $x(t) =$ distance object has fallen at time t.

  It follows from our solution $v(t)$ that

  $$\frac{dx}{dt} = 49 - 49e^{-t/5} \implies x(t) = 49t + 245e^{-t/5} + k$$

  $$x(0) = 0 \implies k = -245 \quad \implies x(t) = 49t + 245e^{-t/5} - 245$$

- Let T be the time of impact. Then

  $$x(T) = 49T + 245e^{-T/5} - 245 = 300$$

- Using a solver, $T \simeq 10.51$ sec, hence

  $$v(10.51) = 49\left(1 - e^{-0.2(10.51)}\right) \approx 43.01\,\text{m/sec}$$

Ch1.3: Classification of Differential Equations

- The main purpose of this course is to discuss properties of solutions of differential equations, and to present methods of finding solutions or approximating them.

- To provide a framework for this discussion, in this section we give several ways of classifying differential equations.

Ordinary Differential Equations

- When the unknown function depends on a single independent variable, only ordinary derivatives appear in the equation.

- In this case the equation is said to be an ordinary differential equations (ODE).

- The equations discussed in the preceding two sections are ordinary differential equations. For example,

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad \frac{dp}{dt} = \frac{p}{2} - 450$$

- Another example of an ordinary differential equation, for charge $Q(t)$ is

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

Partial Differential Equations

- When the unknown function depends on several independent variables, partial derivatives appear in the equation.

- In this case the equation is said to be a partial differential equation (PDE).

- Examples:

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t} \quad \text{(heat equation)}$$

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(wave equation)}$$

Systems of Differential Equations

- Another classification of differential equations depends on the number of unknown functions that are involved.

- If there is a single unknown function to be found, then one equation is sufficient. If there are two or more unknown functions, then a system of equations is required.

- For example, predator-prey equations have the form

$$du/dt = ax - \alpha xy$$

$$dv/dt = -cy + \gamma uy$$

where $x(t)$ and $y(t)$ are the respective populations of prey and predator species. The constants a, c, α, and γ depend on the particular species being studied.

- Systems of equations are discussed in Chapter 7 and 9.

Order of Differential Equations

- The order of a differential equation is the order of the highest derivative that appears in the equation.

- More generally, the equation

$$F(t, y, y', ..., y^{(n)}) = 0 \text{ where } y = u(t)$$

  is an ordinary differential equation of the nth order.

- For example

$$y''' + 2e^t y'' + yy' = t^4$$

  is a third order differential equation, for $y = u(t)$.

- We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)}(t) = f\left(t, y, y', y'', ..., y^{(n-1)}\right)$$

Linear & Nonlinear Differential Equations

- An ordinary differential equation

$$F\left(t, y, y', y'', y''', \ldots, y^{(n)}\right) = 0$$

  is **linear** if F is linear in the variables

$$y, y', y'', y''', \ldots, y^{(n)}$$

- Thus the general linear ODE has the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$$

- Example: Determine whether the equations below are linear or nonlinear.

(1) $y' + 3y = 0$

(2) $y'' + 3e^y y' - 2t = 0$

(3) $y'' + 3y' - 2t^2 = 0$

(4) $\dfrac{d^4 y}{dt^4} - t\dfrac{d^2 y}{dt^2} + 1 = t^2$

(5) $u_{xx} + uu_{yy} = \sin t$

(6) $u_{xx} + \sin(u)u_{yy} = \cos t$

Solutions to Differential Equations

- A solution $\varphi(t)$ to an ordinary differential equation

$$y^{(n)}(t) = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

- satisfies the equation:

$$\varphi^{(n)}(t) = f\left(t, \varphi, \varphi', \varphi'', \ldots, \varphi^{(n-1)}\right)$$

- It is not always easy to find solutions of differential equations. However, it is relatively easy to determine whether a function is actually a solution. Just substitute the function into the equation.

- Example: Verify the following solutions of the ODE

$$y'' + y = 0; \quad y_1(t) = \sin t, \quad y_2(t) = -\cos t, \quad y_3(t) = 2\sin t$$

Solutions to Differential Equations

- There are three important questions in the study of differential equations:
  - Is there a solution? (Existence)
  - If there is a solution, is it unique? (Uniqueness)
  - If there is a solution, how do we find it?

    (Analytical Solution, Numerical Approximation, etc)

Historical Background

- The development of differential equations is a significant part of the general development of mathematics.

- Isaac Newton (1642-1727) was born in England, and is known for his development of calculus and laws of physics (mechanics), series solution to ODEs, 1665 –1690.

- Gottfried Leibniz (1646-1716) was born in Leipzig, Germany. He is known for his development of calculus (1684), notation for the derivative (dy/dx), method separation of variables (1691), first order ODE methods (1694).

The Bernoullis

- Jakob Bernoulli (1654-1705) & Johann Bernoulli (1667-1748) were both raised in Basel, Switzerland.

- They used calculus and integrals in the form of differential equations to solve mechanics problems.

- Daniel Bernoulli (1700-1782), son of Johann, is known for his work on partial differential equations and applications, and Bessel functions.

Leonard Euler (1707-1783)

- Leonard Euler (pronounced "oiler"), was raised in Basel, Switzerland, and was the most prolific mathematician of all time. His collected works fill more than 70 volumes.

- He formulated problems in mechanics into mathematical language and developed methods of solution. "First great work in which analysis is applied to science of movement" (Lagrange).

- He is known also for his work on exactness of first order ODEs (1734), integrating factors (1734), linear equations with constant coefficients (1743), series solutions to differential equations (1750), numerical procedures (1768), PDEs (1768), and calculus of variations (1768).

Joseph-Louis Lagrange (1736-1813).

- Lagrange was raised in Turin, Italy. He was mostly self taught in beginning, and became math professor at age 19.

- Lagrange most famous for his *Analytical Mechanics* (1788) work on Newtonian mechanics.

- Lagrange showed that the general solution of a nth order linear homogenous ODE is a linear combination of n independent solutions (1762-65). He also gave a complete development of variation of parameters (1774-75), and studied PDEs and calculus of variations.

Pierre-Simon de Laplace (1749-1827).

- Laplace was raised in Normandy, France, and was preeminent in celestial mechanics (1799-1825).

- Laplace's equation in PDEs is well known, and he studied it extensively in connection with gravitation attraction.

- The Laplace transform is named after him.

The 1800s

- By the end of the 1700s, many elementary methods of solving ordinary differential equations had been discovered.

- In the 1800s, interest turned to theoretical questions of existence and uniqueness, and series expansions (Ch 5).

- Partial differential equations also became a focus of study, as their role in mathematical physics became clear.

- A classification of certain useful functions arose, called Special Functions. This group included Bessel functions, Chebyshev polynomials, Legendre polynomials, Hermite polynomials, Hankel polynomials.

The 1900s – Present

- Computational results were an essential element in the discovery of "chaos theory".

- In 1961, Edward Lorenz (1917 – 2008), was developing weather prediction models when he observed different results upon restarting a simulation in the middle of the time period using previously computed results.

- He used three-digit approximations rather than restart the computation with the six-digit values that were stored in the computer.

- The common trait that small changes in a problem produce large changes in the solution is one of the defining characteristics of chaos.