ISSE Course, Ritsumeikan University Spring Semester, 2022

Introduction to Differential Equations
Assignment # 1

Date Given: April 11, 2022
Date Due: April 18, 2022

P1. (3 points) Draw a direction field for the differential equation $y' = y(4 - y)$. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency. Note that in this problem the equation is not linear (is not of the form $y' = ay + b$), and the behavior of the solution is somewhat more complicated than for the equations in the text.

Solution: Note that $y(0) = 0$ for $y = 0$ and $y = 4$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 4$. Based on the direction field, $y' > 0$ for $0 < y < 4$; thus solutions with initial values greater than 0 and less that 4 converge to the solution $y(t) = 4$. For $y > 4$, the slopes are negative, and hence solutions with initial values greater than 4 all decrease toward the solution $y(t) = 4$. For $y < 0$, the slopes are all negative; thus solutions with initial values less than 0 diverge from the solution $y(t) = 0$. (see Figure 1).

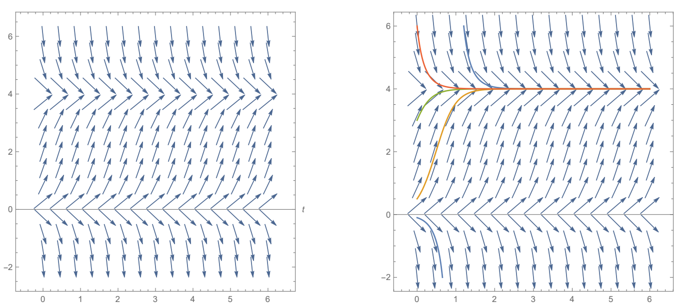

Figure 1: Illustration to problem P1: the direction field (left) and integral curves (right) in the field.

P2. (2 points) Draw a direction field for the differential equation $y' = t + 2y$. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

Solution: All solutions (except $y(0) = -1/4$) diverge from the solution $y(t) = -t/2 - 1/4$ and approach $\pm\infty$ (see Figure 2).

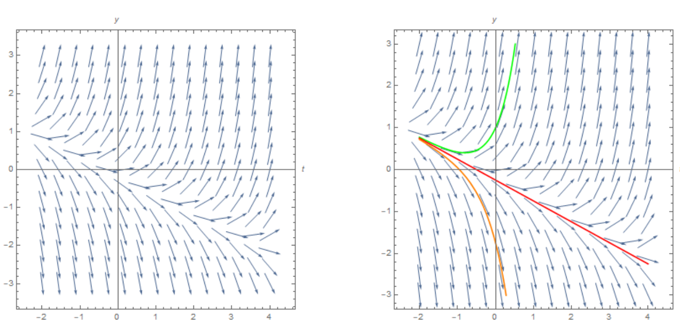

Figure 2: Illustration to problem P2: the direction field (left) and integral curves (right) in the field.

P3. (2 points) Solve each of the following initial value problems and plot the solutions for several values of y_0.

(a) $\mathrm{d}y/\mathrm{d}t = y - 5, \quad y(0) = y_0$.

Then describe in a few words the behavior of the solution.

Solution: The differential equation can be rewritten as

$$\frac{\mathrm{d}y}{y-5} = \mathrm{d}t.$$

Integrating both sides of this equation results in $\ln|y-5| = t + c_1$, or, equivalently, $y - 5 = ce^t$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^t$.

All solutions appear to diverge from the equilibrium solution $y(t) = 5$ (see Figure 3).

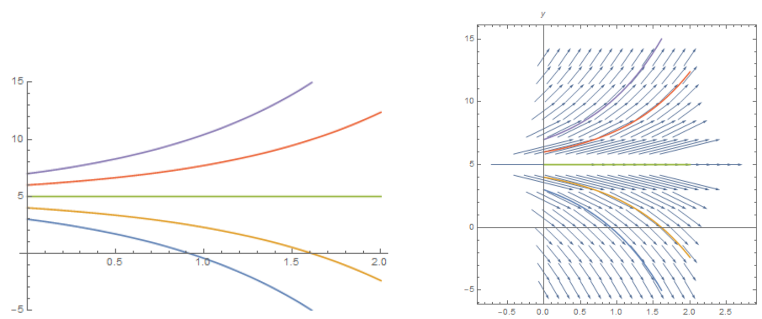

Figure 3: Illustration to problem P3: solutions for different y_0 and (optional graph) the same integral curves in the direction field.

P4. (3 points) Determine the order of the differential equation

(a) $(1+y^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} + \dfrac{\mathrm{d}y}{\mathrm{d}t} + y = e^t$;

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} + ty^2 = 0$;

(c) $\dfrac{\mathrm{d}^3y}{\mathrm{d}t^3} + t\dfrac{\mathrm{d}y}{\mathrm{d}t} + y\cos^2 t = t^3$;

also state whether it is linear or nonlinear.

Solution:

(a) The differential equation has 2nd order since the highest derivative in the equation is of order 2. The equation is nonlinear due to the nonlinear term y^2 (as well as due to the term y^2 multiplying the y'' term).

(b) The differential equation has 1st order since the highest derivative in the equation is of order 1. The equation is nonlinear due to the nonlinear term y^2.

(c) The differential equation has 3rd order since the highest derivative in the equation is of order 3. The equation is linear because the left hand side is a linear function of y and its derivatives, and the right hand side is only a function of t.