(a) Assuming that a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ exists, show that it must satisfy the integral equation

$$\boldsymbol{\phi}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\boldsymbol{\phi}(s)\,ds. \tag{ii}$$

(b) Start with the initial approximation $\boldsymbol{\phi}^{(0)}(t) = \mathbf{x}^0$. Substitute this expression for $\boldsymbol{\phi}(s)$ in the right side of Eq. (ii) and obtain a new approximation $\boldsymbol{\phi}^{(1)}(t)$. Show that

$$\boldsymbol{\phi}^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0. \tag{iii}$$

(c) Repeat this process and thereby obtain a sequence of approximations $\boldsymbol{\phi}^{(0)}$, $\boldsymbol{\phi}^{(1)}$, $\boldsymbol{\phi}^{(2)}, \dots, \boldsymbol{\phi}^{(n)}, \dots$. Use an inductive argument to show that

$$\boldsymbol{\phi}^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0. \tag{iv}$$

(d) Let $n \to \infty$ and show that the solution of the initial value problem (i) is

$$\boldsymbol{\phi}(t) = \exp(\mathbf{A}t)\mathbf{x}^0. \tag{v}$$

7.8 Repeated Eigenvalues

We conclude our consideration of the linear homogeneous system with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

with a discussion of the case in which the matrix $\mathbf{A}$ has a repeated eigenvalue. Recall that in Section 7.3 we stated that a repeated eigenvalue with algebraic multiplicity $m \geq 2$ may have a geometric multiplicity less than m. In other words, there may be fewer than m linearly independent eigenvectors associated with this eigenvalue. The following example illustrates this possibility.

EXAMPLE 1

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}. \tag{2}$$

The eigenvalues r and eigenvectors $\boldsymbol{\xi}$ satisfy the equation $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, or

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2 = 0. \tag{4}$$

Thus the two eigenvalues are $r_1 = 2$ and $r_2 = 2$; that is, the eigenvalue 2 has algebraic multiplicity 2.

To determine the eigenvectors, we must return to Eq. (3) and use for r the value 2. This gives

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

Hence we obtain the single condition $\xi_1 + \xi_2 = 0$, which determines ξ_2 in terms of ξ_1, or vice versa. Thus the eigenvector corresponding to the eigenvalue $r = 2$ is

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{6}$$

or any nonzero multiple of this vector. Observe that there is only one linearly independent eigenvector associated with the double eigenvalue.

Returning to the system (1), suppose that $r = \rho$ is an m-fold root of the characteristic equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{7}$$

Then ρ is an eigenvalue of algebraic multiplicity m of the matrix $\mathbf{A}$. In this event, there are two possibilities: either there are m linearly independent eigenvectors corresponding to the eigenvalue ρ, or else there are fewer than m such eigenvectors.

In the first case, let $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(m)}$ be m linearly independent eigenvectors associated with the eigenvalue ρ of algebraic multiplicity m. Then there are m linearly independent solutions $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{\rho t}, \dots, \mathbf{x}^{(m)}(t) = \boldsymbol{\xi}^{(m)}e^{\rho t}$ of Eq. (1). Thus in this case it makes no difference that the eigenvalue $r = \rho$ is repeated; there is still a fundamental set of solutions of Eq. (1) of the form $\boldsymbol{\xi}e^{rt}$. This case always occurs if the coefficient matrix $\mathbf{A}$ is Hermitian (or real and symmetric).

However, if the coefficient matrix is not Hermitian, then there may be fewer than m independent eigenvectors corresponding to an eigenvalue ρ of algebraic multiplicity m, and if so, there will be fewer than m solutions of Eq. (1) of the form $\boldsymbol{\xi}e^{\rho t}$ associated with this eigenvalue. Therefore, to construct the general solution of Eq. (1), it is necessary to find other solutions of a different form. Recall that a similar situation occurred in Section 3.4 for the linear equation $ay'' + by' + cy = 0$ when the characteristic equation has a double root r. In that case we found one exponential solution $y_1(t) = e^{rt}$, but a second independent solution had the form $y_2(t) = te^{rt}$. With that result in mind, consider the following example.

EXAMPLE 2

Find a fundamental set of solutions of

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \tag{8}$$

and draw a phase portrait for this system.

A direction field for the system (8) is shown in Figure 7.8.1. From this figure it appears that all nonzero solutions depart from the origin.

To solve the system, observe that the coefficient matrix $\mathbf{A}$ is the same as the matrix in Example 1. Thus we know that $r = 2$ is a double eigenvalue and that it has only a single corresponding eigenvector, which we may take as $\boldsymbol{\xi}^T = (1, -1)$. Thus one solution of the system (8) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \tag{9}$$

but there is no second solution of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$.

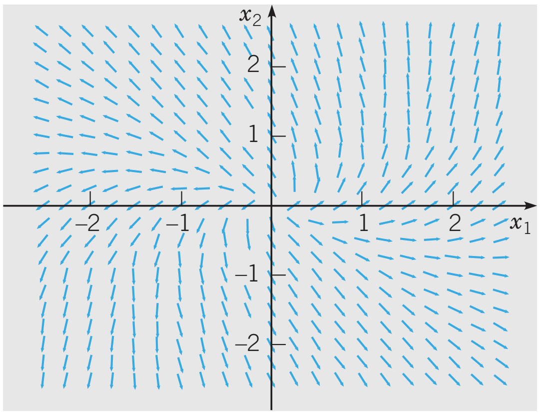


FIGURE 7.8.1 A direction field for the system (8).

Based on the procedure used for second order linear equations in Section 3.4, it may be natural to attempt to find a second independent solution of the system (8) of the form

$$\mathbf{x} = \boldsymbol{\xi} t e^{2t}, \tag{10}$$

where $\boldsymbol{\xi}$ is a constant vector to be determined. Substituting for $\mathbf{x}$ in Eq. (8), we obtain

$$2\boldsymbol{\xi} t e^{2t} + \boldsymbol{\xi} e^{2t} - \mathbf{A}\boldsymbol{\xi} t e^{2t} = \mathbf{0}. \tag{11}$$

For Eq. (11) to be satisfied for all t, it is necessary for the coefficients of te^{2t} and e^{2t} both to be zero. From the term in e^{2t} we find that

$$\boldsymbol{\xi} = \mathbf{0}. \tag{12}$$

Hence there is no nonzero solution of the system (8) of the form (10).

Since Eq. (11) contains terms in both te^{2t} and e^{2t}, it appears that in addition to $\boldsymbol{\xi} te^{2t}$, the second solution must contain a term of the form $\boldsymbol{\eta} e^{2t}$; in other words, we need to assume that

$$\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}, \tag{13}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are constant vectors to be determined. Upon substituting this expression for $\mathbf{x}$ in Eq. (8), we obtain

$$2\boldsymbol{\xi} t e^{2t} + (\boldsymbol{\xi} + 2\boldsymbol{\eta}) e^{2t} = \mathbf{A}(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}). \tag{14}$$

Equating coefficients of te^{2t} and e^{2t} on each side of Eq. (14) gives the two conditions

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \tag{15}$$

and

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi} \tag{16}$$

for the determination of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Equation (15) is satisfied if $\boldsymbol{\xi}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $r = 2$, such as $\boldsymbol{\xi}^T = (1, -1)$. Since $\det(\mathbf{A} - 2\mathbf{I})$ is zero, Eq. (16) is solvable only if the right side $\boldsymbol{\xi}$ satisfies a certain condition. Fortunately, $\boldsymbol{\xi}$ and its multiples are exactly the vectors that allow Eq. (16) to be solved. The augmented matrix for Eq. (16) is

$$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right).$$

The second row of this matrix is proportional to the first, so the system is solvable. We have

$$-\eta_1 - \eta_2 = 1,$$

so if $\eta_1 = k$, where k is arbitrary, then $\eta_2 = -k - 1$. If we write

$$\boldsymbol{\eta} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{17}$$

then by substituting for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in Eq. (13), we obtain

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}. \tag{18}$$

The last term in Eq. (18) is merely a multiple of the first solution $\mathbf{x}^{(1)}(t)$ and may be ignored, but the first two terms constitute a new solution:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}. \tag{19}$$

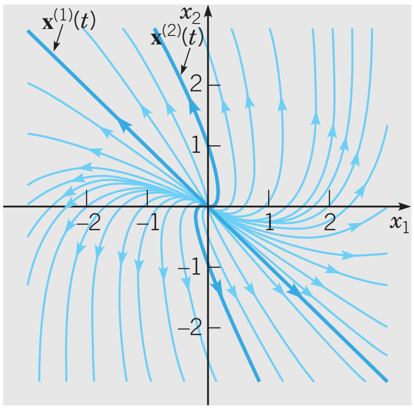
An elementary calculation shows that $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = -e^{4t} \neq 0$, and therefore $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions of the system (8). The general solution is

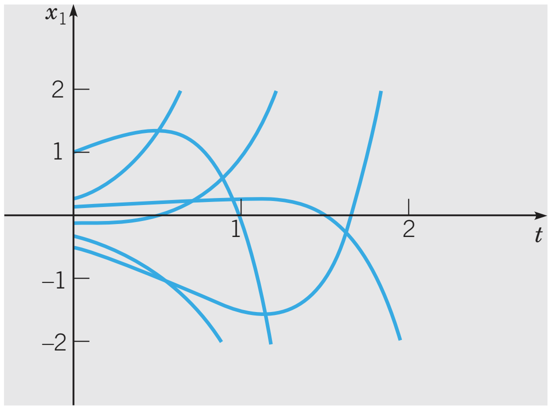
$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \\ &= c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}\right]. \end{aligned} \tag{20}$$

The main features of a phase portrait for the solution (20) follow from the presence of the exponential factor e^{2t} in every term. Therefore $\mathbf{x} \to \mathbf{0}$ as $t \to -\infty$ and, unless both c_1 and c_2 are zero, $\mathbf{x}$ becomes unbounded as $t \to \infty$. If c_1 and c_2 are not both zero, then along any trajectory we have

$$\lim_{t\to-\infty} \frac{x_2(t)}{x_1(t)} = \lim_{t\to-\infty} \frac{-c_1 - c_2 t - c_2}{c_1 + c_2 t} = -1.$$

Therefore, as $t \to -\infty$, every trajectory approaches the origin tangent to the line $x_2 = -x_1$ determined by the eigenvector; this behavior is clearly evident in Figure 7.8.2*a*. Further, as $t \to \infty$, the slope of each trajectory also approaches -1. However, it is possible to show that trajectories do not approach asymptotes as $t \to \infty$. Several trajectories of the system (8) are shown in Figure 7.8.2*a*, and some typical plots of x_1 versus t are shown in Figure 7.8.2*b*. The pattern of trajectories in this figure is typical of 2×2 systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with equal eigenvalues and only one independent eigenvector. The origin is called an **improper node** in this case. If the eigenvalues are negative, then the trajectories are similar but are traversed in the inward direction. An improper node is asymptotically stable or unstable, depending on whether the eigenvalues are negative or positive.





(*a*) (*b*)

FIGURE 7.8.2 (*a*) Phase portrait of the system (8); the origin is an improper node. (*b*) Plots of x_1 versus t for the system (8).

One difference between a system of two first order equations and a single second order equation is evident from the preceding example. For a second order linear equation with a repeated root r_1 of the characteristic equation, a term $ce^{r_1 t}$ in the second solution is not required since it is a multiple of the first solution. On the other hand, for a system of two first order equations, the term $\boldsymbol{\eta} e^{r_1 t}$ of Eq. (13) with $r_1 = 2$ is not, in general, a multiple of the first solution $\boldsymbol{\xi} e^{r_1 t}$, so the term $\boldsymbol{\eta} e^{r_1 t}$ must be retained.

Example 2 is entirely typical of the general case when there is a double eigenvalue and a single associated eigenvector. Consider again the system (1), and suppose that $r = \rho$ is a double eigenvalue of $\mathbf{A}$, but that there is only one corresponding eigenvector $\boldsymbol{\xi}$. Then one solution [similar to Eq. (9)] is

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{\rho t}, \tag{21}$$

where $\boldsymbol{\xi}$ satisfies

$$(\mathbf{A} - \rho \mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{22}$$

By proceeding as in Example 2, we find that a second solution [similar to Eq. (19)] is

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{\rho t} + \boldsymbol{\eta} e^{\rho t}, \tag{23}$$

where $\boldsymbol{\xi}$ satisfies Eq. (22) and $\boldsymbol{\eta}$ is determined from

$$(\mathbf{A} - \rho \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{24}$$

Even though $\det(\mathbf{A} - \rho \mathbf{I}) = 0$, it can be shown that it is always possible to solve Eq. (24) for $\boldsymbol{\eta}$. Note that if we multiply Eq. (24) by $\mathbf{A} - \rho \mathbf{I}$ and use Eq. (22), then we obtain

$$(\mathbf{A} - \rho \mathbf{I})^2 \boldsymbol{\eta} = \mathbf{0}.$$

The vector $\boldsymbol{\eta}$ is called a **generalized eigenvector** of the matrix $\mathbf{A}$ corresponding to the eigenvalue ρ.

Fundamental Matrices. As explained in Section 7.7, fundamental matrices are formed by arranging linearly independent solutions in columns. Thus, for example, a fundamental matrix for the system (8) can be formed from the solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ from Eqs. (9) and (19), respectively:

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ -1 & -1 - t \end{pmatrix}. \tag{25}$$

The particular fundamental matrix $\boldsymbol{\Phi}$ that satisfies $\boldsymbol{\Phi}(0) = \mathbf{I}$ can also be readily found from the relation $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0)$. For Eq. (8) we have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \qquad \boldsymbol{\Psi}^{-1}(0) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \tag{26}$$

and then

$$\begin{aligned} \boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) &= e^{2t} \begin{pmatrix} 1 & t \\ -1 & -1 - t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} 1 - t & -t \\ t & 1 + t \end{pmatrix}. \end{aligned} \tag{27}$$

The latter matrix is also the exponential matrix $\exp(\mathbf{A}t)$.

Jordan Forms. An $n \times n$ matrix $\mathbf{A}$ can be diagonalized as discussed in Section 7.7 only if it has a full complement of n linearly independent eigenvectors. If there is a shortage of eigenvectors (because of repeated eigenvalues), then $\mathbf{A}$ can always be transformed into a nearly diagonal matrix called its Jordan[6] form, which has the eigenvalues of $\mathbf{A}$ on the main diagonal, ones in certain positions on the diagonal above the main diagonal, and zeros elsewhere.

Consider again the matrix $\mathbf{A}$ given by Eq. (2). To transform $\mathbf{A}$ into its Jordan form, we construct the transformation matrix $\mathbf{T}$ with the single eigenvector $\boldsymbol{\xi}$ from Eq. (6) in its first column and the generalized eigenvector $\boldsymbol{\eta}$ from Eq. (17) with $k = 0$ in the second column. Then $\mathbf{T}$ and its inverse are given by

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \qquad \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{28}$$

As you can verify, it follows that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \mathbf{J}. \tag{29}$$

The matrix $\mathbf{J}$ in Eq. (29) is the Jordan form of $\mathbf{A}$. It is typical of all Jordan forms in that it has a 1 above the main diagonal in the column corresponding to the eigenvector that is lacking (and is replaced in $\mathbf{T}$ by the generalized eigenvector).

If we start again from Eq. (1)

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

[6]Marie Ennemond Camille Jordan (1838–1922) was professor at the École Polytechnique and the Collège de France. He is known for his important contributions to analysis and to topology (the Jordan curve theorem) and especially for his foundational work in group theory. The Jordan form of a matrix appeared in his influential book *Traité des substitutions et des équations algébriques*, published in 1870.

the transformation $\mathbf{x} = \mathbf{Ty}$, where $\mathbf{T}$ is given by Eq. (28), produces the system

$$\mathbf{y}' = \mathbf{Jy}, \tag{30}$$

where $\mathbf{J}$ is given by Eq. (29). In scalar form the system (30) is

$$y_1' = 2y_1 + y_2, \qquad y_2' = 2y_2. \tag{31}$$

These equations can be solved readily in reverse order—that is, by starting with the equation for y_2. In this way we obtain

$$y_2 = c_1 e^{2t}, \qquad y_1 = c_1 te^{2t} + c_2 e^{2t}. \tag{32}$$

Thus two independent solutions of the system (30) are

$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \qquad \mathbf{y}^{(2)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t}, \tag{33}$$

and the corresponding fundamental matrix is

$$\hat{\boldsymbol{\Psi}}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}. \tag{34}$$

Since $\hat{\boldsymbol{\Psi}}(0) = \mathbf{I}$, we can also identify the matrix in Eq. (34) as $\exp(\mathbf{J}t)$. The same result can be reached by calculating powers of $\mathbf{J}$ and substituting them into the exponential series (see Problems 20 through 22). To obtain a fundamental matrix for the original system, we now form the product

$$\boldsymbol{\Psi}(t) = \mathbf{T}\exp(\mathbf{J}t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -e^{2t} - te^{2t} \end{pmatrix}, \tag{35}$$

which is the same as the fundamental matrix given in Eq. (25).

We will not discuss $n \times n$ systems $\mathbf{x}' = \mathbf{Ax}$ in more detail here. For large n it is possible that there may be eigenvalues of high algebraic multiplicity m, perhaps with much lower geometric multiplicity q, thus giving rise to $m - q$ generalized eigenvectors. For $n \geq 4$ there may also be repeated complex eigenvalues. A full discussion[7] of the Jordan form of a general $n \times n$ matrix requires a greater background in linear algebra than we assume for most readers of this book. Problems 18 through 22 ask you to explore the use of Jordan forms for systems of three equations.

The amount of arithmetic required in the analysis of a general $n \times n$ system may be prohibitive to do by hand even if n is no greater than 3 or 4. Consequently, suitable computer software should be used routinely in most cases. This does not overcome all difficulties by any means, but it does make many problems much more tractable. Finally, for a set of equations arising from modeling a physical system, it is likely that some of the elements in the coefficient matrix $\mathbf{A}$ result from measurements of some physical quantity. The inevitable uncertainties in such measurements lead to uncertainties in the values of the eigenvalues of $\mathbf{A}$. For example, in such a case it may not be clear whether two eigenvalues are actually equal or are merely close together.

[7]For example, see the books listed in the References at the end of this chapter.

PROBLEMS

In each of Problems 1 through 4:

(a) Draw a direction field and sketch a few trajectories.

(b) Describe how the solutions behave as $t \to \infty$.

(c) Find the general solution of the system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \mathbf{x}$

In each of Problems 5 and 6, find the general solution of the given system of equations.

5. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 7 through 10:

(a) Find the solution of the given initial value problem.

(b) Draw the trajectory of the solution in the x_1x_2-plane, and also draw the graph of x_1 versus t.

7. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

9. $\mathbf{x}' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the solution of the given initial value problem.

(b) Draw the corresponding trajectory in $x_1x_2x_3$-space, and also draw the graph of x_1 versus t.

11. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

In each of Problems 13 and 14, solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

13. $t\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

14. $t\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}$

15. Show that all solutions of the system

$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}$$

approach zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$. Compare this result with that of Problem 37 in Section 3.4.

16. Consider again the electric circuit in Problem 26 of Section 7.6. This circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

(a) Show that the eigenvalues are real and equal if $L = 4R^2C$.

(b) Suppose that $R = 1\ \Omega$, $C = 1$ F, and $L = 4$ H. Suppose also that $I(0) = 1$ A and $V(0) = 2$ V. Find $I(t)$ and $V(t)$.

17. Consider again the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \qquad \text{(i)}$$

that we discussed in Example 2. We found there that $\mathbf{A}$ has a double eigenvalue $r_1 = r_2 = 2$ with a single independent eigenvector $\boldsymbol{\xi}^{(1)} = (1, -1)^T$, or any nonzero multiple thereof. Thus one solution of the system (i) is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{2t}$ and a second independent solution has the form

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t},$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \qquad \text{(ii)}$$

In the text we solved the first equation for $\boldsymbol{\xi}$ and then the second equation for $\boldsymbol{\eta}$. Here we ask you to proceed in the reverse order.

(a) Show that $\boldsymbol{\eta}$ satisfies $(\mathbf{A} - 2\mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}$.

(b) Show that $(\mathbf{A} - 2\mathbf{I})^2 = \mathbf{0}$. Thus the generalized eigenvector $\boldsymbol{\eta}$ can be chosen arbitrarily, except that it must be independent of $\boldsymbol{\xi}^{(1)}$.

(c) Let $\boldsymbol{\eta} = (0, -1)^T$. Then determine $\boldsymbol{\xi}$ from the second of Eqs. (ii) and observe that $\boldsymbol{\xi} = (1, -1)^T = \boldsymbol{\xi}^{(1)}$. This choice of $\boldsymbol{\eta}$ reproduces the solution found in Example 2.

(d) Let $\boldsymbol{\eta} = (1, 0)^T$ and determine the corresponding eigenvector $\boldsymbol{\xi}$.

(e) Let $\boldsymbol{\eta} = (k_1, k_2)^T$, where k_1 and k_2 are arbitrary numbers. Then determine $\boldsymbol{\xi}$. How is it related to the eigenvector $\boldsymbol{\xi}^{(1)}$?

Eigenvalues of Multiplicity 3. If the matrix $\mathbf{A}$ has an eigenvalue of algebraic multiplicity 3, then there may be either one, two, or three corresponding linearly independent eigenvectors. The general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is different, depending on the number of eigenvectors associated with the triple eigenvalue. As noted in the text, there is no difficulty if there are three eigenvectors, since then there are three independent solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$. The following two problems illustrate the solution procedure for a triple eigenvalue with one or two eigenvectors, respectively.

18. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}. \tag{i}$$

(a) Show that $r = 2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix $\mathbf{A}$ and that there is only one corresponding eigenvector, namely,

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (a), write down one solution $\mathbf{x}^{(1)}(t)$ of the system (i). There is no other solution of the purely exponential form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$.

(c) To find a second solution, assume that $\mathbf{x} = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t}$. Show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}.$$

Since $\boldsymbol{\xi}$ has already been found in part (a), solve the second equation for $\boldsymbol{\eta}$. Neglect the multiple of $\boldsymbol{\xi}^{(1)}$ that appears in $\boldsymbol{\eta}$, since it leads only to a multiple of the first solution $\mathbf{x}^{(1)}$. Then write down a second solution $\mathbf{x}^{(2)}(t)$ of the system (i).

(d) To find a third solution, assume that $\mathbf{x} = \boldsymbol{\xi}(t^2/2)e^{2t} + \boldsymbol{\eta}te^{2t} + \boldsymbol{\zeta}e^{2t}$. Show that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}, \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}.$$

The first two equations are the same as in part (c), so solve the third equation for $\boldsymbol{\zeta}$, again neglecting the multiple of $\boldsymbol{\xi}^{(1)}$ that appears. Then write down a third solution $\mathbf{x}^{(3)}(t)$ of the system (i).

(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system (i).

(f) Form a matrix $\mathbf{T}$ with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and the generalized eigenvectors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ in the second and third columns. Then find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

19. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}. \tag{i}$$

(a) Show that $r = 1$ is a triple eigenvalue of the coefficient matrix $\mathbf{A}$ and that there are only two linearly independent eigenvectors, which we may take as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}. \tag{ii}$$

Write down two linearly independent solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ of Eq. (i).

(b) To find a third solution, assume that $\mathbf{x} = \boldsymbol{\xi}te^t + \boldsymbol{\eta}e^t$; then show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ must satisfy

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \tag{iii}$$

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{iv}$$

(c) Equation (iii) is satisfied if $\boldsymbol{\xi}$ is an eigenvector, so one way to proceed is to choose $\boldsymbol{\xi}$ to be a suitable linear combination of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ so that Eq. (iv) is solvable, and then to solve that equation for $\boldsymbol{\eta}$. However, let us proceed in a different way and follow the pattern of Problem 17. First, show that $\boldsymbol{\eta}$ satisfies

$$(\mathbf{A} - \mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}.$$

Further, show that $(\mathbf{A} - \mathbf{I})^2 = \mathbf{0}$. Thus $\boldsymbol{\eta}$ can be chosen arbitrarily, except that it must be independent of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.

(d) A convenient choice for $\boldsymbol{\eta}$ is $\boldsymbol{\eta} = (0, 0, 1)^T$. Find the corresponding $\boldsymbol{\xi}$ from Eq. (iv). Verify that $\boldsymbol{\xi}$ is an eigenvector.

(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system (i).

(f) Form a matrix $\mathbf{T}$ with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and with the eigenvector $\boldsymbol{\xi}$ from part (d) and the generalized eigenvector $\boldsymbol{\eta}$ in the other two columns. Find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

20. Let $\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where λ is an arbitrary real number.

(a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

(b) Use an inductive argument to show that $\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

(c) Determine $\exp(\mathbf{J}t)$.

(d) Use $\exp(\mathbf{J}t)$ to solve the initial value problem $\mathbf{x}' = \mathbf{J}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

21. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

(a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

(b) Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(c) Determine $\exp(\mathbf{J}t)$.

(d) Observe that if you choose $\lambda = 1$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 19(f). Using the matrix $\mathbf{T}$ from Problem 19(f), form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 1$. Is the resulting matrix the same as the fundamental matrix $\boldsymbol{\Psi}(t)$ in Problem 19(e)? If not, explain the discrepancy.

22. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

(a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

(b) Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & [n(n-1)/2]\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(c) Determine $\exp(\mathbf{J}t)$.

(d) Note that if you choose $\lambda = 2$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 18(f). Using the matrix $\mathbf{T}$ from Problem 18(f), form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 2$. The resulting matrix is the same as the fundamental matrix $\mathbf{\Psi}(t)$ in Problem 18(e).

7.9 Nonhomogeneous Linear Systems

In this section we turn to the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \tag{1}$$

where the $n \times n$ matrix $\mathbf{P}(t)$ and $n \times 1$ vector $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. By the same argument as in Section 3.5 (see also Problem 16 in this section), the general solution of Eq. (1) can be expressed as

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t), \tag{2}$$

where $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system (1). We will briefly describe several methods for determining $\mathbf{v}(t)$.

Diagonalization. We begin with systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \tag{3}$$

where $\mathbf{A}$ is an $n \times n$ diagonalizable constant matrix. By diagonalizing the coefficient matrix $\mathbf{A}$, as indicated in Section 7.7, we can transform Eq. (3) into a system of equations that is readily solvable.

Let $\mathbf{T}$ be the matrix whose columns are the eigenvectors $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ of $\mathbf{A}$, and define a new dependent variable $\mathbf{y}$ by

$$\mathbf{x} = \mathbf{T}\mathbf{y}. \tag{4}$$

Then, substituting for $\mathbf{x}$ in Eq. (3), we obtain

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t).$$

When we multiply by $\mathbf{T}^{-1}$, it follows that

$$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{h}(t), \tag{5}$$

where $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$ and where $\mathbf{D}$ is the diagonal matrix whose diagonal entries are the eigenvalues $r_1, \dots, r_n$ of $\mathbf{A}$, arranged in the same order as the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ that appear as columns of $\mathbf{T}$. Equation (5) is a system of *n uncoupled* equations for $y_1(t), \dots, y_n(t)$; as a consequence, the equations can be solved separately. In scalar form, Eq. (5) has the form

$$y_j'(t) = r_j y_j(t) + h_j(t), \qquad j = 1, \dots, n, \tag{6}$$

where $h_j(t)$ is a certain linear combination of $g_1(t), \dots, g_n(t)$. Equation (6) is a first order linear equation and can be solved by the methods of Section 2.1. In fact, we have

$$y_j(t) = e^{r_j t} \int_{t_0}^{t} e^{-r_j s} h_j(s)\, ds + c_j e^{r_j t}, \qquad j = 1, \dots, n, \tag{7}$$

where the c_j are arbitrary constants. Finally, the solution $\mathbf{x}$ of Eq. (3) is obtained from Eq. (4). When multiplied by the transformation matrix $\mathbf{T}$, the second term on the right side of Eq. (7) produces the general solution of the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, while the first term on the right side of Eq. (7) yields a particular solution of the nonhomogeneous system (3).

EXAMPLE 1

Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{8}$$

Proceeding as in Section 7.5, we find that the eigenvalues of the coefficient matrix are $r_1 = -3$ and $r_2 = -1$ and that the corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{9}$$

Thus the general solution of the homogeneous system is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \tag{10}$$

Before writing down the matrix $\mathbf{T}$ of eigenvectors, we recall that eventually we must find $\mathbf{T}^{-1}$. The coefficient matrix $\mathbf{A}$ is real and symmetric, so we can use the result stated just above Example 3 in Section 7.7: $\mathbf{T}^{-1}$ is simply the adjoint or (since $\mathbf{T}$ is real) the transpose of $\mathbf{T}$, provided that the eigenvectors of $\mathbf{A}$ are normalized so that $(\boldsymbol{\xi}, \boldsymbol{\xi}) = 1$. Hence, upon normalizing $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$, we have

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{11}$$

Letting $\mathbf{x} = \mathbf{T}\mathbf{y}$ and substituting for $\mathbf{x}$ in Eq. (8), we obtain the following system of equations for the new dependent variable $\mathbf{y}$:

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} + \frac{1}{\sqrt{2}} \begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}. \tag{12}$$

Thus

$$\begin{aligned} y_1' + 3y_1 &= \sqrt{2}e^{-t} - \frac{3}{\sqrt{2}}t, \\ y_2' + y_2 &= \sqrt{2}e^{-t} + \frac{3}{\sqrt{2}}t. \end{aligned} \tag{13}$$

Each of Eqs. (13) is a first order linear equation and so can be solved by the methods of Section 2.1. In this way we obtain

$$
\begin{aligned}
y_1 &= \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}\left[\left(\frac{t}{3}\right) - \frac{1}{9}\right] + c_1 e^{-3t}, \\
y_2 &= \sqrt{2}te^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2 e^{-t}.
\end{aligned} \tag{14}
$$

Finally, we write the solution in terms of the original variables:

$$
\begin{aligned}
\mathbf{x} = \mathbf{T}\mathbf{y} &= \frac{1}{\sqrt{2}}\begin{pmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} (c_1/\sqrt{2})e^{-3t} + \left[(c_2/\sqrt{2}) + \frac{1}{2}\right]e^{-t} + t - \frac{4}{3} + te^{-t} \\ -(c_1/\sqrt{2})e^{-3t} + \left[(c_2/\sqrt{2}) - \frac{1}{2}\right]e^{-t} + 2t - \frac{5}{3} + te^{-t} \end{pmatrix} \\
&= k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix},
\end{aligned} \tag{15}
$$

where $k_1 = c_1/\sqrt{2}$ and $k_2 = c_2/\sqrt{2}$. The first two terms on the right side of Eq. (15) form the general solution of the homogeneous system corresponding to Eq. (8). The remaining terms are a particular solution of the nonhomogeneous system.

If the coefficient matrix $\mathbf{A}$ in Eq. (3) is not diagonalizable (because of repeated eigenvalues and a shortage of eigenvectors), it can nevertheless be reduced to a Jordan form $\mathbf{J}$ by a suitable transformation matrix $\mathbf{T}$ involving both eigenvectors and generalized eigenvectors. In this case the differential equations for $y_1, \dots, y_n$ are not totally uncoupled since some rows of $\mathbf{J}$ have two nonzero elements: an eigenvalue in the diagonal position and a 1 in the adjacent position to the right. However, the equations for $y_1, \dots, y_n$ can still be solved consecutively, starting with y_n. Then the solution of the original system (3) can be found by the relation $\mathbf{x} = \mathbf{T}\mathbf{y}$.

Undetermined Coefficients. A second way to find a particular solution of the nonhomogeneous system (1) is the method of undetermined coefficients that we first discussed in Section 3.5. To use this method, we assume the form of the solution with some or all of the coefficients unspecified, and then seek to determine these coefficients so as to satisfy the differential equation. As a practical matter, this method is applicable only if the coefficient matrix $\mathbf{P}$ is a constant matrix, and if the components of $\mathbf{g}$ are polynomial, exponential, or sinusoidal functions, or sums or products of these. In these cases the correct form of the solution can be predicted in a simple and systematic manner. The procedure for choosing the form of the solution is substantially the same as that given in Section 3.5 for linear second order equations. The main difference is illustrated by the case of a nonhomogeneous term of the form $\mathbf{u}e^{\lambda t}$, where λ is a simple root of the characteristic equation. In this situation, rather than assuming a solution of the form $\mathbf{a}te^{\lambda t}$, it is necessary to use $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$, where $\mathbf{a}$ and $\mathbf{b}$ are determined by substituting into the differential equation.

EXAMPLE 2

Use the method of undetermined coefficients to find a particular solution of

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{16}$$

This is the same system of equations as in Example 1. To use the method of undetermined coefficients, we write $\mathbf{g}(t)$ in the form

$$\mathbf{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t. \tag{17}$$

Observe that $r = -1$ is an eigenvalue of the coefficient matrix, and therefore we must include both $\mathbf{a}te^{-t}$ and $\mathbf{b}e^{-t}$ in the assumed solution. Thus we assume that

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}, \tag{18}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are vectors to be determined. By substituting Eq. (18) into Eq. (16) and collecting terms, we obtain the following algebraic equations for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$:

$$\begin{aligned} \mathbf{A}\mathbf{a} &= -\mathbf{a}, \\ \mathbf{A}\mathbf{b} &= \mathbf{a} - \mathbf{b} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \\ \mathbf{A}\mathbf{c} &= -\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \\ \mathbf{A}\mathbf{d} &= \mathbf{c}. \end{aligned} \tag{19}$$

From the first of Eqs. (19), we see that $\mathbf{a}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $r = -1$. Thus $\mathbf{a}^T = (\alpha, \alpha)$, where α is any nonzero constant. Then we find that the second of Eqs. (19) can be solved only if $\alpha = 1$ and that in this case,

$$\mathbf{b} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$

for any constant k. The simplest choice is $k = 0$, from which $\mathbf{b}^T = (0, -1)$. Then the third and fourth of Eqs. (19) yield $\mathbf{c}^T = (1, 2)$ and $\mathbf{d}^T = (-\frac{4}{3}, -\frac{5}{3})$, respectively. Finally, from Eq. (18) we obtain the particular solution

$$\mathbf{v}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{21}$$

The particular solution (21) is not identical to the one contained in Eq. (15) of Example 1 because the term in e^{-t} is different. However, if we choose $k = \frac{1}{2}$ in Eq. (20), then $\mathbf{b}^T = (\frac{1}{2}, -\frac{1}{2})$ and the two particular solutions agree.

Variation of Parameters. Now let us turn to more general problems in which the coefficient matrix is not constant or not diagonalizable. Let

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \tag{22}$$

where $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on $\alpha < t < \beta$. Assume that a fundamental matrix $\boldsymbol{\Psi}(t)$ for the corresponding homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{23}$$

has been found. We use the method of variation of parameters to construct a particular solution, and hence the general solution, of the nonhomogeneous system (22).

Since the general solution of the homogeneous system (23) is $\mathbf{\Psi}(t)\mathbf{c}$, it is natural to proceed as in Section 3.6 and to seek a solution of the nonhomogeneous system (22) by replacing the constant vector $\mathbf{c}$ by a vector function $\mathbf{u}(t)$. Thus we assume that

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t), \tag{24}$$

where $\mathbf{u}(t)$ is a vector to be found. Upon differentiating $\mathbf{x}$ as given by Eq. (24) and requiring that Eq. (22) be satisfied, we obtain

$$\mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{g}(t). \tag{25}$$

Since $\mathbf{\Psi}(t)$ is a fundamental matrix, $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$; hence Eq. (25) reduces to

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t). \tag{26}$$

Recall that $\mathbf{\Psi}(t)$ is nonsingular on any interval where $\mathbf{P}$ is continuous. Hence $\mathbf{\Psi}^{-1}(t)$ exists, and therefore

$$\mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)\mathbf{g}(t). \tag{27}$$

Thus for $\mathbf{u}(t)$ we can select any vector from the class of vectors that satisfy Eq. (27). These vectors are determined only up to an arbitrary additive constant vector; therefore, we denote $\mathbf{u}(t)$ by

$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt + \mathbf{c}, \tag{28}$$

where the constant vector $\mathbf{c}$ is arbitrary. If the integrals in Eq. (28) can be evaluated, then the general solution of the system (22) is found by substituting for $\mathbf{u}(t)$ from Eq. (28) in Eq. (24). However, even if the integrals cannot be evaluated, we can still write the general solution of Eq. (22) in the form

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int_{t_1}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\,ds, \tag{29}$$

where t_1 is any point in the interval (α, β). The first term on the right side of Eq. (29) is the general solution of the corresponding homogeneous system (23), and the second term is a particular solution of Eq. (22).

Now let us consider the initial value problem consisting of the differential equation (22) and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}^0. \tag{30}$$

We can find the solution of this problem most conveniently if we choose the lower limit of integration in Eq. (29) to be the initial point t_0. Then the general solution of the differential equation is

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)\,ds. \tag{31}$$

For $t = t_0$ the integral in Eq. (31) is zero, so the initial condition (30) is also satisfied if we choose

$$\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0. \tag{32}$$

Therefore,

$$\mathbf{x} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}^0 + \boldsymbol{\Psi}(t)\int_{t_0}^{t} \boldsymbol{\Psi}^{-1}(s)\mathbf{g}(s)\,ds \tag{33}$$

is the solution of the given initial value problem. Again, although it is helpful to use $\boldsymbol{\Psi}^{-1}$ to write the solutions (29) and (33), it is usually better in particular cases to solve the necessary equations by row reduction than to calculate $\boldsymbol{\Psi}^{-1}$ and substitute into Eqs. (29) and (33).

The solution (33) takes a slightly simpler form if we use the fundamental matrix $\boldsymbol{\Phi}(t)$ satisfying $\boldsymbol{\Phi}(t_0) = \mathbf{I}$. In this case we have

$$\mathbf{x} = \boldsymbol{\Phi}(t)\mathbf{x}^0 + \boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}^{-1}(s)\mathbf{g}(s)\,ds. \tag{34}$$

Equation (34) can be simplified further if the coefficient matrix $\mathbf{P}(t)$ is a constant matrix (see Problem 17).

EXAMPLE 3

Use the method of variation of parameters to find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{35}$$

This is the same system of equations as in Examples 1 and 2.

The general solution of the corresponding homogeneous system was given in Eq. (10). Thus

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \tag{36}$$

is a fundamental matrix. Then the solution $\mathbf{x}$ of Eq. (35) is given by $\mathbf{x} = \boldsymbol{\Psi}(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ satisfies $\boldsymbol{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$, or

$$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}. \tag{37}$$

Solving Eq. (37) by row reduction, we obtain

$$u_1' = e^{2t} - \tfrac{3}{2}te^{3t},$$
$$u_2' = 1 + \tfrac{3}{2}te^{t}.$$

Hence

$$u_1(t) = \tfrac{1}{2}e^{2t} - \tfrac{1}{2}te^{3t} + \tfrac{1}{6}e^{3t} + c_1,$$
$$u_2(t) = t + \tfrac{3}{2}te^{t} - \tfrac{3}{2}e^{t} + c_2,$$

and

$$\begin{aligned}\mathbf{x} &= \boldsymbol{\Psi}(t)\mathbf{u}(t)\\ &= c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix},\end{aligned} \tag{38}$$

which is the same as the solution obtained in Example 1 [Eq. (15)] and is equivalent to the solution obtained in Example 2 [Eq. (21)].

Laplace Transforms. We used the Laplace transform in Chapter 6 to solve linear equations of arbitrary order. It can also be used in very much the same way to solve systems of equations. Since the transform is an integral, the transform of a vector is computed component by component. Thus $\mathcal{L}\{\mathbf{x}(t)\}$ is the vector whose components are the transforms of the respective components of $\mathbf{x}(t)$, and similarly for $\mathcal{L}\{\mathbf{x}'(t)\}$. We will denote $\mathcal{L}\{\mathbf{x}(t)\}$ by $\mathbf{X}(s)$. Then, by an extension of Theorem 6.2.1 to vectors, we also have

$$\mathcal{L}\{\mathbf{x}'(t)\} = s\mathbf{X}(s) - \mathbf{x}(0). \tag{39}$$

EXAMPLE 4

Use the method of Laplace transforms to solve the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{40}$$

This is the same system of equations as in Examples 1 through 3.

We take the Laplace transform of each term in Eq. (40), obtaining

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s), \tag{41}$$

where $\mathbf{G}(s)$ is the transform of $\mathbf{g}(t)$. The transform $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \begin{pmatrix} 2/(s+1) \\ 3/s^2 \end{pmatrix}. \tag{42}$$

To proceed further we need to choose the initial vector $\mathbf{x}(0)$. For simplicity let us choose $\mathbf{x}(0) = \mathbf{0}$. Then Eq. (41) becomes

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{G}(s), \tag{43}$$

where, as usual, $\mathbf{I}$ is the identity matrix. Consequently, $\mathbf{X}(s)$ is given by

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s). \tag{44}$$

The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is called the **transfer matrix** because multiplying it by the transform of the input vector $\mathbf{g}(t)$ yields the transform of the output vector $\mathbf{x}(t)$. In this example we have

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}, \tag{45}$$

and by a straightforward calculation, we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)}\begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}. \tag{46}$$

Then, substituting from Eqs. (42) and (46) in Eq. (44) and carrying out the indicated multiplication, we find that

$$\mathbf{X}(s) = \begin{pmatrix} \dfrac{2(s+2)}{(s+1)^2(s+3)} + \dfrac{3}{s^2(s+1)(s+3)} \\[2ex] \dfrac{2}{(s+1)^2(s+3)} + \dfrac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}. \tag{47}$$

Finally, we need to obtain the solution $\mathbf{x}(t)$ from its transform $\mathbf{X}(s)$. This can be done by expanding the expressions in Eq. (47) in partial fractions and using Table 6.2.1, or (more efficiently) by using appropriate computer software. In any case, after some simplification the result is

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{48}$$

Equation (48) gives the particular solution of the system (40) that satisfies the initial condition $\mathbf{x}(0) = \mathbf{0}$. As a result, it differs slightly from the particular solutions obtained in the preceding three examples. To obtain the general solution of Eq. (40), you must add to the expression in Eq. (48) the general solution (10) of the homogeneous system corresponding to Eq. (40).

Each of the methods for solving nonhomogeneous equations has some advantages and disadvantages. The method of undetermined coefficients requires no integration, but it is limited in scope and may entail the solution of several sets of algebraic equations. The method of diagonalization requires finding the inverse of the transformation matrix and the solution of a set of uncoupled first order linear equations, followed by a matrix multiplication. Its main advantage is that for Hermitian coefficient matrices, the inverse of the transformation matrix can be written down without calculation—a feature that is more important for large systems. The method of Laplace transforms involves a matrix inversion to find the transfer matrix, followed by a multiplication, and finally by the determination of the inverse transform of each term in the resulting expression. It is particularly useful in problems with forcing functions that involve discontinuous or impulsive terms. Variation of parameters is the most general method. On the other hand, it involves the solution of a set of linear algebraic equations with variable coefficients, followed by an integration and a matrix multiplication, so it may also be the most complicated from a computational viewpoint. For many small systems with constant coefficients, such as the one in the examples in this section, all of these methods work well, and there may be little reason to select one over another.

PROBLEMS

In each of Problems 1 through 12 find the general solution of the given system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ \sqrt{3}\, e^{-t} \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$

5. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \qquad t > 0$

6. $\mathbf{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}, \qquad t > 0$

7. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$

8. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$

9. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2t \\ e^t \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

11. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}, \qquad 0 < t < \pi$

12. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}, \qquad \frac{\pi}{2} < t < \pi$

13. The electric circuit shown in Figure 7.9.1 is described by the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} I(t), \tag{i}$$

where x_1 is the current through the inductor, x_2 is the voltage drop across the capacitor, and $I(t)$ is the current supplied by the external source.

(a) Determine a fundamental matrix $\mathbf{\Psi}(t)$ for the homogeneous system corresponding to Eq. (i). Refer to Problem 25 of Section 7.6.

(b) If $I(t) = e^{-t/2}$, determine the solution of the system (i) that also satisfies the initial conditions $\mathbf{x}(0) = \mathbf{0}$.

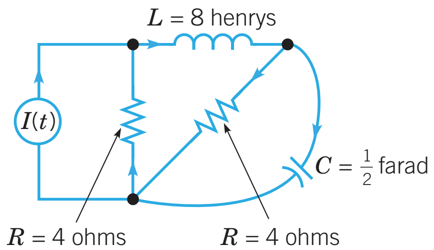


FIGURE 7.9.1 The circuit in Problem 13.

In each of Problems 14 and 15, verify that the given vector is the general solution of the corresponding homogeneous system, and then solve the nonhomogeneous system. Assume that $t > 0$.

14. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 - t^2 \\ 2t \end{pmatrix}, \qquad \mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$

15. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2t \\ t^4 - 1 \end{pmatrix}, \qquad \mathbf{x}^{(c)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2$

16. Let $\mathbf{x} = \boldsymbol{\phi}(t)$ be the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, and let $\mathbf{x} = \mathbf{v}(t)$ be some particular solution of the same system. By considering the difference $\boldsymbol{\phi}(t) - \mathbf{v}(t)$, show that $\boldsymbol{\phi}(t) = \mathbf{u}(t) + \mathbf{v}(t)$, where $\mathbf{u}(t)$ is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

17. Consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(0) = \mathbf{x}^0.$$

(a) By referring to Problem 15(c) in Section 7.7, show that

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t - s)\mathbf{g}(s)\, ds.$$

(b) Show also that

$$\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}^0 + \int_0^t \exp[\mathbf{A}(t-s)]\mathbf{g}(s)\,ds.$$

Compare these results with those of Problem 27 in Section 3.6.

18. Use the Laplace transform to solve the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t) \tag{i}$$

used in the examples in this section. Instead of using zero initial conditions, as in Example 4, let

$$\mathbf{x}(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \tag{ii}$$

where α_1 and α_2 are arbitrary. How must α_1 and α_2 be chosen so that the solution is identical to Eq. (38)?

REFERENCES

Further information on matrices and linear algebra is available in any introductory book on the subject. Here is a representative sample:

Anton, H. and Rorres, C., *Elementary Linear Algebra* (10th ed.) (Hoboken, NJ: Wiley, 2010).

Johnson, L. W., Riess, R. D., and Arnold, J. T., *Introduction to Linear Algebra* (6th ed.) (Boston: Addison-Wesley, 2008).

Kolman, B. and Hill, D. R., *Elementary Linear Algebra* (8th ed.) (Upper Saddle River, NJ: Pearson, 2004).

Lay, D. C., *Linear Algebra and Its Applications*(4th ed.)(Boston: Addison-Wesley, 2012).

Leon, S. J., *Linear Algebra with Applications* (8th ed.) (Upper Saddle River, NJ: Pearson/Prentice-Hall, 2010).

Strang, G., *Linear Algebra and Its Applications* (4th ed.)(Belmont, CA: Thomson, Brooks/Cole, 2006).

A more extended treatment of systems of first order linear equations may be found in many books, including the following:

Coddington, E. A. and Carlson, R., *Linear Ordinary Differential Equations* (Philadelphia, PA: Society for Industrial and Applied Mahematics, 1997).

Hirsch, M. W., Smale, S., and Devaney, R. L., *Differential Equations, Dynamical Systems, and an Introduction to Chaos* (2nd ed.)(San Diego, CA: Academic Press, 2004).

The following book treats elementary differential equations with a particular emphasis on systems of first order equations:

Brannan, J. R. and Boyce, W. E., *Differential Equations: An Introduction to Modern Methods and Applications* (2nd ed.) (New York: Wiley, 2011).