Introduction to Differential Equations
Assignment # 12

Date Given: June 27, 2022
Date Due: July 4, 2022

P1. (2 points)

(a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \boldsymbol{x}$$

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = r^2 - r - 2 = 0 \quad \Longrightarrow \quad r_1 = -1, r_2 = 2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

(b) If the initial condition is a multiple of $(1,2)^{\mathrm{T}}$, then the solution will tend to the origin along the eigenvector $\boldsymbol{\xi}_1$. Likewise, if the initial condition is a multiple of $(2,1)^{\mathrm{T}}$, then the solution will tend away from the origin along the eigenvector $\boldsymbol{\xi}_2$. For $c_2 \neq 0$ all other solutions will tend to infinity asymptotic to $\boldsymbol{\xi}_2$. The direction field and a few trajectories of the system are shown in Figure 1.

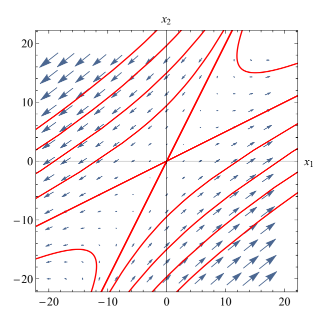

Figure 1: Illustration to problem P1.

P2. (2 points)

(a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}$$

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = r^2 + 4r + 3 = 0 \quad \Longrightarrow \quad r_1 = -1, r_2 = -3.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

(b) If the initial condition is a multiple of $(1, -1)^{\mathrm{T}}$, then the solution will tend to the origin along the eigenvector $\boldsymbol{\xi}_2$. Likewise, if the initial condition is a multiple of $(1, 1)^{\mathrm{T}}$, then the solution will tend to the origin along the eigenvector $\boldsymbol{\xi}_1$. Since e^{-t} is the dominant term as $t \to \infty$, as long as $c_1 \neq 0$, all trajectories approach the origin asymptotic to the eigenvector $\boldsymbol{\xi}_1$.

(c) The direction field and a few trajectories of the system are shown in Figure 2.

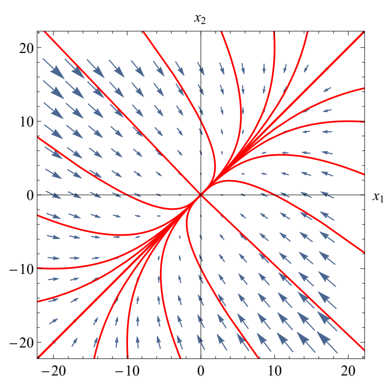

Figure 2: Illustration to problem P2.

P3. (2 points)

(a) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \boldsymbol{x}$$

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(b) Draw a direction field and plot a few trajectories of the system.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 4-r & -3 \\ 8 & -6-r \end{vmatrix} = r^2 + 2r = 0 \quad \Longrightarrow \quad r_1 = -2, r_2 = 0.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \boldsymbol{\xi} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

(b) The entire line along the eigenvector $\boldsymbol{\xi}_2 = (3, 4)^{\mathrm{T}}$ consists of equilibrium points. All other solutions converge. The direction field changes across the line $4x_1 - 3x_2 = 0$ (blue line). Eliminating the exponential terms in the solution, the trajectories are given by $2x_1 - x_2 = 2c_2$ (red lines). The direction field and a few trajectories of the system are shown in Figure 3.

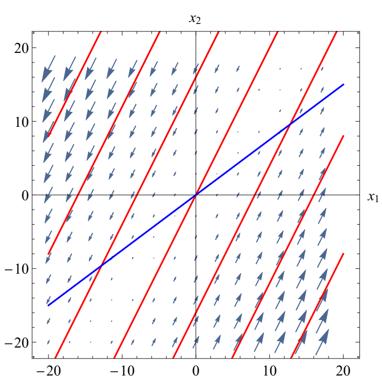

Figure 3: Illustration to problem P3.

P4. (2 points) Find the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \boldsymbol{x}$$

.

Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 3-r & 2 & 4 \\ 2 & -r & 2 \\ 4 & 2 & 3-r \end{vmatrix} = r^3 - 6r^2 - 15r - 8 = 0 \quad \Longrightarrow \quad r_1 = 8, r_2 = r_3 = -1.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system reduces to two equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0, \\ 2\xi_2 - \xi_3 &= 0 \end{aligned}$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

For $r = r_2 = r_3$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This system reduces to only one equation

$$2\xi_1 + \xi_2 + 2\xi_3 = 0.$$

and the two independent solution vectors can be set as

$$\boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \boldsymbol{\xi}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

Hence, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{-t}.$$

P5. (2 points) Solve the initial value problem

$$\boldsymbol{x}' = \begin{bmatrix} 1/2 & 0 \\ 1 & -1/2 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Solution: Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 1/2-r & 0 \\ 1 & -1/2-r \end{vmatrix} = (1/2-r)(-1/2-r) = 0 \quad \Longrightarrow \quad r_1 = -1/2, r_2 = 1/2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the corresponding solution vector can be set as

$$\boldsymbol{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2}$$

If

$$\boldsymbol{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

then $c_1 = 2, c_2 = 3$, and therefore

$$\boldsymbol{x}(t) = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2}.$$