Introduction to Differential Equations
Assignment # 13

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P1.

(a)

$x' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} x$

$x^{(1)}(t) = (a + ib)e^{(\lambda + i\mu)t}$

$= (a + ib)e^{\lambda t}(\cos \mu t + i \sin \mu t$

$x^{(1)}(t) = e^{\lambda t}(a \cos \mu t - b \sin \mu t) + ie^{\lambda t}(a \sin \mu t + b \cos \mu t)$

$u(t) = e^{\lambda t}(a \cos \mu t - b \sin \mu t), v(t) = e^{\lambda t}(a \sin \mu t + b \cos \mu t)$

$\det(A - rI) = \begin{vmatrix} 1 - r & 2 \\ -5 & -1 - r \end{vmatrix} = r^2 + 9 = 0 \Longrightarrow r_1 = -3i, r_2 = 3i$

$(A - r_1 I)\xi = \begin{bmatrix} 1 + 3i & 2 \\ -5 & -1 + 3i \end{bmatrix} \xi = 0 \Longrightarrow \xi_1 = \begin{bmatrix} 1 - 3i \\ -5 \end{bmatrix}$

$(A - r_2 I)\xi = \begin{bmatrix} 1 - 3i & 2 \\ -5 & -1 - 3i \end{bmatrix} \xi = 0 \Longrightarrow \xi_2 = \begin{bmatrix} 1 + 3i \\ -5 \end{bmatrix}$

$a = Re(\xi_1) = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, b = Im(\xi_1) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

$x(t) = c_1 \left(\begin{bmatrix} 1 \\ -5 \end{bmatrix} \cos -3t - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin -3t \right) + c_1 (\begin{bmatrix} 1 \\ -5 \end{bmatrix} \sin -3t +$
$\begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos -3t)$

$= c_1 \begin{bmatrix} \cos -3t + 3 \sin -3t \\ -5 \cos -3t \end{bmatrix} + c_1 \begin{bmatrix} \sin -3t - 3 \cos -3t \\ -5 \sin -3t \end{bmatrix}$

P2.

$x' = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} x$

$\det(A - rI) = \begin{bmatrix} 1 - r & -1 & 2 \\ -1 & 1 - r & 0 \\ -1 & 0 & 1 - r \end{bmatrix} = 3 - 4r + 3r^2 - r^3 = 0 \Longrightarrow$

$r_1 =, r_2 =$

P3.

$x' = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix} x, x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x^{(1)}(t) = (a + ib)e^{(\lambda + i\mu)t}$

$= (a + ib)e^{\lambda t}(\cos \mu t + i \sin \mu t$

$x^{(1)}(t) = e^{\lambda t}(a \cos \mu t - b \sin \mu t) + ie^{\lambda t}(a \sin \mu t + b \cos \mu t)$

$u(t) = e^{\lambda t}(a \cos \mu t - b \sin \mu t), v(t) = e^{\lambda t}(a \sin \mu t + b \cos \mu t)$

$\det(A - rI) = \begin{vmatrix} 6 - r & -1 \\ 5 & 4 - r \end{vmatrix} = r^2 - 10r + 29 = 0 \Longrightarrow r_1 = 5 + 2i, r_2 = 5 - 2i$

$(A - r_1 I)\xi = \begin{bmatrix} 1 - 2i & -1 \\ 5 & -1 - 2i \end{bmatrix} \xi = 0 \Longrightarrow \xi_1 = \begin{bmatrix} 1 + 2i \\ 5 \end{bmatrix}$

$(A - r_2 I)\xi = \begin{bmatrix} 1 + 2i & -1 \\ 5 & -1 + 2i \end{bmatrix} \xi = 0 \Longrightarrow \xi_2 = \begin{bmatrix} 1 - 2i \\ 5 \end{bmatrix}$

$a = Re(\xi_1) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, b = Im(\xi_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$

$x(t) = c_1 e^{5t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t \right) + c_1 e^{5t} (\begin{bmatrix} 1 \\ 5 \end{bmatrix} \sin 2t + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t)$

$= c_1 e^{5t} \begin{bmatrix} \cos 2t - 2 \sin 2t \\ 5 \cos 2t \end{bmatrix} + c_1 e^{5t} \begin{bmatrix} \sin 2t + 2 \cos 2t \\ 5 \sin 2t \end{bmatrix}$

$\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1, \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = -1$

$c_1 = -\frac{1}{2}, c_2 = \frac{3}{10}$

$x(t) = -\frac{1}{2} e^{5t} \begin{bmatrix} \cos 2t - 2 \sin 2t \\ 5 \cos 2t \end{bmatrix} + \frac{3}{10} e^{5t} \begin{bmatrix} \sin 2t + 2 \cos 2t \\ 5 \sin 2t \end{bmatrix}$

$x(t) = e^{5t} \begin{bmatrix} \frac{1}{10} \cos 2t + \frac{13}{10} \sin 2t \\ \frac{3}{2} \sin 2t - \frac{5}{2} \cos 2t \end{bmatrix}$

P5.

$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x$

$\det(A - rI) = \begin{vmatrix} 2 - r & -1 \\ 3 & -2 - r \end{vmatrix} = r^2 - 1 = 0 \Longrightarrow r_1 = 1, r_2 = -1$

$$(A-r_1I)\xi=\begin{bmatrix}1 & -1\\ 3 & -3\end{bmatrix}\xi=0\Longrightarrow \xi_1=\begin{bmatrix}1\\1\end{bmatrix}$$

$$(A-r_2I)\xi=\begin{bmatrix}3 & -1\\ 3 & -1\end{bmatrix}\xi=0\Longrightarrow \xi_2=\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}$$

$$x(t)=c_1\begin{bmatrix}1\\1\end{bmatrix}e^t+c_2\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}e^{-t}$$

$$\Psi(t)=\begin{bmatrix}e^t & \frac{1}{3}e^{-t}\\ e^t & e^{-t}\end{bmatrix}$$

(b)

$$\Psi(0)=\begin{bmatrix}1 & \frac{1}{3}\\ 1 & 1\end{bmatrix}$$

$$\Psi^{-1}(0)=\frac{3}{2}\begin{bmatrix}\frac{3}{2} & -\frac{1}{2}\\ -\frac{3}{2} & \frac{3}{2}\end{bmatrix}$$

$$\Phi(t)=\Psi(t)\Psi^{-1}(0)=\frac{3}{2}\begin{bmatrix}\frac{3}{2}e^t-\frac{1}{6}e^{-t} & \frac{1}{3}e^{t}-\frac{1}{2}e^{-t}\\ \frac{3}{2}e^t-\frac{3}{2}e^{-t} & -\frac{1}{2}e^{-t}+\frac{3}{2}e^{-t}\end{bmatrix}$$

P6.

$$x'=\begin{bmatrix}1 & -1\\ 5 & -3\end{bmatrix}x$$

$$x^{(1)}(t)=(a+ib)e^{(\lambda+i\mu)t}$$

$$=(a+ib)e^{\lambda t}(\cos\mu t+i\sin\mu t$$

$$x^{(1)}(t)=e^{\lambda t}(a\cos\mu t-b\sin\mu t)+ie^{\lambda t}(a\sin\mu t+b\cos\mu t)$$

$$u(t)=e^{\lambda t}(a\cos\mu t-b\sin\mu t),v(t)=e^{\lambda t}(a\sin\mu t+b\cos\mu t)$$

$$\det(A-rI)=\begin{vmatrix}1-r & -1\\ 5 & -3-r\end{vmatrix}=r^2+2r+2=0\Longrightarrow r_1=-1+$$
$$i,r_2=-1-i$$

$$(A-r_1I)\xi=\begin{bmatrix}2-i & -1\\ 5 & -2-i\end{bmatrix}\xi=0\Longrightarrow \xi_1=\begin{bmatrix}2+i\\5\end{bmatrix}$$

$$(A - r_2 I)\xi = \begin{bmatrix} 2+i & -1 \\ 5 & -2-i \end{bmatrix} \xi = 0 \Longrightarrow \xi_2 = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$$

$$a = Re(\xi_1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, b = Im(\xi_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$x(t) = c_1 e^{-t} \left(\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right) + c_1 e^{-t} \left(\begin{bmatrix} 2 \\ 5 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t \right)$$

$$= c_1 e^{-t} \begin{bmatrix} 2\cos t - \sin t \\ 5 \cos t \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} \sin t + \cos t \\ 5 \sin t \end{bmatrix}$$

$$\Psi(t) = e^{-t} \begin{bmatrix} 2\cos t - \sin t & 2\sin t + \cos t \\ 5 \cos t & 5 \sin t \end{bmatrix}$$

(b)

$$\Psi(0) = \begin{bmatrix} 2 & 1 \\ 5 & 0 \end{bmatrix}$$

$$\Psi^{-1}(0) = \frac{1}{5} \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix}$$

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = \frac{1}{5} e^{-t} \begin{bmatrix} 10 \sin t + 5\cos t & -5 \sin t \\ 25 \cos t & 5 \cos t - 10 \sin t \end{bmatrix}$$