31. Prove that $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.

32. In this problem we show that the eigenvalues of a Hermitian matrix $\mathbf{A}$ are real. Let $\mathbf{x}$ be an eigenvector corresponding to the eigenvalue λ.
    (a) Show that $(\mathbf{Ax}, \mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$. *Hint:* See Problem 26(c).
    (b) Show that $\lambda(\mathbf{x}, \mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x})$. *Hint:* Recall that $\mathbf{Ax} = \lambda\mathbf{x}$.
    (c) Show that $\lambda = \overline{\lambda}$; that is, the eigenvalue λ is real.

33. Show that if λ_1 and λ_2 are eigenvalues of a Hermitian matrix $\mathbf{A}$, and if $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
    Hint: Use the results of Problems 26(c) and 32 to show that $(\lambda_1 - \lambda_2)(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$.

34. Show that if λ_1 and λ_2 are eigenvalues of any matrix $\mathbf{A}$, and if $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent.
    Hint: Start from $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \mathbf{0}$; multiply by $\mathbf{A}$ to obtain $c_1\lambda_1\mathbf{x}^{(1)} + c_2\lambda_2\mathbf{x}^{(2)} = \mathbf{0}$. Then show that $c_1 = c_2 = 0$.

7.4 Basic Theory of Systems of First Order Linear Equations

The general theory of a system of n first order linear equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\ \vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{1}$$

closely parallels that of a single linear equation of nth order. The discussion in this section therefore follows the same general lines as that in Sections 3.2 and 4.1. To discuss the system (1) most effectively, we write it in matrix notation. That is, we consider $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ to be components of a vector $\mathbf{x} = \boldsymbol{\phi}(t)$; similarly, $g_1(t), \ldots, g_n(t)$ are components of a vector $\mathbf{g}(t)$, and $p_{11}(t), \ldots, p_{nn}(t)$ are elements of an $n \times n$ matrix $\mathbf{P}(t)$. Equation (1) then takes the form

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{2}$$

The use of vectors and matrices not only saves a great deal of space and facilitates calculations but also emphasizes the similarity between systems of equations and single (scalar) equations.

A vector $\mathbf{x} = \boldsymbol{\phi}(t)$ is said to be a solution of Eq. (2) if its components satisfy the system of equations (1). Throughout this section we assume that $\mathbf{P}$ and $\mathbf{g}$ are continuous on some interval $\alpha < t < \beta$; that is, each of the scalar functions $p_{11}, \ldots, p_{nn}, g_1, \ldots, g_n$ is continuous there. According to Theorem 7.1.2, this is sufficient to guarantee the existence of solutions of Eq. (2) on the interval $\alpha < t < \beta$.

It is convenient to consider first the homogeneous equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{3}$$

obtained from Eq. (2) by setting $\mathbf{g}(t) = \mathbf{0}$. Once the homogeneous equation has been solved, there are several methods that can be used to solve the nonhomogeneous equation (2); this is taken up in Section 7.9. We use the notation

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \quad \ldots, \quad \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}, \quad \ldots \tag{4}$$

to designate specific solutions of the system (3). Note that $x_{ij}(t) = x_i^{(j)}(t)$ refers to the ith component of the jth solution $\mathbf{x}^{(j)}(t)$. The main facts about the structure of solutions of the system (3) are stated in Theorems 7.4.1 to 7.4.5. They closely resemble the corresponding theorems in Sections 3.2 and 4.1; some of the proofs are left to you as exercises.

Theorem 7.4.1 If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system (3), then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

This is the **principle of superposition**; it is proved simply by differentiating $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ and using the fact that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ satisfy Eq. (3). As an example, it can be verified that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \tag{5}$$

satisfy the equation

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}. \tag{6}$$

Then, according to Theorem 7.4.1,

$$\begin{aligned} \mathbf{x} &= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \\ &= c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) \end{aligned} \tag{7}$$

also satisfies Eq. (6).

By repeated application of Theorem 7.4.1, we can conclude that if $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are solutions of Eq. (3), then

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) \tag{8}$$

is also a solution for any constants $c_1, \ldots, c_k$. Thus every finite linear combination of solutions of Eq. (3) is also a solution. The question that now arises is whether all solutions of Eq. (3) can be found in this way. By analogy with previous cases, it is reasonable to expect that for the system (3) of n first order equations, it is sufficient to form linear combinations of n properly chosen solutions. Therefore, let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$

be n solutions of the system (3), and consider the matrix $\mathbf{X}(t)$ whose columns are the vectors $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$:

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}. \tag{9}$$

Recall from Section 7.3 that the columns of $\mathbf{X}(t)$ are linearly independent for a given value of t if and only if $\det \mathbf{X} \neq 0$ for that value of t. This determinant is called the Wronskian of the n solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ and is also denoted by $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}]$; that is,

$$W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t) = \det \mathbf{X}(t). \tag{10}$$

The solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are then linearly independent at a point if and only if $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}]$ is not zero there.

Theorem 7.4.2 If the vector functions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system (3) for each point in the interval $\alpha < t < \beta$, then each solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the system (3) can be expressed as a linear combination of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$

$$\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{11}$$

in exactly one way.

Before proving Theorem 7.4.2, note that according to Theorem 7.4.1, all expressions of the form (11) are solutions of the system (3), while by Theorem 7.4.2 all solutions of Eq. (3) can be written in the form (11). If the constants $c_1, \dots, c_n$ are thought of as arbitrary, then Eq. (11) includes all solutions of the system (3), and it is customary to call it the **general solution.** Any set of solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ of Eq. (3) that is linearly independent at each point in the interval $\alpha < t < \beta$ is said to be a **fundamental set of solutions** for that interval.

To prove Theorem 7.4.2, we will show that any solution $\boldsymbol{\phi}$ of Eq. (3) can be written as $\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ for suitable values of $c_1, \dots, c_n$. Let $t = t_0$ be some point in the interval $\alpha < t < \beta$ and let $\boldsymbol{\xi} = \boldsymbol{\phi}(t_0)$. We now wish to determine whether there is any solution of the form $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ that also satisfies the same initial condition $\mathbf{x}(t_0) = \boldsymbol{\xi}$. That is, we wish to know whether there are values of $c_1, \dots, c_n$ such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0) = \boldsymbol{\xi}, \tag{12}$$

or, in scalar form,

$$\begin{aligned} c_1x_{11}(t_0) + \cdots + c_nx_{1n}(t_0) &= \xi_1, \\ &\ \vdots \\ c_1x_{n1}(t_0) + \cdots + c_nx_{nn}(t_0) &= \xi_n. \end{aligned} \tag{13}$$

The necessary and sufficient condition that Eqs. (13) possess a unique solution $c_1, \dots, c_n$ is precisely the nonvanishing of the determinant of coefficients, which is the Wronskian $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}]$ evaluated at $t = t_0$. The hypothesis that $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent throughout $\alpha < t < \beta$ guarantees that $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}]$ is not zero at $t = t_0$, and therefore there is a (unique) solution of Eq. (3) of the form $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ that also satisfies the initial condition (12). By the uniqueness part of Theorem 7.1.2, this solution is identical to $\boldsymbol{\phi}(t)$, and hence $\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$, as was to be proved.

Theorem 7.4.3 If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are solutions of Eq. (3) on the interval $\alpha < t < \beta$, then in this interval $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ either is identically zero or else never vanishes.

The significance of Theorem 7.4.3 lies in the fact that it relieves us of the necessity of examining $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ at all points in the interval of interest and enables us to determine whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions merely by evaluating their Wronskian at any convenient point in the interval.

Theorem 7.4.3 is proved by first establishing that the Wronskian of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ satisfies the differential equation (see Problem 2)

$$\frac{dW}{dt} = [p_{11}(t) + p_{22}(t) + \cdots + p_{nn}(t)]W. \tag{14}$$

Hence

$$W(t) = c \exp\left\{\int [p_{11}(t) + \cdots + p_{nn}(t)]\, dt\right\}, \tag{15}$$

where c is an arbitrary constant, and the conclusion of the theorem follows immediately. The expression for $W(t)$ in Eq. (15) is known as Abel's formula; note the similarity of this result to Theorem 3.2.7 and especially to Eq. (23) of Section 3.2.

Alternatively, Theorem 7.4.3 can be established by showing that if n solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ of Eq. (3) are linearly dependent at one point $t = t_0$, then they must be linearly dependent at each point in $\alpha < t < \beta$ (see Problem 8). Consequently, if $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent at one point, they must be linearly independent at each point in the interval.

The next theorem states that the system (3) always has at least one fundamental set of solutions.

Theorem 7.4.4 Let

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}^{(n)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix};$$

further, let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be the solutions of the system (3) that satisfy the initial conditions

$$\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \quad \ldots, \quad \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)}, \tag{16}$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions of the system (3).

To prove this theorem, note that the existence and uniqueness of the solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ mentioned in Theorem 7.4.4 are ensured by Theorem 7.1.2. It is not hard to see that the Wronskian of these solutions is equal to 1 when $t = t_0$; therefore $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are a fundamental set of solutions.

Once one fundamental set of solutions has been found, other sets can be generated by forming (independent) linear combinations of the first set. For theoretical purposes, the set given by Theorem 7.4.4 is usually the simplest.

Finally, it may happen (just as for second order linear equations) that a system whose coefficients are all real may give rise to solutions that are complex-valued. In this case, the following theorem is analogous to Theorem 3.2.6 and enables us to obtain real-valued solutions instead.

Theorem 7.4.5 Consider the system (3)

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x},$$

where each element of $\mathbf{P}$ is a real-valued continuous function. If $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex-valued solution of Eq. (3), then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions of this equation.

To prove this result, we substitute $\mathbf{u}(t) + i\mathbf{v}(t)$ for $\mathbf{x}$ in Eq. (3), thereby obtaining

$$\mathbf{x}' - \mathbf{P}(t)\mathbf{x} = \mathbf{u}'(t) - \mathbf{P}(t)\mathbf{u}(t) + i\,[\mathbf{v}'(t) - \mathbf{P}(t)\mathbf{v}(t)] = \mathbf{0}. \tag{17}$$

We have used the assumption that $\mathbf{P}(t)$ is real-valued to separate Eq. (17) into its real and imaginary parts. Since a complex number is zero if and only if its real and imaginary parts are both zero, we conclude that $\mathbf{u}'(t) - \mathbf{P}(t)\mathbf{u}(t) = \mathbf{0}$ and $\mathbf{v}'(t) - \mathbf{P}(t)\mathbf{v}(t) = \mathbf{0}$. Therefore, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions of Eq. (3).

To summarize the results of this section:

1. Any set of n linearly independent solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ constitutes a fundamental set of solutions.
2. Under the conditions given in this section, such fundamental sets always exist.
3. Every solution of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ can be represented as a linear combination of any fundamental set of solutions.

PROBLEMS

1. Prove the generalization of Theorem 7.4.1, as expressed in the sentence that includes Eq. (8), for an arbitrary value of the integer k.
2. In this problem we outline a proof of Theorem 7.4.3 in the case $n = 2$. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be solutions of Eq. (3) for $\alpha < t < \beta$, and let W be the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

   (a) Show that

$$\frac{dW}{dt} = \begin{vmatrix} \dfrac{dx_1^{(1)}}{dt} & \dfrac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \dfrac{dx_2^{(1)}}{dt} & \dfrac{dx_2^{(2)}}{dt} \end{vmatrix}.$$

(b) Using Eq. (3), show that

$$\frac{dW}{dt} = (p_{11} + p_{22})W.$$

(c) Find $W(t)$ by solving the differential equation obtained in part (b). Use this expression to obtain the conclusion stated in Theorem 7.4.3.

(d) Prove Theorem 7.4.3 for an arbitrary value of n by generalizing the procedure of parts (a), (b), and (c).

3. Show that the Wronskians of two fundamental sets of solutions of the system (3) can differ at most by a multiplicative constant.
Hint: Use Eq. (15).

4. If $x_1 = y$ and $x_2 = y'$, then the second order equation

$$y'' + p(t)y' + q(t)y = 0 \tag{i}$$

corresponds to the system

$$x_1' = x_2,$$
$$x_2' = -q(t)x_1 - p(t)x_2. \tag{ii}$$

Show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of Eqs. (ii), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq. (i), then $W[y^{(1)}, y^{(2)}] = cW[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$, where c is a nonzero constant.
Hint: $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

5. Show that the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the sum of any particular solution $\mathbf{x}^{(p)}$ of this equation and the general solution $\mathbf{x}^{(c)}$ of the corresponding homogeneous equation.

6. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.

(a) Compute the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

(b) In what intervals are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?

(c) What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?

(d) Find this system of equations and verify the conclusions of part (c).

7. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, and answer the same questions as in Problem 6.

The following two problems indicate an alternative derivation of Theorem 7.4.2.

8. Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$. Assume that $\mathbf{P}$ is continuous, and let t_0 be an arbitrary point in the given interval. Show that $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent for $\alpha < t < \beta$ if (and only if) $\mathbf{x}^{(1)}(t_0), \dots, \mathbf{x}^{(m)}(t_0)$ are linearly dependent. In other words $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly dependent on the interval (α, β) if they are linearly dependent at any point in it.
Hint: There are constants $c_1, \dots, c_m$ that satisfy $c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}$. Let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$, and use the uniqueness theorem to show that $\mathbf{z}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$.

9. Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ be linearly independent solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, where $\mathbf{P}$ is continuous on $\alpha < t < \beta$.

(a) Show that any solution $\mathbf{x} = \mathbf{z}(t)$ can be written in the form

$$\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

for suitable constants $c_1, \ldots, c_n$.
Hint: Use the result of Problem 12 of Section 7.3, and also Problem 8 above.

(b) Show that the expression for the solution $\mathbf{z}(t)$ in part (a) is unique; that is, if $\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)$, then $k_1 = c_1, \ldots, k_n = c_n$.
Hint: Show that $(k_1 - c_1)\mathbf{x}^{(1)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$, and use the linear independence of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$.

7.5 Homogeneous Linear Systems with Constant Coefficients

We will concentrate most of our attention on systems of homogeneous linear equations with constant coefficients—that is, systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. Unless stated otherwise, we will assume further that all the elements of $\mathbf{A}$ are real (rather than complex) numbers.

If $n = 1$, then the system reduces to a single first order equation

$$\frac{dx}{dt} = ax, \tag{2}$$

whose solution is $x = ce^{at}$. Note that $x = 0$ is the only equilibrium solution if $a \neq 0$. If $a < 0$, then other solutions approach $x = 0$ as t increases, and in this case we say that $x = 0$ is an asymptotically stable equilibrium solution. On the other hand, if $a > 0$, then $x = 0$ is unstable, since other solutions depart from it with increasing t. For systems of n equations, the situation is similar but more complicated. Equilibrium solutions are found by solving $\mathbf{A}\mathbf{x} = \mathbf{0}$. We usually assume that $\det \mathbf{A} \neq 0$, so $\mathbf{x} = \mathbf{0}$ is the only equilibrium solution. An important question is whether other solutions approach this equilibrium solution or depart from it as t increases; in other words, is $\mathbf{x} = \mathbf{0}$ asymptotically stable or unstable? Or are there still other possibilities?

The case $n = 2$ is particularly important and lends itself to visualization in the x_1x_2-plane, called the **phase plane**. By evaluating $\mathbf{A}\mathbf{x}$ at a large number of points and plotting the resulting vectors, we obtain a direction field of tangent vectors to solutions of the system of differential equations. A qualitative understanding of the behavior of solutions can usually be gained from a direction field. More precise information results from including in the plot some solution curves, or trajectories. A plot that shows a representative sample of trajectories for a given system is called a **phase portrait**. A well-constructed phase portrait provides easily understood information about all solutions of a two-dimensional system in a single graphical display. Although creating quantitatively accurate phase portraits requires computer assistance, it is usually possible to sketch qualitatively accurate phase portraits by hand, as we demonstrate in Examples 2 and 3 below.