26. Consider the initial value problem (see Example 5)

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

(a) Solve the initial value problem.

(b) Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.

(c) Determine the smallest value of β for which $y_m \geq 4$.

(d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

27. Consider the equation $ay'' + by' + cy = d$, where a, b, c, and d are constants.

(a) Find all equilibrium, or constant, solutions of this differential equation.

(b) Let y_e denote an equilibrium solution, and let $Y = y - y_e$. Thus Y is the deviation of a solution y from an equilibrium solution. Find the differential equation satisfied by Y.

28. Consider the equation $ay'' + by' + cy = 0$, where a, b, and c are constants with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic equation are:

(a) real, different, and negative.

(b) real with opposite signs.

(c) real, different, and positive.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In turn, this understanding will assist us in finding the solutions of other problems that we will encounter later.

To discuss general properties of linear differential equations, it is helpful to introduce a differential operator notation. Let p and q be continuous functions on an open interval I—that is, for $\alpha < t < \beta$. The cases for $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

Note that $L[\phi]$ is a function on I. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin 3t$, then

$$\begin{aligned} L[\phi](t) &= (\sin 3t)'' + t^2(\sin 3t)' + (1 + t)\sin 3t \\ &= -9\sin 3t + 3t^2\cos 3t + (1 + t)\sin 3t. \end{aligned}$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator.

In this section we study the second order linear homogeneous equation $L[\phi](t) = 0$. Since it is customary to use the symbol y to denote $\phi(t)$, we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{2}$$

With Eq. (2) we associate a set of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{3}$$

where t_0 is any point in the interval I, and y_0 and y_0' are given real numbers. We would like to know whether the initial value problem (2), (3) always has a solution, and whether it may have more than one solution. We would also like to know whether anything can be said about the form and structure of solutions that might be helpful in finding solutions of particular problems. Answers to these questions are contained in the theorems in this section.

The fundamental theoretical result for initial value problems for second order linear equations is stated in Theorem 3.2.1, which is analogous to Theorem 2.4.1 for first order linear equations. The result applies equally well to nonhomogeneous equations, so the theorem is stated in that form.

Theorem 3.2.1 **(Existence and Uniqueness Theorem)**

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{4}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

We emphasize that the theorem says three things:

1. The initial value problem *has* a solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; that is, the solution is *unique*.
3. The solution ϕ is defined *throughout the interval* I where the coefficients are continuous and is at least twice differentiable there.

For some problems some of these assertions are easy to prove. For instance, we found in Example 1 of Section 3.1 that the initial value problem

$$y'' - y = 0, \qquad y(0) = 2, \quad y'(0) = -1 \tag{5}$$

has the solution

$$y = \tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}. \tag{6}$$

The fact that we found a solution certainly establishes that a solution exists for this initial value problem. Further, the solution (6) is twice differentiable, indeed differentiable any number of times, throughout the interval $(-\infty, \infty)$ where the coefficients in the differential equation are continuous. On the other hand, it is not obvious, and is more difficult to show, that the initial value problem (5) has no solutions other than the one given by Eq. (6). Nevertheless, Theorem 3.2.1 states that this solution is indeed the only solution of the initial value problem (5).

For most problems of the form (4), it is not possible to write down a useful expression for the solution. This is a major difference between first order and second order linear equations. Therefore, all parts of the theorem must be proved by general methods that do not involve having such an expression. The proof of Theorem 3.2.1 is fairly difficult, and we do not discuss it here.[2] We will, however, accept Theorem 3.2.1 as true and make use of it whenever necessary.

EXAMPLE 1

Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0, \qquad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

If the given differential equation is written in the form of Eq. (4), then $p(t) = 1/(t-3)$, $q(t) = -(t+3)/t(t-3)$, and $g(t) = 0$. The only points of discontinuity of the coefficients are $t = 0$ and $t = 3$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous is $0 < t < 3$. Thus, this is the longest interval in which Theorem 3.2.1 guarantees that the solution exists.

EXAMPLE 2

Find the unique solution of the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where p and q are continuous in an open interval I containing t_0.

The function $y = \phi(t) = 0$ for all t in I certainly satisfies the differential equation and initial conditions. By the uniqueness part of Theorem 3.2.1, it is the only solution of the given problem.

Let us now assume that y_1 and y_2 are two solutions of Eq. (2); in other words,

$$L[y_1] = y_1'' + py_1' + qy_1 = 0,$$

and similarly for y_2. Then, just as in the examples in Section 3.1, we can generate more solutions by forming linear combinations of y_1 and y_2. We state this result as a theorem.

Theorem 3.2.2 **(Principle of Superposition)**

If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

[2] A proof of Theorem 3.2.1 can be found, for example, in Chapter 6, Section 8 of the book by Coddington listed in the references at the end of this chapter.

A special case of Theorem 3.2.2 occurs if either c_1 or c_2 is zero. Then we conclude that any constant multiple of a solution of Eq. (2) is also a solution.

To prove Theorem 3.2.2, we need only substitute

$$y = c_1y_1(t) + c_2y_2(t) \tag{7}$$

for y in Eq. (2). By calculating the indicated derivatives and rearranging terms, we obtain

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= [c_1y_1 + c_2y_2]'' + p[c_1y_1 + c_2y_2]' + q[c_1y_1 + c_2y_2] \\ &= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2 \\ &= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] \\ &= c_1L[y_1] + c_2L[y_2]. \end{aligned}$$

Since $L[y_1] = 0$ and $L[y_2] = 0$, it follows that $L[c_1y_1 + c_2y_2] = 0$ also. Therefore, regardless of the values of c_1 and c_2, y as given by Eq. (7) satisfies the differential equation (2), and the proof of Theorem 3.2.2 is complete.

Theorem 3.2.2 states that, beginning with only two solutions of Eq. (2), we can construct an infinite family of solutions by means of Eq. (7). The next question is whether all solutions of Eq. (2) are included in Eq. (7) or whether there may be other solutions of a different form. We begin to address this question by examining whether the constants c_1 and c_2 in Eq. (7) can be chosen so as to satisfy the initial conditions (3). These initial conditions require c_1 and c_2 to satisfy the equations

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0'. \end{aligned} \tag{8}$$

The determinant of coefficients of the system (8) is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{9}$$

If $W \neq 0$, then Eqs. (8) have a unique solution (c_1, c_2) regardless of the values of y_0 and y_0'. This solution is given by

$$c_1 = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \qquad c_2 = \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \tag{10}$$

or, in terms of determinants,

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \tag{11}$$

With these values for c_1 and c_2, the linear combination $y = c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions (3) as well as the differential equation (2). Note that the denominator in the expressions for c_1 and c_2 is the nonzero determinant W.

On the other hand, if $W = 0$, then the denominators appearing in Eqs. (10) and (11) are zero. In this case Eqs. (8) have no solution unless y_0 and y_0' have values that also make the numerators in Eqs. (10) and (11) equal to zero. Thus, if $W = 0$, there are many initial conditions that cannot be satisfied no matter how c_1 and c_2 are chosen.

The determinant W is called the **Wronskian**[3] **determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. Sometimes we use the more extended notation $W(y_1, y_2)(t_0)$ to stand for the expression on the right side of Eq. (9), thereby emphasizing that the Wronskian depends on the functions y_1 and y_2, and that it is evaluated at the point t_0. The preceding argument establishes the following result.

Theorem 3.2.3 Suppose that y_1 and y_2 are two solutions of Eq. (2)

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the initial conditions (3)

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1y_1(t) + c_2y_2(t)$$

satisfies the differential equation (2) and the initial conditions (3) if and only if the Wronskian

$$W = y_1y_2' - y_1'y_2$$

is not zero at t_0.

EXAMPLE 3

In Example 2 of Section 3.1 we found that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions of the differential equation

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of y_1 and y_2.

The Wronskian of these two functions is

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}.$$

Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the given differential equation, together with initial conditions prescribed at any value of t. One such initial value problem was solved in Example 3 of Section 3.1.

The next theorem justifies the term "general solution" that we introduced in Section 3.1 for the linear combination $c_1y_1 + c_2y_2$.

Theorem 3.2.4 Suppose that y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Then the family of solutions

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of Eq. (2) if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

[3]Wronskian determinants are named for Jósef Maria Hoëné-Wronski (1776–1853), who was born in Poland but spent most of his life in France. Wronski was a gifted but troubled man, and his life was marked by frequent heated disputes with other individuals and institutions.

Let ϕ be any solution of Eq. (2). To prove the theorem, we must determine whether ϕ is included in the linear combinations $c_1y_1 + c_2y_2$. That is, we must determine whether there are values of the constants c_1 and c_2 that make the linear combination the same as ϕ. Let t_0 be a point where the Wronskian of y_1 and y_2 is nonzero. Then evaluate ϕ and ϕ' at this point and call these values y_0 and y_0', respectively; thus

$$y_0 = \phi(t_0), \qquad y_0' = \phi'(t_0).$$

Next, consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{12}$$

The function ϕ is certainly a solution of this initial value problem. Further, because we are assuming that $W(y_1, y_2)(t_0)$ is nonzero, it is possible (by Theorem 3.2.3) to choose c_1 and c_2 such that $y = c_1y_1(t) + c_2y_2(t)$ is also a solution of the initial value problem (12). In fact, the proper values of c_1 and c_2 are given by Eqs. (10) or (11). The uniqueness part of Theorem 3.2.1 guarantees that these two solutions of the same initial value problem are actually the same function; thus, for the proper choice of c_1 and c_2,

$$\phi(t) = c_1y_1(t) + c_2y_2(t), \tag{13}$$

and therefore ϕ is included in the family of functions $c_1y_1 + c_2y_2$. Finally, since ϕ is an *arbitrary* solution of Eq. (2), it follows that *every* solution of this equation is included in this family.

Now suppose that there is no point t_0 where the Wronskian is nonzero. Thus $W(y_1, y_2)(t_0) = 0$ no matter which point t_0 is selected. Then (by Theorem 3.2.3) there are values of y_0 and y_0' such that the system (8) has no solution for c_1 and c_2. Select a pair of such values and choose the solution $\phi(t)$ of Eq. (2) that satisfies the initial condition (3). Observe that such a solution is guaranteed to exist by Theorem 3.2.l. However, this solution is not included in the family $y = c_1y_1 + c_2y_2$. Thus this linear combination does not include all solutions of Eq. (2) if $W(y_1, y_2) = 0$. This completes the proof of Theorem 3.2.4.

Theorem 3.2.4 states that, if and only if the Wronskian of y_1 and y_2 is not everywhere zero, then the linear combination $c_1y_1 + c_2y_2$ contains all solutions of Eq. (2). It is therefore natural (and we have already done this in the preceding section) to call the expression

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constant coefficients the **general solution** of Eq. (2). The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of Eq. (2) if and only if their Wronskian is nonzero.

We can restate the result of Theorem 3.2.4 in slightly different language: to find the general solution, and therefore all solutions, of an equation of the form (2), we need only find two solutions of the given equation whose Wronskian is nonzero. We did precisely this in several examples in Section 3.1, although there we did not calculate the Wronskians. You should now go back and do that, thereby verifying that all the solutions we called "general solutions" in Section 3.1 do satisfy the necessary Wronskian condition. Alternatively, the following example includes all those mentioned in Section 3.1, as well as many other problems of a similar type.

EXAMPLE 4

Suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of an equation of the form (2). Show that they form a fundamental set of solutions if $r_1 \neq r_2$.

We calculate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)\exp[(r_1 + r_2)t].$$

Since the exponential function is never zero, and since we are assuming that $r_2 - r_1 \neq 0$, it follows that W is nonzero for every value of t. Consequently, y_1 and y_2 form a fundamental set of solutions.

EXAMPLE 5

Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \qquad t > 0. \tag{14}$$

We will show how to solve Eq. (14) later (see Problem 34 in Section 3.3). However, at this stage we can verify by direct substitution that y_1 and y_2 are solutions of the differential equation. Since $y_1'(t) = \frac{1}{2}t^{-1/2}$ and $y_1''(t) = -\frac{1}{4}t^{-3/2}$, we have

$$2t^2(-\tfrac{1}{4}t^{-3/2}) + 3t(\tfrac{1}{2}t^{-1/2}) - t^{1/2} = (-\tfrac{1}{2} + \tfrac{3}{2} - 1)t^{1/2} = 0.$$

Similarly, $y_2'(t) = -t^{-2}$ and $y_2''(t) = 2t^{-3}$, so

$$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4 - 3 - 1)t^{-1} = 0.$$

Next we calculate the Wronskian W of y_1 and y_2:

$$W = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\tfrac{3}{2}t^{-3/2}. \tag{15}$$

Since $W \neq 0$ for $t > 0$, we conclude that y_1 and y_2 form a fundamental set of solutions there.

In several cases we have been able to find a fundamental set of solutions, and therefore the general solution, of a given differential equation. However, this is often a difficult task, and the question arises as to whether a differential equation of the form (2) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

Theorem 3.2.5

Consider the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of Eq. (2) that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let y_2 be the solution of Eq. (2) that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Then y_1 and y_2 form a fundamental set of solutions of Eq. (2).

First observe that the *existence* of the functions y_1 and y_2 is ensured by the existence part of Theorem 3.2.1. To show that they form a fundamental set of solutions, we need only calculate their Wronskian at t_0:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Since their Wronskian is not zero at the point t_0, the functions y_1 and y_2 do form a fundamental set of solutions, thus completing the proof of Theorem 3.2.5.

Note that the potentially difficult part of this proof, demonstrating the existence of a pair of solutions, is taken care of by reference to Theorem 3.2.1. Note also that Theorem 3.2.5 does not address the question of how to find the solutions y_1 and y_2 by solving the specified initial value problems. Nevertheless, it may be reassuring to know that a fundamental set of solutions always exists.

EXAMPLE 6

Find the fundamental set of solutions y_1 and y_2 specified by Theorem 3.2.5 for the differential equation

$$y'' - y = 0, \tag{16}$$

using the initial point $t_0 = 0$.

In Section 3.1 we noted that two solutions of Eq. (16) are $y_1(t) = e^t$ and $y_2(t) = e^{-t}$. The Wronskian of these solutions is $W(y_1, y_2)(t) = -2 \neq 0$, so they form a fundamental set of solutions. However, they are not the fundamental solutions indicated by Theorem 3.2.5 because they do not satisfy the initial conditions mentioned in that theorem at the point $t = 0$.

To find the fundamental solutions specified by the theorem, we need to find the solutions satisfying the proper initial conditions. Let us denote by $y_3(t)$ the solution of Eq. (16) that satisfies the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{17}$$

The general solution of Eq. (16) is

$$y = c_1 e^t + c_2 e^{-t}, \tag{18}$$

and the initial conditions (17) are satisfied if $c_1 = 1/2$ and $c_2 = 1/2$. Thus

$$y_3(t) = \tfrac{1}{2}e^t + \tfrac{1}{2}e^{-t} = \cosh t.$$

Similarly, if $y_4(t)$ satisfies the initial conditions

$$y(0) = 0, \qquad y'(0) = 1, \tag{19}$$

then

$$y_4(t) = \tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t} = \sinh t.$$

Since the Wronskian of y_3 and y_4 is

$$W(y_3, y_4)(t) = \cosh^2 t - \sinh^2 t = 1,$$

these functions also form a fundamental set of solutions, as stated by Theorem 3.2.5. Therefore, the general solution of Eq. (16) can be written as

$$y = k_1 \cosh t + k_2 \sinh t, \tag{20}$$

as well as in the form (18). We have used k_1 and k_2 for the arbitrary constants in Eq. (20) because they are not the same as the constants c_1 and c_2 in Eq. (18). One purpose of this example is to make it clear that a given differential equation has more than one fundamental set of solutions; indeed, it has infinitely many; see Problem 21. As a rule, you should choose the set that is most convenient.

In the next section we will encounter equations that have complex-valued solutions. The following theorem is fundamental in dealing with such equations and their solutions.

Theorem 3.2.6 Consider again the equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous real-valued functions. If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2), then its real part u and its imaginary part v are also solutions of this equation.

To prove this theorem we substitute $u(t) + iv(t)$ for y in $L[y]$, obtaining

$$L[y] = u''(t) + iv''(t) + p(t)[u'(t) + iv'(t)] + q(t)[u(t) + iv(t)]. \tag{21}$$

Then, by separating Eq. (21) into its real and imaginary parts (and this is where we need to know that $p(t)$ and $q(t)$ are real-valued), we find that

$$\begin{aligned} L[y] &= u''(t) + p(t)u'(t) + q(t)u(t) + i[v''(t) + p(t)v'(t) + q(t)v(t)] \\ &= L[u](t) + iL[v](t). \end{aligned}$$

Recall that a complex number is zero if and only if its real and imaginary parts are both zero. We know that $L[y] = 0$ because y is a solution of Eq. (2). Therefore, $L[u](t) = 0$ and $L[v](t) = 0$ also; consequently, u and v are also solutions of Eq. (2), so the theorem is established. We will see examples of the use of Theorem 3.2.6 in Section 3.3.

Incidentally, the complex conjugate $\bar{y}$ of a solution y is also a solution. This is a consequence of Theorem 3.2.2 since $\bar{y} = u(t) - iv(t)$ is a linear combination of two solutions.

Now let us examine further the properties of the Wronskian of two solutions of a second order linear homogeneous differential equation. The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 3.2.7 **(Abel's Theorem)**[4]

If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{22}$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)\,dt\right], \tag{23}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

To prove Abel's theorem, we start by noting that y_1 and y_2 satisfy

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0, \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{aligned} \tag{24}$$

If we multiply the first equation by $-y_2$, multiply the second by y_1, and add the resulting equations, we obtain

$$(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0. \tag{25}$$

Next, we let $W(t) = W(y_1, y_2)(t)$ and observe that

$$W' = y_1 y_2'' - y_1'' y_2. \tag{26}$$

Then we can write Eq. (25) in the form

$$W' + p(t)W = 0. \tag{27}$$

Equation (27) can be solved immediately since it is both a first order linear equation (Section 2.1) and a separable equation (Section 2.2). Thus

$$W(t) = c \exp\left[-\int p(t)\,dt\right], \tag{28}$$

where c is a constant. The value of c depends on which pair of solutions of Eq. (22) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t. This completes the proof of Theorem 3.2.7.

Note that the Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant,

[4]The result in Theorem 3.2.7 was derived by the Norwegian mathematician Niels Henrik Abel (1802–1829) in 1827 and is known as Abel's formula. Abel also showed that there is no general formula for solving a quintic, or fifth degree, polynomial equation in terms of explicit algebraic operations on the coefficients, thereby resolving a question that had been open since the sixteenth century. His greatest contributions, however, were in analysis, particularly in the study of elliptic functions. Unfortunately, his work was not widely noticed until after his death. The distinguished French mathematician Legendre called it a "monument more lasting than bronze."

without solving the differential equation. Further, since under the conditions of Theorem 3.2.7 the Wronskian W is either always zero or never zero, you can determine which case actually occurs by evaluating W at any single convenient value of t.

EXAMPLE 7

In Example 5 we verified that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0. \tag{29}$$

Verify that the Wronskian of y_1 and y_2 is given by Eq. (23).

From the example just cited we know that $W(y_1, y_2)(t) = -(3/2)t^{-3/2}$. To use Eq. (23), we must write the differential equation (29) in the standard form with the coefficient of y'' equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = 3/2t$. Hence

$$W(y_1, y_2)(t) = c\ \exp\left[-\int \frac{3}{2t}\,dt\right] = c\exp\left(-\frac{3}{2}\ln t\right)$$
$$= c\,t^{-3/2}. \tag{30}$$

Equation (30) gives the Wronskian of any pair of solutions of Eq. (29). For the particular solutions given in this example, we must choose $c = -3/2$.

Summary. We can summarize the discussion in this section as follows: to find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \qquad \alpha < t < \beta,$$

we must first find two functions y_1 and y_2 that satisfy the differential equation in $\alpha < t < \beta$. Then we must make sure that there is a point in the interval where the Wronskian W of y_1 and y_2 is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions, and the general solution is

$$y = c_1y_1(t) + c_2y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a point in $\alpha < t < \beta$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

PROBLEMS

In each of Problems 1 through 6, find the Wronskian of the given pair of functions.

1. $e^{2t}, \quad e^{-3t/2}$
2. $\cos t, \quad \sin t$
3. $e^{-2t}, \quad te^{-2t}$
4. $x, \quad xe^x$
5. $e^t\sin t, \quad e^t\cos t$
6. $\cos^2\theta, \quad 1 + \cos 2\theta$

In each of Problems 7 through 12, determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

7. $ty'' + 3y = t, \qquad y(1) = 1, \quad y'(1) = 2$
8. $(t-1)y'' - 3ty' + 4y = \sin t, \qquad y(-2) = 2, \quad y'(-2) = 1$
9. $t(t-4)y'' + 3ty' + 4y = 2, \qquad y(3) = 0, \quad y'(3) = -1$
10. $y'' + (\cos t)y' + 3(\ln|t|)y = 0, \qquad y(2) = 3, \quad y'(2) = 1$

11. $(x-3)y'' + xy' + (\ln|x|)y = 0, \quad y(1) = 0, \quad y'(1) = 1$
12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2$

13. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $y = c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.
14. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.
15. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, then $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.
16. Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients? Explain your answer.
17. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.
18. If the Wronskian W of f and g is t^2e^t, and if $f(t) = t$, find $g(t)$.
19. If $W(f, g)$ is the Wronskian of f and g, and if $u = 2f - g$, $v = f + 2g$, find the Wronskian $W(u, v)$ of u and v in terms of $W(f, g)$.
20. If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g$, $v = f - g$, find the Wronskian of u and v.
21. Assume that y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ and let $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$, where a_1, a_2, b_1, and b_2 are any constants. Show that

$$W(y_3, y_4) = (a_1b_2 - a_2b_1)W(y_1, y_2).$$

Are y_3 and y_4 also a fundamental set of solutions? Why or why not?

In each of Problems 22 and 23, find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

22. $y'' + y' - 2y = 0, \quad t_0 = 0$
23. $y'' + 4y' + 3y = 0, \quad t_0 = 1$

In each of Problems 24 through 27, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

24. $y'' + 4y = 0; \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
25. $y'' - 2y' + y = 0; \quad y_1(t) = e^t, \quad y_2(t) = te^t$
26. $x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0; \quad y_1(x) = x, \quad y_2(x) = xe^x$
27. $(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi; \quad y_1(x) = x, \quad y_2(x) = \sin x$

28. Consider the equation $y'' - y' - 2y = 0$.
    (a) Show that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions.
    (b) Let $y_3(t) = -2e^{2t}$, $y_4(t) = y_1(t) + 2y_2(t)$, and $y_5(t) = 2y_1(t) - 2y_3(t)$. Are $y_3(t)$, $y_4(t)$, and $y_5(t)$ also solutions of the given differential equation?
    (c) Determine whether each of the following pairs forms a fundamental set of solutions: $[y_1(t), y_3(t)]$; $[y_2(t), y_3(t)]$; $[y_1(t), y_4(t)]$; $[y_4(t), y_5(t)]$.

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

29. $t^2y'' - t(t+2)y' + (t+2)y = 0$
30. $(\cos t)y'' + (\sin t)y' - ty = 0$
31. $x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad$ Bessel's equation
32. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad$ Legendre's equation

33. Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = c/p(t)$, where c is a constant.
34. If the differential equation $ty'' + 2y' + te^t y = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.
35. If the differential equation $t^2y'' - 2y' + (3 + t)y = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(4)$.
36. If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients p and q?
37. If f, g, and h are differentiable functions, show that $W(fg, fh) = f^2 W(g, h)$.

In Problems 38 through 40, assume that p and q are continuous and that the functions y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I.

38. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.
39. Prove that if y_1 and y_2 have maxima or minima at the same point in I, then they cannot be a fundamental set of solutions on that interval.
40. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.

41. **Exact Equations.** The equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

is said to be exact if it can be written in the form

$$[P(x)y']' + [f(x)y]' = 0,$$

where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated once immediately, resulting in a first order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is

$$P''(x) - Q'(x) + R(x) = 0.$$

It can be shown that this is also a sufficient condition.

In each of Problems 42 through 45, use the result of Problem 41 to determine whether the given equation is exact. If it is, then solve the equation.

42. $y'' + xy' + y = 0$
43. $y'' + 3x^2y' + xy = 0$
44. $xy'' - (\cos x)y' + (\sin x)y = 0, \quad x > 0$
45. $x^2y'' + xy' - y = 0, \quad x > 0$

46. **The Adjoint Equation.** If a second order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$$

can be written in the form

$$[\mu(x)P(x)y']' + [f(x)y]' = 0.$$

By equating coefficients in these two equations and eliminating $f(x)$, show that the function μ must satisfy

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

This equation is known as the adjoint of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second order equation.

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

47. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation
48. $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, Legendre's equation
49. $y'' - xy = 0$, Airy's equation
50. For the second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$, show that the adjoint of the adjoint equation is the original equation.
51. A second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be self-adjoint if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 47 through 49 is self-adjoint.

3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

We showed in Section 3.1 that if the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of Eq. (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq. (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]. \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from Eq. (5),

$$y_1(3) = e^{-3+6i}. \tag{6}$$