(b) Multiply the result of part (a) by $e^{-(r_2-r_1)t}$ and differentiate with respect to t to obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3-r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n-r_2)t} = 0.$$

(c) Continue the procedure from parts (a) and (b), eventually obtaining

$$c_n(r_n - r_{n-1}) \cdots (r_n - r_1)e^{(r_n-r_{n-1})t} = 0.$$

Hence $c_n = 0$, and therefore,

$$c_1 e^{r_1 t} + \cdots + c_{n-1}e^{r_{n-1}t} = 0.$$

(d) Repeat the preceding argument to show that $c_{n-1} = 0$. In a similar way it follows that $c_{n-2} = \cdots = c_1 = 0$. Thus the functions $e^{r_1 t}, \ldots, e^{r_n t}$ are linearly independent.

41. In this problem we indicate one way to show that if $r = r_1$ is a root of multiplicity s of the characteristic polynomial $Z(r)$, then $e^{r_1 t},\ te^{r_1 t}, \ldots,\ t^{s-1}e^{r_1 t}$ are solutions of Eq. (1). This problem extends to nth order equations the method for second order equations given in Problem 22 of Section 3.4. We start from Eq. (2) in the text

$$L[e^{rt}] = e^{rt}Z(r) \tag{i}$$

and differentiate repeatedly with respect to r, setting $r = r_1$ after each differentiation.

(a) Observe that if r_1 is a root of multiplicity s, then $Z(r) = (r - r_1)^s q(r)$, where $q(r)$ is a polynomial of degree $n - s$ and $q(r_1) \neq 0$. Show that $Z(r_1), Z'(r_1), \ldots, Z^{(s-1)}(r_1)$ are all zero, but $Z^{(s)}(r_1) \neq 0$.

(b) By differentiating Eq. (i) repeatedly with respect to r, show that

$$\frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}],$$
$$\vdots$$
$$\frac{\partial^{s-1}}{\partial r^{s-1}}L[e^{rt}] = L[t^{s-1}e^{rt}].$$

(c) Show that $e^{r_1 t}, te^{r_1 t}, \ldots, t^{s-1}e^{r_1 t}$ are solutions of Eq. (1).

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = g(t) \tag{1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. Although the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when it is applicable.

Just as for the second order linear equation, when the constant coefficient linear differential operator L is applied to a polynomial $A_0 t^m + A_1 t^{m-1} + \cdots + A_m$, an exponential function $e^{\alpha t}$, a sine function $\sin \beta t$, or a cosine function $\cos \beta t$, the result is a polynomial, an exponential function, or a linear combination of sine and cosine functions, respectively. Hence, if $g(t)$ is a sum of polynomials, exponentials, sines, and cosines, or products of such functions, we can expect that it is possible to find $Y(t)$ by choosing a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants. The constants are then determined by substituting the assumed expression into Eq. (1).

The main difference in using this method for higher order equations stems from the fact that roots of the characteristic polynomial equation may have multiplicity greater than 2. Consequently, terms proposed for the nonhomogeneous part of the solution may need to be multiplied by higher powers of t to make them different from terms in the solution of the corresponding homogeneous equation. The following examples illustrate this. In these examples we have omitted numerous straightforward algebraic steps, because our main goal is to show how to arrive at the correct form for the assumed solution.

EXAMPLE 1

Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t. \tag{2}$$

The characteristic polynomial for the homogeneous equation corresponding to Eq. (2) is

$$r^3 - 3r^2 + 3r - 1 = (r - 1)^3,$$

so the general solution of the homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t. \tag{3}$$

To find a particular solution $Y(t)$ of Eq. (2), we start by assuming that $Y(t) = Ae^t$. However, since e^t, te^t, and $t^2 e^t$ are all solutions of the homogeneous equation, we must multiply this initial choice by t^3. Thus our final assumption is that $Y(t) = At^3 e^t$, where A is an undetermined coefficient. To find the correct value for A, we differentiate $Y(t)$ three times, substitute for y and its derivatives in Eq. (2), and collect terms in the resulting equation. In this way we obtain

$$6Ae^t = 4e^t.$$

Thus $A = \frac{2}{3}$ and the particular solution is

$$Y(t) = \tfrac{2}{3} t^3 e^t. \tag{4}$$

The general solution of Eq. (2) is the sum of $y_c(t)$ from Eq. (3) and $Y(t)$ from Eq. (4):

$$y = c_1 e^t + c_2 te^t + c_3 t^2 e^t + \tfrac{2}{3} t^3 e^t.$$

EXAMPLE 2

Find a particular solution of the equation

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t. \tag{5}$$

The general solution of the homogeneous equation was found in Example 3 of Section 4.2; it is

$$y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t, \tag{6}$$

corresponding to the roots $r = i, i, -i,$ and $-i$ of the characteristic equation. Our initial assumption for a particular solution is $Y(t) = A\sin t + B\cos t$, but we must multiply this choice by t^2 to make it different from all solutions of the homogeneous equation. Thus our final assumption is

$$Y(t) = At^2 \sin t + Bt^2 \cos t.$$

Next, we differentiate $Y(t)$ four times, substitute into the differential equation (4), and collect terms, obtaining finally

$$-8A\sin t - 8B\cos t = 3\sin t - 5\cos t.$$

Thus $A = -\frac{3}{8}$, $B = \frac{5}{8}$, and the particular solution of Eq. (4) is

$$Y(t) = -\tfrac{3}{8}t^2 \sin t + \tfrac{5}{8}t^2 \cos t. \tag{7}$$

If $g(t)$ is a sum of several terms, it may be easier in practice to compute separately the particular solution corresponding to each term in $g(t)$. As for the second order equation, the particular solution of the complete problem is the sum of the particular solutions of the individual component problems. This is illustrated in the following example.

EXAMPLE 3

Find a particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t}. \tag{8}$$

First we solve the homogeneous equation. The characteristic equation is $r^3 - 4r = 0$, and the roots are $r = 0, \pm 2$; hence

$$y_c(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

We can write a particular solution of Eq. (8) as the sum of particular solutions of the differential equations

$$y''' - 4y' = t, \qquad y''' - 4y' = 3\cos t, \qquad y''' - 4y' = e^{-2t}.$$

Our initial choice for a particular solution $Y_1(t)$ of the first equation is $A_0 t + A_1$, but a constant is a solution of the homogeneous equation, so we multiply by t. Thus

$$Y_1(t) = t(A_0 t + A_1).$$

For the second equation we choose

$$Y_2(t) = B\cos t + C\sin t,$$

and there is no need to modify this initial choice since $\sin t$ and $\cos t$ are not solutions of the homogeneous equation. Finally, for the third equation, since e^{-2t} is a solution of the homogeneous equation, we assume that

$$Y_3(t) = Ete^{-2t}.$$

The constants are determined by substituting into the individual differential equations; they are $A_0 = -\frac{1}{8}$, $A_1 = 0$, $B = 0$, $C = -\frac{3}{5}$, and $E = \frac{1}{8}$. Hence a particular solution of Eq. (8) is

$$Y(t) = -\tfrac{1}{8}t^2 - \tfrac{3}{5}\sin t + \tfrac{1}{8}te^{-2t}. \tag{9}$$

You should keep in mind that the amount of algebra required to calculate the coefficients may be quite substantial for higher order equations, especially if the nonhomogeneous term is even moderately complicated. A computer algebra system can be extremely helpful in executing these algebraic calculations.

The method of undetermined coefficients can be used whenever it is possible to guess the correct form for $Y(t)$. However, this is usually impossible for differential equations not having constant coefficients, or for nonhomogeneous terms other than the type described previously. For more complicated problems we can use the method of variation of parameters, which is discussed in the next section.

PROBLEMS

In each of Problems 1 through 8, determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$
2. $y^{(4)} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$
4. $y''' - y' = 2\sin t$
5. $y^{(4)} - 4y'' = t^2 + e^t$
6. $y^{(4)} + 2y'' + y = 3 + \cos 2t$
7. $y^{(6)} + y''' = t$
8. $y^{(4)} + y''' = \sin 2t$

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + 4y' = t;\qquad y(0) = y'(0) = 0,\quad y''(0) = 1$
10. $y^{(4)} + 2y'' + y = 3t + 4;\qquad y(0) = y'(0) = 0,\quad y''(0) = y'''(0) = 1$
11. $y''' - 3y'' + 2y' = t + e^t;\qquad y(0) = 1,\quad y'(0) = -\frac{1}{4},\quad y''(0) = -\frac{3}{2}$
12. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t};\qquad y(0) = 3,\quad y'(0) = 0,$ $y''(0) = -1,\quad y'''(0) = 2$

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

13. $y''' - 2y'' + y' = t^3 + 2e^t$
14. $y''' - y' = te^{-t} + 2\cos t$
15. $y^{(4)} - 2y'' + y = e^t + \sin t$
16. $y^{(4)} + 4y'' = \sin 2t + te^t + 4$
17. $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$
18. $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$

19. Consider the nonhomogeneous nth order linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t), \tag{i}$$

where $a_0, \ldots, a_n$ are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t}(b_0 t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t}u(t)$ reduces Eq. (i) to the form

$$k_0 u^{(n)} + k_1 u^{(n-1)} + \cdots + k_n u = b_0 t^m + \cdots + b_m, \tag{ii}$$

where $k_0, \ldots, k_n$ are constants. Determine k_0 and k_n in terms of the a's and α. Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Method of Annihilators. In Problems 20 through 22, we consider another way of arriving at the proper form of $Y(t)$ for use in the method of undetermined coefficients. The procedure is based on the observation that exponential, polynomial, or sinusoidal terms (or sums and products of such terms) can be viewed as solutions of certain linear homogeneous differential equations with constant coefficients. It is convenient to use the symbol D for d/dt. Then, for example, e^{-t} is a solution of $(D+1)y = 0$; the differential operator $D + 1$ is said to *annihilate*, or to be an *annihilator* of, e^{-t}. In the same way, $D^2 + 4$ is an annihilator of $\sin 2t$ or $\cos 2t$, $(D-3)^2 = D^2 - 6D + 9$ is an annihilator of e^{3t} or te^{3t}, and so forth.

20. Show that linear differential operators with constant coefficients obey the commutative law. That is, show that

$$(D-a)(D-b)f = (D-b)(D-a)f$$

for any twice-differentiable function f and any constants a and b. The result extends at once to any finite number of factors.

21. Consider the problem of finding the form of a particular solution $Y(t)$ of

$$(D-2)^3(D+1)Y = 3e^{2t} - te^{-t}, \tag{i}$$

where the left side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.

(a) Show that $D-2$ and $(D+1)^2$, respectively, are annihilators of the terms on the right side of Eq. (i), and that the combined operator $(D-2)(D+1)^2$ annihilates both terms on the right side of Eq. (i) simultaneously.

(b) Apply the operator $(D-2)(D+1)^2$ to Eq. (i) and use the result of Problem 20 to obtain

$$(D-2)^4(D+1)^3Y = 0. \tag{ii}$$

Thus Y is a solution of the homogeneous equation (ii). By solving Eq. (ii), show that

$$Y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4t^3e^{2t} + c_5e^{-t} + c_6te^{-t} + c_7t^2e^{-t}, \tag{iii}$$

where $c_1, \ldots, c_7$ are constants, as yet undetermined.

(c) Observe that e^{2t}, te^{2t}, t^2e^{2t}, and e^{-t} are solutions of the homogeneous equation corresponding to Eq. (i); hence these terms are not useful in solving the nonhomogeneous equation. Therefore, choose c_1, c_2, c_3, and c_5 to be zero in Eq. (iii), so that

$$Y(t) = c_4t^3e^{2t} + c_6te^{-t} + c_7t^2e^{-t}. \tag{iv}$$

This is the form of the particular solution Y of Eq. (i). The values of the coefficients c_4, c_6, and c_7 can be found by substituting from Eq. (iv) in the differential equation (i).

Summary. Suppose that

$$L(D)y = g(t), \tag{v}$$

where $L(D)$ is a linear differential operator with constant coefficients, and $g(t)$ is a sum or product of exponential, polynomial, or sinusoidal terms. To find the form of a particular solution of Eq. (v), you can proceed as follows:

(a) Find a differential operator $H(D)$ with constant coefficients that annihilates $g(t)$—that is, an operator such that $H(D)g(t) = 0$.

(b) Apply $H(D)$ to Eq. (v), obtaining

$$H(D)L(D)y = 0, \tag{vi}$$

which is a homogeneous equation of higher order.

(c) Solve Eq. (vi).

(d) Eliminate from the solution found in step (c) the terms that also appear in the solution of $L(D)y = 0$. The remaining terms constitute the correct form of a particular solution of Eq. (v).

22. Use the method of annihilators to find the form of a particular solution $Y(t)$ for each of the equations in Problems 13 through 18. Do not evaluate the coefficients.

4.4 The Method of Variation of Parameters

The method of variation of parameters for determining a particular solution of the nonhomogeneous nth order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{1}$$

is a direct extension of the method for the second order differential equation (see Section 3.6). As before, to use the method of variation of parameters, it is first necessary to solve the corresponding homogeneous differential equation. In general, this may be difficult unless the coefficients are constants. However, the method of variation of parameters is still more general than the method of undetermined coefficients in that it leads to an expression for the particular solution for *any* continuous function g, whereas the method of undetermined coefficients is restricted in practice to a limited class of functions g.

Suppose then that we know a fundamental set of solutions $y_1, y_2, \ldots, y_n$ of the homogeneous equation. Then the general solution of the homogeneous equation is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t). \tag{2}$$

The method of variation of parameters for determining a particular solution of Eq. (1) rests on the possibility of determining n functions $u_1, u_2, \ldots, u_n$ such that $Y(t)$ is of the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t). \tag{3}$$

Since we have n functions to determine, we will have to specify n conditions. One of these is clearly that Y satisfy Eq. (1). The other $n-1$ conditions are chosen so as to make the calculations as simple as possible. Since we can hardly expect a simplification in determining Y if we must solve high order differential equations for $u_1, \ldots, u_n$, it is natural to impose conditions to suppress the terms that lead to higher derivatives of $u_1, \ldots, u_n$. From Eq. (3) we obtain

$$Y' = (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n), \tag{4}$$

where we have omitted the independent variable t on which each function in Eq. (4) depends. Thus the first condition that we impose is that

$$u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0. \tag{5}$$

It follows that the expression (4) for Y' reduces to

$$Y' = u_1y_1' + u_2y_2' + \cdots + u_ny_n'. \tag{6}$$

We continue this process by calculating the successive derivatives $Y'', \ldots, Y^{(n-1)}$. After each differentiation we set equal to zero the sum of terms involving derivatives of $u_1, \ldots, u_n$. In this way we obtain $n-2$ further conditions similar to Eq. (5); that is,

$$u_1'y_1^{(m)} + u_2'y_2^{(m)} + \cdots + u_n'y_n^{(m)} = 0, \qquad m = 1, 2, \ldots, n-2. \tag{7}$$

As a result of these conditions, it follows that the expressions for $Y'', \ldots, Y^{(n-1)}$ reduce to

$$Y^{(m)} = u_1y_1^{(m)} + u_2y_2^{(m)} + \cdots + u_ny_n^{(m)}, \qquad m = 2, 3, \ldots, n-1, \tag{8}$$

Finally, we need to impose the condition that Y must be a solution of Eq. (1). By differentiating $Y^{(n-1)}$ from Eq. (8), we obtain

$$Y^{(n)} = (u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}). \tag{9}$$

To satisfy the differential equation we substitute for Y and its derivatives in Eq. (1) from Eqs. (3), (6), (8), and (9). Then we group the terms involving each of the functions $y_1, \dots, y_n$ and their derivatives. It then follows that most of the terms in the equation drop out because each of $y_1, \dots, y_n$ is a solution of Eq. (1) and therefore $L[y_i] = 0,\ i = 1, 2, \dots, n$. The remaining terms yield the relation

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g. \tag{10}$$

Equation (10), Eq. (5), and the $n - 2$ equations (7) provide n simultaneous linear nonhomogeneous algebraic equations for $u_1', u_2', \dots, u_n'$:

$$\begin{aligned} y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0, \\ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0, \\ y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' &= 0, \\ &\ \vdots \\ y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} u_n' &= g. \end{aligned} \tag{11}$$

The system (11) is a linear algebraic system for the unknown quantities $u_1', \dots, u_n'$. By solving this system and then integrating the resulting expressions, you can obtain the coefficients $u_1, \dots, u_n$. A sufficient condition for the existence of a solution of the system of equations (11) is that the determinant of coefficients is nonzero for each value of t. However, the determinant of coefficients is precisely $W(y_1, y_2, \dots, y_n)$, and it is nowhere zero since $y_1, \dots, y_n$ is a fundamental set of solutions of the homogeneous equation. Hence it is possible to determine $u_1', \dots, u_n'$. Using Cramer's[3] rule, we can write the solution of the system of equations (11) in the form

$$u_m'(t) = \frac{g(t) W_m(t)}{W(t)}, \qquad m = 1, 2, \dots, n. \tag{12}$$

Here $W(t) = W(y_1, y_2, \dots, y_n)(t)$, and W_m is the determinant obtained from W by replacing the mth column by the column $(0, 0, \dots, 0, 1)$. With this notation a particular solution of Eq. (1) is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)}\, ds, \tag{13}$$

where t_0 is arbitrary. Although the procedure is straightforward, the algebraic computations involved in determining $Y(t)$ from Eq. (13) become more and more

[3]Cramer's rule is credited to the Swiss mathematician Gabriel Cramer (1704–1752), professor at the Académie de Calvin in Geneva, who published it in a general form (but without proof) in 1750. For small systems the result had been known earlier.

complicated as n increases. In some cases the calculations may be simplified to some extent by using Abel's identity (Problem 20 of Section 4.1),

$$W(t) = W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\, dt\right].$$

The constant c can be determined by evaluating W at some convenient point.

EXAMPLE 1

Given that $y_1(t) = e^t$, $y_2(t) = te^t$, and $y_3(t) = e^{-t}$ are solutions of the homogeneous equation corresponding to

$$y''' - y'' - y' + y = g(t), \tag{14}$$

determine a particular solution of Eq. (14) in terms of an integral.

We use Eq. (13). First, we have

$$W(t) = W(e^t, te^t, e^{-t})(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix}.$$

Factoring e^t from each of the first two columns and e^{-t} from the third column, we obtain

$$W(t) = e^t \begin{vmatrix} 1 & t & 1 \\ 1 & t+1 & -1 \\ 1 & t+2 & 1 \end{vmatrix}.$$

Then, by subtracting the first row from the second and third rows, we have

$$W(t) = e^t \begin{vmatrix} 1 & t & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 0 \end{vmatrix}.$$

Finally, evaluating the latter determinant by minors associated with the first column, we find that

$$W(t) = 4e^t.$$

Next,

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix}.$$

Using minors associated with the first column, we obtain

$$W_1(t) = \begin{vmatrix} te^t & e^{-t} \\ (t+1)e^t & -e^{-t} \end{vmatrix} = -2t - 1.$$

In a similar way,

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = -\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = 2$$

and

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t}.$$

Substituting these results in Eq. (13), we have

$$\begin{aligned} Y(t) &= e^t \int_{t_0}^{t} \frac{g(s)(-1-2s)}{4e^s}\, ds + te^t \int_{t_0}^{t} \frac{g(s)(2)}{4e^s}\, ds + e^{-t} \int_{t_0}^{t} \frac{g(s)e^{2s}}{4e^s}\, ds \\ &= \frac{1}{4} \int_{t_0}^{t} \left\{ e^{t-s}[-1 + 2(t-s)] + e^{-(t-s)} \right\} g(s)\, ds. \qquad (15) \end{aligned}$$

Depending on the specific function $g(t)$, it may or may not be possible to evaluate the integrals in Eq. (15) in terms of elementary functions.

PROBLEMS

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

1. $y''' + y' = \tan t, \quad -\pi/2 < t < \pi/2$
2. $y''' - y' = t$
3. $y''' - 2y'' - y' + 2y = e^{4t}$
4. $y''' + y' = \sec t, \quad -\pi/2 < t < \pi/2$
5. $y''' - y'' + y' - y = e^{-t} \sin t$
6. $y^{(4)} + 2y'' + y = \sin t$

In each of Problems 7 and 8, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

7. $y''' - y'' + y' - y = \sec t, \quad -\pi/2 < t < \pi/2$
8. $y''' - y' = \csc t, \quad 0 < t < \pi$

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + y' = \sec t; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -2$
10. $y^{(4)} + 2y'' + y = \sin t; \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 1$
11. $y''' - y'' + y' - y = \sec t; \quad y(0) = 2, \quad y'(0) = -1, \quad y''(0) = 1$
12. $y''' - y' = \csc t; \quad y(\pi/2) = 2, \quad y'(\pi/2) = 1, \quad y''(\pi/2) = -1$

13. Given that x, x^2, and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4, \qquad x > 0,$$

determine a particular solution.

14. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - y'' + y' - y = g(t).$$

15. Find a formula involving integrals for a particular solution of the differential equation

$$y^{(4)} - y = g(t).$$

Hint: The functions $\sin t$, $\cos t$, $\sinh t$, and $\cosh t$ form a fundamental set of solutions of the homogeneous equation.

16. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - 3y'' + 3y' - y = g(t).$$

If $g(t) = t^{-2}e^t$, determine $Y(t)$.

17. Find a formula involving integrals for a particular solution of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = g(x), \qquad x > 0.$$

Hint: Verify that x, x^2, and x^3 are solutions of the homogeneous equation.

REFERENCES

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