Introduction to Differential Equations
Assignment # 9

Date Given: June 6, 2022
Date Due: June 13, 2022

P1. (1 points) Use integration by parts to find the Laplace transform of $f(t) = t^n e^{at}$, where a is a real constant, and n is a non-negative integer.

Solution: Denote the Laplace transform $\mathcal{L}[f(t)]$ as $F_n(s)$. Then, using integration by parts, we get

$$F_n(s) = \int_0^\infty e^{-st}e^{at}t^n \mathrm{d}t = \int_0^\infty t^n e^{-(s-a)t}\mathrm{d}t = \int_0^\infty t^n \mathrm{d}\left(-\frac{1}{s-a}e^{-(s-a)t}\right) =$$
$$-\left.\frac{t^n e^{-(s-a)t}}{s-a}\right|_0^\infty + \frac{n}{s-a}\int_0^\infty t^{n-1}e^{-(s-a)t}\mathrm{d}t = \frac{n}{s-a}F_{n-1}(s)$$

Therefore we obtain by induction

$$F_n(s) = \frac{n}{s-a}F_{n-1}(s) = \frac{n(n-1)}{(s-a)^2}F_{n-2}(s) = \ldots = \frac{n!}{(s-a)^{n-1}}F_1(s) = \frac{n!}{(s-a)^n}F_0(s)$$

Now, since

$$F_0(s) = \int_0^\infty e^{-(s-a)t}\mathrm{d}t = \frac{1}{s-a},$$

we have

$$F_n(s) = \frac{n!}{(s-a)^{n+1}}$$

P2. (1 point) Find the Laplace transform of

$$f(t) = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 0, & \text{if } 1 \le t < \infty \end{cases}$$

Solution: Using the fact that $f(t) = 0$ when $t \ge 1$ we obtain (after integration by parts) that

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)\mathrm{d}t = \int_0^1 te^{-st}\mathrm{d}t = \left[-\frac{e^{-st}}{s}t\right]_0^1 + \int_0^1 \frac{e^{-st}}{s}\mathrm{d}t =$$
$$-\frac{e^{-s}}{s} + \left[-\frac{e^{-st}}{s^2}\right]_0^1 = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} = \frac{1-e^{-s}(s+1)}{s^2}$$

P3. (1 point) Find the inverse Laplace transform of $F(s) = \dfrac{2s+1}{s^2-2s-2}$

Solution: The problem is solved by using partial fractions and algebra to manipulate the given function into a form matching one of the functions appearing in the middle column of Table 6.2.1.
Completing the square in the denominator, we have

$$\frac{2s+1}{s^2-2s-2} = \frac{2s+1}{(s-1)^2-3} = \frac{2(s-1)}{(s-1)^2-(\sqrt{3})^2} + \frac{3}{(s-1)^2-(\sqrt{3})^2}$$

Therefore, we establish (from lines 7,8, and 14, in Table 6.2.1) that

$$\mathcal{L}^{-1}[Y(s)] = 2e^t\cosh(\sqrt{3}t) + \sqrt{3}e^t\sinh(\sqrt{3}t)$$

P4. (1 point) Find the inverse Laplace transform of $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2+4)}$

Solution: Using partial fractions,

$$\frac{8s^2 - 4s + 12}{s(s^2+4)} = 3\frac{1}{s} + 5\frac{s}{s^2+4} - 2\frac{2}{s^2+4}$$

Hence

$$\mathcal{L}^{-1}\left[Y(s)\right] = 3 + 5\cos 2t - 2\sin 2t$$

P5. (2 points) Use the Laplace transform to solve the following initial value problem: $y'' - 2y' + 4y = 0$, $y(0) = 2, y'(0) = 0$.

Solution: Taking the Laplace transform of the differential equation, using Eq. (1) and Eq. (3) of Section 6.2, we obtain $\left[s^2Y(s) - sy(0) - y'(0)\right] - 2\left[sY(s) - y(0)\right] + 4Y(s) = 0$, Using the initial conditions and solving for $Y(s)$, we obtain $Y(s) = (2s-4)/(s^2-2s+4)$. Completing the square in the denominator, we have

$$Y(s) = \frac{2s-4}{s^2-2s+4} = \frac{2s-4}{(s-1)^2+3} = \frac{2(s-1)}{(s-1)^2+3} - \frac{2}{(s-1)^2+3}$$

which (using line 14 in Table 6.2.1) gives

$$y(t) = 2e^t\cos(\sqrt{3}t) - \frac{2}{\sqrt{3}}e^t\sin(\sqrt{3}t)$$

P6. (2 points) Use the Laplace transform to solve the following initial value problem: $y'' - 2y' + 2y = e^{-t}$, $y(0) = 0, y'(0) = 1$.

Solution: Taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 2\left[sY(s) - y(0)\right] + 2Y(s) = \frac{1}{s+1}$$

Applying the initial conditions

$$s^2Y(s) - 2sY(s) + 2Y(s) - 1 = \frac{1}{s+1}$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2-2s+2} + \frac{1}{(s^2-2s+2)(s+1)}$$

Using partial fractions on the second term, we have

$$\frac{1}{(s^2-2s+2)(s+1} = \frac{1}{5}\frac{1}{s+1} + \frac{1}{5}\frac{3-s}{s^2-2s+2}$$

Therefore, we can write

$$Y(s) = \frac{1}{5}\frac{1}{s+1} + \frac{1}{5}\frac{8-s}{s^2-2s+2}$$

Completing the square in the denominator for the last term, we have

$$\frac{8-s}{s^2-2s+2} = -\frac{(s-1)-7}{(s-1)^2+1}$$

Therefore,

$$Y(s) = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)-7}{(s-1)^2+1}$$

Hence the solution of the initial value problem is

$$y(t) = \frac{1}{5}\left(e^{-t} - e^t\cos t + 7e^t\sin t\right)$$

P7. (2 points) Find the Laplace transform $Y(s) = \mathcal{L}[y]$ of the solution of the following initial value problem

$$y'' + 4y = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 1, & \text{if } 1 \le t < \infty \end{cases}$$

$y(0) = 0, y'(0) = 0.$

Solution: By the definition of the Laplace transform

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t) \mathrm{d}t = \int_0^1 te^{-st} \mathrm{d}t + \int_1^\infty e^{-st} \mathrm{d}t$$

The first integral is

$$\int_0^1 te^{-st} \mathrm{d}t = \left[-\frac{e^{-st}}{s} t\right]_0^1 + \int_0^1 \frac{e^{-st}}{s} \mathrm{d}t = -\frac{e^{-s}}{s} + \left[-\frac{e^{-st}}{s^2}\right]_0^1 = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s},$$

and the second one is

$$\int_1^\infty e^{-st} \mathrm{d}t = \left[-\frac{e^{-st}}{s}\right]_1^\infty = \frac{e^{-s}}{s}.$$

Therefore

$$\mathcal{L}\left[f(t)\right] = \frac{1 - e^{-s}}{s^2}$$

Now, taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 4Y(s) = \frac{1 - e^{-s}}{s^2}$$

Therefore, using the initial conditions and solving for $Y(s)$, we get

$$Y(s) = \frac{1 - e^{-s}}{s^2(s^2 + 4)}$$