Introduction to Differential Equations
Assignment # 8

Date Given: May 30, 2022
Date Due: June 6, 2022

P1. (1 point) Use the method of undetermined coefficients to find the general solution of the differential equation $y''' - 6y'' = 3 - \cos t$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^3 - 6r^2 = 0$ with roots $r_1 = r_2 = 0, r_3 = 6$. The solution of the homogeneous equation is $y_c(t) = c_1 + c_2 t + c_3 e^{6t}$. We can set $Y(t) = At^2 + B\cos t + C\sin t$. Substitution into the given equation gives $-12A = 3, 6B - C = -1, B + 6C = 0$. Then $A = -1/4, B = -6/37, C = 1/37$, and therefore $Y(t) = -\frac{1}{4}t^2 - \frac{6}{37}\cos t + \frac{1}{37}\sin t$. Thus the general solution is

$$y(t) = c_1 + c_2 t + c_3 e^{6t} - \frac{1}{4}t^2 - \frac{6}{37}\cos t + \frac{1}{37}\sin t.$$

P2. (1 point) Use the method of undetermined coefficients to find the general solution of the differential equation $y''' - 3y'' + 3y' - y = t - 4e^t$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^3 - 3r^2 + 3r - 1 = 0$ with roots $r_1 = r_2 = r_3 = 1$. The solution of the homogeneous equation is $y_c(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t$. We can set $Y(t) = At + B + Ct^3 e^t$. Substitution into the given equation gives $-A = 1, 3A - B = 0, 6C = -4$. Then $A = -1, B = -3, C = -2/3$, and therefore $Y(t) = -t - 3 - \frac{2}{3}t^3 e^t$. Thus the general solution is

$$y(t) = c_1 e^t + c_2 te^t + c_3 t^2 e^t - t - 3 - \frac{2}{3}t^3 e^t.$$

P3. (1 point) Use the method of undetermined coefficients to find the general solution of the differential equation $y^{(4)} - y'' = 4t + 2te^{-t}$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^4 - r^2 = 0$ with roots $r_1 = 0, r_2 = 0, r_3 = 1, r_4 = -1$. The solution of the homogeneous equation is $y_c(t) = c_1 + c_2 t + c_3 e^t + c_4 e^{-t}$. We can set $Y(t) = At^3 + Bt^2 + (Ct^2 + Dt)e^{-t}$. Substitution into the given equation gives $-6A = 4, -2B = 0, 10C - 2D = 0, -4C = 2$. Then $A = -2/3, B = 0, C = -1/2, D = -5/2$, and therefore $Y(t) = -\frac{2}{3}t^3 + (-\frac{1}{2}t^2 - \frac{5}{2}t)e^{-t}$. Thus the general solution is

$$y(t) = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} - \frac{2}{3}t^3 - \left(\frac{1}{2}t^2 + \frac{5}{2}t\right)e^{-t}.$$

P4. (2 points) Use the method of undetermined coefficients to find the general solution of the differential equation $y''' + 4y' = t$, $y(0) = 0$, $y'(0) = 0$, $y''(1) = 0$. Then plot a graph of the solution.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r(r^2 + 4) = 0$. The roots are $r_1 = 0$, $r_2 = 2\imath$, and $r_3 = 2\imath$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2 \sin 2t + c_3 \cos 2t$. Since $g(t)$ is a solution of the homogeneous problem, set $Y(t) = t(At + B)$. Substitution into the differential equation results in $A = 1/8$ and $B = 0$. Thus the general solution is $y(t) = c_1 + c_2 \cos 2t + c_3 \sin 2t + t^2/8$. Applying the initial conditions at this point ($t = 0$), we obtain that $y(0) = c_1 + c_2 = 0$, $y'(0) = 2c_3 = 0$, and $y''(0) = -4c_2 + 1/4 = 0$. This gives $c_1 = 3/16$, $c_2 = -3/16$, $c_3 = 0$. Thus the solution is

$$y(t) = \frac{3}{16} - \frac{3}{16}\cos 2t + t^2/8.$$

We can see that for $t = \pi, 2\pi, \ldots$ the graph of the solution (plotted in blue in Figure 1) will be tangent to $t^4/8$ (plotted in yellow) and for large t the graph can be approximated by $t^2/8$.

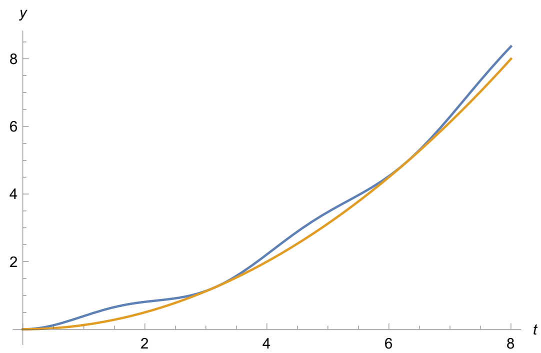

Figure 1: Illustration to problem P4.

P5. (2 points) Use the method of variation of parameters to find the general solution of the differential equation $y''' - 3y'' + 2y' = \frac{e^{2t}}{1+e^t}$.

Solution:

(a) The characteristic equation $r^3 - 3r^2 + 2r = 0$ has three roots, $r_1 = 0, r_2 = 1, r_3 = 2$. The solution of the homogeneous equation is $y_c(t) = c_1 + c_2 et + c_3 e^{2t}$. The functions $y_1(t) = 1$ and $y_2(t) = et$ and $y_3(t) = e^{2t}$ form a fundamental set of solutions. The Wronskian of these functions is

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & e^t & e^{2t} \\ 0 & e^t & 2e^{2t} \\ 0 & e^t & 4e^{2t} \end{vmatrix} = 2e^{3t}.$$

(b) Here $g(t) = \frac{e^{2t}}{1+e^t}$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t)$, in which

$$u_1'(t) = W_1/W = \frac{1}{2e^{3t}} \begin{vmatrix} 0 & e^t & e^{2t} \\ 0 & e^t & 2e^{2t} \\ \frac{e^{2t}}{1+e^t} & e^t & 4e^{2t} \end{vmatrix} = \frac{e^{2t}}{2(1+e^t)}$$

$$u_2'(t) = W_2/W = \frac{1}{2e^{3t}} \begin{vmatrix} 1 & 0 & e^{2t} \\ 0 & 0 & 2e^{2t} \\ 0 & \frac{e^{2t}}{1+e^t} & 4e^{2t} \end{vmatrix} = -\frac{e^t}{1+e^t},$$

$$u_3'(t) = W_3/W = \frac{1}{2e^{3t}} \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & \frac{e^{2t}}{1+e^t} \end{vmatrix} = \frac{1}{2(1+e^t)}.$$

Therefore,

$$\begin{aligned} u_1(t) &= \int \left(\frac{e^{2t}}{2(1+e^t)}\right) \mathrm{d}t = \tfrac{1}{2}\left(e^t - \ln(1+e^t)\right), \\ u_2(t) &= -\int \left(\frac{e^t}{1+e^t}\right) \mathrm{d}t = -\ln(1+e^t), \\ u_3(t) &= \int \left(\frac{1}{2(1+e^t)}\right) \mathrm{d}t = \tfrac{1}{2}\left(t - \ln(1+e^t)\right). \end{aligned}$$

Hence the particular solution is

$$\begin{aligned} Y(t) = \frac{1}{2}\left(e^t - \ln(1+e^t)\right) - e^t \ln(1+e^t) + e^{2t}\frac{1}{2}\left(t - \ln(1+e^t)\right) \\ = \frac{1}{2}\left(e^t + te^{2t}\right) - \frac{1}{2}\left(1 + 2e^t + e^{2t}\right)\ln(1+e^t) \\ = \frac{1}{2}\left\{e^t(1+te^t) - (1+e^t)^2 \ln(1+e^t)\right\}. \end{aligned}$$

The general solution is given by

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} + \frac{1}{2}\left\{e^t(1+te^t) - (1+e^t)^2 \ln(1+e^t)\right\}.$$

P6. (2 points) Use the method of variation of parameters to determine the general solution of the differential equation $y''' + y' = \sec t$, $-\pi/2 < t < \pi/2$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r(r^2+1) = 0$. The roots are $r_1 = 0$, $r_2 = \imath$, and $r_3 = -\imath$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2 \cos t + c_3 \sin t$. The Wronskian is evaluated as

$$W(1, \cos t, \sin t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1.$$

Now compute the three determinants

$$W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1, \quad W_2 = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t, \quad W_3 = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t.$$

The solution of the system of equations is

$$u_1'(t) = \frac{\sec t W_1(t)}{W(t)} = \sec t, \quad u_2'(t) = \frac{\sec t W_2(t)}{W(t)} = -1, \quad u_3'(t) = \frac{\sec t W_3(t)}{W(t)} = -\sin t / \cos t.$$

Hence $u_1(t) = \ln(\sec t + \tan t)$, $u_2(t) = -t$, and $u_3(t) = \ln(\cos t)$. The particular solution becomes $Y(t) = \ln(\sec t + \tan t) - t\cos t + \sin t \ln(\cos t)$. The constant (-1) is a solution of the homogeneous equation, therefore the general solution is

$$y(t) = c_1 + c_2 \cos t + c_3 \sin t + \ln(\sec t + \tan t) - t\cos t + \sin t \ln(\cos t).$$