## 7.6 Complex Eigenvalues

In this section we consider again a system of $n$ linear homogeneous equations with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where the coefficient matrix $\mathbf{A}$ is real-valued. If we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$, then it follows, as in Section 7.5, that $r$ must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix $\mathbf{A}$. Recall that the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ are the roots of the characteristic equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0 \tag{2}$$

and that the corresponding eigenvectors satisfy

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{3}$$

If $\mathbf{A}$ is real, then the coefficients in the polynomial equation (2) for $r$ are real, and any complex eigenvalues must occur in conjugate pairs. For example, if $r_1 = \lambda + i\mu$, where $\lambda$ and $\mu$ are real, is an eigenvalue of $\mathbf{A}$, then so is $r_2 = \lambda - i\mu$. To explore the effect of complex eigenvalues, we begin with an example.

**EXAMPLE 1**

Find a fundamental set of real-valued solutions of the system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}. \tag{4}$$

Plot a phase portrait and graphs of components of typical solutions.

A direction field for the system (4) is shown in Figure 7.6.1. This plot suggests that the trajectories in the phase plane spiral clockwise toward the origin.

To find a fundamental set of solutions, we assume that

$$\mathbf{x} = \boldsymbol{\xi}e^{rt} \tag{5}$$

and obtain the set of linear algebraic equations

$$\begin{pmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

for the eigenvalues and eigenvectors of $\mathbf{A}$. The characteristic equation is

$$\begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = r^2 + r + \tfrac{5}{4} = 0; \tag{7}$$

therefore the eigenvalues are $r_1 = -\frac{1}{2} + i$ and $r_2 = -\frac{1}{2} - i$. From Eq. (6) a straightforward calculation shows that the corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{8}$$

Observe that the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are also complex conjugates. Hence a fundamental set of solutions of the system (4) is

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1/2+i)t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1/2-i)t}. \tag{9}$$

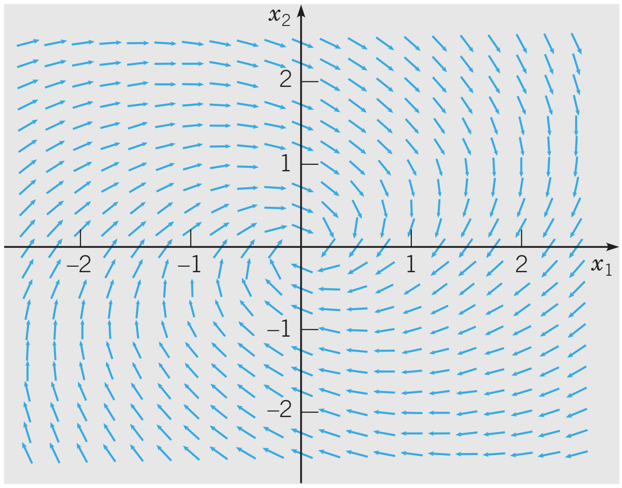



**FIGURE 7.6.1** A direction field for the system (4).

To obtain a set of real-valued solutions, we can (by Theorem 7.4.5) choose the real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. In fact,

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t/2}(\cos t + i \sin t) = \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + i \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}. \tag{10}$$

Hence a set of real-valued solutions of (Eq. 4) is

$$\mathbf{u}(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \mathbf{v}(t) = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \tag{11}$$

To verify that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, we compute their Wronskian:

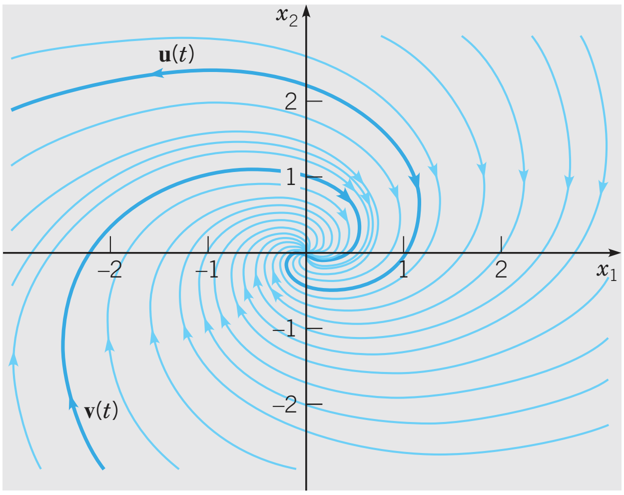
$$\begin{aligned} W(\mathbf{u}, \mathbf{v})(t) &= \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} \\ &= e^{-t}(\cos^2 t + \sin^2 t) = e^{-t}. \end{aligned}$$

The Wronskian $W(\mathbf{u}, \mathbf{v})(t)$ is never zero, so it follows that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ constitute a fundamental set of (real-valued) solutions of the system (4).

The graphs of the solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are shown in Figure 7.6.2*a*. Since

$$\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the graphs of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ pass through the points $(1, 0)$ and $(0, 1)$, respectively. Other solutions of the system (4) are linear combinations of $\mathbf{u}(t)$ and $\mathbf{v}(t)$, and graphs of a few of these solutions are also shown in Figure 7.6.2*a*; this figure is a phase portrait for the system (4). Each trajectory approaches the origin along a spiral path as $t \to \infty$, making infinitely many circuits about the origin; this is due to the fact that the solutions (11) are products of decaying exponential and sine or cosine factors. Some typical graphs of $x_1$ versus $t$ are shown in Figure 7.6.2*b*; each one represents a decaying oscillation in time.



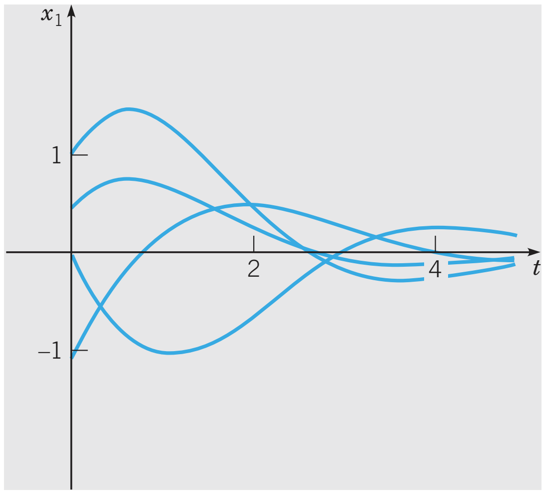




**FIGURE 7.6.2** (*a*) A phase portrait for the system (4); the origin is a spiral point. (*b*) Plots of $x_1$ versus $t$ for the system (4); graphs of $x_2$ versus $t$ are similar.

Figure 7.6.2*a* is typical of all $2 \times 2$ systems $\mathbf{x}' = \mathbf{Ax}$ whose eigenvalues are complex with negative real part. The origin is called a **spiral point** and is asymptotically stable because all trajectories approach it as $t$ increases. For a system whose eigenvalues have a positive real part, the trajectories are similar to those in Figure 7.6.2*a*, but the direction of motion is away from the origin, and the trajectories become unbounded. In this case, the origin is unstable. If the real part of the eigenvalues is zero, then the trajectories neither approach the origin nor become unbounded but instead repeatedly traverse a closed curve about the origin. Examples of this behavior can be seen in Figures 7.6.3*b* and 7.6.4*b* below. In this case the origin is called a **center** and is said to be stable, but not asymptotically stable. In all three cases, the direction of motion may be either clockwise, as in this example, or counterclockwise, depending on the elements of the coefficient matrix **A**.

The phase portrait in Figure 7.6.2*a* was drawn by a computer, but it is possible to produce a useful sketch of the phase portrait by hand. We have noted that when the eigenvalues $\lambda \pm i\mu$ are complex, then the trajectories either spiral in ($\lambda < 0$), spiral out ($\lambda > 0$), or repeatedly traverse a closed curve ($\lambda = 0$). To determine whether the direction of motion is clockwise or counterclockwise, we only need to determine the direction of motion at a single convenient point. For instance, in the system (4) we might choose $\mathbf{x} = (0, 1)^T$. Then $\mathbf{Ax} = (1, -\frac{1}{2})^T$. Thus at the point $(0, 1)$ in the phase plane the tangent vector $\mathbf{x}'$ to the trajectory at that point has a positive $x_1$-component and therefore is directed from the second quadrant into the first. The direction of motion is therefore clockwise for the trajectories of this system.

Returning to the general equation (1)

$$\mathbf{x}' = \mathbf{Ax},$$

we can proceed just as in the example. Suppose that there is a pair of complex conjugate eigenvalues, $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$. Then the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are also complex conjugates. To see that this is so, recall that

$r_1$ and $\boldsymbol{\xi}^{(1)}$ satisfy

$$(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}. \tag{12}$$

On taking the complex conjugate of this equation and noting that **A** and **I** are real-valued, we obtain

$$(\mathbf{A} - \bar{r}_1\mathbf{I})\bar{\boldsymbol{\xi}}^{(1)} = \mathbf{0}, \tag{13}$$

where $\bar{r}_1$ and $\bar{\boldsymbol{\xi}}^{(1)}$ are the complex conjugates of $r_1$ and $\boldsymbol{\xi}^{(1)}$, respectively. In other words, $r_2 = \bar{r}_1$ is also an eigenvalue, and $\boldsymbol{\xi}^{(2)} = \bar{\boldsymbol{\xi}}^{(1)}$ is a corresponding eigenvector. The corresponding solutions

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{r_1 t}, \qquad \mathbf{x}^{(2)}(t) = \bar{\boldsymbol{\xi}}^{(1)}e^{\bar{r}_1 t} \tag{14}$$

of the differential equation (1) are then complex conjugates of each other. Therefore, as in Example 1, we can find two real-valued solutions of Eq. (1) corresponding to the eigenvalues $r_1$ and $r_2$ by taking the real and imaginary parts of $\mathbf{x}^{(1)}(t)$ or $\mathbf{x}^{(2)}(t)$ given by Eq. (14).

Let us write $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$, where **a** and **b** are real; then we have

$$\begin{aligned}\mathbf{x}^{(1)}(t) &= (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} \\ &= (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t).\end{aligned} \tag{15}$$

Upon separating $\mathbf{x}^{(1)}(t)$ into its real and imaginary parts, we obtain

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t). \tag{16}$$

If we write $\mathbf{x}^{(1)}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$, then the vectors

$$\begin{aligned}\mathbf{u}(t) &= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t), \\ \mathbf{v}(t) &= e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)\end{aligned} \tag{17}$$

are real-valued solutions of Eq. (1). It is possible to show that **u** and **v** are linearly independent solutions (see Problem 27).

For example, suppose that the matrix **A** has two complex eigenvalues $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, and that $r_3, \dots, r_n$ are all real and distinct. Let the corresponding eigenvectors be $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$, $\boldsymbol{\xi}^{(2)} = \mathbf{a} - i\mathbf{b}$, $\boldsymbol{\xi}^{(3)}, \dots, \boldsymbol{\xi}^{(n)}$. Then the general solution of Eq. (1) is

$$\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t) + c_3\boldsymbol{\xi}^{(3)}e^{r_3 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}, \tag{18}$$

where $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are given by Eqs. (17). We emphasize that this analysis applies only if the coefficient matrix **A** in Eq. (1) is real, for it is only then that complex eigenvalues and eigenvectors must occur in conjugate pairs.

For $2 \times 2$ systems with real coefficients, we have now completed our description of the three main cases that can occur.

1. Eigenvalues are real and have opposite signs; $\mathbf{x} = \mathbf{0}$ is a saddle point.
2. Eigenvalues are real and have the same sign but are unequal; $\mathbf{x} = \mathbf{0}$ is a node.
3. Eigenvalues are complex with nonzero real part; $\mathbf{x} = \mathbf{0}$ is a spiral point.

Other possibilities are of less importance and occur as transitions between two of the cases just listed. For example, a zero eigenvalue occurs during the transition between a saddle point and a node. Purely imaginary eigenvalues occur during a transition between asymptotically stable and unstable spiral points. Finally, real and equal eigenvalues appear during the transition between nodes and spiral points.

**EXAMPLE 2**

The system

$$\mathbf{x}' = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x} \tag{19}$$

contains a parameter $\alpha$. Describe how the solutions depend qualitatively on $\alpha$; in particular, find the critical values of $\alpha$ at which the qualitative behavior of the trajectories in the phase plane changes markedly.

The behavior of the trajectories is controlled by the eigenvalues of the coefficient matrix. The characteristic equation is

$$r^2 - \alpha r + 4 = 0, \tag{20}$$

so the eigenvalues are

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}. \tag{21}$$

From Eq. (21) it follows that the eigenvalues are complex conjugates for $-4 < \alpha < 4$ and are real otherwise. Thus two critical values are $\alpha = -4$ and $\alpha = 4$, where the eigenvalues change from real to complex, or vice versa. For $\alpha < -4$ both eigenvalues are negative, so all trajectories approach the origin, which is an asymptotically stable node. For $\alpha > 4$ both eigenvalues are positive, so the origin is again a node, this time unstable; all trajectories (except $\mathbf{x} = \mathbf{0}$) become unbounded. In the intermediate range, $-4 < \alpha < 4$, the eigenvalues are complex and the trajectories are spirals. However, for $-4 < \alpha < 0$ the real part of the eigenvalues is negative, the spirals are directed inward, and the origin is asymptotically stable, whereas for $0 < \alpha < 4$ the real part of the eigenvalues is positive and the origin is unstable. Thus $\alpha = 0$ is also a critical value where the direction of the spirals changes from inward to outward. For this value of $\alpha$, the origin is a center and the trajectories are closed curves about the origin, corresponding to solutions that are periodic in time. The other critical values, $\alpha = \pm 4$, yield eigenvalues that are real and equal. In this case the origin is again a node, but the phase portrait differs somewhat from those in Section 7.5. We take up this case in Section 7.8.

***A Multiple Spring–Mass System.*** Consider the system of two masses and three springs shown in Figure 7.1.1, whose equations of motion are given by Eqs. (1) in Section 7.1. If we assume that there are no external forces, then $F_1(t) = 0, F_2(t) = 0$, and the resulting equations are

$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= -(k_1 + k_2)x_1 + k_2 x_2, \\ m_2 \frac{d^2 x_2}{dt^2} &= k_2 x_1 - (k_2 + k_3)x_2. \end{aligned} \tag{22}$$

These equations can be solved as a system of two second order equations (see Problem 29), but, as is consistent with our approach in this chapter, we will transform them into a system of four first order equations. Let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, and $y_4 = x_2'$. Then

$$y_1' = y_3, \qquad y_2' = y_4, \tag{23}$$

and, from Eqs. (22),

$$m_1 y_3' = -(k_1 + k_2)y_1 + k_2 y_2, \qquad m_2 y_4' = k_2 y_1 - (k_2 + k_3)y_2. \tag{24}$$

The following example deals with a particular case of this two-mass, three-spring system.

**EXAMPLE 3**

Suppose that $m_1 = 2$, $m_2 = 9/4$, $k_1 = 1$, $k_2 = 3$, and $k_3 = 15/4$ in Eqs. (23) and (24) so that these equations become

$$y_1' = y_3, \qquad y_2' = y_4, \qquad y_3' = -2y_1 + \tfrac{3}{2}y_2, \qquad y_4' = \tfrac{4}{3}y_1 - 3y_2. \tag{25}$$

Analyze the possible motions described by Eqs. (25), and draw graphs showing typical behavior.

We can write the system (25) in matrix form as

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{pmatrix} \mathbf{y} = \mathbf{A}\mathbf{y}. \tag{26}$$

Keep in mind that $y_1$ and $y_2$ are the positions of the two masses, relative to their equilibrium positions, and that $y_3$ and $y_4$ are their velocities. We assume, as usual, that $\mathbf{y} = \boldsymbol{\xi} e^{rt}$, where $r$ must be an eigenvalue of the matrix $\mathbf{A}$ and $\boldsymbol{\xi}$ a corresponding eigenvector. It is possible, though a bit tedious, to find the eigenvalues and eigenvectors of $\mathbf{A}$ by hand, but it is easy with appropriate computer software. The characteristic polynomial of $\mathbf{A}$ is

$$r^4 + 5r^2 + 4 = (r^2 + 1)(r^2 + 4), \tag{27}$$

so the eigenvalues are $r_1 = i$, $r_2 = -i$, $r_3 = 2i$, and $r_4 = -2i$. The corresponding eigenvectors are

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \qquad \boldsymbol{\xi}^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix}. \tag{28}$$

The complex-valued solutions $\boldsymbol{\xi}^{(1)}e^{it}$ and $\boldsymbol{\xi}^{(2)}e^{-it}$ are complex conjugates, so two real-valued solutions can be found by finding the real and imaginary parts of either of them. For instance, we have

$$\begin{aligned} \boldsymbol{\xi}^{(1)}e^{it} &= \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) \\ &= \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + i \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} = \mathbf{u}^{(1)}(t) + i\mathbf{v}^{(1)}(t). \end{aligned} \tag{29}$$

In a similar way, we obtain

$$\boldsymbol{\xi}^{(3)}e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}(\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + i \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} = \mathbf{u}^{(2)}(t) + i\mathbf{v}^{(2)}(t). \qquad (30)$$

We leave it to you to verify that $\mathbf{u}^{(1)}$, $\mathbf{v}^{(1)}$, $\mathbf{u}^{(2)}$, and $\mathbf{v}^{(2)}$ are linearly independent and therefore form a fundamental set of solutions. Thus the general solution of Eq. (26) is

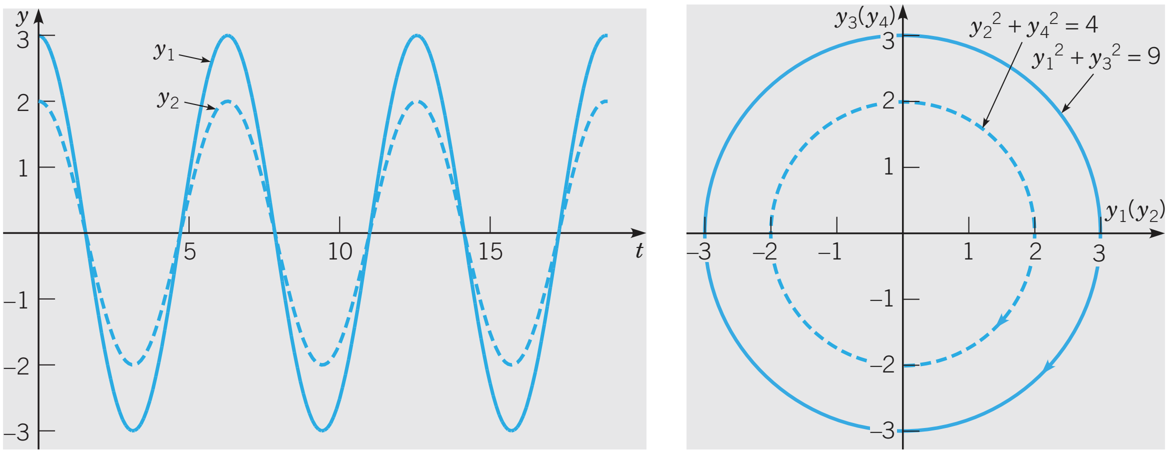
$$\mathbf{y} = c_1 \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} + c_3 \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + c_4 \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix}, \qquad (31)$$

where $c_1, c_2, c_3$, and $c_4$ are arbitrary constants.

The phase space for this system is four-dimensional, and each solution, obtained by a particular set of values for $c_1, \dots, c_4$ in Eq. (31), corresponds to a trajectory in this space. Since each solution, given by Eq. (31), is periodic with period $2\pi$, each trajectory is a closed curve. No matter where the trajectory starts at $t = 0$, it returns to that point at $t = 2\pi$, $t = 4\pi$, and so forth, repeatedly traversing the same curve in each time interval of length $2\pi$. We do not attempt to show any of these four-dimensional trajectories here. Instead, in the figures below we show projections of certain trajectories in the $y_1y_3$- or $y_2y_4$-plane, thereby showing the motion of each mass separately.

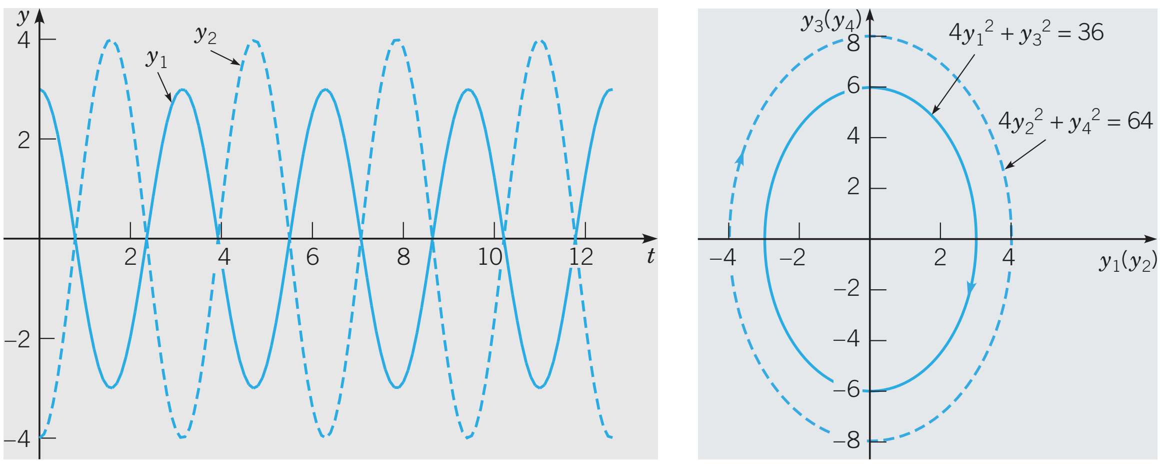
The first two terms on the right side of Eq. (31) describe motions with frequency 1 and period $2\pi$. Note that $y_2 = (2/3)y_1$ in these terms and that $y_4 = (2/3)y_3$. This means that the two masses move back and forth together, always going in the same direction, but with the second mass moving only two-thirds as far as the first mass. If we focus on the solution $\mathbf{u}^{(1)}(t)$ and plot $y_1$ versus $t$ and $y_2$ versus $t$ on the same axes, we obtain the cosine graphs of amplitude 3 and 2, respectively, shown in Figure 7.6.3*a*. The trajectory of the first mass in the $y_1y_3$-plane lies on the circle of radius 3 shown in Figure 7.6.3*b*, traversed clockwise starting at the point $(3, 0)$ and completing a circuit in time $2\pi$. Also shown in this figure is the trajectory of the second mass in the $y_2y_4$-plane, which lies on the circle of radius 2, also traversed clockwise starting at $(2, 0)$ and also completing a circuit in time $2\pi$. The origin is a center in the respective $y_1y_3$- and $y_2y_4$-planes. Similar graphs (with an appropriate shift in time) are obtained from $\mathbf{v}^{(1)}$ or from a linear combination of $\mathbf{u}^{(1)}$ and $\mathbf{v}^{(1)}$.

The remaining terms on the right side of Eq. (31) describe motions with frequency 2 and period $\pi$. Observe that in this case, $y_2 = -(4/3)y_1$ and $y_4 = -(4/3)y_3$. This means that the two masses are always moving in opposite directions and that the second mass moves four-thirds as far as the first mass. If we look only at $\mathbf{u}^{(2)}(t)$ and plot $y_1$ versus $t$ and $y_2$ versus $t$ on the same axes, we obtain Figure 7.6.4*a*. There is a phase difference of $\pi$, and the amplitude of $y_2$ is four-thirds that of $y_1$, confirming the preceding statements about the motions of the masses. Figure 7.6.4*b* shows a superposition of the trajectories for the two masses in their respective phase planes. Both graphs are ellipses, the inner one corresponding to the first mass and the outer one to the second. The trajectory on the inner ellipse starts at $(3, 0)$, and the trajectory on the outer ellipse starts at $(-4, 0)$. Both are traversed clockwise, and a circuit is completed in time $\pi$. The origin is a center in the respective $y_1y_3$- and $y_2y_4$-planes. Once again, similar graphs are obtained from $\mathbf{v}^{(2)}$ or from a linear combination of $\mathbf{u}^{(2)}$ and $\mathbf{v}^{(2)}$.




(a) (b)

**FIGURE 7.6.3** (a) A plot of $y_1$ versus $t$ and $y_2$ versus $t$ for the solution $\mathbf{u}^{(1)}(t)$. (b) Superposition of projections of trajectories in the $y_1y_3$- and $y_2y_4$-planes for the solution $\mathbf{u}^{(1)}(t)$.



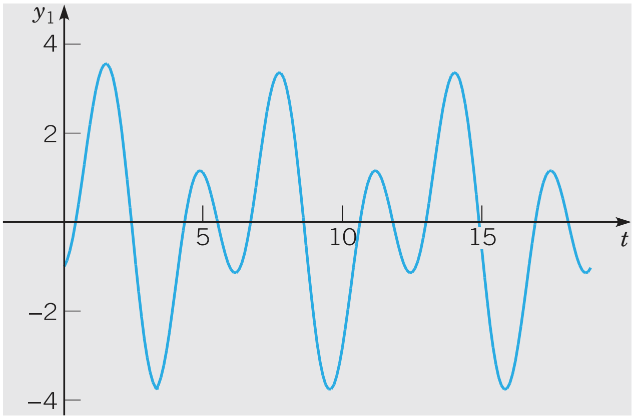

(a) (b)

**FIGURE 7.6.4** (a) A plot of $y_1$ versus $t$ and $y_2$ versus $t$ for the solution $\mathbf{u}^{(2)}(t)$. (b) Superposition of projections of trajectories in the $y_1y_3$- and $y_2y_4$-planes for the solution $\mathbf{u}^{(2)}(t)$.

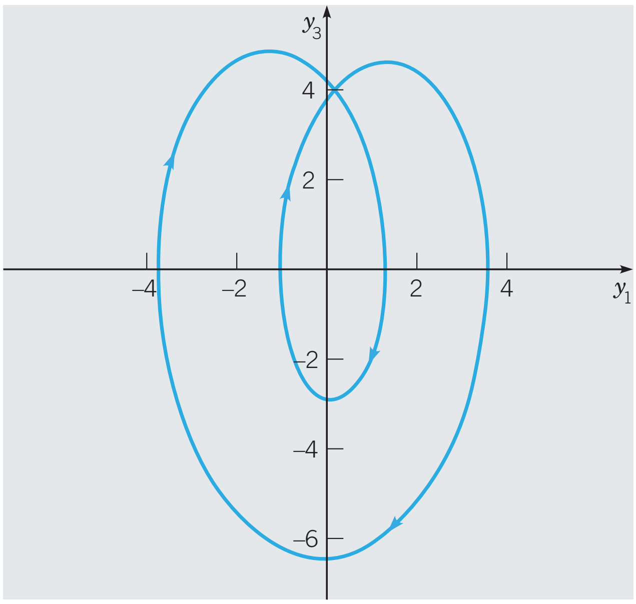
The types of motion described in the two preceding paragraphs are called **fundamental modes** of vibration for the two-mass system. Each of them results from fairly special initial conditions. For example, to obtain the fundamental mode of frequency 1, both of the constants $c_3$ and $c_4$ in Eq. (31) must be zero. This occurs only for initial conditions in which

$3y_2(0) = 2y_1(0)$ and $3y_4(0) = 2y_3(0)$. Similarly, the mode of frequency 2 is obtained only when both of the constants $c_1$ and $c_2$ in Eq. (31) are zero—that is, when the initial conditions are such that $3y_2(0) = -4y_1(0)$ and $3y_4(0) = -4y_3(0)$.

For more general initial conditions the solution is a combination of the two fundamental modes. A plot of $y_1$ versus $t$ for a typical case is shown in Figure 7.6.5*a*, and the projection of the corresponding trajectory in the $y_1y_3$-plane is shown in Figure 7.6.5*b*. Observe that this latter figure may be a bit misleading in that it shows the projection of the trajectory crossing itself. This cannot be the case for the actual trajectory in four dimensions, because it would violate the general uniqueness theorem: there cannot be two different solutions issuing from the same initial point.




(*a*)




(*b*)

**FIGURE 7.6.5** A solution of the system (25) satisfying the initial condition $\mathbf{y}(0) = (-1, 4, 1, 1)^T$. (*a*) A plot of $y_1$ versus $t$. (*b*) The projection of the trajectory in the $y_1y_3$-plane. As stated in the text, the actual trajectory in four dimensions does not intersect itself.

## PROBLEMS

In each of Problems 1 through 6:

(a) Express the general solution of the given system of equations in terms of real-valued functions.

(b) Also draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 7 and 8, express the general solution of the given system of equations in terms of real-valued functions.

7. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 9 and 10, find the solution of the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

9. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the $x_1x_2$-plane.

(c) For your trajectory in part (b), draw the graphs of $x_1$ versus $t$ and of $x_2$ versus $t$.

(d) For your trajectory in part (b), draw the corresponding graph in three-dimensional $tx_1x_2$-space.

11. $\mathbf{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix} \mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{pmatrix} \mathbf{x}$

In each of Problems 13 through 20, the coefficient matrix contains a parameter $\alpha$. In each of these problems:

(a) Determine the eigenvalues in terms of $\alpha$.

(b) Find the critical value or values of $\alpha$ where the qualitative nature of the phase portrait for the system changes.

(c) Draw a phase portrait for a value of $\alpha$ slightly below, and for another value slightly above, each critical value.

13. $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$

15. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}$

16. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$

17. $\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}$

18. $\mathbf{x}' = \begin{pmatrix} 3 & \alpha \\ -6 & -4 \end{pmatrix} \mathbf{x}$

19. $\mathbf{x}' = \begin{pmatrix} \alpha & 10 \\ -1 & -4 \end{pmatrix} \mathbf{x}$

20. $\mathbf{x}' = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix} \mathbf{x}$

In each of Problems 21 and 22, solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

21. $t\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 23 and 24:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the $x_1x_2$-plane. Also draw the trajectories in the $x_1x_3$- and $x_2x_3$-planes.

(c) For the initial point in part (b), draw the corresponding trajectory in $x_1x_2x_3$-space.

23. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x}$

24. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \mathbf{x}$

25. Consider the electric circuit shown in Figure 7.6.6. Suppose that $R_1 = R_2 = 4\ \Omega$, $C = \frac{1}{2}$ F, and $L = 8$ H.

(a) Show that this circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}, \tag{i}$$

where $I$ is the current through the inductor and $V$ is the voltage drop across the capacitor. *Hint:* See Problem 20 of Section 7.1.

(b) Find the general solution of Eqs. (i) in terms of real-valued functions.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 3$ V.

(d) Determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

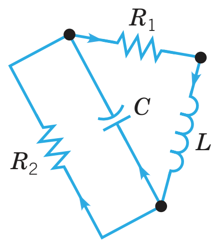



**FIGURE 7.6.6** The circuit in Problem 25.

26. The electric circuit shown in Figure 7.6.7 is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}, \tag{i}$$

where $I$ is the current through the inductor and $V$ is the voltage drop across the capacitor. These differential equations were derived in Problem 19 of Section 7.1.

(a) Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$; show that they are complex conjugates if $L < 4R^2C$.

(b) Suppose that $R = 1\ \Omega$, $C = \frac{1}{2}$ F, and $L = 1$ H. Find the general solution of the system (i) in this case.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 1$ V.

(d) For the circuit of part (b) determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

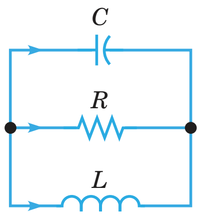



**FIGURE 7.6.7** The circuit in Problem 26.

27. In this problem we indicate how to show that $\mathbf{u}(t)$ and $\mathbf{v}$(t), as given by Eqs. (17), are linearly independent. Let $r_1 = \lambda + i\mu$ and $\bar{r}_1 = \lambda - i\mu$ be a pair of conjugate eigenvalues of the coefficient matrix $\mathbf{A}$ of Eq. (1); let $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ and $\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{a} - i\mathbf{b}$ be the corresponding eigenvectors. Recall that it was stated in Section 7.3 that two different eigenvalues have linearly independent eigenvectors, so if $r_1 \neq \bar{r}_1$, then $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}}^{(1)}$ are linearly independent.

(a) First we show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Consider the equation $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}}^{(1)}$, and then show that $(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{0}$.

(b) Show that $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$ and then that $c_1 = 0$ and $c_2 = 0$. Consequently, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.

(c) To show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, where $t_0$ is an arbitrary point. Rewrite this equation in terms of $\mathbf{a}$ and $\mathbf{b}$, and then proceed as in part (b) to show that $c_1 = 0$ and $c_2 = 0$. Hence $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent at the arbitrary point $t_0$. Therefore, they are linearly independent at every point and on every interval.

28. A mass $m$ on a spring with constant $k$ satisfies the differential equation (see Section 3.7)

$$mu'' + ku = 0,$$

where $u(t)$ is the displacement at time $t$ of the mass from its equilibrium position.

(a) Let $x_1 = u$, $x_2 = u'$, and show that the resulting system is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} \mathbf{x}.$$

(b) Find the eigenvalues of the matrix for the system in part (a).

(c) Sketch several trajectories of the system. Choose one of your trajectories, and sketch the corresponding graphs of $x_1$ versus $t$ and $x_2$ versus $t$. Sketch both graphs on one set of axes.

(d) What is the relation between the eigenvalues of the coefficient matrix and the natural frequency of the spring–mass system?

29. Consider the two-mass, three-spring system of Example 3 in the text. Instead of converting the problem into a system of four first order equations, we indicate here how to proceed directly from Eqs. (22).

(a) Show that Eqs. (22) can be written in the form

$$\mathbf{x}'' = \begin{pmatrix} -2 & \frac{3}{2} \\ \frac{4}{3} & -3 \end{pmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}. \tag{i}$$

(b) Assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ and show that

$$(\mathbf{A} - r^2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

Note that $r^2$ (rather than $r$) is an eigenvalue of $\mathbf{A}$ corresponding to an eigenvector $\boldsymbol{\xi}$.

(c) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(d) Write down expressions for $x_1$ and $x_2$. There should be four arbitrary constants in these expressions.

(e) By differentiating the results from part (d), write down expressions for $x_1'$ and $x_2'$. Your results from parts (d) and (e) should agree with Eq. (31) in the text.

30. Consider the two-mass, three-spring system whose equations of motion are Eqs. (22) in the text. Let $m_1 = 1, m_2 = 4/3, k_1 = 1, k_2 = 3$, and $k_3 = 4/3$.

(a) As in the text, convert the system to four first order equations of the form $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Determine the coefficient matrix $\mathbf{A}$.

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(c) Write down the general solution of the system.

(d) Describe the fundamental modes of vibration. For each fundamental mode draw graphs of $y_1$ versus $t$ and $y_2$ versus $t$. Also draw the corresponding trajectories in the $y_1y_3$- and $y_2y_4$-planes.

(e) Consider the initial conditions $\mathbf{y}(0) = (2, 1, 0, 0)^T$. Evaluate the arbitrary constants in the general solution in part (c). What is the period of the motion in this case? Plot graphs of $y_1$ versus $t$ and $y_2$ versus $t$. Also plot the corresponding trajectories in the $y_1y_3$- and $y_2y_4$-planes. Be sure you understand how the trajectories are traversed for a full period.

(f) Consider other initial conditions of your own choice, and plot graphs similar to those requested in part (e).

31. Consider the two-mass, three-spring system whose equations of motion are Eqs. (22) in the text. Let $m_1 = m_2 = 1$ and $k_1 = k_2 = k_3 = 1$.

(a) As in the text, convert the system to four first order equations of the form $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Determine the coefficient matrix $\mathbf{A}$.

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(c) Write down the general solution of the system.

(d) Describe the fundamental modes of vibration. For each fundamental mode draw graphs of $y_1$ versus $t$ and $y_2$ versus $t$. Also draw the corresponding trajectories in the $y_1y_3$- and $y_2y_4$-planes.

(e) Consider the initial conditions $\mathbf{y}(0) = (-1, 3, 0, 0)^T$. Evaluate the arbitrary constants in the general solution in part (c). Plot $y_1$ versus $t$ and $y_2$ versus $t$. Do you think the solution is periodic? Also draw the trajectories in the $y_1y_3$- and $y_2y_4$-planes.

(f) Consider other initial conditions of your own choice, and plot graphs similar to those requested in part (e).

# 7.7 Fundamental Matrices

The structure of the solutions of systems of linear differential equations can be further illuminated by introducing the idea of a fundamental matrix. Suppose that $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{1}$$

on some interval $\alpha < t < \beta$. Then the matrix

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}, \tag{2}$$

whose columns are the vectors $\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)$, is said to be a **fundamental matrix** for the system (1). Note that a fundamental matrix is nonsingular since its columns are linearly independent vectors.

**EXAMPLE 1**

Find a fundamental matrix for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}. \tag{3}$$

In Example 2 of Section 7.5, we found that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

are linearly independent solutions of Eq. (3). Thus a fundamental matrix for the system (3) is

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}. \tag{4}$$

The solution of an initial value problem can be written very compactly in terms of a fundamental matrix. The general solution of Eq. (1) is

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{5}$$

or, in terms of $\boldsymbol{\Psi}(t)$,

$$\mathbf{x} = \boldsymbol{\Psi}(t)\mathbf{c}, \tag{6}$$

where $\mathbf{c}$ is a constant vector with arbitrary components $c_1, \dots, c_n$. For an initial value problem consisting of the differential equation (1) and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}^0, \tag{7}$$

where $t_0$ is a given point in $\alpha < t < \beta$ and $\mathbf{x}^0$ is a given initial vector, it is only necessary to choose the vector $\mathbf{c}$ in Eq. (6) so as to satisfy the initial condition (7). Hence $\mathbf{c}$ must satisfy

$$\boldsymbol{\Psi}(t_0)\mathbf{c} = \mathbf{x}^0. \tag{8}$$

Therefore, since $\boldsymbol{\Psi}(t_0)$ is nonsingular,

$$\mathbf{c} = \boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}^0 \tag{9}$$

and

$$\mathbf{x} = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)\mathbf{x}^0 \tag{10}$$

is the solution of the initial value problem (1), (7). We emphasize, however, that to solve a given initial value problem, we would ordinarily solve Eq. (8) by row reduction and then substitute for $\mathbf{c}$ in Eq. (6), rather than compute $\boldsymbol{\Psi}^{-1}(t_0)$ and use Eq. (10).

Recall that each column of the fundamental matrix $\boldsymbol{\Psi}$ is a solution of Eq. (1). It follows that $\boldsymbol{\Psi}$ satisfies the matrix differential equation

$$\boldsymbol{\Psi}' = \mathbf{P}(t)\boldsymbol{\Psi}. \tag{11}$$

This relation is readily confirmed by comparing the two sides of Eq. (11) column by column.

Sometimes it is convenient to make use of the special fundamental matrix, denoted by $\boldsymbol{\Phi}(t)$, whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ designated in Theorem 7.4.4. Besides the differential equation (1), these vectors satisfy the initial conditions

$$\mathbf{x}^{(j)}(t_0) = \mathbf{e}^{(j)}, \tag{12}$$

where $\mathbf{e}^{(j)}$ is the unit vector, defined in Theorem 7.4.4, with a one in the $j$th position and zeros elsewhere. Thus $\boldsymbol{\Phi}(t)$ has the property that

$$\boldsymbol{\Phi}(t_0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{I}. \tag{13}$$

We will always reserve the symbol $\boldsymbol{\Phi}$ to denote the fundamental matrix satisfying the initial condition (13) and use $\boldsymbol{\Psi}$ when an arbitrary fundamental matrix is intended. In terms of $\boldsymbol{\Phi}(t)$, the solution of the initial value problem (1), (7) is even simpler in appearance; since $\boldsymbol{\Phi}^{-1}(t_0) = \mathbf{I}$, it follows from Eq. (10) that

$$\mathbf{x} = \boldsymbol{\Phi}(t)\mathbf{x}^0. \tag{14}$$

Although the fundamental matrix $\boldsymbol{\Phi}(t)$ is often more complicated than $\boldsymbol{\Psi}(t)$, it is especially helpful if the same system of differential equations is to be solved repeatedly subject to many different initial conditions. This corresponds to a given physical system that can be started from many different initial states. If the fundamental matrix $\boldsymbol{\Phi}(t)$ has been determined, then the solution for each set of initial conditions can be found simply by matrix multiplication, as indicated by Eq. (14). The matrix $\boldsymbol{\Phi}(t)$ thus represents a transformation of the initial conditions $\mathbf{x}^0$ into the solution $\mathbf{x}(t)$ at an arbitrary time $t$. Comparing Eqs. (10) and (14) makes it clear that $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)$.

**EXAMPLE 2**

For the system (3)

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

in Example 1, find the fundamental matrix $\mathbf{\Phi}$ such that $\mathbf{\Phi}(0) = \mathbf{I}$.

The columns of $\mathbf{\Phi}$ are solutions of Eq. (3) that satisfy the initial conditions

$$\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{15}$$

Since the general solution of Eq. (3) is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t},$$

we can find the solution satisfying the first set of these initial conditions by choosing $c_1 = c_2 = \frac{1}{2}$; similarly, we obtain the solution satisfying the second set of initial conditions by choosing $c_1 = \frac{1}{4}$ and $c_2 = -\frac{1}{4}$. Hence

$$\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}. \tag{16}$$

Note that the elements of $\mathbf{\Phi}(t)$ are more complicated than those of the fundamental matrix $\mathbf{\Psi}(t)$ given by Eq. (4); however, it is now easy to determine the solution corresponding to any set of initial conditions.

***The Matrix* exp(A*t*).** Recall that the solution of the scalar initial value problem

$$x' = ax, \qquad x(0) = x_0, \tag{17}$$

where $a$ is a constant, is

$$x = x_0 \exp(at). \tag{18}$$

Now consider the corresponding initial value problem for an $n \times n$ system, namely,

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}^0, \tag{19}$$

where $\mathbf{A}$ is a constant matrix. Applying the results of this section to the problem (19), we can write its solution as

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0, \tag{20}$$

where $\mathbf{\Phi}(0) = \mathbf{I}$. Comparing the problems (17) and (19), and their solutions, suggests that the matrix $\mathbf{\Phi}(t)$ might have an exponential character. We now explore this possibility.

The scalar exponential function $\exp(at)$ can be represented by the power series

$$\exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}, \tag{21}$$

which converges for all $t$. Let us now replace the scalar $a$ by the $n \times n$ constant matrix $\mathbf{A}$ and consider the corresponding series

$$\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots. \tag{22}$$

Each term in the series (22) is an $n \times n$ matrix. It is possible to show that each element of this matrix sum converges for all $t$ as $n \to \infty$. Thus the series (22) defines as its sum a new matrix, which we denote by $\exp(\mathbf{A}t)$; that is,

$$\exp(\mathbf{A}t) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}, \tag{23}$$

analogous to the expansion (21) of the scalar function $\exp(at)$.

By differentiating the series (23) term by term, we obtain

$$\frac{d}{dt}[\exp(\mathbf{A}t)] = \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^{n-1}}{(n-1)!} = \mathbf{A}\left[\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}\right] = \mathbf{A}\exp(\mathbf{A}t). \tag{24}$$

Therefore, $\exp(\mathbf{A}t)$ satisfies the differential equation

$$\frac{d}{dt}\exp(\mathbf{A}t) = \mathbf{A}\exp(\mathbf{A}t). \tag{25}$$

Further, by setting $t = 0$ in Eq. (23) we find that $\exp(\mathbf{A}t)$ satisfies the initial condition

$$\exp(\mathbf{A}t)\Big|_{t=0} = \mathbf{I}. \tag{26}$$

The fundamental matrix $\mathbf{\Phi}$ satisfies the same initial value problem as $\exp(\mathbf{A}t)$, namely,

$$\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}, \qquad \mathbf{\Phi}(0) = \mathbf{I}. \tag{27}$$

Then, by the uniqueness part of Theorem 7.1.2 (extended to matrix differential equations), we conclude that $\exp(\mathbf{A}t)$ and the fundamental matrix $\mathbf{\Phi}(t)$ are the same. Thus we can write the solution of the initial value problem (19) in the form

$$\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}^0, \tag{28}$$

which is analogous to the solution (18) of the initial value problem (17).

In order to justify more conclusively the use of $\exp(\mathbf{A}t)$ for the sum of the series (22), we should demonstrate that this matrix function does indeed have the properties we associate with the exponential function. One way to do this is outlined in Problem 15.

***Diagonalizable Matrices.*** The basic reason why a system of linear (algebraic or differential) equations presents some difficulty is that the equations are usually *coupled*. In other words, some or all of the equations involve more than one—typically all—of the unknown variables. Hence the equations in the system must be solved *simultaneously*. In contrast, if each equation involves only a single variable, then each equation can be solved independently of all the others, which is a much easier task. This observation suggests that one way to solve a system of equations might be by transforming it into an equivalent *uncoupled* system in which each equation contains only one unknown variable. This corresponds to transforming the coefficient matrix $\mathbf{A}$ into a *diagonal* matrix.

Eigenvectors are useful in accomplishing such a transformation. Suppose that the $n \times n$ matrix $\mathbf{A}$ has a full set of $n$ linearly independent eigenvectors. Recall that this will certainly be the case if the eigenvalues of $\mathbf{A}$ are all different, or if $\mathbf{A}$ is Hermitian.

Letting $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ denote these eigenvectors and $\lambda_1, \dots, \lambda_n$ the corresponding eigenvalues, form the matrix $\mathbf{T}$ whose columns are the eigenvectors—that is,

$$\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}. \tag{29}$$

Since the columns of $\mathbf{T}$ are linearly independent vectors, $\det \mathbf{T} \neq 0$; hence $\mathbf{T}$ is nonsingular and $\mathbf{T}^{-1}$ exists. A straightforward calculation shows that the columns of the matrix $\mathbf{AT}$ are just the vectors $\mathbf{A}\boldsymbol{\xi}^{(1)}, \dots, \mathbf{A}\boldsymbol{\xi}^{(n)}$. Since $\mathbf{A}\boldsymbol{\xi}^{(k)} = \lambda_k \boldsymbol{\xi}^{(k)}$, it follows that

$$\mathbf{AT} = \begin{pmatrix} \lambda_1 \xi_1^{(1)} & \cdots & \lambda_n \xi_1^{(n)} \\ \vdots & & \vdots \\ \lambda_1 \xi_n^{(1)} & \cdots & \lambda_n \xi_n^{(n)} \end{pmatrix} = \mathbf{TD}, \tag{30}$$

where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{31}$$

is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$. From Eq. (30) it follows that

$$\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}. \tag{32}$$

Thus, if the eigenvalues and eigenvectors of $\mathbf{A}$ are known, $\mathbf{A}$ can be transformed into a diagonal matrix by the process shown in Eq. (32). This process is known as a **similarity transformation**, and Eq. (32) is summed up in words by saying that $\mathbf{A}$ is **similar** to the diagonal matrix $\mathbf{D}$. Alternatively, we may say that $\mathbf{A}$ is **diagonalizable**. Observe that a similarity transformation leaves the eigenvalues of $\mathbf{A}$ unchanged and transforms its eigenvectors into the coordinate vectors $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(n)}$.

If $\mathbf{A}$ is Hermitian, then the determination of $\mathbf{T}^{-1}$ is very simple. The eigenvectors $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ of $\mathbf{A}$ are known to be mutually orthogonal, so let us choose them so that they are also normalized by $(\boldsymbol{\xi}^{(i)}, \boldsymbol{\xi}^{(i)}) = 1$ for each $i$. Then it is easy to verify that $\mathbf{T}^{-1} = \mathbf{T}^*$; in other words, the inverse of $\mathbf{T}$ is the same as its adjoint (the transpose of its complex conjugate).

Finally, we note that if $\mathbf{A}$ has fewer than $n$ linearly independent eigenvectors, then there is no matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$. In this case, $\mathbf{A}$ is not similar to a diagonal matrix and is not diagonalizable.

**EXAMPLE 3**

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}. \tag{33}$$

Find the similarity transformation matrix $\mathbf{T}$ and show that $\mathbf{A}$ can be diagonalized.

In Example 2 of Section 7.5, we found that the eigenvalues and eigenvectors of $\mathbf{A}$ are

$$r_1 = 3, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = -1, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{34}$$

Thus the transformation matrix $\mathbf{T}$ and its inverse $\mathbf{T}^{-1}$ are

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}; \qquad \mathbf{T}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}. \tag{35}$$

Consequently, you can check that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{D}. \tag{36}$$

Now let us turn again to the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{37}$$

where $\mathbf{A}$ is a constant matrix. In Sections 7.5 and 7.6, we described how to solve such a system by starting from the assumption that $\mathbf{x} = \boldsymbol{\xi}e^{rt}$. Now we provide another viewpoint, one based on diagonalizing the coefficient matrix $\mathbf{A}$.

According to the results stated just above, it is possible to diagonalize $\mathbf{A}$ whenever $\mathbf{A}$ has a full set of $n$ linearly independent eigenvectors. Let $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ be eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues $r_1, \dots, r_n$ and form the transformation matrix $\mathbf{T}$ whose columns are $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$. Then, defining a new dependent variable $\mathbf{y}$ by the relation

$$\mathbf{x} = \mathbf{T}\mathbf{y}, \tag{38}$$

we have from Eq. (37) that

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y}. \tag{39}$$

Multiplying by $\mathbf{T}^{-1}$, we then obtain

$$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y}, \tag{40}$$

or, using Eq. (32),

$$\mathbf{y}' = \mathbf{D}\mathbf{y}. \tag{41}$$

Recall that $\mathbf{D}$ is the diagonal matrix with the eigenvalues $r_1, \dots, r_n$ of $\mathbf{A}$ along the diagonal. A fundamental matrix for the system (41) is the diagonal matrix (see Problem 16)

$$\mathbf{Q}(t) = \exp(\mathbf{D}t) = \begin{pmatrix} e^{r_1 t} & 0 & \dots & 0 \\ 0 & e^{r_2 t} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{r_n t} \end{pmatrix}. \tag{42}$$

A fundamental matrix $\boldsymbol{\Psi}$ for the system (37) is then found from $\mathbf{Q}$ by the transformation (38)

$$\boldsymbol{\Psi} = \mathbf{T}\mathbf{Q}; \tag{43}$$

that is,

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} \xi_1^{(1)} e^{r_1 t} & \dots & \xi_1^{(n)} e^{r_n t} \\ \vdots & & \vdots \\ \xi_n^{(1)} e^{r_1 t} & \dots & \xi_n^{(n)} e^{r_n t} \end{pmatrix}. \tag{44}$$

The columns of $\mathbf{\Psi}(t)$ are the same as the solutions in Eq. (27) of Section 7.5. Thus the diagonalization procedure does not offer any computational advantage over the method of Section 7.5, since in either case it is necessary to calculate the eigenvalues and eigenvectors of the coefficient matrix in the system of differential equations.

**EXAMPLE 4**

Consider again the system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{45}$$

where $\mathbf{A}$ is given by Eq. (33). Using the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T}$ is given by Eq. (35), you can reduce the system (45) to the diagonal system

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} = \mathbf{D}\mathbf{y}. \tag{46}$$

Obtain a fundamental matrix for the system (46), and then transform it to obtain a fundamental matrix for the original system (45).

By multiplying $\mathbf{D}$ repeatedly with itself, we find that

$$\mathbf{D}^2 = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}^3 = \begin{pmatrix} 27 & 0 \\ 0 & -1 \end{pmatrix}, \quad \ldots. \tag{47}$$

Therefore, it follows from Eq. (23) that $\exp(\mathbf{D}t)$ is a diagonal matrix with the entries $e^{3t}$ and $e^{-t}$ on the diagonal; that is,

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{48}$$

Finally, we obtain the required fundamental matrix $\mathbf{\Psi}(t)$ by multiplying $\mathbf{T}$ and $\exp(\mathbf{D}t)$:

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}. \tag{49}$$

Observe that this fundamental matrix is the same as the one found in Example 1.

## PROBLEMS

In each of Problems 1 through 10:

(a) Find a fundamental matrix for the given system of equations.

(b) Also find the fundamental matrix $\mathbf{\Phi}(t)$ satisfying $\mathbf{\Phi}(0) = \mathbf{I}$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

7. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

9. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \mathbf{x}$

10. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$

11. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

by using the fundamental matrix $\mathbf{\Phi}(t)$ found in Problem 3.

12. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

by using the fundamental matrix $\mathbf{\Phi}(t)$ found in Problem 6.

13. Show that $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$, where $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ are as defined in this section.

14. The fundamental matrix $\mathbf{\Phi}(t)$ for the system (3) was found in Example 2. Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$ by multiplying $\mathbf{\Phi}(t)$ and $\mathbf{\Phi}(s)$.

15. Let $\mathbf{\Phi}(t)$ denote the fundamental matrix satisfying $\mathbf{\Phi}' = \mathbf{A\Phi}$, $\mathbf{\Phi}(0) = \mathbf{I}$. In the text we also denoted this matrix by $\exp(\mathbf{A}t)$. In this problem we show that $\mathbf{\Phi}$ does indeed have the principal algebraic properties associated with the exponential function.

(a) Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$; that is, show that $\exp(\mathbf{A}t)\exp(\mathbf{A}s) = \exp[\mathbf{A}(t+s)]$. *Hint:* Show that if $s$ is fixed and $t$ is variable, then both $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ and $\mathbf{\Phi}(t+s)$ satisfy the initial value problem $\mathbf{Z}' = \mathbf{AZ}$, $\mathbf{Z}(0) = \mathbf{\Phi}(s)$.

(b) Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{I}$; that is, $\exp(\mathbf{A}t)\exp[\mathbf{A}(-t)] = \mathbf{I}$. Then show that $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$.

(c) Show that $\mathbf{\Phi}(t-s) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

16. Show that if $\mathbf{A}$ is a diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$, then $\exp(\mathbf{A}t)$ is also a diagonal matrix with diagonal elements $\exp(a_1 t), \exp(a_2 t), \ldots, \exp(a_n t)$.

17. Consider an oscillator satisfying the initial value problem

$$u'' + \omega^2 u = 0, \qquad u(0) = u_0, \quad u'(0) = v_0. \tag{i}$$

(a) Let $x_1 = u, x_2 = u'$, and transform Eqs. (i) into the form

$$\mathbf{x}' = \mathbf{Ax}, \qquad \mathbf{x}(0) = \mathbf{x}^0. \tag{ii}$$

(b) By using the series (23), show that

$$\exp \mathbf{A}t = \mathbf{I}\cos\omega t + \mathbf{A}\frac{\sin\omega t}{\omega}. \tag{iii}$$

(c) Find the solution of the initial value problem (ii).

18. The method of successive approximations (see Section 2.8) can also be applied to systems of equations. For example, consider the initial value problem

$$\mathbf{x}' = \mathbf{Ax}, \qquad \mathbf{x}(0) = \mathbf{x}^0, \tag{i}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{x}^0$ is a prescribed vector.

(a) Assuming that a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ exists, show that it must satisfy the integral equation

$$\boldsymbol{\phi}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\boldsymbol{\phi}(s)\,ds. \tag{ii}$$

(b) Start with the initial approximation $\boldsymbol{\phi}^{(0)}(t) = \mathbf{x}^0$. Substitute this expression for $\boldsymbol{\phi}(s)$ in the right side of Eq. (ii) and obtain a new approximation $\boldsymbol{\phi}^{(1)}(t)$. Show that

$$\boldsymbol{\phi}^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0. \tag{iii}$$

(c) Repeat this process and thereby obtain a sequence of approximations $\boldsymbol{\phi}^{(0)}$, $\boldsymbol{\phi}^{(1)}$, $\boldsymbol{\phi}^{(2)}, \dots, \boldsymbol{\phi}^{(n)}, \dots$. Use an inductive argument to show that

$$\boldsymbol{\phi}^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0. \tag{iv}$$

(d) Let $n \to \infty$ and show that the solution of the initial value problem (i) is

$$\boldsymbol{\phi}(t) = \exp(\mathbf{A}t)\mathbf{x}^0. \tag{v}$$

# 7.8 Repeated Eigenvalues

We conclude our consideration of the linear homogeneous system with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

with a discussion of the case in which the matrix $\mathbf{A}$ has a repeated eigenvalue. Recall that in Section 7.3 we stated that a repeated eigenvalue with algebraic multiplicity $m \geq 2$ may have a geometric multiplicity less than $m$. In other words, there may be fewer than $m$ linearly independent eigenvectors associated with this eigenvalue. The following example illustrates this possibility.

**EXAMPLE 1**

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}. \tag{2}$$

The eigenvalues $r$ and eigenvectors $\boldsymbol{\xi}$ satisfy the equation $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, or

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2 = 0. \tag{4}$$

Thus the two eigenvalues are $r_1 = 2$ and $r_2 = 2$; that is, the eigenvalue 2 has algebraic multiplicity 2.