Ch 3.1: 2nd Order Linear Homogeneous Equations-Constant Coefficients

- A **second order ordinary differential equation** has the general form
$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$
where *f* is some given function.
- This equation is said to be **linear** if *f* is linear in *y* and *y′*:
$$y'' + p(t)y' + q(t)y = g(t)$$
Otherwise the equation is said to be **nonlinear**.
- A second order linear equation often appears as
$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$
- If $g(t)$ or $G(t) = 0$ for all t, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.

Homogeneous Equations, Initial Values

- In Sections 3.5 and 3.6, we will see that once a solution to a homogeneous equation is found, then it is possible to solve the corresponding nonhomogeneous equation, or at least express the solution in terms of an integral.
- The focus of this chapter is thus on homogeneous equations; and in particular, those with constant coefficients:

$$ay'' + by' + cy = 0$$

We will examine the variable coefficient case in Chapter 5.

- Initial conditions typically take the form

$$y(t_0) = y_0,\ y'(t_0) = y'_0$$

- Thus solution passes through (t_0, y_0), and the slope of solution at (t_0, y_0) is equal to y_0'.

Example 1: Infinitely Many Solutions (1 of 3)

- Consider the second order linear differential equation

$$y'' - y = 0$$

- Two solutions of this equation are

$$y_1(t) = e^t, \quad y_2(t) = e^{-t}$$

- Other solutions include

$$y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$$

- Based on these observations, we see that there are infinitely many solutions of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- It will be shown in Section 3.2 that all solutions of the differential equation above can be expressed in this form.

Example 1: Initial Conditions (2 of 3)

- Now consider the following initial value problem for our equation:

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

- We have found a general solution of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- Using the initial equations,

$$\left.\begin{aligned} y(0) = c_1 + c_2 = 2 \\ y'(0) = c_1 - c_2 = -1 \end{aligned}\right\} \Rightarrow c_1 = \frac{1}{2},\ c_2 = \frac{3}{2}$$

- Thus

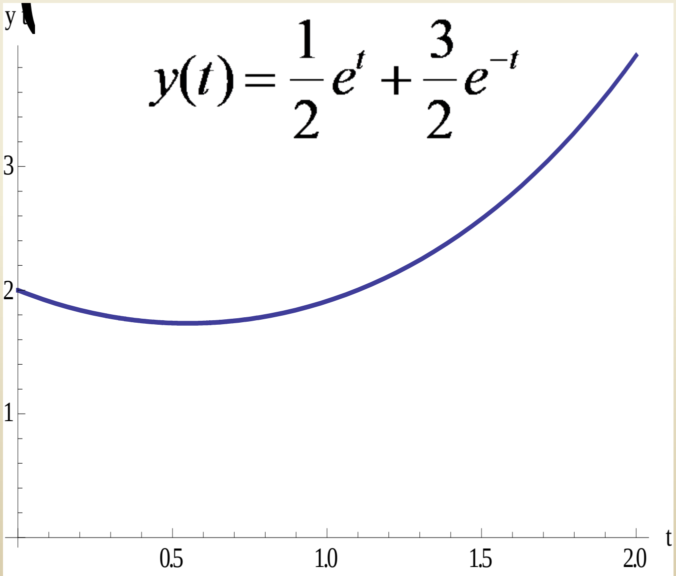
$$y(t) = \frac{1}{2} e^t + \frac{3}{2} e^{-t}$$

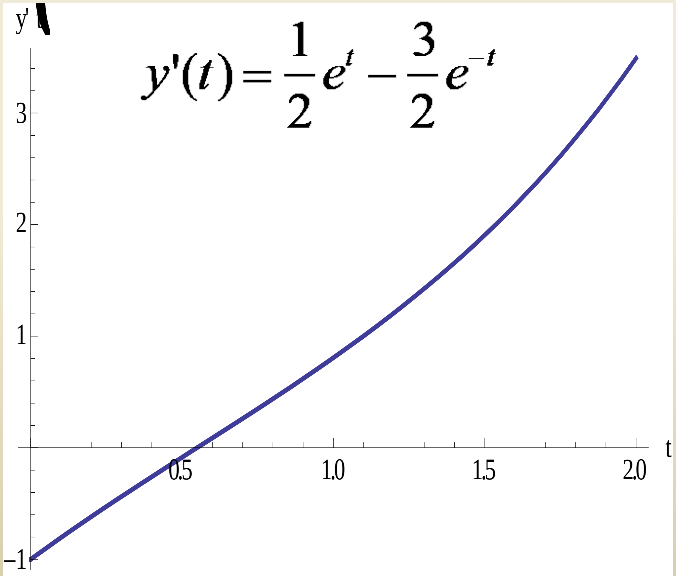
Example 1: Solution Graphs (3 of 3)

- Our initial value problem and solution are

$$y'' - y = 0, \quad y(0) = 2, \; y'(0) = -1 \; \Rightarrow \; y(t) = \frac{1}{2}e^{t} + \frac{3}{2}e^{-t}$$

- Graphs of both y(t) and y′(t) are given below. Observe that both initial conditions are satisfied.





Characteristic Equation

- To solve the 2nd order equation with constant coefficients,

$$ay'' + by' + cy = 0,$$

  we begin by assuming a solution of the form $y = e^{rt}$.
- Substituting this into the differential equation, we obtain

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

- Simplifying,

$$e^{rt}(ar^2 + br + c) = 0$$

  and hence

$$ar^2 + br + c = 0$$

- This last equation is called the **characteristic equation** of the differential equation.
- We then solve for *r* by factoring or using quadratic formula.

General Solution

- Using the quadratic formula on the characteristic equation
$$ar^2 + br + c = 0,$$
we obtain two solutions, r_1 and r_2.
- There are three possible results:
  - The roots r_1, r_2 are real and $r_1 \neq r_2$.
  - The roots r_1, r_2 are real and $r_1 = r_2$.
  - The roots r_1, r_2 are complex.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In this section, we will assume r_1, r_2 are real and $r_1 \neq r_2$.
- In this case, the general solution has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Initial Conditions

- For the initial value problem

$$ay'' + by' + cy = 0, \;\; y(t_0) = y_0, \;\; y'(t_0) = y'_0,$$

we use the general solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

together with the initial conditions to find c_1 and c_2. That is,

$$\left.\begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y'_0 \end{aligned}\right\} \Rightarrow c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \; c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} e^{-r_2 t_0}$$

- Since we are assuming $r_1 \neq r_2$, it follows that a solution of the form $y = e^{rt}$ to the above initial value problem will always exist, for any set of initial conditions.

Example 2 (General Solution)

- Consider the linear differential equation

$$y'' + 5y' + 6y = 0$$

- Assuming an exponential solution leads to the characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 5r + 6 = 0 \;\Leftrightarrow\; (r+2)(r+3) = 0$$

- Factoring the characteristic equation yields two solutions: $r_1 = -2$ and $r_2 = -3$
- Therefore, the general solution to this differential equation has the form

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

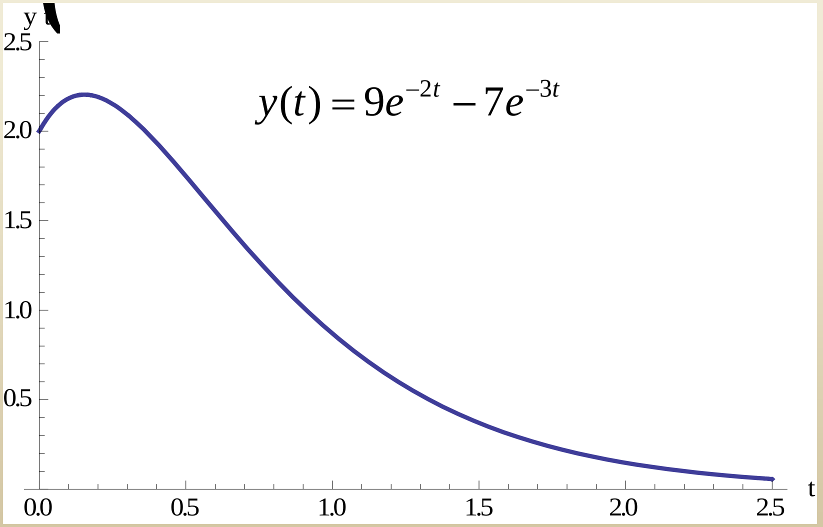
Example 3 (Particular Solution)

- Consider the initial value problem

$$y'' + 5y' + 6y = 0,\ y(0) = 2,\ y'(0) = 3$$

- From the preceding example, we know the general solution has the form: $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$
- With derivative: $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$
- Using the initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 - 3c_2 &= 3 \end{aligned}\right\} \Rightarrow c_1 = 9, c_2 = -7$$



- Thus $y(t) = 9e^{-2t} - 7e^{-3t}$

Example 4: Initial Value Problem

- Consider the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \; y'(0) = \frac{1}{2}$$

- Then

$$y(t) = e^{rt} \quad \Rightarrow \quad 4r^2 - 8r + 3 = 0 \Leftrightarrow (2r-3)(2r-1) = 0$$

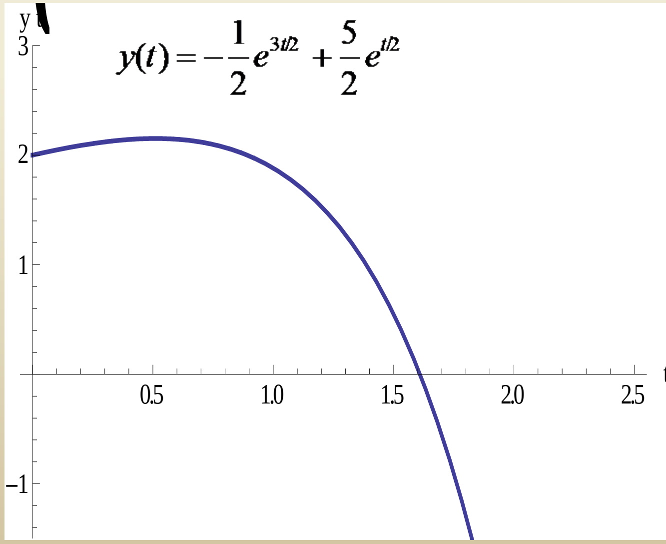
- Factoring yields two solutions, $r_1 = \dfrac{3}{2}$ and $r_2 = \dfrac{1}{2}$
- The general solution has the form

$$y(t) = c_1 e^{3t/2} + c_2 e^{t/2}$$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ \frac{3}{2}c_1 + \frac{1}{2}c_2 &= \frac{1}{2} \end{aligned}\right\} \Rightarrow c_1 = -\frac{1}{2}, \; c_2 = \frac{5}{2}$$

- Thus $\quad y(t) = -\dfrac{1}{2}e^{3t/2} + \dfrac{5}{2}e^{t/2}$



Example 5: Find Maximum Value

- For the initial value problem in Example 3, to find the maximum value attained by the solution, we set $y'(t) = 0$ and solve for t:

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

$$y'(t) = -18e^{-2t} + 21e^{-3t} \overset{set}{=} 0$$

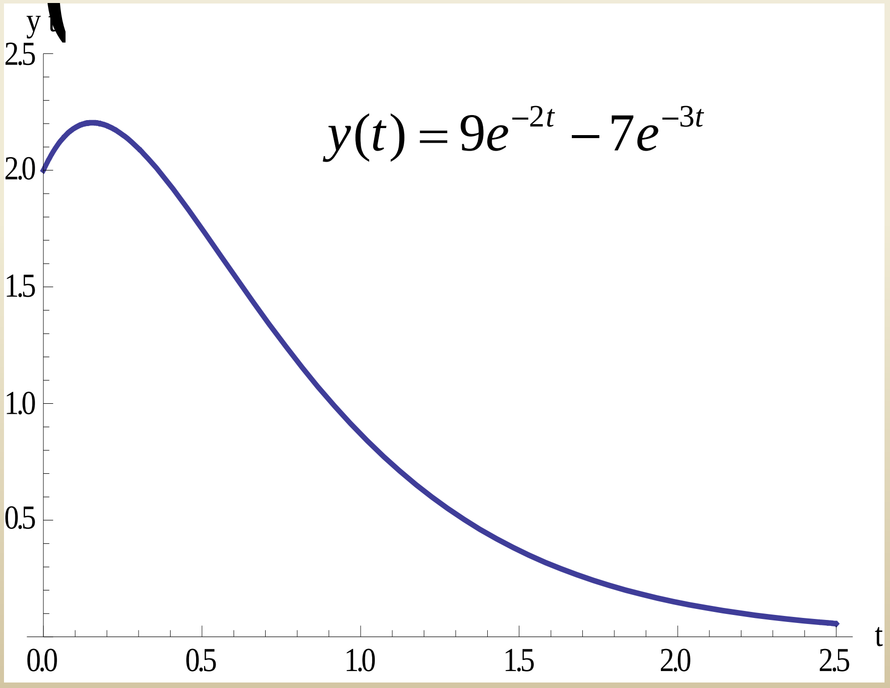
$$6e^{-2t} = 7e^{-3t}$$

$$e^{t} = 7/6$$

$$t = \ln(7/6)$$

$$t \approx 0.1542$$

$$y \approx 2.204$$



Ch 3.2: Fundamental Solutions of Linear Homogeneous Equations

- Let *p*, *q* be continuous functions on an interval $I = (\alpha, \beta)$ which could be infinite. For any function *y* that is twice differentiable on *I*, define the differential operator *L* by

$$L[y] = y'' + py' + qy$$

- Note that *L*[*y*] is a function on *I*, with output value

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t)$$

- For example,

$$p(t) = t^2,\ q(t) = e^{2t},\ y(t) = \sin(t),\ \ I = (0, 2\pi)$$
$$L[y](t) = -\sin(t) + t^2\cos(t) + 2e^{2t}\sin(t)$$

Differential Operator Notation

- In this section we will discuss the second order linear homogeneous equation $L[y](t) = 0$, along with initial conditions as indicated below:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

$$y(t_0) = y_0, \; y'(t_0) = y_1$$

- We would like to know if there are solutions to this initial value problem, and if so, are they unique.
- Also, we would like to know what can be said about the form and structure of solutions that might be helpful in finding solutions to particular problems.
- These questions are addressed in the theorems of this section.

Theorem 3.2.1 (Existence and Uniqueness)

- Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y'_0$$

- where *p*, *q*, and *g* are continuous on an open interval *I* that contains t_0. Then there exists a unique solution $y = \phi(t)$ on *I*.

- Note: While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression for the solution. This is a major difference between first and second order linear equations.

Example 1

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y_1$$

- Consider the second order linear initial value problem
$$(t^2 - 3t)y'' + ty' - (t+3)y = 0,\ y(1) = 2,\ y'(1) = 1$$
- Writing the differential equation in the form :
$$y'' + p(t)y' + q(t)y = g(t)$$
$$p(t) = \frac{1}{t-3},\ q(t) = -\frac{t+3}{t(t-3)} \text{ and } g(t) = 0$$
- The only points of discontinuity for these coefficients are $t = 0$ and $t = 3$. So the longest open interval containing the initial point $t = 1$ in which all the coefficients are continuous is $0 < t < 3$
- Therefore, the longest interval in which Theorem 3.2.1 guarantees the existence of the solution is $0 < t < 3$

Example 2

- Consider the second order linear initial value problem
$$y'' + p(t)y' + q(t)y = 0,\ y(0) = 0,\ y'(0) = 0$$
where *p*, *q* are continuous on an open interval *I* containing t_0.

- In light of the initial conditions, note that *y* = 0 is a solution to this homogeneous initial value problem.

- Since the hypotheses of Theorem 3.2.1 are satisfied, it follows that *y* = 0 is the only solution of this problem.

Theorem 3.2.2 (Principle of Superposition)

- If $y_1$and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

  then the linear combination $c_1y_1 + y_2c_2$ is also a solution, for all constants c_1 and c_2.

- To prove this theorem, substitute $c_1y_1 + y_2c_2$ in for y in the equation above, and use the fact that y_1 and y_2 are solutions.
- Thus for any two solutions y_1 and y_2, we can construct an infinite family of solutions, each of the form $y = c_1y_1 + c_2 y_2$.
- Can all solutions can be written this way, or do some solutions have a different form altogether? To answer this question, we use the Wronskian determinant.

The Wronskian Determinant (1 of 3)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

- From Theorem 3.2.2, we know that $y = c_1 y_1 + c_2 y_2$ is a solution to this equation.
- Next, find coefficients such that $y = c_1 y_1 + c_2 y_2$ satisfies the initial conditions

$$y(t_0) = y_0, \; y'(t_0) = y'_0$$

- To do so, we need to solve the following equations:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y'_1(t_0) + c_2 y'_2(t_0) &= y'_0 \end{aligned}$$

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

The Wronskian Determinant

- Solving the equations, we obtain

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$
$$c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

- In terms of determinants (by the Cramer rule):

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

The Wronskian Determinant (3 of 3)

- In order for these formulas to be valid, the determinant *W* in the denominator cannot be zero:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$

- *W* is called the **Wronskian determinant**, or more simply, the Wronskian of the solutions y_1 and y_2. We will sometimes use the notation

$$W(y_1, y_2)(t_0)$$

Theorem 3.2.3

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)y' + q(t)y = 0$$
with the initial conditions
$$y(t_0) = y_0,\ y'(t_0) = y'_0$$
Then it is always possible to choose constants c_1, c_2 so that
$$y = c_1 y_1(t) + c_2 y_2(t)$$
satisfies the differential equation and initial conditions if and ony if the Wronskian
$$W = y_1 y'_2 - y'_1 y_2$$
is not zero at the point t_0

Example 3

- In Example 2 of Section 3.1, we found that
$$y_1(t) = e^{-2t} \text{and } y_2(t) = e^{-3t}$$
were solutions to the differential equation
$$y'' + 5y' + 6y = 0$$
- The Wronskian of these two functions is
$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$
- Since *W* is nonzero for all values of *t*, the functions can be used to construct solutions of the differential y_1 and y_2 equation with initial conditions at any value of *t*

Theorem 3.2.4 (Fundamental Solutions)

- Suppose y_1 and y_2 are solutions to the equation

  $$L[y] = y'' + p(t)y' + q(t)y = 0.$$

  Then the family of solutions

  $$y = c_1 y_1 + c_2 y_2$$

  with arbitrary coefficients c_1, c_2 includes every solution to the differential equation if an only if there is a point t_0 such that $W(y_1,y_2)(t_0) \neq 0$, .

- The expression $y = c_1 y_1 + c_2 y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.

Example 4

- Consider the general second order linear equation below, with the two solutions indicated:

$$y'' + p(t)\,y' + q(t)\,y = 0$$

- Suppose the functions below are solutions to this equation:

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \quad r_1 \neq r_2$$

- The Wronskian of $y_1$and y_2 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0 \text{ for all } t.$$

- Thus $y_1$and y_2 form a fundamental set of solutions to the equation, and can be used to construct all of its solutions.
- The general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Example 5: Solutions (1 of 2)

- Consider the following differential equation:

$$2t^2y'' + 3t\,y' - y = 0, \; t > 0$$

- Show that the functions below are fundamental solutions:

$$y_1 = t^{1/2}, y_2 = t^{-1}$$

- To show this, first substitute y_1 into the equation:

$$2t^2\left(\frac{-t^{-3/2}}{4}\right) + 3t\left(\frac{t^{-1/2}}{2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0$$

- Thus y_1 is a indeed a solution of the differential equation.
- Similarly, y_2 is also a solution:

$$2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = (4 - 3 - 1)t^{-1} = 0$$

Example 5: Fundamental Solutions (2 of 2)

- Recall that

$$y_1 = t^{1/2}, y_2 = t^{-1}$$

- To show that y_1 and y_2 form a fundamental set of solutions, we evaluate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2}$$

- Since $W \neq 0$ for $t > 0$, y_1 and y_2 form a fundamental set of solutions for the differential equation

$$2t^2 y'' + 3t\, y' - y = 0, \; t > 0$$

Theorem 3.2.5: Existence of Fundamental Set of Solutions

- Consider the differential equation below, whose coefficients *p* and *q* are continuous on some open interval *I*:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

- Let t_0 be a point in *I*, and y_1 and y_2 solutions of the equation with y_1 satisfying initial conditions

$$y_1(t_0) = 1, \; y'_1(t_0) = 0$$

and y_2 satisfying initial conditions

$$y_2(t_0) = 0, \; y'_2(t_0) = 1$$

- Then y_1 and y_2 form a fundamental set of solutions to the given differential equation.

Example 6: Apply Theorem 3.2.5 (1 of 3)

- Find the fundamental set specified by Theorem 3.2.5 for the differential equation and initial point

$$y'' - y = 0, \quad t_0 = 0$$

- In Section 3.1, we found two solutions of this equation:

$$y_1 = e^t, y_2 = e^{-t}$$

The Wronskian of these solutions is $W(y_1, y_2)(t_0) = -2 \neq 0$ so they form a fundamental set of solutions.
- But these two solutions do not satisfy the initial conditions stated in Theorem 3.2.5, and thus they do not form the fundamental set of solutions mentioned in that theorem.
- Let y_3 and y_4 be the fundamental solutions of Thm 3.2.5.

$$y_3(0) = 1, \ y_3'(0) = 0; \quad y_4(0) = 0, \ y_4'(0) = 1$$

Example 6: General Solution (2 of 3)

- Since y_1 and y_2 form a fundamental set of solutions,

$$y_3 = c_1 e^t + c_2 e^{-t}, \quad y_3(0) = 1, y_3'(0) = 0$$

$$y_4 = d_1 e^t + d_2 e^{-t}, \quad y_4(0) = 0, y_4'(0) = 1$$

- Solving each equation, we obtain

$$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t), \quad y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t)$$

- The Wronskian of y_3 and y_4 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1 \neq 0$$

- Thus y_3, y_4 form the fundamental set of solutions indicated in Theorem 3.2.5, with general solution in this case

$$y(t) = k_1 \cosh(t) + k_2 \sinh(t)$$

Example 6: Many Fundamental Solution Sets (3 of 3)

- Thus

$$S_1 = \left\{ e^t, e^{-t} \right\}, \quad S_2 = \left\{ \cosh t, \sinh t \right\}$$

both form fundamental solution sets to the differential equation and initial point

$$y'' - y = 0, \quad t_0 = 0$$

- In general, a differential equation will have infinitely many different fundamental solution sets. Typically, we pick the one that is most convenient or useful.

Theorem 3.2.6

Consider again the equation (2):

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where *p* and *q* are continuous real-valued functions.

If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2), then its real part *u* and its imaginary part *v* are also solutions of this equation.

Theorem 3.2.7 (Abel's Theorem)

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)y' + q(t)y = 0$$
where p and q are continuous on some open interval I. Then the $W[y_1,y_2](t)$ is given by
$$W[y_1, y_2](t) = ce^{-\int p(t)dt}$$
where c is a constant that depends on y_1 and y_2 but not on t.

- Note that $W[y_1,y_2](t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Example 7 Apply Abel's Theorem

- Recall the following differential equation and its solutions:

  $$2t^2y''+3ty'-y=0,\ t>0$$ with solutions $y_1=t^{1/2}, y_2=t^{-1}$

- We computed the Wronskian for these solutions to be

  $$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=-\frac{3}{2}t^{-3/2}=-\frac{3}{2\sqrt{t^3}}$$

- Writing the differential equation in the standard form

  $$y''+\frac{3}{2t}y'-\frac{1}{2t^2}y=0,\ t>0$$

- So $p(t)=\dfrac{3}{2t}$ and the Wronskian given by Thm.3.2.6 is

  $$W[y_1,y_2](t)=ce^{-\int\frac{3}{2t}dt}=ce^{-\frac{3}{2}\ln t}=ct^{-3/2}$$

- This is the Wronskian for any pair of fundamental solutions. For the solutions given above, we must let c = –3/2

Summary

- To find a general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \;\; \alpha < t < \beta$$

  we first find two solutions y_1 and y_2.
- Then make sure there is a point t_0 in the interval such that $W[y_1, y_2](t_0) \neq 0$.
- It follows that y_1 and y_2 form a fundamental set of solutions to the equation, with general solution $y = c_1 y_1 + c_2 y_2$.
- If initial conditions are prescribed at a point t_0 in the interval where $W \neq 0$, then c_1 and c_2 can be chosen to satisfy those conditions.