CHAPTER

7

Systems of First Order Linear Equations

There are many physical problems that involve a number of separate elements linked together in some manner. For example, electrical networks have this character, as do many problems in mechanics and in other fields. In these and similar cases, the corresponding mathematical problem consists of a *system* of two or more differential equations, which can always be written as first order equations. In this chapter we focus on systems of first order *linear* equations, and in particular equations having constant coefficients, utilizing some of the elementary aspects of linear algebra to unify the presentation. In many respects this chapter follows the same lines as the treatment of second order linear equations in Chapter 3.

7.1 Introduction

Systems of simultaneous ordinary differential equations arise naturally in problems involving several dependent variables, each of which is a function of the same single independent variable. We will denote the independent variable by t and will let $x_1, x_2, x_3, \ldots$ represent dependent variables that are functions of t. Differentiation with respect to t will be denoted by a prime.

For example, consider the spring–mass system in Figure 7.1.1. The two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constrained by the three springs whose constants are k_1, k_2, and k_3, respectively. Using arguments similar to those in Section 3.7, we find the following equations for the coordinates x_1 and x_2 of the two masses:

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= k_2(x_2 - x_1) - k_1x_1 + F_1(t) \\ &= -(k_1 + k_2)x_1 + k_2x_2 + F_1(t), \\ m_2\frac{d^2x_2}{dt^2} &= -k_3x_2 - k_2(x_2 - x_1) + F_2(t) \\ &= k_2x_1 - (k_2 + k_3)x_2 + F_2(t). \end{aligned} \tag{1}$$

A derivation of Eqs. (1) is outlined in Problem 17.

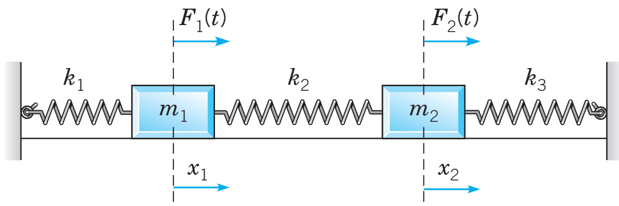

FIGURE 7.1.1 A two-mass, three-spring system.

Next, consider the parallel LRC circuit shown in Figure 7.1.2. Let V be the voltage drop across the capacitor and I the current through the inductor. Then, referring to Section 3.7 and to Problem 19 of this section, we can show that the voltage and current are described by the system of equations

$$\begin{aligned} \frac{dI}{dt} &= \frac{V}{L}, \\ \frac{dV}{dt} &= -\frac{I}{C} - \frac{V}{RC}, \end{aligned} \tag{2}$$

where L is the inductance, C is the capacitance, and R is the resistance.

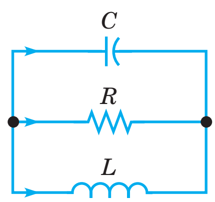

FIGURE 7.1.2 A parallel LRC circuit.

One reason why systems of first order equations are particularly important is that equations of higher order can always be transformed into such systems. This is usually required if a numerical approach is planned, because almost all codes for generating numerical approximations to solutions of differential equations are written for systems of first order equations. The following example illustrates how easy it is to make the transformation.

EXAMPLE 1

The motion of a certain spring–mass system (see Example 3 of Section 3.7) is described by the second order differential equation

$$u'' + 0.125u' + u = 0. \tag{3}$$

Rewrite this equation as a system of first order equations.

Let $x_1 = u$ and $x_2 = u'$. Then it follows that $x_1' = x_2$. Further, $u'' = x_2'$. Then, by substituting for u, u', and u'' in Eq. (3), we obtain

$$x_2' + 0.125x_2 + x_1 = 0.$$

Thus x_1 and x_2 satisfy the following system of two first order differential equations:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 - 0.125x_2. \end{aligned} \tag{4}$$

The general equation of motion of a spring–mass system

$$mu'' + \gamma u' + ku = F(t) \tag{5}$$

can be transformed into a system of first order equations in the same manner. If we let $x_1 = u$ and $x_2 = u'$, and proceed as in Example 1, we quickly obtain the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -(k/m)x_1 - (\gamma/m)x_2 + F(t)/m. \end{aligned} \tag{6}$$

To transform an arbitrary nth order equation

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}) \tag{7}$$

into a system of n first order equations, we extend the method of Example 1 by introducing the variables $x_1, x_2, \ldots, x_n$ defined by

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \ldots, \quad x_n = y^{(n-1)}. \tag{8}$$

It then follows immediately that

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\ \vdots \\ x_{n-1}' &= x_n, \end{aligned} \tag{9}$$

and, from Eq. (7),

$$x_n' = F(t, x_1, x_2, \ldots, x_n). \tag{10}$$

Equations (9) and (10) are a special case of the more general system

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n), \\ x_2' &= F_2(t, x_1, x_2, \ldots, x_n), \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots, x_n). \end{aligned} \tag{11}$$

In a similar way, the system (1) can be reduced to a system of four first order equations of the form (11), and the system (2) is already in this form. In fact, systems of the form (11) include almost all cases of interest, so much of the more advanced theory of differential equations is devoted to such systems.

A **solution** of the system (11) on the interval $I: \alpha < t < \beta$ is a set of n functions

$$x_1 = \phi_1(t), \quad x_2 = \phi_2(t), \quad \ldots, \quad x_n = \phi_n(t) \tag{12}$$

that are differentiable at all points in the interval I and that satisfy the system of equations (11) at all points in this interval. In addition to the given system of differential equations, there may also be given initial conditions of the form

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0, \tag{13}$$

where t_0 is a specified value of t in I, and $x_1^0, \ldots, x_n^0$ are prescribed numbers. The differential equations (11) and the initial conditions (13) together form an initial value problem.

A solution (12) can be viewed as a set of parametric equations in an n-dimensional space. For a given value of t, Eqs. (12) give values for the coordinates $x_1, \ldots, x_n$ of a point in the space. As t changes, the coordinates in general also change. The collection of points corresponding to $\alpha < t < \beta$ forms a curve in the space. It is often helpful to think of the curve as the trajectory, or path, of a particle moving in accordance with the system of differential equations (11). The initial conditions (13) determine the starting point of the moving particle.

The following conditions on $F_1, F_2, \ldots, F_n$, which are easily checked in specific problems, are sufficient to ensure that the initial value problem (11), (13) has a unique solution. Theorem 7.1.1 is analogous to Theorem 2.4.2, the existence and uniqueness theorem for a single first order equation.

Theorem 7.1.1 Let each of the functions $F_1, \ldots, F_n$ and the partial derivatives $\partial F_1/\partial x_1, \ldots, \partial F_1/\partial x_n, \ldots, \partial F_n/\partial x_1, \ldots, \partial F_n/\partial x_n$ be continuous in a region R of $tx_1x_2 \cdots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1, \ldots, \alpha_n < x_n < \beta_n$, and let the point $(t_0, x_1^0, x_2^0, \ldots, x_n^0)$ be in R. Then there is an interval $|t - t_0| < h$ in which there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system of differential equations (11) that also satisfies the initial conditions (13).

The proof of this theorem can be constructed by generalizing the argument in Section 2.8, but we do not give it here. However, note that, in the hypotheses of the theorem, nothing is said about the partial derivatives of $F_1, \ldots, F_n$ with respect to the independent variable t. Also, in the conclusion, the length $2h$ of the interval in which the solution exists is not specified exactly, and in some cases it may be very short. Finally, the same result can be established on the basis of somewhat weaker but more complicated hypotheses, so the theorem as stated is not the most general one known, and the given conditions are sufficient, but not necessary, for the conclusion to hold.

If each of the functions $F_1, \ldots, F_n$ in Eqs. (11) is a linear function of the dependent variables $x_1, \ldots, x_n$, then the system of equations is said to be **linear**; otherwise, it is **nonlinear**. Thus the most general system of n first order linear equations has the form

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t),\\
x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t),\\
&\ \ \vdots\\
x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t).
\end{aligned} \tag{14}$$

If each of the functions $g_1(t), \ldots, g_n(t)$ is zero for all t in the interval I, then the system (14) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**. Observe that the systems (1) and (2) are both linear. The system (1) is nonhomogeneous unless $F_1(t) = F_2(t) = 0$, while the system (2) is homogeneous. For the linear system (14), the existence and uniqueness theorem is simpler and also has a stronger conclusion. It is analogous to Theorems 2.4.1 and 3.2.1.

Theorem 7.1.2 If the functions $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an open interval I: $\alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system (14) that also satisfies the initial conditions (13), where t_0 is any point in I, and $x_1^0, \ldots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval I.

Note that, in contrast to the situation for a nonlinear system, the existence and uniqueness of the solution of a linear system are guaranteed throughout the interval in which the hypotheses are satisfied. Furthermore, for a linear system the initial values $x_1^0, \ldots, x_n^0$ at $t = t_0$ are completely arbitrary, whereas in the nonlinear case the initial point must lie in the region R defined in Theorem 7.1.1.

The rest of this chapter is devoted to systems of linear first order equations (nonlinear systems are included in the discussion in Chapters 8 and 9). Our presentation makes use of matrix notation and assumes that you have some familiarity with the properties of matrices. The basic facts about matrices are summarized in Sections 7.2 and 7.3, and some more advanced material is reviewed as needed in later sections.

PROBLEMS

In each of Problems 1 through 4, transform the given equation into a system of first order equations.

1. $u'' + 0.5u' + 2u = 0$
2. $u'' + 0.5u' + 2u = 3\sin t$
3. $t^2u'' + tu' + (t^2 - 0.25)u = 0$
4. $u^{(4)} - u = 0$

In each of Problems 5 and 6, transform the given initial value problem into an initial value problem for two first order equations.

5. $u'' + 0.25u' + 4u = 2\cos 3t, \qquad u(0) = 1, \quad u'(0) = -2$
6. $u'' + p(t)u' + q(t)u = g(t), \qquad u(0) = u_0, \quad u'(0) = u_0'$
7. Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system

$$x_1' = -2x_1 + x_2, \qquad x_2' = x_1 - 2x_2.$$

(a) Solve the first equation for x_2 and substitute into the second equation, thereby obtaining a second order equation for x_1. Solve this equation for x_1 and then determine x_2 also.

(b) Find the solution of the given system that also satisfies the initial conditions $x_1(0) = 2$, $x_2(0) = 3$.

(c) Sketch the curve, for $t \geq 0$, given parametrically by the expressions for x_1 and x_2 obtained in part (b).

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find x_1 and x_2 that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

8. $x_1' = 3x_1 - 2x_2, \qquad x_1(0) = 3$
   $x_2' = 2x_1 - 2x_2, \qquad x_2(0) = \frac{1}{2}$
9. $x_1' = 1.25x_1 + 0.75x_2, \qquad x_1(0) = -2$
   $x_2' = 0.75x_1 + 1.25x_2, \qquad x_2(0) = 1$
10. $x_1' = x_1 - 2x_2, \qquad x_1(0) = -1$
    $x_2' = 3x_1 - 4x_2, \qquad x_2(0) = 2$
11. $x_1' = 2x_2, \qquad x_1(0) = 3$
    $x_2' = -2x_1, \qquad x_2(0) = 4$
12. $x_1' = -0.5x_1 + 2x_2, \qquad x_1(0) = -2$
    $x_2' = -2x_1 - 0.5x_2, \qquad x_2(0) = 2$
13. Transform Eqs. (2) for the parallel circuit into a single second order equation.
14. Show that if a_{11}, a_{12}, a_{21}, and a_{22} are constants with a_{12} and a_{21} not both zero, and if the functions g_1 and g_2 are differentiable, then the initial value problem

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + g_1(t), \qquad x_1(0) = x_1^0 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + g_2(t), \qquad x_2(0) = x_2^0 \end{aligned}$$

can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if $a_{11}, \ldots, a_{22}$ are functions of t?

15. Consider the linear homogeneous system

$$\begin{aligned} x' &= p_{11}(t)x + p_{12}(t)y, \\ y' &= p_{21}(t)x + p_{22}(t)y. \end{aligned}$$

Show that if $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system, then $x = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2. This is the principle of superposition.

16. Let $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ be any two solutions of the linear nonhomogeneous system

$$x' = p_{11}(t)x + p_{12}(t)y + g_1(t),$$

$$y' = p_{21}(t)x + p_{22}(t)y + g_2(t).$$

Show that $x = x_1(t) - x_2(t)$, $y = y_1(t) - y_2(t)$ is a solution of the corresponding homogeneous system.

17. Equations (1) can be derived by drawing a free-body diagram showing the forces acting on each mass. Figure 7.1.3*a* shows the situation when the displacements x_1 and x_2 of the two masses are both positive (to the right) and $x_2 > x_1$. Then springs 1 and 2 are elongated and spring 3 is compressed, giving rise to forces as shown in Figure 7.1.3*b*. Use Newton's law ($F = ma$) to derive Eqs. (1).

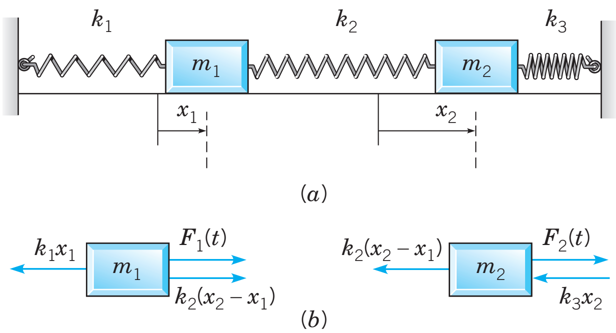

FIGURE 7.1.3 (a) The displacements x_1 and x_2 are both positive.
(b) The free-body diagram for the spring–mass system.

18. Transform the system (1) into a system of first order equations by letting $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, and $y_4 = x_2'$.

Electric Circuits. The theory of electric circuits, such as that shown in Figure 7.1.2, consisting of inductors, resistors, and capacitors, is based on Kirchhoff's laws: (1) The net flow of current into each node (or junction) is zero, and (2) the net voltage drop around each closed loop is zero. In addition to Kirchhoff's laws, we also have the relation between the current I in amperes through each circuit element and the voltage drop V in volts across the element:

$$V = RI, \qquad R = \text{resistance in ohms;}$$

$$C\frac{dV}{dt} = I, \qquad C = \text{capacitance in farads}^1\text{;}$$

$$L\frac{dI}{dt} = V, \qquad L = \text{inductance in henrys.}$$

Kirchhoff's laws and the current–voltage relation for each circuit element provide a system of algebraic and differential equations from which the voltage and current throughout the circuit can be determined. Problems 19 through 21 illustrate the procedure just described.

[1] Actual capacitors typically have capacitances measured in microfarads. We use farad as the unit for numerical convenience.

19. Consider the circuit shown in Figure 7.1.2. Let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Likewise, let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrarily chosen directions in which currents and voltage drops will be taken to be positive.

(a) Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0. \tag{i}$$

In a similar way, show that

$$V_2 - V_3 = 0. \tag{ii}$$

(b) Applying Kirchhoff's first law to either node in the circuit, show that

$$I_1 + I_2 + I_3 = 0. \tag{iii}$$

(c) Use the current–voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \qquad V_2 = RI_2, \qquad LI_3' = V_3. \tag{iv}$$

(d) Eliminate V_2, V_3, I_1, and I_2 among Eqs. (i) through (iv) to obtain

$$CV_1' = -I_3 - \frac{V_1}{R}, \qquad LI_3' = V_1. \tag{v}$$

Observe that if we omit the subscripts in Eqs. (v), then we have the system (2) of this section.

20. Consider the circuit shown in Figure 7.1.4. Use the method outlined in Problem 19 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$\frac{dI}{dt} = -I - V, \qquad \frac{dV}{dt} = 2I - V.$$

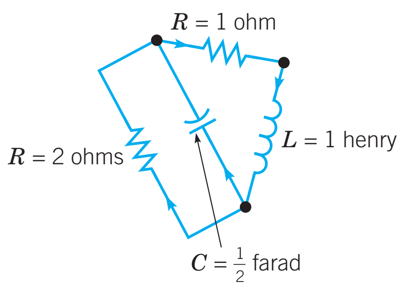

FIGURE 7.1.4 The circuit in Problem 20.

21. Consider the circuit shown in Figure 7.1.5. Use the method outlined in Problem 19 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$L\frac{dI}{dt} = -R_1 I - V, \qquad C\frac{dV}{dt} = I - \frac{V}{R_2}.$$

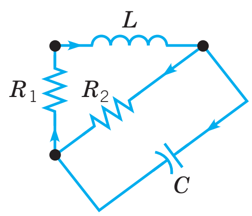

FIGURE 7.1.5 The circuit in Problem 21.

22. Consider the two interconnected tanks shown in Figure 7.1.6. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is nonhomogeneous.

(b) Find the values of Q_1 and Q_2 for which the system is in equilibrium—that is, does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?

(c) Let $x_1 = Q_1(t) - Q_1^E$ and $x_2 = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.

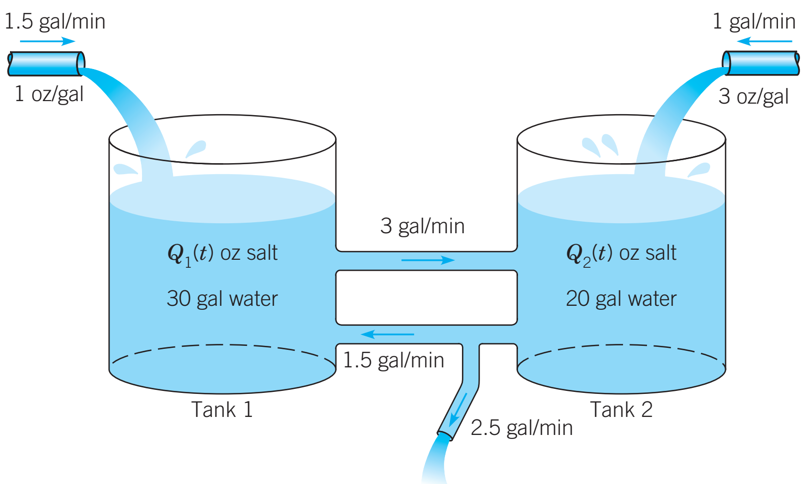

FIGURE 7.1.6 Two interconnected tanks (Problem 22).

23. Consider two interconnected tanks similar to those in Figure 7.1.6. Initially, Tank 1 contains 60 gal of water and Q_1^0 oz of salt, and Tank 2 contains 100 gal of water and Q_2^0 oz of salt. Water containing q_1 oz/gal of salt flows into Tank 1 at a rate of 3 gal/min. The mixture in Tank 1 flows out at a rate of 4 gal/min, of which half flows into Tank 2, while the remainder leaves the system. Water containing q_2 oz/gal of salt also flows into Tank 2 from the outside at the rate of 1 gal/min. The mixture in Tank 2 leaves it at a rate of 3 gal/min, of which some flows back into Tank 1 at a rate of 1 gal/min, while the rest leaves the system.

(a) Draw a diagram that depicts the flow process described above. Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions for Q_1 and Q_2 that model the flow process.

(b) Find the equilibrium values Q_1^E and Q_2^E in terms of the concentrations q_1 and q_2.

(c) Is it possible (by adjusting q_1 and q_2) to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state?

(d) Describe which equilibrium states are possible for this system for various values of q_1 and q_2.

7.2 Review of Matrices

For both theoretical and computational reasons, it is advisable to bring some of the results of matrix algebra[2] to bear on the initial value problem for a system of linear differential equations. For reference purposes, this section and the next are devoted to a brief summary of the facts that will be needed later. More details can be found in any elementary book on linear algebra. We assume, however, that you are familiar with determinants and how to evaluate them.

We designate matrices by boldfaced capitals $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ..., occasionally using boldfaced Greek capitals $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, ... A matrix $\mathbf{A}$ consists of a rectangular array of numbers, or elements, arranged in m rows and n columns—that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{1}$$

We speak of $\mathbf{A}$ as an $m \times n$ matrix. Although later in the chapter we will often assume that the elements of certain matrices are real numbers, in this section we assume that

[2]The properties of matrices were first extensively explored in 1858 in a paper by the English algebraist Arthur Cayley (1821–1895), although the word "matrix" was introduced by his good friend James Sylvester (1814–1897) in 1850. Cayley did some of his best mathematical work while practicing law from 1849 to 1863; he then became professor of mathematics at Cambridge, a position he held for the rest of his life. After Cayley's groundbreaking work, the development of matrix theory proceeded rapidly, with significant contributions by Charles Hermite, Georg Frobenius, and Camille Jordan, among others.