Introduction to Differential Equations
Assignment # 10

Date Given: June 13, 2022
Date Due: June 20, 2022

P1. (1 point) Find the Laplace transform of

$$f(t) = \begin{cases} 0, & \text{if } t < 1 \\ t^2 - 2t + 2, & \text{if } t \geq 1 \end{cases}$$

Solution: Using the Heaviside function and completing the square, we can write $f(t) = \left((t-1)^2 + 1\right) u_1(t)$. The Laplace transform has the property that $\mathcal{L}[u_c(t)\, f(t-c)] = e^{-cs}\mathcal{L}[f(t)]$. Hence

$$\mathcal{L}\left[(t-1)^2 + 1\right] u_1(t) = e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$$

P2. (1 point) Find the inverse Laplace transform of $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$.

Solution: First, consider the function

$$G(s) = \frac{1}{s^2 + s - 2}$$

Factoring the denominator,

$$G(s) = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}$$

It follows that

$$\mathcal{L}^{-1}\left[G(s)\right] = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s}G(s)\right] = \left(\frac{1}{3}e^{t-2} - \frac{1}{3}e^{-2(t-2)}\right)u_2(t)$$

P3. (2 points) Find the solution of the following initial value problem

$$y'' + y = g(t), \quad y(0) = 0, y'(0) = 1, \quad g(t) = \begin{cases} t/2, & \text{if } 0 \leq t < 6 \\ 3, & \text{if } t \geq 6 \end{cases}$$

Draw the graphs of the forcing function and of the solution.

Solution: Let $g(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + Y(s) = \mathcal{L}\left[g(t)\right].$$

Applying the initial conditions, we have

$$s^2 Y(s) + Y(s) - 1 = \mathcal{L}\left[g(t)\right].$$

The forcing function can be written as

$$g(t) = \frac{t}{2}\left\{1 - u_6(t)\right\} + 3u_6(t) = \frac{t}{2} - \frac{1}{2}(t-6)u_6(t),$$

with its Laplace transform

$$\mathcal{L}\left[g(t)\right] = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}$$

Using partial fractions,

$$\frac{1}{2s^2(s^2+1)} = \frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right).$$

Taking the inverse transform, term-by-term, and using Theorem 6.3.1, the solution of the initial value problem is Hence the solution of the initial value problem is

$$y(t) = \sin t + \frac{1}{2}\left(t - \sin t\right) - \frac{1}{2}\left\{(t-6) - \sin(t-6)\right\} u_6(t),$$

that is

$$y(t) = \frac{1}{2}\left(t + \sin t\right) - \frac{1}{2}\left\{(t-6) - \sin(t-6)\right\} u_6(t).$$

Graphs of the solution and the forcing function are shown in Figure 1.

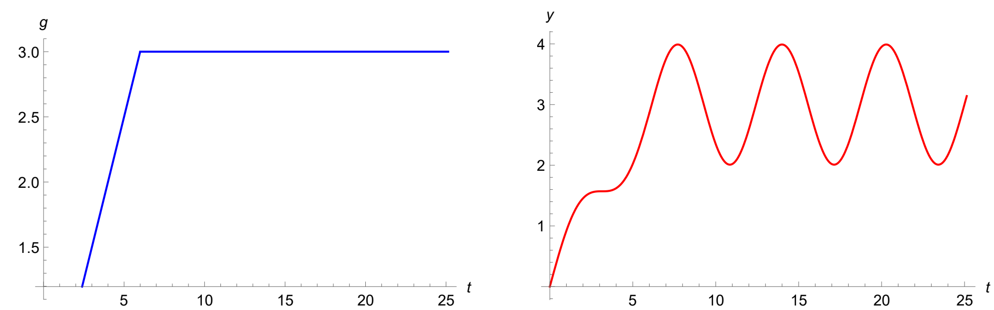

Figure 1: Illustration to problem P3. The solution increases, in response to the ramp input, and thereafter oscillates about a mean value of $y_m = 3$.

P4. (2 points) Find the solution of the following initial value problem

$$y'' + y = u_\pi(t) - u_{3\pi}(t), \quad y(0) = 0, y'(0) = 0.$$

Draw the graphs of the forcing function and of the solution.

Solution: Taking the Laplace transform of both sides of the differential equation, we obtain

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions, we have

$$s^2Y(s) + Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{e^{-\pi s}}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2+1)} = \left(\frac{1}{s} - \frac{s}{s^2+1}\right).$$

Taking the inverse transform, term-by-term, and using Theorem 6.3.1, the solution of the initial value problem is

$$y(t) = \{1 - \cos(t-\pi)\}\, u_\pi(t) - \{1 - \cos(t - 3\pi)\}\, u_{3\pi}(t)$$

that is

$$y(t) = (1 + \cos t)\,\{u_\pi(t) - u_{3\pi}(t)\}$$

Graphs of the solution and the forcing function are shown in Figure 2.

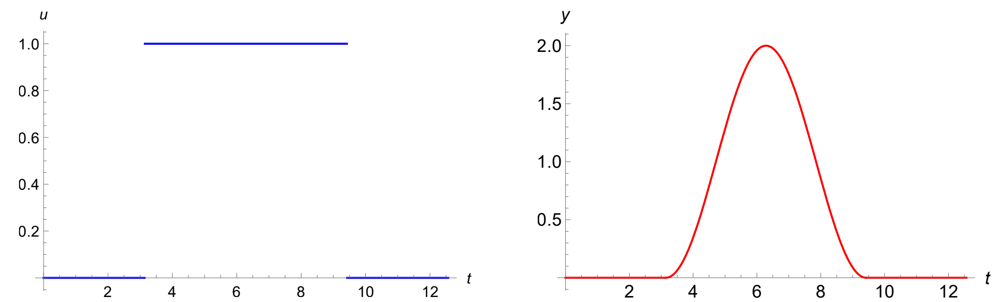

Figure 2: Illustration to problem P4. Since there is no damping term, the solution responds immediately to the forcing input. There is a temporary oscillation about $y = 1$.

P5. (2 points) Find the solution of the following initial value problem $y'' + 2y' + 2y = \delta(t-\pi)$; $y(0) = 1, y'(0) = 0$. Draw a graph of the solution.

Solution: Let $Y(s) = \mathcal{L}[y]$ and take the Laplace transform of the differential equation. We arrive at

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] + 2Y(s) = e^{-\pi s}.$$

Applying the initial conditions, we have

$$\left[s^2Y(s) - s\right] + 2\left[sY(s) - 1\right] + 2Y(s) = e^{-\pi s},$$

which can be written as

$$\left[s^2 + 2s + 2\right] Y(s) - s - 2 = e^{-\pi s}.$$

Therefore

$$Y(s) = \frac{s+2+e^{-\pi s}}{s^2+2s+2} = \frac{s+2+e^{-\pi s}}{(s+1)^2+1} =$$

$$\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} + \frac{e^{-\pi s}}{(s+1)^2+1}.$$

Therefore

$$y(t) = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)\, e^{-(t-\pi)} \sin(t-\pi).$$

A graph of the solution is shown in Figure 3.

P6. (2 points) Find the solution of the following initial value problem $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$; $y(0) = 0, y'(0) = 1/2$. Draw a graph of the solution.

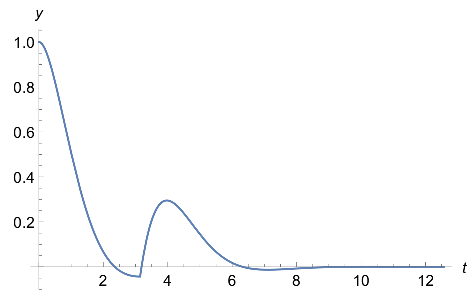

Figure 3: Illustration to problem P5.

Solution: Let $Y(s) = \mathcal{L}[y]$ and take the Laplace transform of the differential equation. We arrive at

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2Y(s) = e^{-5s} + \frac{e^{-10s}}{s}.$$

Applying the initial conditions, we get

$$Y(s) = \frac{1/2 + e^{-5s}}{s^2 + 3s + 2} + \frac{e^{-10s}}{s(s^2 + 3s + 2)}.$$

Therefore

$$Y(s) = \left(\frac{1}{2} + e^{-5s}\right)\left[\frac{1}{s+1} - \frac{1}{s+2}\right] + e^{-10s}\left[\frac{1}{2(s+2)} - \frac{1}{s+1} + \frac{1}{2s}\right].$$

Thus

$$y(t) = \frac{1}{2}\left(e^{-t} - e^{-2t}\right) + u_5(t)\left(e^{-(t-5)} - e^{-2(t-5)}\right) + u_{10}(t)\left(\frac{1}{2}e^{-2(t-10)} - e^{-(t-10)} + \frac{1}{2}\right).$$

A graph of the solution is shown in Figure 4.

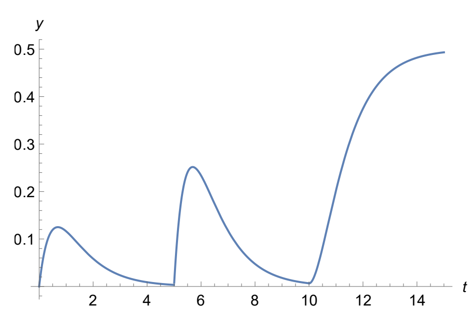

Figure 4: Illustration to problem P6.