Introduction to Differential Equations
Sample problems # 13

Date Given: July 4, 2022

P1. (a) Express the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \boldsymbol{x}$$

in terms of real-valued functions

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} -1-r & -4 \\ 1 & -1-r \end{vmatrix} = r^2 + 2r + 5 = 0 \quad \Longrightarrow \quad r_1 = -1 + 2\imath, r_2 = -1 - 2\imath.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -2\imath & -4 \\ 1 & -2\imath \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 2\imath \\ 1 \end{bmatrix}.$$

For $r = r_2 = \bar{r}_1$, we have $\boldsymbol{\xi}_2 = \bar{\boldsymbol{\xi}}_1 = (-2\imath, 1)^{\mathrm{T}}$. Next,

$$\boldsymbol{a} = \mathrm{Re}(\boldsymbol{\xi}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \mathrm{Im}(\boldsymbol{\xi}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$\boldsymbol{u}(t) = e^{-t}(\boldsymbol{a}\cos 2t - \boldsymbol{b}\sin 2t)$, $\boldsymbol{v}(t) = e^{-t}(\boldsymbol{a}\sin 2t + \boldsymbol{b}\cos 2t)$, and the general solution

$$\boldsymbol{x}(t) = c_1\boldsymbol{u}(t) + c_2\boldsymbol{v}(t)$$

is

$$\boldsymbol{x}(t) = c_1 e^{-t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right) + c_2 e^{-t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t\right) =$$
$$c_1 e^{-t}\begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}.$$

(b) The direction field and a few trajectories of the system are shown in Figure 1. As $\to \infty$ the trajectories converge to the origin.

P2. (a) Express the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} 2 & -5/2 \\ 9/5 & -1 \end{bmatrix} \boldsymbol{x}$$

in terms of real-valued functions

(b) Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $\to \infty$.

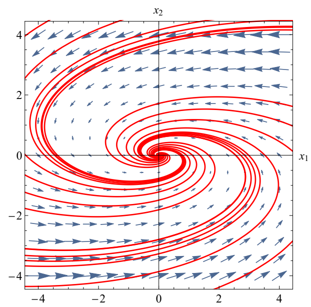

Figure 1: Illustration to problem P1.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 2-r & -5/2 \\ 9/5 & -1-r \end{vmatrix} = r^2 - r + 5/2 = 0 \quad \Longrightarrow \quad r_1 = (1+3\imath)/2, r_2 = (1-3\imath)/2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 3/2 - 3\imath/2 & -5/2 \\ 9/5 & -3/2 - 3\imath/2 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\xi}_1 = \begin{bmatrix} 5(1+\imath) \\ 6 \end{bmatrix}.$$

For $r = r_2 = \bar{r}_1$, we have $\boldsymbol{\xi}_2 = \bar{\boldsymbol{\xi}}_1 = (5(1-\imath), 1)^{\mathrm{T}}$. Next,

$$\boldsymbol{a} = \mathrm{Re}(\boldsymbol{\xi}_1) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad \boldsymbol{b} = \mathrm{Im}(\boldsymbol{\xi}_1) = \begin{bmatrix} 5 \\ 0 \end{bmatrix},$$

$\boldsymbol{u}(t) = e^{t/2}(\boldsymbol{a}\cos\frac{3t}{2} - \boldsymbol{b}\sin\frac{3t}{2})$, $\boldsymbol{v}(t) = e^{t/2}(\boldsymbol{a}\sin\frac{3t}{2} + \boldsymbol{b}\cos\frac{3t}{2})$, and the general solution

$$\boldsymbol{x}(t) = c_1\boldsymbol{u}(t) + c_2\boldsymbol{v}(t)$$

is

$$\boldsymbol{x}(t) = c_1 e^{t/2}\left(\begin{bmatrix} 5 \\ 6 \end{bmatrix}\cos\frac{3t}{2} - \begin{bmatrix} 5 \\ 0 \end{bmatrix}\sin\frac{3t}{2}\right) + c_2 e^{t/2}\left(\begin{bmatrix} 5 \\ 6 \end{bmatrix}\sin\frac{3t}{2} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}\cos\frac{3t}{2}\right) =$$
$$c_1 e^{t/2}\begin{bmatrix} 5(\cos\frac{3t}{2} - \sin\frac{3t}{2}) \\ 6\cos\frac{3t}{2} \end{bmatrix} + c_2 e^{t/2}\begin{bmatrix} 5(\sin\frac{3t}{2} + \cos\frac{3t}{2}) \\ 6\sin\frac{3t}{2} \end{bmatrix}.$$

(b) The direction field and a few trajectories of the system are shown in Figure 2. As $\to \infty$ the trajectories diverge from the origin.

P3. Express the general solution of the system of equations

$$\boldsymbol{x}' = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \boldsymbol{x}$$

in terms of real-valued functions.

Solution:

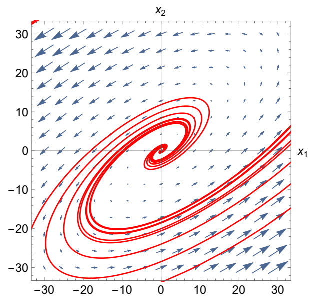

Figure 2: Illustration to problem P2.

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{bmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & -r \end{bmatrix} = r^3 + 4r^2 + 7r + 6 = 0 \implies$$

$$r_1 = -2,\ r_2 = -1 - \sqrt{2}\imath,\ r_3 = -1 + \sqrt{2}\imath.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \implies \boldsymbol{\xi}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -2+\sqrt{2}\imath & 0 & 2 \\ 1 & \sqrt{2}\imath & 0 \\ -2 & -1 & 1+\sqrt{2}\imath \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \implies \boldsymbol{\xi}_2 = \begin{bmatrix} -\sqrt{2}\imath \\ 1 \\ -1-\sqrt{2}\imath \end{bmatrix}.$$

For $r = r_3 = \bar{r}_2$, we have $\boldsymbol{\xi}_3 = \bar{\boldsymbol{\xi}}_2 = (\sqrt{2}\imath, 1, -1+\sqrt{2}\imath)^{\mathrm{T}}$. Next,

$$\boldsymbol{a} = \operatorname{Re}(\boldsymbol{\xi}_2) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{b} = \operatorname{Im}(\boldsymbol{\xi}_2) = -\sqrt{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$\boldsymbol{u}(t) = e^{-t}(\boldsymbol{a}\cos\sqrt{2}t - \boldsymbol{b}\sin\sqrt{2}t)$, $\boldsymbol{v}(t) = e^{-t}(\boldsymbol{a}\sin\sqrt{2}t + \boldsymbol{b}\cos\sqrt{2}t)$, and the general solution

$$\boldsymbol{x}(t) = c_1\boldsymbol{\xi}_1 e^{-2t} + c_2\boldsymbol{u}(t) + c_3\boldsymbol{v}(t)$$

is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \sqrt{2}\sin\sqrt{2}t \\ -\cos\sqrt{2}t \\ \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -\sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t + \sin\sqrt{2}t \end{bmatrix} e^{-t}.$$

P4. (a) Find the solution of the initial value problem

$$\boldsymbol{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Describe the behavior of the solution as $t \to \infty$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A}-r\boldsymbol{I}) = \begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = r^2+2r+2=0 \quad\Longrightarrow\quad r_1=-1+\imath, r_2=-1-\imath.$$

Find the eigenvectors. For $r=r_1$,

$$(\boldsymbol{A}-r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 2-\imath & -5 \\ 1 & -2-\imath \end{bmatrix}\boldsymbol{\xi}=\boldsymbol{0} \quad\Longrightarrow\quad \boldsymbol{\xi}_1 = \begin{bmatrix} 2+\imath \\ 1 \end{bmatrix}.$$

For $r=r_2=\bar{r}_1$, we have $\boldsymbol{\xi}_2=\bar{\boldsymbol{\xi}}_1=(2-\imath,1)^{\mathrm{T}}$. Next,

$$\boldsymbol{a}=\operatorname{Re}(\boldsymbol{\xi}_1)=\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}=\operatorname{Im}(\boldsymbol{\xi}_1)=\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$\boldsymbol{u}(t)=e^{-t}(\boldsymbol{a}\cos t-\boldsymbol{b}\sin t)$, $\boldsymbol{v}(t)=e^{-t}(\boldsymbol{a}\sin t+\boldsymbol{b}\cos t)$, and the general solution

$$\boldsymbol{x}(t)=c_1\boldsymbol{u}(t)+c_2\boldsymbol{v}(t)$$

is

$$\boldsymbol{x}(t)=c_1e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos t-\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right)+c_2e^{-t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin t+\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t\right)=$$

$$c_1e^{-t}\begin{bmatrix} 2\cos t-\sin t \\ \cos t \end{bmatrix}+c_2e^{-t}\begin{bmatrix} 2\sin t+\cos t \\ \sin t \end{bmatrix}.$$

Invoking the initial conditions we obtain the system of equations $2c_1+c_2=1$ and $c_1=1$. Therefore, $c_2=-1$, and

$$\boldsymbol{x}(t)=e^{-t}\begin{bmatrix} \cos t-3\sin t \\ \cos t-\sin t \end{bmatrix}.$$

(b) The solution is a spiral that tends to zero as $t\to\infty$ due to the e^{-t} term

P5. The system of differential equations is given as

$$\boldsymbol{x}'=\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}\boldsymbol{x},$$

where α is a constant parameter.

(a) Determine the eigenvalues in terms of α.

(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.

(c) Draw a phase portrait for a value of α slightly below, and for another value slightly above, each critical value.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A}-r\boldsymbol{I}) = \begin{vmatrix} \alpha-r & 1 \\ -1 & \alpha-r \end{vmatrix} = r^2-2\alpha r+1+\alpha^2=0 \quad\Longrightarrow\quad r_1=\alpha+\imath, r_2=\alpha-\imath.$$

(b) When $\alpha<0$ and $\alpha>0$, the equilibrium point $(0,0)$ is a stable spiral and an unstable spiral, respectively. The equilibrium point is a center when $\alpha=0$.

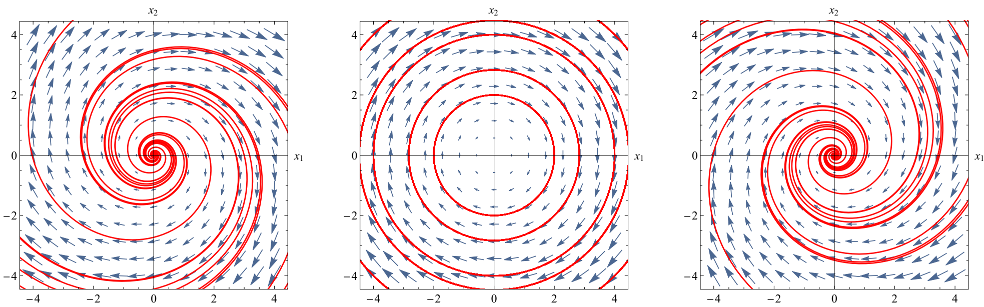

Figure 3: Illustration to problem P5 for $\alpha = 1/4$ (left), $\alpha = 0$ (middle), $\alpha = -1/4$ (right).

(c) The phase portraits for a value of α slightly below, and for another value slightly above the critical value are shown in Figure 3.

P6. The system of differential equations is given as

$$\boldsymbol{x}' = \begin{bmatrix} 0 & -5 \\ 1 & \alpha \end{bmatrix} \boldsymbol{x},$$

where α is a constant parameter.

(a) Determine the eigenvalues in terms of α.

(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.

(c) Draw a phase portrait for a value of α slightly below, and for another value slightly above, each critical value.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} -r & -5 \\ 1 & \alpha - r \end{vmatrix} = r^2 - \alpha r + 5 = 0 \quad \Longrightarrow \quad r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}.$$

(b) Note that the roots are complex when $-\sqrt{2} < \alpha < \sqrt{20}$. For the case when $\alpha \in (-\sqrt{20}, 0)$, the equilibrium point $(0, 0)$ is a stable spiral. On the other hand, $\alpha \in (0, \sqrt{20})$, the equilibrium point is an unstable spiral. For the case $\alpha = 0$, the roots are purely imaginary, so the equilibrium point is a center. When $\alpha^2 > 20$, the roots are real and distinct. The equilibrium point becomes a node, with its stability dependent on the sign of α. Finally, the case $\alpha^2 = 20$ marks the transition from spirals to nodes.

(c) The phase portraits for a value of α slightly below, and for another value slightly above the critical values ($\alpha = -\sqrt{20}$ and $\alpha = 0$) are shown in Figure 4.

P7. (a) Find a fundamental matrix for the system of equations.

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \boldsymbol{x}.$$

(b) Find the fundamental matrix $\boldsymbol{\Phi}(t)$ satisfying $\boldsymbol{\Phi}(0) = \boldsymbol{I}$.

Solution:

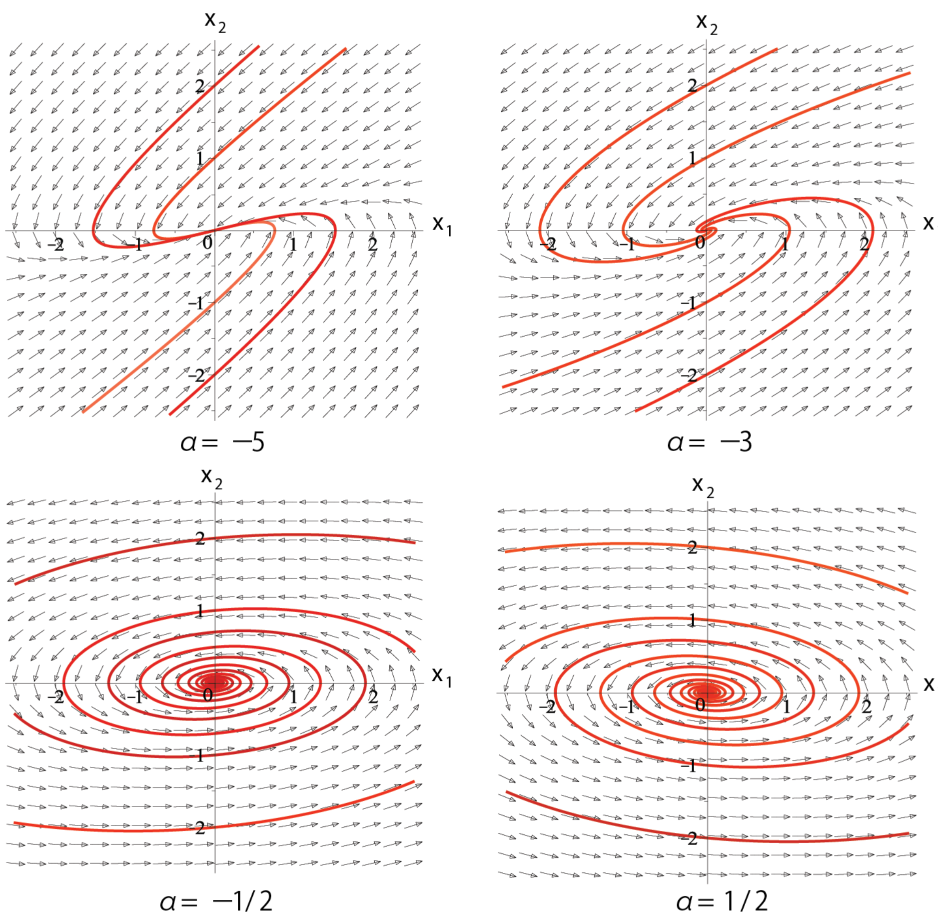

Figure 4: Illustration to problem P6.

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = r^2 - r - 2 = 0 \implies r_1 = -1, r_2 = 2.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \implies \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $r = r_2$,

$$(\boldsymbol{A} - r_2\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \implies \boldsymbol{\xi}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since the eigenvalues are real and distinct, the general solution is

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{bmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{bmatrix}$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{\Psi}^{-1}(0) = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix},$$

so that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{3} \begin{bmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{bmatrix}.$$

P8. (a) Find a fundamental matrix for the system of equations.

$$\boldsymbol{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \boldsymbol{x}.$$

(b) Find the fundamental matrix $\boldsymbol{\Phi}(t)$ satisfying $\boldsymbol{\Phi}(0) = \boldsymbol{I}$.

Solution:

(a) Find the eigenvalues. The characteristic equation

$$\det(\boldsymbol{A} - r\boldsymbol{I}) = \begin{vmatrix} -1-r & -4 \\ 1 & -1-r \end{vmatrix} = r^2 + 2r + 5 = 0 \implies r_1 = -1 + 2\imath, \ \ r_2 = -1 - 2\imath.$$

Find the eigenvectors. For $r = r_1$,

$$(\boldsymbol{A} - r_1\boldsymbol{I})\boldsymbol{\xi} = \begin{bmatrix} -2\imath & -4 \\ 1 & -2\imath \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{0} \implies \boldsymbol{\xi}_1 = \begin{bmatrix} 2\imath \\ 1 \end{bmatrix}.$$

For $r = r_2 = \bar{r}_1$, we have $\boldsymbol{\xi}_2 = \bar{\boldsymbol{\xi}}_1 = (-2\imath, 1)^{\mathrm{T}}$. Next,

$$\boldsymbol{a} = \mathrm{Re}(\boldsymbol{\xi}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \mathrm{Im}(\boldsymbol{\xi}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$\boldsymbol{u}(t) = e^{-t}(\boldsymbol{a}\cos 2t - \boldsymbol{b}\sin 2t)$, $\boldsymbol{v}(t) = e^{-t}(\boldsymbol{a}\sin 2t + \boldsymbol{b}\cos 2t)$, and the general solution

$$\boldsymbol{x}(t) = c_1\boldsymbol{u}(t) + c_2\boldsymbol{v}(t)$$

is

$$\boldsymbol{x}(t) = c_1 e^{-t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right) + c_2 e^{-t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t\right) =$$
$$c_1 e^{-t}\begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = e^{-t}\begin{bmatrix} -2\sin 2t & 2\cos 2t \\ \cos 2t & \sin 2t \end{bmatrix}$$

(b) We now have

$$\boldsymbol{\Psi}(0) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\Psi}^{-1}(0) = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix},$$

so that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{2}e^{-t}\begin{bmatrix} 2\cos 2t & -4\sin 2t \\ \sin 2t & 2\cos 2t \end{bmatrix}.$$