

Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hôpital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point  $P$  and to orient the axes as shown in Figure 2.3.6. The lower point  $Q$  has coordinates  $(x_0, y_0)$ . It is then possible to show that the curve of minimum time is given by a function  $y = \phi(x)$  that satisfies the differential equation

$$(1 + y'^2)y = k^2, \quad (i)$$

where  $k^2$  is a certain positive constant to be determined later.

(a) Solve Eq. (i) for  $y'$ . Why is it necessary to choose the positive square root?

(b) Introduce the new variable  $t$  by the relation

$$y = k^2 \sin^2 t. \quad (ii)$$

Show that the equation found in part (a) then takes the form

$$2k^2 \sin^2 t \, dt = dx. \quad (iii)$$

(c) Letting  $\theta = 2t$ , show that the solution of Eq. (iii) for which  $x = 0$  when  $y = 0$  is given by

$$x = k^2(\theta - \sin \theta)/2, \quad y = k^2(1 - \cos \theta)/2. \quad (iv)$$

Equations (iv) are parametric equations of the solution of Eq. (i) that passes through  $(0, 0)$ . The graph of Eqs. (iv) is called a **cycloid**.

(d) If we make a proper choice of the constant  $k$ , then the cycloid also passes through the point  $(x_0, y_0)$  and is the solution of the brachistochrone problem. Find  $k$  if  $x_0 = 1$  and  $y_0 = 2$ .

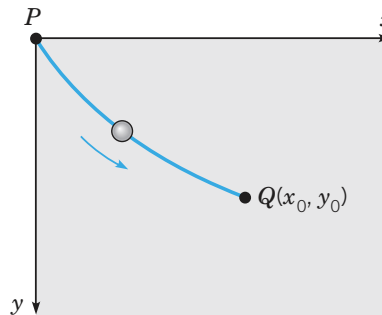


FIGURE 2.3.6 The brachistochrone.

## 2.4 Differences Between Linear and Nonlinear Equations

Up to now, we have been primarily concerned with showing that first order differential equations can be used to investigate many different kinds of problems in the natural sciences, and with presenting methods of solving such equations if they are either linear or separable. Now it is time to turn our attention to some more general

questions about differential equations and to explore in more detail some important ways in which nonlinear equations differ from linear ones.

**Existence and Uniqueness of Solutions.** So far, we have discussed a number of initial value problems, each of which had a solution and apparently only one solution. That raises the question of whether this is true of all initial value problems for first order equations. In other words, does every initial value problem have exactly one solution? This may be an important question even for nonmathematicians. If you encounter an initial value problem in the course of investigating some physical problem, you might want to know that it has a solution before spending very much time and effort in trying to find it. Further, if you are successful in finding one solution, you might be interested in knowing whether you should continue a search for other possible solutions or whether you can be sure that there are no other solutions. For linear equations, the answers to these questions are given by the following fundamental theorem.

**Theorem 2.4.1** If the functions  $p$  and  $g$  are continuous on an open interval  $I: \alpha < t < \beta$  containing the point  $t = t_0$ , then there exists a unique function  $y = \phi(t)$  that satisfies the differential equation

$$y' + p(t)y = g(t) \quad (1)$$

for each  $t$  in  $I$ , and that also satisfies the initial condition

$$y(t_0) = y_0, \quad (2)$$

where  $y_0$  is an arbitrary prescribed initial value.

Observe that Theorem 2.4.1 states that the given initial value problem *has* a solution and also that the problem has *only one* solution. In other words, the theorem asserts both the *existence* and the *uniqueness* of the solution of the initial value problem (1), (2). In addition, it states that the solution exists throughout any interval  $I$  containing the initial point  $t_0$  in which the coefficients  $p$  and  $g$  are continuous. That is, the solution can be discontinuous or fail to exist only at points where at least one of  $p$  and  $g$  is discontinuous. Such points can often be identified at a glance.

The proof of this theorem is partly contained in the discussion in Section 2.1 leading to the formula [Eq. (32) in Section 2.1]

$$\mu(t)y = \int \mu(t)g(t) dt + c, \quad (3)$$

where [Eq. (30) in Section 2.1]

$$\mu(t) = \exp \int p(t) dt. \quad (4)$$

The derivation in Section 2.1 shows that if Eq. (1) has a solution, then it must be given by Eq. (3). By looking slightly more closely at that derivation, we can also conclude that the differential equation (1) must indeed have a solution. Since  $p$  is

continuous for  $\alpha < t < \beta$ , it follows that  $\mu$  is defined in this interval and is a nonzero differentiable function. Upon multiplying Eq. (1) by  $\mu(t)$ , we obtain

$$[\mu(t)y]' = \mu(t)g(t). \quad (5)$$

Since both  $\mu$  and  $g$  are continuous, the function  $\mu g$  is integrable, and Eq. (3) follows from Eq. (5). Further, the integral of  $\mu g$  is differentiable, so  $y$  as given by Eq. (3) exists and is differentiable throughout the interval  $\alpha < t < \beta$ . By substituting the expression for  $y$  from Eq. (3) into either Eq. (1) or Eq. (5), you can verify that this expression satisfies the differential equation throughout the interval  $\alpha < t < \beta$ . Finally, the initial condition (2) determines the constant  $c$  uniquely, so there is only one solution of the initial value problem; this completes the proof.

Equation (4) determines the integrating factor  $\mu(t)$  only up to a multiplicative factor that depends on the lower limit of integration. If we choose this lower limit to be  $t_0$ , then

$$\mu(t) = \exp \int_{t_0}^t p(s) ds, \quad (6)$$

and it follows that  $\mu(t_0) = 1$ . Using the integrating factor given by Eq. (6), and choosing the lower limit of integration in Eq. (3) also to be  $t_0$ , we obtain the general solution of Eq. (1) in the form

$$y = \frac{1}{\mu(t)} \left[ \int_{t_0}^t \mu(s)g(s) ds + c \right]. \quad (7)$$

To satisfy the initial condition (2), we must choose  $c = y_0$ . Thus the solution of the initial value problem (1), (2) is

$$y = \frac{1}{\mu(t)} \left[ \int_{t_0}^t \mu(s)g(s) ds + y_0 \right], \quad (8)$$

where  $\mu(t)$  is given by Eq. (6).

Turning now to nonlinear differential equations, we must replace Theorem 2.4.1 by a more general theorem, such as the one that follows.

### Theorem 2.4.2

Let the functions  $f$  and  $\partial f/\partial y$  be continuous in some rectangle  $\alpha < t < \beta$ ,  $\gamma < y < \delta$  containing the point  $(t_0, y_0)$ . Then, in some interval  $t_0 - h < t < t_0 + h$  contained in  $\alpha < t < \beta$ , there is a unique solution  $y = \phi(t)$  of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0. \quad (9)$$

Observe that the hypotheses in Theorem 2.4.2 reduce to those in Theorem 2.4.1 if the differential equation is linear. In this case

$$f(t, y) = -p(t)y + g(t) \quad \text{and} \quad \partial f(t, y)/\partial y = -p(t),$$

so the continuity of  $f$  and  $\partial f/\partial y$  is equivalent to the continuity of  $p$  and  $g$ .

The proof of Theorem 2.4.1 was comparatively simple because it could be based on the expression (3) that gives the solution of an arbitrary linear equation. There is no corresponding expression for the solution of the differential equation (9), so

the proof of Theorem 2.4.2 is much more difficult. It is discussed to some extent in Section 2.8 and in greater depth in more advanced books on differential equations.

Here we note that the conditions stated in Theorem 2.4.2 are sufficient to guarantee the existence of a unique solution of the initial value problem (9) in some interval  $t_0 - h < t < t_0 + h$ , but they are not necessary. That is, the conclusion remains true under slightly weaker hypotheses about the function  $f$ . In fact, the existence of a solution (but not its uniqueness) can be established on the basis of the continuity of  $f$  alone.

An important geometrical consequence of the uniqueness parts of Theorems 2.4.1 and 2.4.2 is that the graphs of two solutions cannot intersect each other. Otherwise, there would be two solutions that satisfy the initial condition corresponding to the point of intersection, in contradiction to Theorem 2.4.1 or 2.4.2.

We now consider some examples.

### EXAMPLE 1

Use Theorem 2.4.1 to find an interval in which the initial value problem

$$ty' + 2y = 4t^2, \quad (10)$$

$$y(1) = 2 \quad (11)$$

has a unique solution.

Rewriting Eq. (10) in the standard form (1), we have

$$y' + (2/t)y = 4t,$$

so  $p(t) = 2/t$  and  $g(t) = 4t$ . Thus, for this equation,  $g$  is continuous for all  $t$ , while  $p$  is continuous only for  $t < 0$  or for  $t > 0$ . The interval  $t > 0$  contains the initial point; consequently, Theorem 2.4.1 guarantees that the problem (10), (11) has a unique solution on the interval  $0 < t < \infty$ . In Example 4 of Section 2.1 we found the solution of this initial value problem to be

$$y = t^2 + \frac{1}{t^2}, \quad t > 0. \quad (12)$$

Now suppose that the initial condition (11) is changed to  $y(-1) = 2$ . Then Theorem 2.4.1 asserts the existence of a unique solution for  $t < 0$ . As you can readily verify, the solution is again given by Eq. (12), but now on the interval  $-\infty < t < 0$ .

### EXAMPLE 2

Apply Theorem 2.4.2 to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1. \quad (13)$$

Note that Theorem 2.4.1 is not applicable to this problem since the differential equation is nonlinear. To apply Theorem 2.4.2, observe that

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x, y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2}.$$

Thus each of these functions is continuous everywhere except on the line  $y = 1$ . Consequently, a rectangle can be drawn about the initial point  $(0, -1)$  in which both  $f$  and  $\partial f / \partial y$  are continuous. Therefore, Theorem 2.4.2 guarantees that the initial value problem has a unique solution in some interval about  $x = 0$ . However, even though the rectangle can be stretched infinitely far in both the positive and the negative  $x$  directions, this does not necessarily mean that the solution exists for all  $x$ . Indeed, the initial value problem (13) was solved in Example 2 of Section 2.2, and the solution exists only for  $x > -2$ .

Now suppose we change the initial condition to  $y(0) = 1$ . The initial point now lies on the line  $y = 1$ , so no rectangle can be drawn about it within which  $f$  and  $\partial f/\partial y$  are continuous. Consequently, Theorem 2.4.2 says nothing about possible solutions of this modified problem. However, if we separate the variables and integrate, as in Section 2.2, we find that

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

Further, if  $x = 0$  and  $y = 1$ , then  $c = -1$ . Finally, by solving for  $y$ , we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}. \quad (14)$$

Equation (14) provides two functions that satisfy the given differential equation for  $x > 0$  and also satisfy the initial condition  $y(0) = 1$ .

### EXAMPLE 3

Consider the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0 \quad (15)$$

for  $t \geq 0$ . Apply Theorem 2.4.2 to this initial value problem and then solve the problem.

The function  $f(t, y) = y^{1/3}$  is continuous everywhere, but  $\partial f/\partial y$  does not exist when  $y = 0$ , and hence it is not continuous there. Thus Theorem 2.4.2 does not apply to this problem, and no conclusion can be drawn from it. However, by the remark following Theorem 2.4.2, the continuity of  $f$  does ensure the existence of solutions, though not their uniqueness.

To understand the situation more clearly, we must actually solve the problem, which is easy to do since the differential equation is separable. Thus we have

$$y^{-1/3} dy = dt,$$

so

$$\frac{3}{2}y^{2/3} = t + c$$

and

$$y = \left[\frac{2}{3}(t + c)\right]^{3/2}.$$

The initial condition is satisfied if  $c = 0$ , so

$$y = \phi_1(t) = \left(\frac{2}{3}t\right)^{3/2}, \quad t \geq 0 \quad (16)$$

satisfies both of Eqs. (15). On the other hand, the function

$$y = \phi_2(t) = -\left(\frac{2}{3}t\right)^{3/2}, \quad t \geq 0 \quad (17)$$

is also a solution of the initial value problem. Moreover, the function

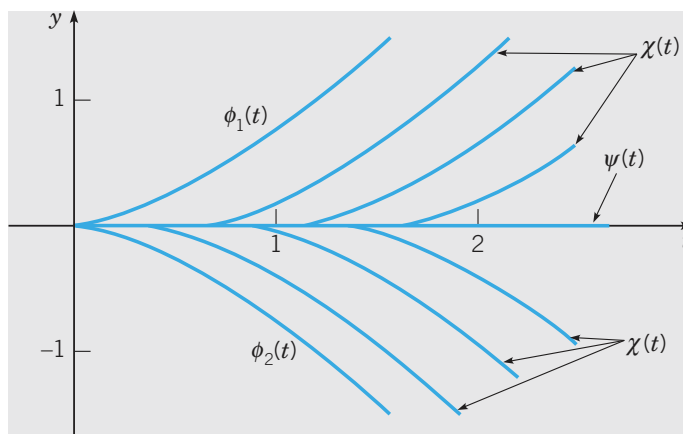
$$y = \psi(t) = 0, \quad t \geq 0 \quad (18)$$

is yet another solution. Indeed, for an arbitrary positive  $t_0$ , the functions

$$y = \chi(t) = \begin{cases} 0, & \text{if } 0 \leq t < t_0, \\ \pm \left[\frac{2}{3}(t - t_0)\right]^{3/2}, & \text{if } t \geq t_0 \end{cases} \quad (19)$$

are continuous, are differentiable (in particular at  $t = t_0$ ), and are solutions of the initial value problem (15). Hence this problem has an infinite family of solutions; see Figure 2.4.1, where a few of these solutions are shown.

As already noted, the nonuniqueness of the solutions of the problem (15) does not contradict the existence and uniqueness theorem, since the theorem is not applicable if the initial point lies on the  $t$ -axis. If  $(t_0, y_0)$  is any point not on the  $t$ -axis, however, then the theorem guarantees that there is a unique solution of the differential equation  $y' = y^{1/3}$  passing through  $(t_0, y_0)$ .



**FIGURE 2.4.1** Several solutions of the initial value problem  $y' = y^{1/3}$ ,  $y(0) = 0$ .

**Interval of Definition.** According to Theorem 2.4.1, the solution of a linear equation (1)

$$y' + p(t)y = g(t),$$

subject to the initial condition  $y(t_0) = y_0$ , exists throughout any interval about  $t = t_0$  in which the functions  $p$  and  $g$  are continuous. Thus vertical asymptotes or other discontinuities in the solution can occur only at points of discontinuity of  $p$  or  $g$ . For instance, the solutions in Example 1 (with one exception) are asymptotic to the  $y$ -axis, corresponding to the discontinuity at  $t = 0$  in the coefficient  $p(t) = 2/t$ , but none of the solutions has any other point where it fails to exist and to be differentiable. The one exceptional solution shows that solutions may sometimes remain continuous even at points of discontinuity of the coefficients.

On the other hand, for a nonlinear initial value problem satisfying the hypotheses of Theorem 2.4.2, the interval in which a solution exists may be difficult to determine. The solution  $y = \phi(t)$  is certain to exist as long as the point  $[t, \phi(t)]$  remains within a region in which the hypotheses of Theorem 2.4.2 are satisfied. This is what determines the value of  $h$  in that theorem. However, since  $\phi(t)$  is usually not known, it may be impossible to locate the point  $[t, \phi(t)]$  with respect to this region. In any case, the interval in which a solution exists may have no simple relationship to the function  $f$  in the differential equation  $y' = f(t, y)$ . This is illustrated by the following example.

#### EXAMPLE 4

Solve the initial value problem

$$y' = y^2, \quad y(0) = 1, \quad (20)$$

and determine the interval in which the solution exists.

Theorem 2.4.2 guarantees that this problem has a unique solution since  $f(t, y) = y^2$  and  $\partial f / \partial y = 2y$  are continuous everywhere. To find the solution, we separate the variables and integrate with the result that

$$y^{-2} dy = dt \quad (21)$$

and

$$-y^{-1} = t + c.$$

Then, solving for  $y$ , we have

$$y = -\frac{1}{t + c}. \quad (22)$$

To satisfy the initial condition, we must choose  $c = -1$ , so

$$y = \frac{1}{1 - t} \quad (23)$$

is the solution of the given initial value problem. Clearly, the solution becomes unbounded as  $t \rightarrow 1$ ; therefore, the solution exists only in the interval  $-\infty < t < 1$ . There is no indication from the differential equation itself, however, that the point  $t = 1$  is in any way remarkable. Moreover, if the initial condition is replaced by

$$y(0) = y_0, \quad (24)$$

then the constant  $c$  in Eq. (22) must be chosen to be  $c = -1/y_0$ , and it follows that

$$y = \frac{y_0}{1 - y_0 t} \quad (25)$$

is the solution of the initial value problem with the initial condition (24). Observe that the solution (25) becomes unbounded as  $t \rightarrow 1/y_0$ , so the interval of existence of the solution is  $-\infty < t < 1/y_0$  if  $y_0 > 0$ , and is  $1/y_0 < t < \infty$  if  $y_0 < 0$ . This example illustrates another feature of initial value problems for nonlinear equations: the singularities of the solution may depend in an essential way on the initial conditions as well as on the differential equation.

**General Solution.** Another way in which linear and nonlinear equations differ concerns the concept of a general solution. For a first order linear equation it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant. For nonlinear equations this may not be the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant. For instance, for the differential equation  $y' = y^2$  in Example 4, the expression in Eq. (22) contains an arbitrary constant but does not include all solutions of the differential equation. To show this, observe that the function  $y = 0$  for all  $t$  is certainly a solution of the differential equation, but it cannot be obtained from Eq. (22) by assigning a value to  $c$ . In this example we might anticipate that something of this sort might happen, because to rewrite the original differential equation in the form (21), we must require that  $y$  is not zero. However, the existence of “additional” solutions is not uncommon for nonlinear equations; a less obvious example is given in Problem 22. Thus we will use the term “general solution” only when discussing linear equations.

**Implicit Solutions.** Recall again that for an initial value problem for a first order linear equation, Eq. (8) provides an explicit formula for the solution  $y = \phi(t)$ . As long as the necessary antiderivatives can be found, the value of the solution at any point can be determined merely by substituting the appropriate value of  $t$  into the equation. The situation for nonlinear equations is much less satisfactory. Usually, the best that we can hope for is to find an equation

$$F(t, y) = 0 \quad (26)$$

involving  $t$  and  $y$  that is satisfied by the solution  $y = \phi(t)$ . Even this can be done only for differential equations of certain particular types, of which separable equations are the most important. The equation (26) is called an integral, or first integral, of the differential equation, and (as we have already noted) its graph is an integral curve, or perhaps a family of integral curves. Equation (26), assuming it can be found, defines the solution implicitly; that is, for each value of  $t$  we must solve Eq. (26) to find the corresponding value of  $y$ . If Eq. (26) is simple enough, it may be possible to solve it for  $y$  by analytical means and thereby obtain an explicit formula for the solution. However, more frequently this will not be possible, and you will have to resort to a numerical calculation to determine (approximately) the value of  $y$  for a given value of  $t$ . Once several pairs of values of  $t$  and  $y$  have been calculated, it is often helpful to plot them and then to sketch the integral curve that passes through them. You should arrange for a computer to do this for you, if possible.

Examples 2, 3, and 4 are nonlinear problems in which it is easy to solve for an explicit formula for the solution  $y = \phi(t)$ . On the other hand, Examples 1 and 3 in Section 2.2 are cases in which it is better to leave the solution in implicit form and to use numerical means to evaluate it for particular values of the independent variable. The latter situation is more typical; unless the implicit relation is quadratic in  $y$  or has some other particularly simple form, it is unlikely that it can be solved exactly by analytical methods. Indeed, more often than not, it is impossible even to find an implicit expression for the solution of a first order nonlinear equation.

**Graphical or Numerical Construction of Integral Curves.** Because of the difficulty in obtaining exact analytical solutions of nonlinear differential equations, methods that yield approximate solutions or other qualitative information about solutions are of correspondingly greater importance. We have already described, in Section 1.1, how the direction field of a differential equation can be constructed. The direction field can often show the qualitative form of solutions and can also be helpful in identifying regions of the  $ty$ -plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigation. Graphical methods for first order equations are discussed further in Section 2.5. An introduction to numerical methods for first order equations is given in Section 2.7, and a systematic discussion of numerical methods appears in Chapter 8. However, it is not necessary to study the numerical algorithms themselves in order to use effectively one of the many software packages that generate and plot numerical approximations to solutions of initial value problems.

**Summary.** The linear equation  $y' + p(t)y = g(t)$  has several nice properties that can be summarized in the following statements:

1. Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. A particular solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.
2. There is an expression for the solution, namely, Eq. (7) or Eq. (8). Moreover, although it involves two integrations, the expression is an explicit one for the solution  $y = \phi(t)$  rather than an equation that defines  $\phi$  implicitly.



3. The possible points of discontinuity, or singularities, of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all  $t$ , then the solution also exists and is differentiable for all  $t$ .

None of these statements is true, in general, of nonlinear equations. Although a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. If you are able to integrate a nonlinear equation, you are likely to obtain an equation defining solutions implicitly rather than explicitly. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition as well as on the differential equation.

## PROBLEMS

In each of Problems 1 through 6, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1.  $(t - 3)y' + (\ln t)y = 2t$ ,  $y(1) = 2$
2.  $t(t - 4)y' + y = 0$ ,  $y(2) = 1$
3.  $y' + (\tan t)y = \sin t$ ,  $y(\pi) = 0$
4.  $(4 - t^2)y' + 2ty = 3t^2$ ,  $y(-3) = 1$
5.  $(4 - t^2)y' + 2ty = 3t^2$ ,  $y(1) = -3$
6.  $(\ln t)y' + y = \cot t$ ,  $y(2) = 3$

In each of Problems 7 through 12, state where in the  $ty$ -plane the hypotheses of Theorem 2.4.2 are satisfied.

7.  $y' = \frac{t - y}{2t + 5y}$
8.  $y' = (1 - t^2 - y^2)^{1/2}$
9.  $y' = \frac{\ln |ty|}{1 - t^2 + y^2}$
10.  $y' = (t^2 + y^2)^{3/2}$
11.  $\frac{dy}{dt} = \frac{1 + t^2}{3y - y^2}$
12.  $\frac{dy}{dt} = \frac{(\cot t)y}{1 + y}$

In each of Problems 13 through 16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value  $y_0$ .

13.  $y' = -4t/y$ ,  $y(0) = y_0$
14.  $y' = 2ty^2$ ,  $y(0) = y_0$
15.  $y' + y^3 = 0$ ,  $y(0) = y_0$
16.  $y' = t^2/y(1 + t^3)$ ,  $y(0) = y_0$

In each of Problems 17 through 20, draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as  $t$  increases and how their behavior depends on the initial value  $y_0$  when  $t = 0$ .

17.  $y' = ty(3 - y)$
18.  $y' = y(3 - ty)$
19.  $y' = -y(3 - ty)$
20.  $y' = t - 1 - y^2$
21. Consider the initial value problem  $y' = y^{1/3}$ ,  $y(0) = 0$  from Example 3 in the text.
  - (a) Is there a solution that passes through the point  $(1, 1)$ ? If so, find it.
  - (b) Is there a solution that passes through the point  $(2, 1)$ ? If so, find it.
  - (c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at  $t = 2$ .

22. (a) Verify that both  $y_1(t) = 1 - t$  and  $y_2(t) = -t^2/4$  are solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1.$$

Where are these solutions valid?

- (b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.
- (c) Show that  $y = ct + c^2$ , where  $c$  is an arbitrary constant, satisfies the differential equation in part (a) for  $t \geq -2c$ . If  $c = -1$ , the initial condition is also satisfied, and the solution  $y = y_1(t)$  is obtained. Show that there is no choice of  $c$  that gives the second solution  $y = y_2(t)$ .
23. (a) Show that  $\phi(t) = e^{2t}$  is a solution of  $y' - 2y = 0$  and that  $y = c\phi(t)$  is also a solution of this equation for any value of the constant  $c$ .
- (b) Show that  $\phi(t) = 1/t$  is a solution of  $y' + y^2 = 0$  for  $t > 0$  but that  $y = c\phi(t)$  is not a solution of this equation unless  $c = 0$  or  $c = 1$ . Note that the equation of part (b) is nonlinear, while that of part (a) is linear.
24. Show that if  $y = \phi(t)$  is a solution of  $y' + p(t)y = 0$ , then  $y = c\phi(t)$  is also a solution for any value of the constant  $c$ .
25. Let  $y = y_1(t)$  be a solution of

$$y' + p(t)y = 0, \tag{i}$$

and let  $y = y_2(t)$  be a solution of

$$y' + p(t)y = g(t). \tag{ii}$$

Show that  $y = y_1(t) + y_2(t)$  is also a solution of Eq. (ii).

26. (a) Show that the solution (7) of the general linear equation (1) can be written in the form

$$y = cy_1(t) + y_2(t), \tag{i}$$

where  $c$  is an arbitrary constant.

- (b) Show that  $y_1$  is a solution of the differential equation

$$y' + p(t)y = 0, \tag{ii}$$

corresponding to  $g(t) = 0$ .

- (c) Show that  $y_2$  is a solution of the full linear equation (1). We see later (for example, in Section 3.5) that solutions of higher order linear equations have a pattern similar to Eq. (i).

**Bernoulli Equations.** Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$y' + p(t)y = q(t)y^n,$$

and is called a Bernoulli equation after Jakob Bernoulli. Problems 27 through 31 deal with equations of this type.

27. (a) Solve Bernoulli's equation when  $n = 0$ ; when  $n = 1$ .
- (b) Show that if  $n \neq 0, 1$ , then the substitution  $v = y^{1-n}$  reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

In each of Problems 28 through 31, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

28.  $t^2 y' + 2ty - y^3 = 0$ ,  $t > 0$

29.  $y' = ry - ky^2$ ,  $r > 0$  and  $k > 0$ . This equation is important in population dynamics and is discussed in detail in Section 2.5.

30.  $y' = \epsilon y - \sigma y^3$ ,  $\epsilon > 0$  and  $\sigma > 0$ . This equation occurs in the study of the stability of fluid flow.

31.  $dy/dt = (\Gamma \cos t + T)y - y^3$ , where  $\Gamma$  and  $T$  are constants. This equation also occurs in the study of the stability of fluid flow.

**Discontinuous Coefficients.** Linear differential equations sometimes occur in which one or both of the functions  $p$  and  $g$  have jump discontinuities. If  $t_0$  is such a point of discontinuity, then it is necessary to solve the equation separately for  $t < t_0$  and  $t > t_0$ . Afterward, the two solutions are matched so that  $y$  is continuous at  $t_0$ ; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make  $y'$  continuous at  $t_0$ .

32. Solve the initial value problem

$$y' + 2y = g(t), \quad y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases}$$

33. Solve the initial value problem

$$y' + p(t)y = 0, \quad y(0) = 1,$$

where

$$p(t) = \begin{cases} 2, & 0 \leq t \leq 1, \\ 1, & t > 1. \end{cases}$$

## 2.5 Autonomous Equations and Population Dynamics

An important class of first order equations consists of those in which the independent variable does not appear explicitly. Such equations are called **autonomous** and have the form

$$dy/dt = f(y). \quad (1)$$

We will discuss these equations in the context of the growth or decline of the population of a given species, an important issue in fields ranging from medicine to ecology to global economics. A number of other applications are mentioned in some of the problems. Recall that in Sections 1.1 and 1.2 we considered the special case of Eq. (1) in which  $f(y) = ay + b$ .

Equation (1) is separable, so the discussion in Section 2.2 is applicable to it, but the main purpose of this section is to show how geometrical methods can be used to obtain important qualitative information directly from the differential equation without