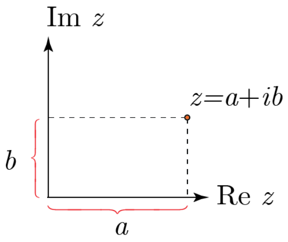
Complex numbers

Definition



- complex number $z = a + \imath b$
- $\imath$ the imaginary unit, $\imath^2 = -1$
- a is the real part of z
- b is the imaginary part of z
  - $\bar{z} = a - \imath b$ is the complex conjugate of z

    $$z\bar{z} = (a+\imath b)\,(a-\imath b) = a^2 - (\imath b)^2 = a^2 + b^2 \quad \text{is a real number}$$

  - modulus of z

    $$|z| = \sqrt{z\bar{z}} = \sqrt{(a^2 + b^2)}$$

  - $\bar{\bar{z}} = z$ and

    $$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2\imath}$$

Complex numbers: operations

- addition

$$z_1 + z_2 = (a_1 + \imath b_1) + (a_2 + \imath b_2) = (a_1 + a_2) + \imath(b_1 + b_2)$$

- multiplication

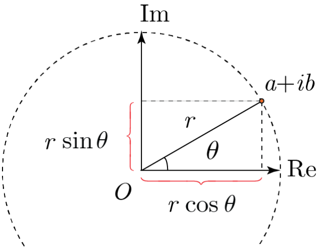
$$z_1 z_2 = (a_1 + \imath b_1)(a_2 + \imath b_2) = (a_1 a_2 - b_1 b_2) + \imath(a_1 b_2 + a_2 b_1)$$

- division

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\bar{z}_2}{\bar{z}_2} = \frac{a_1 + \imath b_1}{a_2 + \imath b_2}\frac{a_2 - \imath b_2}{a_2 - \imath b_2} = \frac{(a_1 + \imath b_1)(a_2 - \imath b_2)}{a_2^2 + b_2^2}$$
$$= \frac{(a_1 a_2 + b_1 b_2) + \imath(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}$$

Complex numbers: polar form

Definition



- similar to polar coordinates
- $r = |z| = \sqrt{a^2 + b^2}$ is the modulus of z
- $\theta = \arg(z)$ is the argument of z $(\tan\theta = a/b)$

- Polar representation

$$z = a + \imath b = r\cos\theta + \imath r\sin\theta = r\left(\cos\theta + \imath\sin\theta\right) = re^{\imath\theta}$$

- Euler's formula

$$\boxed{\cos\theta + \imath\sin\theta = e^{\imath\theta}}$$

$$\boxed{-1 = e^{\imath\pi}}$$

Complex numbers: Euler's formula

- proof (by Taylor expansion)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

- set $x = \imath\theta$ and use identities $\imath^2 = -1, \imath^3 = -\imath, \imath^4 = 1, \ldots$

$$e^{\imath\theta} = 1 + \imath\theta + \frac{1}{2!}(\imath\theta)^2 + \frac{1}{3!}(\imath\theta)^3 + \frac{1}{4!}(\imath\theta)^4 + \frac{1}{5!}(\imath\theta)^5 \ldots$$
$$= \underbrace{\left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \ldots\right)}_{\cos\theta} + \imath \underbrace{\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots\right)}_{\sin\theta}$$

- exponential representation

$$z = a + \imath b = e^{\rho + \imath\theta} = e^{\rho}e^{\imath\theta} = e^{\rho}\left(\cos\theta + \imath\sin\theta\right)$$

- modulus $|z| = |e^{\rho}||e^{\imath\theta}| = e^{\rho}$ since

$$|e^{\imath\theta}| = |\cos\theta + \imath\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Complex numbers: expressions for $\cos$ and $\sin$

- Let z be a complex number of unit modulus:

$$z = e^{\imath\theta} = \cos\theta + \imath\sin\theta \qquad \bar{z} = e^{-\imath\theta} = \cos\theta - \imath\sin\theta$$

- Expressions for $\cos\theta$ and $\sin\theta$

$$\cos\theta = \mathrm{Re}(e^{\imath\theta}) = \mathrm{Re}(z) = \frac{z+\bar{z}}{2} = \frac{e^{\imath\theta}+e^{-\imath\theta}}{2}$$

$$\sin\theta = \mathrm{Im}(e^{\imath\theta}) = \mathrm{Im}(z) = \frac{z-\bar{z}}{2\imath} = \frac{e^{\imath\theta}-e^{-\imath\theta}}{2\imath}$$

- Recall that hyperbolic sine and cosine are defined as

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \qquad \sinh x = \frac{e^{x}-e^{-x}}{2}$$

- Then

$$\cos\theta = \cosh(\imath\theta) \qquad \sin\theta = \frac{\sinh(\imath\theta)}{\imath} = -\imath\sinh(\imath\theta)$$

Complex numbers: computation of some integrals

- Compute

$$\boxed{\int e^{px} \cos qx \, \mathrm{d}x}$$

without integrating by parts (p and q are real numbers)

- Note that $\cos qx = \mathrm{Re}\left\{e^{\imath qx}\right\}$. Therefore

$$\int e^{px} \cos qx \, \mathrm{d}x = \int e^{px} \mathrm{Re}\left\{e^{\imath qx}\right\} \, \mathrm{d}x = \mathrm{Re}\left\{\int e^{(p+\imath q)x} \, \mathrm{d}x\right\} =$$

$$\mathrm{Re}\left\{\frac{e^{(p+\imath q)x}}{p+\imath q}\right\} = \mathrm{Re}\left\{\frac{p-\imath q}{p-\imath q}\,\frac{e^{(p+\imath q)x}}{p+\imath q}\right\} = \mathrm{Re}\left\{\frac{(p-\imath q)\, e^{(p+\imath q)x}}{p^2+q^2}\right\} =$$

$$\frac{e^{px}}{p^2+q^2}\mathrm{Re}\left\{(p-\imath q)e^{\imath qx}\right\} = \frac{e^{px}}{p^2+q^2}\mathrm{Re}\left\{(p-\imath q)(\cos qx + \imath \sin qx)\right\} =$$

$$\frac{e^{px}}{p^2+q^2}\mathrm{Re}\left\{(p\cos qx + q \sin qx) + \imath(p \sin qx - q\cos qx)\right\} =$$

$$\boxed{\frac{e^{px}(p\cos qx + q\sin qx)}{p^2+q^2}}$$

Complex numbers: computation of some integrals

- Compute

$$\boxed{\int e^{px}\sin qx\,\mathrm{d}x}$$

without integrating by parts (p and q are real numbers)

- Note that $\sin qx = \operatorname{Im}\{e^{\imath qx}\}$. Therefore

$$\int e^{px}\sin qx\,\mathrm{d}x = \int e^{px}\operatorname{Im}\{e^{\imath qx}\}\,\mathrm{d}x = \operatorname{Im}\left\{\int e^{(p+\imath q)x}\,\mathrm{d}x\right\} =$$

$$\operatorname{Im}\left\{\frac{e^{(p+\imath q)x}}{p+\imath q}\right\} = \operatorname{Im}\left\{\frac{p-\imath q}{p-\imath q}\,\frac{e^{(p+\imath q)x}}{p+\imath q}\right\} = \operatorname{Im}\left\{\frac{(p-\imath q)\,e^{(p+\imath q)x}}{p^2+q^2}\right\} =$$

$$\frac{e^{px}}{p^2+q^2}\operatorname{Im}\{(p-\imath q)e^{\imath qx}\} = \frac{e^{px}}{p^2+q^2}\operatorname{Im}\{(p-\imath q)(\cos qx+\imath\sin qx)\} =$$

$$\frac{e^{px}}{p^2+q^2}\operatorname{Im}\{(p\cos qx+q\sin qx)+\imath(p\sin qx-q\cos qx)\} =$$

$$\boxed{\frac{e^{px}(p\sin qx-q\cos qx)}{p^2+q^2}}$$

Complex numbers: computation of some derivatives

- Compute
$$\boxed{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\cos qx\right)}$$
where p and q are real numbers, and n is integer
- Note that $\cos qx = \mathrm{Re}\left\{e^{\imath qx}\right\}$. Therefore
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\cos qx\right) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\mathrm{Re}\left\{e^{\imath qx}\right\}\right) = \mathrm{Re}\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{(p+\imath q)x}\right)\right\} =$$
$$\mathrm{Re}\left\{(p+\imath q)^n e^{(p+\imath q)x}\right\} = e^{px}\mathrm{Re}\left\{(p+\imath q)^n e^{\imath qx}\right\}$$
- Define φ such that $\cos\varphi = \frac{p}{\sqrt{p^2+q^2}}$ and $\sin\varphi = \frac{q}{\sqrt{p^2+q^2}}$.

  Then $\boxed{p+\imath q = \sqrt{p^2+q^2}\,(\cos\varphi + \imath\sin\varphi) = (p^2+q^2)^{\frac{1}{2}}e^{\imath\varphi}}$
$$\Longrightarrow \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\cos qx\right) = e^{px}\mathrm{Re}\left\{(p^2+q^2)^{\frac{n}{2}}e^{\imath n\varphi}e^{\imath qx}\right\} =$$
$$e^{px}(p^2+q^2)^{\frac{n}{2}}\mathrm{Re}\left\{e^{\imath(n\varphi+qx)}\right\} = \boxed{e^{px}(p^2+q^2)^{\frac{n}{2}}\cos(n\varphi+qx)}$$

Complex numbers: computation of some derivatives

- Compute

$$\boxed{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\sin qx\right)}$$

where p and q are real numbers, and n is integer

- Note that $\sin qx = \operatorname{Im}\{e^{\imath qx}\}$. Therefore

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\sin qx\right) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\operatorname{Im}\{e^{\imath qx}\}\right) = \operatorname{Im}\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{(p+\imath q)x}\right)\right\} =$$

$$\operatorname{Im}\left\{(p+\imath q)^n e^{(p+\imath q)x}\right\} = e^{px}\operatorname{Im}\{(p+\imath q)^n e^{\imath qx}\}$$

- Define φ such that $\cos\varphi = \frac{p}{\sqrt{p^2+q^2}}$ and $\sin\varphi = \frac{q}{\sqrt{p^2+q^2}}$.

Then $\boxed{p+\imath q = \sqrt{p^2+q^2}\left(\cos\varphi + \imath\sin\varphi\right) = (p^2+q^2)^{\frac{1}{2}}e^{\imath\varphi}}$

$$\Longrightarrow \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\sin qx\right) = e^{px}\operatorname{Im}\left\{(p^2+q^2)^{\frac{n}{2}}e^{\imath n\varphi}e^{\imath qx}\right\} =$$

$$e^{px}(p^2+q^2)^{\frac{n}{2}}\operatorname{Im}\left\{e^{\imath(n\varphi+qx)}\right\} = \boxed{e^{px}(p^2+q^2)^{\frac{n}{2}}\sin(n\varphi+qx)}$$

Complex numbers: computation of some derivatives

- We have defined

$$\boxed{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\sin qx\right) = e^{px}(p^2+q^2)^{\frac{n}{2}}\sin(n\varphi+qx)}$$

$$\boxed{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{px}\cos qx\right) = e^{px}(p^2+q^2)^{\frac{n}{2}}\cos(n\varphi+qx)}$$

- Example: $p=4, q=3 \Longrightarrow \sqrt{p^2+q^2}=5$ and $\tan\varphi = 3/4$

$$\boxed{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{4x}\sin 3x\right) = 5^n e^{4x}\sin(3x+n\arctan(3/4))}$$

- Example: $p=1, q=1 \Longrightarrow \sqrt{p^2+q^2}=\sqrt{2}$ and $\tan\varphi = 1$

$$\boxed{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{x}\cos x\right) = (\sqrt{2})^n e^{x}\cos(x+n\pi/4)}$$

Complex numbers: powers

- from $z = re^{\imath\theta}$ we have

$$z^n = r^n \left(e^{\imath\theta}\right)^n = r^n e^{\imath\theta n} = r^n \left(\cos\theta n + \imath \sin\theta n\right)$$

- de Moivre's formula: from $\left(e^{\imath\theta}\right)^n = e^{\imath\theta n}$ we have

$$\boxed{(\cos\theta + \imath\sin\theta)^n = \cos\theta n + \imath\sin\theta n}$$

  - example ($n = 2$)

$$\begin{aligned}(\cos\theta + \imath\sin\theta)^2 &= \left(\cos^2\theta - \sin^2\theta\right) + \imath\left(2\sin\theta\cos\theta\right)\\ &= \cos 2\theta + \imath\sin 2\theta\end{aligned}$$

  - example ($n = 3$)

$$\begin{aligned}(\cos\theta + \imath\sin\theta)^3 &= \left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + \imath\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)\\ &= \cos 3\theta + \imath\sin 3\theta\end{aligned}$$

Complex numbers: extraction of roots

- from $z = re^{\imath\theta}$ we have

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(e^{\imath\theta}\right)^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\imath\frac{\theta}{n}} = r^{\frac{1}{n}} \left(\cos\frac{\theta}{n} + \imath\sin\frac{\theta}{n}\right)$$

- but $\sin$ and $\cos$ are periodic functions and z does not change if $\theta \to \theta + 2\pi k$, therefore

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\frac{\theta + 2\pi k}{n} + \imath\sin\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1$$

- a number λ is said to be an n-th root of complex number $z = re^{\imath\theta}$ if $\boxed{\lambda^n = z}$, and we write $\lambda = \sqrt[n]{z}$, that is

$$\boxed{\lambda = r^{\frac{1}{n}} \left(\cos\frac{\theta + 2\pi k}{n} + \imath\sin\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1}$$

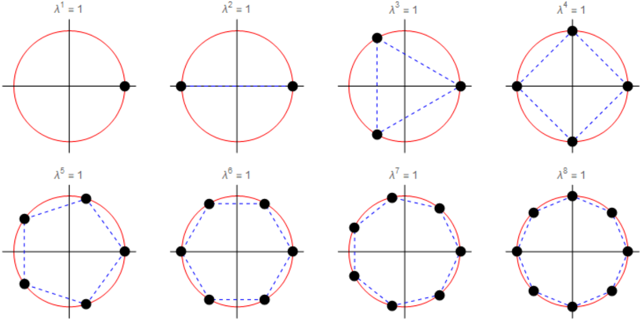
- every complex number has exactly n distinct n-th roots.

Complex numbers: example, $\boxed{\lambda^n = 1}$ (roots of unity)

- Here $z = re^{\imath\theta} = r(\cos\theta + \imath\sin\theta) = 1$, so $r = 1$ and $\theta = 0$.

$$\boxed{\lambda = \left(\cos\frac{2\pi k}{n} + \imath\sin\frac{2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1}$$

- $n = 1$: $\lambda = 1$
- $n = 2$: $\lambda = 1$, $\lambda = -1$
- $n = 3$: $\lambda = 1$, $\lambda = \left(\cos\frac{2\pi}{3} + \imath\sin\frac{2\pi}{3}\right)$, $\lambda = \left(\cos\frac{4\pi}{3} + \imath\sin\frac{4\pi}{3}\right)$

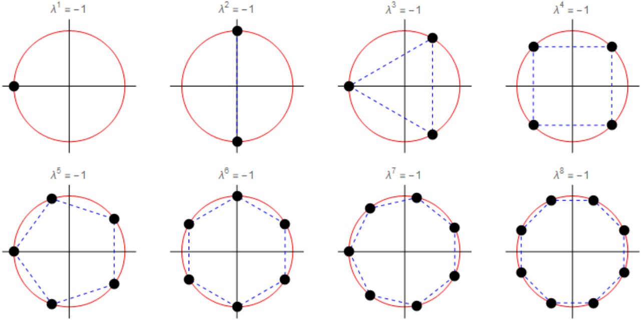


Complex numbers: example, $\boxed{\lambda^n = -1}$

- Here $z = re^{\imath\theta} = r(\cos\theta + \imath\sin\theta) = -1$, so $r = 1$ and $\theta = \pi$.

$$\boxed{\lambda = \left(\cos\frac{\pi + 2\pi k}{n} + \imath\sin\frac{\pi + 2\pi k}{n}\right), \quad k = 0, 1, \dots, n-1}$$

- $n = 1$: $\lambda = -1$
- $n = 2$: $\lambda = \imath$, $\lambda = -\imath$
- $n = 3$: $\lambda = \left(\cos\frac{\pi}{3} + \imath\sin\frac{\pi}{3}\right)$, $\lambda = -1$, $\lambda = \left(\cos\frac{5\pi}{3} + \imath\sin\frac{5\pi}{3}\right)$



Complex numbers: example, $\boxed{\lambda^n = 1 + \imath}$

- $z = re^{\imath\theta} = r(\cos\theta + \imath\sin\theta) = 1 + \imath$, so $r = \sqrt{2}, \theta = \pi/4$.

$$\boxed{\lambda = 2^{\frac{1}{2n}}\left(\cos\frac{\frac{\pi}{4} + 2\pi k}{n} + \imath\sin\frac{\frac{\pi}{4} + 2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1}$$

- $n = 1$: $\lambda = 2^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + \imath\sin\frac{\pi}{4}\right) = 1 + \imath$
- $n = 2$: $\lambda = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + \imath\sin\frac{\pi}{8}\right)$, $\lambda = 2^{\frac{1}{4}}\left(\cos\frac{9\pi}{8} + \imath\sin\frac{9\pi}{8}\right)$
- $n = 3$: $\lambda = 2^{\frac{1}{6}}\left(\cos\frac{\pi}{12} + \imath\sin\frac{\pi}{12}\right)$, $\lambda = 2^{\frac{1}{6}}\left(\cos\frac{9\pi}{12} + \imath\sin\frac{9\pi}{12}\right)$, $\lambda = 2^{\frac{1}{6}}\left(\cos\frac{17\pi}{12} + \imath\sin\frac{17\pi}{12}\right)$