Introduction to Differential Equations
Sample problems # 8

Date Given: May 30, 2022

P1. Use the method of undetermined coefficients to find the general solution of the differential equation $y''' - 2y'' - 4y' + 8y = 6te^{2t}$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^3 - 2r^2 - r + 8 = 0$ with roots $r_1 = r_2 = 2, r_3 = -2$. The solution of the homogeneous equation is $y_c(t) = c_1e^{2t} + c_2te^{2t} + c_3e^{2t}$. We can set $Y(t) = t^2(At + B)e^{2t}$. Substitution into the given equation gives $24A = 6, 6A + 8B = 0$. Then $A = 1/4, B = -3/16$, and therefore $Y(t) = \left(\frac{1}{4}t^3 - \frac{3}{16}t^2\right)e^{2t}$. Thus the general solution is

$$y(t) = c_1e^{2t} + c_2te^{2t} + c_3e^{2t} + \left(\frac{1}{4}t^3 - \frac{3}{16}t^2\right)e^{2t}.$$

P2. Use the method of undetermined coefficients to find the general solution of the differential equation $y''' - y'' - 4y' + 4y = 5 - e^t + e^{2t}$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^3 - r^2 - 4r + 1 = 0$ with roots $r_1 = 1, r_2 = 2, r_3 = -2$. The solution of the homogeneous equation is $y_c(t) = c_1e^t + c_2e^{2t} + c_3e^{-2t}$. We can set $Y(t) = A + Bte^t + Cte^{2t}$. Substitution into the given equation gives $4A = 5, -3B = -1, 4C = 1$. Then $A = 5/4, B = 1/3, C = 1/4$, and therefore $Y(t) = \frac{5}{4} + \frac{1}{3}te^t + \frac{1}{4}te^{2t}$. Thus the general solution is

$$y(t) = c_1e^t + c_2e^{2t} + c_3e^{-2t} + \frac{5}{4} + \frac{1}{3}te^t + \frac{1}{4}te^{2t}.$$

P3. Use the method of undetermined coefficients to find the general solution of the differential equation $y^{(4)} + 2y'' + y = (t-1)^2$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^4 + 2r^2 + 1 = 0$ with roots $r_1 = \imath, r_2 = \imath, r_3 = -\imath, r_4 = -\imath$. The solution of the homogeneous equation is $y_c(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t$. We can set $Y(t) = At^2 + Bt + C$. Substitution into the given equation gives $A = 1, B = -2, 4A + C = 1$. Then $A = 1, B = -2, C = -3$, and therefore $Y(t) = t^2 - 2t - 3$. Thus the general solution is

$$y(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t + t^2 - 2t - 3.$$

P4. Use the method of undetermined coefficients to find the general solution of the differential equation $y''' - y' = 2\sin t$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r(r+1)(r-1) = 0$. The solution of the homogeneous equation is $y_c(t) = c_1 + c_2e^t + c_3e^{-t}$. Since $g(t) = 2\sin t$ is not a solution of the homogeneous problem, we can set $Y(t) = A\cos t + B\sin t$. Substitution into the given equation results in $A = 1$ and $B = 0$. Thus the general solution is

$$y(t) = \cos t + c_1 + c_2e^t + c_3e^{-t}.$$

P5. Use the method of undetermined coefficients to find the general solution of the differential equation $y^{(6)} + y''' = t$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1)=0$. Thus the homogeneous solution is

$$y_c(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + e^{t/2}\left(c_5\cos(\sqrt{3}t/2) + c_6\sin(\sqrt{3}t/2)\right).$$

Note the $g(t)=t$ is a solution of the homogenous problem. Consider a particular solution of the form $Y(t)=t^3(At+B)$. Substitution into the given differential equation gives us that $A=1/24$ and $B=0$. Thus the general solution

$$y(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + e^{t/2}\left(c_5\cos(\sqrt{3}t/2) + c_6\sin(\sqrt{3}t/2)\right) + t^4/24.$$

P6. Use the method of undetermined coefficients to find the general solution of the differential equation $y^{(4)} + y''' = \sin 2t$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r+1)=0$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A\cos 2t + B\sin 2t$. Substitution into the differential equation results in $A=1/40$ and $B=1/20$. Thus the general solution is

$$y(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + \frac{1}{40}\cos 2t + \frac{1}{20}\sin 2t.$$

P7. Use the method of variation of parameters to find the general solution of the differential equation $y''' + y' = \tan t$ for $-\pi/2 < t < \pi/2$.

Solution:

(a) The characteristic equation $r^3 + r = 0$ has three roots, $r_1 = 0, r_2 = \imath, r_3 = -\imath$. The solution of the homogeneous equation is $y_c(t) = c_1 + c_2\cos t + c_3\sin t$. The functions $y_1(t)=1$ and $y_2(t)=\cos t$ and $y_3(t)=\sin t$ form a fundamental set of solutions. The Wronskian of these functions is

$$W(y_1,y_2,y_3)(t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1.$$

(b) Here $g(t)=\tan t$. Using the method of variation of parameters, the particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t)$, in which

$$u_1'(t) = W_1/W = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \tan t & -\cos t & -\sin t \end{vmatrix} = \tan t,$$

$$u_2'(t) = W_2/W = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & \tan t & -\sin t \end{vmatrix} = -\sin t,$$

$$u_3'(t) = W_3/W = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \tan t \end{vmatrix} = -\sin t\tan t = \frac{\cos^2 t - 1}{\cos t} = \cos t - \sec t.$$

Therefore,

$$\begin{aligned} u_1(t) &= \int \tan t\,\mathrm{d}t = -\ln|\cos t|, \\ u_2(t) &= -\int \sin t\,\mathrm{d}t = \cos t, \\ u_3(t) &= \int(\cos t - \sec t)\mathrm{d}t = \sin t - \ln|\sec t + \tan t|. \end{aligned}$$

Hence the particular solution is $Y(t) = -\ln|\cos t| + \cos^2 t + \sin^2 t - \sin t\ln|\sec t + \tan t|$. The general solution is given by

$$y(t) = c_1 + c_2\cos t + c_3\sin t - \ln|\cos t| + 1 - \sin t\ln|\sec t + \tan t|.$$

P8. Use the method of variation of parameters to determine the general solution of the differential equation $y''' - y' = t$.

Solution: The characteristic equation corresponding to the homogeneous problem can be written as $r(r^2-1) = 0$. The roots are $r_1 = 0$, $r_2 = 1$, and $r_3 = -1$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$. The Wronskian is evaluated as

$$W(1, e^t, e^{-t}) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2.$$

Now compute the three determinants

$$W_1 = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2, \quad W_2 = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t}, \quad W_3 = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = e^t.$$

The solution of the system of equations is

$$u_1'(t) = \frac{tW_1(t)}{W(t)} = -t, \quad u_2'(t) = \frac{tW_2(t)}{W(t)} = te^{-t}/2, \quad u_3'(t) = \frac{tW_3(t)}{W(t)} = te^t/2.$$

Hence $u_1(t) = -t^2/2$, $u_2(t) = -e^{-t}(t+1)/2$, and $u_3(t) = e^t(t-1)/2$. The particular solution becomes $Y(t) = -t^2/2 - (t+1)/2 + (t-1)/2 = -t^2/2 - 1$. The constant $(-1)$is a solution of the homogeneous equation, therefore the general solution is

$$y(t) = (c_1 - 1) + c_2 e^t + c_3 e^{-t} - t^2/2 = \tilde{c}_1 + c_2 e^t + c_3 e^{-t} - t^2/2.$$