Introduction to Differential Equations
Assignment # 7

Date Given: May 23, 2022
Date Due: May 30, 2022

P1. (1 point) Find the general solution of the differential equation $y''' - 6y'' + 12y' - 8y = 0$.

Solution: The characteristic equation is $r^3 - 6r^2 + 12r - 8 = 0$. The roots are $r_1 = 2$, $r_2 = 2$, $r_3 = 2$. Note that the three roots are repeated. Thus

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t}$$

P2. (1 point) Find the general solution of the differential equation $16y^{(4)} + 24y'' + 9y = 0$.

Solution: The characteristic equation is $16r^4 + 24r^2 + 9 = 0$. The roots are $r_1 = \sqrt{3}\imath/2$, $r_2 = \sqrt{3}\imath/2$, $r_3 = -\sqrt{3}\imath/2$, $r_4 = -\sqrt{3}\imath/2$. Note that there are repeated roots. Thus

$$y(t) = c_1 \cos(\sqrt{3}t/2) + c_2 t \cos(\sqrt{3}t/2) + c_3 \sin(\sqrt{3}t/2) + c_4 t \sin(\sqrt{3}t/2)$$

P3. (1 point) Find the general solution of the differential equation $y^{(6)} + y = 0$.

Solution: The characteristic equation is $r^6 + 1 = 0$. The roots are given by $r = (-1)^{1/6}$, that is the six sixth roots of $-1 = \cos\pi + \imath \sin\pi$. They are $e^{\imath(\frac{\pi + 2\pi k}{6})}$, $k = 0, 1, \ldots, 5$. Explicitly, $r_1 = (\sqrt{3} - \imath)/2$, $r_2 = (\sqrt{3} + \imath)/2$, $r_3 = \imath$, $r_4 = -\imath$, $r_5 = (-\sqrt{3} + \imath)/2$, $r_6 = (-\sqrt{3} - \imath)/2$. Note that they are three pairs of complex conjugate roots. Thus

$$y(t) = e^{\sqrt{3}\,t/2}\left(c_1 \cos\frac{t}{2} + c_2 \sin\frac{t}{2}\right) + (c_3 \cos t + c_4 \sin t) + e^{-\sqrt{3}\,t/2}\left(c_5 \cos\frac{t}{2} + c_6 \sin\frac{t}{2}\right).$$

P4. (1 point) Find the general solution of the differential equation $y^{(4)} + 2y^{(2)} + y = 0$.

Solution: The characteristic equation can be written as $r^4 + 2r^2 + 1 = (r^2 + 1)^2 = 0$. The repeated roots are $\imath$ and $-\imath$. Each of these roots has multiplicity two. The general solution is

$$y(t) = (c_1 \cos t + c_2 \sin t) + t\,(c_3 \cos t + c_4 \sin t)\,.$$

P5. (1 point) Find the solution of the initial value problem $y''' + 12'' + 36y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 6$.

Solution: The characteristic equation is $r^3 + 12r^2 + 36r = 0$, with roots $r_1 = 0$, $r_2 = -6$, and $r_3 = -6$. Hence the general solution is $y(t) = c_1 + c_2 e^{-6t} + c_3 t e^{-6t}$. Invoking the initial conditions, we obtain the system of equations $c_1 + c_2 = 0, -6c_2 + c_3 = 1, 36c_2 - 12c_3 = 6$, with solution $c_1 = 1/2, c_2 = -1/2, c_3 = -2$. Therefore the solution of the initial value problem is

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-6t} - 2te^{-6t}.$$

P6. (1 point) Find a differential equation whose general solution is $y = c_1 + c_2 e^{2t}\cos(5t) + c_3 e^{2t}\sin(5t)$.

Solution: An algebraic equation with roots $r_1 = 0$, $r_2 = 2 + 5\imath$ and $r_3 = 2 + 5\imath$ is $(r - r_1)(r - r_2)(r - r_3) = r^3 - 4r^2 + 29r = 0$. This is the characteristic equation for the differential equation

$$y''' - 4y'' + 29y' = 0.$$