Ch 6.1: Definition of Laplace Transform

- Many practical engineering problems involve mechanical or electrical systems acted upon by discontinuous or impulsive forcing terms.
- For such problems the methods described in Chapter 3 are difficult to apply.
- In this chapter we use the Laplace transform to convert a problem for an unknown function *f* into a simpler problem for *F*, solve for *F*, and then recover *f* from its transform *F*.
- Given a known function $K(s,t)$, an **integral transform** of a function *f* is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t) dt, \quad \infty \le \alpha < \beta \le \infty$$

Improper Integrals

- The Laplace transform will involve an integral from zero to infinity. Such an integral is a type of improper integral.
- An improper integral over an unbounded interval is defined as the limit of an integral over a finite interval

$$\int_a^\infty f(t)dt = \lim_{A\to\infty}\int_a^A f(t)dt$$

where A is a positive real number.

- If the integral from a to A exists for each $A > a$ and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise, the integral is said to **diverge** or fail to exist.

Example 1

- Consider the following improper integral.

$$\int_0^\infty e^{ct}dt$$

- We can evaluate this integral as follows:

$$\int_0^\infty e^{ct}dt = \lim_{A\to\infty}\int_0^A e^{ct}dt = \lim_{A\to\infty}\frac{1}{c}\left(e^{cA}-1\right)$$

- Note that if $c = 0$, then $e^{ct} = 1$. Thus the following two cases hold:

$$\int_0^\infty e^{ct}dt = -\frac{1}{c}, \text{ if } c<0; \text{ and}$$

$$\int_0^\infty e^{ct}dt \text{ diverges, if } c\geq 0.$$

Example 2

- Consider the following improper integral.

$$\int_1^{\infty} 1/t \; dt$$

- We can evaluate this integral as follows:

$$\int_1^{\infty} 1/t \, dt = \lim_{A \to \infty} \int_1^{A} 1/t \, dt = \lim_{A \to \infty} (\ln A) \to \infty$$

- Therefore, the improper integral diverges.

Example 3

- Consider the following improper integral.

$$\int_1^{\infty} t^{-p}\, dt$$

- From Example 2, this integral diverges at $p = 1$
- We can evaluate this integral for $p \neq 1$ as follows:

$$\int_1^{\infty} t^{-p}\, dt = \lim_{A\to\infty}\int_1^{A} t^{-p}\, dt = \lim_{A\to\infty}\frac{1}{1-p}\left(A^{1-p}-1\right)$$

- The improper integral diverges at $p = 1$ and

$$\text{If } p>1,\ \lim_{A\to\infty}\frac{1}{1-p}\left(A^{1-p}-1\right)=\frac{1}{p-1}$$

$$\text{If } p<1,\ \lim_{A\to\infty}\frac{1}{1-p}\left(A^{1-p}-1\right)\to\infty$$

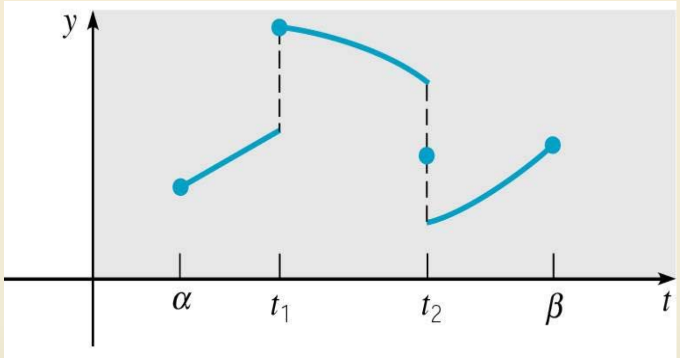
Piecewise Continuous Functions

- A function f is **piecewise continuous** on an interval $[a, b]$ if this interval can be partitioned by a finite number of points $a = t_0 < t_1 < \ldots < t_n = b$ such that

  (1) f is continuous on each (t_k, t_{k+1})

  (2) $\left|\lim_{t \to t_k^+} f(t)\right| < \infty, \quad k = 0, \ldots, n-1$

  (3) $\left|\lim_{t \to t_{k+1}^-} f(t)\right| < \infty, \quad k = 1, \ldots, n$



- In other words, f is piecewise continuous on $[a, b]$ if it is continuous there except for a finite number of jump discontinuities.

The Laplace Transform

- Let f be a function defined for $t \geq 0$, and satisfies certain conditions to be named later.
- The **Laplace Transform of *f*** is defined as an **integral transform**:
$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$
- The kernel function is $K(s,t) = e^{-st}$.
- Since solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations.
- Note that the Laplace Transform is defined by an improper integral, and thus must be checked for convergence.
- On the next few slides, we review examples of improper integrals and piecewise continuous functions.

Theorem 6.1.2

- Suppose that f is a function for which the following hold:

  (1) f is piecewise continuous on $[0, b]$ for all $b > 0$.

  (2) $| f(t) | \leq Ke^{at}$ when $t \geq M$, for constants a, K, M, with K, $M > 0$.

- Then the Laplace Transform of f exists for $s > a$.

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt \text{ finite}$$

- Note: A function f that satisfies the conditions specified above is said to to have **exponential order** as $t \to \infty$.

Example 4

- Let $f(t) = 1$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$
\begin{aligned}
L\{1\} &= \int_0^\infty e^{-st}\,dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}\,dt \\
&= -\lim_{b\to\infty} \left.\frac{e^{-st}}{s}\right|_0^b \\
&= \frac{1}{s}, \quad s > 0
\end{aligned}
$$

Example 5

- Let $f(t) = e^{at}$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$
\begin{aligned}
L\left\{e^{at}\right\} &= \int_0^\infty e^{-st} e^{at} dt \\
&= \lim_{b \to \infty} \int_0^b e^{-(s-a)t} dt \\
&= -\lim_{b \to \infty} \left. \frac{e^{-(s-a)t}}{s-a} \right|_0^b \\
&= \frac{1}{s-a}, \quad s > a
\end{aligned}
$$

Example 6

- Consider the following piecewise-defined function *f*

$$f(t)=\begin{cases}1, & 0\le t\le 1\\ k, & t=1\\ 0 & t>1\end{cases}$$

where k is a constant. This represents a unit impulse.

- Noting that $f(t)$ is piecewisecontinuous, we can compute its Laplace transform

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^1 e^{-st}dt=\frac{1-e^{-s}}{s},\qquad s>0$$

- Observe that this result does not depend on k, the function value at the point of discontinuity.

Example 7

- Let $f(t) = \sin(at)$ for $t \geq 0$. Using integration by parts twice, the Laplace transform $F(s)$ of f is found as follows:

$$F(s) = L\{\sin(at)\} = \int_0^\infty e^{-st}\sin at\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\sin at\,dt$$

$$= \lim_{b\to\infty}\left[-(e^{-st}\cos at)/a\Big|_0^b - \frac{s}{a}\int_0^b e^{-st}\cos at\right]$$

$$= \frac{1}{a} - \frac{s}{a}\lim_{b\to\infty}\left[\int_0^b e^{-st}\cos at\right]$$

$$= \frac{1}{a} - \frac{s}{a}\lim_{b\to\infty}\left[(e^{-st}\sin at)/a\Big|_0^b + \frac{s}{a}\int_0^b e^{-st}\sin at\right]$$

$$= \frac{1}{a} - \frac{s^2}{a^2}F(s) \;\Rightarrow\; F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

Linearity of the Laplace Transform

- Suppose f and g are functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively.
- Then, for s greater than the maximum of a_1 and a_2, the Laplace transform of $c_1 f(t) + c_2 g(t)$ exists. That is,

$$L\left\{c_1 f(t) + c_2 g(t)\right\} = \int_0^\infty e^{-st}\left[c_1 f(t) + c_2 g(t)\right]dt \quad \text{is finite}$$

with

$$\begin{aligned} L\left\{c_1 f(t) + c_2 g(t)\right\} &= c_1 \int_0^\infty e^{-st} f(t)dt + c_2 \int_0^\infty e^{-st} g(t)dt \\ &= c_1 L\left\{f(t)\right\} + c_2 L\left\{g(t)\right\} \end{aligned}$$

Example 8

- Let $f(t) = 5e^{-2t} - 3\sin(4t)$ for $t \geq 0$.
- Then by linearity of the Laplace transform, and using results of previous examples, the Laplace transform $F(s)$ of f is:

$$
\begin{aligned}
F(s) &= L\{f(t)\} \\
&= L\left\{5e^{-2t} - 3\sin(4t)\right\} \\
&= 5L\left\{e^{-2t}\right\} - 3L\left\{\sin(4t)\right\} \\
&= \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0
\end{aligned}
$$

Ch 6.2: Solution of Initial Value Problems

- The Laplace transform is named for the French mathematician Laplace, who studied this transform in 1782.
- The techniques described in this chapter were developed primarily by Oliver Heaviside (1850-1925), an English electrical engineer.
- In this section we see how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients.
- The Laplace transform is useful in solving these differential equations because the transform of f' is related in a simple way to the transform of f, as stated in Theorem 6.2.1.

Theorem 6.2.1

- Suppose that f is a function for which the following hold:
  (1) f is continuous and f' is piecewise continuous on $[0, b]$ for all $b > 0$.
  (2) $|f(t)| \le Ke^{at}$ when $t \ge M$, for constants a, K, M, with $K, M > 0$.
- Then the Laplace Transform of f' exists for $s > a$, with

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- **Proof** (outline): For f and f' continuous on $[0, b]$, we have

$$\begin{aligned}\lim_{b\to\infty}\int_0^b e^{-st} f'(t)dt &= \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^b - \int_0^b (-s)e^{-st} f(t)dt\right] \\ &= \lim_{b\to\infty}\left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt\right]\end{aligned}$$

- Similarly for f' piecewise continuous on $[0, b]$, see text.

The Laplace Transform of f'

- Thus if f and f' satisfy the hypotheses of Theorem 6.2.1, then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- Now suppose f' and f'' satisfy the conditions specified for f and f' of Theorem 6.2.1. We then obtain

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s\left[sL\{f(t)\} - f(0)\right] - f'(0) \\ &= s^2 L\{f(t)\} - sf(0) - f'(0) \end{aligned}$$

- Similarly, we can derive an expression for $L\{f^{(n)}\}$, provided f and its derivatives satisfy suitable conditions. This result is given in Corollary 6.2.2

Corollary 6.2.2

- Suppose that f is a function for which the following hold:

  (1) $f, f', f'', \ldots, f^{(n-1)}$ are continuous, and $f^{(n)}$ piecewise continuous, on $[0, b]$ for all $b > 0$.

  (2) $|f(t)| \le Ke^{at}$, $|f'(t)| \le Ke^{at}$, $\ldots$, $|f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$, for constants a, K, M, with $K, M > 0$.

Then the Laplace Transform of $f^{(n)}$ exists for $s > a$, with

$$L\left\{f^{(n)}(t)\right\} = s^n L\left\{f(t)\right\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Example 1: Chapter 3 Method (1 of 4)

- Consider the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = 1, \;\; y'(0) = 0$$

- Recall from Section 3.1:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 - r - 2 = 0 \;\Leftrightarrow\; (r-2)(r+1) = 0$$

- Thus r_1 = -2 and r_2 = -3, and general solution has the form

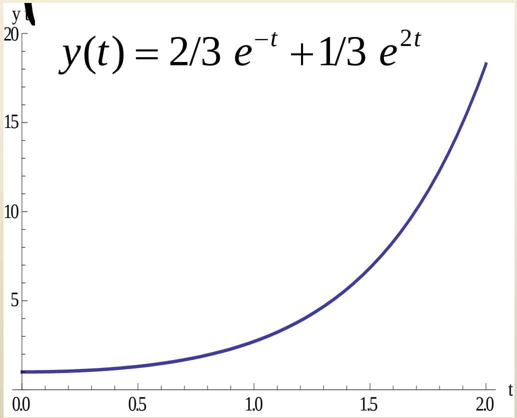
$$y(t) = c_1 e^{-t} + c_2 e^{2t}$$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + 2c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 2/3 \;,\; c_2 = 1/3$$

- Thus

$$y(t) = 2/3\; e^{-t} + 1/3\; e^{2t}$$



- We now solve this problem using Laplace Transforms.

$$y'' - y' - 2y = 0, \quad y(0) = 1, \;\; y'(0) = 0$$

Example 1: Laplace Transform Method (2 of 4)

- Assume that our IVP has a solution ϕ and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{y'' - y' - 2y\} = L\{y''\} - L\{y'\} - 2L\{y\} = L\{0\} = 0$$

and hence

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] - \left[sL\{y\} - y(0)\right] - 2L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 - s - 2\right)Y(s) - (s-1)y(0) - y'(0) = 0$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 - s - 2\right)Y(s) - (s-1) = 0$$

- Thus

$$L\{y\} = Y(s) = \frac{s-1}{(s-2)(s+1)}$$

Example 1: Partial Fractions (3 of 4)

- Using partial fraction decomposition, $Y(s)$ can be rewritten:

$$\frac{s-1}{(s-2)(s+1)} = \frac{a}{(s-2)} + \frac{b}{(s+1)}$$

$$s-1 = a(s+1) + b(s-2)$$

$$s-1 = (a+b)s + (a-2b)$$

$$a+b=1, \;\; a-2b=-1$$

$$a = 1/3 \;,\; b = 2/3$$

- Thus

$$L\{y\} = Y(s) = \frac{1/3}{(s-2)} + \frac{2/3}{(s+1)}$$

Example 1: Solution (4 of 4)

- Recall from Section 6.1:

$$L\left\{e^{at}\right\} = F(s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a$$

- Thus

$$Y(s) = \frac{1/3}{(s-2)} + \frac{2/3}{(s+1)} = 1/3\ L\{e^{2t}\} + 2/3\ L\{e^{-t}\}, \ \ s > 2$$

- Recalling $Y(s) = L\{y\}$, we have

$$L\{y\} = L\{2/3\ e^{-t} + 1/3\ e^{2t}\}$$

and hence

$$y(t) = 2/3\ e^{-t} + 1/3\ e^{2t}$$

General Laplace Transform Method

- Consider the constant coefficient equation

$$ay'' + by' + cy = f(t)$$

- Assume that this equation has a solution $y = \phi(t)$, and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{ay'' + by' + cy\} = aL\{y''\} + bL\{y'\} + cL\{y\} = L\{f(t)\}$$

- If we let $Y(s) = L\{y\}$ and $F(s) = L\{ f \}$, then

$$a\left[s^2 L\{y\} - sy(0) - y'(0)\right] + b\left[sL\{y\} - y(0)\right] + cL\{y\} = F(s)$$

$$\left(as^2 + bs + c\right)Y(s) - \left(as + b\right)y(0) - ay'(0) = F(s)$$

$$Y(s) = \frac{\left(as + b\right)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

Algebraic Problem

- Thus the differential equation has been transformed into the the algebraic equation

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

for which we seek $y = \phi(t)$ such that $L\{\phi(t)\} = Y(s)$.

- Note that we do not need to solve the homogeneous and nonhomogeneous equations separately, nor do we have a separate step for using the initial conditions to determine the values of the coefficients in the general solution.

Characteristic Polynomial

- Using the Laplace transform, our initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \; y'(0) = y_0'$$

becomes

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

- The polynomial in the denominator is the characteristic polynomial associated with the differential equation.
- The partial fraction expansion of $Y(s)$ used to determine ϕ requires us to find the roots of the characteristic equation.
- For higher order equations, this may be difficult, especially if the roots are irrational or complex.

Inverse Problem

- The main difficulty in using the Laplace transform method is determining the function $y = \phi(t)$ such that $L\{\phi(t)\} = Y(s)$.
- This is an inverse problem, in which we try to find ϕ such that $\phi(t) = L^{-1}\{Y(s)\}$.
- There is a general formula for L^{-1}, but it requires knowledge of the theory of functions of a complex variable, and we do not consider it here.
- It can be shown that if f is continuous with $L\{f(t)\} = F(s)$, then f is the **unique** continuous function with $f(t) = L^{-1}\{F(s)\}$.
- Table 6.2.1 in the text lists many of the functions and their transforms that are encountered in this chapter.

Linearity of the Inverse Transform

- Frequently a Laplace transform $F(s)$ can be expressed as

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

- Let

$$f_1(t) = L^{-1}\{F_1(s)\}, \ldots, f_n(t) = L^{-1}\{F_n(s)\}$$

- Then the function

$$f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)$$

has the Laplace transform $F(s)$, since L is linear.

- By the uniqueness result of the previous slide, no other continuous function f has the same transform $F(s)$.
- Thus L^{-1} is a linear operator with

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\{F_1(s)\} + \cdots + L^{-1}\{F_n(s)\}$$

Example 2: Nonhomogeneous Problem (1 of 2)

- Consider the initial value problem

$$y'' + y = \sin 2t, \quad y(0) = 2, \;\; y'(0) = 1$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + L\{y\} = 2/(s^2 + 4)$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 + 1\right)Y(s) - sy(0) - y'(0) = 2/(s^2 + 4)$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 + 1\right)Y(s) - 2s - 1 = 2/(s^2 + 4)$$

- Thus

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}$$

Example 2: Solution (2 of 2)

- Using partial fractions,

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

- Then

$$\begin{aligned} 2s^3 + s^2 + 8s + 6 &= (As+B)(s^2+4) + (Cs+D)(s^2+1) \\ &= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D) \end{aligned}$$

- Solving, we obtain $A = 2$, $B = 5/3$, $C = 0$, and $D = -2/3$. Thus

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

- Hence

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Example 3: Solving a 4^{th} Order IVP (1 of 2)

- Consider the initial value problem

$$y^{(4)} - y = 0, \qquad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)\right] + L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$ and substituting the initial values, we have

$$Y(s) = \frac{s^2}{(s^4 - 1)} = \frac{s^2}{(s^2 - 1)(s^2 + 1)}$$

- Using partial fractions

$$Y(s) = \frac{s^2}{(s^2 - 1)(s^2 + 1)} = \frac{as + b}{(s^2 - 1)} + \frac{cs + d}{(s^2 + 1)}$$

- Thus

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2$$

$$y^{(4)} - y = 0, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0$$

Example 3: Solving a 4th Order IVP (2 of 2)

- In the expression: $(as+b)(s^2+1)+(cs+d)(s^2-1)=s^2$
- Setting s = 1 and s = -1 enables us to solve for *a* and *b*:

$$2(a+b)=1 \text{ and } 2(-a+b)=1 \Rightarrow a=0,\, b=1/2$$

- Setting s = 0, *b* – *d* = 0, so *d* = 1/2
- Equating the coefficients of s^3 in the first expression gives *a* + *c* = 0, so *c* = 0
- Thus $Y(s)=\dfrac{1/2}{(s^2-1)}+\dfrac{1/2}{(s^2+1)}$
- Using Table 6.2.1, the solution is

$$y(t)=\frac{\sinh t+\sin t}{2}$$