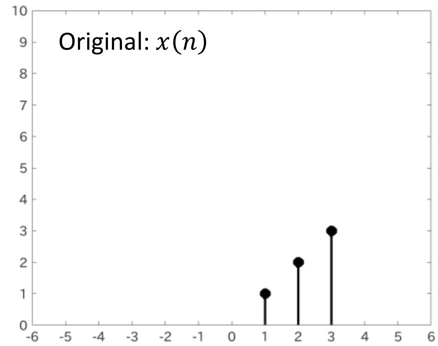
Solutions to Practice Midterm Exam

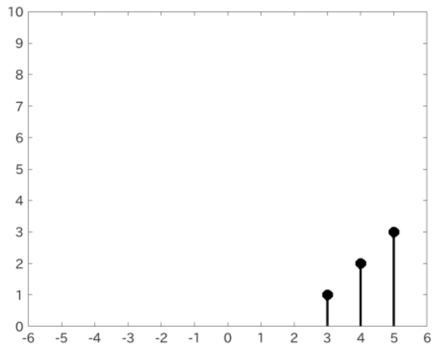
DSP

Spring 2021

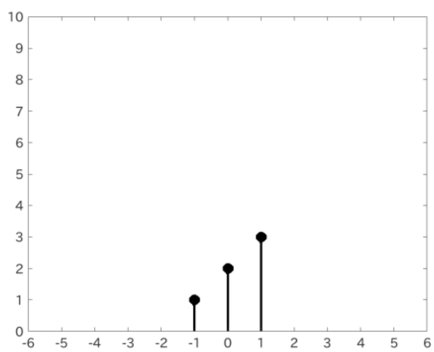
Question 1



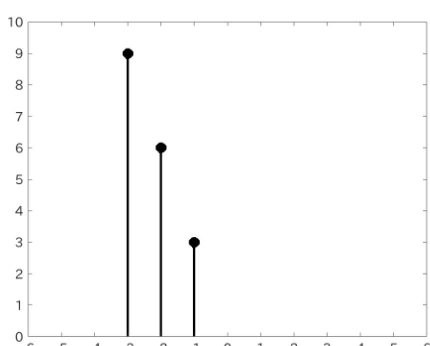
(a)



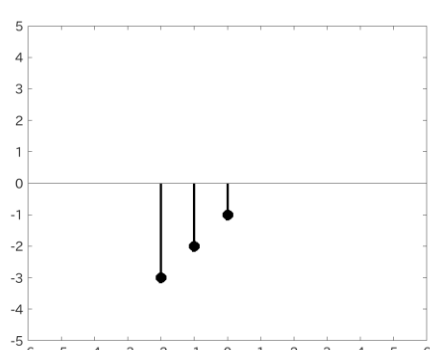
(b)



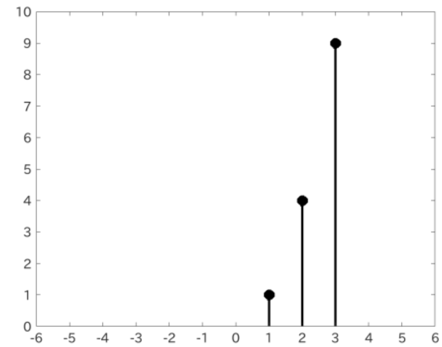
(c)



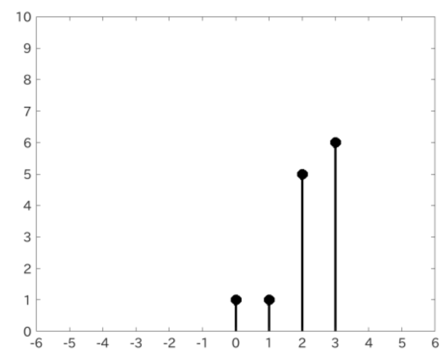
(d)



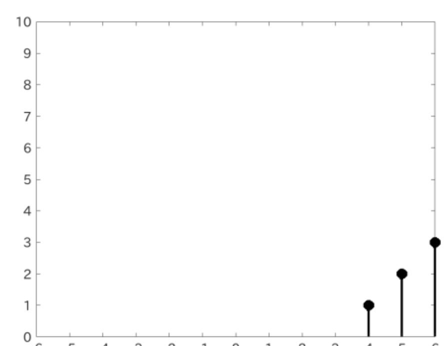
(e)



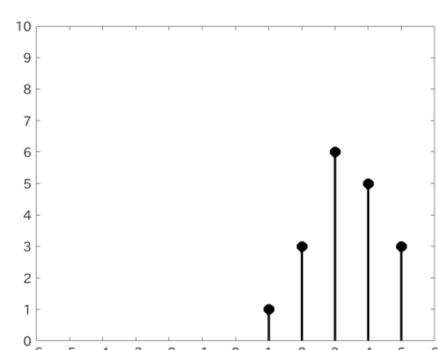
(f)



(g)



(h)



Question 2

Q2

$x(n) = 2\,\delta(n+2) + \delta(n) + 2\,\delta(n-2)$ | use $\delta(n-n_0) \overset{DTFT}{\longleftrightarrow} e^{-j\omega n_0}$

(a) $2x(n) = 4\,\delta(n+2) + 2\,\delta(n) + 4\,\delta(n-2)$

$$2x(n) \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) = 4e^{+j\omega.2} + 2e^{j\omega.0} + 4e^{-j\omega 2}$$

$$= 4(e^{j\omega.2} + e^{-j\omega.2}) + 2.1$$

$$= 4 \cdot 2\cos 2\omega + 2 = 2 + 8\cos(2\omega)$$

(b) $x(n-2) = 2\delta(n) + \delta(n-2) + 2\,\delta(n-4)$

$$\overset{dtFT}{\longleftrightarrow} X(e^{j\omega}) = 2e^{-j\omega.0} + e^{-j\omega 2} + 2e^{-j\omega.4}$$

$$= e^{-j\omega.2}\left[2e^{j\omega 2} + 1 + 2e^{-j\omega 2}\right]$$

$$= e^{-j\omega.2}\left[1 + 2\left(e^{j\omega 2} + e^{-j\omega 2}\right)\right]$$

$$= e^{-j\omega.2}\left[1 + 4\cos 2\omega\right]$$

(c) $3(x(-n)) = 6\,\delta(-n+2) + 3\,\delta(-n) + 6\,\delta(-n-2)$

$$= 6\,\delta(n-2) + 3\,\delta(n) + 6\,\delta(n+2)$$

$$= \frac{3}{2}\left[2x(n)\right]$$

$$\Rightarrow DTFT[3x(-n)] = \frac{3}{2}\, DTFT[2x(n)]$$

$$= \frac{3}{2}\left[2 + 8\cos 2\omega\right] = 3 + 12\cos 2\omega$$

(d) $-x(-n+1) = -2\,\delta(-n+1+2) - \delta(-n+1) - 2\,\delta(-n+1-2)$

$$= -2\,\delta(-n+3) - \delta(-n+1) - 2\,\delta(-n-1)$$

$$= -2\,\delta(n-3) - \delta(n-1) - 2\,\delta(n+1)$$

$$\overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) = -2e^{-j\omega.3} - e^{-j\omega.1} - 2e^{+j\omega}$$

$$= -e^{-j\omega}\left(2e^{-j\omega 2} + 1 + 2e^{+j\omega.2}\right)$$

$$= -e^{-j\omega}\left(1 + 4\cos 2\omega\right)$$

Q2

(e) $x^2(n).u(n)$: $x^2(n) = 4\delta(n+2) + \delta(n) + 4\delta(n-2)$

$\to x^2(n).u(n) = \delta(n) + 4\delta(n-2)$

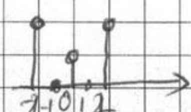
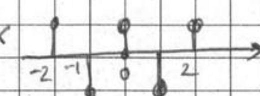
$\leftrightarrow X(e^{jw}) = 1 + 4e^{-jw.2}$

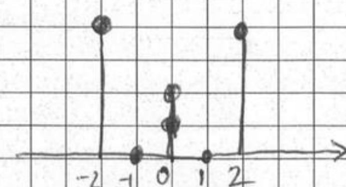
(f) $x(n+1)\,u(n-1) = [2\delta(n+3) + \delta(n+1) + 2\delta(n-1)] \times$

$\times\ [\delta(n-1) + \delta(n-2) + \delta(n-3) \cdots]$

$= 2\delta(n-1)$

$\xleftrightarrow{DTFT} 2e^{-jw}$

(g) $2\,x(n)\cos(\pi n)$ $\quad 2\times$  $\times$ 

$=+$ 

$= 4\delta(n+2) + 2\delta(n) + 4\delta(n-2)$

$= 2(x(n)) \xleftrightarrow{DTFT} 2 + 8\cos(2w)$ from (a)

(h) $x(2n) = 2\delta(2n+2) + \delta(2n) + 2\delta(2n-2)$

$= 2\delta(n+1) + \delta(n) + 2\delta(n-1)$

$\xleftrightarrow{DTFT} X(e^{jw}) = 2e^{jw} + 1 + 2e^{-jw}$

$= 1 + 2(e^{jw} + e^{-jw})$

$= 1 + 4\cos \cancel{2w}$

$1 + 4\cos w$

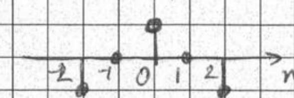
Question 3

Q3

$x(n) = \cos\left(\frac{\pi}{2} n\right) \, \mathrm{rect}_5(n+2)$

(a) $\mathrm{rect}_2(n+2)$: keeps 5 lollipos of $\cos\left(\frac{\pi}{2} n\right)$

→ sketch:



⇒ Length 5 Energy: $\sum_{n=-\infty}^{\infty} x^2(n) = (-1)^2 + 0^2 + 1^2 + 0^2 + (-1)^2 = 3$

(b) DTFT formula

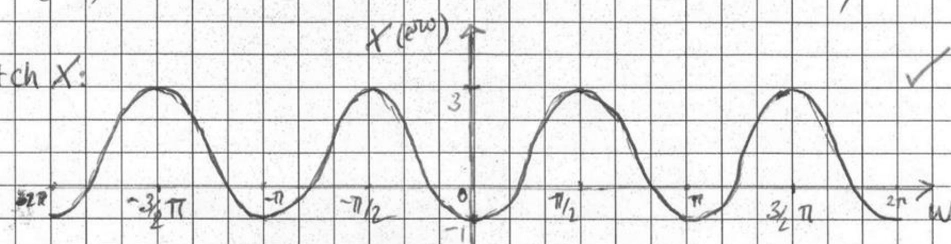
$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$

$x(n)$ non-zero at $n = -2; 0; 2$ → $X(e^{j\omega}) = x(-2) e^{2j\omega} + x(0) e^{0} + x(2) e^{-j\omega 2}$

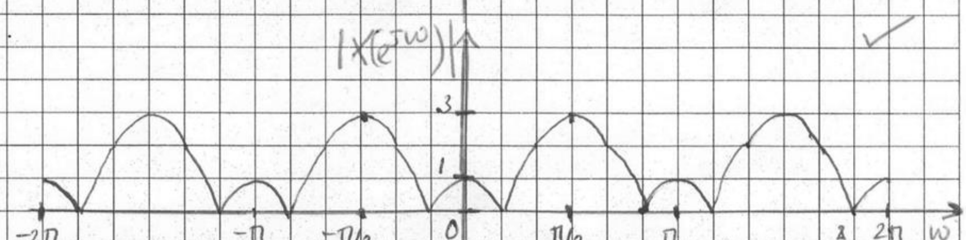
→ $X(e^{j\omega}) = -e^{j2\omega} + 1 - e^{-j\omega 2} = 1 - 2\cos(2\omega)$

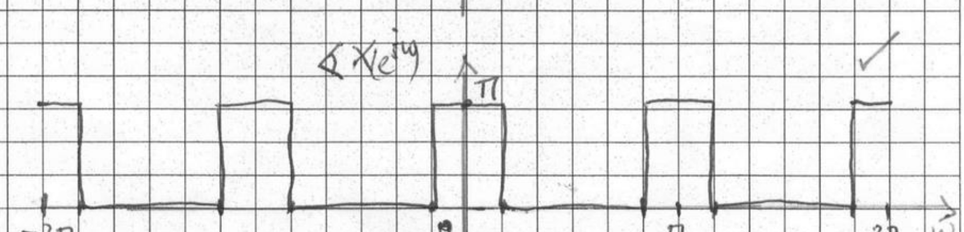
(c)

Sketch X:



(d)





$X(e^{j\omega})$ is purely Real → $|X(e^{j\omega})| = \begin{cases} 1 - \cos 2\omega & \text{if } 1 - \cos 2\omega \geq 0 \\ \cos 2\omega - 1 & \text{if } 1 - \cos 2\omega < 0 \end{cases}$

⇒ $\measuredangle X(e^{j\omega}) = \begin{cases} 0 & \text{if } 1 - \cos 2\omega \geq 0 \\ \pi & \text{if } 1 - \cos 2\omega < 0 \end{cases}$

Q3

(e) Use table and frequency-shift property:

$$\cos\left(\frac{\pi}{2}n\right) = \frac{1}{2}e^{-j\pi/2\,n} + \frac{1}{2}e^{j\pi/2\,n}$$

$$X(n) = \frac{1}{2}e^{-j\pi/2\,n}\cdot \text{rect}_5(n+2) + \frac{1}{2}e^{j\pi/2\,n}\cdot \text{rect}_5(n+2)$$

$$\text{rect}_5(n+2) \xleftrightarrow{DTFT} \frac{\sin\frac{5}{2}\omega}{\sin\frac{1}{2}\omega}$$

$$\Rightarrow X(e^{j\omega}) = \frac{1}{2}\frac{\sin\frac{5}{2}(\omega-\frac{\pi}{2})}{\sin\frac{1}{2}(\omega-\frac{\pi}{2})} + \frac{1}{2}\frac{\sin\frac{5}{2}(\omega+\frac{\pi}{2})}{\sin\frac{1}{2}(\omega+\frac{\pi}{2})}$$

$$= \frac{1}{2}\cdot\frac{\sin(\frac{5\omega}{2}-\frac{5\pi}{4})}{\sin(\frac{\omega}{2}-\frac{\pi}{4})} + \frac{1}{2}\cdot\frac{\sin(\frac{5\omega}{2}+\frac{5\pi}{4})}{\sin(\frac{\omega}{2}+\frac{\pi}{4})}$$ ✓

(f) Multiplication-in-time : circular convolution.

$$x_1(n) = \cos\frac{\pi}{2}n \xleftrightarrow{DTFT} X_1(e^{j\omega}) = \pi\left[\delta(\omega+\tfrac{\pi}{2} + k2\pi) + \delta(\omega-\tfrac{\pi}{2} + k2\pi)\right]$$

$$x_2(n) = \text{rect}_5(n+2) \longleftrightarrow \frac{\sin\frac{5}{2}\omega}{\sin\frac{\omega}{2}} = X_2(e^{j\omega})$$

circular convolution.

$$x_1(n)\cdot x_2(n) \xleftrightarrow{DTFT} X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$$

$$= \frac{1}{2\pi}\; X_1(e^{j\omega}) * X_2(e^{j\omega})$$

one period.

$$= \frac{1}{2\pi}\cdot\left[\pi\left(\delta(\omega+\tfrac{\pi}{2}) + \delta(\omega-\tfrac{\pi}{2})\right)\right] * \frac{\sin\frac{5}{2}\omega}{\sin\frac{\omega}{2}}$$

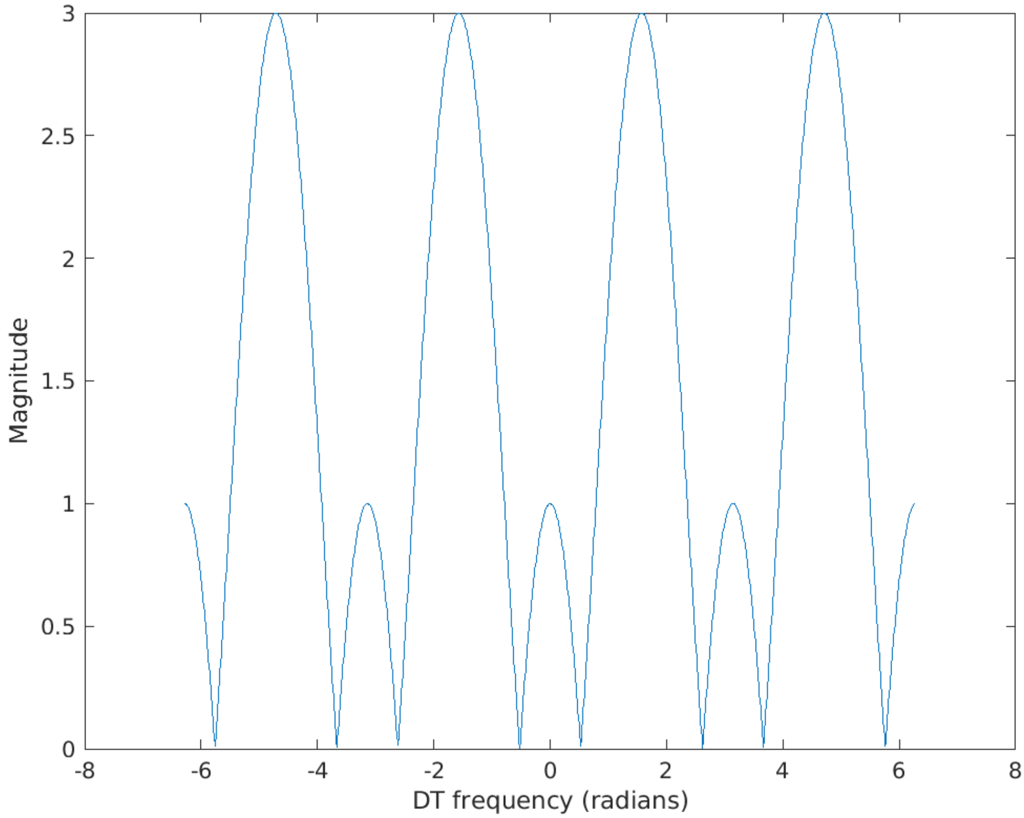
$$= \frac{1}{2}\left[\frac{\sin\frac{5}{2}(\omega+\frac{\pi}{2})}{\sin\frac{1}{2}(\omega+\frac{\pi}{2})} + \frac{\sin\frac{5}{2}(\omega-\frac{\pi}{2})}{\sin\frac{1}{2}(\omega-\pi/2)}\right]$$ ✓

(g) Verify by Matlab

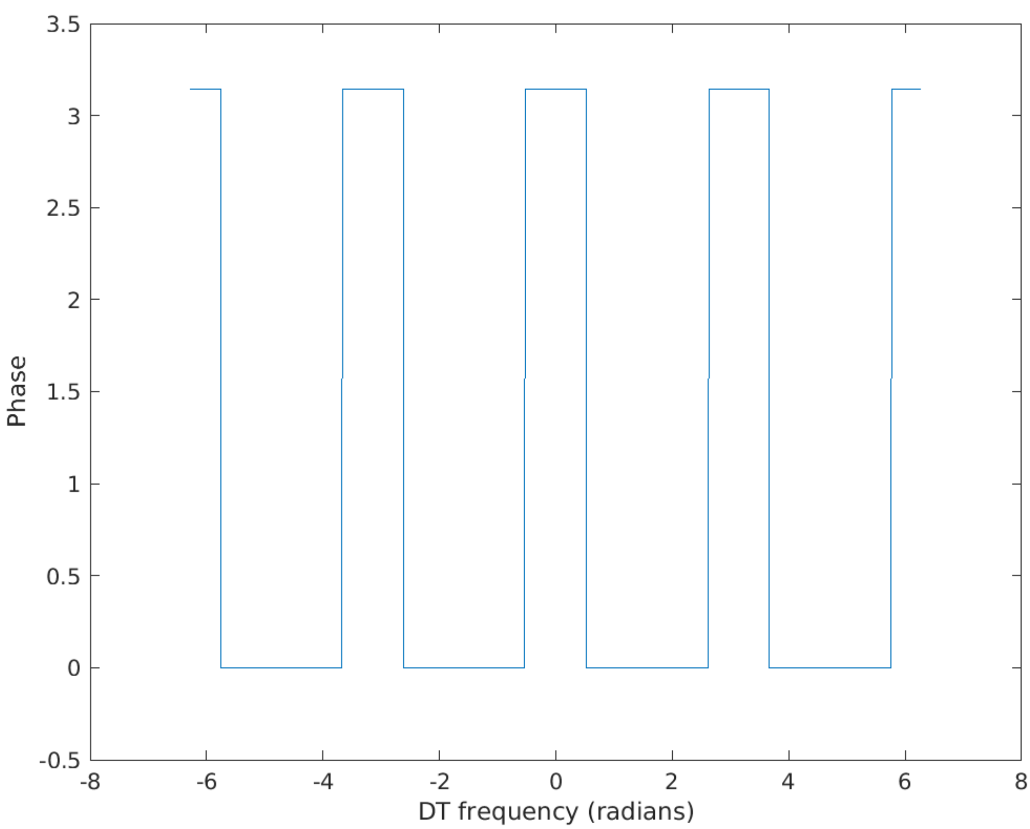
(plots not included here—try this on your own in Matlab)

Matlab code for Question 3(g)

```
n = -2:2;
c = cos(pi/2*n);
r = [1 1 1 1 1];
x = c.*r;

% Note that you can directly do:
%   x = cos(pi/2*n)
% because the cosine is already time-limited by the limited range
% of n, and because the rectangle is all ones.

w = -2*pi:0.01:2*pi;
X = freqz(x, 1, w);
% Note: The X that freqz() computes assumes that the input signal
% starts at n=0.  To account for the fact that our n-range starts
% at -2, we just need to use the DTFT's time-shift property
% (i.e., just multiply the original DTFT by e^(-j*w*2)
X = exp(-j*w*n(1)).*X;

figure(1);
plot(w, abs(X));
xlabel('DT frequency (radians)');
ylabel('Magnitude');

figure(2);
% Although we could simply use the following line to plot the
% phase spectrum...
%   plot(w, angle(X));
% it makes a graph that shows phase shifts of both +pi and -pi,
% which is a bit hard to see. Because -pi is the same phase shift
% as +pi (remember, both are 180 deg), let's make a cleaner phase
% plot that changes (anything close to) -pi into +pi:
phs = angle(X);
phs(abs(phs-(-pi))<1e-3) = pi;
plot(w, phs);
xlabel('DT frequency (radians)');
ylabel('Phase');
```



Magnitude
3
2.5
2
1.5
1
0.5
0
-8
-6
-4
-2
0
2
4
6
8
DT frequency (radians)



Phase
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-8
-6
-4
-2
0
2
4
6
8
DT frequency (radians)