Digital Signal Processing

Spring Semester 2022

Frequency-Based Analysis, Part 3

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Last time's learning objectives

- Compute the DTFTs of basic signals on paper
  - Shifted impulse
  - Combination of (scaled, shifted) impulses
  - Rectangles
- Write a signal's spectrum in terms of magnitude and phase
  - How to interpret the magnitude spectrum
  - How changes to a signal affects its spectrum

Today's learning objectives

From **today's lecture**, you should **be able to**…

- Simplify DTFT computation by…
  - Using a table of transform pairs
  - Using DTFT properties

Discrete-time Fourier transform (DTFT)

Transforms a signal from the **time domain** into the **frequency domain** (and vice-versa)

time-domain $\boxed{x(n) \leftrightarrow X(e^{j\omega})}$ frequency domain

Forward DTFT: $$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$$

Inverse DTFT: $$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$$

Ex: $x(n) = \delta(n)$

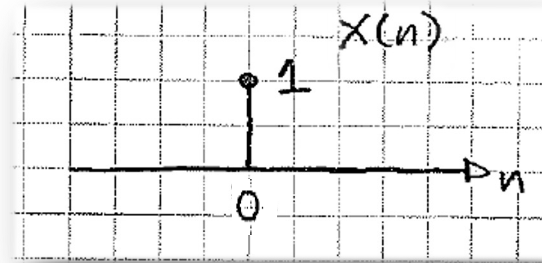
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n}$$

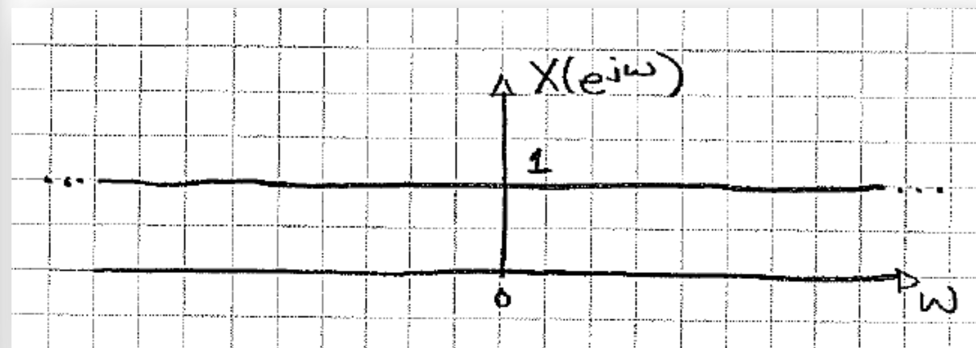
$$= \overset{1}{\cancel{\delta(0)}}\, e^{-j\omega 0}$$

$$= 1$$

Time domain $x(n)$



Frequency domain $X\left(e^{j\omega}\right)$



Ex: $x(n) = \delta(n - n_0)$

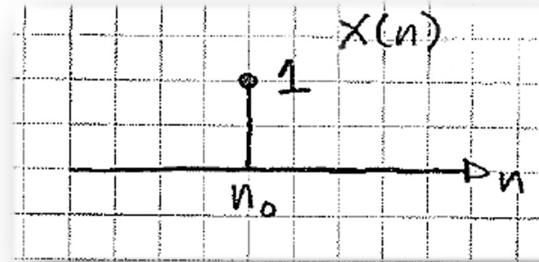
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-n_0)\, e^{-j\omega n}$$

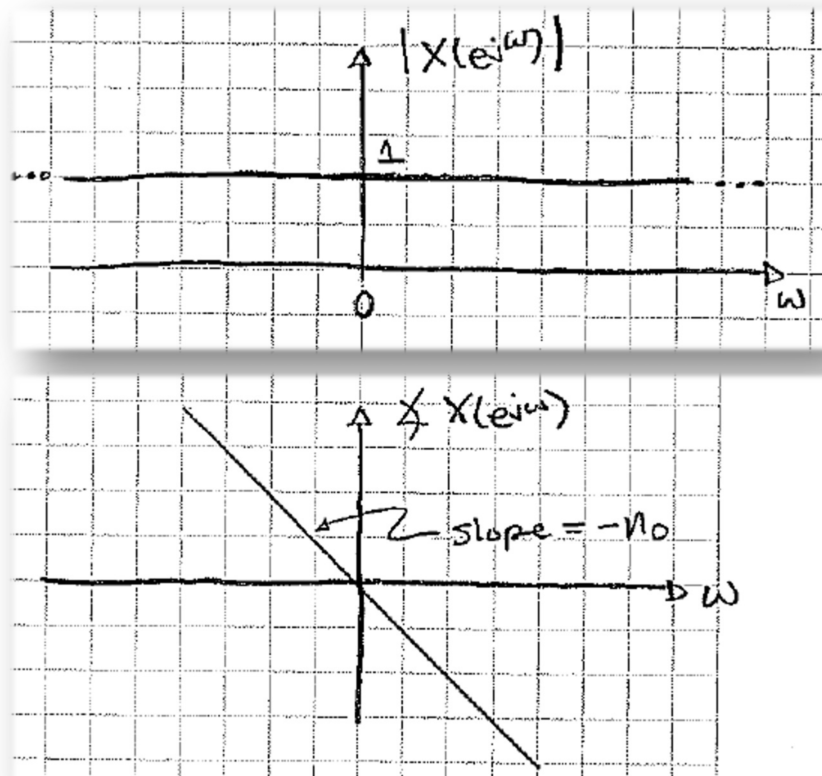
$$= \underbrace{\delta(0)}_{1}\, e^{-j\omega n_0}$$

$$= e^{-j\omega n_0}$$

Time domain $x(n)$



Frequency domain $X(e^{j\omega})$



Ex: $x(n) = \delta(n) + \delta(n-1)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta(n) + \delta(n-1)\right) e^{-j\omega n}$$

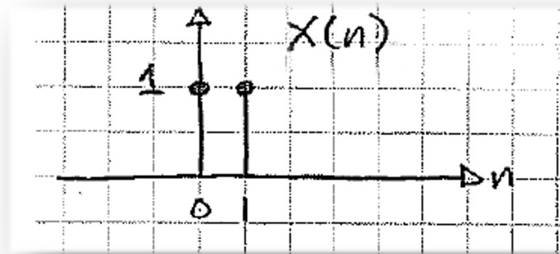
$$= \delta(0)\, e^{-j\omega 0} + \delta(0)\, e^{-j\omega 1}$$

$$= 1 + e^{-j\omega}$$

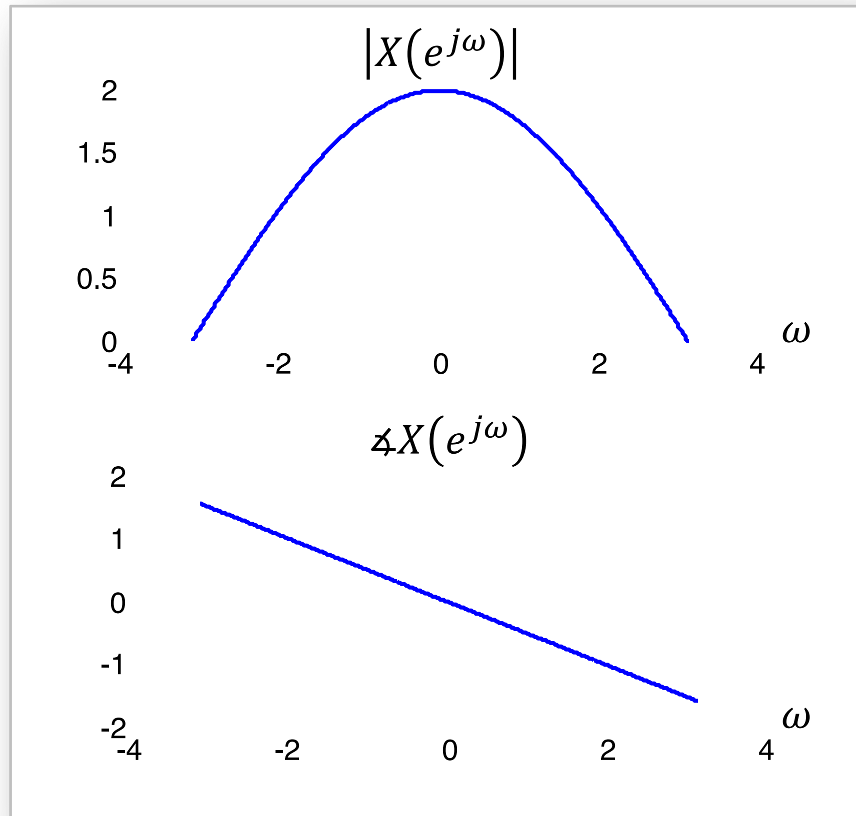
$$= e^{-j0.5\omega}\left(e^{j0.5\omega} + e^{-j0.5\omega}\right)$$

$$= e^{-j0.5\omega}\left(2\cos(0.5\omega)\right)$$

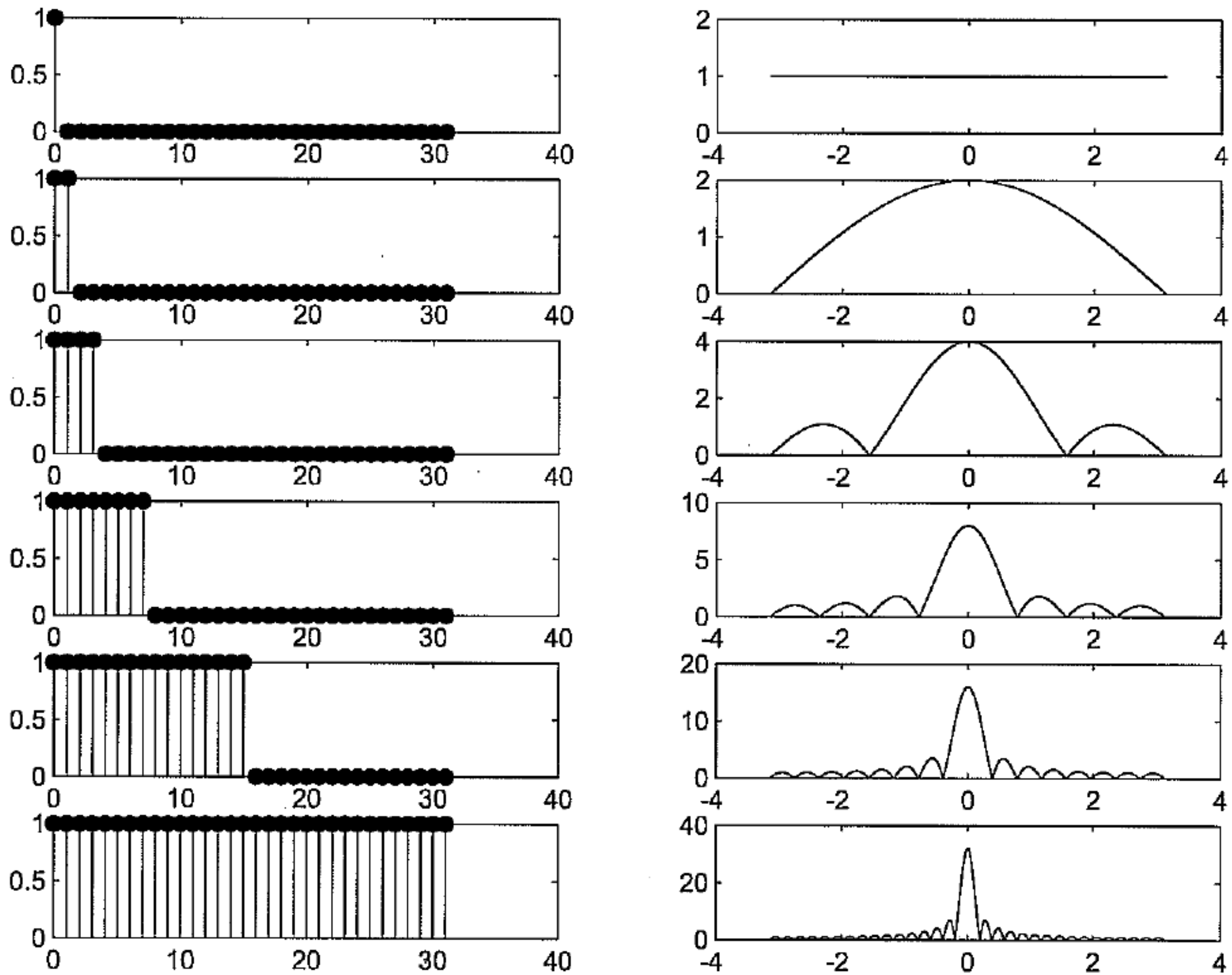
Time domain $x(n)$



Frequency domain $X\left(e^{j\omega}\right)$



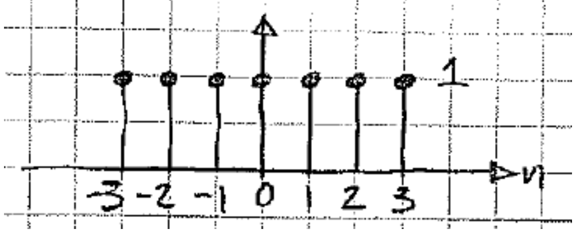
Rectangle in time $\xleftrightarrow{\text{DTFT}}$ Periodic sinc in frequency



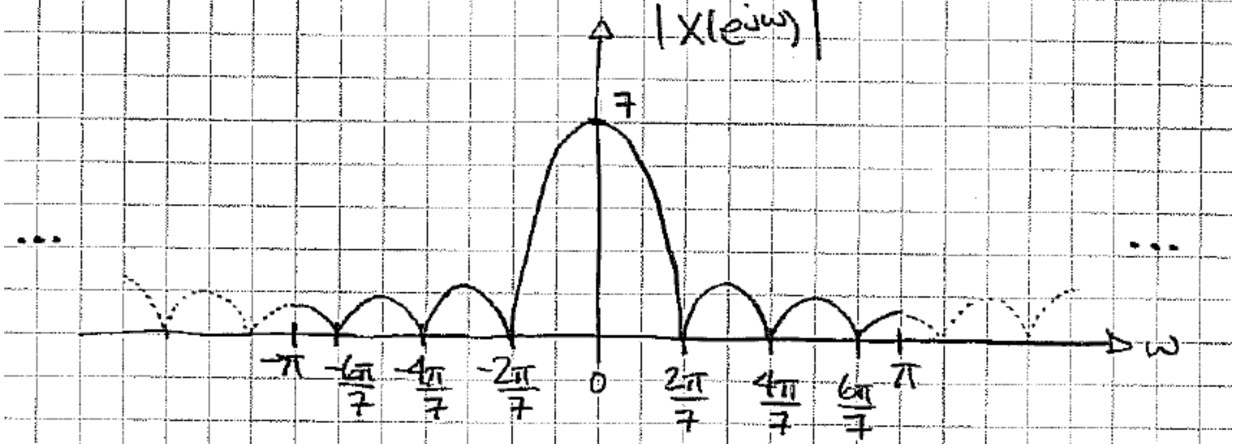
(Note: Plots are of the magnitude spectrum)

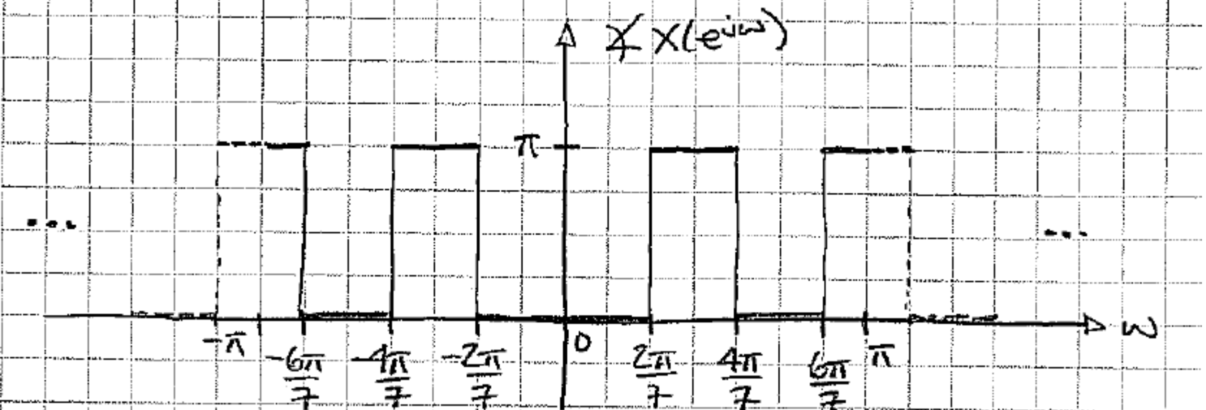
Rectangle $\overset{DTFT}{\longleftrightarrow}$ Periodic Sinc

$$x(n) = \mathrm{rect}_7(n+3)$$



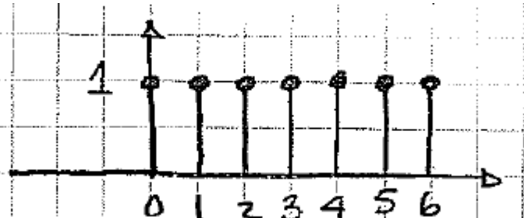
$$X(e^{j\omega}) = \frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$





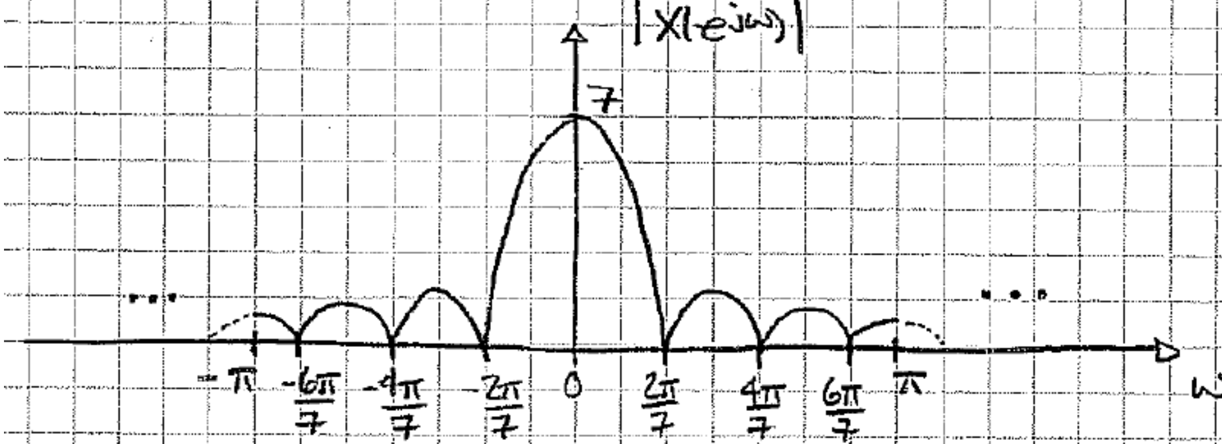
Rectangle $\overset{DTFT}{\longleftrightarrow}$ Periodic Sinc

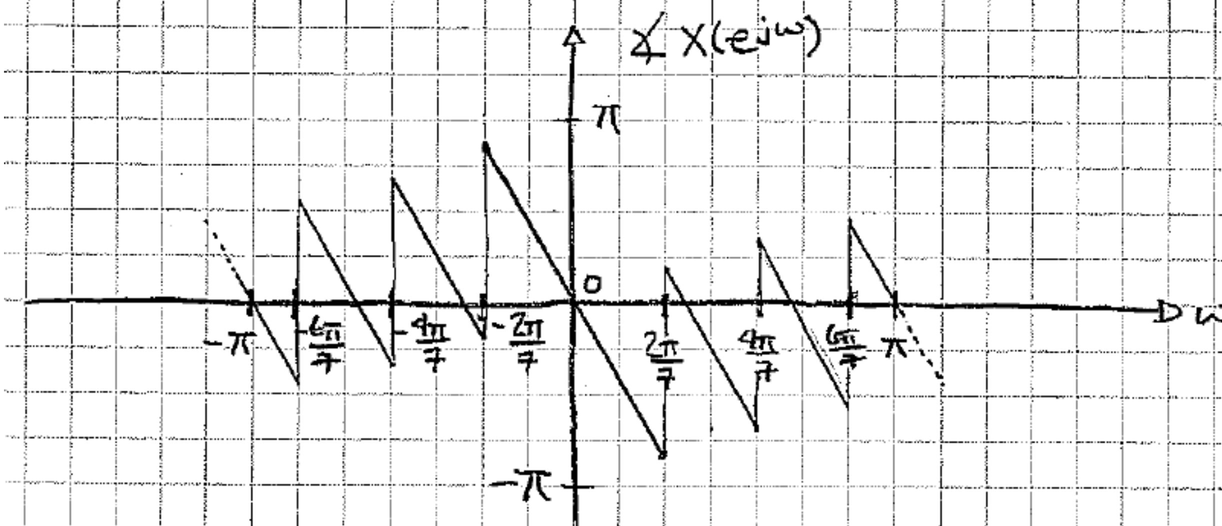
$x(n) = \text{rect}_7(n)$



$X(e^{j\omega})$

$= e^{-j\omega 3} \dfrac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$





Last week's in-class activity

Compute the DTFT of the following signals:

1. $x(n) = \delta(n+1) - \delta(n-1)$
2. $y(n) = \delta(n+1) + 2\delta(n) - \delta(n-1)$

Question: We can see that $y(n) = x(n) + 2\delta(n)$. Please write $Y\left(e^{j\omega}\right)$ in terms of $X\left(e^{j\omega}\right)$?

Ex: $y(n) = \delta(n+1) + 2\delta(n) - \delta(n-1) = x(n) + 2\delta(n)$

$Y(e^{j\omega}) = X(e^{j\omega}) + 2$ By using the linearity property of the DTFT

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\omega n} = \sum_{n=-1}^{1} [\delta(n+1) + 2\delta(n) - \delta(n-1)]e^{-j\omega n}$$

$$= [\delta(-1+1) + 2\delta(-1) - \delta(-1-1)]e^{-j\omega(-1)} \qquad n=-1$$
$$+ [\delta(0+1) + 2\delta(0) - \delta(0-1)]e^{-j\omega 0} \qquad n=0$$
$$+ [\delta(1+1) + 2\delta(1) - \delta(1-1)]e^{-j\omega 1} \qquad n=1$$

$$= [\delta(0) + 2\delta(-1) - \delta(-2)]e^{j\omega 1}$$
$$+ [\delta(1) + 2\delta(0) - \delta(-1)]e^{-j\omega 0}$$
$$+ [\delta(2) + 2\delta(1) - \delta(0)]e^{-j\omega 1}$$

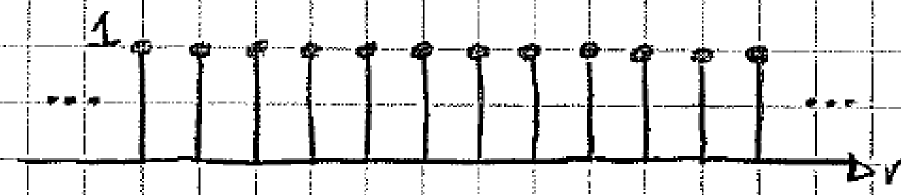
$$= [1 + 0 - 0]e^{j\omega 1}$$
$$+ [0 + 2 - 0]e^{-j\omega 0}$$
$$+ [0 + 0 - 1]e^{-j\omega 1}$$

$$= e^{j\omega 1} + 2 - e^{-j\omega 1} \quad = 2j\sin(\omega) + 2$$

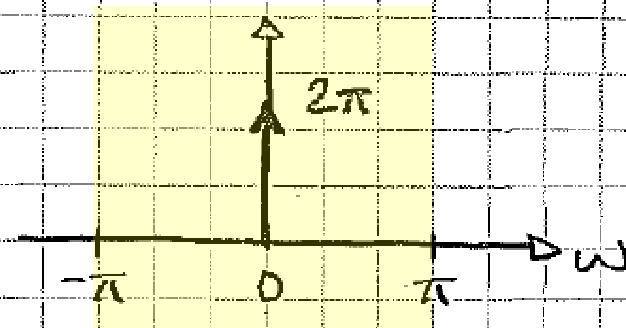
Today

1. Other important DTFT examples
2. Table of transform pairs
3. DTFT properties

Ex: $x(n) = 1$



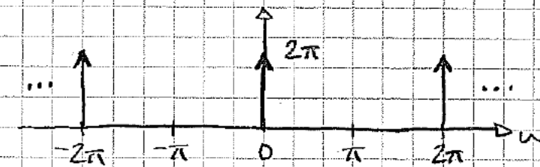
$X(e^{j\omega}) =$



Ex: $x(n) = 1$

Claim:

$X(e^{j\omega}) =$



$$= \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - k2\pi)$$

Check: $$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - k2\pi)\; e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega)\, e^{j\omega n}\, d\omega$$

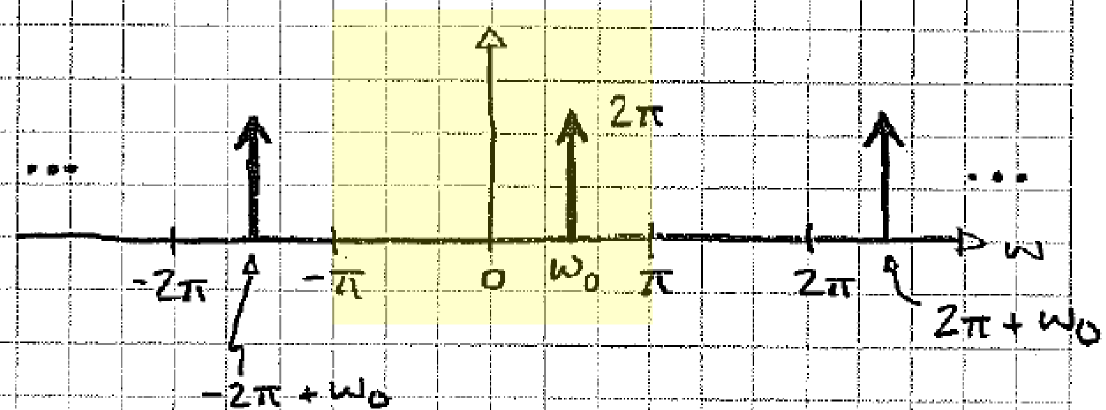
$$= e^{j0n} \int_{-\pi}^{\pi} \delta(\omega)\, d\omega$$

$$= 1 \quad \checkmark$$

Ex: $x(n) = e^{j\omega_0 n}, \quad -\pi < \omega_0 \leq \pi$

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi)$$

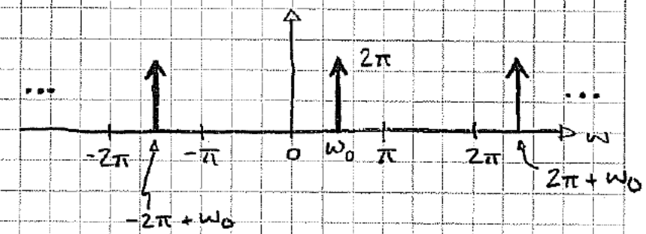
$= \ldots$



Ex: $x(n) = e^{j\omega_0 n}, \quad -\pi < \omega_0 \leq \pi$

Claim: $X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi)$

$=$



$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

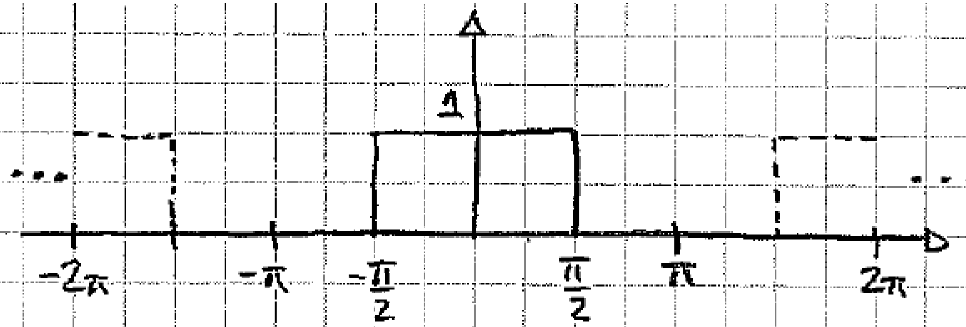
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi)\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi)\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega$$

$$= e^{j\omega_0 n}\int_{-\pi}^{\pi} \delta(\omega - \omega_0)\, d\omega$$

$$= e^{j\omega_0 n} \quad \checkmark$$

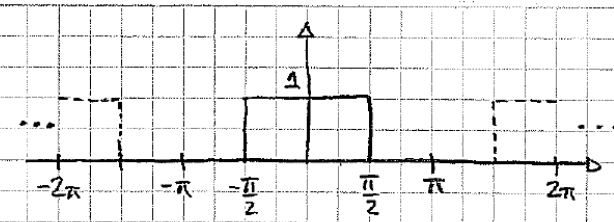
Ex: $X(e^{jw}) =$ 

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{jwn}\, dw$$

$$\vdots$$

$$= \frac{\frac{1}{2}\sin\left(\frac{\pi}{2}n\right)}{\frac{\pi}{2}n} = \frac{1}{2}\,\mathrm{sinc}\left(\frac{\pi}{2}n\right)$$

Ex: $X(e^{j\omega}) = $ 

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jn}\cdot e^{j\omega n}\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jn}\left(e^{-j\frac{\pi}{2}n} - e^{j\frac{\pi}{2}n}\right)$$

$$= \frac{1}{2j\pi n}\cdot\left(2j\sin\left(\frac{\pi}{2}n\right)\right)$$

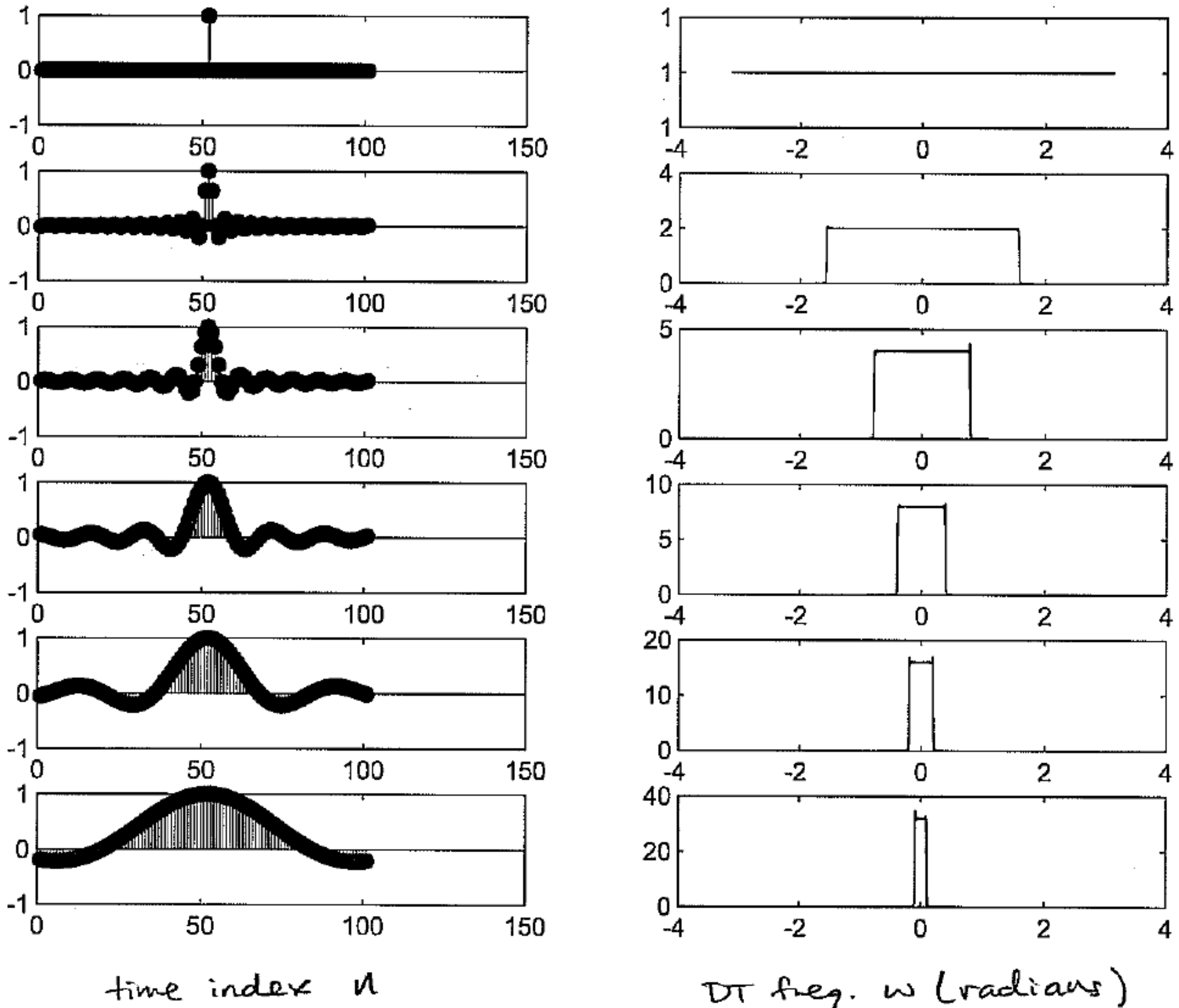
$$= \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}$$

$$= \frac{\frac{1}{2}\sin\left(\frac{\pi}{2}n\right)}{\frac{\pi}{2}n}$$

$$= \frac{1}{2}\,\mathrm{sinc}\left(\frac{\pi}{2}n\right)$$

Note: $\mathrm{sinc}(\alpha n)\Big|_{n=0} = 1$

Sinc in time $\xleftrightarrow{\text{DTFT}}$ Periodic Rectangle in frequency



Today

1. Other important DTFT examples

2. Table of transform pairs

3. DTFT properties

Continuous time

Constant		Impulse
$1 \ \forall t$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$2\pi\delta(\Omega)$
Unit impulse		Constant
$\delta(t)$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$1 \ \forall \Omega$
Shifted impulse		Complex exponential
$\delta(t-t_0)$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$e^{-jt_0\Omega}$
Complex exponential		Shifted impulse
$e^{j\Omega_0 t}$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$2\pi\delta(\Omega-\Omega_0)$
Cosine / sine		Pair of shifted impulses
$\cos(\Omega_0 t)$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$\pi[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)]$
$\sin(\Omega_0 t)$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$\frac{\pi}{j}[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)]$
Rectangle		Sinc
$\mathrm{rect}\left(\frac{t}{2\tau}\right)$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$2\tau\,\mathrm{sinc}(\tau\Omega)$
Sinc		Rectangle
$\mathrm{sinc}(\Omega_c t)$	$\overset{\text{CTFT}}{\longleftrightarrow}$	$\frac{\pi}{\Omega_c}\mathrm{rect}\left(\frac{\Omega}{2\Omega_c}\right)$

Discrete time

Constant		Impulse (train)
$1 \ \forall n$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-k2\pi)$
Unit impulse		Constant
$\delta(n)$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$1 \ \forall \omega$
Shifted impulse		Complex exponential
$\delta(n-n_0)$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$e^{-jn_0\omega}$
Complex exponential		Shifted impulse (train)
$e^{j\omega_0 n}$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0-k2\pi)$
Cosine / sine		Pair of shifted impulses (train)
$\cos(\omega_0 n)$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$\pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0-k2\pi) + \delta(\omega+\omega_0-k2\pi)$
$\sin(\omega_0 n)$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$\frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0-k2\pi) - \delta(\omega+\omega_0-k2\pi)$
Rectangle		Sinc (periodic)
$\mathrm{rect}_N(n)$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$e^{-j\omega\left(\frac{N-1}{2}\right)}\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$
Sinc		Rectangle (periodic)
$\frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c n)$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$\sum_{k=-\infty}^{\infty}\mathrm{rect}\left(\frac{\omega-k2\pi}{2\omega_c}\right)$

Discrete time

<u>Constant</u>

$$1 \;\forall n \quad \overset{\text{DTFT}}{\longleftrightarrow}$$

<u>Impulse (train)</u>

$$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$$

<u>Unit impulse</u>

$$\delta(n) \quad \overset{\text{DTFT}}{\longleftrightarrow}$$

<u>Constant</u>

$$1 \;\forall \omega$$

<u>Shifted impulse</u>

$$\delta(n-n_0) \quad \overset{\text{DTFT}}{\longleftrightarrow}$$

<u>Complex exponential</u>

$$e^{-jn_0\omega}$$

<u>Complex exponential</u>

$$e^{j\omega_0 n} \quad \overset{\text{DTFT}}{\longleftrightarrow}$$

<u>Shifted impulse (train)</u>

$$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi)$$

Discrete time

<u>Constant</u> <u>Impulse (train)</u>

$$1 \ \forall n \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$$

<u>Unit impulse</u> <u>Constant</u>

$$\delta(n) \xleftrightarrow{\text{DTFT}} 1 \ \forall \omega$$

<u>Shifted impulse</u> <u>Complex exponential</u>

$$\delta(n-n_0) \xleftrightarrow{\text{DTFT}} e^{-jn_0\omega}$$

<u>Complex exponential</u> <u>Shifted impulse (train)</u>

$$e^{j\omega_0 n} \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi)$$

Cosine / sine — **Pair of shifted impulses (train)**

$$\cos(\omega_0 n) \overset{DTFT}{\longleftrightarrow} \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi) + \delta(\omega + \omega_0 - k2\pi)$$

$$\sin(\omega_0 n) \overset{DTFT}{\longleftrightarrow} \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - k2\pi) - \delta(\omega + \omega_0 - k2\pi)$$

Rectangle — **Sinc (periodic)**

$$\text{rect}_N(n) \overset{DTFT}{\longleftrightarrow} e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

Sinc — **Rectangle (periodic)**

$$\frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \overset{DTFT}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega - k2\pi}{2\omega_c}\right)$$

Cosine / sine — Pair of shifted impulses (train)

$$\cos(\omega_0 n) \xleftrightarrow{\text{DTFT}} \pi \sum_{k=-\infty}^{\infty} \delta(\omega-\omega_0-k2\pi) + \delta(\omega+\omega_0-k2\pi)$$

$$\sin(\omega_0 n) \xleftrightarrow{\text{DTFT}} \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \delta(\omega-\omega_0-k2\pi) - \delta(\omega+\omega_0-k2\pi)$$

Rectangle — Sinc (periodic)

$$\mathrm{rect}_N(n) \xleftrightarrow{\text{DTFT}} e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

Sinc — Rectangle (periodic)

$$\frac{\omega_c}{\pi}\mathrm{sinc}(\omega_c n) \xleftrightarrow{\text{DTFT}} \sum_{k=-\infty}^{\infty} \mathrm{rect}\left(\frac{\omega-k2\pi}{2\omega_c}\right)$$

Today

1. Other important DTFT examples

2. Table of transform pairs

3. DTFT properties

Properties of the DTFT

① **Linearity:** $\text{DTFT}\left[\alpha_1 x_1(n) + \alpha_2 x_2(n)\right]$

$$= \alpha_1 \, \text{DTFT}\left[x_1(n)\right] + \alpha_2 \, \text{DTFT}\left[x_2(n)\right]$$

Example of using the linearity property:

$$x(n) = 2\delta(n-1) + 5\text{rect}_3(n)$$

$$\big\Updownarrow \text{DTFT} \qquad\qquad \big\Updownarrow \text{DTFT}$$

$$X\left(e^{j\omega}\right) = 2e^{-j\omega 1} + 5e^{-j\omega 1}\frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

Shifted impulse | Complex exponential

$$\delta(n-n_0) \xleftrightarrow{\text{DTFT}} e^{-jn_0\omega}$$

Rectangle | Sinc (periodic)

$$\text{rect}_N(n) \xleftrightarrow{\text{DTFT}} e^{-j\omega\left(\frac{N-1}{2}\right)}\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

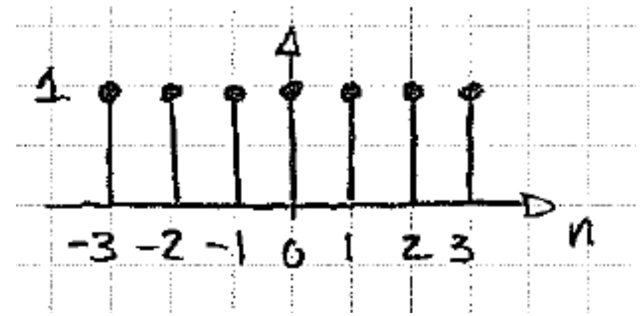
Properties of the DTFT

(2) **Time Shift**: If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

Then $x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$

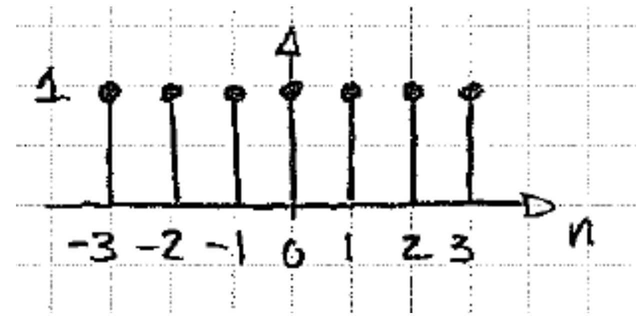
Example of using the time-shift property:

$$x(n) = \mathrm{rect}_7(n+2)$$



Example of using the time-shift property:

$$x(n) = \mathrm{rect}_7(n+2)$$



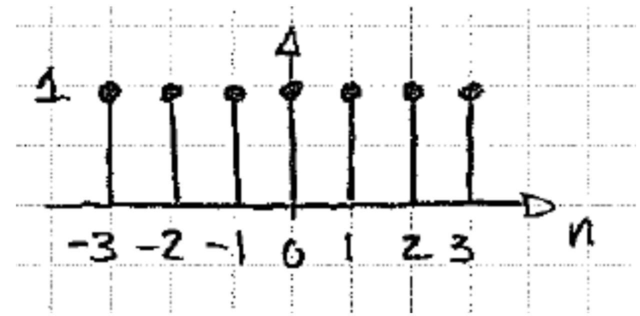
From table, we know:

$$\mathrm{rect}_7(n) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega 2} \frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

Rectangle		Sinc (periodic)
$\mathrm{rect}_N(n)$	$\overset{\text{DTFT}}{\longleftrightarrow}$	$e^{-j\omega\left(\frac{N-1}{2}\right)} \dfrac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$

Example of using the time-shift property:

$$x(n) = \mathrm{rect}_7(n+2)$$



From table, we know:

$$\mathrm{rect}_7(n) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega 2}\frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

(2) Time Shift: If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j\omega})$

Then $x(n-n_0) \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$

$$\mathrm{rect}_7(n+2) \overset{\text{DTFT}}{\longleftrightarrow} e^{+j\omega 2}e^{-j\omega 2}\frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} = \frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

Properties of the DTFT

(3) Frequency shift: If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

Then $e^{j\omega_0 n} x(n) \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$

Example of using the frequency-shift property:

$$x(n) = e^{j\pi n}\text{rect}_7(n+2)$$

We know:

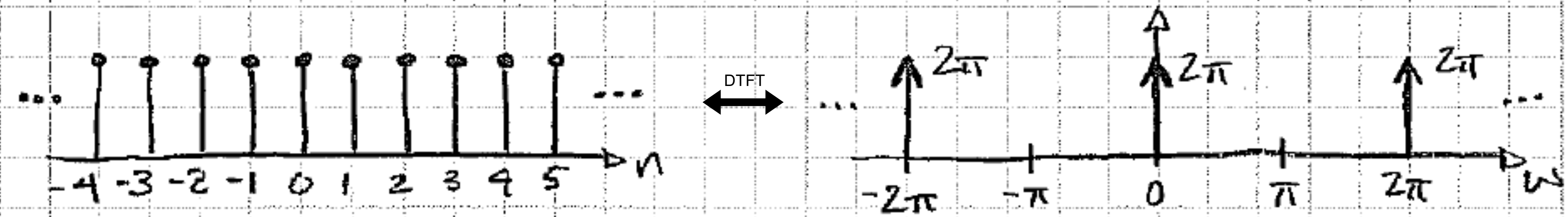
$$\text{rect}_7(n+2) \xleftrightarrow{\text{DTFT}} \frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

(3) **Frequency shift:** If $x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$

Then $e^{j\omega_0 n}\, x(n) \xleftrightarrow{\text{DTFT}} X(e^{j(\omega-\omega_0)})$

$$e^{j\pi n}\text{rect}_7(n+2) \xleftrightarrow{\text{DTFT}} \frac{\sin\left(\frac{7}{2}(\omega-\pi)\right)}{\sin\left(\frac{1}{2}(\omega-\pi)\right)}$$

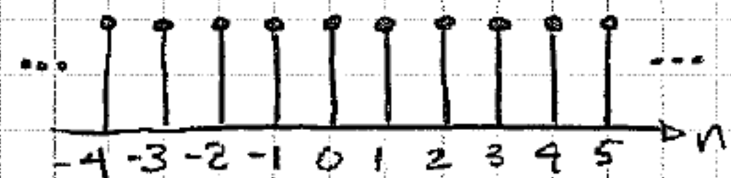
Ex: We know $x(n) = 1 \overset{DTFT}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$

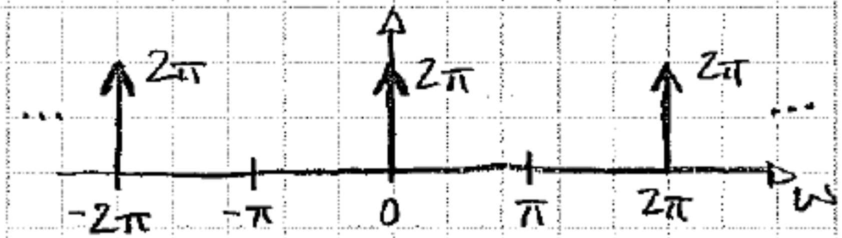


By the freq. shift property

$$e^{j\pi n}\, x(n) \overset{DTFT}{\longleftrightarrow} X(e^{j(\omega-\pi)})$$

Ex: We know $x(n) = 1 \overset{\text{DTFT}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$



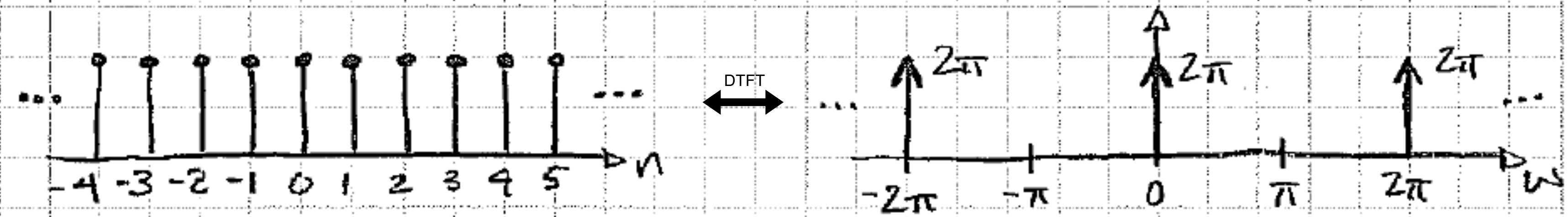


By the freq. shift property

$$e^{j\pi n}\, x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j(\omega-\pi)})$$

$$e^{j\pi n}\, 1 \overset{\text{DTFT}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi - k2\pi)$$

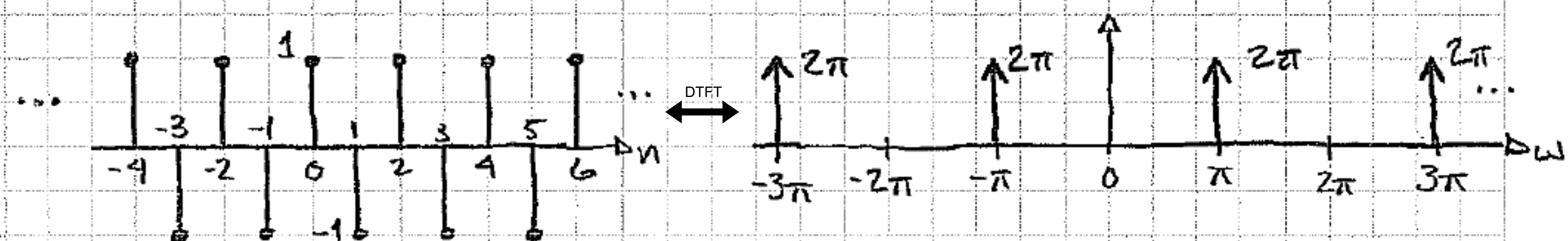
Ex: We know $x(n) = 1 \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$



By the freq. shift property

$$e^{j\pi n}\, x(n) \xleftrightarrow{\text{DTFT}} X(e^{j(\omega-\pi)})$$

$$e^{j\pi n}\, 1 \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi - k2\pi)$$



Properties of the DTFT

4. <u>Multiplication in time:</u> If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

Then $x_1(n)\, x_2(n) \xleftrightarrow{DTFT} \underbrace{\frac{1}{2\pi} \int_{\text{one period}} X_1(e^{ju})\, X_2(e^{j(\omega-u)})\, du}_{\text{periodic convolution}}$

$\frac{1}{2\pi} X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$

Properties of the DTFT

5 <u>Convolution in time:</u> If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

Then $x_1(n) * x_2(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})X_2(e^{j\omega})$

Properties of the DTFT

6 Parseval's theorem: If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(e^{j\omega})$

$x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(e^{j\omega})$

Then $$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega$$

If $x_1(n) = x_2(n) = x(n)$

$$\underbrace{\sum_{n=-\infty}^{\infty} |x_1(n)|^2}_{\text{energy in time}} = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega}_{\text{energy (per period) in frequency}}$$

Ex:

$$x(n) = \cos(\pi n)\ \mathrm{rect}_3(n+1)$$