Digital Signal Processing

Spring Semester 2022

Frequency-Based Analysis, Part 1

Damon M. Chandler <chandler@fc.ritsumei.ac.jp>
Visual Information Engineering Laboratory
Ritsumeikan University

Last time’s learning objectives

- Describe filtering (what it is and why it’s used)
- Describe the relationship between filtering and convolution
- Perform basic convolution between two signals

Operations on DT signals

Convolution

$$y(n) = x_1(n) * x_2(n)$$

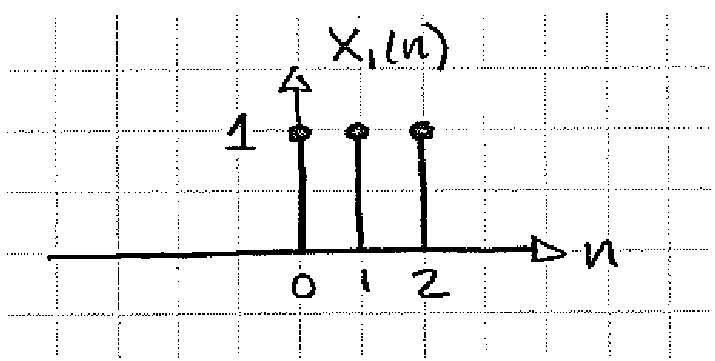
$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k)$$

Convolution Example 1

$$y(n) = x_1(n) * x_2(n)$$

$=$



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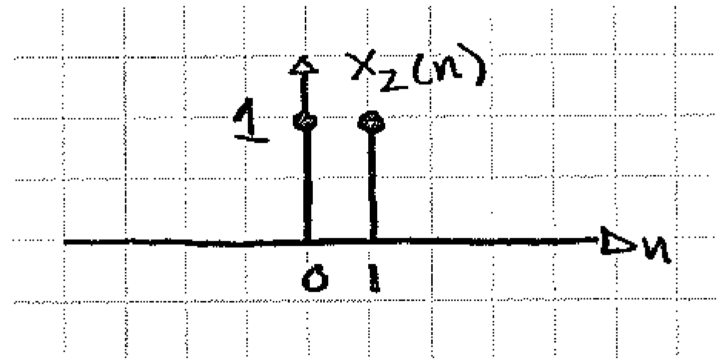

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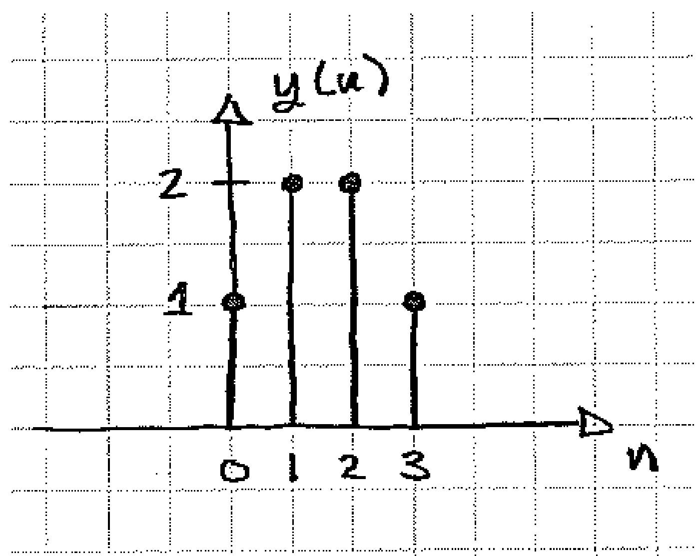
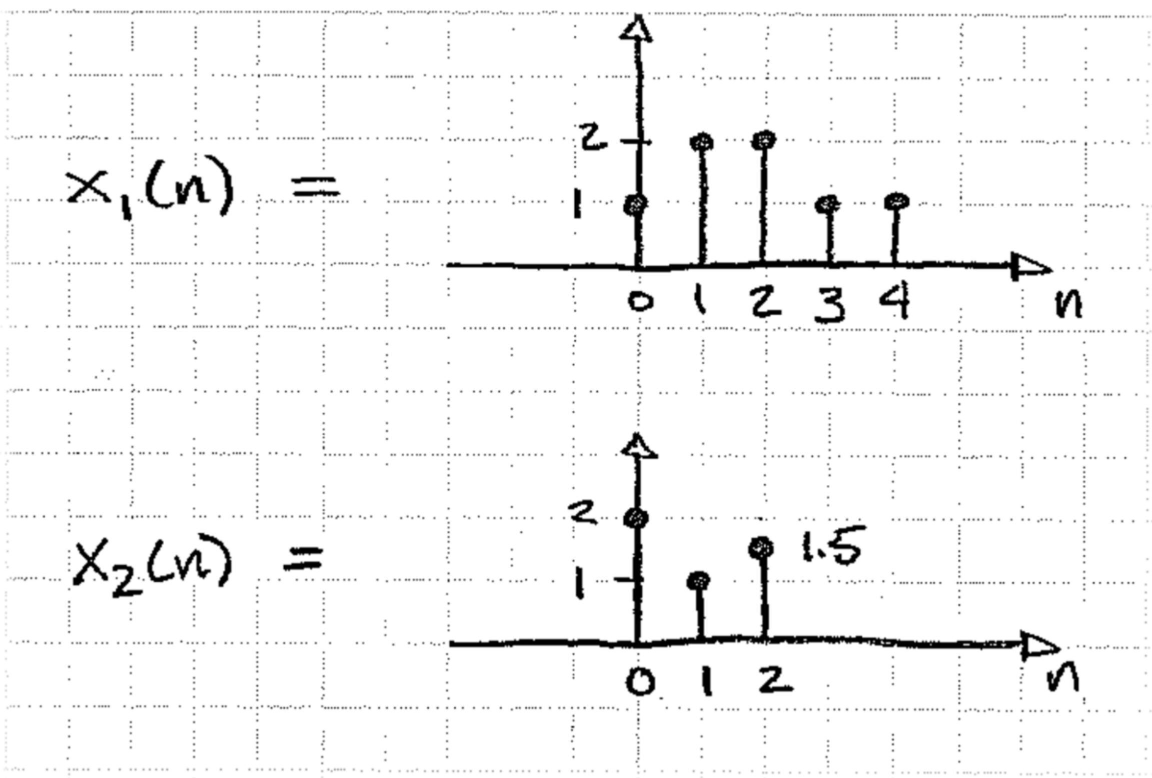
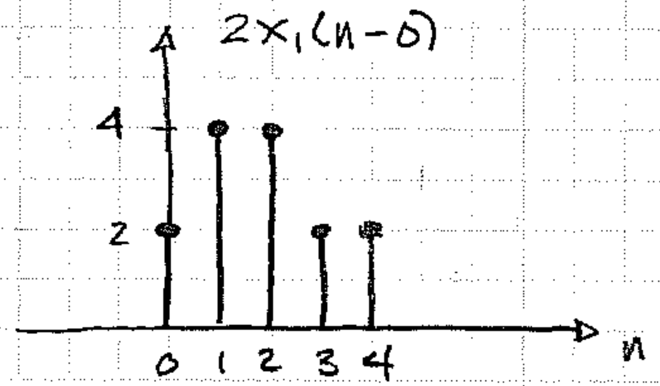
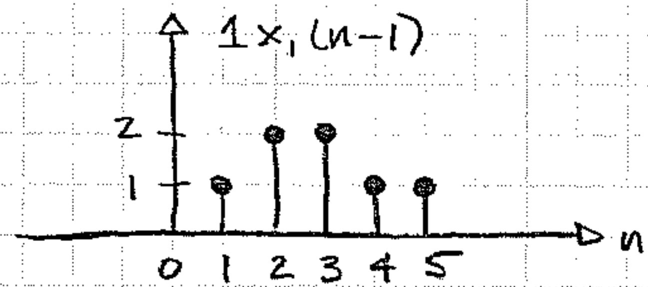


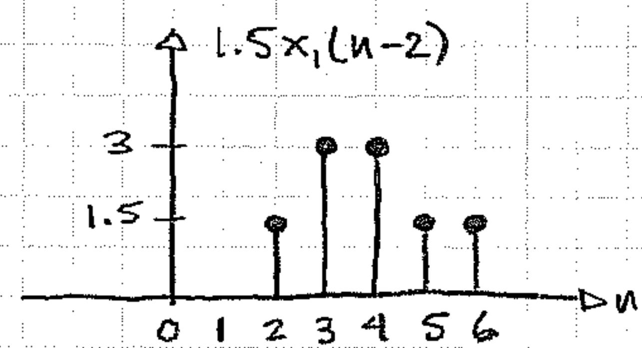
Last time's in-class activity

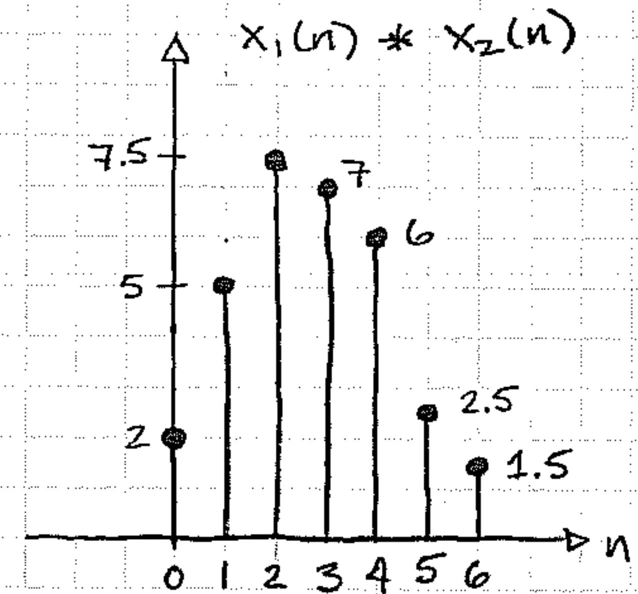
Given the following two signals, compute $x_1(n) * x_2(n)$:




2x1(n-0)
4
2
0 1 2 3 4
n


1x1(n-1)
2
1
0 1 2 3 4 5
n


1.5x1(n-2)
3
1.5
0 1 2 3 4 5 6
n


x1(n) * x2(n)
7.5
7
6
5
2.5
2
1.5
0 1 2 3 4 5 6
n

Properties of convolution

① Commutative property

$$x_1(n) * x_2(n) = x_2(n) * x_1(n)$$

② Associative property

$$[x_1(n) * x_2(n)] * x_3(n)$$
$$= x_1(n) * [x_2(n) * x_3(n)]$$

③ Distributive property

$$x_1(n) * [x_2(n) + x_3(n)]$$
$$= x_1(n) * x_2(n) + x_1(n) * x_3(n)$$

Properties of convolution

④ Identity property

$$x(n) * \delta(n) = x(n)$$

⑤ Translation property

$$x(n) * \delta(n - n_0) = x(n - n_0)$$

Convolution Example 3

Ex: $x_1(n) = \left(\frac{3}{4}\right)^n u(n)$

$x_2(n) = u(n)$

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Convolution Example 3

<u>Ex:</u> $x_1(n) = \left(\frac{3}{4}\right)^n u(n)$

$x_2(n) = u(n)$

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{3}{4}\right)^k u(k)\, u(n-k)$$

$$= \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k u(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$$

Convolution Example 3

$$= \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$$

Note: $\displaystyle \sum_{k=a}^{b} c^k = \frac{c^a - c^{b+1}}{1-c}, \quad b \geq a$

Convolution Example 3

$$= \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$$

Note: $\displaystyle\sum_{k=a}^{b} c^k = \frac{c^a - c^{b+1}}{1-c}, \quad b \geq a$

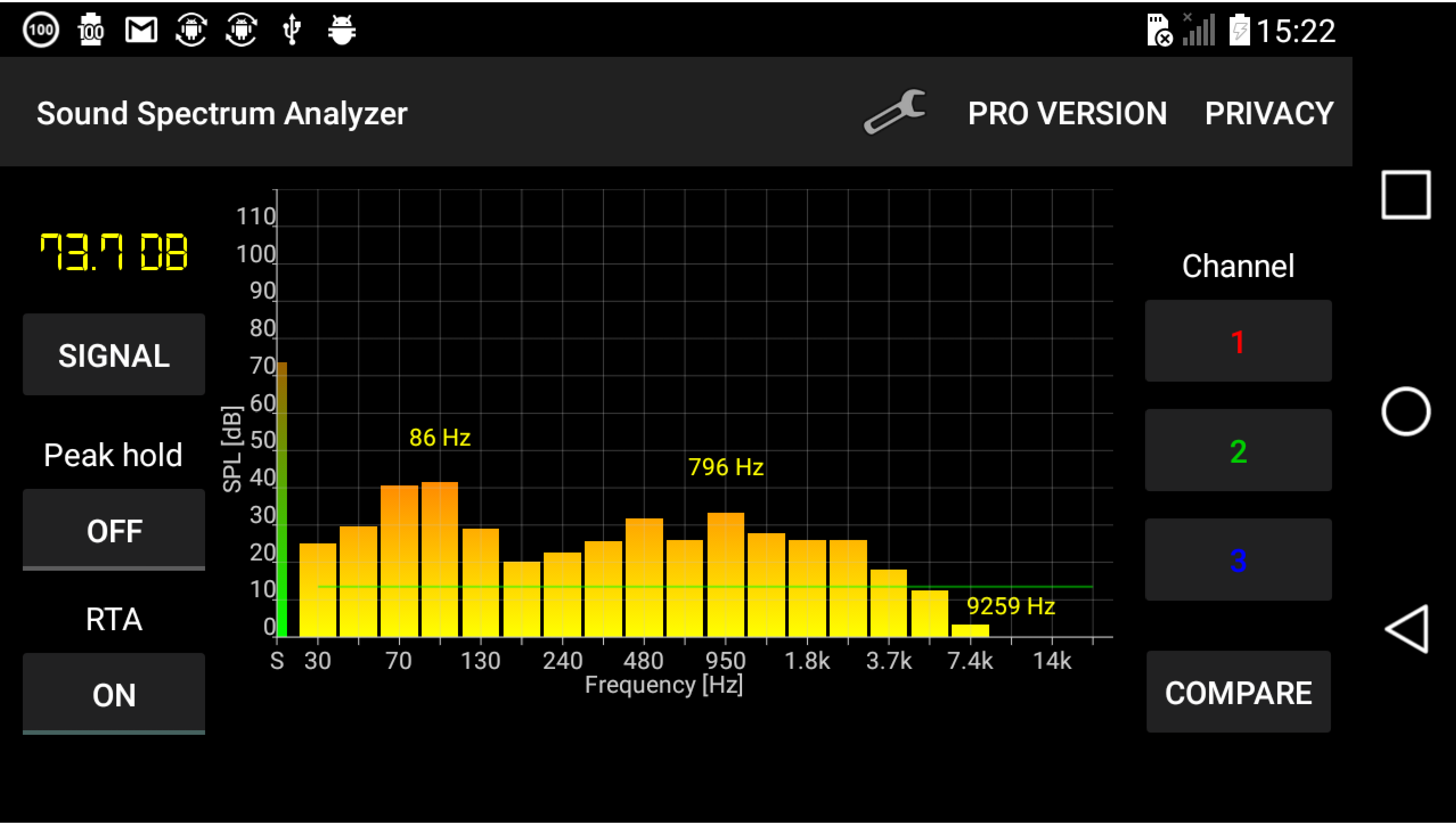
$$= \frac{\left(\frac{3}{4}\right)^0 - \left(\frac{3}{4}\right)^{n+1}}{1 - \frac{3}{4}}$$

$$= \frac{1 - \left(\frac{3}{4}\right)^{n+1}}{1/4} = 4 - 3\left(\frac{3}{4}\right)^n$$

Why is convolution useful?

- Used for **analysis** and **filtering**
  - to **isolate** particular features/frequencies
  - to **remove** noise
  - **enhance** particular features/frequencies
  - **interpolation** (e.g., rate changing, resizing)
  - and **many** other uses in encryption, compression, transmission, etc.
- We will return to filtering later in the course

Ex: Audio DSP software



Today's learning objectives

From **today's lecture**, you should **be able to**…

- Explain the term "DT frequency"
- Explain what is the "spectrum" of a signal
- Use the Discrete-Time Fourier Transform (DTFT) to compute a signal's spectrum

Today

1. What is DT frequency?

2. Discrete-time Fourier transform (DTFT)

3. Some DTFT examples

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What is DT frequency?

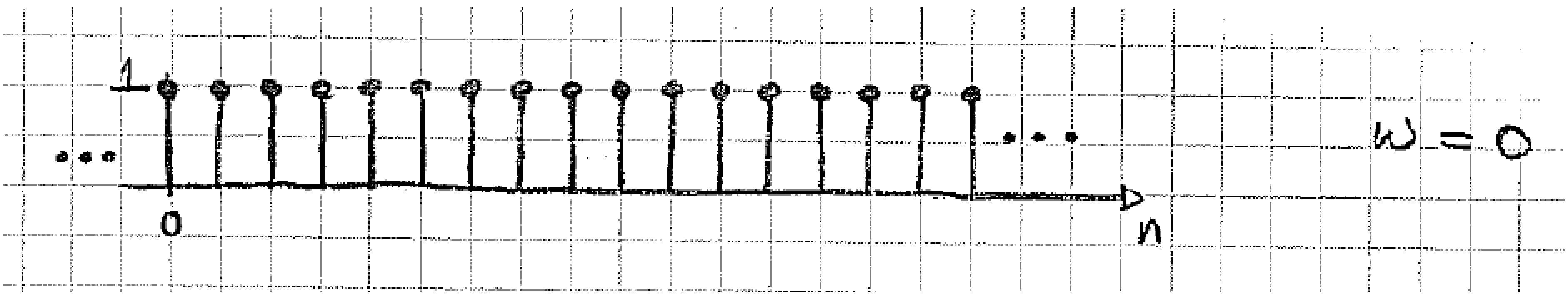
What is frequency?

Answer: A measure of how frequently a signal changes

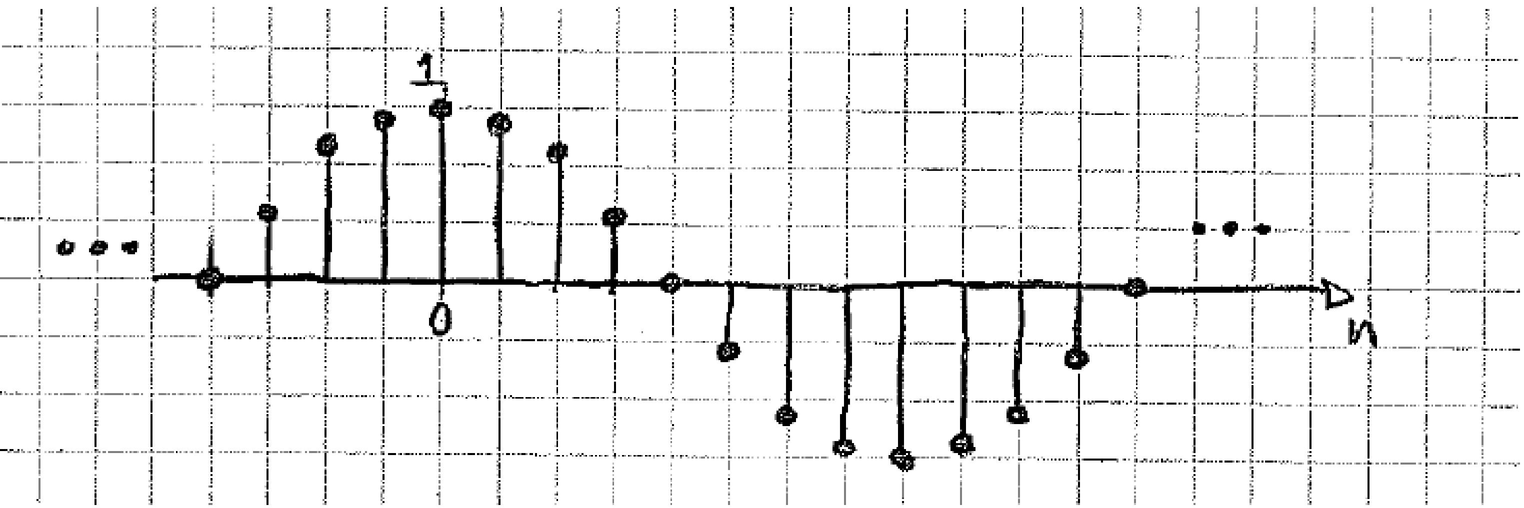
→ CT frequency, Ω, describes how frequently a signal changes over time

→ DT frequency, ω, describes how frequently a signal changes over index

What is DT frequency?



What is DT frequency?



$\omega = \ ?$

What is DT frequency?

1 cycle / 16 samples

(Recall: 1 cycle = 2π radians)

2π radians / 16 samples

π radians / 8 samples

$\boxed{\rightarrow \ \omega = \pi/8 \text{ radians/sample}}$

What is DT frequency?

1 cycle / 16 samples

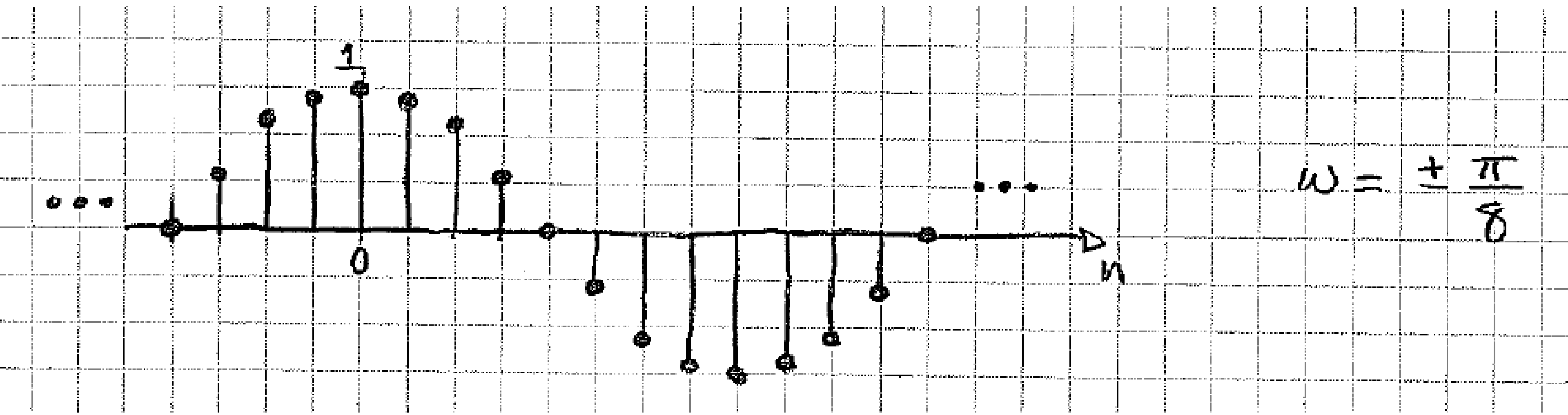
(Recall: 1 cycle = 2π radians)

2π radians / 16 samples

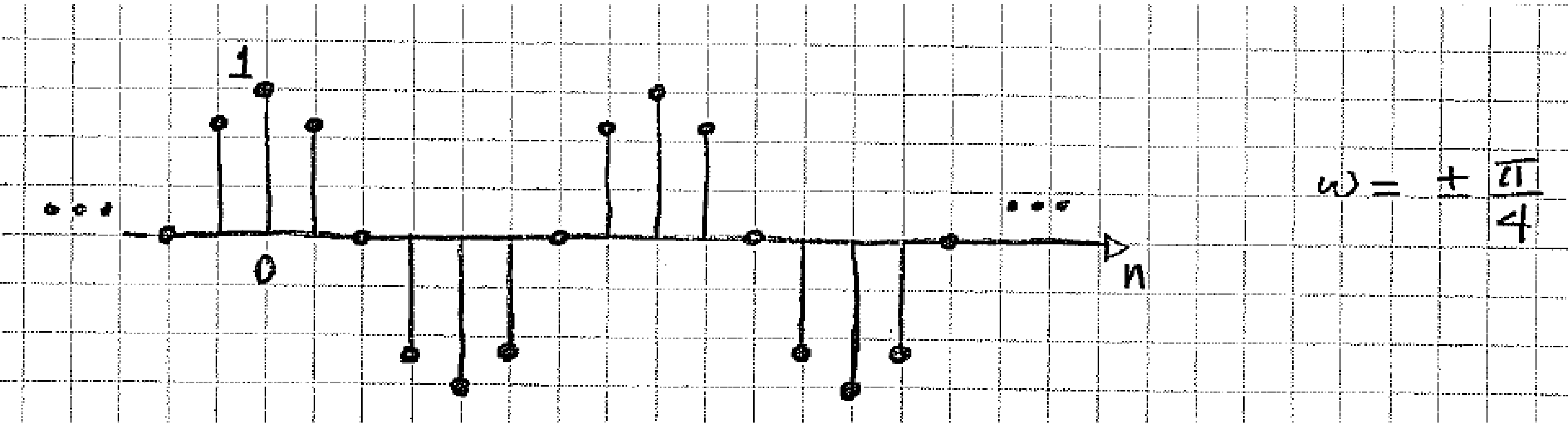
π radians / 8 samples

$\rightarrow$ $\omega = \pi/8$ radians

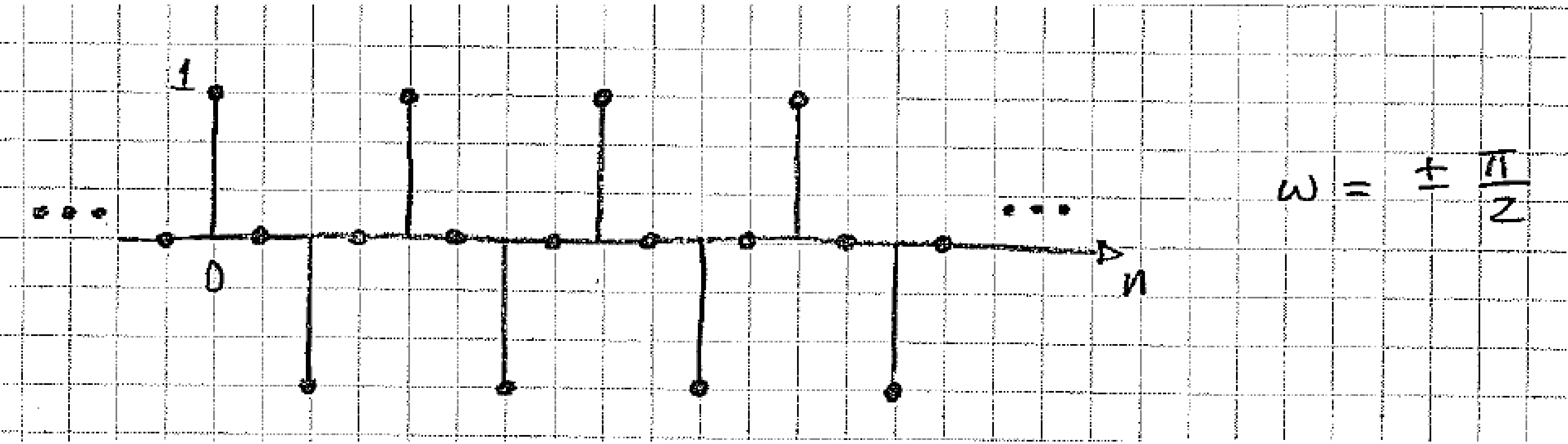
What is DT frequency?



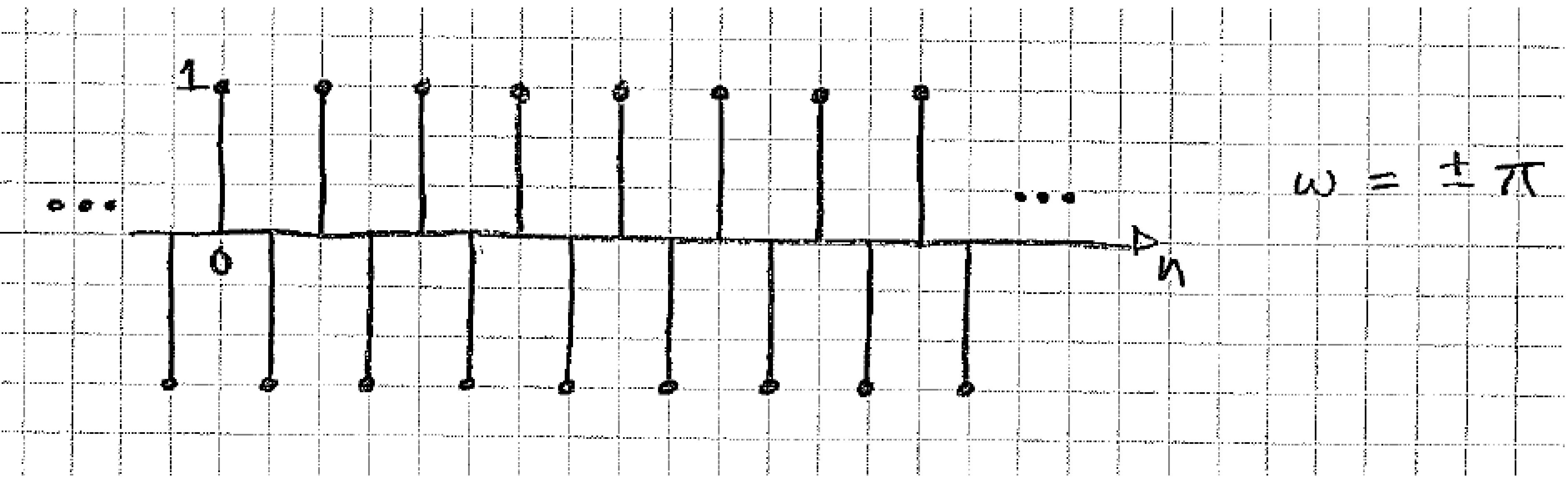
What is DT frequency?

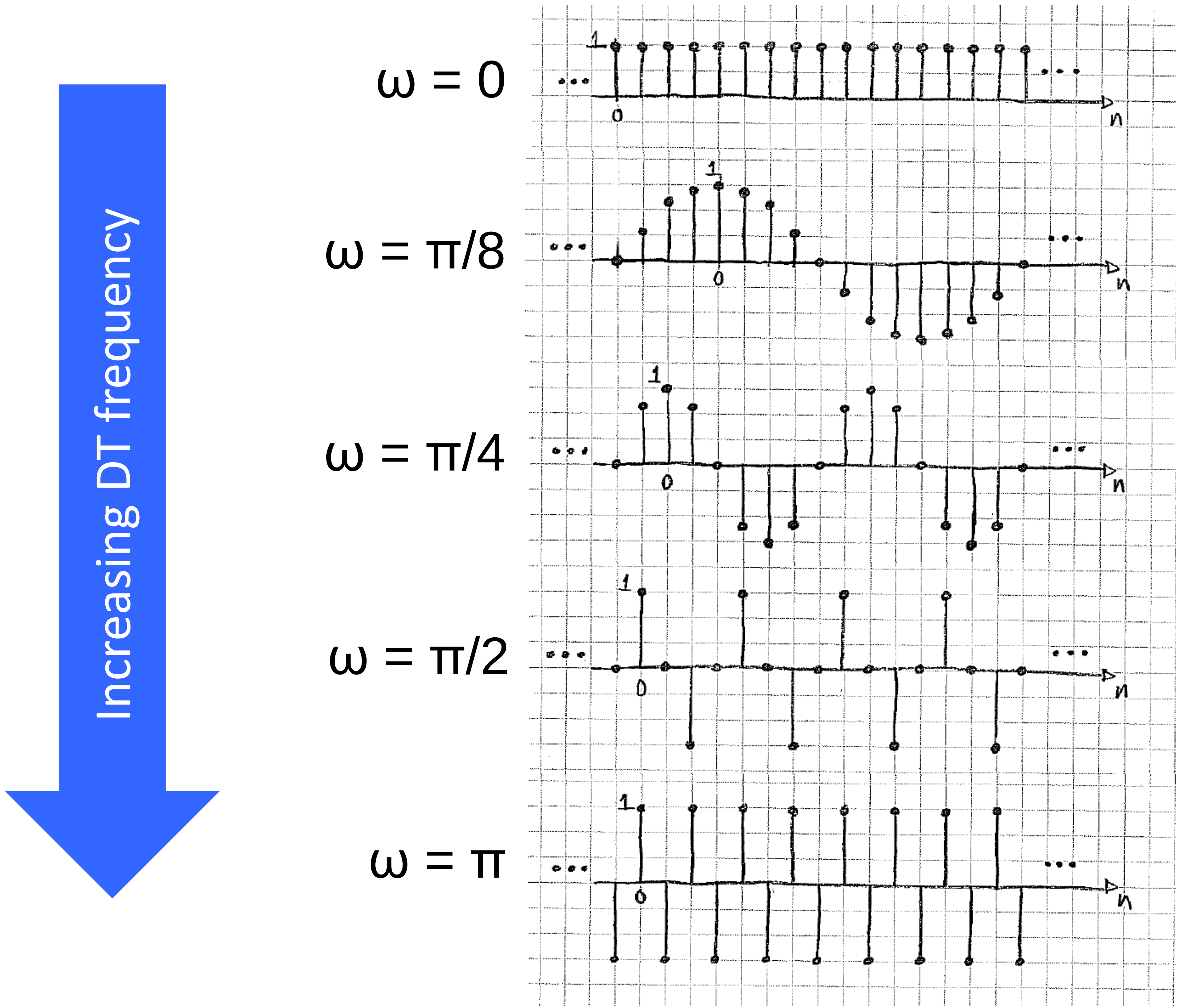


What is DT frequency?



What is DT frequency?




Increasing DT frequency
ω = 0
ω = π/8
ω = π/4
ω = π/2
ω = π
1
0
n

Today

1. What is DT frequency?

2. Discrete-time Fourier transform (DTFT)

3. Some DTFT examples

Discrete-time Fourier transform (DTFT)

Transforms a signal from the **time domain** into the **frequency domain**

time-domain $\boxed{x(n) \leftrightarrow X(\omega)}$ frequency domain

$x(n)$ specifies signal value as a function of index n (unitless)

$X(\omega)$ specifies signal magnitude and phase as a function of DT frequency ω (radians)

CTFT vs. DTFT

In CT: $X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$ (CTFT)

In DT: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$ (DTFT)

Notation: $X(\omega) = X(e^{j\omega})$ ← most texts use the $e^{j\omega}$ argument, so we will too

Discrete-time Fourier transform (DTFT)

Forward DTFT: $\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$

Inverse DTFT: $\boxed{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$

Discrete-time Fourier transform (DTFT)

Forward DTFT: $$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$$

Inverse DTFT: $$\boxed{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$$

Notation: $x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$, $\text{DTFT}[x(n)] = X(e^{j\omega})$

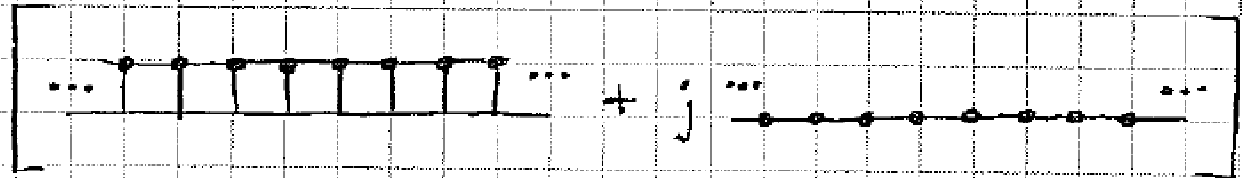
Discrete-time Fourier transform (DTFT)

Look at the inverse DTFT

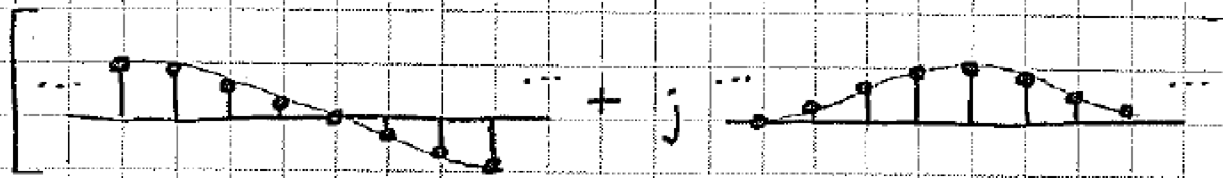
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

What is this equation saying?

Our signal $x(n)$ is a weighted "sum" (integral) of sinusoids (complex exponentials)

$x(n) = X(e^{j0}) \times$ 

$+ \cdots$

$+ X(e^{j\frac{\pi}{8}}) \times$ 

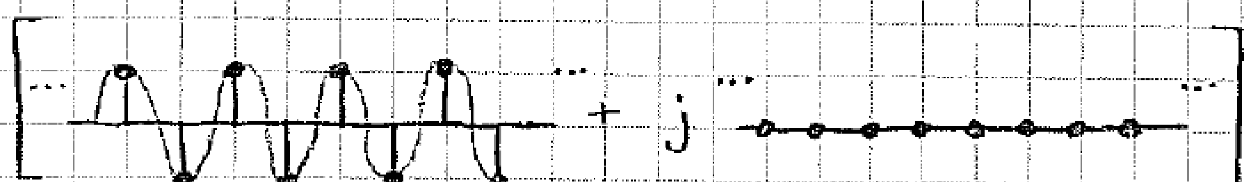
$\cdots$

$+ X(e^{j\frac{\pi}{4}}) \times$ 

$\cdots$

$+ X(e^{j\frac{\pi}{2}}) \times$ 

$\cdots$

$+ X(e^{j\pi}) \times$ 

Continuum

Discrete-time Fourier transform (DTFT)

Three very important notes about the DTFT…

Note 1: $X(e^{j\omega})$ is a continuous function of ω

Note 2: $X(e^{j\omega})$ is generally complex

$|X(e^{j\omega})|$ specifies height , $\measuredangle X(e^{j\omega})$ specifies phase shift

Note 3: $X(e^{j\omega})$ is a periodic function of ω with period 2π

Proof of Note 3

[that $X(e^{j\omega})$ repeats with a period of 2π]

Note 3: $X(e^{j\omega})$ is a periodic function of ω with period 2π

DTFT formula

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

After a shift of 2π (or 4π, 8π, etc.)…

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} e^{-j2\pi n} = X(e^{j\omega})$$

This follows from the fact that DT complex exponentials are limited to the frequency range $(-\pi, \pi]$.

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Today's in-class activity

Compute the DTFT of the following signals:

1. $x(n) = \delta(n+1) - \delta(n-1)$
2. $y(n) = \delta(n+1) + 2\delta(n) - \delta(n-1)$

Question: We can see that $y(n) = x(n) + 2\delta(n)$. Please write $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$?