Digital Signal Processing

Spring Semester 2022

Frequency-Based Analysis, Part 2

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Last time's learning objectives

- Explain the term "DT frequency"
  - ➢ A measure of **how frequently a signal changes** per time
  - ➢ Usually represented by **variable $\boldsymbol{\omega}$** with units radians/sample and over the range $(-\pi, \pi]$
  - ➢ Sines, cosines, and $e^{j\omega_0}$ fluctuate at only one $\pm$ frequency
  - ➢ Other signals fluctuate over a range (a "spectrum") of frequencies
- Explain what is the "spectrum" of a signal
  - ➢ A function that represents how much **energy (and delay)** a signal has **at each frequency**
  - ➢ Computed by using the DTFT
- Use the Discrete-Time Fourier Transform (DTFT) to compute a signal's spectrum

Today's learning objectives

From **today's lecture**, you should **be able to**…

- Compute the DTFTs of basic signals on paper
- Write a signal's spectrum in terms of magnitude and phase

Discrete-time Fourier transform (DTFT)

Transforms a signal from the **time domain** into the **frequency domain** (and vice-versa)

time-domain $\boxed{x(n) \leftrightarrow X(\omega)}$ frequency domain

Forward DTFT: $\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$

Inverse DTFT: $\boxed{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$

The **DTFT** basically **answers this question**:

How much of each frequency does the signal $x(n)$ contain?

Not just these frequencies, but all frequencies in the continuous range $\omega = -\pi$ to π.

$X\left(e^{j\omega}\right)$ contains the answer:

Magnitude
$\left|X\left(e^{j\omega}\right)\right|$ tells the *amount*

Phase
$\angle X\left(e^{j\omega}\right)$ tells the *shift*

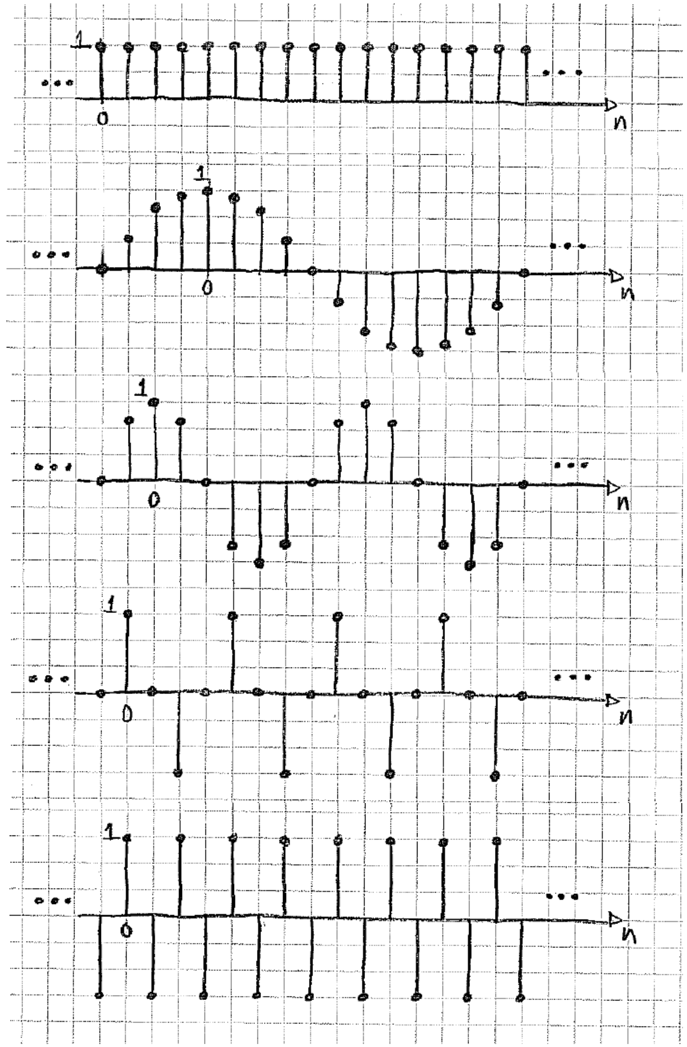
ω = 0

ω = π/8

ω = π/4

ω = π/2

ω = π



The **DTFT** basically **answers this question**:

How much of $\omega = 0$ does $x(n)$ contain?

→ $X(e^{j\omega})\big|_{\omega=0} = X(e^{j0})$

How much of $\omega = \frac{2\pi}{3}$ does $x(n)$ contain?

→ $X(e^{j\omega})\big|_{\omega=\frac{2\pi}{3}} = X\left(e^{j\frac{2\pi}{3}}\right)$

How much of $\omega = -\frac{\pi}{5}$ does $x(n)$ contain?

→ $X(e^{j\omega})\big|_{\omega=\frac{-\pi}{5}} = X\left(e^{j\frac{-\pi}{5}}\right)$

and so on…

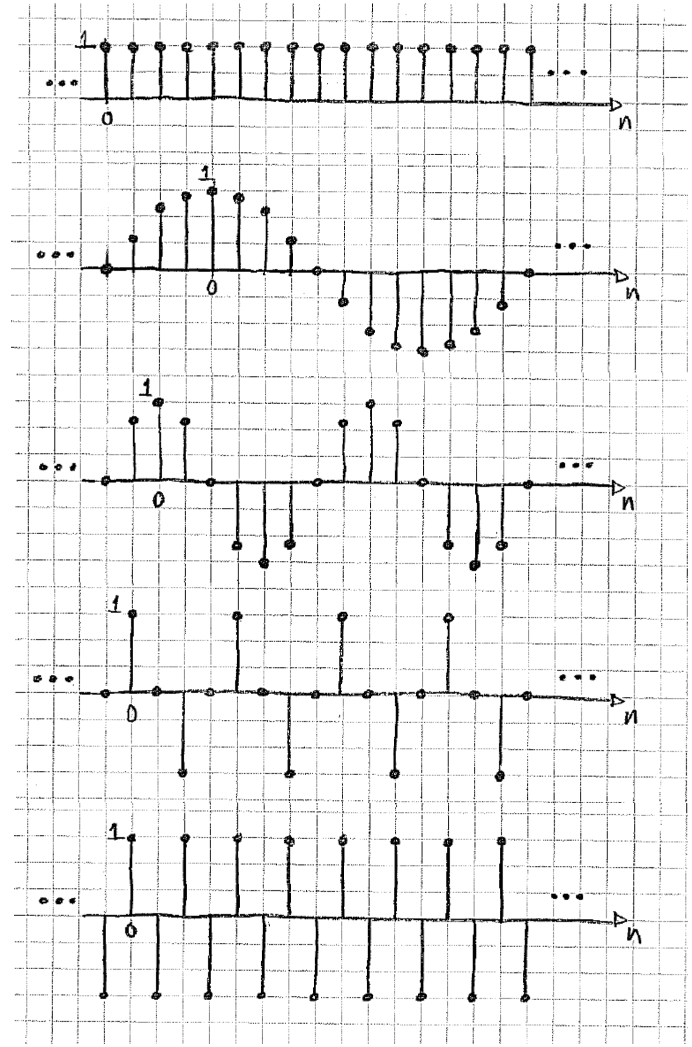
ω = 0

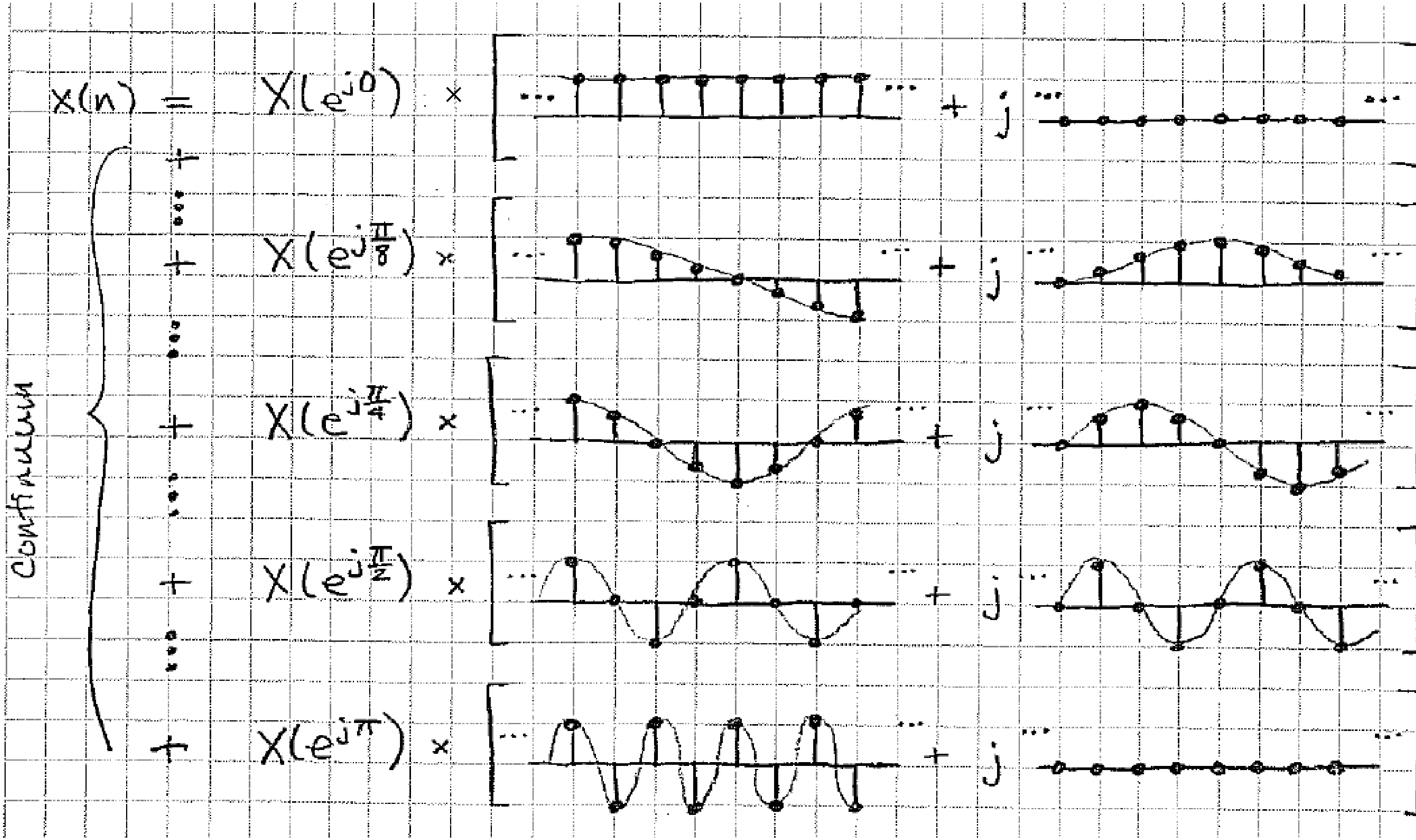
ω = π/8

ω = π/4

ω = π/2

ω = π




$x(n) = X(e^{j0}) \times [\;\cdots \quad \cdots + j \cdots \quad \cdots\;]$
$+ \; \vdots$
$+ X(e^{j\frac{\pi}{8}}) \times [\;\cdots \quad \cdots + j \cdots \quad \cdots\;]$
$\vdots$
$+ X(e^{j\frac{\pi}{4}}) \times [\;\cdots \quad \cdots + j \cdots \quad \cdots\;]$
$\vdots$
$+ X(e^{j\frac{\pi}{2}}) \times [\;\cdots \quad \cdots + j \cdots \quad \cdots\;]$
$\vdots$
$+ X(e^{j\pi}) \times [\;\cdots \quad \cdots + j \cdots \quad \cdots\;]$
continuum

Ex: $x(n) = \delta(n)$

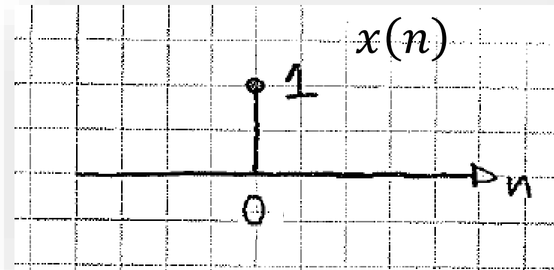
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n}$$

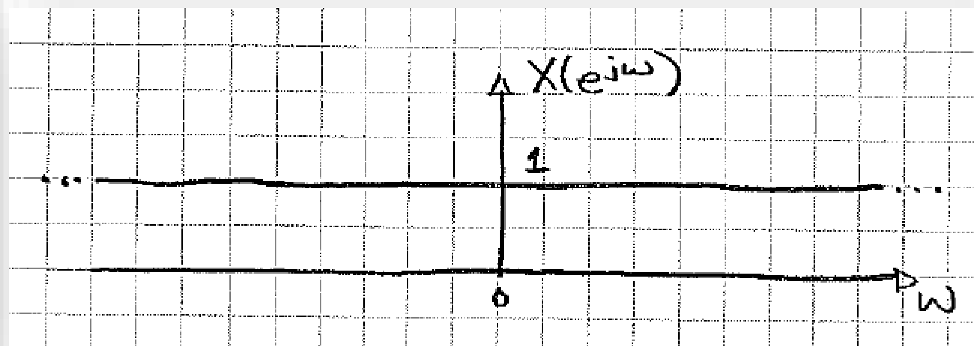
$$= \overset{1}{\cancel{\delta(0)}}\, e^{-j\omega 0}$$

$$= \underline{1}$$

Time domain $x(n)$



Frequency domain $X(e^{j\omega})$



Last week’s in-class activity

Compute the DTFT of the following signals:

1. $x(n) = \delta(n+1) - \delta(n-1)$
2. $y(n) = \delta(n+1) + 2\delta(n) - \delta(n-1)$

Question: We can see that $y(n) = x(n) + 2\delta(n)$. Please write $Y\left(e^{j\omega}\right)$ in terms of $X\left(e^{j\omega}\right)$.

Ex: $x(n) = \delta(n+1) - \delta(n-1)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-1}^{1} [\delta(n+1) - \delta(n-1)] e^{-j\omega n}$$

$$= [\delta(-1+1) - \delta(-1-1)] e^{-j\omega(-1)} \qquad n = -1$$
$$+ [\delta(0+1) - \delta(0-1)] e^{-j\omega 0} \qquad n = 0$$
$$+ [\delta(1+1) - \delta(1-1)] e^{-j\omega 1} \qquad n = 1$$

$$= [\delta(0) - \delta(-2)] e^{j\omega 1}$$
$$+ [\delta(1) - \delta(-1)] e^{-j\omega 0}$$
$$+ [\delta(2) - \delta(0)] e^{-j\omega 1}$$

$$= [1 - 0] e^{j\omega 1}$$
$$+ [0 - 0] e^{-j\omega 0}$$
$$+ [0 - 1] e^{-j\omega 1}$$

$$= e^{j\omega 1} - e^{-j\omega 1} \quad = 2j \sin(\omega)$$

Note: $e^{ja} - e^{-ja} = 2j \sin(a)$
https://en.wikipedia.org/wiki/Euler%27s_formula#Relationship_to_trigonometry

Ex: $y(n) = \delta(n+1) + 2\delta(n) - \delta(n-1) = x(n) + 2\delta(n)$

$Y(e^{j\omega}) = X(e^{j\omega}) + 2$ By using the linearity property of the DTFT

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\omega n} = \sum_{n=-1}^{1} [\delta(n+1) + 2\delta(n) - \delta(n-1)]e^{-j\omega n}$$

$$= [\delta(-1+1) + 2\delta(-1) - \delta(-1-1)]e^{-j\omega(-1)} \qquad n=-1$$
$$+ [\delta(0+1) + 2\delta(0) - \delta(0-1)]e^{-j\omega 0} \qquad n=0$$
$$+ [\delta(1+1) + 2\delta(1) - \delta(1-1)]e^{-j\omega 1} \qquad n=1$$

$$= [\delta(0) + 2\delta(-1) - \delta(-2)]e^{j\omega 1}$$
$$+ [\delta(1) + 2\delta(0) - \delta(-1)]e^{-j\omega 0}$$
$$+ [\delta(2) + 2\delta(1) - \delta(0)]e^{-j\omega 1}$$

$$= [1 + 0 - 0]e^{j\omega 1}$$
$$+ [0 + 2 - 0]e^{-j\omega 0}$$
$$+ [0 + 0 - 1]e^{-j\omega 1}$$

$$= e^{j\omega 1} + 2 - e^{-j\omega 1} \quad = 2j\sin(\omega) + 2$$

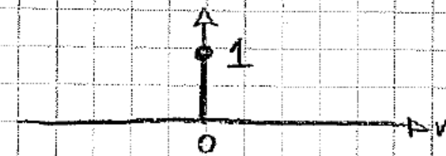
Today

1. More DTFT examples

2. ~~Computing the DTFT in Matlab~~ (next time)

3. ~~DTFT properties~~ (next time)

Ex: $x(n) = \delta(n)$



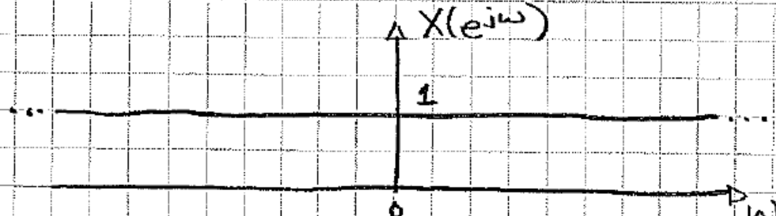
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n}$$

$$= \overset{1}{\cancel{\delta(0)}}\, e^{-j\omega 0}$$

$$= 1$$

Recall: $x(n)\,\delta(n) = x(0)\,\delta(n)$



Ex: $x(n) = \delta(n - n_0)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-n_0)\, e^{-j\omega n}$$

$$= \cancelto{1}{\delta(0)}\, e^{-j\omega n_0}$$

$$= e^{-j\omega n_0}$$

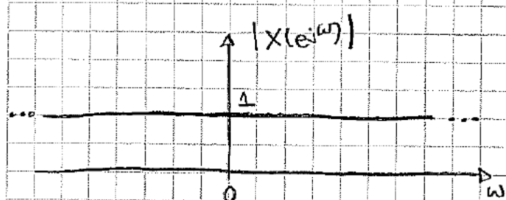
$$X(e^{j\omega}) = e^{-j\omega n_0} \longrightarrow |X(e^{j\omega})| = |e^{-j\omega n_0}| = 1$$

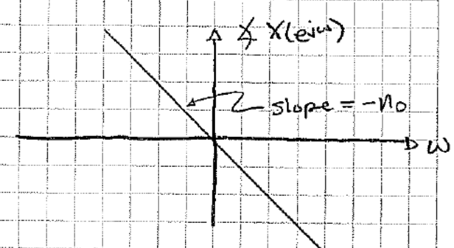
$$\longrightarrow \measuredangle X(e^{j\omega}) = \operatorname{atan}\left(\frac{\operatorname{Im}\{e^{-j\omega n_0}\}}{\operatorname{Re}\{e^{-j\omega n_0}\}}\right)$$

$$= \operatorname{atan}\left(\frac{-\sin(\omega n_0)}{\cos(\omega n_0)}\right)$$

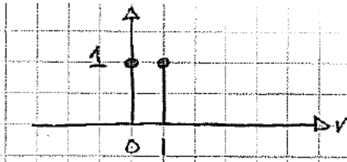
$$= \operatorname{atan}(-\tan(\omega n_0))$$

$$= -\omega n_0$$





Ex: $x(n) = \delta(n) + \delta(n-1)$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\delta(n) + \delta(n-1) \right) e^{-j\omega n}$$

$$= \delta(0)\, e^{-j\omega 0} + \delta(0)\, e^{-j\omega 1}$$

$$= 1 + e^{-j\omega}$$

Looking ahead: The DTFT is a linear transform

$$\mathrm{DTFT}\left[x_1(n) + x_2(n) \right] = \mathrm{DTFT}\left[x_1(n) \right] + \mathrm{DTFT}\left[x_2(n) \right]$$

Previously, $\delta(n) \overset{\mathrm{DTFT}}{\longleftrightarrow} 1$

$\delta(n-n_0) \overset{\mathrm{DTFT}}{\longleftrightarrow} e^{-j\omega n_0}$

Thus, $\delta(n) + \delta(n-1) \overset{\mathrm{DTFT}}{\longleftrightarrow} 1 + e^{-j\omega}$

How to compute the magnitude and phase of $1 + e^{-j\omega}$

$X(e^{j\omega}) = 1 + e^{-j\omega}$

$= e^{-j0.5\omega} e^{j0.5\omega} + e^{-j\omega}$ Re-write 1 as $e^{-j0.5\omega} e^{j0.5\omega}$

$= e^{-j0.5\omega}\left(e^{j0.5\omega} + e^{-j0.5\omega}\right)$ Factor out $e^{-j0.5\omega}$

$= e^{-j0.5\omega}(2\cos(0.5\omega))$ Note that $e^{ja} + e^{-ja} = 2\cos(a)$

https://en.wikipedia.org/wiki/Euler%27s_formula#Relationship_to_trigonometry

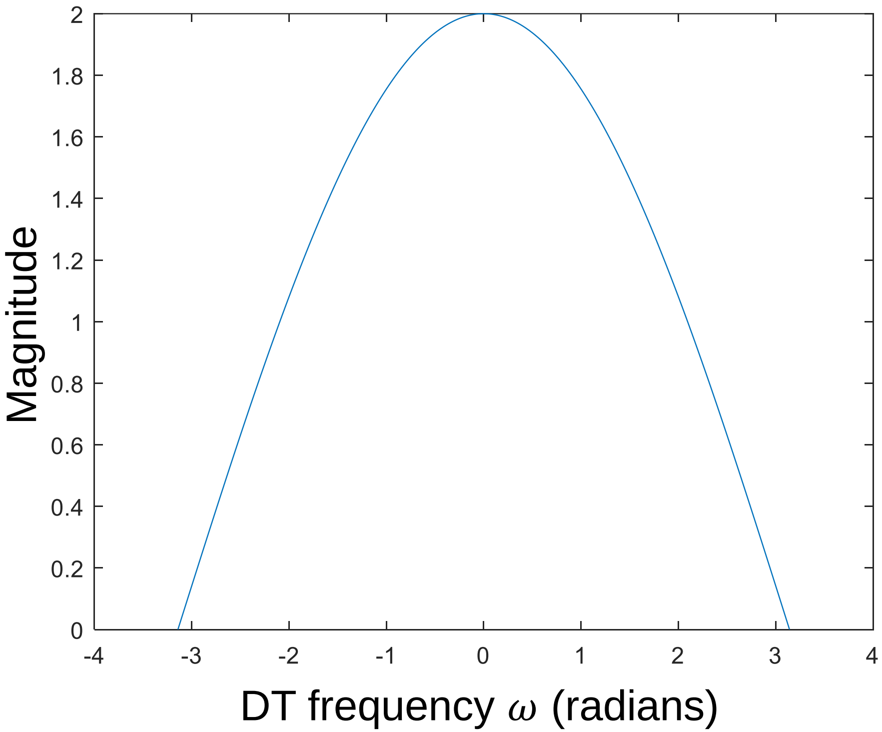
So, $X(e^{j\omega}) = 2\cos(0.5\omega)\, e^{-j0.5\omega}$

thus, $\left|X(e^{j\omega})\right| = |2\cos(0.5\omega)|$

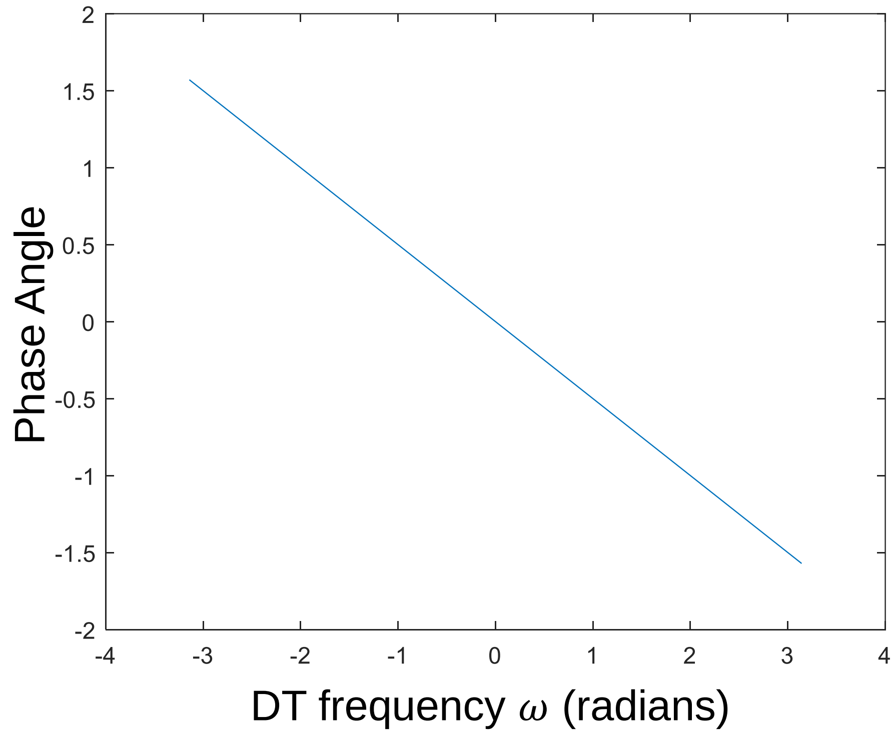
$= 2\cos(0.5\omega)$ for $\omega \in (-\pi, \pi]$

and $\measuredangle X(e^{j\omega}) = -0.5\omega$ for $\omega \in (-\pi, \pi]$

Magnitude Spectrum



Phase Spectrum

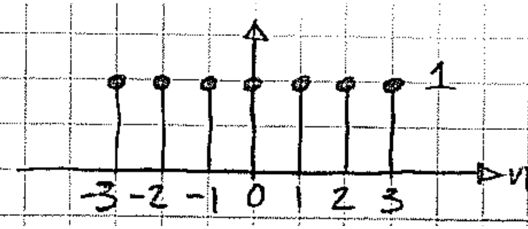


Matlab code to generate the plots:

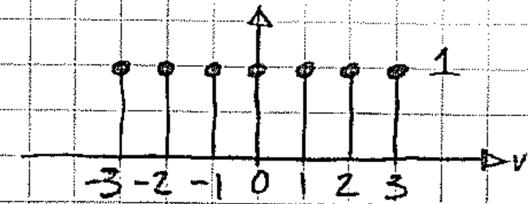
```
w = [-pi:0.001:pi];
X_mag = 2*cos(0.5*w);
X_phs = -0.5*w;

figure(1); plot(w, X_mag);
figure(2); plot(w, X_phs);
```

__Ex:__ $x(n) = \text{rect}_7(n+3)$

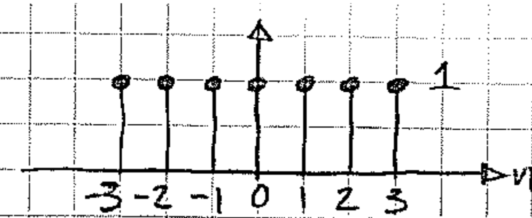


Ex: $x(n) = \text{rect}_7(n+3)$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Ex: $x(n) = \mathrm{rect}_7(n+3)$

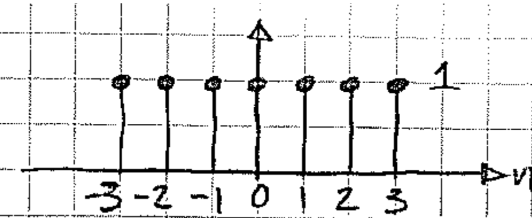


$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-3}^{3} e^{-j\omega n}$$

Recall: $\displaystyle\sum_{n=a}^{b} c^n = \frac{c^a - c^b \cdot c}{1 - c}$

Ex: $x(n) = \text{rect}_7(n+3)$



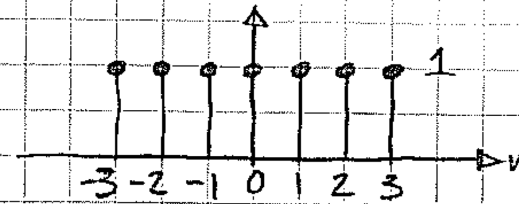
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-3}^{3} e^{-j\omega n}$$

Recall: $\displaystyle\sum_{n=a}^{b} c^n = \frac{c^a - c^b \cdot c}{1 - c}$

$$= \frac{e^{j\omega 3} - e^{-j\omega 4}}{1 - e^{-j\omega}}$$

Ex: $x(n) = \text{rect}_7(n+3)$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-3}^{3} e^{-j\omega n}$$

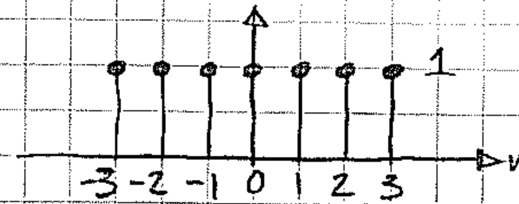
Recall: $\displaystyle\sum_{n=a}^{b} c^n = \frac{c^a - c^b \cdot c}{1 - c}$

$$= \frac{e^{j\omega 3} - e^{-j\omega 4}}{1 - e^{-j\omega}}$$

$\frac{1}{2}(j\omega 3 - j\omega 4) = -j\omega/2$

$\frac{1}{2}(0 - j\omega) = -j\omega/2$

Ex: $x(n) = \text{rect}_7(n+3)$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-3}^{3} e^{-j\omega n}$$

Recall: $\displaystyle\sum_{n=a}^{b} c^n = \frac{c^a - c^b \cdot c}{1 - c}$

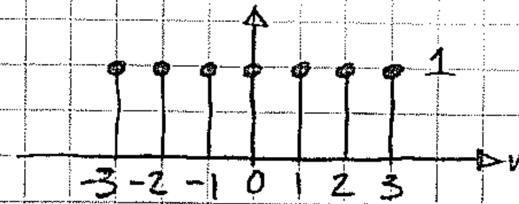
$$= \frac{e^{j\omega 3} - e^{-j\omega 4}}{1 - e^{-j\omega}}$$

$\frac{1}{2}(j\omega 3 - j\omega 4) = -j\omega/2$

$\frac{1}{2}(0 - j\omega) = -j\omega/2$

$$= \frac{e^{-j\omega\frac{1}{2}}\left(e^{j\omega 3.5} - e^{-j\omega 3.5}\right)}{e^{-j\omega\frac{1}{2}}\left(e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}\right)}$$

Ex: $x(n) = \text{rect}_7(n+3)$



$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-3}^{3} e^{-jwn} \qquad \underline{\text{Recall:}} \quad \sum_{n=a}^{b} c^n = \frac{c^a - c^b \cdot c}{1 - c}$$

$$= \frac{e^{jw3} - e^{-jw4}}{1 - e^{-jw}}$$

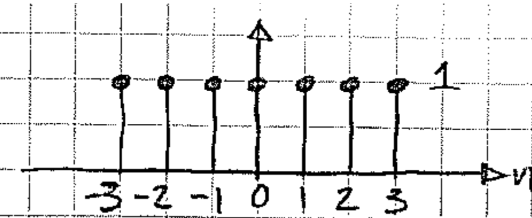
$\frac{1}{2}(j\omega 3 - j\omega 4) = -j\omega/2$

$\frac{1}{2}(0 - j\omega) = -j\omega/2$

$$= \frac{e^{-jw\frac{1}{2}}\left(e^{jw3.5} - e^{-jw3.5}\right)}{e^{-jw\frac{1}{2}}\left(e^{jw\frac{1}{2}} - e^{-jw\frac{1}{2}}\right)}$$

$$= \frac{\cancel{2/j}\, \sin\left(\frac{7}{2}w\right)}{\cancel{2/j}\, \sin\left(\frac{1}{2}w\right)}$$

Ex: $x(n) = \text{rect}_7(n+3)$



$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-3}^{3} e^{-jwn} \qquad \text{Recall: } \sum_{n=a}^{b} c^n = \frac{c^a - c^b \cdot c}{1 - c}$$

$$= \frac{e^{jw3} - e^{-jw4}}{1 - e^{-jw}}$$

$\frac{1}{2}(j\omega 3 - j\omega 4) = -j\omega/2$

$\frac{1}{2}(0 - j\omega) = -j\omega/2$

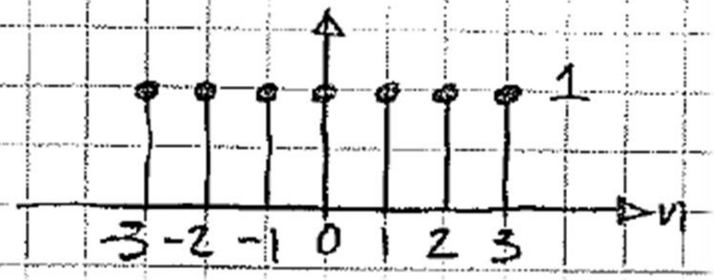
$$= \frac{e^{-jw\frac{1}{2}}\left(e^{jw3.5} - e^{-jw3.5}\right)}{e^{-jw\frac{1}{2}}\left(e^{jw\frac{1}{2}} - e^{-jw\frac{1}{2}}\right)}$$

$$= \frac{\sin(\tfrac{7}{2}\,\omega)}{\sin(\tfrac{1}{2}\,\omega)}$$

Special form: "periodic sinc"

Rectangle in time

$$x(n) = \mathrm{rect}_7(n+3)$$



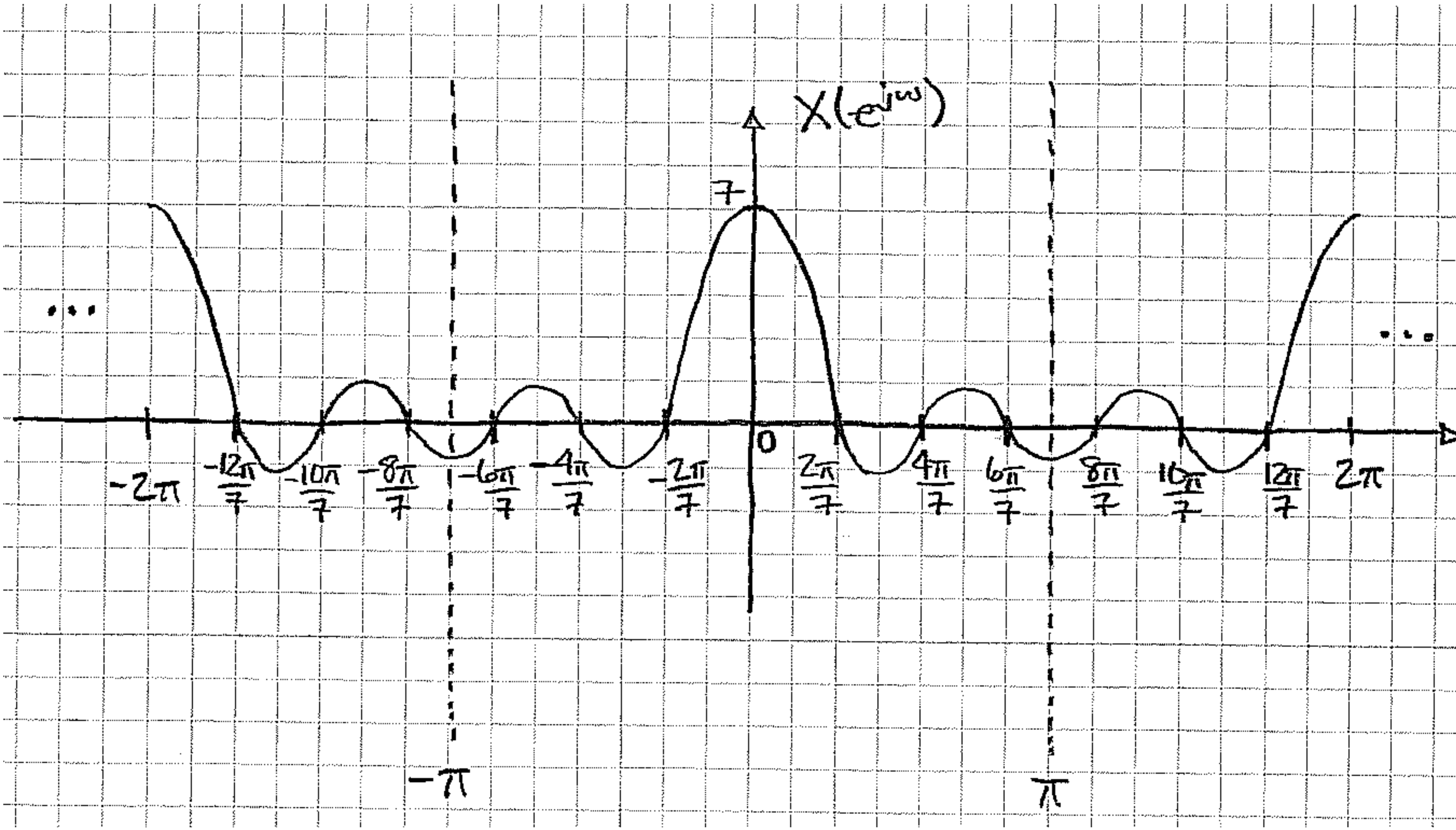
Periodic sinc in frequency

$$X(e^{j\omega}) = \frac{\sin(\frac{7}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

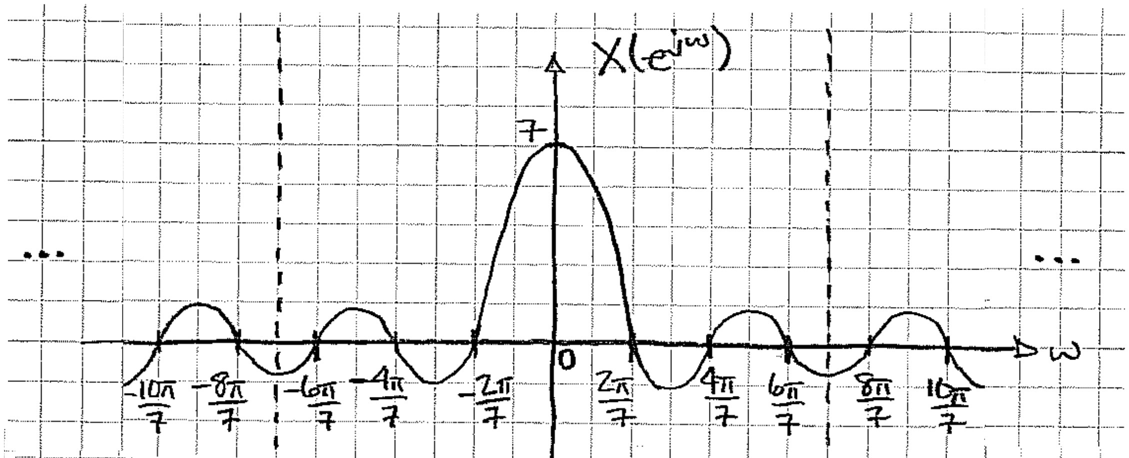
Special form:
"periodic sinc"

Periodic sinc in frequency

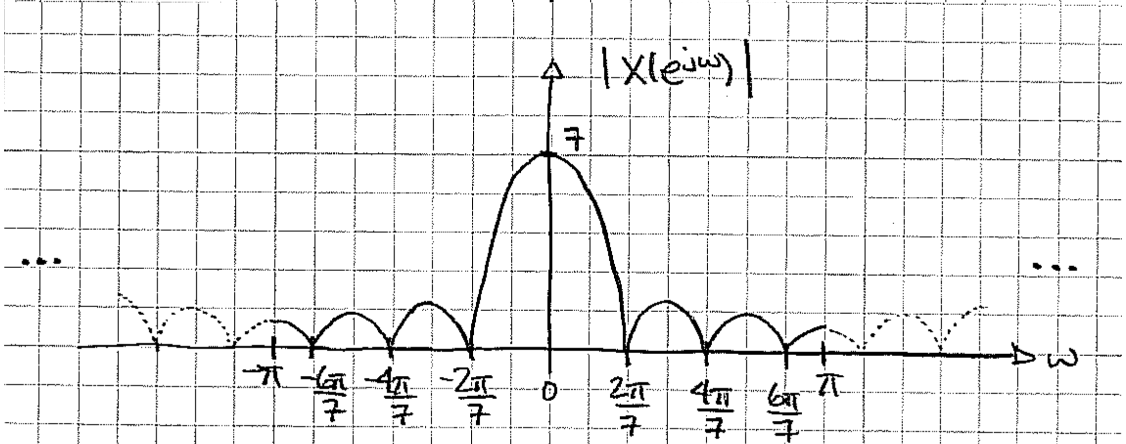
$$X(e^{j\omega}) = \frac{\sin(\tfrac{7}{2}\omega)}{\sin(\tfrac{1}{2}\omega)}$$



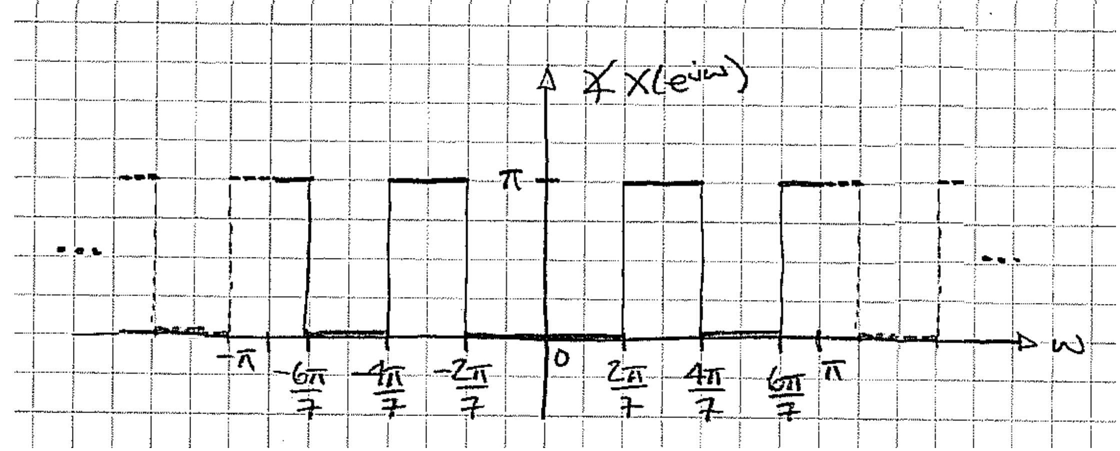
Regular spectrum



Magnitude spectrum



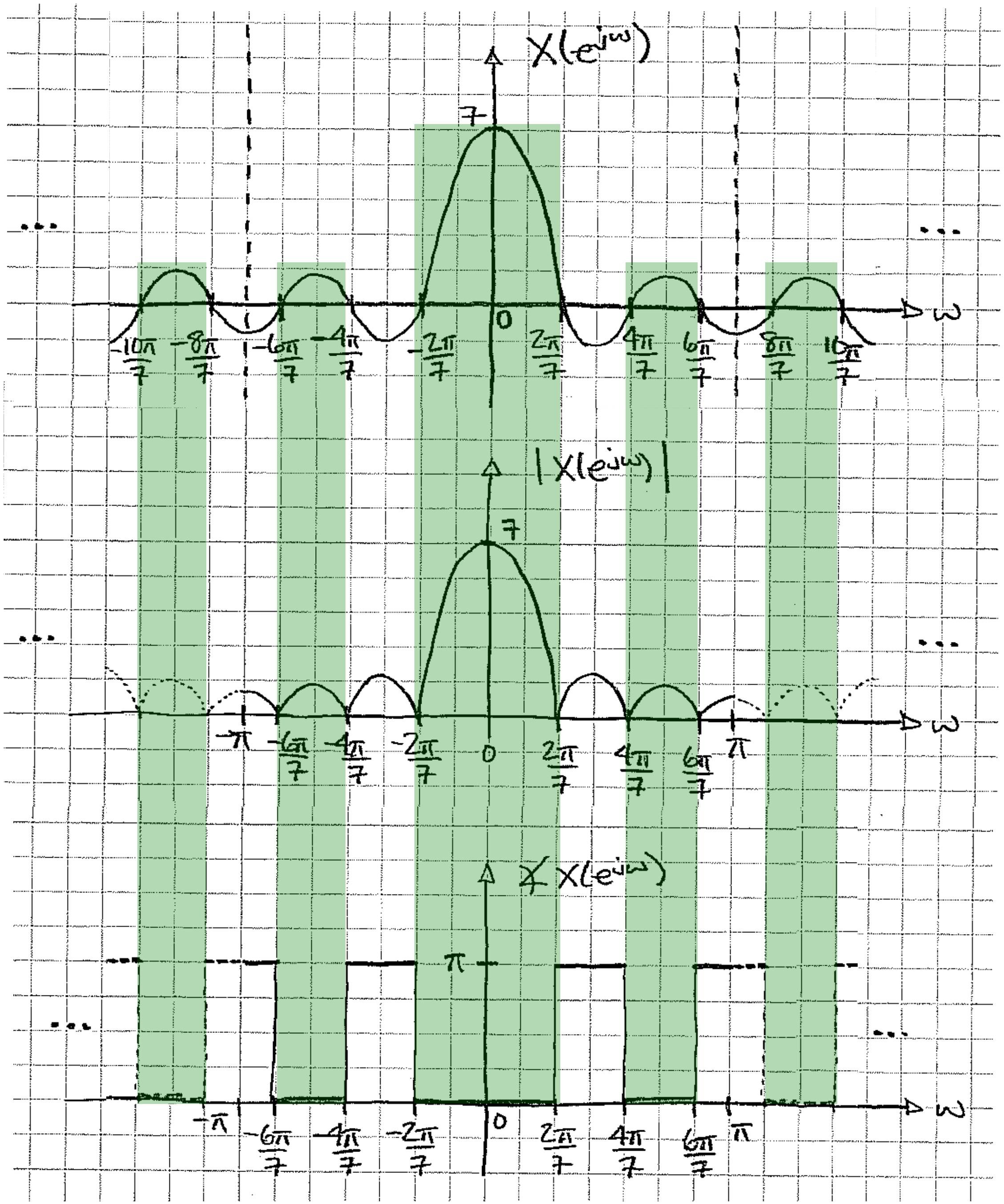
Phase spectrum



Regular spectrum

Magnitude spectrum

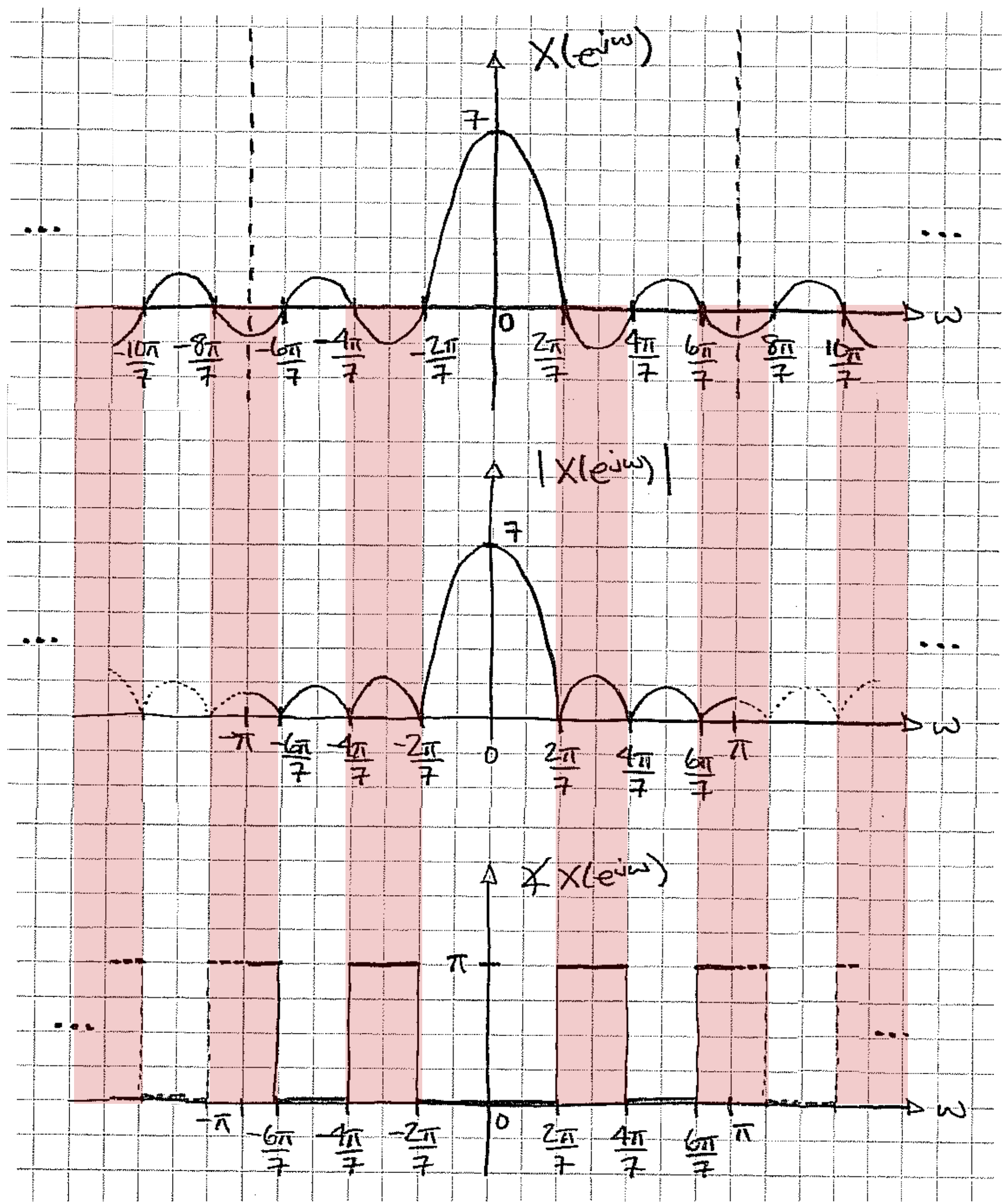
Phase spectrum



Regular spectrum

Magnitude spectrum

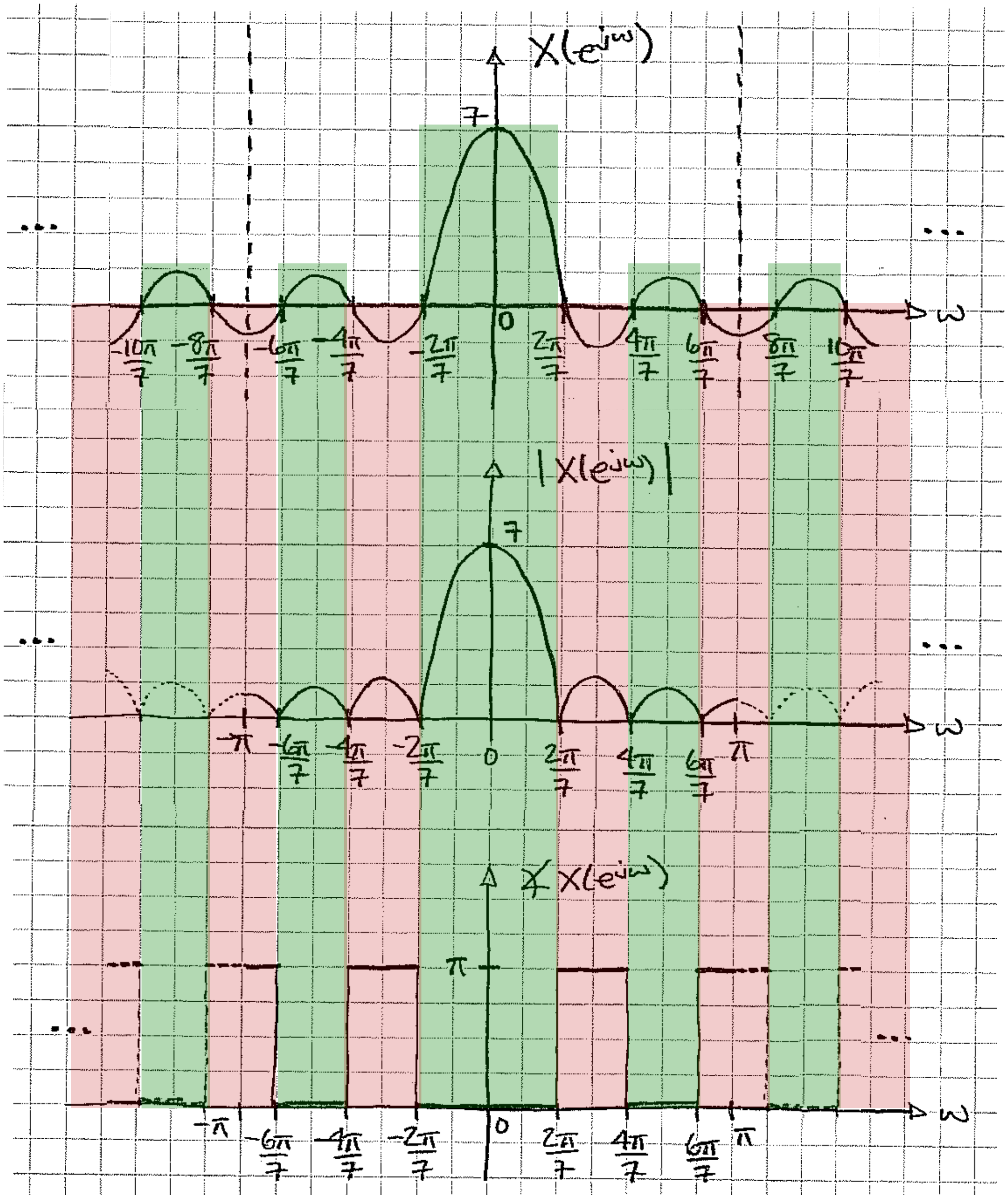
Phase spectrum



Regular spectrum

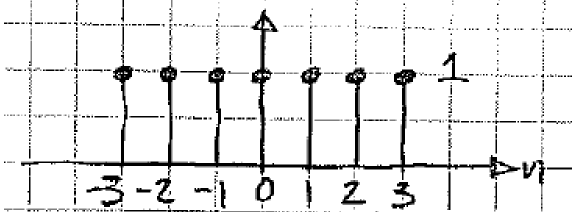
Magnitude spectrum

Phase spectrum

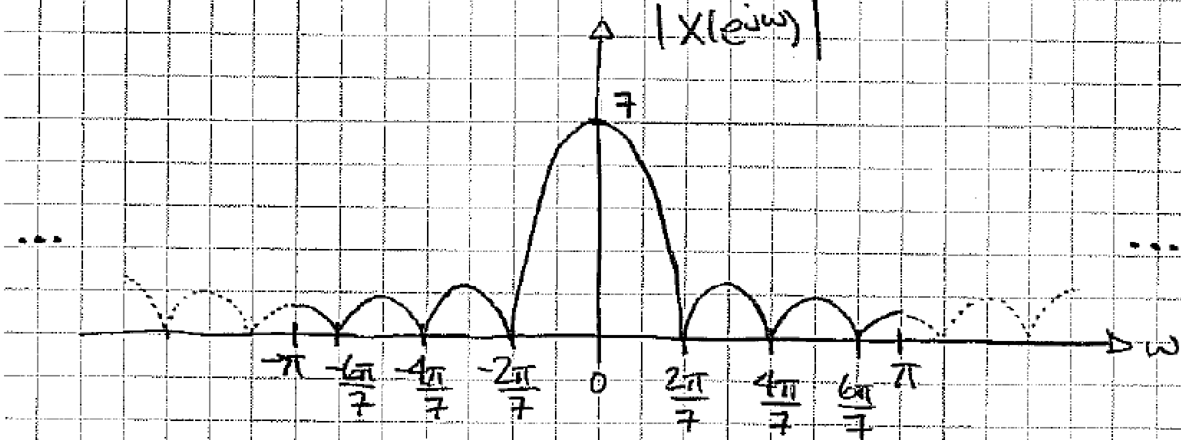


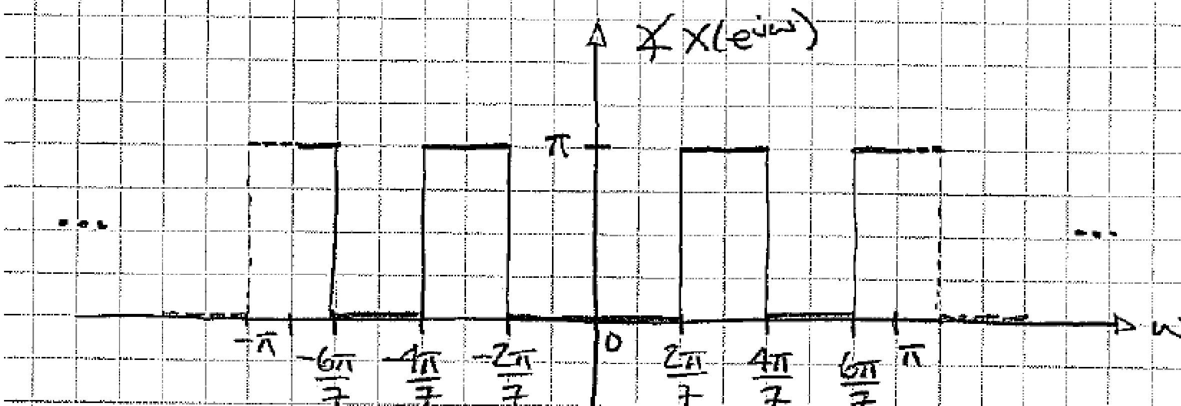
Rectangle $\xleftrightarrow{DTFT}$ Periodic Sinc

$$x(n) = \text{rect}_7(n+3)$$



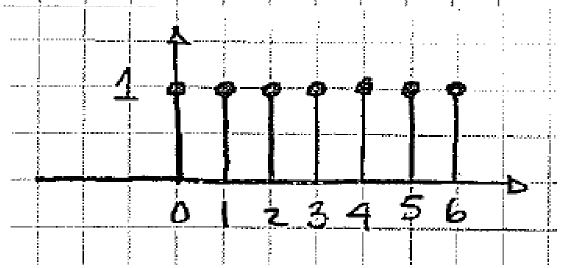
$$X(e^{j\omega}) = \frac{\sin(\tfrac{7}{2}\omega)}{\sin(\tfrac{1}{2}\omega)}$$





Rectangle $\overset{DTFT}{\longleftrightarrow}$ Periodic Sinc

$$x(n) = \mathrm{rect}_7(n)$$



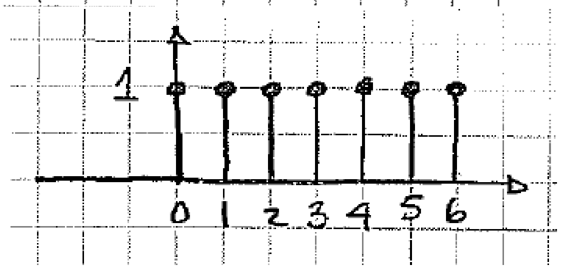
$$X(e^{j\omega}) = e^{-j\omega 3}\,\frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

$$\left|X(e^{j\omega})\right| = \left|\frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right|$$

$$\measuredangle X(e^{j\omega}) = \begin{cases} -3\omega, & \dfrac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} > 0 \\[2ex] -3\omega \pm \pi, & \dfrac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} < 0 \end{cases}$$

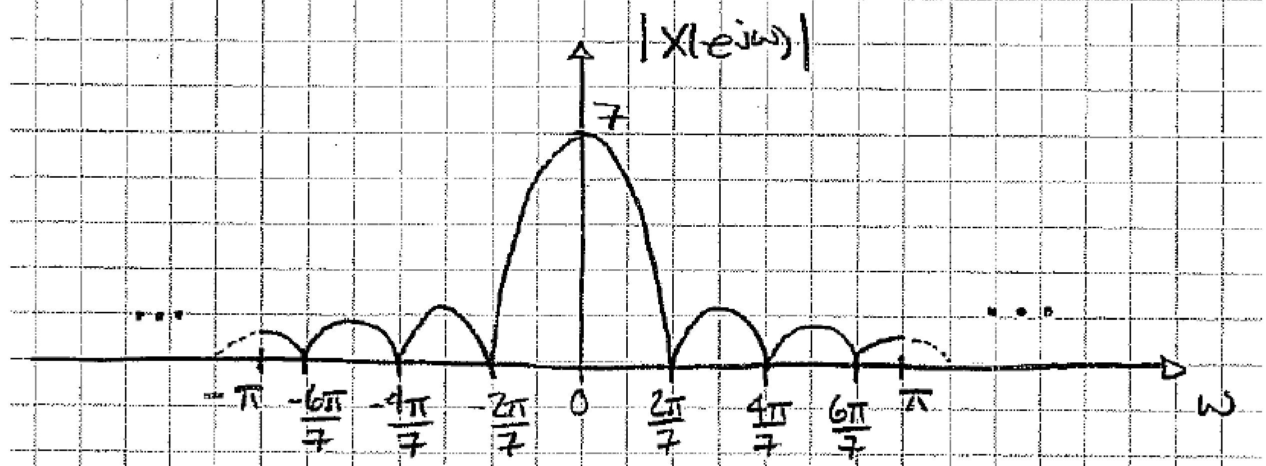
Rectangle $\overset{DTFT}{\longleftrightarrow}$ Periodic Sinc

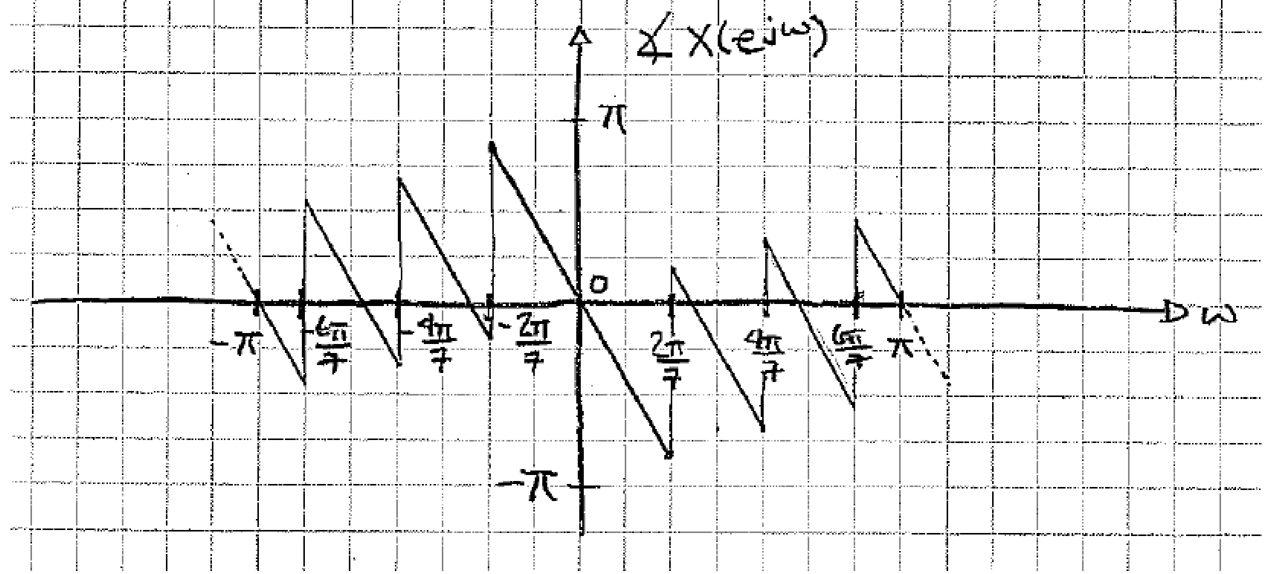
$x(n) = \mathrm{rect}_7(n)$



$X(e^{j\omega})$

$= e^{-j\omega 3} \dfrac{\sin(\frac{7}{2}\omega)}{\sin(\frac{1}{2}\omega)}$

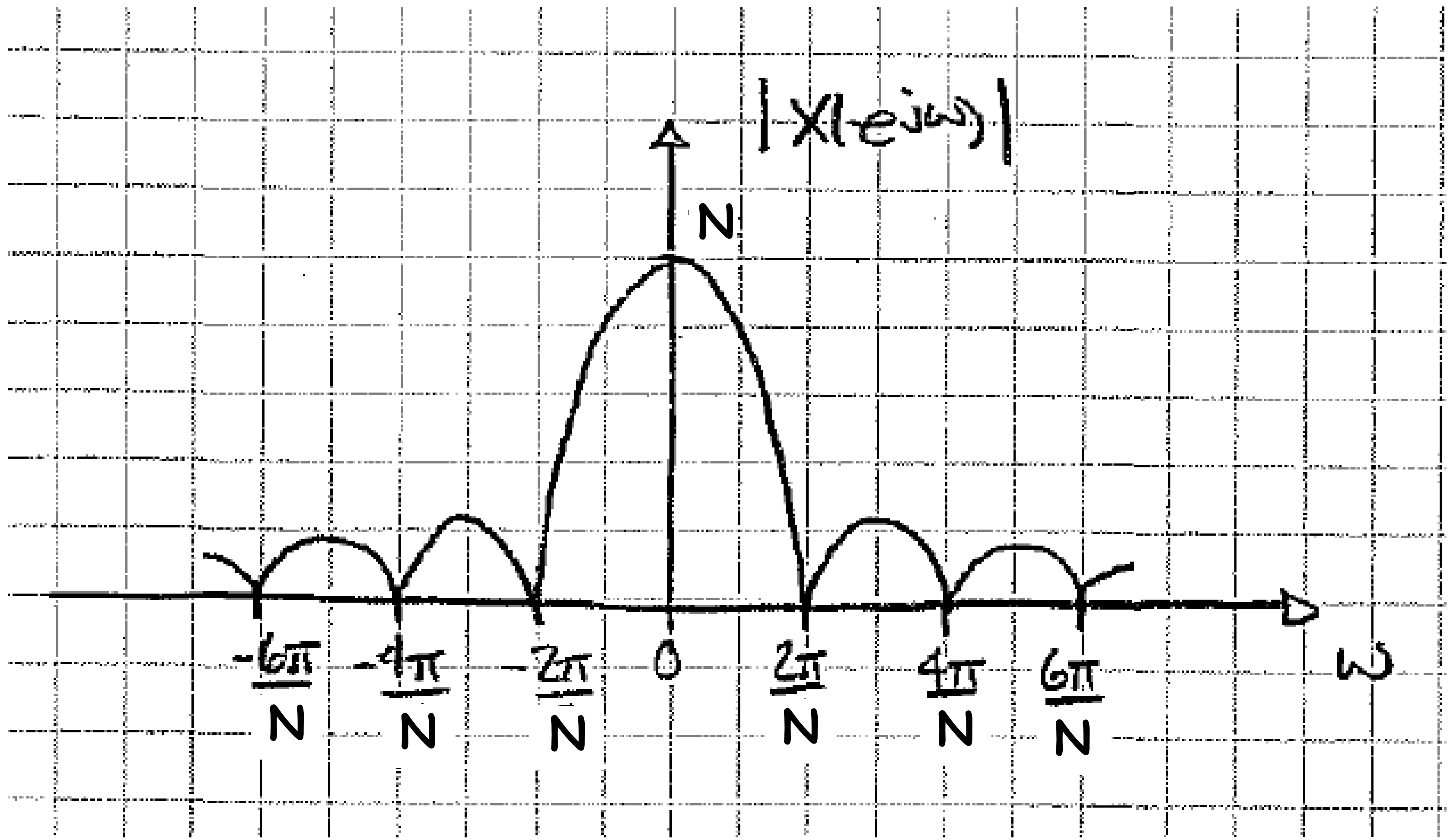




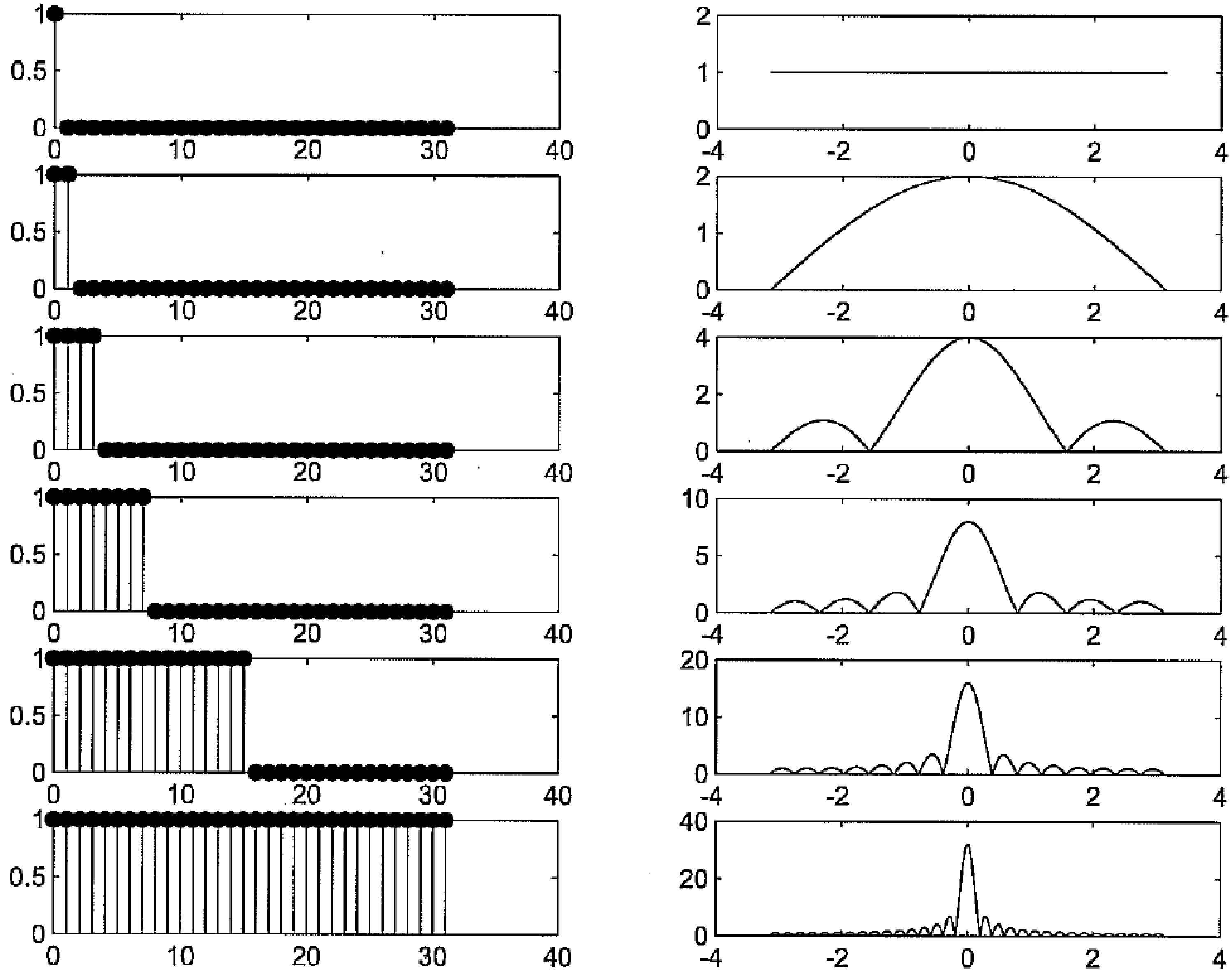
The DTFT of a general rectangle

$$\mathrm{rect}_N(n+n_0) \xleftrightarrow{\text{DTFT}} e^{-jw\left(\frac{N-1}{2}-n_0\right)} \frac{\sin\left(\frac{N}{2}w\right)}{\sin\left(\frac{1}{2}w\right)}$$

The DTFT of a general rectangle



Rectangle in time $\xleftrightarrow{\text{DTFT}}$ Periodic sinc in frequency



time index n DT freq. ω (radians)