Digital Signal Processing

Spring Semester 2022

Fundamental Signal Processing, Part 2

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Last time’s learning objectives

- Perform basic signal measurements
  - How long is the signal?
  - How strong is the signal?
  - How often does it repeat (if at all)?
  - Sample statistics that you would perform on any array of data (mean, variance, etc.)
- Perform basic math/operations on signals

Today’s learning objectives

From **today’s lecture**, you should **be able to**…

- Describe filtering (what it is and why it’s used)
- Describe the relationship between filtering and convolution
- Perform basic convolution between two signals

Outline

1. Filtering
2. Convolution
   1. Definition
   2. Examples
   3. Properties

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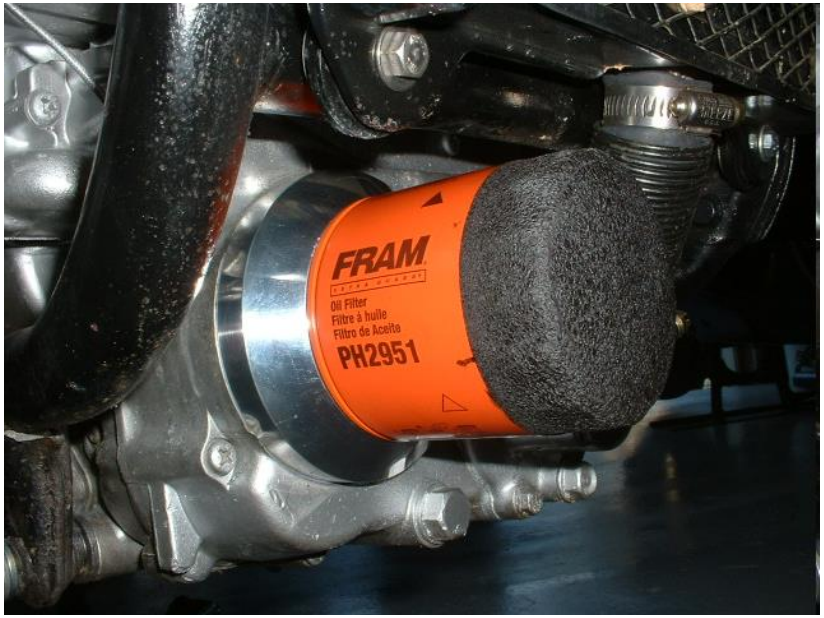
Not a signal filter

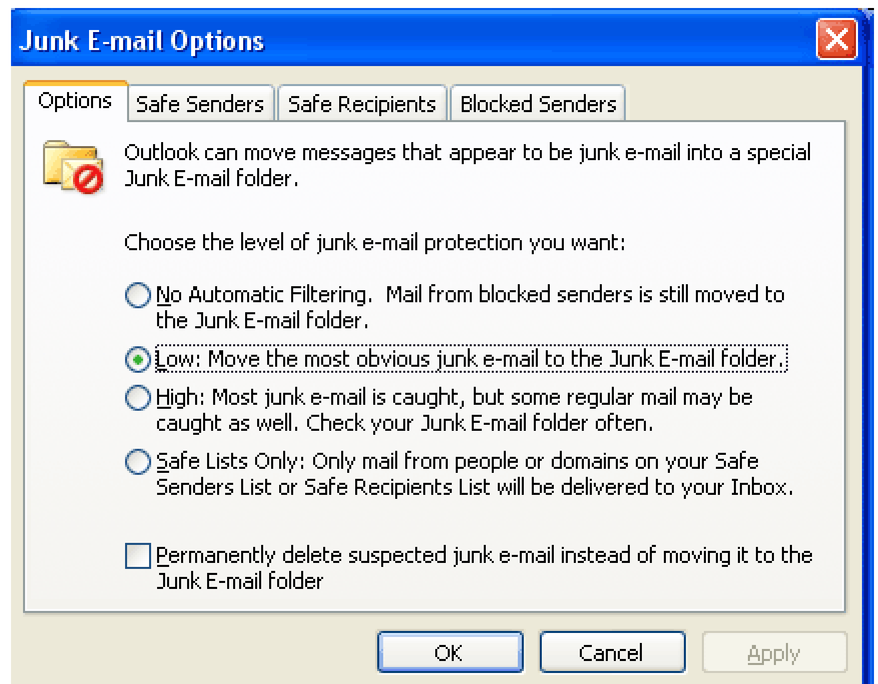
Not a signal filter



Not a signal filter

A signal filter?

SpamAssassin



A signal filter…

- Removes unwanted frequencies
- Attenuates certain frequencies relative to others (passive or active filter)
- Boosts certain frequencies relative to others (active filter only)

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The convolution operator

Convolution

$$y(n) = x_1(n) * x_2(n)$$

The convolution operator

Convolution

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

The convolution operator

Convolution

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$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k)$$

The convolution operator

Convolution

In CT

$$y(t) = x_1(t) * x_2(t)$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

In DT

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Observe: In both cases, convolution involves adding up scaled, time-shifted versions of x_2

The convolution operator

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

The convolution operator

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} \cancel{x_1(k)}\, x_2(n-k)$$

The convolution operator

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} \cancel{x_1(k)}\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x_2(n-k)$$

The convolution operator

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

The convolution operator

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \alpha_k\, x_2(n-k), \quad \text{where } \alpha_k = x_1(k)$$

The convolution operator

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\alpha_k}_{\text{scaling factor}} \underbrace{x_2(n-k)}_{\text{time-shifted version of } x_2(n)}, \quad \text{where } \alpha_k = x_1(k)$$

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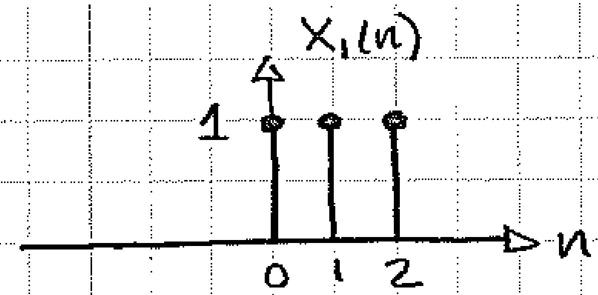
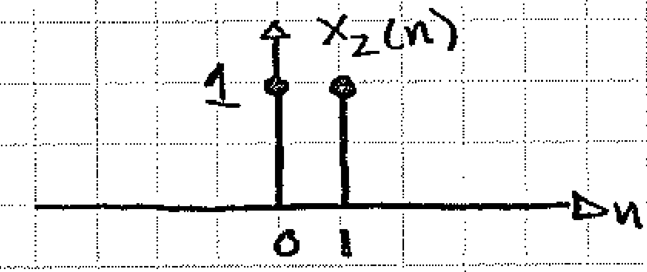
Convolution Example 1

Ex: $x_1(n) = \mathrm{rect}_3(n)$



$x_2(n) = \mathrm{rect}_2(n)$



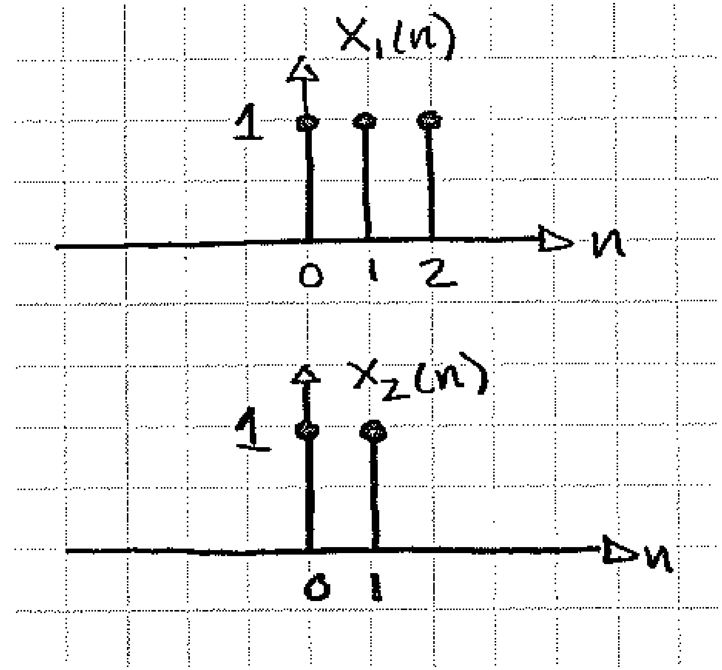
Goal: Compute $y(n) = x_1(n) * x_2(n)$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Convolution Example 1

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$



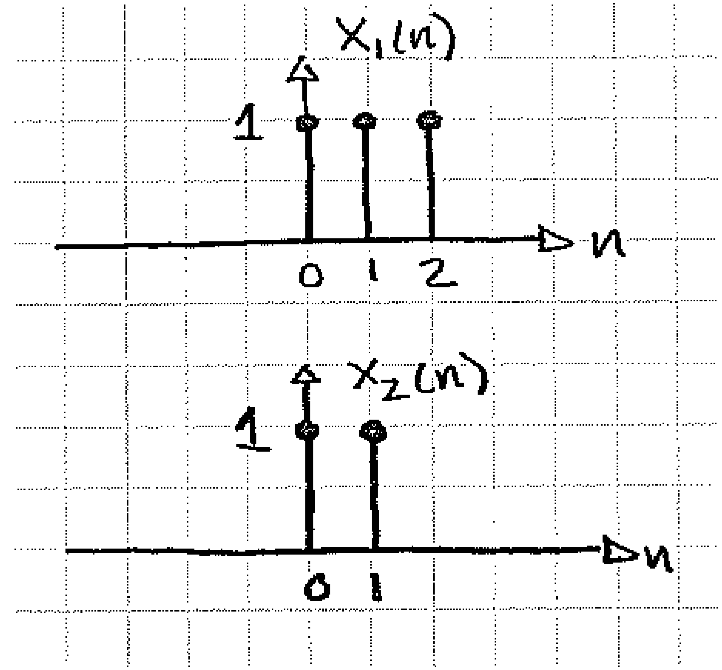
$$\begin{aligned} = \ldots + \quad & x_1(-2)x_2(n-(-2)) && + \\ & x_1(-1)x_2(n-(-1)) && + \\ & x_1(0)x_2(n-0) && + \\ & x_1(1)x_2(n-1) && + \\ & x_1(2)x_2(n-2) && + \\ & x_1(3)x_2(n-3) && + \ \ldots \end{aligned}$$

Convolution Example 1

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

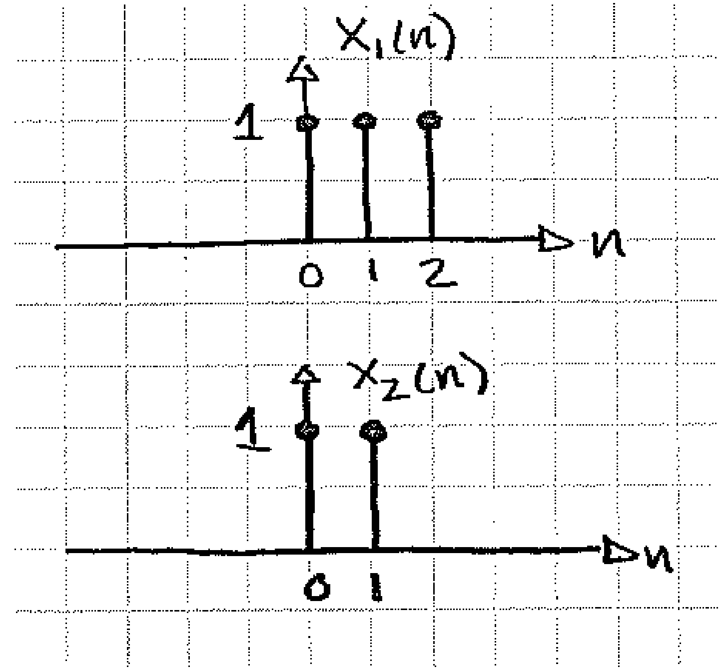
$$
\begin{aligned}
= \ldots + \quad & \overset{0}{x_1(-2)}\, x_2(n+2) \quad + \\
& \overset{0}{x_1(-1)}\, x_2(n+1) \quad + \\
& \overset{1}{x_1(0)}\, x_2(n) \quad + \\
& \overset{1}{x_1(1)}\, x_2(n-1) \quad + \\
& \overset{1}{x_1(2)}\, x_2(n-2) \quad + \\
& \overset{0}{x_1(3)}\, x_2(n-3) \quad + \quad \ldots
\end{aligned}
$$



Convolution Example 1

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$



$=$

$$1x_2(n) \qquad +$$

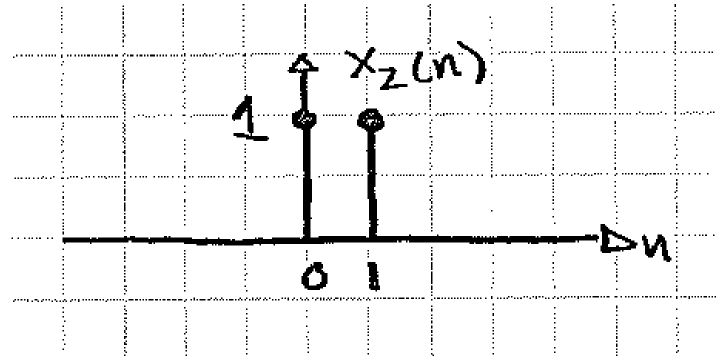
$$1x_2(n-1) \qquad +$$

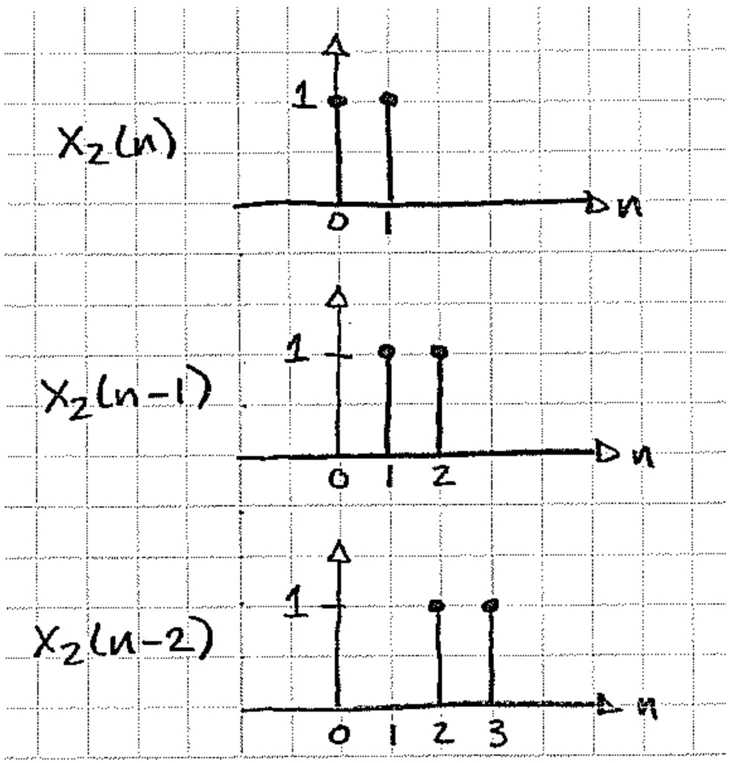
$$1x_2(n-2) \qquad +$$

Convolution Example 1

$$y(n) = x_1(n) * x_2(n)$$

$$= x_2(n) + x_2(n-1) + x_2(n-2)$$

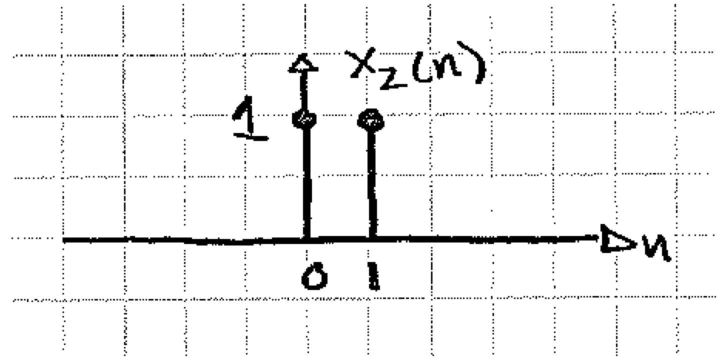


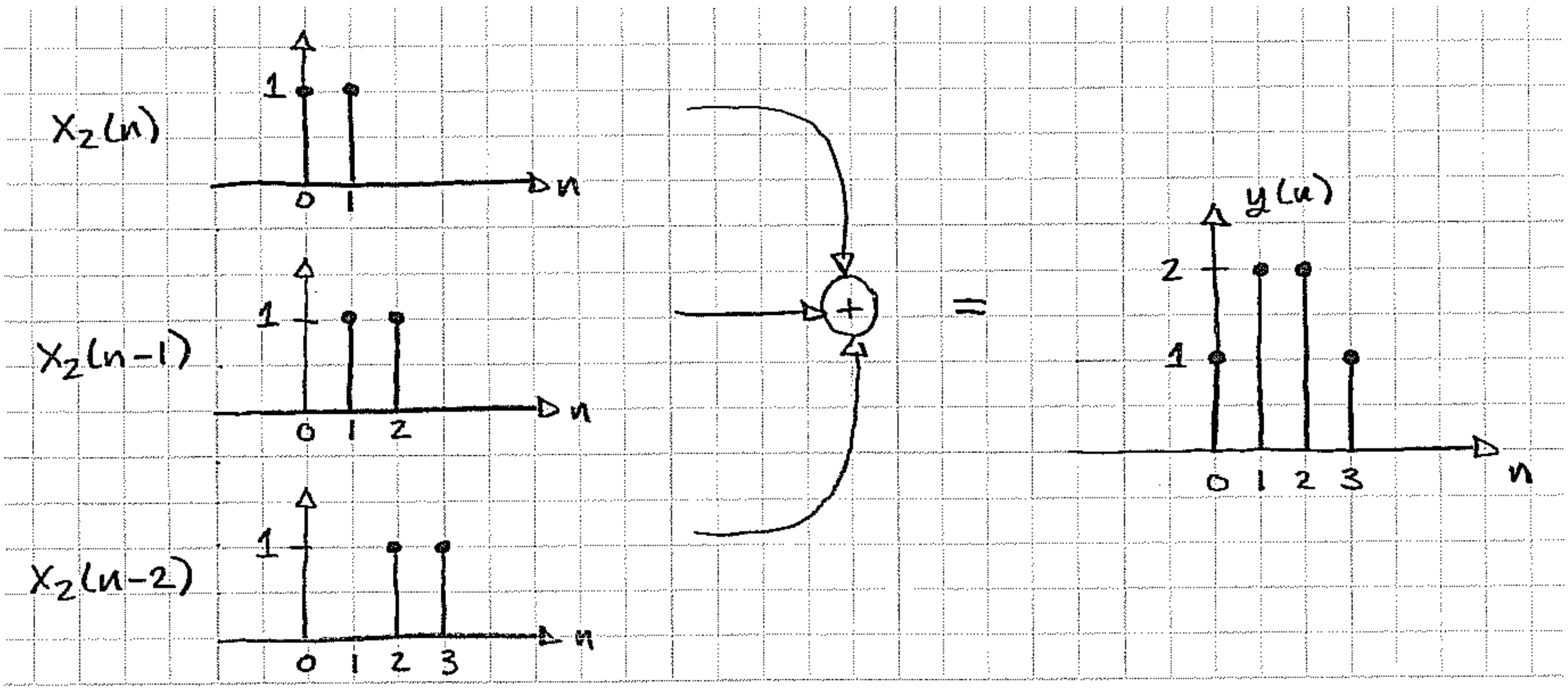


Convolution Example 1

$y(n) = x_1(n) * x_2(n)$

$$= x_2(n) + x_2(n-1) + x_2(n-2)$$

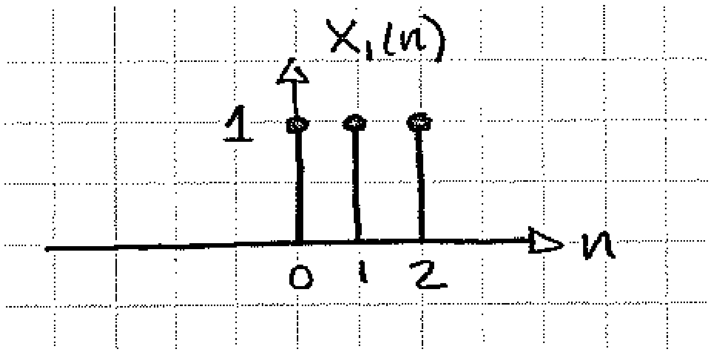




Convolution Example 1

$$y(n) = x_1(n) * x_2(n)$$

$=$



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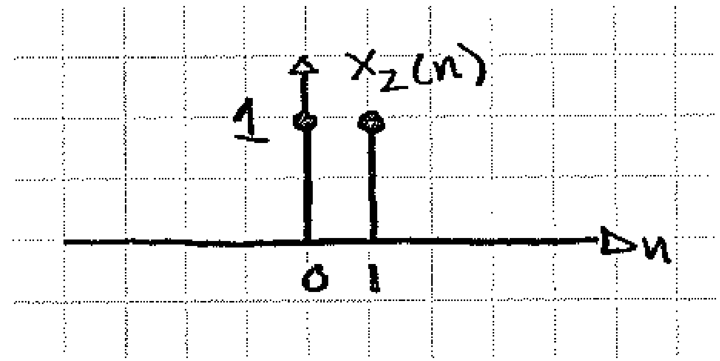

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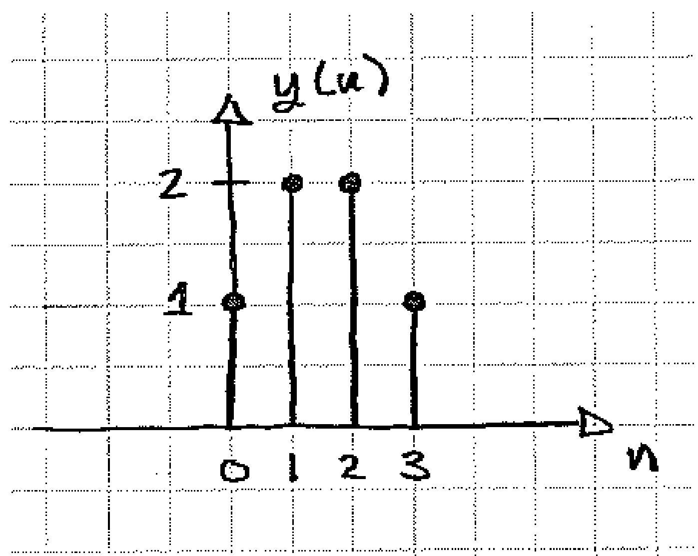
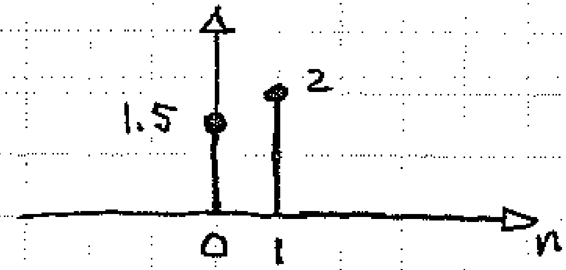
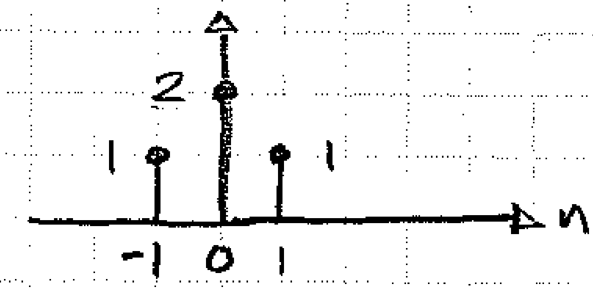


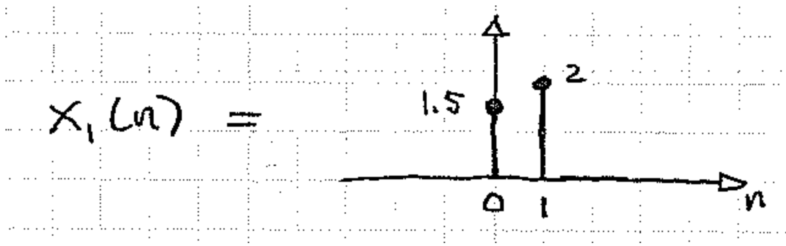
Convolution Example 2

Ex:

$x_1(n) =$ 

$x_2(n) =$ 

Convolution Example 2



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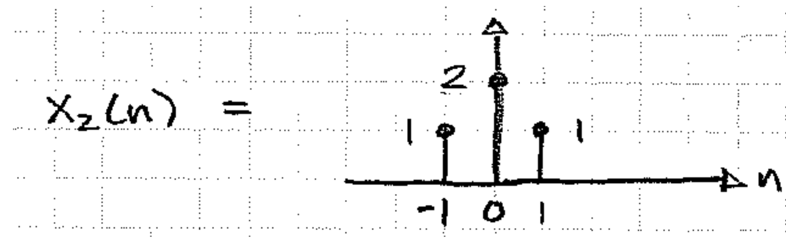
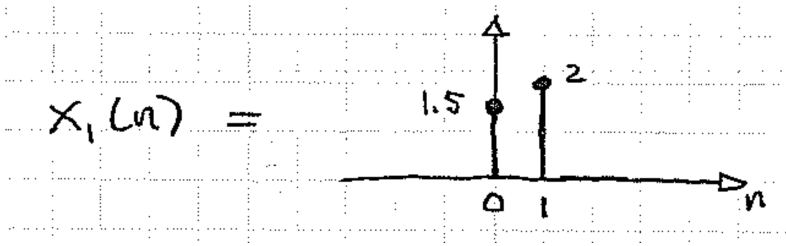


Convolution Example 2



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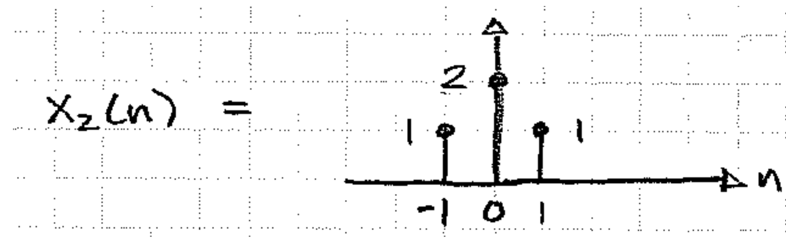
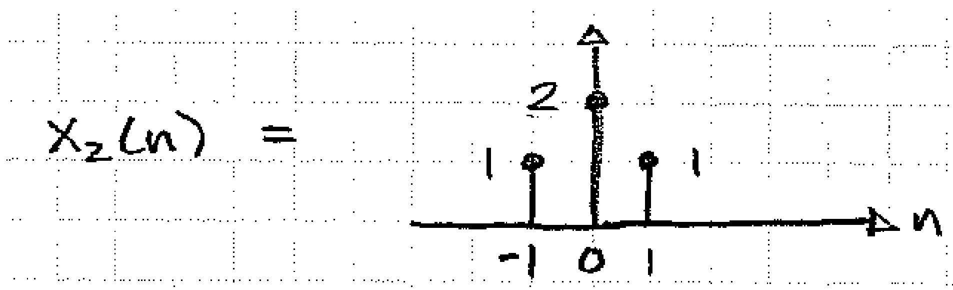
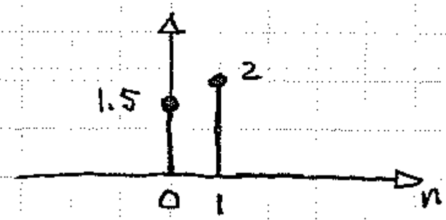


Step 1: Choose one signal to scale/shift



Convolution Example 2

$x_1(n) =$ 

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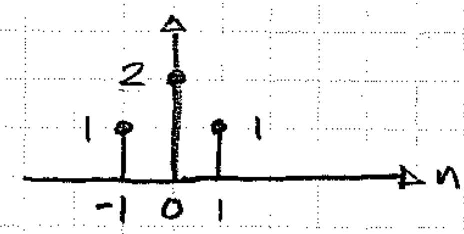
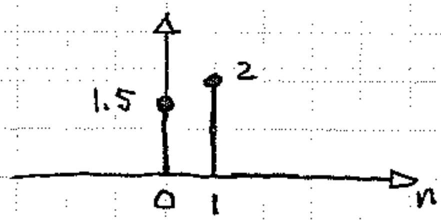
$x_2(n) =$ 

Step 2: Draw scaled/shifted versions

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\alpha_k}_{\text{scaling factor}} \underbrace{x_2(n-k)}_{\text{time-shifted version of } x_2(n)}, \quad \text{where } \alpha_k = x_1(k)$$

Convolution Example 2

$x_1(n) =$ 

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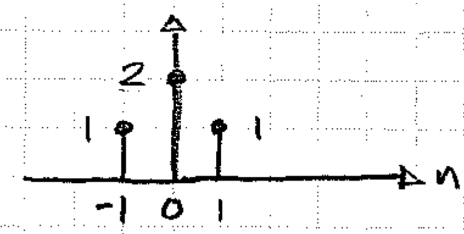
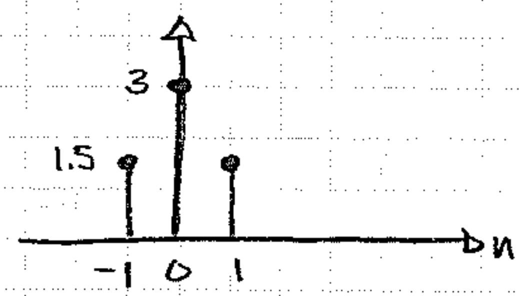
$x_2(n) =$ 

Step 2: Draw scaled/shifted versions

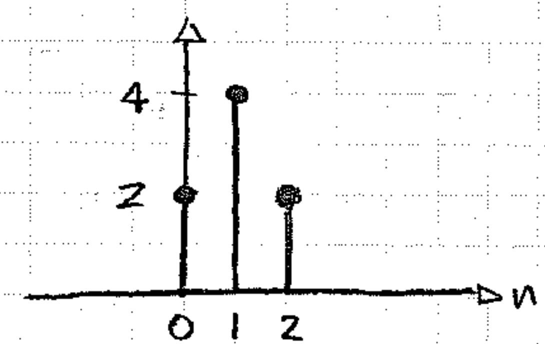
$x_1(0)$

$\downarrow$

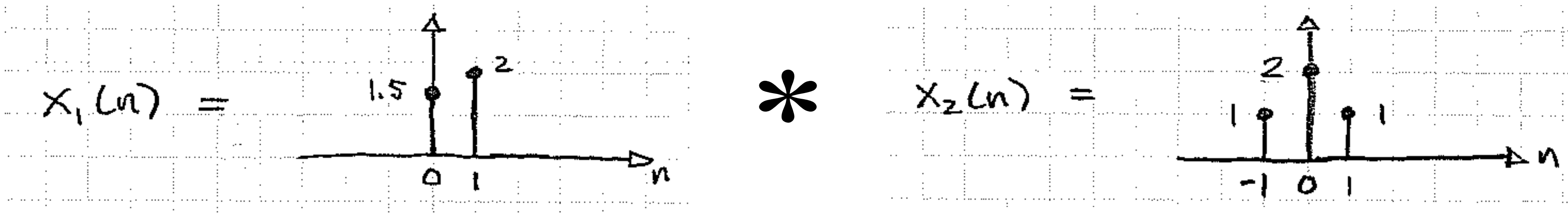
$\alpha_0 x_2(n-0) =$ 

$x_1(1)$

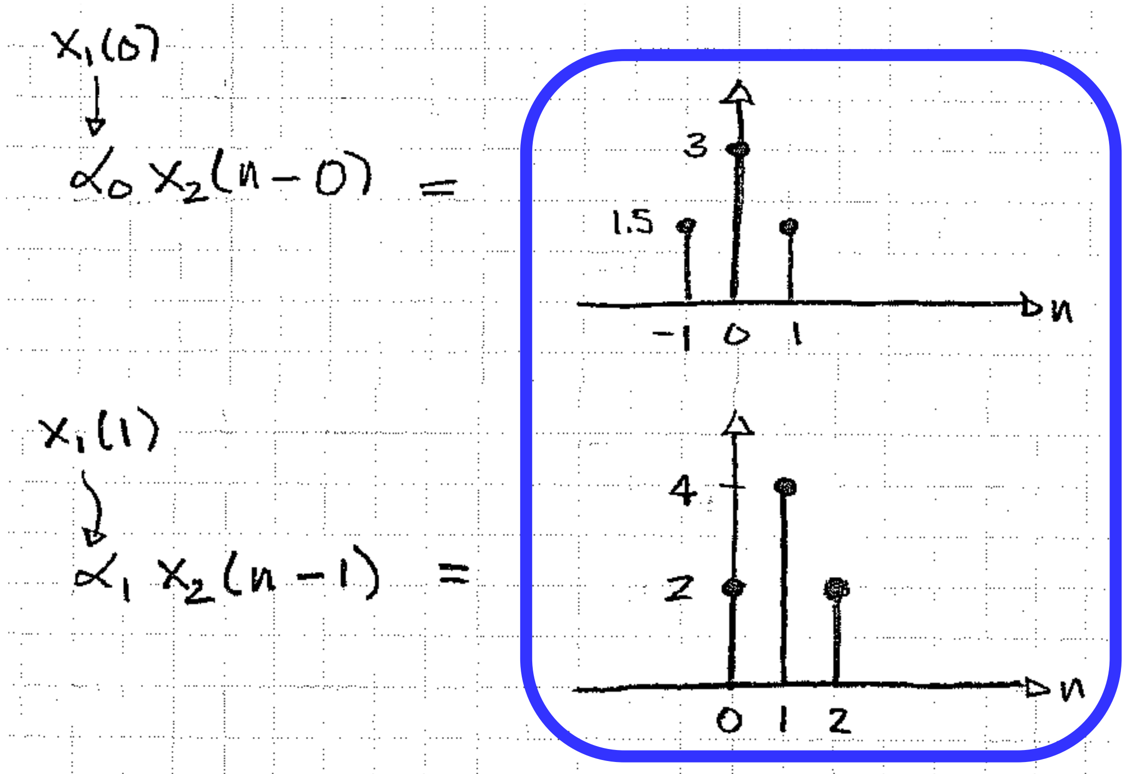
$\downarrow$

$\alpha_1 x_2(n-1) =$ 

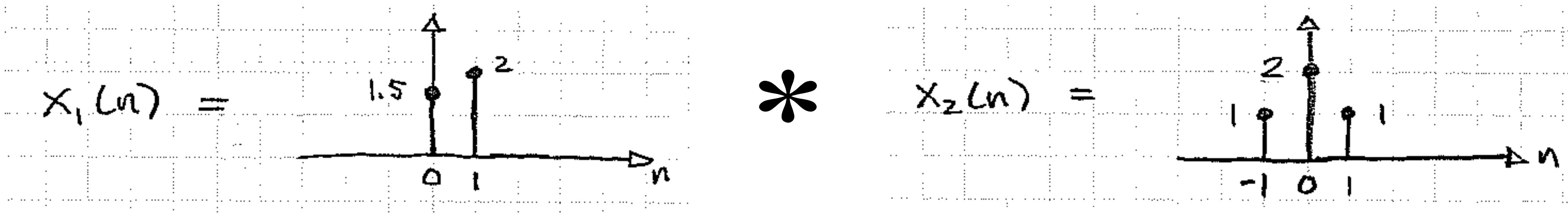
Convolution Example 2



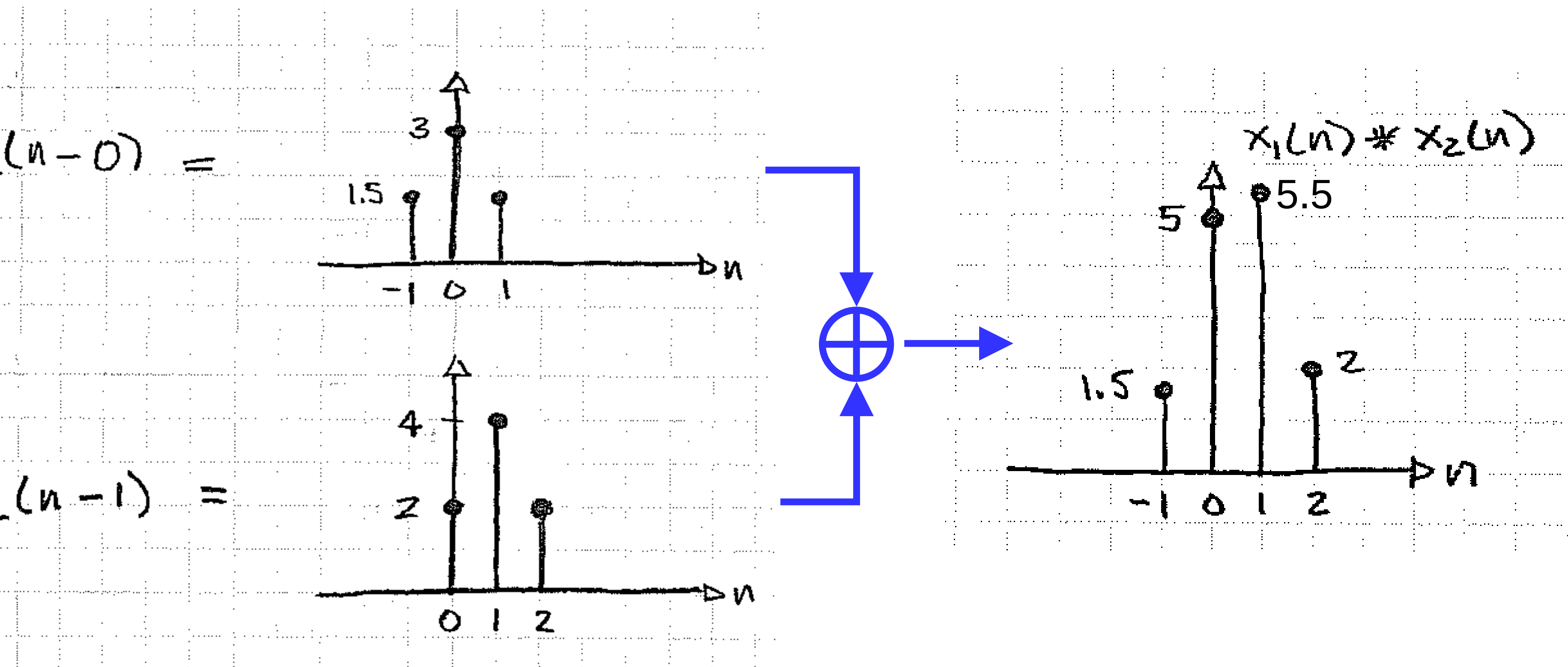
Step 3: Stack and add



Convolution Example 2



Step 3: Stack and add



Convolution Example 3

Ex: $x_1(n) = \left(\frac{3}{4}\right)^n u(n)$

$x_2(n) = u(n)$

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Convolution Example 3

Ex: $x_1(n) = \left(\frac{3}{4}\right)^n u(n)$

$x_2(n) = u(n)$

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{3}{4}\right)^k u(k)\, u(n-k)$$

$$= \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k u(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$$

Convolution Example 3

$$= \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$$

Note: $\displaystyle\sum_{k=a}^{b} c^k = \frac{c^a - c^{b+1}}{1-c}, \quad b \geq a$

$$= \frac{\left(\frac{3}{4}\right)^0 - \left(\frac{3}{4}\right)^{n+1}}{1 - \frac{3}{4}}$$

$$= \frac{1 - \left(\frac{3}{4}\right)^{n+1}}{1/4} = 4 - 3\left(\frac{3}{4}\right)^n$$

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Properties of convolution

① Commutative property

$$x_1(n) * x_2(n) = x_2(n) * x_1(n)$$

② Associative property

$$[x_1(n) * x_2(n)] * x_3(n)$$
$$= x_1(n) * [x_2(n) * x_3(n)]$$

③ Distributive property

$$x_1(n) * [x_2(n) + x_3(n)]$$
$$= x_1(n) * x_2(n) + x_1(n) * x_3(n)$$

Properties of convolution

④ Identity property

$$x(n) * \delta(n) = x(n)$$

⑤ Translation property

$$x(n) * \delta(n - n_0) = x(n - n_0)$$

Today's in-class activity

Given the following two signals, compute $x_1(n) * x_2(n)$: