Digital Signal Processing

Spring Semester 2022

Frequency-Based Analysis, Part 5

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Last time’s learning objectives

- Using Matlab as a fancy calculator
- Defining and plotting signals in Matlab
- Math operations with signals in Matlab
- Computing and visualizing spectra in Matlab

Today's learning objectives

From **today's lecture**, you should **be able to**…

- Explain how the Discrete/Fast Fourier Transform (DFT/FFT) differs from the DTFT
- Compute the DFT/FFT on paper
- Compute the DFT/FFT in Matlab (and interpret the results)

Discrete-time Fourier transform (DTFT)

Transforms a signal from the **time domain** into the **frequency domain** (and vice-versa)

time-domain $\boxed{x(n) \leftrightarrow X(\omega)}$ frequency domain

Forward DTFT: $\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$

Inverse DTFT: $\boxed{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$

What is the DFT?

- DTFT (Discrete-Time Fourier Transform):

$$X(e^{j\omega}) = \sum_n x(n)e^{-j\omega n}$$

- DFT/FFT (Discrete Fourier Transform):

$$X(k) = \sum_n x(n)e^{-j\left(\frac{2\pi k}{N}\right)n}$$

$$\omega = \frac{2\pi k}{N} \quad \text{for } k = 0, 1, 2, \dots, N-1$$

Take-home message: The DFT is a sampled version of the DTFT (i.e., it is the DTFT evaluated only at particular values of ω).

Given a DT signal $x(n)$, we can compute the DFT via either:

On paper

① First compute the DTFT, and then sample in frequency

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

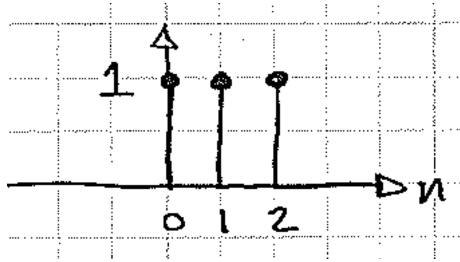
$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k,\; k = 0, 1, \ldots, N-1}$$

On computer

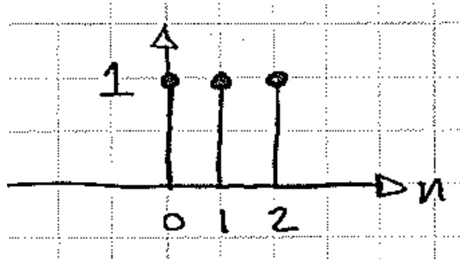
② Using the DFT equation directly

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}k\right)n}, \quad 0 \le k < N$$

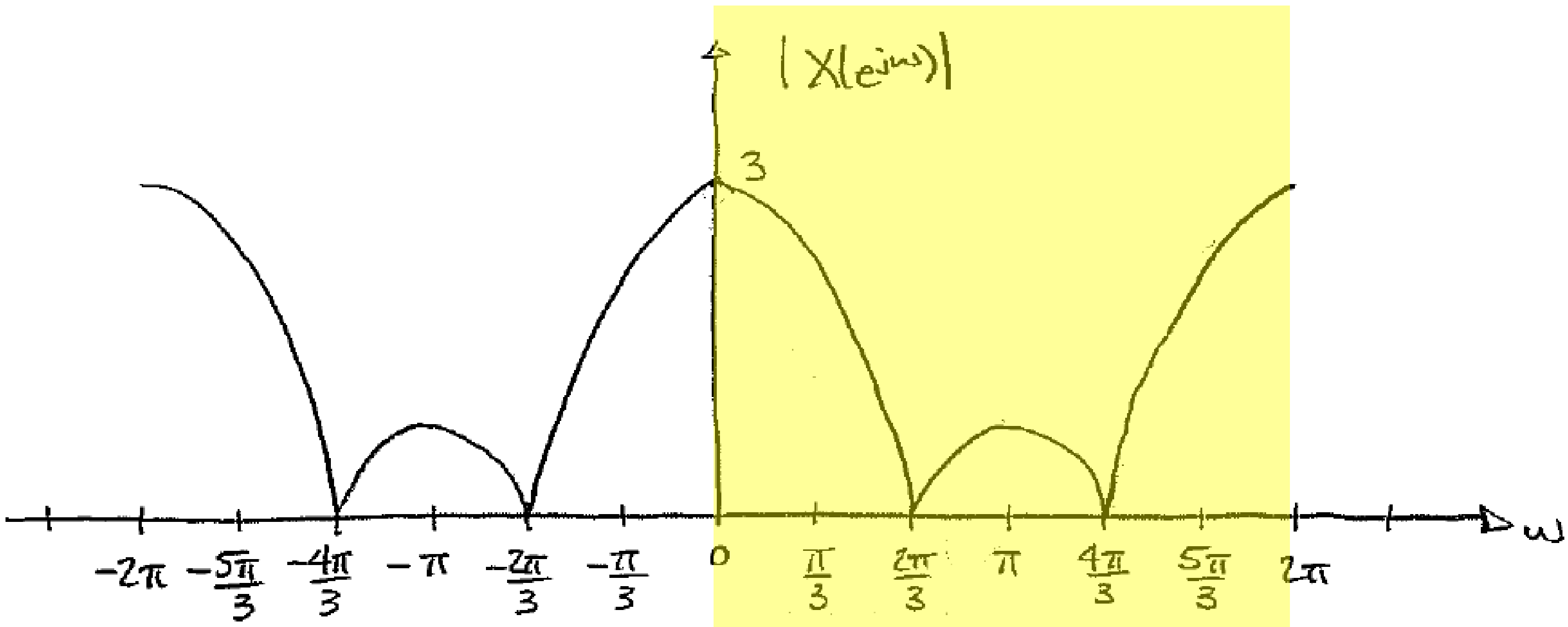
$x(n) =$

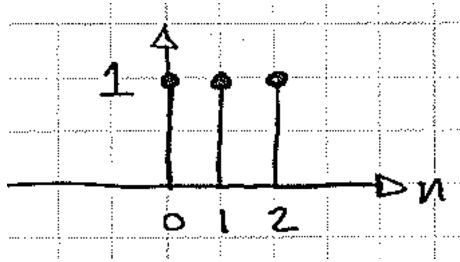


$x(n) =$



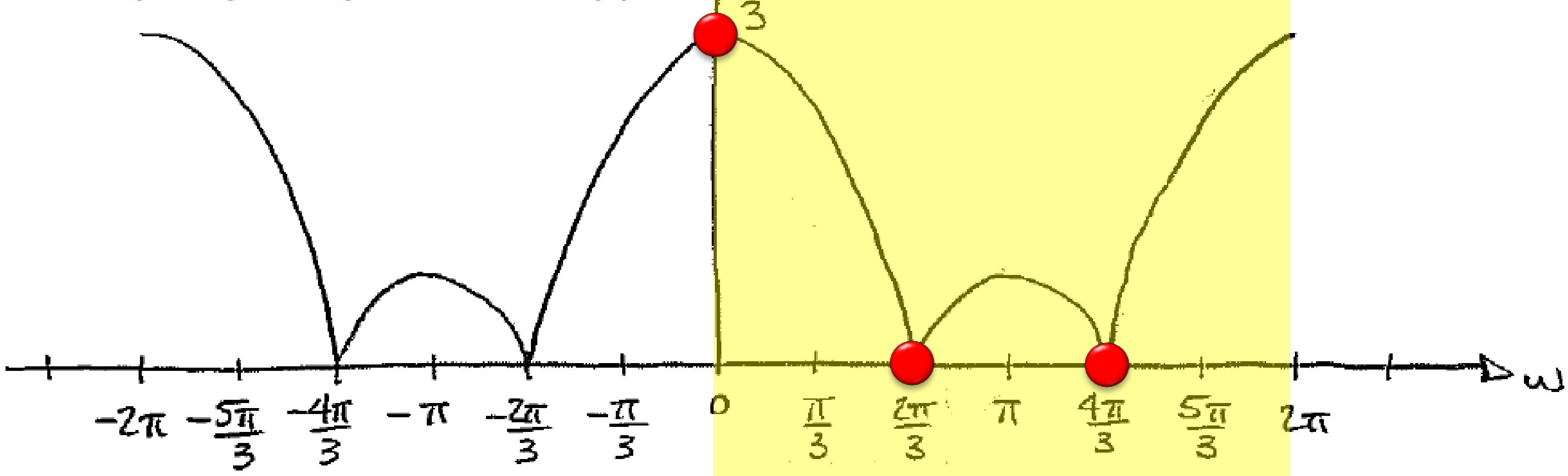
$$X(e^{j\omega}) = e^{-j\omega}\,\frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$



$x(n) =$ 

$$X(e^{j\omega}) = e^{-j\omega} \frac{\sin(\frac{3}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

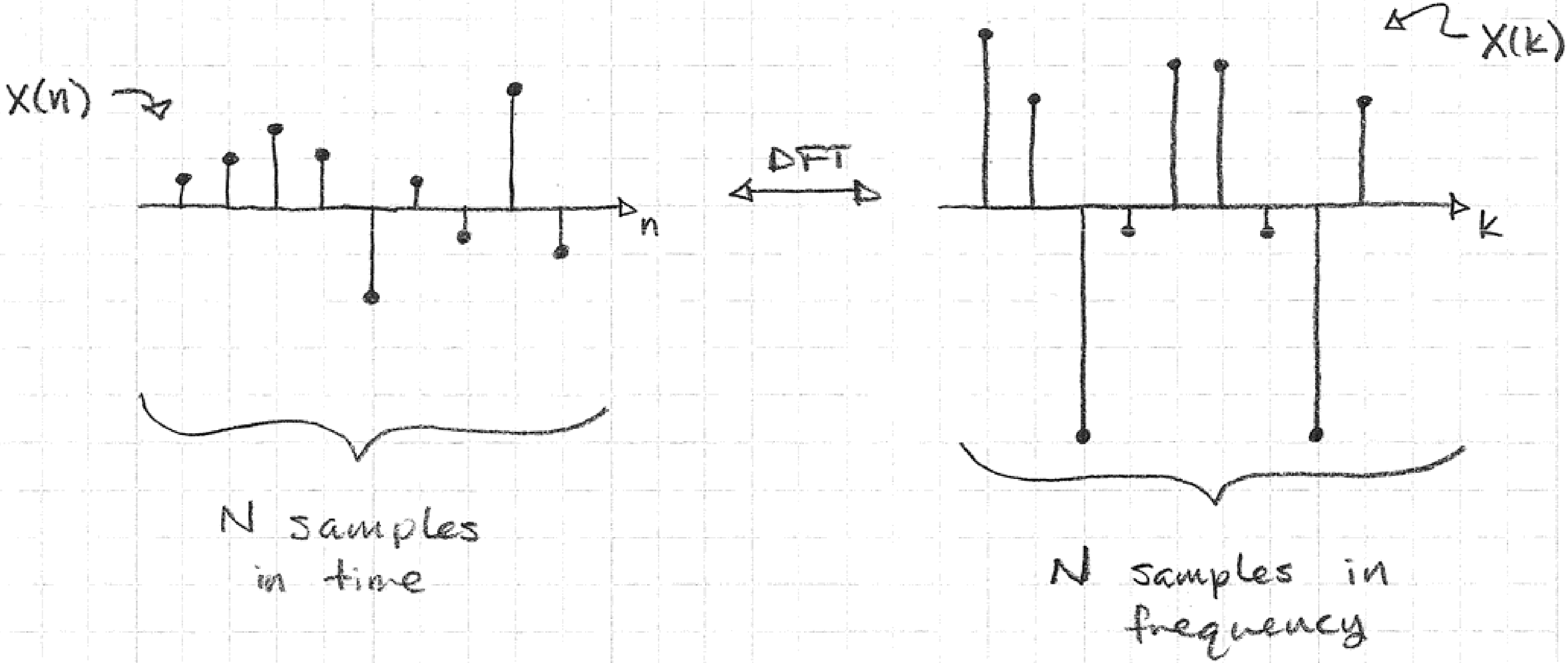
The red dots indicate the samples (of the DTFT) obtained when computing a length-3 DFT of x(n).

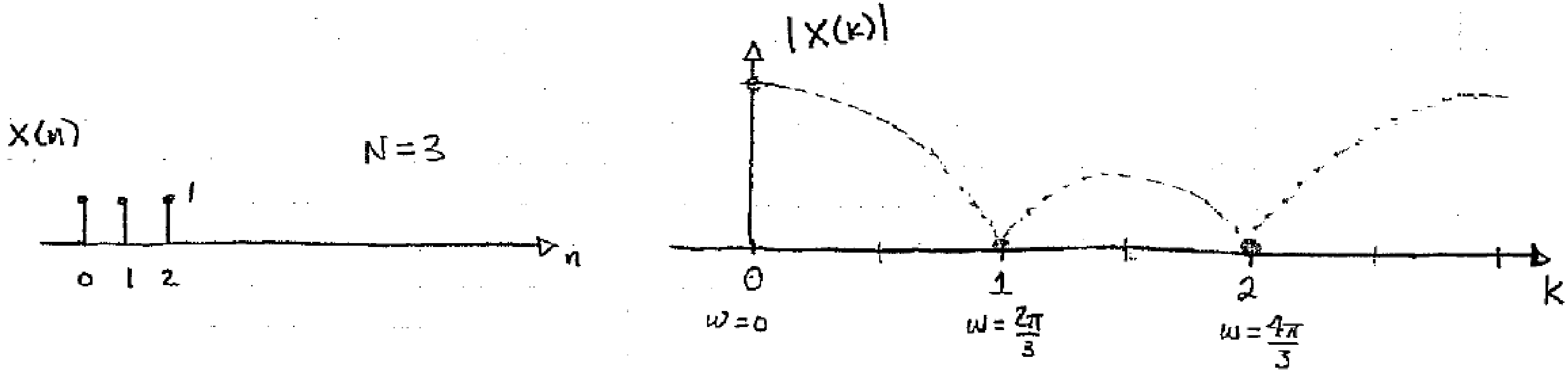


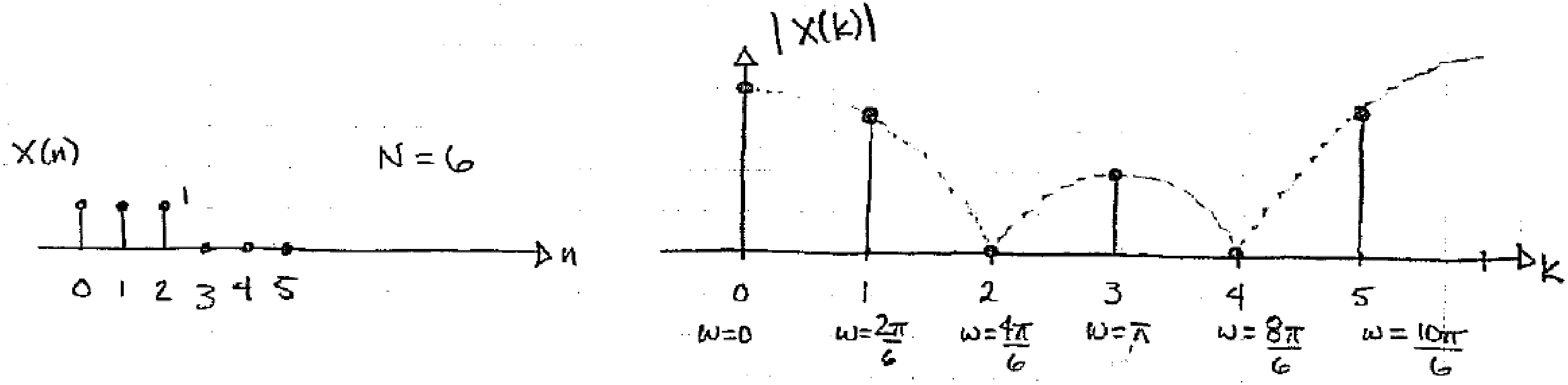
$$\boxed{X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k,\; k = 0, 1, \ldots, N-1}}$$

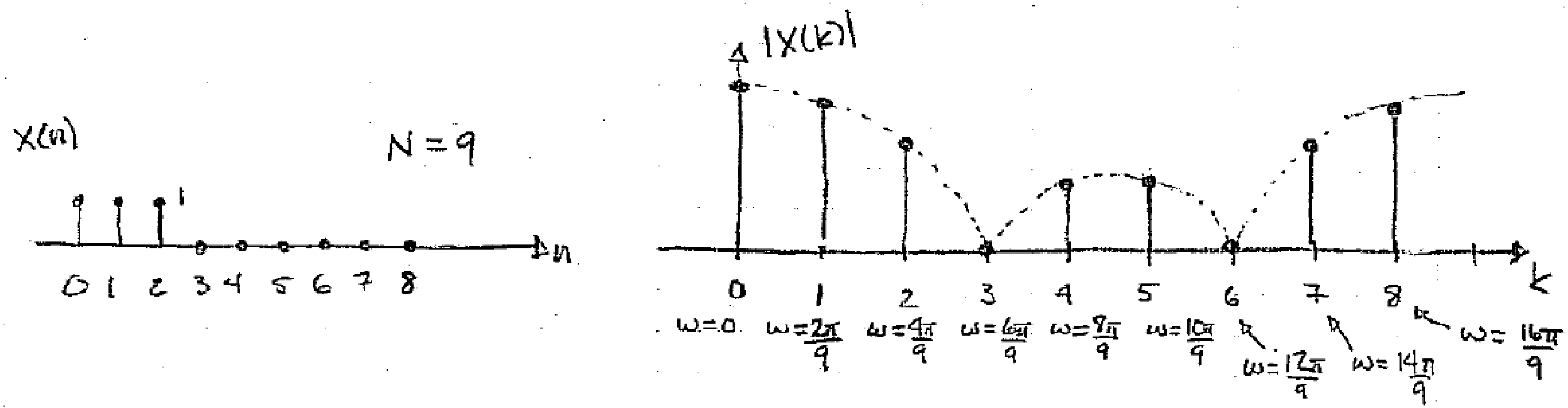
DFT = Discrete Fourier Transform

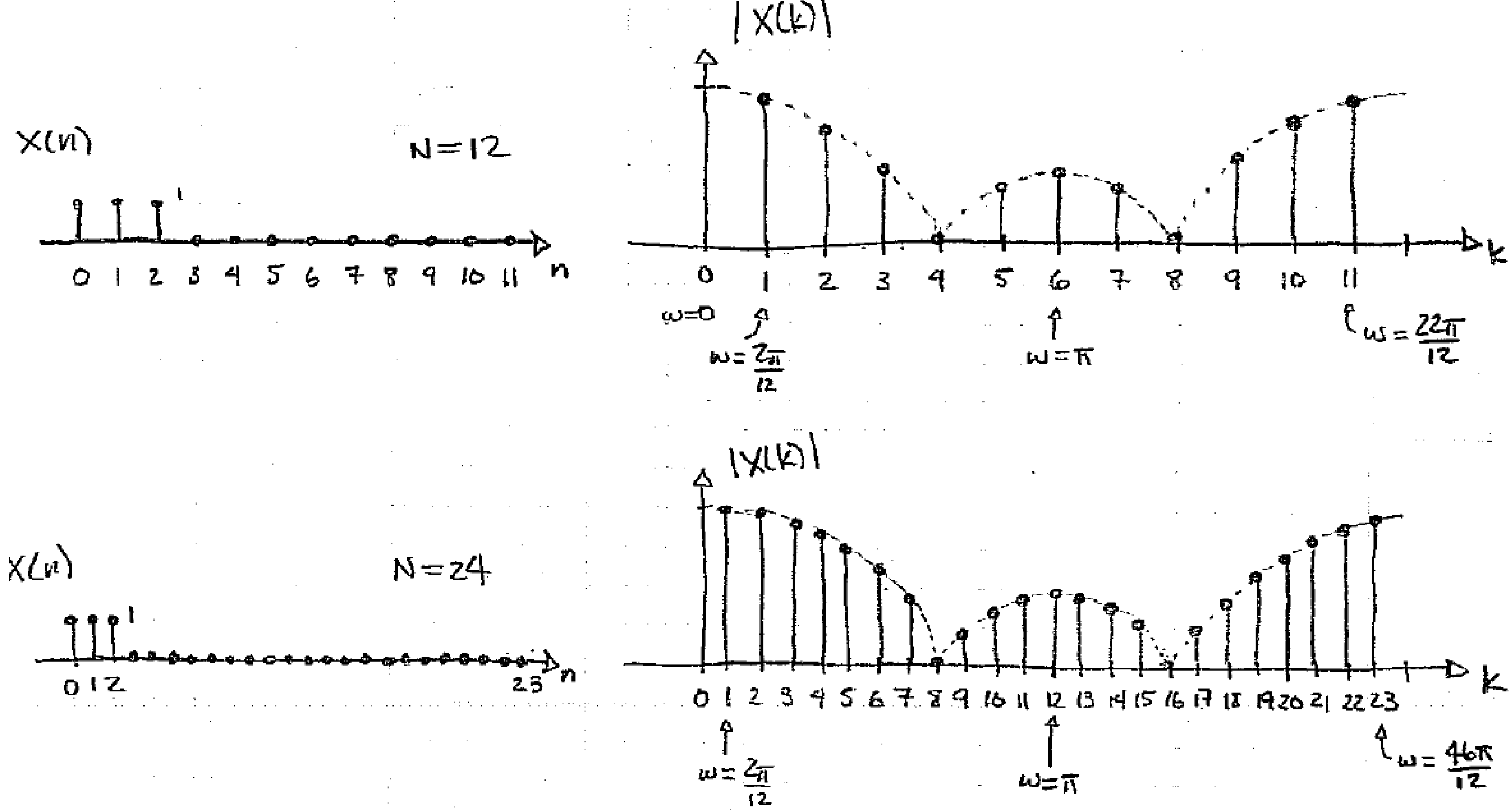
– Transforms a finite-length discrete-time signal into a finite-length discrete frequency representation




x(n)
N=3
1
0 1 2
n
|X(k)|
0
1
2
k
ω=0
ω=2π/3
ω=4π/3


x(n)
N=6
1
0 1 2 3 4 5
n
|X(k)|
0 1 2 3 4 5
k
ω=0
ω=2π/6
ω=4π/6
ω=π
ω=8π/6
ω=10π/6

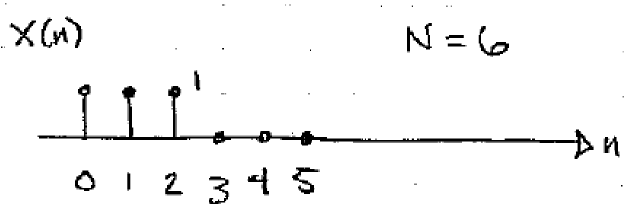

x(n)
N=9
1
0 1 2 3 4 5 6 7 8
n
|X(k)|
0 1 2 3 4 5 6 7 8
k
ω=0
ω=2π/9
ω=4π/9
ω=6π/9
ω=8π/9
ω=10π/9
ω=12π/9
ω=14π/9
ω=16π/9

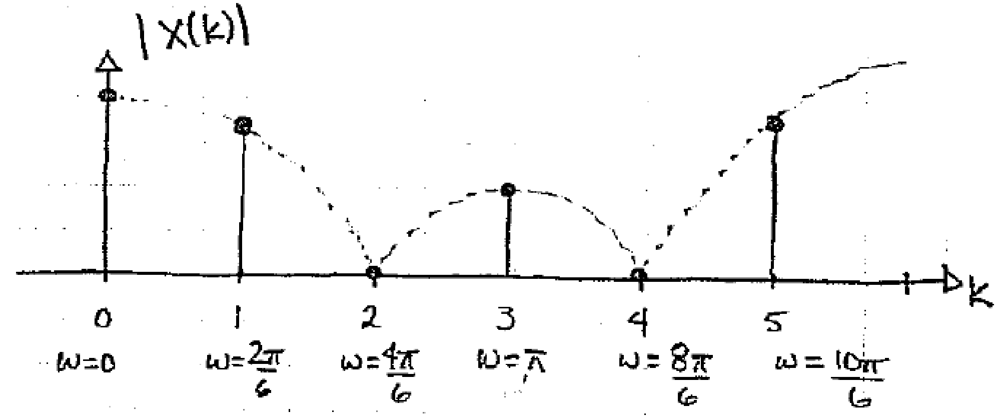


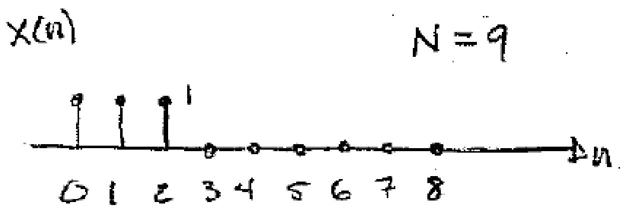
X(n)
N=12
1
0 1 2 3 4 5 6 7 8 9 10 11 n
|X(k)|
0 1 2 3 4 5 6 7 8 9 10 11 k
ω=0
ω=2π/12
ω=π
ω=22π/12
X(n)
N=24
1
0 1 2
23 n
|X(k)|
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 k
ω=2π/12
ω=π
ω=46π/12

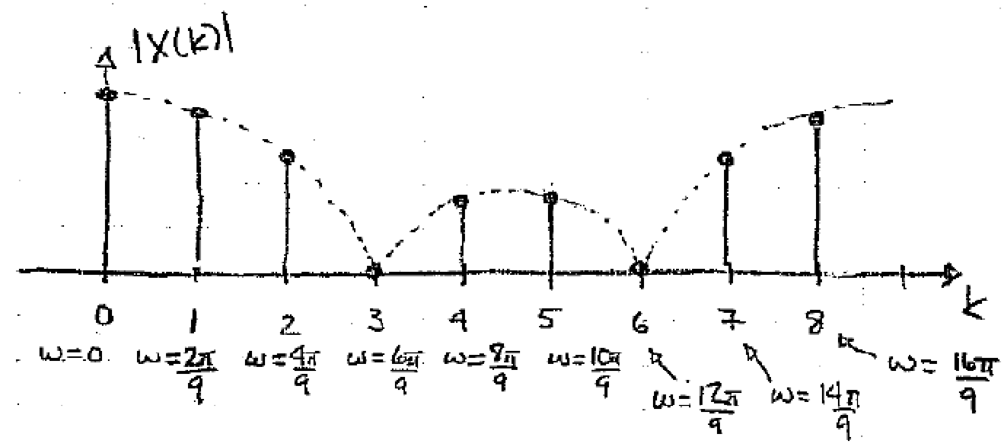
Note that if N is even (e.g., $N=6$, $N=12$) you'll get a DFT sample @ $k = N/2 \rightarrow \omega = \pi$

If N is odd, you get two samples spaced equally about $\omega = \pi$ @ $k = \frac{N-1}{2}$ and $k = \frac{N+1}{2}$









Ex: $x(n) = \delta(n) \quad , \quad 0 \le n < N$

Ex: $x(n) = \delta(n)$, $0 \le n < N$

Approach 1: Sample the DTFT

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

Ex: $x(n) = \delta(n), \quad 0 \le n < N$

Approach 1: Sample the DTFT

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} = 1$$

<u>Ex:</u> $x(n) = \delta(n) \quad , \quad 0 \le n < N$

<u>Approach 2:</u> Use the DFT formula

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}k\right)n}$$

<u>Ex:</u> $x(n) = \delta(n)$, $0 \le n < N$

<u>Approach 2:</u> Use the DFT formula

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi}{N}k\right)n}$$

$$= \sum_{n=0}^{N-1} \delta(n) e^{-j\left(\frac{2\pi}{N}k\right)n}$$

Ex: $x(n) = \delta(n)$, $0 \le n < N$

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$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi}{N}k\right)n}$$

$$= \sum_{n=0}^{N-1} \delta(n) e^{-j\left(\frac{2\pi}{N}k\right)n}$$

$$= e^{-j\left(\frac{2\pi}{N}k\right)0} \quad = 1$$

Ex: $x(n) = \delta(n - n_0)$, $0 \le n < N$, $n_0 < N$

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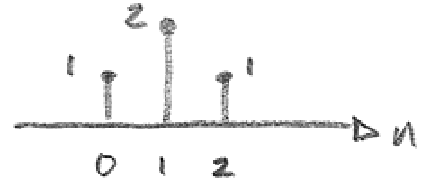
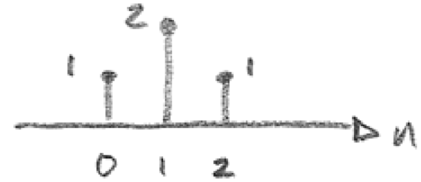
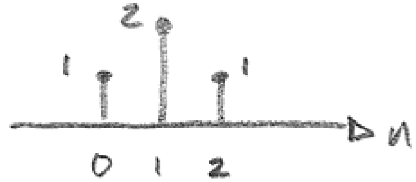
Ex: $x(n) =$ 

<u>Ex:</u> $x(n) =$ 

<u>Approach 1:</u> Sample the DTFT

$$x(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$$

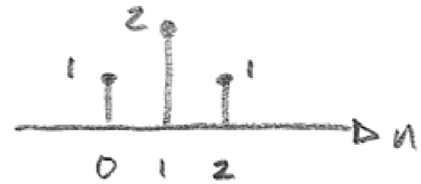
Ex: $x(n) =$



Approach 1: Sample the DTFT

$$x(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$$

$$X(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

Ex: $x(n) =$ 

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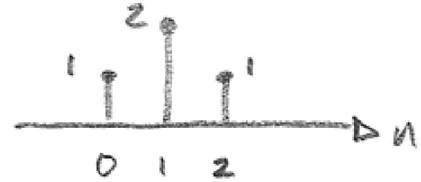
Ex: $x(n) =$ 

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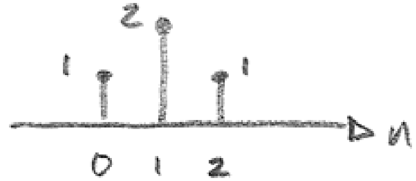
$$= e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= e^{-j\omega}\left(2 + 2\cos(\omega)\right)$$

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{3}k}$$

Ex: $x(n) =$ 

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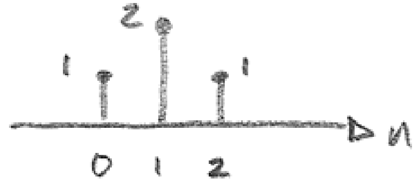
$$= e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= e^{-j\omega}\left(2 + 2\cos(\omega)\right)$$

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{3}k}$$

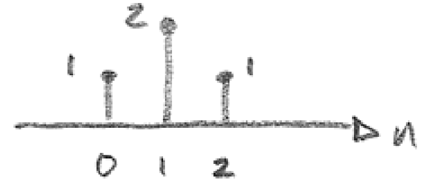
$$= e^{-j\frac{2\pi}{3}k}\left(2 + 2\cos\left(\tfrac{2\pi}{3}k\right)\right)$$

Ex: $x(n) =$



Approach 2: Use the DFT equation

$$X(k) = \sum_{n=0}^{2} x(n)\, e^{-j\left(\frac{2\pi}{3}k\right)n}$$

Ex: $x(n) =$ 

Approach 2: Use the DFT equation

$$X(k) = \sum_{n=0}^{2} x(n)\, e^{-j\left(\frac{2\pi}{3}k\right)n}$$

$$= e^{-j\left(\frac{2\pi}{3}k\right)0} + 2e^{-j\left(\frac{2\pi}{3}k\right)1} + e^{-j\left(\frac{2\pi}{3}k\right)2}$$

$$= 1 + 2e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k}$$

$$= e^{-j\frac{2\pi}{3}k}\left(e^{j\frac{2\pi}{3}k} + 2 + e^{-j\frac{2\pi}{3}k}\right)$$

$$= e^{-j\frac{2\pi}{3}k}\left(2 + 2\cos\left(\frac{2\pi}{3}k\right)\right)$$