Digital Signal Processing

Spring Semester 2022

Digital Systems, Part 2

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Last time's learning objectives

- Explain the terms "digital system" and "digital filter"
  - **Digital system**: Takes as input a digital signal and does something with it (processing) to yield another signal
  - **Digital filter**: Type of digital system that "shapes" a signal to achieve a desired effect
- List the ways to characterize a filter
  - Via an equation for computing output given input:
    - **Difference equation**: $y(n) = \text{some function of } x(n)$
  - Via how filter responds to a "ping" (a single pulse):
    - **Impulse response**: $h(n)$
  - Via how filter responds to "tones" (single-frequency sines/cosines)
    - **Frequency response**: $H\left(e^{j\omega}\right)$

Today's learning objectives

From **today's lecture**, you should **be able to**…

- Use convolution to filter a signal
- Characterize a filter in terms of its frequency response

How do you process a signal?

- How is a filter characterized?
  - Two ways:
    1. By examining how the filter changes an impulse
       - Called the filter's "***impulse response***"
    2. By examining how the filter changes sines/cosines of various frequencies
       - Called the filter's "***frequency response***"
- How is a filter applied?
  - Two ways:
    1. Convolution in the time domain
    2. Multiplication in the frequency domain

Impulse response of LTI systems

LTI systems (filters) can be characterized by how they change an impulse:

$$x(n) = \delta(n) \longrightarrow \boxed{\text{LTI system}} \longrightarrow y(n) = h(n)$$

Impulse input

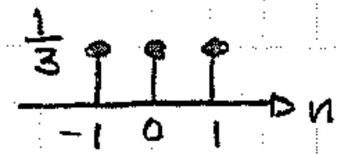
response to impulse

We call $h(n)$ the "impulse response" of the system.

Impulse response of LTI systems

Ex: Moving-average filter (length-3)

$$h(n) = \frac{1}{3}\delta(n+1) + \frac{1}{3}\delta(n) + \frac{1}{3}\delta(n-1) =$$



Earlier, $y(n) = \frac{1}{3}\big(x(n+1) + x(n) + x(n-1)\big)$

Impulse response of LTI systems

Ex: Moving-average filter (length-3)

$$h(n) = \frac{1}{3}\delta(n+1) + \frac{1}{3}\delta(n) + \frac{1}{3}\delta(n-1) \quad =$$

Earlier, $y(n) = \frac{1}{3}\Big(x(n+1) + x(n) + x(n-1) \Big)$

Here, $y(n) = x(n) * h(n)$

$$= x(n) * \frac{1}{3}\Big(\delta(n+1) + \delta(n) + \delta(n-1) \Big)$$

$$= \frac{1}{3}\Big(x(n+1) + x(n) + x(n-1) \Big)$$

Impulse response of LTI systems

Notation:

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

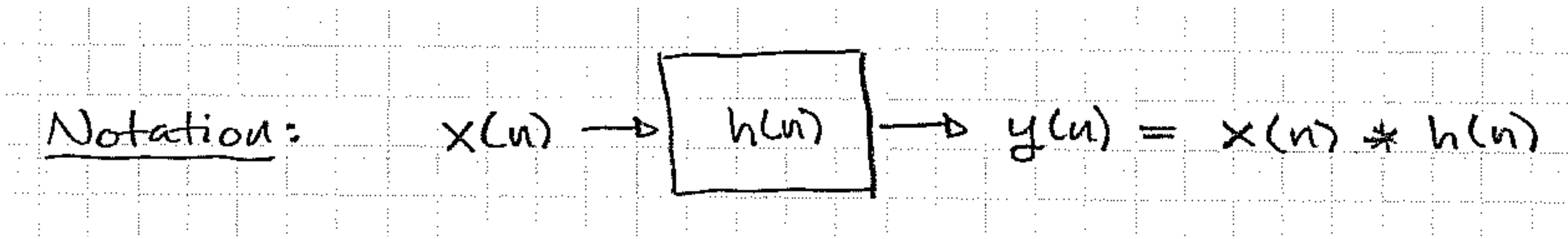
Other variants:

$$x(n) \longrightarrow \boxed{h} \longrightarrow y(n)$$

$$x(n) \longrightarrow \boxed{H} \longrightarrow y(n)$$

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow y(n)$$

Impulse response of LTI systems



In Matlab:

- Use the **conv** function

```
y = conv(x, h);
```

- Use the **filter** function

```
y = filter(h, 1, x);
```

In Python:

- Use the **convolve** function from the **numpy** library

```python
from numpy import convolve
y = convolve(x, h)
```

- Use the **lfilter** function from the **scipy.signal** library

```python
from scipy.signal import lfilter
y = lfilter(b=h, a=1, x=x)
```

How do you process a signal?

- How is a filter characterized?
  - Two ways:
    1. By examining how the filter changes an impulse ✓
       - Called the filter's "***impulse response***"
    2. By examining how the filter changes sines/cosines of various frequencies
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- How is a filter applied?
  - Two ways:
    1. Convolution in the time domain ✓
    2. Multiplication in the frequency domain

After-class Activity

$$y(n) = 0.3x(n) + 0.5x(n-1) + 0.3x(n-2)$$

- What's the filter's impulse response?

  $h(n) = 0.3\delta(n) + 0.5\delta(n-1) + 0.3\delta(n-2)$

- Is the filter memoryless?

  No, requires storing past samples $x(n-1)$ and $x(n-2)$

- Is the filter causal?

  Yes, does not require future samples

- What type of filter is it?

  Lowpass filter

- Does the filter induce phase distortion?

  No, it is a linear-phase filter (because $h(n)$ is symmetric)

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Frequency Response

1. What is frequency response (and how to compute it)?
2. Example: Frequency response of a moving-average filter

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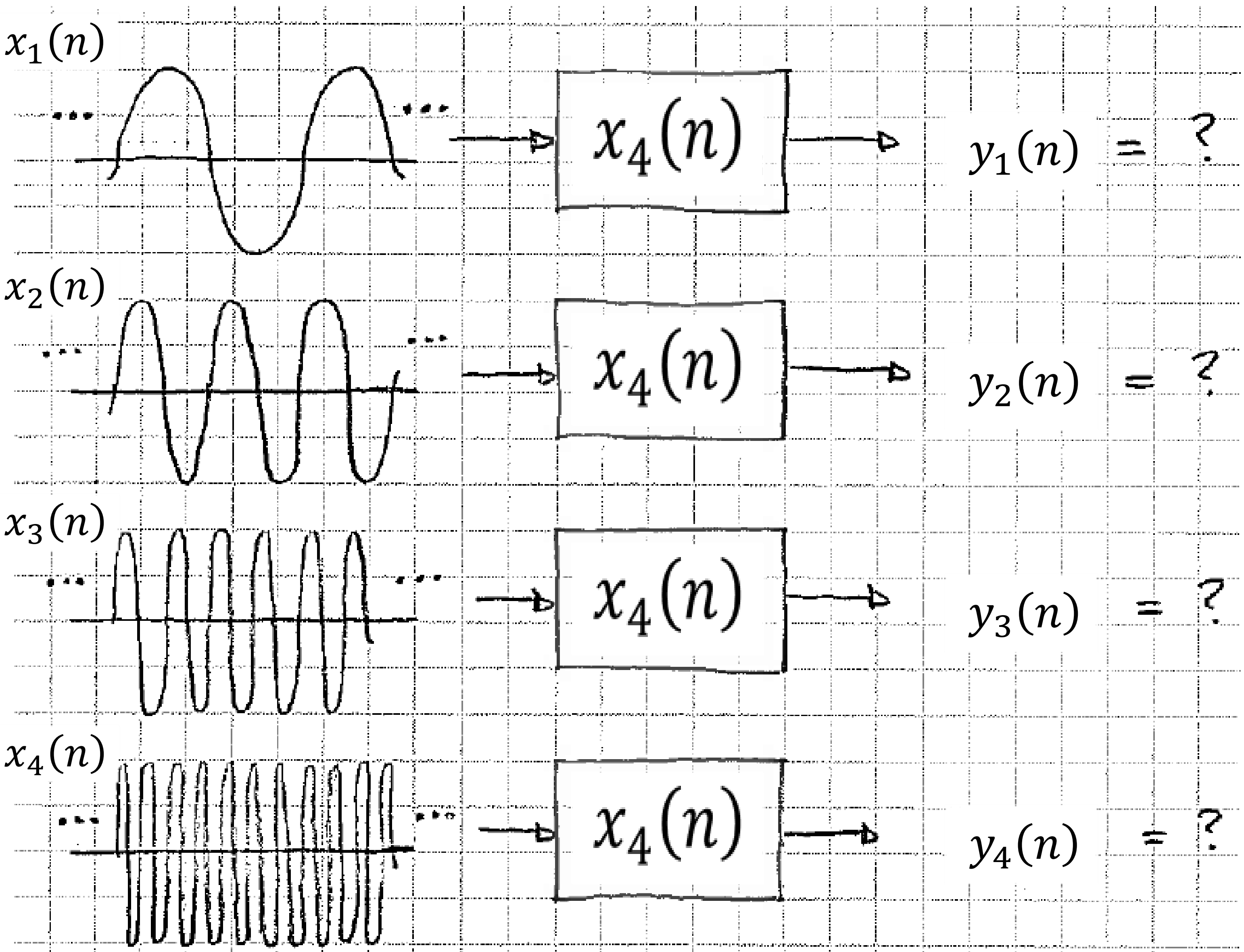
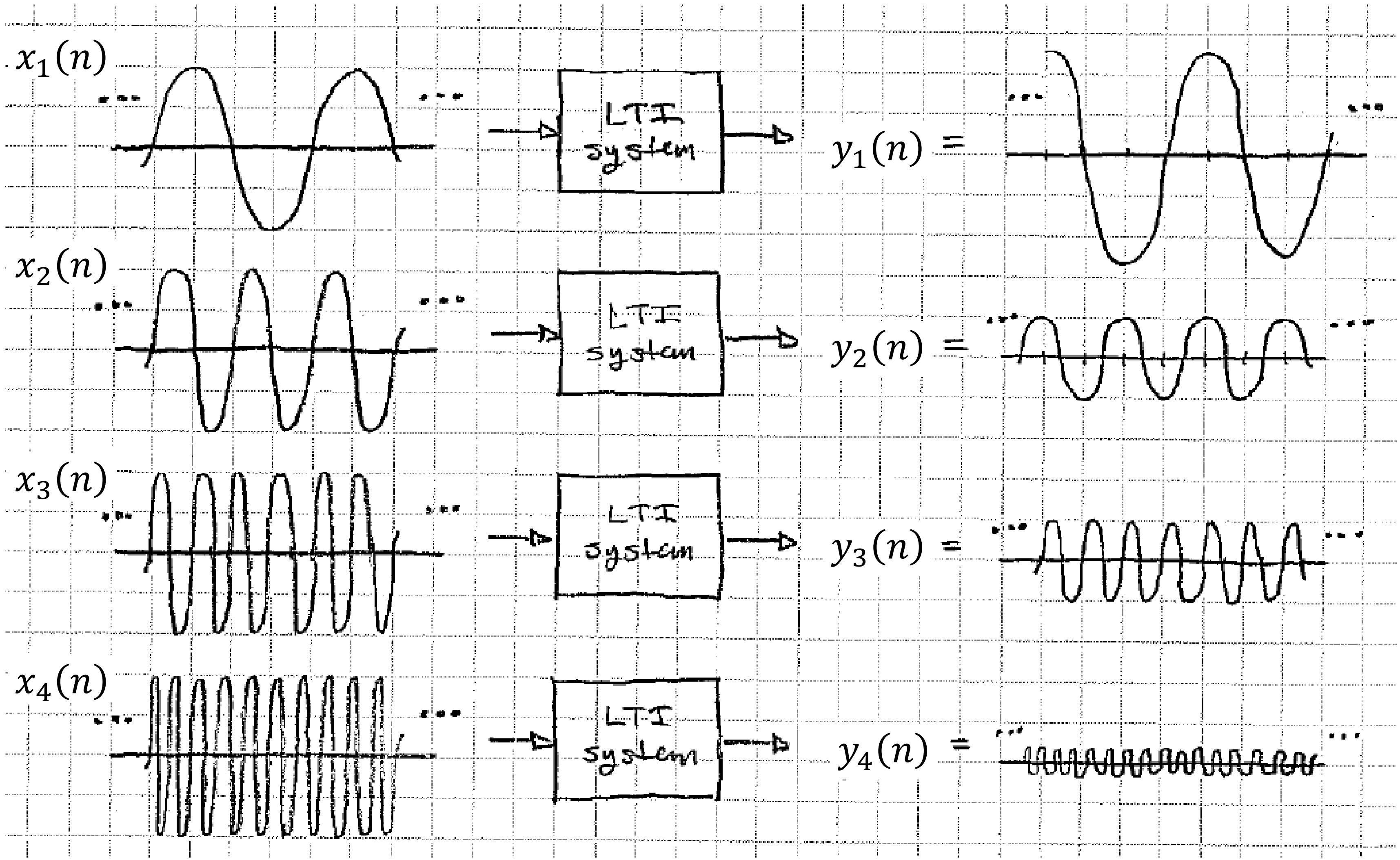
Frequency response



Frequency response



Frequency response of LTI DT systems

DT complex exponentials are eigenfunctions of DT LTI systems

$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \begin{pmatrix}\text{magnitude}\\ \text{scaling}\end{pmatrix} e^{j\left(\omega_0 n + \substack{\text{phase}\\ \text{shift}}\right)}$$

$$e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

$$\cos(\omega_0 n) = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$$

$$\sin(\omega_0 n) = \frac{1}{2j}e^{j\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 n}$$

Frequency response of LTI DT systems

DT complex exponentials are eigenfunctions of DT LTI systems

$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \begin{pmatrix}\text{magnitude} \\ \text{scaling}\end{pmatrix} e^{j\left(\omega_0 n + \substack{\text{phase} \\ \text{shift}}\right)}$$

Sinusoid in, sinusoid out

Frequency response of LTI DT systems

DT complex exponentials are eigenfunctions of DT LTI systems

$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \left(\begin{matrix}\text{magnitude}\\ \text{scaling}\end{matrix}\right) e^{j\left(\omega_0 n + \begin{matrix}\text{phase}\\ \text{shift}\end{matrix}\right)}$$

Frequency response of LTI DT systems

DT complex exponentials are eigenfunctions of DT LTI systems

$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \left(\begin{array}{c}\text{magnitude}\\ \text{scaling}\end{array}\right) e^{j(\omega_0 n + \text{phase shift})}$$

$$h(n) \xleftrightarrow{\text{DTFT}} H(e^{j\omega})$$

frequency response

Frequency response of LTI DT systems

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frequency response

$$\underbrace{|H(e^{j\omega_0})|}_{\text{magnitude scaling}} \qquad \underbrace{\measuredangle H(e^{j\omega_0})}_{\text{phase shift}}$$

Frequency response of LTI DT systems

DT complex exponentials are eigenfunctions of DT LTI systems

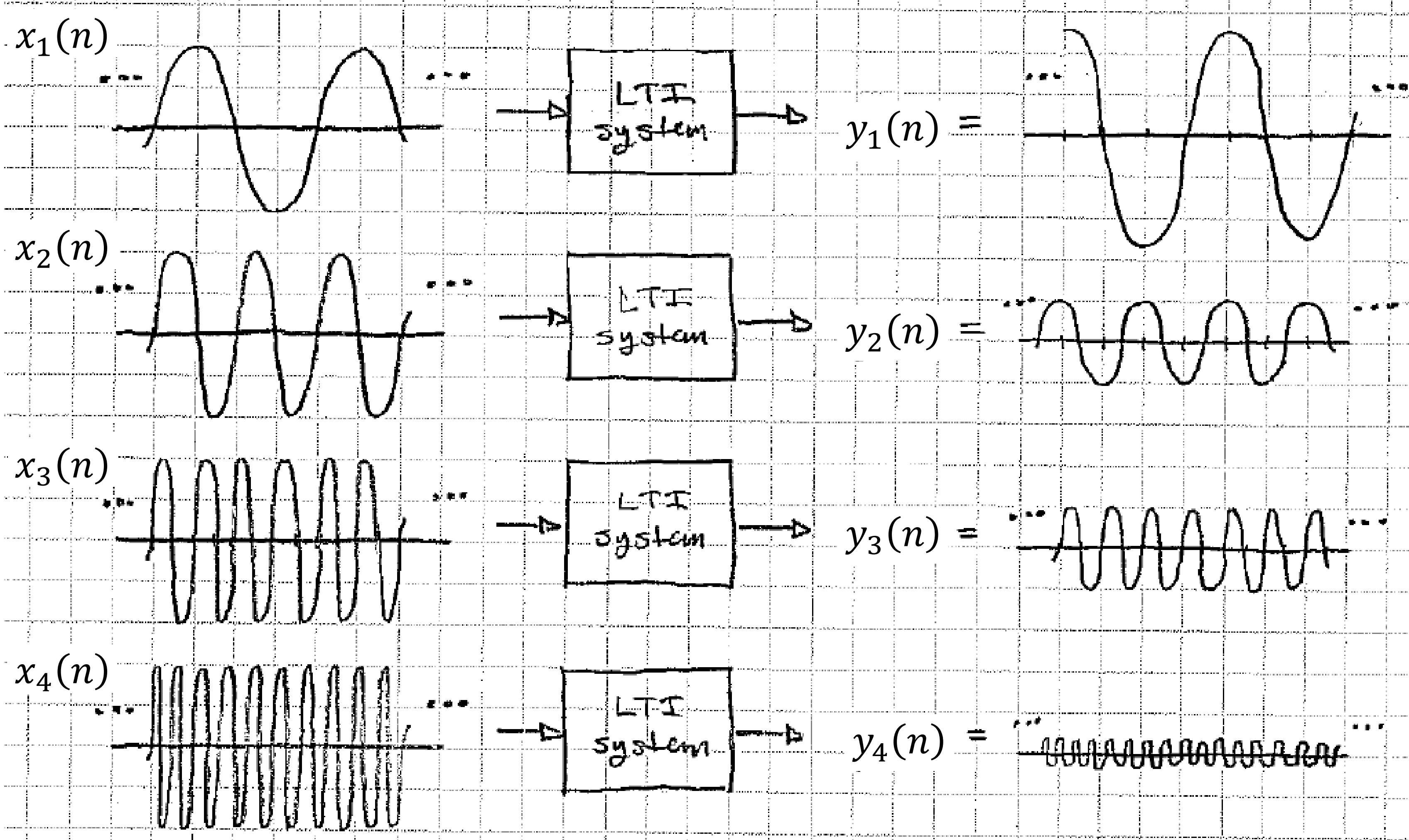
$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = \left(\text{magnitude scaling}\right) e^{j(\omega_0 n + \text{phase shift})}$$

$$y(n) = H(e^{j\omega_0})\, e^{j\omega_0 n}$$

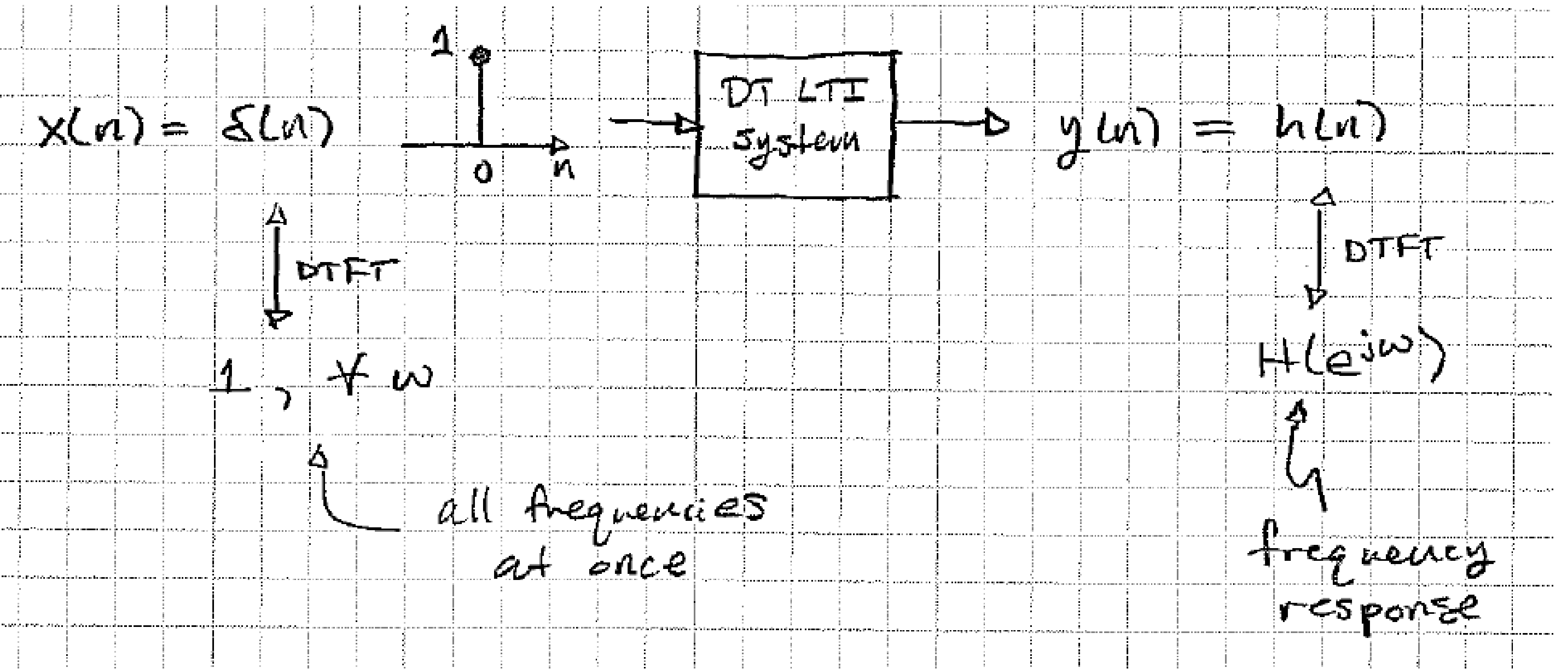
$$= \underbrace{\left| H(e^{j\omega_0}) \right|}_{\text{magnitude scaling}} e^{j(\omega_0 n + \underbrace{\angle H(e^{j\omega_0})}_{\text{phase shift}})}$$

Frequency response

So, instead of probing your system like this...



Frequency response of DT systems



Frequency response of DT systems

Take-home message: Given impulse response $h(n)$, we can compute frequency response $H(e^{j\omega})$ by computing the DTFT of $h(n)$.

Magnitude: $|H(e^{j\omega})|$ tells you by how much the system changes the height of $e^{j\omega n}$

Phase: $\measuredangle H(e^{j\omega})$ tells you by how much the system shifts the phase of $e^{j\omega n}$

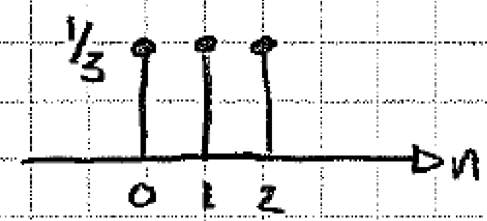
Frequency Response

1. What is frequency response (and how to compute it)?
2. Example: Frequency response of a moving-average filter

Ex: Length-3 MA filter

Ex: Length-3 moving-average filter

$$h(n) = \frac{1}{3}\,\mathrm{rect}_3(n)$$



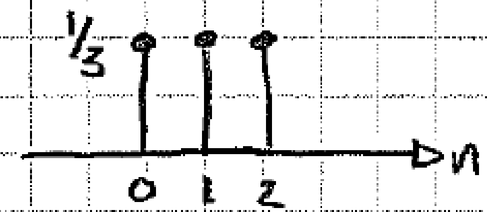
Given input $x_1(n) = e^{j0n}$, $y_1(n) = ?$

$x_2(n) = e^{j\frac{\pi}{3}n}$, $y_2(n) = ?$

$x_3(n) = e^{j\frac{2\pi}{3}n}$, $y_3(n) = ?$

Ex: Length-3 moving-average filter

$$h(n) = \frac{1}{3}\,\mathrm{rect}_3(n)$$



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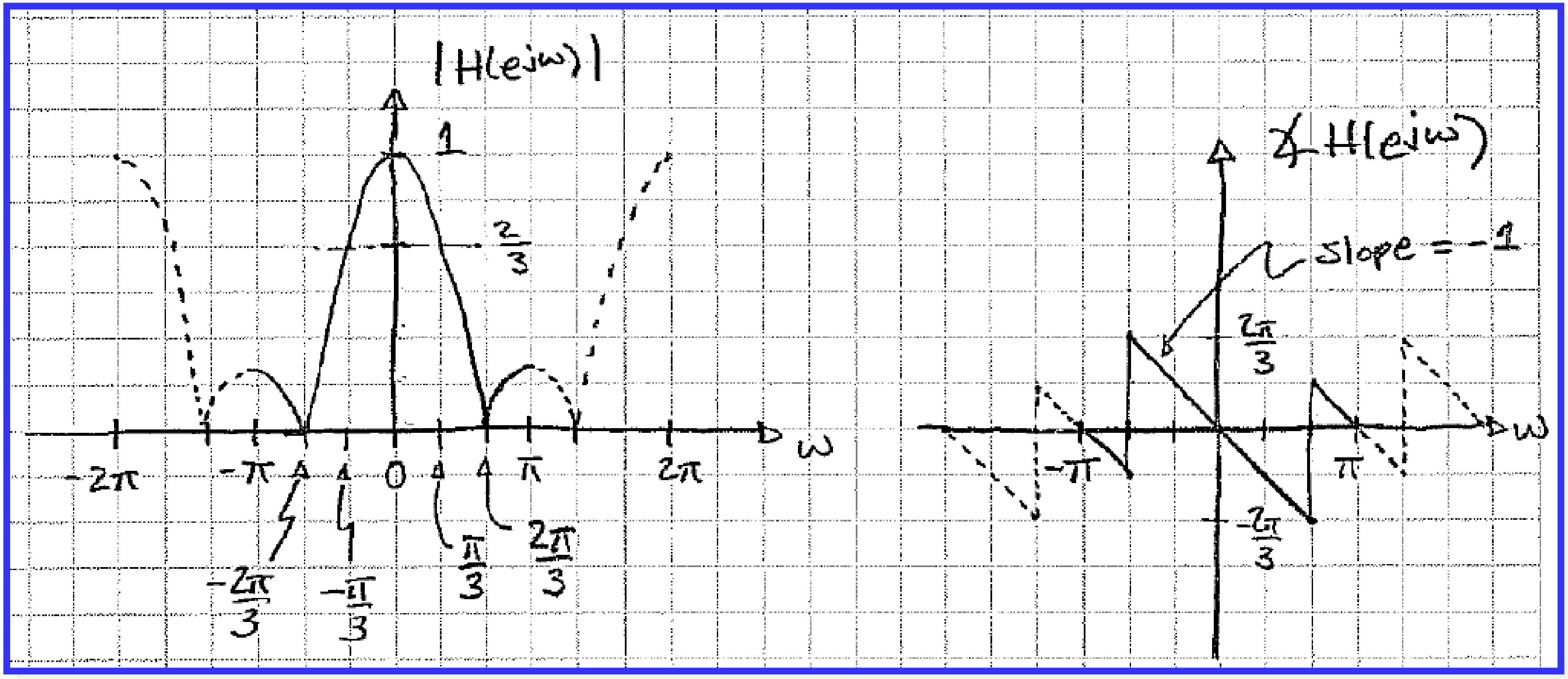
$x_2(n) = e^{j\frac{\pi}{3}n}$, $y_2(n) = ?$

$x_3(n) = e^{j\frac{2\pi}{3}n}$, $y_3(n) = ?$

$$\boxed{H(e^{j\omega}) = \frac{1}{3}e^{-j\omega}\,\frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}}$$

Example: Length-3 MA filter



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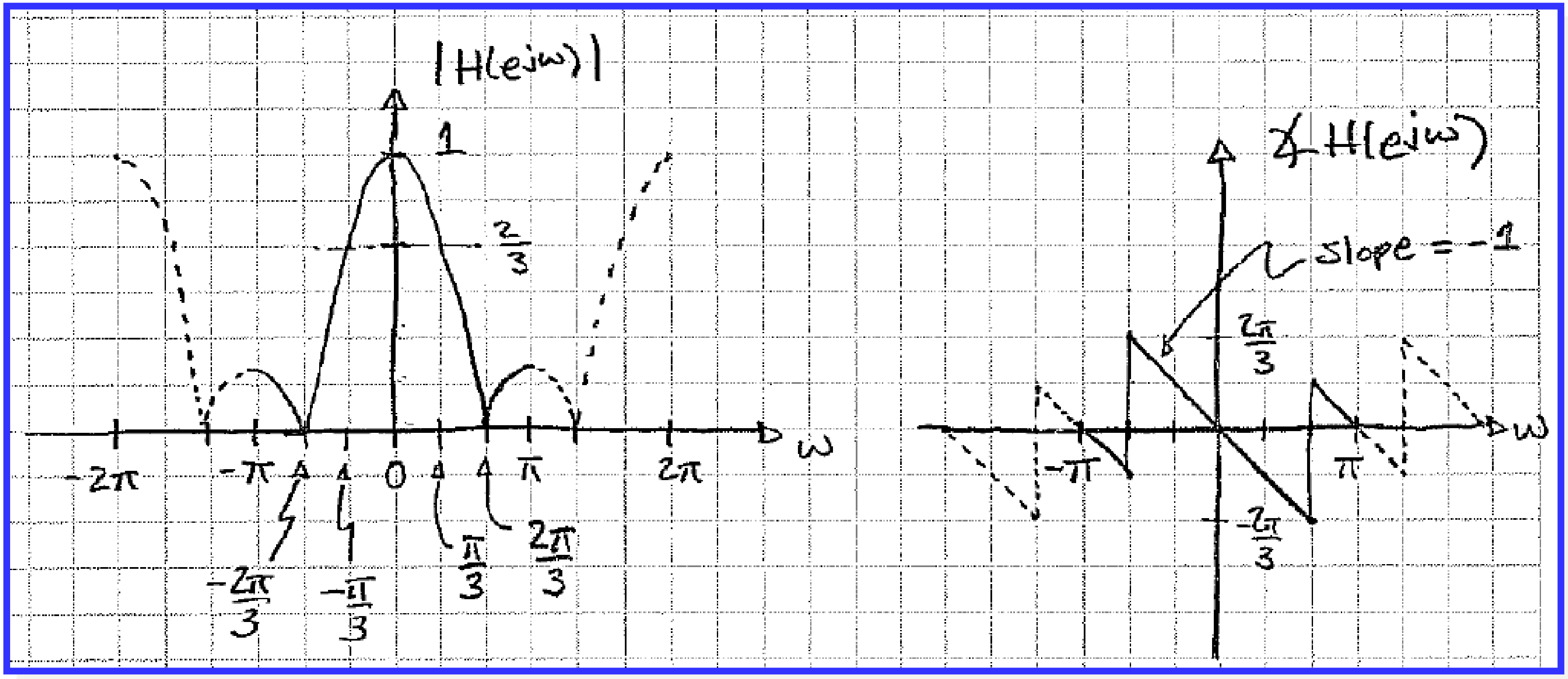
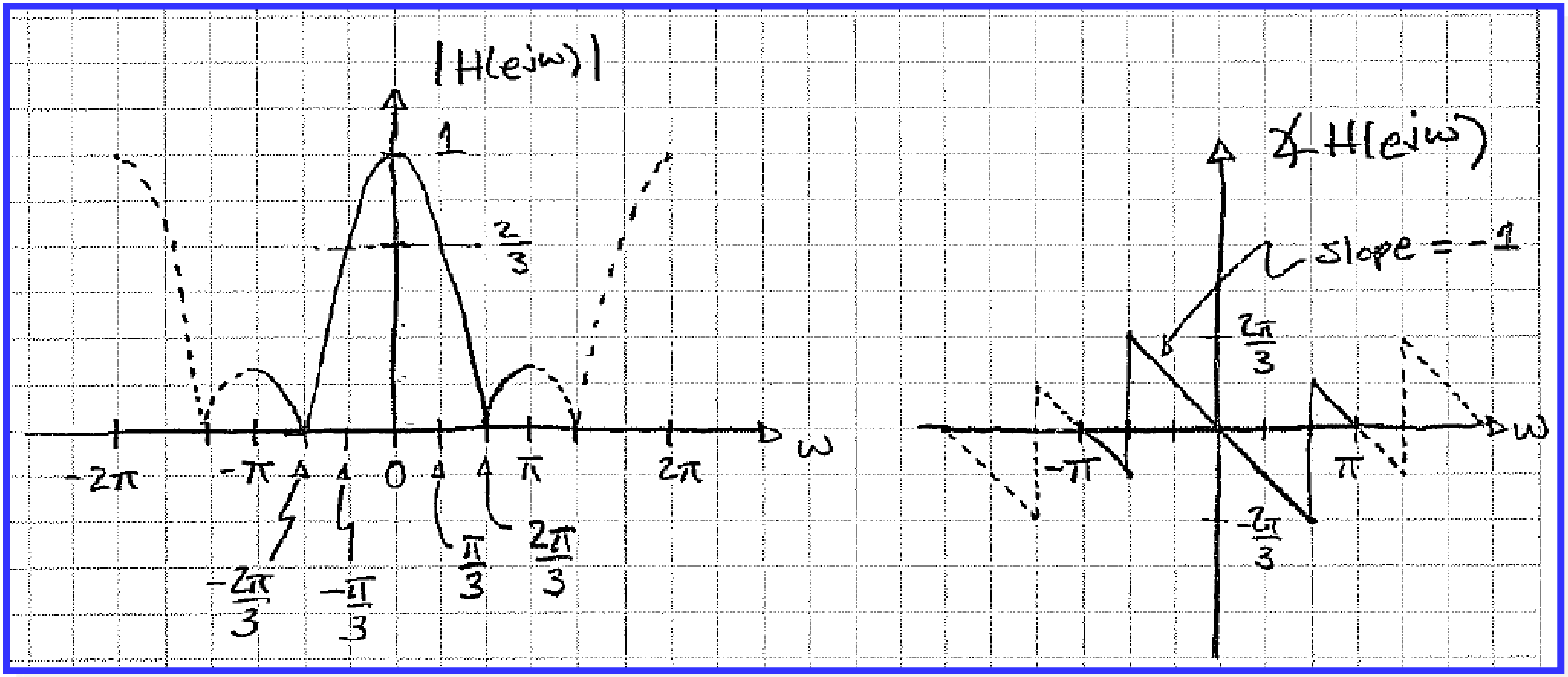


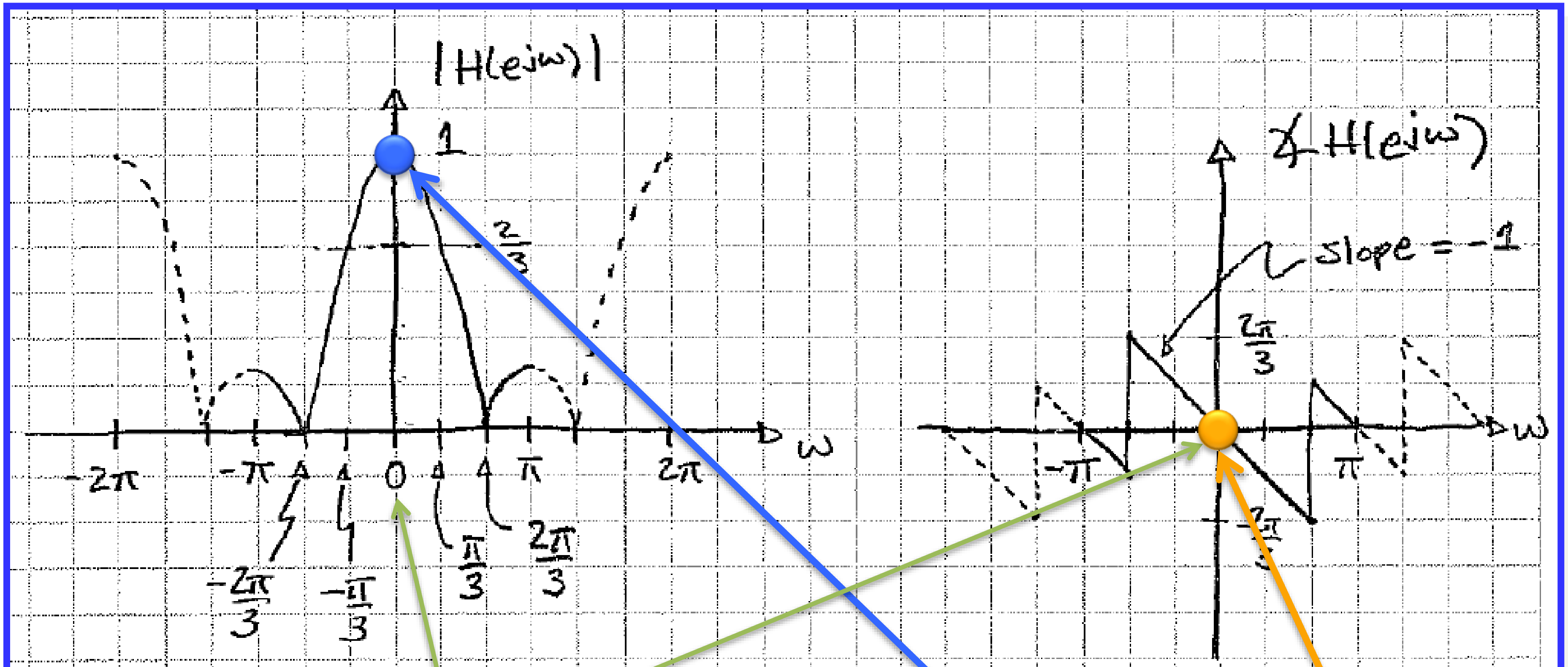
$$x(n) = e^{j0n} \qquad y(n) = ?$$

Example: Length-3 MA filter



$$x(n) = e^{j0n} \qquad y(n) = \text{Mag}\, e^{j0n} e^{j\,\text{Phase}}$$

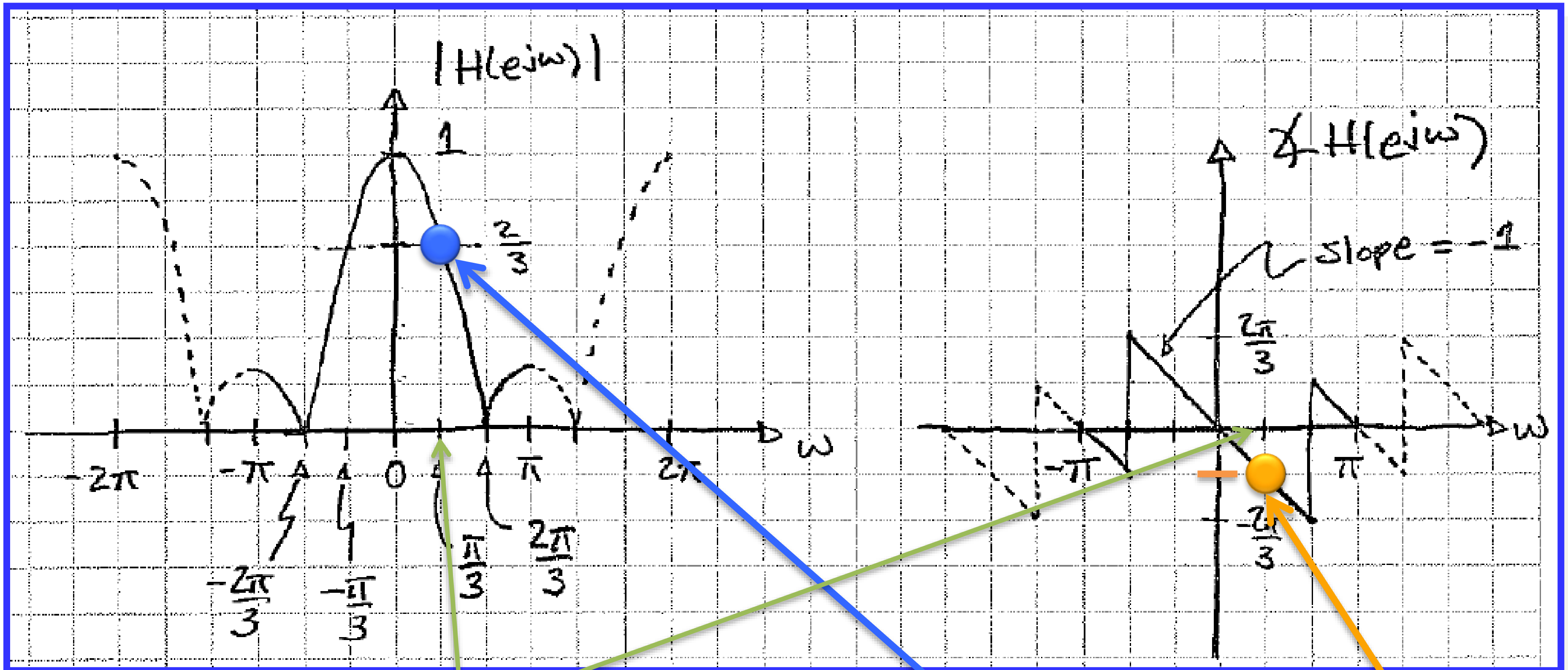
Example: Length-3 MA filter



$x(n) = e^{j0n}$ $\quad$ $y(n) = \boxed{1}\, e^{j0n} e^{j\boxed{0}}$

$= e^{j0n}$

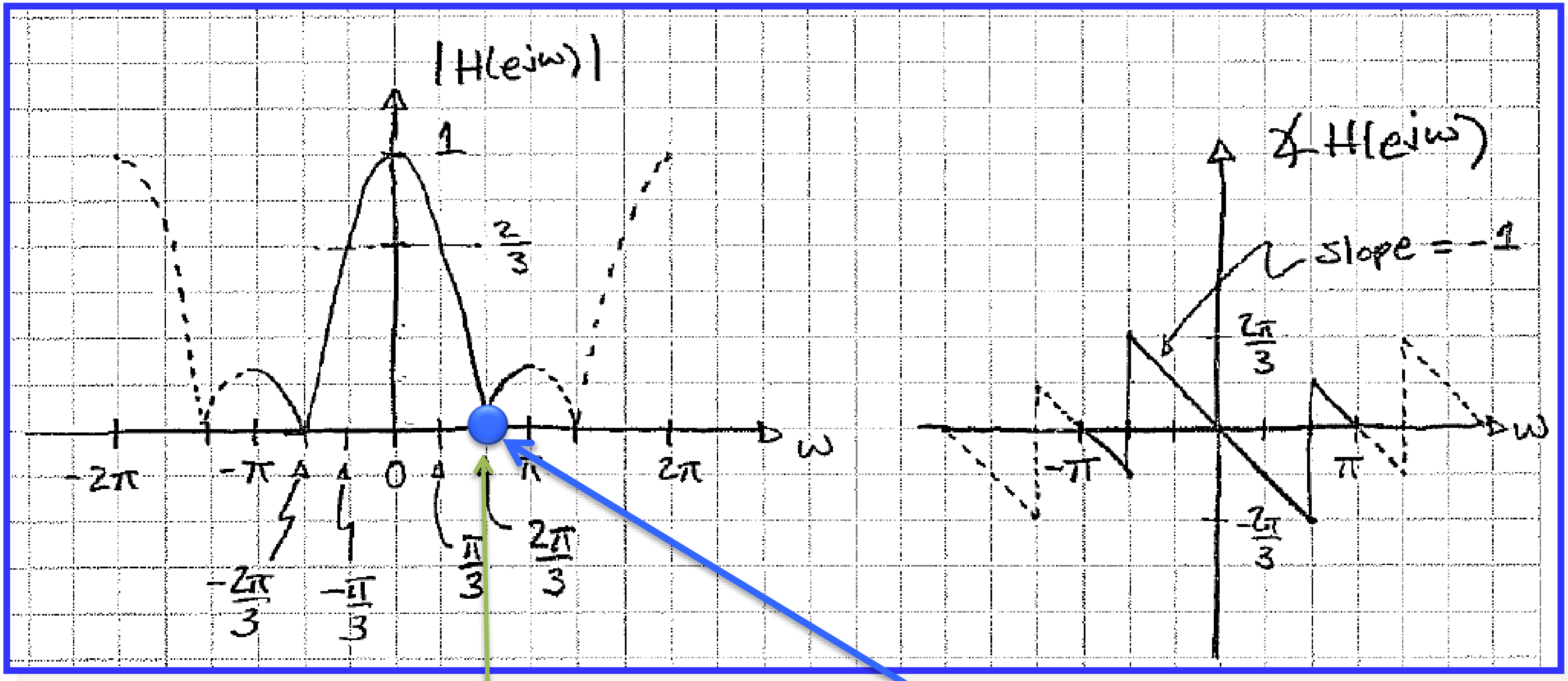
Example: Length-3 MA filter



$$x(n) = \boxed{e^{j\frac{\pi}{3}n}} \qquad y(n) = \boxed{{}^{2}/_{3}}\ \boxed{e^{j\frac{\pi}{3}n}}\, e^{j\boxed{(-\pi/3)}}$$

$$= \frac{2}{3}\, e^{j\frac{\pi}{3}(n-1)}$$

Example: Length-3 MA filter



$$x(n) = e^{j\frac{2\pi}{3}n} \qquad y(n) = \boxed{0}\, e^{j\frac{2\pi}{3}n} e^{j\,\text{who cares}}$$

$$= 0$$

Summary

- **Impulse response:** $h(n)$
  - How the system responds to an impulse
- **Frequency response:** $H\left(e^{j\omega}\right)$
  - How the system responds to various frequencies ($\sin(\omega n)$, $\cos(\omega n)$, $e^{j\omega n}$)
  - $H\left(e^{j\omega}\right) = DTFT[\ h(n)\ ]$
  - **Magnitude of $\boldsymbol{H\left(e^{j\omega_0}\right)}$** = how system will change amplitude at frequency ω_0
  - **Phase of $\boldsymbol{H\left(e^{j\omega_0}\right)}$** = how system will change phase (i.e., what time-shift) at frequency ω_0