Lecture plan

- Course outline and references
- Evaluation system
- Historical notes
- Classification and keywords
- Newtonian mechanics
- Units
- Vectors and vector operations

Preliminary

- Course: Physics for Computer Science
  - Lectures (2 credits)
  - Exercises (2 credits)
- Main textbook
  - J.L Meriam and L.E. Kraige, “Engineering mechanics. Dynamics,” Wiley
- additional references
  - L.E. Goodman and W.H. Warner, “Dynamics,” Springer, Dover, 2001
  - O.M. O’Reily, “Engineering Dynamics: A Primer,” Springer, 2nd Edition, 2010

Preliminary

- Evaluation for the lecture course:
  - Mid and end-term tests (60%)
  - Homework quizzes and assignments (30%)
  - Attendance (10%)
- Evaluation for the exercise course:
  - Mid and end-term test (60%)
  - Weekly assignments (homework problems mainly from the textbook) (30%)
  - Attendance (10%)

Historical notes

The founding fathers

- Galileo Galilei [1564–1642]
- Sir Isaac Newton [1642–1727]
- Leonhard Euler [1707–1783]
- Jean Le Rond d'Alembert [1717–1783]
- Joseph-Louis Lagrange [1736–1813]
- Carl Friedrich Gauss [1777–1855]
- Gaspard Gustave de Coriolis [1792–1843]
- Sir William Rowan Hamilton [1805–1865]
- Paul Emile Appell [1855–1930]
- Sergey Chaplygin [1869–1942]

Outline of the course

Fundamentals of kinematics and dynamics

- Vector calculus
- Kinematics of particle, cylindrical and spherical coordinates
- Kinematics of relative motion
- Force and moment of force, work and energy, potential energy and conservative systems
- Dynamics of system of points: change linear and angular momentum, change of kinetic energy
- Rigid body dynamics: center of masses and inertia tensor
- Rigid body dynamics: Newton-Euler motion equations

Preliminaries

Definition

- Statics – the study of objects in equilibrium
- Dynamics – the study of objects with accelerated motion
  - Kinematics – treats only the geometric aspects of the motion
  - Kinetics – analysis of forces causing the motion

Keywords

- material point
- rigid body
- coordinates and coordinate frames
- velocities and accelerations
- forces and moments of forces
- constraints and reactions forces
- degrees of freedom

Newton's Laws

Definition

1. A particle in isolation moves with constant velocity
2. The acceleration of a particle relative to an inertial reference frame is equal to the force per unit mass applied to the particle

$$\boldsymbol{F} = m\boldsymbol{a}$$

3. The forces of action and reaction between interacting bodies are equal in magnitude and opposite in direction.

SI system

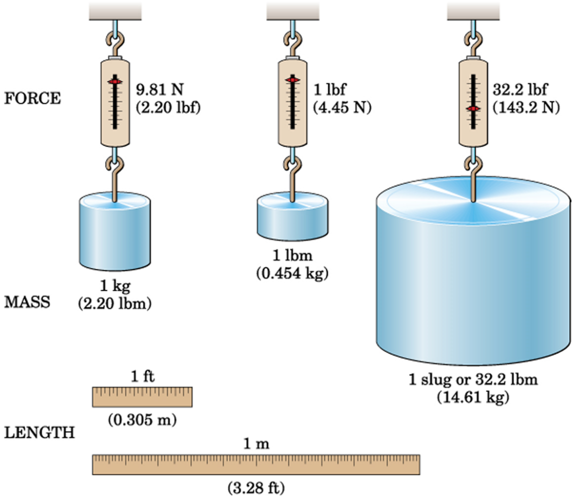
Base units

- Length $[L]$ — meter (m)
- Mass $[M]$ — kilogram (kg)
- Time $[T]$ — second (s)

Derived units

- Velocity $[L/T]$ and acceleration $[L/T^2]$
- Force $[ML/T^2]$ — Newton (N) $(1\mathrm{N} = 1\mathrm{kg} \cdot \mathrm{m/s}^2)$
- Moment of force $[ML^2/T^2]$ — $(\mathrm{N} \cdot \mathrm{m})$
- Energy, work $[ML^2/T^2]$ — Joule [J] $(1\mathrm{J} = 1\mathrm{N} \cdot \mathrm{m})$
- Power $[ML^2/T^3]$ — Watt [W] $(1\mathrm{W} = 1\mathrm{J/s} = 1\mathrm{N} \cdot \mathrm{m/s})$
- Angle $[L^0]$ — radian (dimensionless)
- Angular velocity $[L^0/T]$ and acceleration $[L^0/T^2]$

US units

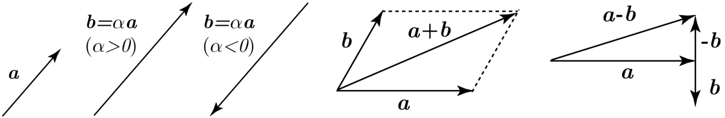

FORCE
9.81 N
(2.20 lbf)
1 lbf
(4.45 N)
32.2 lbf
(143.2 N)
MASS
1 kg
(2.20 lbm)
1 lbm
(0.454 kg)
1 slug or 32.2 lbm
(14.61 kg)
LENGTH
1 ft
(0.305 m)
1 m
(3.28 ft)

Vectors and vector operations

- Physical quantities
  - scalars (work, energy, ...)
  - vectors (force, velocity, ...)
  - tensors (stiffness, stress, ...)
- Vectors and linear combinations
- Lengths and dot products
- Cross product

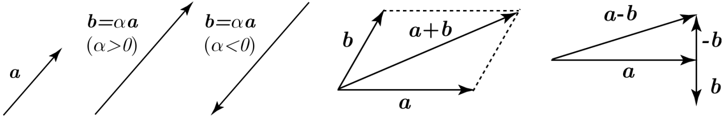
Vectors

- Vectors as oriented arrows (coordinate free-objects)
  - Basic vector operations

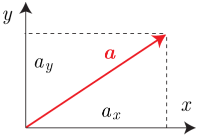


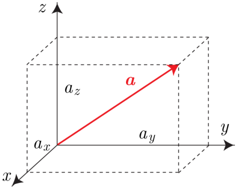
Vectors

- Vectors as oriented arrows (coordinate free-objects)
  - Basic vector operations



- Vectors as sets of numbers
  - Coordinates of a vector





$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Vectors of arbitrary dimensions

- Let n be the dimension of the space
  - in our course $n = 3$, $n = 2$, or $n = 1$
- Coordinates of the vector $\boldsymbol{a}$

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

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- Notation:
  - $\boldsymbol{a} \in \mathbb{R}^n$ if the components of $\boldsymbol{a}$ are real numbers

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- Notation:
  - $\boldsymbol{a} \in \mathbb{R}^n$ if the components of $\boldsymbol{a}$ are real numbers
- Inline writing:
  - To save writing space, in the inline writing the above vector $\boldsymbol{a}$ will be written as $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$

Vector operations

Vector operations

- vector addition

$$\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

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- scalar multiplication

$$c\boldsymbol{a} = c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

Vector operations

- vector addition

$$\boldsymbol{a}+\boldsymbol{b}=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}+\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

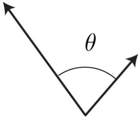
- scalar multiplication

$$c\boldsymbol{a}=c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}=\begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

Linear combinations of vectors: $c\boldsymbol{a}+d\boldsymbol{b}$

Dot product and lengths

- Dot (scalar, inner) product
- Length of a vector
- Unit vectors
- Angle between vectors



Dot product and lengths

Definition

If $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ are vectors in $\mathbb{R}^n$, then the dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

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Definition

If $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ is a vector in $\mathbb{R}^n$, its length $\| \boldsymbol{a} \|$ is defined as

$$\| \boldsymbol{a} \| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

Dot product and lengths

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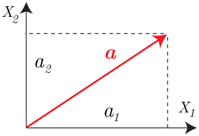

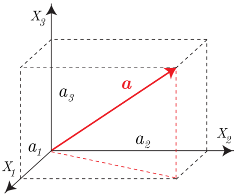

Unit vectors

Definition

A vector $\boldsymbol{a} \in \mathbb{R}^n$ is a unit vector if $\| \boldsymbol{a} \| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = 1$ and therefore

$$\boldsymbol{a} \cdot \boldsymbol{a} = a_1^2 + a_2^2 + \ldots + a_n^2 = 1$$

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Examples

- in $\mathbb{R}^2$: $(1, 0)$, $(0, 1)$
- in $\mathbb{R}^3$: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$

Unit vectors

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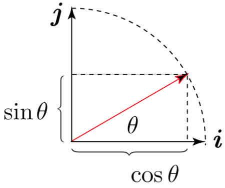
Examples

- in $\mathbb{R}^2$: $(1, 0)$, $(0, 1)$
- in $\mathbb{R}^3$: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$
- in $\mathbb{R}^2$: $(\cos\theta, \sin\theta)$ is a unit vector $(\sin^2\theta + \cos^2\theta = 1)$



Angle between two vectors

- Represent $\boldsymbol{a} = \| \boldsymbol{a} \| \, \boldsymbol{i}_a$ and $\boldsymbol{b} = \| \boldsymbol{b} \| \, \boldsymbol{i}_b$
- $\boldsymbol{i}_a = (\cos\alpha, \sin\alpha)$ and $\boldsymbol{i}_b = (\cos\beta, \sin\beta)$

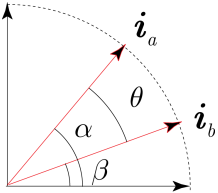
Angle between two vectors

- Represent $\boldsymbol{a} = \| \boldsymbol{a} \| \boldsymbol{i}_a$ and $\boldsymbol{b} = \| \boldsymbol{b} \| \boldsymbol{i}_b$
- $\boldsymbol{i}_a = (\cos\alpha, \sin\alpha)$ and $\boldsymbol{i}_b = (\cos\beta, \sin\beta)$
- $\boldsymbol{a} \cdot \boldsymbol{b} = \| \boldsymbol{a} \| \| \boldsymbol{b} \| \boldsymbol{i}_a \cdot \boldsymbol{i}_b$

Angle between two vectors

- Represent $\boldsymbol{a} = \| \boldsymbol{a} \| \, \boldsymbol{i}_a$ and $\boldsymbol{b} = \| \boldsymbol{b} \| \, \boldsymbol{i}_b$
- $\boldsymbol{i}_a = (\cos\alpha, \sin\alpha)$ and $\boldsymbol{i}_b = (\cos\beta, \sin\beta)$
- $\boldsymbol{a} \cdot \boldsymbol{b} = \| \boldsymbol{a} \| \| \boldsymbol{b} \| \, \boldsymbol{i}_a \cdot \boldsymbol{i}_b$

$$\boldsymbol{i}_a \cdot \boldsymbol{i}_b = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta) = \cos\theta$$



Angle between two vectors

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- $\boldsymbol{i}_a = (\cos\alpha, \sin\alpha)$ and $\boldsymbol{i}_b = (\cos\beta, \sin\beta)$
- $\boldsymbol{a} \cdot \boldsymbol{b} = \| \boldsymbol{a} \| \| \boldsymbol{b} \| \boldsymbol{i}_a \cdot \boldsymbol{i}_b$

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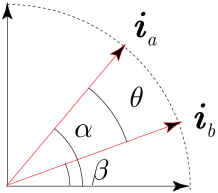
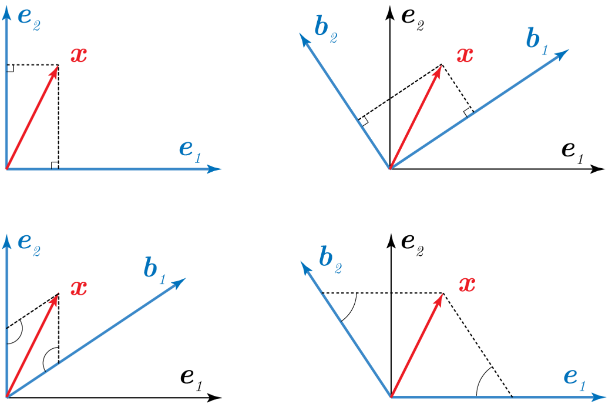


- $\boldsymbol{a} \cdot \boldsymbol{b} = \| \boldsymbol{a} \| \| \boldsymbol{b} \| \cos\theta$

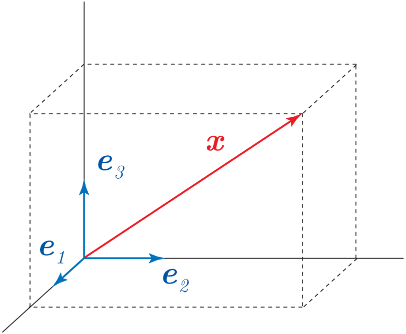
$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\| \boldsymbol{a} \| \| \boldsymbol{b} \|}$$

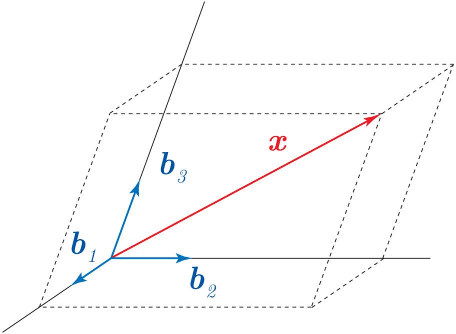
Bases (coordinate systems) and coordinates in $\mathbb{R}^2$



- vectors can be represented in different coordinate systems
- the coordinate systems may not necessarily be perpendicular

Bases and coordinates in $\mathbb{R}^3$





$$\boxed{\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = \tilde{x}_1\boldsymbol{b}_1 + \tilde{x}_2\boldsymbol{b}_2 + \tilde{x}_3\boldsymbol{b}_3}$$

- the same vector has different coordinates in different bases
- x_1, x_2, x_3 – coordinates of $\boldsymbol{x}$ in the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$
- $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ – coordinates of $\boldsymbol{x}$ in the basis $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$

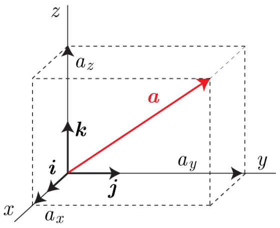
Vectors and vectors products

- Vectors in $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ basis
- Cross product

Triple vectors products

- Double vector product
- Mixed product

Vectors we will work with

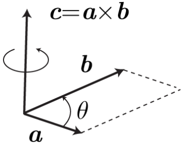


Definition

- components and basis vectors: $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$
- magnitude (length): $|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$
- directions

$$\cos(\widehat{\boldsymbol{a},\boldsymbol{i}}) = \frac{a_x}{|\boldsymbol{a}|}, \quad \cos(\widehat{\boldsymbol{a},\boldsymbol{j}}) = \frac{a_y}{|\boldsymbol{a}|}, \quad \cos(\widehat{\boldsymbol{a},\boldsymbol{k}}) = \frac{a_z}{|\boldsymbol{a}|}.$$

Vector product (cross product) $\boldsymbol{a} \times \boldsymbol{b}$



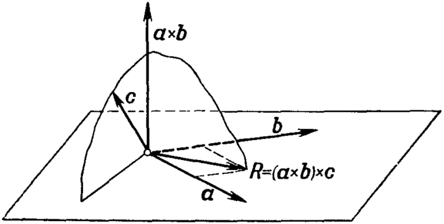
- $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ (area of the parallelogram built on $\boldsymbol{a}$ and $\boldsymbol{b}$)
- antisymmetry: $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \quad \Longrightarrow \boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$
- cross products of unit vectors
  - $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{0}, \quad \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{0}, \quad \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$
  - $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$
- computation: $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}, \; \boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$

$$\boldsymbol{a} \times \boldsymbol{b} = (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) \times (b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}) =$$

$$(a_yb_z - a_zb_y)\boldsymbol{i} + (a_zb_x - a_xb_z)\boldsymbol{j} + (a_xb_y - a_yb_x)\boldsymbol{k}$$

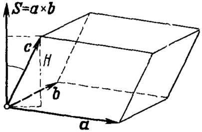
Double vector product $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$

- Compute cross products twice, first $\boldsymbol{a} \times \boldsymbol{b}$ and then $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$
- Shorter formula $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$



- $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$
- $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$ (Jacobi identity)

Triple product (mixed product) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$



- geometric meaning: $|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$ is the volume of the parallelepiped built on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.
- computation
  - $\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}$
  - $\boldsymbol{b} = b_x \boldsymbol{i} + b_y \boldsymbol{j} + b_z \boldsymbol{k}$
  - $\boldsymbol{c} = c_x \boldsymbol{i} + c_y \boldsymbol{j} + c_z \boldsymbol{k}$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \det \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

- $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a})$

Simple vector equations

Example 1

Given the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and the scalar λ find the vector $\boldsymbol{x}$ from

$$\begin{cases} \boldsymbol{a} \times \boldsymbol{x} & = & \boldsymbol{b} \\ \boldsymbol{c} \cdot \boldsymbol{x} & = & \lambda \end{cases}$$

- using linear algebra is possible but a bit messy (4 equations with 3 unknowns)
- assumptions: $\boldsymbol{a} \perp \boldsymbol{b}$ (compatibility) and $\boldsymbol{a} \cdot \boldsymbol{c} \neq 0$
- $(\boldsymbol{a} \times \boldsymbol{x}) \times \boldsymbol{c} = -\boldsymbol{a}(\boldsymbol{x} \cdot \boldsymbol{c}) + \boldsymbol{x}(\boldsymbol{a} \cdot \boldsymbol{c}) = -\boldsymbol{a}\lambda + \boldsymbol{x}(\boldsymbol{a} \cdot \boldsymbol{c}) = \boldsymbol{b} \times \boldsymbol{c}$

$$\boxed{\boldsymbol{x} = \frac{\lambda \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{c}}{\boldsymbol{a} \cdot \boldsymbol{c}}}$$

Simple vector equations

Example 2

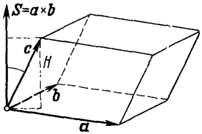
Given the non-coplanar vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and the scalars α, β, γ find the vector $\boldsymbol{x}$ from the following system

$$\boldsymbol{a} \cdot \boldsymbol{x} = \alpha, \quad \boldsymbol{b} \cdot \boldsymbol{x} = \beta, \quad \boldsymbol{c} \cdot \boldsymbol{x} = \gamma$$

- from the first two equations we have $(\boldsymbol{a} \cdot \boldsymbol{x})\boldsymbol{b} = \alpha \boldsymbol{b}$ and $-(\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{a} = -\beta \boldsymbol{a}$.
- summing up these equations, one gets $(\boldsymbol{a} \cdot \boldsymbol{x})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{a} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{x} = \alpha \boldsymbol{b} - \beta \boldsymbol{a}$.
- considering the last equation together with $\boldsymbol{c} \cdot \boldsymbol{x} = \gamma$ and using Example 1, one obtains

$$\boldsymbol{x} = \frac{(\alpha \boldsymbol{b} - \beta \boldsymbol{a}) \times \boldsymbol{c} + (\boldsymbol{a} \times \boldsymbol{b})\gamma}{(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}} = \frac{\alpha(\boldsymbol{b} \times \boldsymbol{c}) + \beta(\boldsymbol{c} \times \boldsymbol{a}) + \gamma(\boldsymbol{a} \times \boldsymbol{b})}{(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}}$$

Triple product (mixed product) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$



- geometric meaning: $|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$ is the volume of the parallelepiped built on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.
- computation
  - $\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}$
  - $\boldsymbol{b} = b_x \boldsymbol{i} + b_y \boldsymbol{j} + b_z \boldsymbol{k}$
  - $\boldsymbol{c} = c_x \boldsymbol{i} + c_y \boldsymbol{j} + c_z \boldsymbol{k}$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \det \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

- $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a})$