Work and Potential Energy

Example

$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} = xy^2\boldsymbol{i} + x^2y\boldsymbol{j} + z^2\boldsymbol{k}$

- Check if the force is potential or not:

$$\text{curl}\boldsymbol{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\boldsymbol{k}$$
$$= 0\boldsymbol{i} + 0\boldsymbol{j} + (2xy - 2xy)\boldsymbol{k} = \boldsymbol{0}$$

- So, $\boldsymbol{F}$ is a potential force.

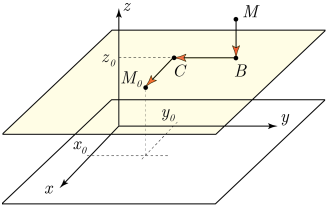
Work and Potential Energy

Example

- Find the potential energy.
  - Select the point $M_0 = (x_0, y_0, z_0)$. Often (but not always!), when appropriate, we select $x_0 = y_0 = z_0 = 0$.
  - Define the potential energy

$$V(x, y, z) = U_{MM_0} = \int_M^{M_0} F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z$$

  - Compute the potential energy as the work done along *any feasible* path. The simplest way is to do it along the path, constructed from straight lines parallel to coordinates axes, from the current position $M = (x, y, z)$ to $M_0 = (x_0, y_0, z_0)$



Work and Potential Energy

Example

- Calculation of the potential energy. Let $B = (x, y, z_0)$, $C = (x, y_0, z_0)$. Then $U_{MM_0} = U_{MB} + U_{BC} + U_{CM_0}$.
  - Moving parallel to z axis along MB: Here $x =$ const, $\mathrm{d}x = 0$, $y =$ const, $\mathrm{d}y = 0 \Longrightarrow U_{MB} = \int_z^{z_0} F_z(x, y, z)\mathrm{d}z = \frac{1}{3}(z_0^3 - z^3)$
  - Moving parallel to y axis along BC: Here $z = z_0 =$ const, $\mathrm{d}z = 0$, $x =$ const, $\mathrm{d}x = 0 \Longrightarrow$ $U_{BC} = \int_y^{y_0} F_y(x, y, z_0)\mathrm{d}y = \frac{1}{2}x^2(y_0^2 - y^2)$
  - Moving parallel to x axis along CM_0: Here $z = z_0 =$ const, $\mathrm{d}z = 0$, $y = y_0 =$ const, $\mathrm{d}y = 0 \Longrightarrow$ $U_{CM_0} = \int_x^{x_0} F_x(x, y_0, z_0)\mathrm{d}x = \frac{1}{2}y_0^2(x_0^2 - x^2)$
- Final expression:

$$V(x, y, z) = -\frac{1}{2}x^2y^2 - \frac{1}{3}z^3 + \underbrace{\frac{1}{2}y_0^2x_0^2 + \frac{1}{3}z_0^3}_{\text{const}}$$

- Check out that $\frac{\partial V}{\partial x} = -F_x$, $\frac{\partial V}{\partial y} = -F_y$, $\frac{\partial V}{\partial z} = -F_z$

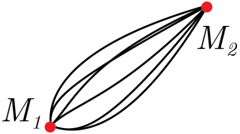
Work and Potential Energy

Example

- Note: given two points, $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ the work done when moving from M_1 to M_2 (on any path) is

$$U_{M_1 M_2} = V(M_1) - V(M_2) = V(x_1, y_1, z_1) - V(x_2, y_2, z_2)$$

so, no need to parameterize the path and compute integrals when $V(x, y, z)$ is known.



- When $M_1 = M_2$ (work along a closed path) $U_{M_1 M_2} = 0$.