Lecture plan

Review

2D motion in polar coordinates

- Definition of the polar coordinates
- Velocity and acceleration in the polar coordinates
- The idea of moving coordinate systems
- Sample problems

Summary

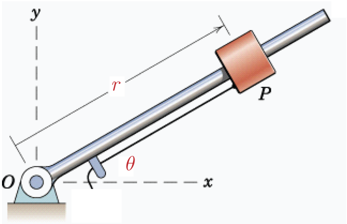
Review and the purpose

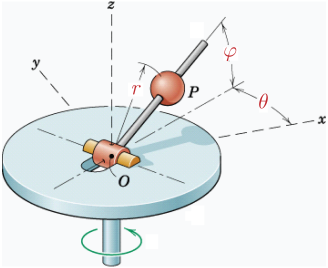
What we have learnt so far

- Vectors and vectors operations
- Kinematics in Cartesian (xy or xyz) coordinates
- Kinematics in path variables ($n - t$ coordinates)

What we are going to learn today

- Motion in non-Cartesian coordinates
- Motivating examples

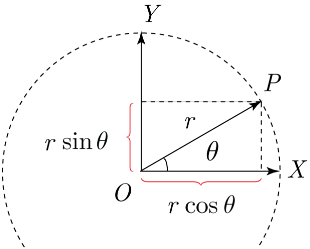




- We start with planar motion and consider polar coordinates

Polar coordinates

Definition



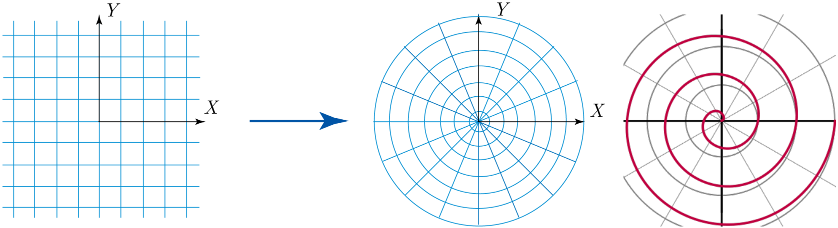
$$\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta}$$

Inverse transformation

$$r^2 = x^2 + y^2, \quad \theta = \arctan(y/x)$$

Coordinate lines



Velocity in polar coordinates

- By direct differentiation of the position

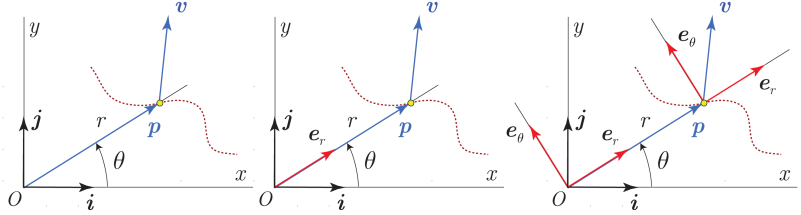
$$\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j}$$

  we obtain

$$\begin{aligned}\boldsymbol{v} = \frac{\mathrm{d}\overrightarrow{OP}}{\mathrm{d}t} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} = \frac{\mathrm{d}x(r,\theta)}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}y(r,\theta)}{\mathrm{d}t}\boldsymbol{i} = \\ \left(\dot{r}\cos\theta - r\sin\theta\ \dot{\theta}\right)\boldsymbol{i} + \left((\dot{r}\sin\theta + r\cos\theta\ \dot{\theta}\right)\boldsymbol{j}\end{aligned}$$

- What is the geometric meaning of this mathematical expression?

Velocity in polar coordinates



- Velocity of the particle

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{\mathrm{d}}{\mathrm{d}t}(r\boldsymbol{e}_r) = \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r = \boxed{\dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta}$$

- The radial $\boxed{v_r = \dot{r}}$ and transverse $\boxed{v_\theta = r\dot{\theta}}$ components of $\boldsymbol{v}$

- Moving coordinate system (rotating with angular velocity $\dot{\theta}$)

$$\boldsymbol{e}_r(\theta) = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} \qquad \boldsymbol{e}_\theta(\theta) = \boldsymbol{e}_r(\theta + \pi/2) = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$$

- The time derivatives are $\boxed{\dot{\boldsymbol{e}}_r = \dot{\theta}\,\boldsymbol{e}_\theta}$ and $\boxed{\dot{\boldsymbol{e}}_\theta = -\dot{\theta}\,\boldsymbol{e}_r}$

Velocity in polar coordinates

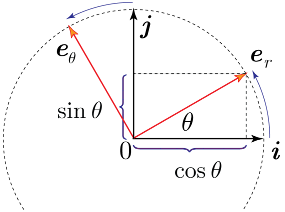
- Answer to the originally posed question. What we obtained by direct differentiation, can be rearranged as

$$\boldsymbol{v} = \left(\dot r\cos\theta - r\sin\theta\;\dot\theta\right)\boldsymbol{i} + \left(\dot r\sin\theta + r\cos\theta\;\dot\theta\right)\boldsymbol{j}$$

$$= \dot r\,(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}) + r\dot\theta\,(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}) = \boxed{\dot r\boldsymbol{e}_r + r\dot\theta\boldsymbol{e}_\theta}$$

- New coordinate system

$$\boldsymbol{e}_r(\theta) = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$$
$$\boldsymbol{e}_\theta(\theta) = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$$

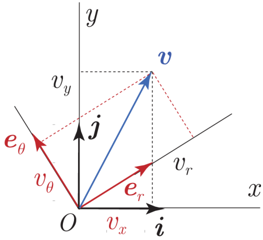


- The unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are "moving" vectors (configuration-dependent) and their time derivatives are $\boxed{\dot{\boldsymbol{e}}_r = \dot\theta\boldsymbol{e}_\theta}$ and $\boxed{\dot{\boldsymbol{e}}_\theta = -\dot\theta\boldsymbol{e}_r}$

Velocity in polar coordinates

$$\begin{cases} \boldsymbol{e}_r = \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \\ \boldsymbol{e}_\theta = -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} \end{cases}$$

$$\begin{cases} \boldsymbol{i} = \cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta \\ \boldsymbol{j} = \sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta \end{cases}$$



- Velocity is a physical quantity that can be expressed in different coordinate systems $\boxed{\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta}$
- Multiply $\boldsymbol{v}$ by $\boldsymbol{i}$ and by $\boldsymbol{j}$

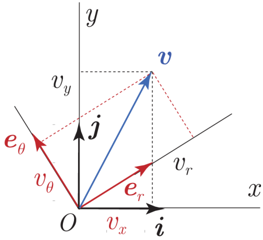
$$v_x = \boldsymbol{v} \cdot \boldsymbol{i} = v_r(\boldsymbol{e}_r \cdot \boldsymbol{i}) + v_\theta(\boldsymbol{e}_\theta \cdot \boldsymbol{i}) = v_r \cos\theta - v_\theta \sin\theta$$
$$v_y = \boldsymbol{v} \cdot \boldsymbol{j} = v_r(\boldsymbol{e}_r \cdot \boldsymbol{j}) + v_\theta(\boldsymbol{e}_\theta \cdot \boldsymbol{j}) = v_r \sin\theta + v_\theta \cos\theta$$

- Multiply $\boldsymbol{v}$ by $\boldsymbol{e}_r$ and by $\boldsymbol{e}_\theta$

$$v_r = \boldsymbol{v} \cdot \boldsymbol{e}_r = v_x(\boldsymbol{i} \cdot \boldsymbol{e}_r) + v_y(\boldsymbol{j} \cdot \boldsymbol{e}_r) = v_x \cos\theta + v_y \sin\theta$$
$$v_\theta = \boldsymbol{v} \cdot \boldsymbol{e}_\theta = v_x(\boldsymbol{i} \cdot \boldsymbol{e}_\theta) + v_y(\boldsymbol{j} \cdot \boldsymbol{e}_\theta) = -v_x \sin\theta + v_y \cos\theta$$

Velocity in polar coordinates

- $v_x = \dot{x}$ and $v_y = \dot{y}$ are the components of $\boldsymbol{v}$ in the fixed frame $\boldsymbol{i}, \boldsymbol{j}$
- $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$ are the components of $\boldsymbol{v}$ in the moving frame $\boldsymbol{e}_r, \boldsymbol{e}_\theta$



- Velocity is a physical quantity and its magnitude (the length) *does not depend* on the choice of the frame of reference
- The magnitude of the velocity (the speed) $v = |\boldsymbol{v}| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$

$$v = \sqrt{(v_x\boldsymbol{i} + v_y\boldsymbol{j}) \cdot (v_x\boldsymbol{i} + v_y\boldsymbol{j})} = \sqrt{v_x^2 + v_y^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$v = \sqrt{(v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta) \cdot (v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta)} = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$$

Acceleration in polar coordinates

- Time derivatives of the unit base vectors

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta, \quad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\boldsymbol{e}_r$$

- Position of the particle $\boxed{\boldsymbol{p} = r\boldsymbol{e}_r}$
- Velocity of the particle

$$\begin{aligned} \dot{\boldsymbol{p}} &= \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r \\ &= \boxed{\dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta} \end{aligned}$$

- Acceleration of the particle

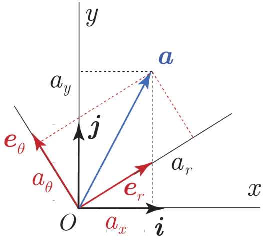
$$\begin{aligned} \boldsymbol{a} = \dot{\boldsymbol{v}} = \tfrac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta) &= \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r + (r\ddot{\theta} + \dot{r}\dot{\theta})\boldsymbol{e}_\theta + r\dot{\theta}\dot{\boldsymbol{e}}_\theta \\ &= \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\theta}\boldsymbol{e}_\theta + (r\ddot{\theta} + \dot{r}\dot{\theta})\boldsymbol{e}_\theta - r\dot{\theta}^2\boldsymbol{e}_r \\ &= \boxed{(\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta} \end{aligned}$$

- The radial and transverse components of the acceleration

$$a_r = (\ddot{r} - r\dot{\theta}^2), \quad a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Acceleration in polar coordinates

- $a_x = \ddot{x}$ and $a_y = \ddot{y}$ are the components of $\boldsymbol{a}$ in the fixed frame $\boldsymbol{i}, \boldsymbol{j}$
- $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ are the components of $\boldsymbol{a}$ in the moving frame $\boldsymbol{e}_r, \boldsymbol{e}_\theta$



- Acceleration is a physical quantity and its magnitude (the length) *does not depend* on the choice of the frame of reference
- The magnitude of the acceleration $a = |\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$

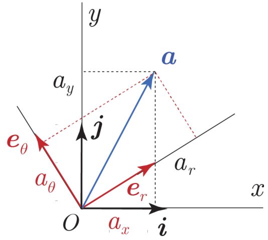
$$a = \sqrt{(a_x\boldsymbol{i} + a_y\boldsymbol{j}) \cdot (a_x\boldsymbol{i} + a_y\boldsymbol{j})} = \sqrt{a_x^2 + a_y^2} = \sqrt{\ddot{x}^2 + \ddot{y}^2}$$

$$a = \sqrt{(a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta) \cdot (a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta)} = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$

Acceleration in polar coordinates

$$\begin{cases} \boldsymbol{e}_r &= \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} \\ \boldsymbol{e}_\theta &= -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} \end{cases}$$

$$\begin{cases} \boldsymbol{i} &= \cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta \\ \boldsymbol{j} &= \sin\theta\boldsymbol{e}_r + \cos\theta\boldsymbol{e}_\theta \end{cases}$$



- Acceleration is a physical quantity that can be expressed in different coordinate systems $\boxed{\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} = a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta}$
- Multiply $\boldsymbol{v}$ by $\boldsymbol{i}$ and by $\boldsymbol{j}$

$$\begin{aligned} a_x &= \boldsymbol{a}\cdot\boldsymbol{i} = a_r(\boldsymbol{e}_r\cdot\boldsymbol{i}) + a_\theta(\boldsymbol{e}_\theta\cdot\boldsymbol{i}) = a_r\cos\theta - a_\theta\sin\theta \\ a_y &= \boldsymbol{a}\cdot\boldsymbol{j} = a_r(\boldsymbol{e}_r\cdot\boldsymbol{j}) + a_\theta(\boldsymbol{e}_\theta\cdot\boldsymbol{j}) = a_r\sin\theta + a_\theta\cos\theta \end{aligned}$$

- Multiply $\boldsymbol{v}$ by $\boldsymbol{e}_r$ and by $\boldsymbol{e}_\theta$

$$\begin{aligned} a_r &= \boldsymbol{a}\cdot\boldsymbol{e}_r = a_x(\boldsymbol{i}\cdot\boldsymbol{e}_r) + a_y(\boldsymbol{j}\cdot\boldsymbol{e}_r) = a_x\cos\theta + a_y\sin\theta \\ a_\theta &= \boldsymbol{a}\cdot\boldsymbol{e}_\theta = a_x(\boldsymbol{i}\cdot\boldsymbol{e}_\theta) + a_y(\boldsymbol{j}\cdot\boldsymbol{e}_\theta) = -a_x\sin\theta + a_y\cos\theta \end{aligned}$$

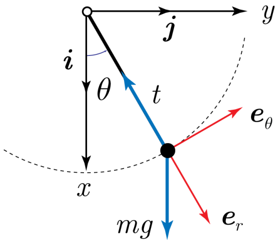
Motion equations in polar coordinates

- Newton's 2nd law, $m\boldsymbol{a} = \boldsymbol{F}$, can be written (in the moving frame $\boldsymbol{e}_r, \boldsymbol{e}_\theta$) as

$$m(a_r \boldsymbol{e}_r + a_\theta \boldsymbol{e}_\theta) = F_r \boldsymbol{e}_r + F_\theta \boldsymbol{e}_\theta \quad \Longleftrightarrow \quad \begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= F_r \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= F_\theta \end{aligned}$$

- Example: mathematical pendulum

$$\begin{aligned} r &= \text{const}, \dot{r} = \ddot{r} = 0, \\ \boldsymbol{F} &= \boldsymbol{G} + \boldsymbol{T} \quad \text{(total force)} \\ \boldsymbol{T} &= -t\boldsymbol{e}_r \\ \boldsymbol{G} &= mg\boldsymbol{i} \\ &= mg(\cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta) \end{aligned}$$



- Resulting motion equations

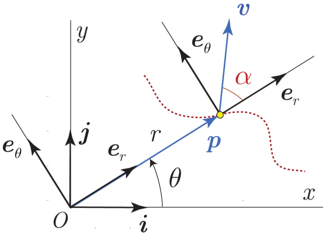
$$\begin{aligned} -mr\dot{\theta}^2 &= -t + mg\cos\theta & &\Longrightarrow \quad t = mr\dot{\theta}^2 + mg\cos\theta \\ mr\ddot{\theta} &= -mg\sin\theta & &\Longrightarrow \quad \boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0} \end{aligned}$$

Summary

What we have learnt today

- Motion description in polar coordinates
- Moving frames $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$
- Computation of velocity and acceleration
- Application to dynamic equations
- The use of non-rectangular coordinates is a good choice if the particle motion features a circular pattern
- Physical quantities (velocities, accelerations, forces) can be expressed in the moving frames

Illustrative example



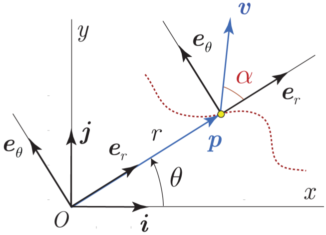
A particle moves in such a way that the angle between the velocity and position vectors is always constant. Define the trajectory of the particle if the initial conditions are specified as $r(0) = r_0$ and $\theta(0) = \theta_0$.

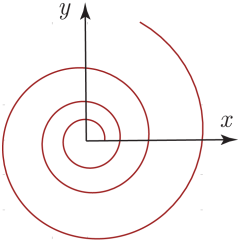
Solution

- In polar coordinates $\boldsymbol{p} = r\boldsymbol{e}_r$ and $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$.
- Since the angle α between $\boldsymbol{v}$ and $\boldsymbol{e}_r$ is constant, we have

$$\frac{v_r}{v_\theta} = \cot\alpha \Longrightarrow \frac{\dot{r}}{r\dot{\theta}} = \frac{\mathrm{d}r}{\mathrm{d}t}\Big/\frac{r\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}r}{r\mathrm{d}\theta} = \cot\alpha$$

Illustrative example





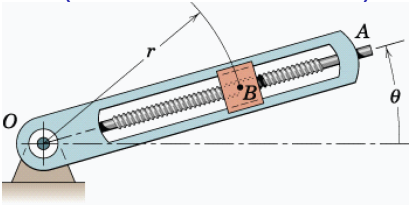
- The last equation can arranged as $\boxed{\dfrac{\mathrm{d}r}{r} = \mathrm{d}\theta \cot\alpha}$
- This equation can be integrated as

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{r} = \int_{\theta_0}^{\theta} \mathrm{d}\theta \cot\alpha = \cot\alpha \int_{\theta_0}^{\theta} \mathrm{d}\theta \quad \Longrightarrow \quad \ln\frac{r}{r_0} = \cot\alpha\,(\theta - \theta_0)$$

- The trajectory is a logarithmic spiral

$$\boxed{r(\theta) = r_0 e^{\cot\alpha\,(\theta - \theta_0)}}$$

Sample problem (from the textbook)



Rotation of the radially slotted arm is governed by $\theta(t) = 0.2t + 0.02t^3$. Simultaneously, the power screw in the arm drives the slider B according to $r(t) = 0.2 + 0.04t^2$. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when $t = 3\mathrm{s}$.

Solution

- Define the polar coordinates and their derivatives

$$
\begin{aligned}
r(t) &= 0.2 + 0.04t^2 & \qquad \theta(t) &= 0.2t + 0.02t^3 \\
\dot{r}(t) &= 0.08t & \qquad \dot{\theta}(t) &= 0.2 + 0.06t^2 \\
\ddot{r}(t) &= 0.08 & \qquad \ddot{\theta}(t) &= 0.12t
\end{aligned}
$$

Sample problem: calculations

- For the instant $t = 3\text{s}$ we have

$$
\begin{array}{lll}
r(3) = 0.56\text{m} & \dot{r}(3) = 0.24\text{m/s} & \ddot{r}(3) = 0.08\text{m/s}^2 \\
\theta(3) = 1.14\text{rad} = 65.3° & \dot{\theta}(3) = 0.74\text{rad/s} & \ddot{\theta}(3) = 0.36\text{rad/s}^2
\end{array}
$$

- Velocity components and magnitude

$$
\begin{array}{lll}
v_r = \dot{r} & \text{for } \ t = 3\text{s} & v_r(3) = 0.24\text{m/s} \\
v_\theta = r\dot{\theta} & \text{for } \ t = 3\text{s} & v_\theta(3) = 0.414\text{m/s} \\
v = \sqrt{v_r^2 + v_\theta^2} & \text{for } \ t = 3\text{s} & v(3) = 0.479\text{m/s}
\end{array}
$$

- Acceleration components and magnitude

$$
\begin{array}{lll}
a_r = \ddot{r} - r\dot{\theta}^2 & \text{for } \ t = 3\text{s} & a_r(3) = -0.227\text{m/s}^2 \\
a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} & \text{for } \ t = 3\text{s} & a_\theta(3) = 0.557\text{m/s}^2 \\
a = \sqrt{a_r^2 + a_\theta^2} & \text{for } \ t = 3\text{s} & a(3) = 0.601\text{m/s}^2
\end{array}
$$

Sample problem: visualization