Lecture plan

Review: moving coordinate system in polar coordinates

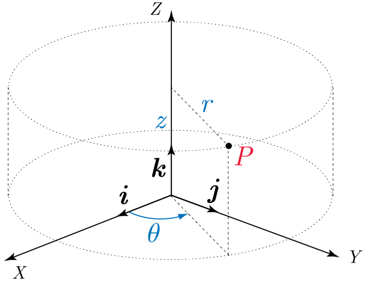
Today: extension to 3D motion

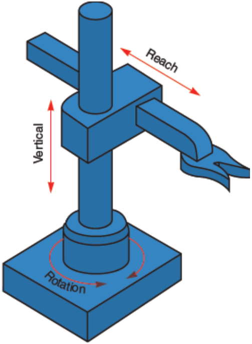
- Cylindrical coordinates
  - Definition
  - Velocity and acceleration in cylindrical coordinates
  - Illustrative examples
- Spherical coordinates
  - Definition
  - Velocity and acceleration in spherical coordinates
  - Illustrative examples

Summary

Cylindrical coordinates: definition and examples

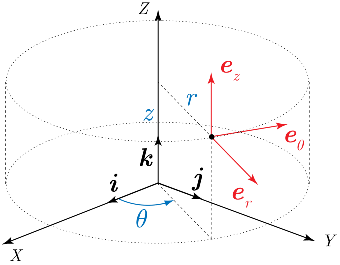
$$\boldsymbol{p} = \begin{bmatrix} x(r,\theta) \\ y(r,\theta) \\ z \end{bmatrix}$$





Cylindrical coordinates: definition

$$\boldsymbol{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$$



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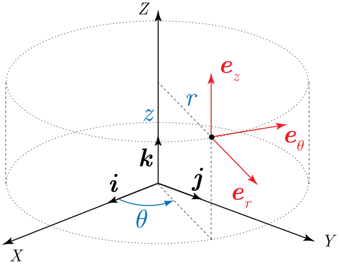

Moving frame: $\boldsymbol{e}_r = \frac{\partial \boldsymbol{p}/\partial r}{|\partial \boldsymbol{p}/\partial r|}$, $\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{p}/\partial \theta}{|\partial \boldsymbol{p}/\partial \theta|}$, $\boldsymbol{e}_z = \frac{\partial \boldsymbol{p}/\partial z}{|\partial \boldsymbol{p}/\partial z|}$.

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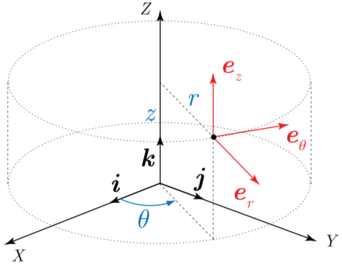

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$$\boldsymbol{e}_r = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \quad \boldsymbol{e}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \quad \boldsymbol{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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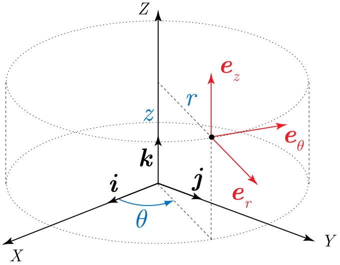

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Basis unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ are mutually perpendicular, and
$\boldsymbol{e}_r \times \boldsymbol{e}_\theta = \boldsymbol{e}_z, \quad \boldsymbol{e}_\theta \times \boldsymbol{e}_z = \boldsymbol{e}_r, \quad \boldsymbol{e}_z \times \boldsymbol{e}_r = \boldsymbol{e}_\theta$

Cylindrical coordinates: velocity & acceleration

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- Velocity of the particle

$$\boldsymbol{v} = \dot{\boldsymbol{p}} \quad = \quad \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r + \dot{z}\boldsymbol{e}_z + z\dot{\boldsymbol{e}}_z$$

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- Acceleration of the particle

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r + (r\ddot{\theta} + \dot{r}\dot{\theta})\boldsymbol{e}_\theta + r\dot{\theta}\dot{\boldsymbol{e}}_\theta + \ddot{z}\boldsymbol{e}_z + \dot{z}\dot{\boldsymbol{e}}_z$$

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&= \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\theta}\boldsymbol{e}_\theta + (r\ddot{\theta} + \dot{r}\dot{\theta})\boldsymbol{e}_\theta - r\dot{\theta}^2\boldsymbol{e}_r + \ddot{z}\boldsymbol{e}_z
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&= \boxed{(\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{e}_z}
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Cylindrical coordinates: magnitudes

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- The components of the acceleration

$$
\begin{aligned}
v_r &= \dot{r} \\
v_\theta &= r\dot{\theta} \\
v_z &= \dot{z}
\end{aligned}
$$

- The magnitude of the velocity (the speed)

$$v = |\boldsymbol{v}| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \sqrt{v_r^2 + v_\theta^2 + v_z^2}$$

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a_r &= \ddot{r} - r\dot{\theta}^2 \\
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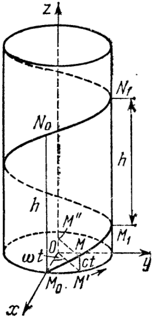
Illustrative example: Helix



Parameters of helix:

period $\boxed{T = 2\pi/\omega}$, pitch $\boxed{h = cT}$ (z-displacement over time T),

helix angle (lead angle) $\boxed{\tan\gamma = h/2\pi R}$

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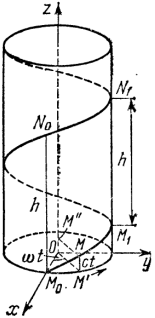

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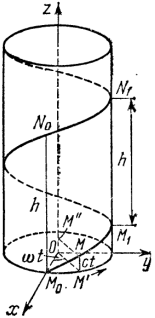

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- $\overrightarrow{OM} = R\cos\omega t\boldsymbol{i} + R\sin\omega t\boldsymbol{j} + ct\boldsymbol{k}$

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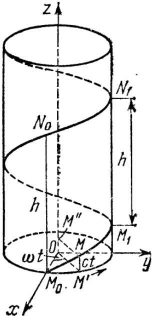

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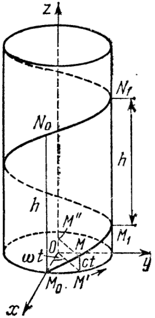

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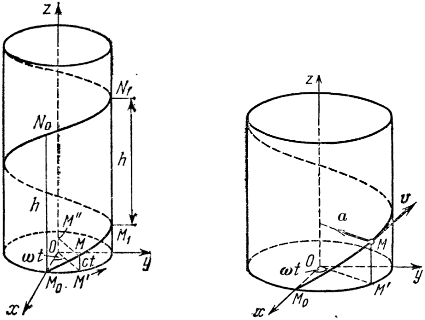

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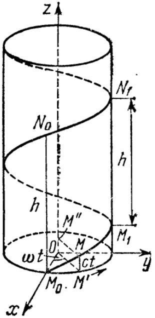

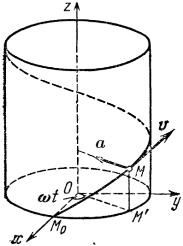

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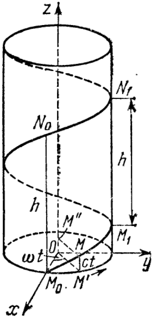

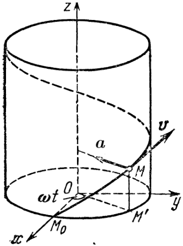

In cylindrical coordinates: $r = R$, $\theta = \omega t$, $z = ct$.

First derivatives $\dot{r} = 0$, $\dot{\theta} = \omega$, $\dot{z} = c$.

Second derivatives $\ddot{r} = 0$, $\ddot{\theta} = 0$, $\ddot{z} = 0$.

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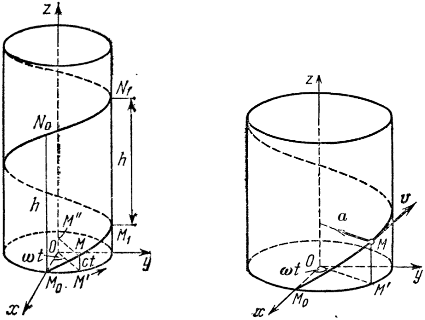

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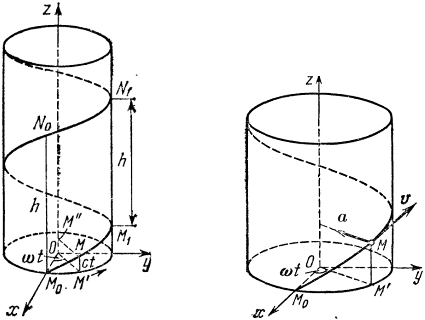

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- $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z = R\omega\boldsymbol{e}_\theta + c\boldsymbol{e}_z$

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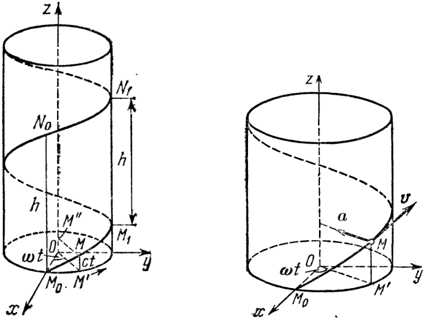

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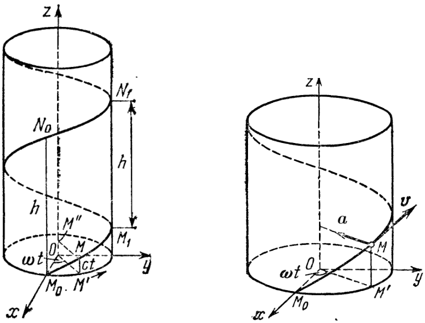
- $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z = R\omega\boldsymbol{e}_\theta + c\boldsymbol{e}_z$
- $\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{e}_z = -R\omega^2\boldsymbol{e}_r$

Illustrative example



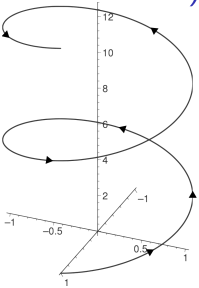
In cylindrical coordinates: $r = R$, $\theta = \omega t$, $z = ct$.

First derivatives $\dot{r} = 0$, $\dot{\theta} = \omega$, $\dot{z} = c$.

Second derivatives $\ddot{r} = 0$, $\ddot{\theta} = 0$, $\ddot{z} = 0$.

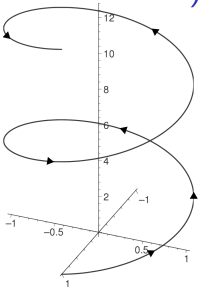
- $\overrightarrow{OM} = r\boldsymbol{e}_r + z\boldsymbol{e}_z = R\boldsymbol{e}_r + ct\boldsymbol{e}_z$
- $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z = R\omega\boldsymbol{e}_\theta + c\boldsymbol{e}_z$
- $\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{e}_z = -R\omega^2\boldsymbol{e}_r$
- Features: $v = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} =$ const, $a = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}} =$ const,
  $\boldsymbol{a} = \boldsymbol{e}_z \times \boldsymbol{v} \implies \boldsymbol{a} \perp \boldsymbol{v}$.

Illustrative example (generalization)



We assumed that $r = R$, $\theta = \omega t$, $z = ct$, where $\omega = \text{const}$.

Illustrative example (generalization)



We assumed that $r = R$, $\theta = \omega t$, $z = ct$, where $\omega =$ const.

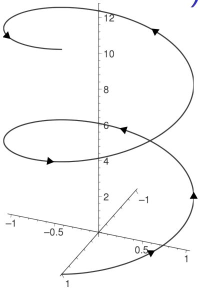
But if the angular velocity is not constant,

$r = R$, $\theta = \theta(t)$, $z = \alpha\theta(t)$.

First derivatives $\dot{r} = 0$, $\dot{\theta} \neq$ const, $\dot{z} \neq$ const.

Second derivatives $\ddot{r} = 0$, $\ddot{\theta} \neq$ const, $\ddot{z} \neq$ const.

Illustrative example (generalization)



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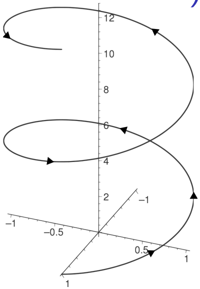
- $\overrightarrow{OM} = R\boldsymbol{e}_r + z\boldsymbol{e}_z$

Illustrative example (generalization)



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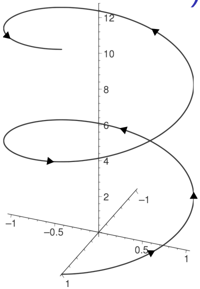
- $\boldsymbol{v} = R\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z$

Illustrative example (generalization)



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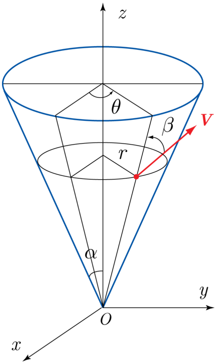
First derivatives $\dot{r} = 0$, $\dot{\theta} \neq \text{const}$, $\dot{z} \neq \text{const}$.

Second derivatives $\ddot{r} = 0$, $\ddot{\theta} \neq \text{const}$, $\ddot{z} \neq \text{const}$.

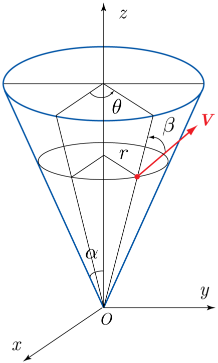
- $\overrightarrow{OM} = R\boldsymbol{e}_r + z\boldsymbol{e}_z$
- $\boldsymbol{v} = R\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z$
- $\boldsymbol{a} = -R\dot{\theta}^2\boldsymbol{e}_r + R\ddot{\theta}\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{e}_z$

Illustrative example



A particle moves on a circular cone. The angle between the ruling of the cone and the cone's axis is α. At any instant of time the particle crosses the ruling line under the same constant angle β. Define the trajectory of the particle.

Illustrative example

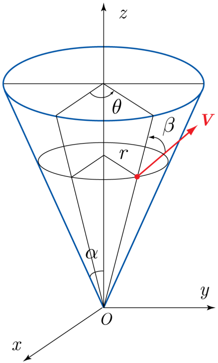


A particle moves on a circular cone. The angle between the ruling of the cone and the cone's axis is α. At any instant of time the particle crosses the ruling line under the same constant angle β. Define the trajectory of the particle.

- In cylindrical coordinates

$$\overrightarrow{OP} = r\boldsymbol{e}_r + z\boldsymbol{e}_z, \quad \boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z.$$

Illustrative example



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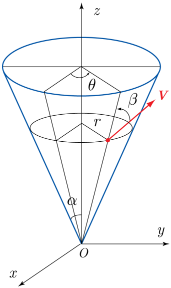
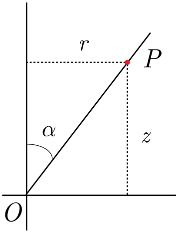
- In cylindrical coordinates

$$\overrightarrow{OP} = r\boldsymbol{e}_r + z\boldsymbol{e}_z, \quad \boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{e}_z.$$

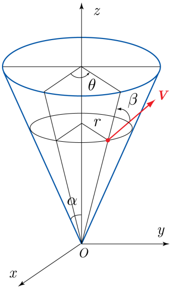
- What is the ruling angle?

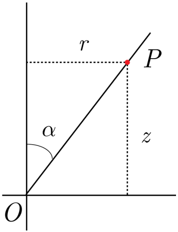
Illustrative example





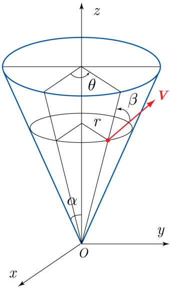
Illustrative example

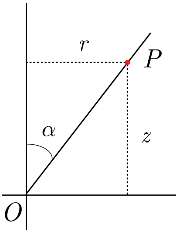




- Since the particle moves on the cone, we have $z = r/\tan\alpha$.

Illustrative example



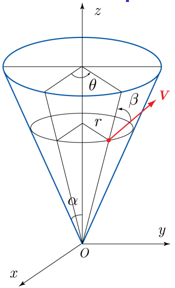


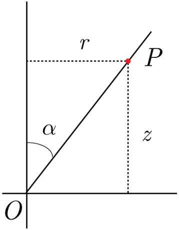
- Since the particle moves on the cone, we have $z = r/\tan\alpha$.
- The unit vector along the ruling passing through point P

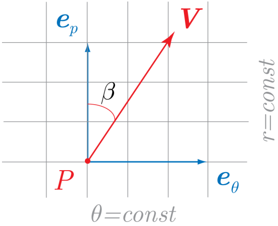
$$\boldsymbol{e}_p = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{\overrightarrow{OP}}{\sqrt{r^2 + \dfrac{r^2}{\tan^2\alpha}}} = \frac{\overrightarrow{OP}\sin\alpha}{r} =$$

$$\frac{\sin\alpha(r\boldsymbol{e}_r + z\boldsymbol{e}_z)}{r} = \frac{\sin\alpha(r\boldsymbol{e}_r + \dfrac{r}{\tan\alpha}\boldsymbol{e}_z)}{r} = \sin\alpha\boldsymbol{e}_r + \cos\alpha\boldsymbol{e}_z$$

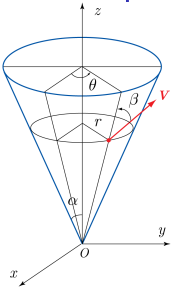
Illustrative example

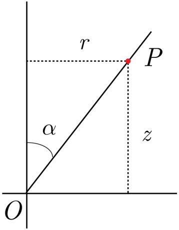


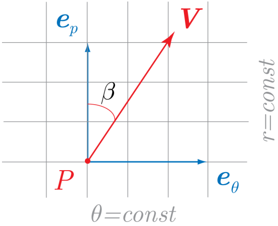




Illustrative example



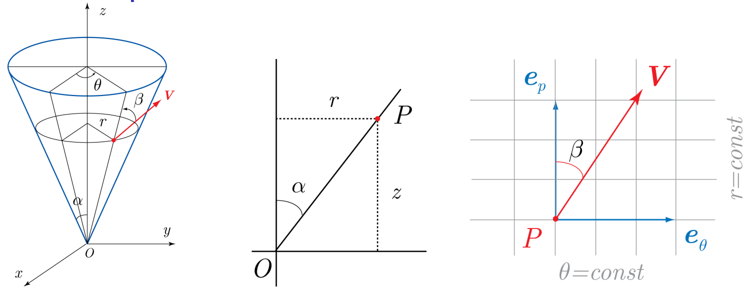




- Projecting the velocity vector onto the line trough $\boldsymbol{e}_p$ gives

$$v_p = \boldsymbol{v} \cdot \boldsymbol{e}_p = \dot{r} \sin\alpha + \dot{z} \cos\alpha = \dot{r} \sin\alpha + \dot{r} \frac{\cos\alpha}{\tan\alpha} = \frac{\dot{r}}{\sin\alpha}$$

Illustrative example



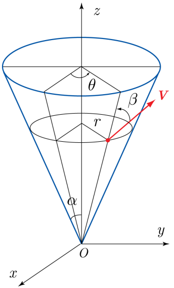
- Projecting the velocity vector onto the line trough $\boldsymbol{e}_p$ gives

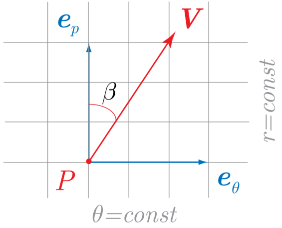
$$v_p = \boldsymbol{v} \cdot \boldsymbol{e}_p = \dot{r} \sin \alpha + \dot{z} \cos \alpha = \dot{r} \sin \alpha + \dot{r} \frac{\cos \alpha}{\tan \alpha} = \frac{\dot{r}}{\sin \alpha}$$

- Since $v_\theta = \boldsymbol{v} \cdot \boldsymbol{e}_\theta = r\dot{\theta}$, and the angle β is constant,

$$\frac{v_\theta}{v_p} = \frac{r\dot{\theta}}{\dot{r}/\sin \alpha} = \tan \beta \quad \Longrightarrow \quad \frac{\mathrm{d}r}{r} = \frac{\sin \alpha}{\tan \beta} \mathrm{d}\theta$$

Illustrative example

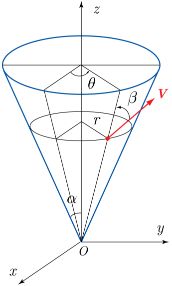


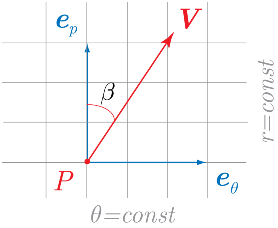


- Assume that $\theta(0) = \theta_0$ and $r(\theta_0) = r_0$. Then

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{r} = \frac{\sin\alpha}{\tan\beta}\int_{\theta_0}^{\theta} \mathrm{d}\theta \quad \Longrightarrow \quad \ln r - \ln r_0 = \frac{\sin\alpha}{\tan\beta}(\theta - \theta_0)$$

Illustrative example





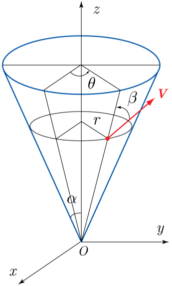
- Assume that $\theta(0) = \theta_0$ and $r(\theta_0) = r_0$. Then

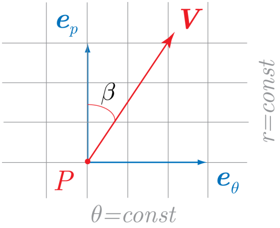
$$\int_{r_0}^{r} \frac{\mathrm{d}r}{r} = \frac{\sin\alpha}{\tan\beta} \int_{\theta_0}^{\theta} \mathrm{d}\theta \quad \Longrightarrow \quad \ln r - \ln r_0 = \frac{\sin\alpha}{\tan\beta}(\theta - \theta_0)$$

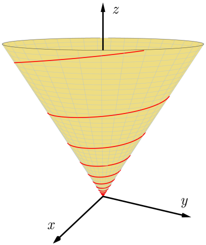
- Therefore

$$\boxed{r(\theta) = r_0\, e^{\frac{\sin\alpha}{\tan\beta}(\theta - \theta_0)}}$$

Illustrative example







- Assume that $\theta(0) = \theta_0$ and $r(\theta_0) = r_0$. Then

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{r} = \frac{\sin\alpha}{\tan\beta} \int_{\theta_0}^{\theta} \mathrm{d}\theta \quad \Longrightarrow \quad \ln r - \ln r_0 = \frac{\sin\alpha}{\tan\beta}(\theta - \theta_0)$$

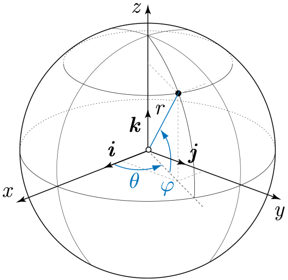
- Therefore

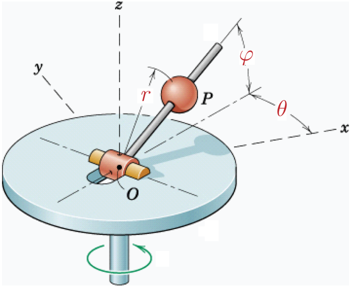
$$r(\theta) = r_0\, e^{\dfrac{\sin\alpha}{\tan\beta}(\theta - \theta_0)}$$

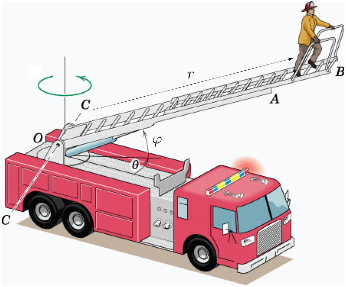
- Since $z = r/\tan\alpha$, the trajectory is a conical spiral.

Spherical coordinates

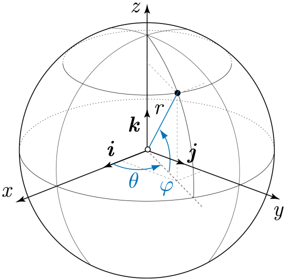
$$\boldsymbol{p} = \begin{bmatrix} x(r,\theta,\varphi) \\ y(r,\theta,\varphi) \\ z(r,\theta,\varphi) \end{bmatrix}$$





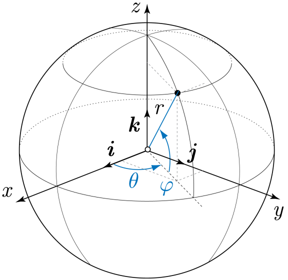


Spherical coordinates: computation



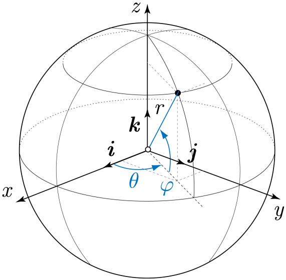
$\boldsymbol{p} =$

Spherical coordinates: computation



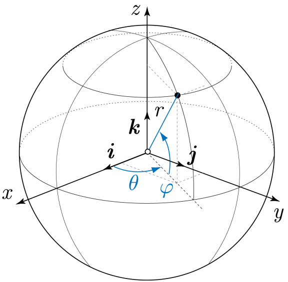
$$\boldsymbol{p} = \begin{bmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{bmatrix}$$

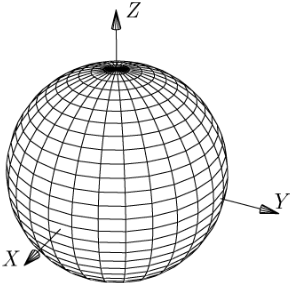
Spherical coordinates: computation



$$\boldsymbol{p} = \begin{bmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{bmatrix} = r\cos\varphi\cos\theta\boldsymbol{i} + r\cos\varphi\sin\theta\boldsymbol{j} + r\sin\varphi\boldsymbol{k}$$

Spherical coordinates: computation





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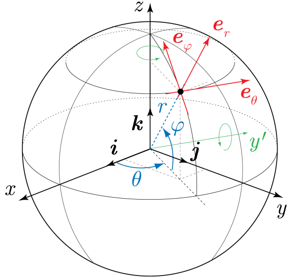
The coordinate lines for a fixed $r = $ const:

- the meridians (the lines of constant longitude $\theta = $ const).
- the parallels (the lines of constant latitude $\varphi = $ const).

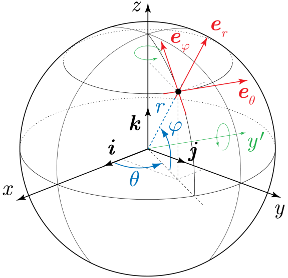
Spherical coordinates: the moving frame

$$\boldsymbol{p} = \begin{bmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{bmatrix}$$



Spherical coordinates: the moving frame

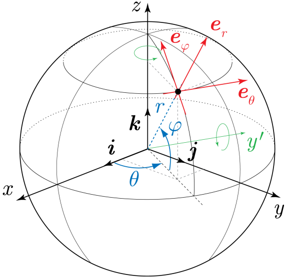
$$\boldsymbol{p} = \begin{bmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{bmatrix}$$



Moving frame: $\boldsymbol{e}_r = \dfrac{\partial\boldsymbol{p}/\partial r}{|\partial\boldsymbol{p}/\partial r|}$, $\boldsymbol{e}_\theta = \dfrac{\partial\boldsymbol{p}/\partial\theta}{|\partial\boldsymbol{p}/\partial\theta|}$, $\boldsymbol{e}_\varphi = \dfrac{\partial\boldsymbol{p}/\partial\varphi}{|\partial\boldsymbol{p}/\partial\varphi|}$.

Spherical coordinates: the moving frame

$$
\boldsymbol{p} = \begin{bmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{bmatrix}
$$

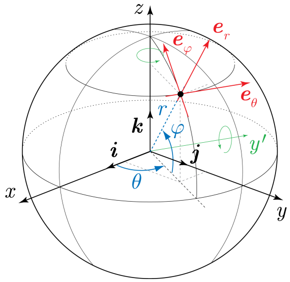


Moving frame: $\boldsymbol{e}_r = \dfrac{\partial \boldsymbol{p}/\partial r}{|\partial \boldsymbol{p}/\partial r|}$, $\boldsymbol{e}_\theta = \dfrac{\partial \boldsymbol{p}/\partial \theta}{|\partial \boldsymbol{p}/\partial \theta|}$, $\boldsymbol{e}_\varphi = \dfrac{\partial \boldsymbol{p}/\partial \varphi}{|\partial \boldsymbol{p}/\partial \varphi|}$.

$$
\boldsymbol{e}_r = \begin{bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ \sin\varphi \end{bmatrix} \quad \boldsymbol{e}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \quad \boldsymbol{e}_\varphi = \begin{bmatrix} -\sin\varphi\cos\theta \\ -\sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}
$$

Spherical coordinates: the moving frame

$$\boldsymbol{p} = \begin{bmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{bmatrix}$$



Moving frame: $\boldsymbol{e}_r = \dfrac{\partial\boldsymbol{p}/\partial r}{|\partial\boldsymbol{p}/\partial r|}$, $\boldsymbol{e}_\theta = \dfrac{\partial\boldsymbol{p}/\partial\theta}{|\partial\boldsymbol{p}/\partial\theta|}$, $\boldsymbol{e}_\varphi = \dfrac{\partial\boldsymbol{p}/\partial\varphi}{|\partial\boldsymbol{p}/\partial\varphi|}$.

$$\boldsymbol{e}_r = \begin{bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ \sin\varphi \end{bmatrix} \quad \boldsymbol{e}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \quad \boldsymbol{e}_\varphi = \begin{bmatrix} -\sin\varphi\cos\theta \\ -\sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}$$

Basis unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ are mutually perpendicular, and

$$\boldsymbol{e}_r \times \boldsymbol{e}_\theta = \boldsymbol{e}_\varphi, \qquad \boldsymbol{e}_\theta \times \boldsymbol{e}_\varphi = \boldsymbol{e}_r, \qquad \boldsymbol{e}_\varphi \times \boldsymbol{e}_r = \boldsymbol{e}_\theta$$

Spherical coordinates: angular velocity

- Angular velocity of the moving frame $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ is

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$$\alpha = \boldsymbol{e}_r \cdot \boldsymbol{k} \qquad \beta = \boldsymbol{e}_\theta \cdot \boldsymbol{k} \qquad \gamma = \boldsymbol{e}_\varphi \cdot \boldsymbol{k}$$

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$$\begin{aligned}
&\boldsymbol{k} = \alpha\boldsymbol{e}_r + \beta\boldsymbol{e}_\theta + \gamma\boldsymbol{e}_\varphi \\
&\alpha = \boldsymbol{e}_r \cdot \boldsymbol{k} = \sin\varphi, \quad \beta = \boldsymbol{e}_\theta \cdot \boldsymbol{k} = 0, \quad \gamma = \boldsymbol{e}_\varphi \cdot \boldsymbol{k} = \cos\varphi \\
&\boldsymbol{k} = \sin\varphi\boldsymbol{e}_r + \cos\varphi\boldsymbol{e}_\varphi
\end{aligned}$$

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$$\alpha = \boldsymbol{e}_r \cdot \boldsymbol{k} = \sin\varphi, \quad \beta = \boldsymbol{e}_\theta \cdot \boldsymbol{k} = 0, \quad \gamma = \boldsymbol{e}_\varphi \cdot \boldsymbol{k} = \cos\varphi$$
$$\boldsymbol{k} = \sin\varphi\boldsymbol{e}_r + \cos\varphi\boldsymbol{e}_\varphi$$

- Finally, we have

$$\boxed{\boldsymbol{\omega} = \dot{\theta}\sin\varphi\boldsymbol{e}_r - \dot{\varphi}\boldsymbol{e}_\theta + \dot{\theta}\cos\varphi\boldsymbol{e}_\varphi}$$

Spherical coordinates: derivatives of moving unit vectors

$$\dot{\boldsymbol{e}}_r = \boldsymbol{\omega} \times \boldsymbol{e}_r \quad =$$

$$\dot{\boldsymbol{e}}_\theta = \boldsymbol{\omega} \times \boldsymbol{e}_\theta \quad =$$

$$\dot{\boldsymbol{e}}_\varphi = \boldsymbol{\omega} \times \boldsymbol{e}_\varphi \quad =$$

Spherical coordinates: derivatives of moving unit vectors

$$\dot{\boldsymbol{e}}_r = \boldsymbol{\omega} \times \boldsymbol{e}_r \quad = \quad (\dot{\theta} \sin\varphi \boldsymbol{e}_r - \dot{\varphi} \boldsymbol{e}_\theta + \dot{\theta} \cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_r$$

$$\dot{\boldsymbol{e}}_\theta = \boldsymbol{\omega} \times \boldsymbol{e}_\theta \quad =$$

$$\dot{\boldsymbol{e}}_\varphi = \boldsymbol{\omega} \times \boldsymbol{e}_\varphi \quad =$$

Spherical coordinates: derivatives of moving unit vectors

$$\begin{aligned}
\dot{\boldsymbol{e}}_r = \boldsymbol{\omega} \times \boldsymbol{e}_r \quad &= \quad (\dot\theta \sin\varphi \boldsymbol{e}_r - \dot\varphi \boldsymbol{e}_\theta + \dot\theta \cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_r \\
&= \quad \dot\varphi \boldsymbol{e}_\varphi + \dot\theta \cos\varphi \boldsymbol{e}_\theta
\end{aligned}$$

$$\dot{\boldsymbol{e}}_\theta = \boldsymbol{\omega} \times \boldsymbol{e}_\theta \quad =$$

$$\dot{\boldsymbol{e}}_\varphi = \boldsymbol{\omega} \times \boldsymbol{e}_\varphi \quad =$$

Spherical coordinates: derivatives of moving unit vectors

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\dot{\boldsymbol{e}}_r = \boldsymbol{\omega} \times \boldsymbol{e}_r &= (\dot{\theta} \sin\varphi \boldsymbol{e}_r - \dot{\varphi} \boldsymbol{e}_\theta + \dot{\theta} \cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_r \\
&= \dot{\varphi} \boldsymbol{e}_\varphi + \dot{\theta} \cos\varphi \boldsymbol{e}_\theta
\end{aligned}$$

$$\dot{\boldsymbol{e}}_\theta = \boldsymbol{\omega} \times \boldsymbol{e}_\theta = (\dot{\theta} \sin\varphi \boldsymbol{e}_r - \dot{\varphi} \boldsymbol{e}_\theta + \dot{\theta} \cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_\theta$$

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Spherical coordinates: derivatives of moving unit vectors

$$\begin{aligned}
\dot{\boldsymbol{e}}_r = \boldsymbol{\omega} \times \boldsymbol{e}_r &= (\dot{\theta}\sin\varphi \boldsymbol{e}_r - \dot{\varphi}\boldsymbol{e}_\theta + \dot{\theta}\cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_r \\
&= \dot{\varphi}\boldsymbol{e}_\varphi + \dot{\theta}\cos\varphi \boldsymbol{e}_\theta
\end{aligned}$$

$$\begin{aligned}
\dot{\boldsymbol{e}}_\theta = \boldsymbol{\omega} \times \boldsymbol{e}_\theta &= (\dot{\theta}\sin\varphi \boldsymbol{e}_r - \dot{\varphi}\boldsymbol{e}_\theta + \dot{\theta}\cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_\theta \\
&= \dot{\theta}\sin\varphi \boldsymbol{e}_\varphi - \dot{\theta}\cos\varphi \boldsymbol{e}_r
\end{aligned}$$

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Spherical coordinates: derivatives of moving unit vectors

$$\begin{aligned}
\dot{\boldsymbol{e}}_r = \boldsymbol{\omega} \times \boldsymbol{e}_r &= (\dot{\theta}\sin\varphi\boldsymbol{e}_r - \dot{\varphi}\boldsymbol{e}_\theta + \dot{\theta}\cos\varphi\boldsymbol{e}_\varphi) \times \boldsymbol{e}_r \\
&= \boxed{\dot{\varphi}\boldsymbol{e}_\varphi + \dot{\theta}\cos\varphi\boldsymbol{e}_\theta}
\end{aligned}$$

$$\begin{aligned}
\dot{\boldsymbol{e}}_\theta = \boldsymbol{\omega} \times \boldsymbol{e}_\theta &= (\dot{\theta}\sin\varphi\boldsymbol{e}_r - \dot{\varphi}\boldsymbol{e}_\theta + \dot{\theta}\cos\varphi\boldsymbol{e}_\varphi) \times \boldsymbol{e}_\theta \\
&= \boxed{\dot{\theta}\sin\varphi\boldsymbol{e}_\varphi - \dot{\theta}\cos\varphi\boldsymbol{e}_r}
\end{aligned}$$

$$\dot{\boldsymbol{e}}_\varphi = \boldsymbol{\omega} \times \boldsymbol{e}_\varphi = (\dot{\theta}\sin\varphi\boldsymbol{e}_r - \dot{\varphi}\boldsymbol{e}_\theta + \dot{\theta}\cos\varphi\boldsymbol{e}_\varphi) \times \boldsymbol{e}_\varphi$$

Spherical coordinates: derivatives of moving unit vectors

$$
\begin{aligned}
\dot{\boldsymbol{e}}_r = \boldsymbol{\omega} \times \boldsymbol{e}_r &= (\dot\theta \sin\varphi \boldsymbol{e}_r - \dot\varphi \boldsymbol{e}_\theta + \dot\theta \cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_r \\
&= \dot\varphi \boldsymbol{e}_\varphi + \dot\theta \cos\varphi \boldsymbol{e}_\theta
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{e}}_\theta = \boldsymbol{\omega} \times \boldsymbol{e}_\theta &= (\dot\theta \sin\varphi \boldsymbol{e}_r - \dot\varphi \boldsymbol{e}_\theta + \dot\theta \cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_\theta \\
&= \dot\theta \sin\varphi \boldsymbol{e}_\varphi - \dot\theta \cos\varphi \boldsymbol{e}_r
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{e}}_\varphi = \boldsymbol{\omega} \times \boldsymbol{e}_\varphi &= (\dot\theta \sin\varphi \boldsymbol{e}_r - \dot\varphi \boldsymbol{e}_\theta + \dot\theta \cos\varphi \boldsymbol{e}_\varphi) \times \boldsymbol{e}_\varphi \\
&= -\dot\theta \sin\varphi \boldsymbol{e}_\theta - \dot\varphi \boldsymbol{e}_r
\end{aligned}
$$

Spherical coordinates: velocity

- Position of the particle

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- Position of the particle $\boldsymbol{p} = r\boldsymbol{e}_r$

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- Position of the particle $\boldsymbol{p} = r\boldsymbol{e}_r$
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- Position of the particle $\boldsymbol{p} = r\boldsymbol{e}_r$
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$$\begin{aligned} \boldsymbol{v} = \dot{\boldsymbol{p}} &= \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r \\ &= \dot{r}\boldsymbol{e}_r + r\boldsymbol{\omega} \times \boldsymbol{e}_r \end{aligned}$$

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&= \dot{r}\boldsymbol{e}_r + r\boldsymbol{\omega} \times \boldsymbol{e}_r \\
&= \dot{r}\boldsymbol{e}_r + r(\dot{\theta}\sin\varphi\boldsymbol{e}_r - \dot{\varphi}\boldsymbol{e}_\theta + \dot{\theta}\cos\varphi\boldsymbol{e}_\varphi) \times \boldsymbol{e}_r
\end{aligned}$$

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&= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\cos\varphi\boldsymbol{e}_\theta + r\dot{\varphi}\boldsymbol{e}_\varphi
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&= \dot{r}\boldsymbol{e}_r + r\dot{\theta}\cos\varphi\boldsymbol{e}_\theta + r\dot{\varphi}\boldsymbol{e}_\varphi \\
&\triangleq v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi
\end{aligned}
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\end{aligned}
$$

- The radial and two angular components of the velocity

$$
v_r = \dot{r}, \quad v_\theta = r\dot{\theta}\cos\varphi, \quad v_\varphi = r\dot{\varphi}
$$

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- The radial and two angular components of the velocity

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta}\cos\varphi, \quad v_\varphi = r\dot{\varphi}$$

- The magnitude of the velocity

$$v = |\boldsymbol{v}| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \sqrt{v_r^2 + v_\theta^2 + v_\varphi^2}$$

Spherical coordinates: acceleration

- Velocity of the particle $\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\cos\varphi\boldsymbol{e}_\theta + r\dot{\varphi}\boldsymbol{e}_\varphi$

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- Acceleration of the particle

$$
\begin{aligned}
\boldsymbol{a} = \dot{\boldsymbol{v}} &= \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r \\
&+ \left(\dot{r}\dot{\theta}\cos\varphi + r\ddot{\theta}\cos\varphi - r\dot{\theta}\dot{\varphi}\sin\varphi\right)\boldsymbol{e}_\theta + r\dot{\theta}\cos\varphi\dot{\boldsymbol{e}}_\theta \\
&+ \left(\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\boldsymbol{e}_\varphi + r\dot{\varphi}\dot{\boldsymbol{e}}_\varphi
\end{aligned}
$$

Spherical coordinates: acceleration

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\boldsymbol{a} = \dot{\boldsymbol{v}} &= \ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r \\
&+ \left(\dot{r}\dot{\theta}\cos\varphi + r\ddot{\theta}\cos\varphi - r\dot{\theta}\dot{\varphi}\sin\varphi\right)\boldsymbol{e}_\theta + r\dot{\theta}\cos\varphi\dot{\boldsymbol{e}}_\theta \\
&+ \left(\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\boldsymbol{e}_\varphi + r\dot{\varphi}\dot{\boldsymbol{e}}_\varphi
\end{aligned}$$

- Finally

$$\begin{aligned}
\boldsymbol{a} &= \left(\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2\cos^2\varphi\right)\boldsymbol{e}_r \\
&+ \left(r\ddot{\theta}\cos\varphi + 2\dot{r}\dot{\theta}\cos\varphi - 2r\dot{\theta}\dot{\varphi}\sin\varphi\right)\boldsymbol{e}_\theta \\
&+ \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r\dot{\theta}^2\sin\varphi\cos\varphi\right)\boldsymbol{e}_\varphi
\end{aligned}$$

Spherical coordinates: acceleration

- Since $\boldsymbol{a} \triangleq a_r \boldsymbol{e}_r + a_\theta \boldsymbol{e}_\theta + a_\varphi \boldsymbol{e}_\varphi$ and

$$
\begin{aligned}
\boldsymbol{a} \quad = \quad & \left(\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2 \cos^2\varphi\right) \boldsymbol{e}_r \\
+ \quad & \left(r\ddot{\theta}\cos\varphi + 2\dot{r}\dot{\theta}\cos\varphi - 2r\dot{\theta}\dot{\varphi}\sin\varphi\right) \boldsymbol{e}_\theta \\
+ \quad & \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r\dot{\theta}^2 \sin\varphi\cos\varphi\right) \boldsymbol{e}_\varphi
\end{aligned}
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$$\begin{aligned}\boldsymbol{a} &= \left(\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2 \cos^2\varphi\right)\boldsymbol{e}_r \\ &+ \left(r\ddot{\theta}\cos\varphi + 2\dot{r}\dot{\theta}\cos\varphi - 2r\dot{\theta}\dot{\varphi}\sin\varphi\right)\boldsymbol{e}_\theta \\ &+ \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r\dot{\theta}^2\sin\varphi\cos\varphi\right)\boldsymbol{e}_\varphi\end{aligned}$$

- The radial and two angular components of the acceleration

$$\begin{aligned}a_r &= \ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2\cos^2\varphi \\ a_\theta &= r\ddot{\theta}\cos\varphi + 2\dot{r}\dot{\theta}\cos\varphi - 2r\dot{\theta}\dot{\varphi}\sin\varphi \\ a_\varphi &= r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r\dot{\theta}^2\sin\varphi\cos\varphi\end{aligned}$$

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$$\begin{aligned}\boldsymbol{a} &= \left(\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2\cos^2\varphi\right)\boldsymbol{e}_r \\ &+ \left(r\ddot{\theta}\cos\varphi + 2\dot{r}\dot{\theta}\cos\varphi - 2r\dot{\theta}\dot{\varphi}\sin\varphi\right)\boldsymbol{e}_\theta \\ &+ \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r\dot{\theta}^2\sin\varphi\cos\varphi\right)\boldsymbol{e}_\varphi\end{aligned}$$

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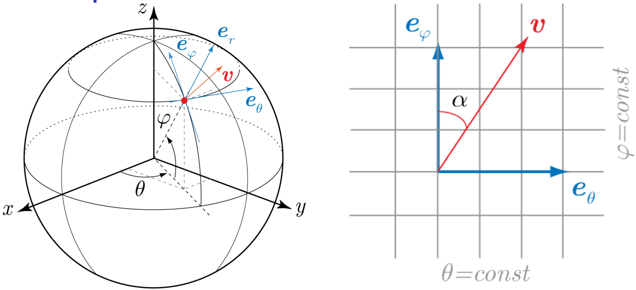
- The magnitude of the acceleration

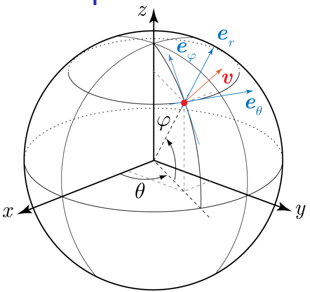
$$a = |\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}} = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2} = \sqrt{a_r^2 + a_\theta^2 + a_\varphi^2}$$

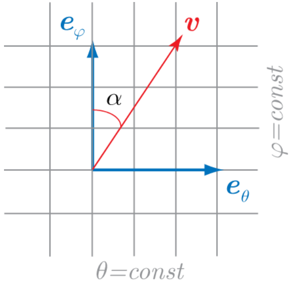
Illustrative example



A particle moves on a sphere in such a way that it crosses all the meridians (coordinate lines of longitude $\theta = \text{const}$) at the same constant angle α. Find the trajectory of the particle.

Illustrative example





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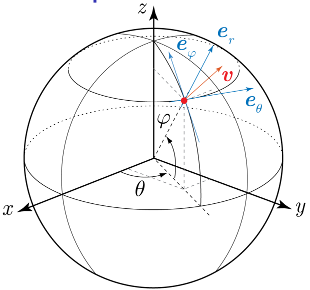
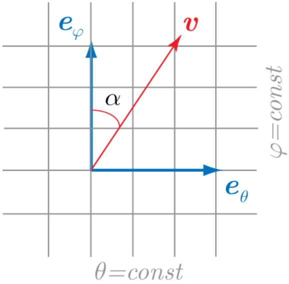
- For $r = \text{const}$ we have $\boldsymbol{v} = \dot{\boldsymbol{p}} = r\dot{\theta}\cos\varphi\boldsymbol{e}_\theta + r\dot{\varphi}\boldsymbol{e}_\varphi$.

Illustrative example



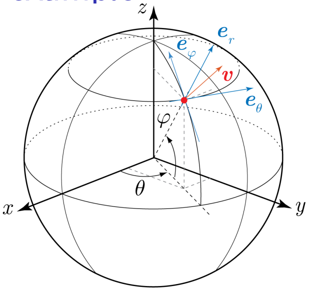


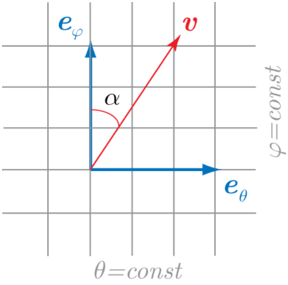
A particle moves on a sphere in such a way that it crosses all the meridians (coordinate lines of longitude $\theta = \text{const}$) at the same constant angle α. Find the trajectory of the particle.

- For $r = \text{const}$ we have $\boldsymbol{v} = \dot{\boldsymbol{p}} = r\dot{\theta}\cos\varphi \boldsymbol{e}_\theta + r\dot{\varphi}\boldsymbol{e}_\varphi$.
- Since the crossing angle $\alpha = \text{const}$ we have:

$$\frac{v_\varphi}{v_\theta} = \frac{r\dot{\varphi}}{r\dot{\theta}\cos\varphi} = \cot\alpha \quad\Longrightarrow\quad \frac{\mathrm{d}\varphi}{\cos\varphi} = \cot\alpha \mathrm{d}\theta$$

Illustrative example

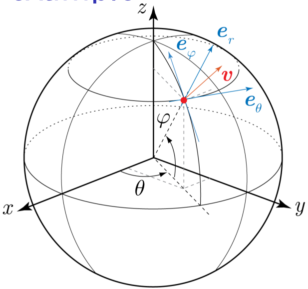


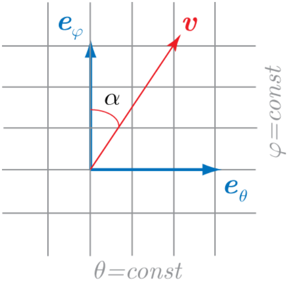


- Assume that at the equatorial line ($\varphi = 0$) the longitude of the particle $\theta = \theta_0$. Then

$$\int_0^\varphi \frac{\mathrm{d}\varphi}{\cos\varphi} = \int_{\theta_0}^\theta \cot\alpha \mathrm{d}\theta = (\theta - \theta_0)\cot\alpha$$

Illustrative example





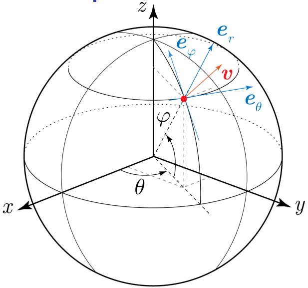
- Assume that at the equatorial line ($\varphi = 0$) the longitude of the particle $\theta = \theta_0$. Then

$$\int_0^\varphi \frac{d\varphi}{\cos\varphi} = \int_{\theta_0}^\theta \cot\alpha d\theta = (\theta - \theta_0)\cot\alpha$$

- On the other hand

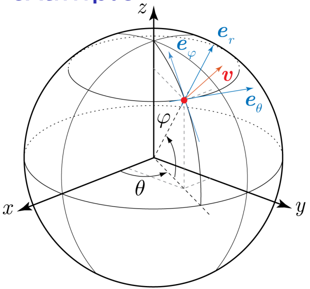
$$\int_0^\varphi \frac{d\varphi}{\cos\varphi} = \ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} = \ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$$

Illustrative example



$$\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=(\theta-\theta_0)\cot\alpha \implies \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=e^{(\theta-\theta_0)\cot\alpha}$$

Illustrative example

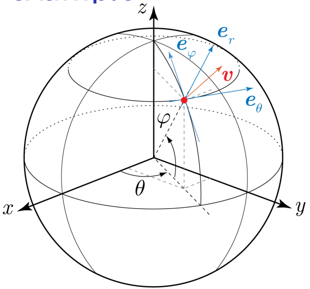


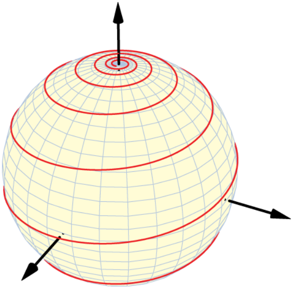
$$\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=(\theta-\theta_0)\cot\alpha \implies \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=e^{(\theta-\theta_0)\cot\alpha}$$

North and South poles:

- if $\cot\alpha > 0$, $\lim_{\theta\to\infty}\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\infty \implies \lim_{\theta\to\infty}\varphi=\frac{\pi}{2}$
- if $\cot\alpha < 0$, $\lim_{\theta\to\infty}\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=0 \implies \lim_{\theta\to\infty}\varphi=-\frac{\pi}{2}$

Illustrative example





$$\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=(\theta-\theta_0)\cot\alpha \implies \tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=e^{(\theta-\theta_0)\cot\alpha}$$

North and South poles:

- if $\cot\alpha>0$, $\displaystyle\lim_{\theta\to\infty}\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\infty \implies \lim_{\theta\to\infty}\varphi=\frac{\pi}{2}$
- if $\cot\alpha<0$, $\displaystyle\lim_{\theta\to\infty}\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=0 \implies \lim_{\theta\to\infty}\varphi=-\frac{\pi}{2}$

The trajectory is a spherical spiral called *loxodrome*