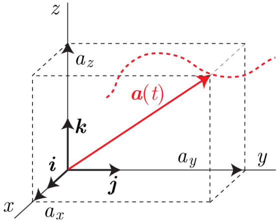
Time-dependent vectors



Definition

- time-dependent vector: $\boldsymbol{a} = a_x(t)\boldsymbol{i} + a_y(t)\boldsymbol{j} + a_z(t)\boldsymbol{k}$
  - $a_x(t), a_y(t), a_z(t)$ are called components (coordinates)
  - $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are called basis vectors
- magnitude (length): $|\boldsymbol{a}| = \sqrt{\boldsymbol{a}(t) \cdot \boldsymbol{a}(t)} = \sqrt{a_x^2 + a_y^2 + a_z^2}$
- directions (with respect to the basis vectors)

$$\cos(\widehat{\boldsymbol{a}(t), \boldsymbol{i}}) = \frac{a_x}{|\boldsymbol{a}|}, \quad \cos(\widehat{\boldsymbol{a}(t), \boldsymbol{j}}) = \frac{a_y}{|\boldsymbol{a}|}, \quad \cos(\widehat{\boldsymbol{a}(t), \boldsymbol{k}}) = \frac{a_z}{|\boldsymbol{a}|}.$$

Vector calculus

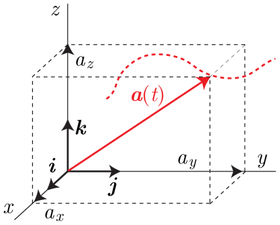
Differentiation and integration of vectors

- $$\frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{a}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{a}(t+\Delta t) - \boldsymbol{a}(t)}{\Delta t}$$
- If $\dfrac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} = \boldsymbol{b}(t) \quad \Longrightarrow \quad \boldsymbol{b}(t) = \boldsymbol{b}(t_0) + \displaystyle\int_{t_0}^{t} \boldsymbol{a}(\tau)\mathrm{d}\tau$

Differentiation of vector functions

- $$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{a}(t) + \boldsymbol{b}(t)\right\} = \frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{b}(t)}{\mathrm{d}t}$$
- If $\alpha(t)$ is a scalar function of time, then
  $$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\alpha(t)\boldsymbol{a}(t)\right\} = \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}\boldsymbol{a}(t) + \alpha(t)\frac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t}$$

Vector calculus



Operations in terms of vector components

$$\boldsymbol{a} = a_x(t)\boldsymbol{i} + a_y(t)\boldsymbol{j} + a_z(t)\boldsymbol{k}$$

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \frac{\mathrm{d}a_x}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}a_y}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}a_z}{\mathrm{d}t}\boldsymbol{k} + \underbrace{a_x\frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}t} + a_y\frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}t} + a_z\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}}_{\mathbf{0} \text{ since } \boldsymbol{i},\boldsymbol{j},\boldsymbol{k} \text{ constant}}$$

Vector calculus

Chain rule

If $\alpha(t)$ is a scalar function of time and $\boldsymbol{a}$ depends on α, then

- $\dfrac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{a}(\alpha(t))\right\} = \dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}\alpha}\dfrac{\mathrm{d}\alpha}{\mathrm{d}t}$

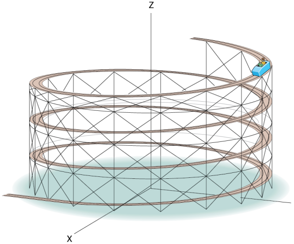
Derivative of dot and cross products

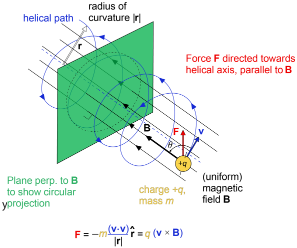
- $\dfrac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{a}(t) \cdot \boldsymbol{b}(t)\right\} = \dfrac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} \cdot \boldsymbol{b}(t) + \boldsymbol{a}(t) \cdot \dfrac{\mathrm{d}\boldsymbol{b}(t)}{\mathrm{d}t}$
- $\dfrac{\mathrm{d}}{\mathrm{d}t}\left\{\boldsymbol{a}(t) \times \boldsymbol{b}(t)\right\} = \dfrac{\mathrm{d}\boldsymbol{a}(t)}{\mathrm{d}t} \times \boldsymbol{b}(t) + \boldsymbol{a}(t) \times \dfrac{\mathrm{d}\boldsymbol{b}(t)}{\mathrm{d}t}$

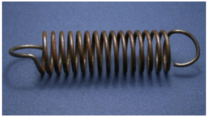
Derivative of a vector of constant length

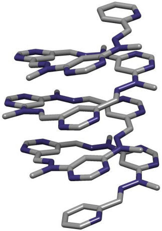
If $|\boldsymbol{a}|$ is constant then $\boldsymbol{a} \cdot \boldsymbol{a} = \text{const} \Longrightarrow \dfrac{\mathrm{d}(\boldsymbol{a} \cdot \boldsymbol{a})}{\mathrm{d}t} =$
$\dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = 2\boldsymbol{a} \cdot \dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = 0 \Longrightarrow \boldsymbol{a} \cdot \dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = 0.$ Thus, $\boldsymbol{a}$ and $\dfrac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}$ are perpendicular.

Helix


z
y
x

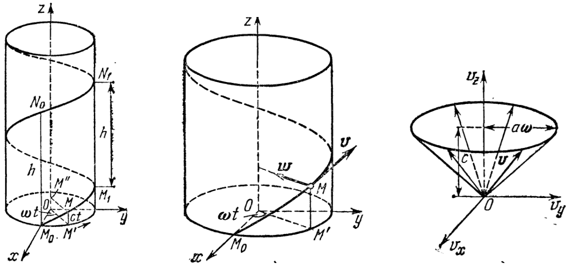

helical path
radius of curvature |**r**|
r
Force **F** directed towards helical axis, parallel to **B**
B
F
v
θ
+q
Plane perp. to **B** to show circular projection
charge +q, mass m
(uniform) magnetic field **B**
$\mathbf{F} = -m\frac{(\mathbf{v}\cdot\mathbf{v})}{|\mathbf{r}|}\hat{\mathbf{r}} = q\,(\mathbf{v}\times\mathbf{B})$





Motion along a helix

Parameters
of the curve:
period $T = 2\pi/\omega$
pitch $h = cT$



Position

$$\boldsymbol{r} = \begin{bmatrix} a\cos\omega t \\ a\sin\omega t \\ ct \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} -a\omega\sin\omega t \\ a\omega\cos\omega t \\ c \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} -a\omega^2\cos\omega t \\ -a\omega^2\sin\omega t \\ 0 \end{bmatrix}$$

Features: $\boldsymbol{v}\cdot\boldsymbol{v} = a^2\omega^2 + c^2 =$ const, $\boldsymbol{v}\cdot\boldsymbol{k} =$ const, (these follows from $\boldsymbol{w} = \boldsymbol{k}\times\boldsymbol{v}$). Also, $\ddot{x} = -\omega^2 x$ and $\ddot{y} = -\omega^2 y$