Physics for Computer Science
Sample Problems # 1

Date Given: April 6, 2022

P1. Compute $\boldsymbol{a} \cdot \boldsymbol{b}$ for

(a) $\boldsymbol{a} = (3, 2, -5)$ and $\boldsymbol{b} = (10, 1, 2)$

(b) $\boldsymbol{a} = (1, 0, 3)$ and $\boldsymbol{b} = (-4, 15, 1)$

(c) $\boldsymbol{a} = (2, 1, 5)$ and $\boldsymbol{b} = (7, -9, -1)$

Solution:

(a) $\boldsymbol{a} \cdot \boldsymbol{b} = (3 \times 10 + 2 \times 1 - 5 \times 2) = 20$

(b) $\boldsymbol{a} \cdot \boldsymbol{b} = (-1 \times 4 + 0 \times 15 + 3 \times 1) = -1$

(c) $\boldsymbol{a} \cdot \boldsymbol{b} = (2 \times 7 - 1 \times 9 - 5 \times 1) = 0$

P2. If $\boldsymbol{v} + \boldsymbol{w} = (5, 1)$ and $\boldsymbol{v} - \boldsymbol{w} = (1, 5)$, compute and draw $\boldsymbol{v}$ and $\boldsymbol{w}$.

Solution: Let $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$. Then, we have: $v_1 + w_1 = 5, v_2 + w_2 = 1$, and $v_1 - w_1 = 1, v_2 - w_2 = 5$. By solving these equations one finds $v_1 = 3, v_2 = 3$, and $w_1 = 2, w_2 = -2$. The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are drawn in Figure 1.

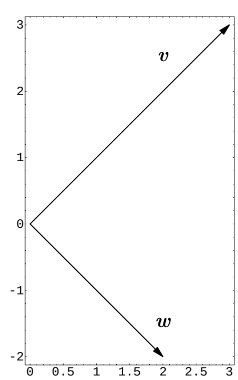

Figure 1: Illustration to Problem P2.

P3. The parallelogram in Figure 2 has diagonal $\boldsymbol{v} + \boldsymbol{w}$. What is the other diagonal? What is the sum of the two diagonals? Draw that vector sum.

Solution: The other diagonal is $\boldsymbol{v} - \boldsymbol{w}$ and therefore $(\boldsymbol{v} + \boldsymbol{w})$+$(\boldsymbol{v} - \boldsymbol{w})$=$2\boldsymbol{v}$. Note that in vector notation the other diagonal can also be shown as $\boldsymbol{w} - \boldsymbol{v}$. In this case $(\boldsymbol{v} + \boldsymbol{w})$+$(\boldsymbol{w} - \boldsymbol{v})$=$2\boldsymbol{w}$.

P4. Find two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ that are perpendicular to $(1, 0, 1)$ and to each other.

Solution: There are many ways to go about this. One way would be to write $\boldsymbol{v} = (v_1, v_2, v_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$ and then to write down the equations that $\boldsymbol{v}$ and $\boldsymbol{w}$ must satisfy. These are: $\boldsymbol{v} \cdot (1, 0, 1) = 0$, $\boldsymbol{w} \cdot (1, 0, 1) = 0$, and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. This gives a system of three equations with six unknowns (the v_i and w_i), that you could then try to solve to find the set of all possible $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfying these requirements.

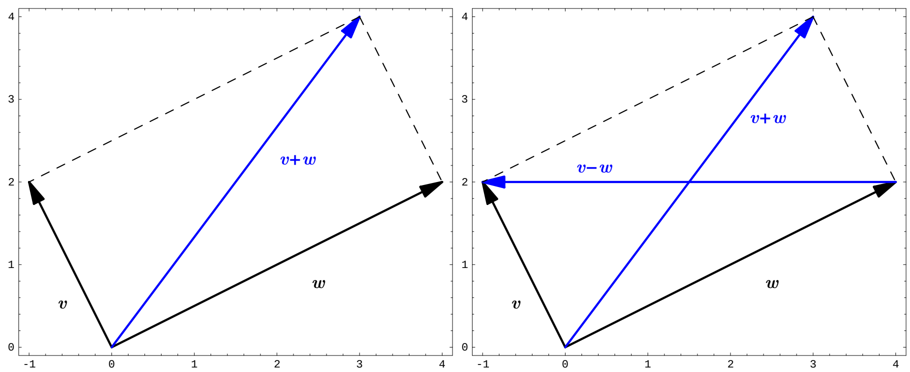

Figure 2: Illustration to Problem P3.

On the other hand, the problem just asks you to find one specific choice of $\boldsymbol{v}$ and $\boldsymbol{w}$, not all possible choices. It is easiest first to fix appropriately the components of $\boldsymbol{v}$ and proceed then by inspection: $\boldsymbol{v} = (1, 1, -1)$ is clearly perpendicular to $(1, 0, 1)$ since their dot product is $\boldsymbol{v}\cdot(1, 0, 1) = 1\times1+1\times0-1\times1 = 0$. So, we just have to find a $\boldsymbol{w} = (w_1, w_2, w_3)$ that is perpendicular to both of these vectors. This $\boldsymbol{w}$ must satisfy $\boldsymbol{w}\cdot(1, 0, 1) = w_1 + w_3 = 0$ and $\boldsymbol{v}\cdot\boldsymbol{w} = (1, 1, -1)\cdot\boldsymbol{w} = w_1 + w_2 - w_3 = 0$. There are infinitely many solutions to this system of equations; one particular solution is $\boldsymbol{w} = (-1, 2, 1)$. There are many others.

P5. Find a distance from a line through vector $\boldsymbol{a}$ to the point M given by vector $\boldsymbol{b}$ as shown in Figure 3.

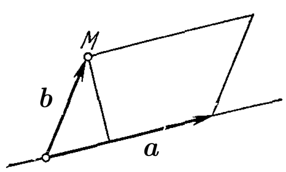

Figure 3: Illustration to Problem P5.

Solution: The area of the parallelogram built on the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $S = |\boldsymbol{a}\times\boldsymbol{b}|$. On the other hand, the same area $S = |a|h$ where h is the distance from point M to the line through the vector $\boldsymbol{a}$. Therefore $h = |\boldsymbol{a}\times\boldsymbol{b}|/|\boldsymbol{a}|$.

P6. Find the angle between the line through vector $\boldsymbol{c}$ and the plane spanned by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in Figure 4.

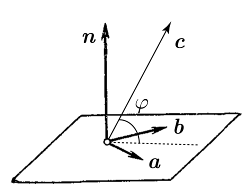

Figure 4: Illustration to Problem P6.

Solution: The normal vector to the plane spanned by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{n} = \boldsymbol{a}\times\boldsymbol{b}$. The angle between $\boldsymbol{n}$ and $\boldsymbol{c}$ is defined as $\cos(\frac{\pi}{2} - \varphi) = \frac{\boldsymbol{n}\cdot\boldsymbol{c}}{|\boldsymbol{n}||\boldsymbol{c}|}$. Therefore $\sin\varphi = \frac{\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})}{|\boldsymbol{c}||\boldsymbol{a}\times\boldsymbol{b}|}$.

P7. The plane π_1 is formed by the vectors $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$, and the plane π_2 is formed by the vectors $\boldsymbol{a}_2$ and $\boldsymbol{b}_2$ (see Figure 5). Find the vector along the line of intersection of π_1 and π_2.

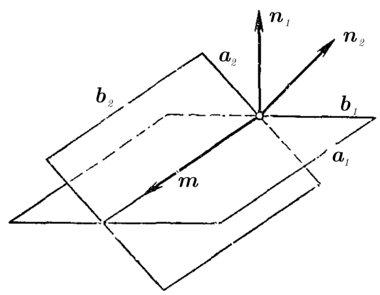

Figure 5: Illustration to Problem P7.

Solution: The normal vectors to the planes are $\boldsymbol{n}_1 = \boldsymbol{a}_1 \times \boldsymbol{b}_1$ and $\boldsymbol{n}_2 = \boldsymbol{a}_2 \times \boldsymbol{b}_2$. Therefore, the vector along the intersection line is defined as

$$\boldsymbol{m} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = (\boldsymbol{a}_1 \times \boldsymbol{b}_1) \times (\boldsymbol{a}_2 \times \boldsymbol{b}_2).$$

P8. Find the volume of a tetrahedron built on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ as shown in Figure 6.

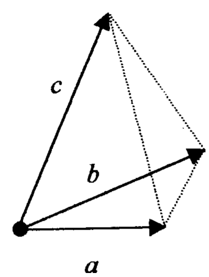

Figure 6: Illustration to Problem P8.

Solution: The volume of an arbitrary tetrahedron is one sixth the volume of the parallelepiped built on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Therefore

$$V_{\text{tetrahedron}} = \frac{1}{6} \left| \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \right|$$

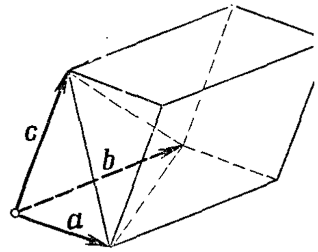

Figure 7: Illustration to Problem P8.