SECTION B WORK AND ENERGY

3/6 Work and Kinetic Energy

In the previous two articles, we applied Newton's second law $\mathbf{F} = m\mathbf{a}$ to various problems of particle motion to establish the instantaneous relationship between the net force acting on a particle and the resulting acceleration of the particle. When we needed to determine the change in velocity or the corresponding displacement of the particle, we integrated the computed acceleration by using the appropriate kinematic equations.

There are two general classes of problems in which the cumulative effects of unbalanced forces acting on a particle are of interest to us. These cases involve (1) integration of the forces with respect to the displacement of the particle and (2) integration of the forces with respect to the time they are applied. We may incorporate the results of these integrations directly into the governing equations of motion so that it becomes unnecessary to solve directly for the acceleration. Integration with respect to displacement leads to the equations of work and energy, which are the subject of this article. Integration with respect to time leads to the equations of impulse and momentum, discussed in Section C.

Definition of Work

We now develop the quantitative meaning of the term "work."* Figure 3/2a shows a force $\mathbf{F}$ acting on a particle at A which moves along the path shown. The position vector $\mathbf{r}$ measured from some convenient origin O locates the particle as it passes point A, and $d\mathbf{r}$ is the differential displacement associated with an infinitesimal movement from A to A'. The work done by the force $\mathbf{F}$ during the displacement $d\mathbf{r}$ is defined as

$$dU = \mathbf{F}\cdot d\mathbf{r}$$

The magnitude of this dot product is $dU = F\,ds \cos\alpha$, where α is the angle between $\mathbf{F}$ and $d\mathbf{r}$ and where ds is the magnitude of $d\mathbf{r}$. This expression may be interpreted as the displacement multiplied by the force component $F_t = F \cos\alpha$ in the direction of the displacement, as represented by the dashed lines in Fig. 3/2b. Alternatively, the work dU may be interpreted as the force multiplied by the displacement component $ds \cos\alpha$ in the direction of the force, as represented by the full lines in Fig. 3/2b.

With this definition of work, it should be noted that the component $F_n = F \sin\alpha$ normal to the displacement does no work. Thus, the work dU may be written as

$$dU = F_t\,ds$$

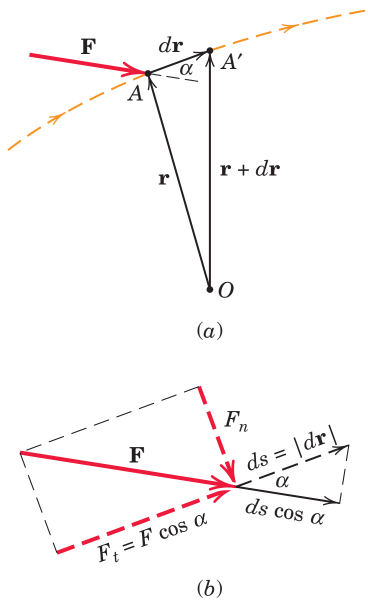


Figure 3/2

*The concept of work was also developed in the study of virtual work in Chapter 7 of *Vol. 1 Statics*.

Work is positive if the working component F_t is in the direction of the displacement and negative if it is in the opposite direction. Forces which do work are termed *active forces*. Constraint forces which do no work are termed *reactive forces*.

Units of Work

The SI units of work are those of force (N) times displacement (m) or N·m. This unit is given the special name *joule* (J), which is defined as the work done by a force of 1 N acting through a distance of 1 m in the direction of the force. Consistent use of the joule for work (and energy) rather than the units N·m will avoid possible ambiguity with the units of moment of a force or torque, which are also written N·m.

In the U.S. customary system, work has the units of ft-lb. Dimensionally, work and moment are the same. In order to distinguish between the two quantities, it is recommended that work be expressed as foot pounds (ft-lb) and moment as pound feet (lb-ft). It should be noted that work is a scalar as given by the dot product and involves the product of a force and a distance, both measured along the same line. Moment, on the other hand, is a vector as given by the cross product and involves the product of force and distance measured at right angles to the force.

Calculation of Work

During a finite movement of the point of application of a force, the force does an amount of work equal to

$$U = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (F_x\,dx + F_y\,dy + F_z\,dz)$$

or

$$U = \int_{s_1}^{s_2} F_t\,ds$$

In order to carry out this integration, it is necessary to know the relations between the force components and their respective coordinates or the relation between F_t and s. If the functional relationship is not known as a mathematical expression which can be integrated but is specified in the form of approximate or experimental data, then we can compute the work by carrying out a numerical or graphical integration as represented by the area under the curve of F_t versus s, as shown in Fig. 3/3.

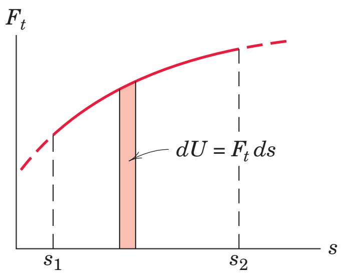


Figure 3/3

Examples of Work

When work must be calculated, we may always begin with the definition of work, $U = \int \mathbf{F}\cdot d\mathbf{r}$, insert appropriate vector expressions for the force $\mathbf{F}$ and the differential displacement vector $d\mathbf{r}$, and carry out the required integration. With some experience, simple work calculations, such as those associated with constant forces, may be performed by inspection. We now formally compute the work associated with three frequently occurring forces: constant forces, spring forces, and weights.

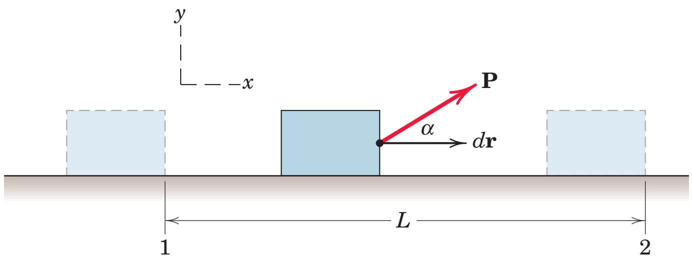


Figure 3/4

(1) Work Associated with a Constant External Force. Consider the constant force **P** applied to the body as it moves from position 1 to position 2, Fig. 3/4. With the force **P** and the differential displacement $d\mathbf{r}$ written as vectors, the work done on the body by the force is

$$U_{1\text{-}2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 [(P\cos\alpha)\mathbf{i} + (P\sin\alpha)\mathbf{j}] \cdot dx\,\mathbf{i}$$

$$= \int_{x_1}^{x_2} P\cos\alpha\, dx = P\cos\alpha(x_2 - x_1) = PL\cos\alpha \qquad \textbf{(3/9)}$$

As previously discussed, this work expression may be interpreted as the force component $P\cos\alpha$ times the distance L traveled. Should α be between 90° and 270°, the work would be negative. The force component $P\sin\alpha$ normal to the displacement does no work.

(2) Work Associated with a Spring Force. We consider here the common linear spring of stiffness k where the force required to stretch or compress the spring is proportional to the deformation x, as shown in Fig. 3/5*a*. We wish to determine the work done on the body by the spring force as the body undergoes an arbitrary displacement from an initial position x_1 to a final position x_2. The force exerted by the spring on the body is $\mathbf{F} = -kx\mathbf{i}$, as shown in Fig. 3/5*b*. From the definition of work, we have

$$U_{1\text{-}2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (-kx\mathbf{i}) \cdot dx\,\mathbf{i} = -\int_{x_1}^{x_2} kx\, dx = \frac{1}{2}k({x_1}^2 - {x_2}^2) \qquad \textbf{(3/10)}$$

If the initial position is the position of zero spring deformation so that $x_1 = 0$, then the work is negative for any final position $x_2 \neq 0$. This is verified by recognizing that if the body begins at the undeformed spring position and then moves to the right, the spring force is to the left; if the body begins at $x_1 = 0$ and moves to the left, the spring force is to the right. On the other hand, if we move from an arbitrary initial position $x_1 \neq 0$ to the undeformed final position $x_2 = 0$, we see that the work is positive. In any movement toward the undeformed spring position, the spring force and the displacement are in the same direction.

In the *general* case, of course, neither x_1 nor x_2 is zero. The magnitude of the work is equal to the shaded trapezoidal area of Fig. 3/5*a*. In calculating the work done on a body by a spring force, care must be

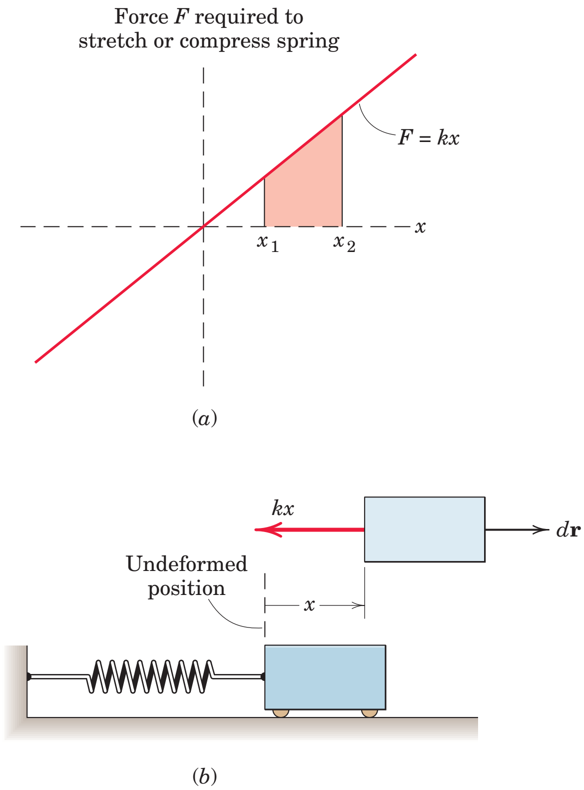


Figure 3/5

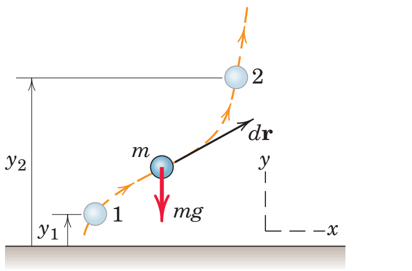
taken to ensure that the units of k and x are consistent. If x is in meters (or feet), k must be in N/m (or lb/ft). In addition, be sure to recognize that the variable x represents a deformation from the unstretched spring length and *not* the total length of the spring.

The expression $F = kx$ is actually a static relationship which is true only when elements of the spring have no acceleration. The dynamic behavior of a spring when its mass is accounted for is a fairly complex problem which will not be treated here. We shall assume that the mass of the spring is small compared with the masses of other accelerating parts of the system, in which case the linear static relationship will not involve appreciable error.

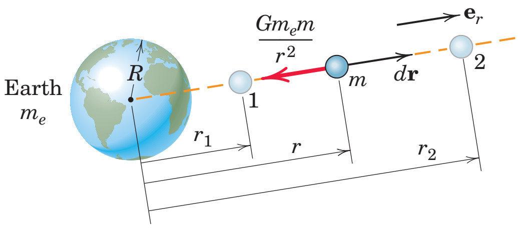
(3) Work Associated with Weight. *Case (a) g = constant.* If the altitude variation is sufficiently small so that the acceleration of gravity g may be considered constant, the work done by the weight mg of the body shown in Fig. 3/6a as the body is displaced from an arbitrary altitude y_1 to a final altitude y_2 is

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (-mg\mathbf{j})\cdot(dx\mathbf{i} + dy\mathbf{j})$$

$$= -mg\int_{y_1}^{y_2} dy = -mg(y_2 - y_1) \qquad \textbf{(3/11)}$$



(*a*)



(*b*)

Figure 3/6

We see that horizontal movement does not contribute to this work. We also note that if the body rises (perhaps due to other forces not shown), then $(y_2 - y_1) > 0$ and this work is negative. If the body falls, $(y_2 - y_1) < 0$ and the work is positive.

Case (b) $g \neq$ constant. If large changes in altitude occur, then the weight (gravitational force) is no longer constant. We must therefore use the gravitational law (Eq. 1/2) and express the weight as a variable force of magnitude $F = \dfrac{Gm_e m}{r^2}$, as indicated in Fig. 3/6*b*. Using the radial coordinate shown in the figure allows the work to be expressed as

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{-Gm_e m}{r^2}\,\mathbf{e}_r\cdot dr\,\mathbf{e}_r = -Gm_e m\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= Gm_e m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad \textbf{(3/12)}$$

where the equivalence $Gm_e = gR^2$ was established in Art. 1/5, with g representing the acceleration of gravity at the earth's surface and R representing the radius of the earth. The student should verify that if a body rises to a higher altitude $(r_2 > r_1)$, this work is negative, as it was in case (*a*). If the body falls to a lower altitude $(r_2 < r_1)$, the work is positive. Be sure to realize that r represents a radial distance from the center of the earth and not an altitude $h = r - R$ above the surface of the earth. As in case (*a*), had we considered a transverse displacement in addition to the radial displacement shown in Fig. 3/6*b*, we would have concluded that the transverse displacement, because it is perpendicular to the weight, does not contribute to the work.

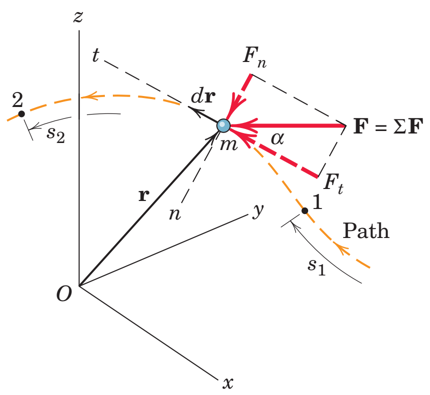


Figure 3/7

Work and Curvilinear Motion

We now consider the work done on a particle of mass m, Fig. 3/7, moving along a curved path under the action of the force $\mathbf{F}$, which stands for the resultant $\Sigma\mathbf{F}$ of all forces acting on the particle. The position of m is specified by the position vector $\mathbf{r}$, and its displacement along its path during the time dt is represented by the change $d\mathbf{r}$ in its position vector. The work done by $\mathbf{F}$ during a finite movement of the particle from point 1 to point 2 is

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_{s_1}^{s_2} F_t\, ds$$

where the limits specify the initial and final end points of the motion.

When we substitute Newton's second law $\mathbf{F} = m\mathbf{a}$, the expression for the work of all forces becomes

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 m\mathbf{a}\cdot d\mathbf{r}$$

But $\mathbf{a}\cdot d\mathbf{r} = a_t\, ds$, where a_t is the tangential component of the acceleration of m. In terms of the velocity v of the particle, Eq. 2/3 gives $a_t ds = v\, dv$. Thus, the expression for the work of $\mathbf{F}$ becomes

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_{v_1}^{v_2} mv\, dv = \frac{1}{2}m({v_2}^2 - {v_1}^2) \tag{3/13}$$

where the integration is carried out between points 1 and 2 along the curve, at which points the velocities have the magnitudes v_1 and v_2, respectively.

Principle of Work and Kinetic Energy

The *kinetic energy* T of the particle is defined as

$$\boxed{T = \frac{1}{2}mv^2} \tag{3/14}$$

and is the total work which must be done on the particle to bring it from a state of rest to a velocity v. Kinetic energy T is a scalar quantity with the units of N·m or joules (J) in SI units and ft-lb in U.S. customary units. Kinetic energy is *always* positive, regardless of the direction of the velocity.

Equation 3/13 may be restated as

$$U_{1\text{-}2} = T_2 - T_1 = \Delta T \tag{3/15}$$

which is the *work-energy equation* for a particle. The equation states that the *total work done* by all forces acting on a particle as it moves from point 1 to point 2 equals the corresponding *change in kinetic energy* of the particle. Although T is always positive, the change ΔT may be positive, negative, or zero. When written in this concise form, Eq. 3/15 tells us that the work always results in a *change* of kinetic energy.

Alternatively, the work-energy relation may be expressed as the initial kinetic energy T_1 plus the work done $U_{1\text{-}2}$ equals the final kinetic energy T_2, or

$$\boxed{T_1 + U_{1\text{-}2} = T_2} \qquad \textbf{(3/15}\boldsymbol{a}\textbf{)}$$

When written in this form, the terms correspond to the natural sequence of events. Clearly, the two forms 3/15 and 3/15*a* are equivalent.

Advantages of the Work-Energy Method

We now see from Eq. 3/15 that a major advantage of the method of work and energy is that it avoids the necessity of computing the acceleration and leads directly to the velocity changes as functions of the forces which do work. Further, the work-energy equation involves only those forces which do work and thus give rise to changes in the magnitude of the velocities.

We consider now a system of two particles joined together by a connection which is frictionless and incapable of any deformation. The forces in the connection are equal and opposite, and their points of application necessarily have identical displacement components in the direction of the forces. Therefore, the net work done by these internal forces is zero during any movement of the system. Thus, Eq. 3/15 is applicable to the entire system, where $U_{1\text{-}2}$ is the total or net work done on the system by forces external to it and ΔT is the change, $T_2 - T_1$, in the total kinetic energy of the system. The total kinetic energy is the sum of the kinetic energies of both elements of the system. We thus see that another advantage of the work-energy method is that it enables us to analyze a system of particles joined in the manner described without dismembering the system.

Application of the work-energy method requires isolation of the particle or system under consideration. For a single particle you should draw a *free-body diagram* showing all externally applied forces. For a system of particles rigidly connected without springs, draw an *active-force diagram* showing only those external forces which do work (active forces) on the entire system.*

Power

The capacity of a machine is measured by the time rate at which it can do work or deliver energy. The total work or energy output is not a measure of this capacity since a motor, no matter how small, can deliver a large amount of energy if given sufficient time. On the other hand, a large and powerful machine is required to deliver a large amount of energy in a short period of time. Thus, the capacity of a machine is rated by its *power*, which is defined as the *time rate of doing work*.

*The active-force diagram was introduced in the method of virtual work in statics. See Chapter 7 of *Vol. 1 Statics*.

The power which must be produced by a bike rider depends on the bicycle speed and the propulsive force which is exerted by the supporting surface on the rear wheel. The driving force depends on the slope being negotiated.

Accordingly, the power P developed by a force $\mathbf{F}$ which does an amount of work U is $P = dU/dt = \mathbf{F}\cdot d\mathbf{r}/dt$. Because $d\mathbf{r}/dt$ is the velocity $\mathbf{v}$ of the point of application of the force, we have

$$\boxed{P = \mathbf{F}\cdot\mathbf{v}} \qquad \textbf{(3/16)}$$

Power is clearly a scalar quantity, and in SI it has the units of N·m/s = J/s. The special unit for power is the *watt* (W), which equals one joule per second (J/s). In U.S. customary units, the unit for mechanical power is the *horsepower* (hp). These units and their numerical equivalences are

$$1 \text{ W} = 1 \text{ J/s}$$

$$1 \text{ hp} = 550 \text{ ft-lb/sec} = 33{,}000 \text{ ft-lb/min}$$

$$1 \text{ hp} = 746 \text{ W} = 0.746 \text{ kW}$$

Efficiency

The ratio of the work done *by* a machine to the work done *on* the machine during the same time interval is called the *mechanical efficiency* e_m of the machine. This definition assumes that the machine operates uniformly so that there is no accumulation or depletion of energy within it. Efficiency is always less than unity since every device operates with some loss of energy and since energy cannot be created within the machine. In mechanical devices which involve moving parts, there will always be some loss of energy due to the negative work of kinetic friction forces. This work is converted to heat energy which, in turn, is dissipated to the surroundings. The mechanical efficiency at any instant of time may be expressed in terms of mechanical power P by

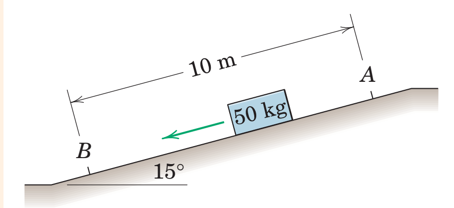
$$e_m = \frac{P_{\text{output}}}{P_{\text{input}}} \qquad \textbf{(3/17)}$$

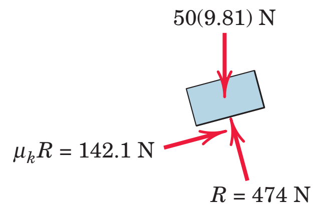
In addition to energy loss by mechanical friction, there may also be electrical and thermal energy loss, in which case, the *electrical efficiency* e_e and *thermal efficiency* e_t are also involved. The *overall efficiency* e in such instances is

$$e = e_m e_e e_t$$

Sample Problem 3/11

Calculate the velocity v of the 50-kg crate when it reaches the bottom of the chute at B if it is given an initial velocity of 4 m/s down the chute at A. The coefficient of kinetic friction is 0.30.





Solution. The free-body diagram of the crate is drawn and includes the normal force R and the kinetic friction force F calculated in the usual manner. The work done by the weight is positive, whereas that done by the friction force is negative. The total work done on the crate during the motion is

① $[U = Fs]$ $\qquad U_{1\text{-}2} = 50(9.81)(10 \sin 15°) - 142.1(10) = -151.9 \text{ J}$

The work-energy equation gives

$$[T_1 + U_{1\text{-}2} = T_2] \qquad \frac{1}{2}mv_1{}^2 + U_{1\text{-}2} = \frac{1}{2}mv_2{}^2$$

$$\frac{1}{2}(50)(4)^2 - 151.9 = \frac{1}{2}(50)v_2{}^2$$

$$v_2 = 3.15 \text{ m/s} \qquad \textit{Ans.}$$

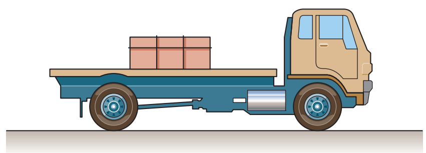
Since the net work done is negative, we obtain a decrease in the kinetic energy.

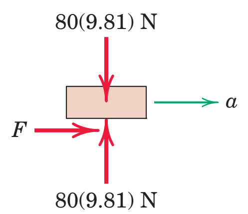
Helpful Hint

① The work due to the weight depends only on the *vertical* distance traveled.

Sample Problem 3/12

The flatbed truck, which carries an 80-kg crate, starts from rest and attains a speed of 72 km/h in a distance of 75 m on a level road with constant acceleration. Calculate the work done by the friction force acting on the crate during this interval if the static and kinetic coefficients of friction between the crate and the truck bed are (a) 0.30 and 0.28, respectively, or (b) 0.25 and 0.20, respectively.





Solution. If the crate does not slip on the bed, its acceleration will be that of the truck, which is

$$[v^2 = 2as] \qquad a = \frac{v^2}{2s} = \frac{(72/3.6)^2}{2(75)} = 2.67 \text{ m/s}^2$$

Case (a). This acceleration requires a friction force on the block of

$$[F = ma] \qquad F = 80(2.67) = 213 \text{ N}$$

which is less than the maximum possible value of $\mu_s N = 0.30(80)(9.81) = 235$ N. Therefore, the crate does not slip and the work done by the actual static friction force of 213 N is

① $[U = Fs]$ $\qquad U_{1\text{-}2} = 213(75) = 16\,000 \text{ J} \quad \text{or} \quad 16 \text{ kJ}$ $\qquad$ *Ans.*

Case (b). For $\mu_s = 0.25$, the maximum possible friction force is $0.25(80)(9.81) = 196.2$ N, which is slightly less than the value of 213 N required for no slipping. Therefore, we conclude that the crate slips, and the friction force is governed by the kinetic coefficient and is $F = 0.20(80)(9.81) = 157.0$ N. The acceleration becomes

$$[F = ma] \qquad a = F/m = 157.0/80 = 1.962 \text{ m/s}^2$$

The distances traveled by the crate and the truck are in proportion to their accelerations. Thus, the crate has a displacement of $(1.962/2.67)75 = 55.2$ m, and the work done by kinetic friction is

② $[U = Fs]$ $\qquad U_{1\text{-}2} = 157.0(55.2) = 8660 \text{ J} \quad \text{or} \quad 8.66 \text{ kJ}$ $\qquad$ *Ans.*

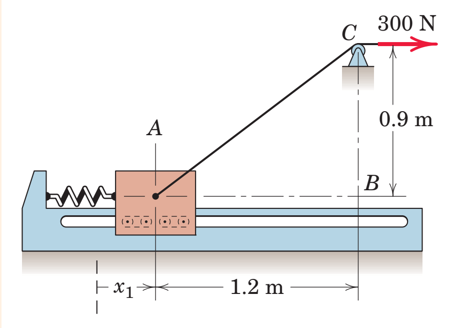
Helpful Hints

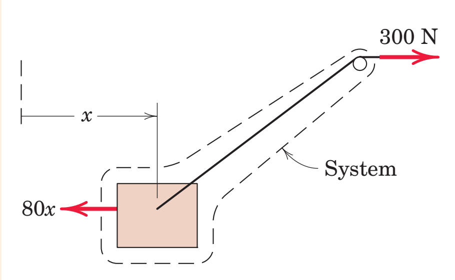
① We note that static friction forces do no work when the contacting surfaces are both at rest. When they are in motion, however, as in this problem, the static friction force acting on the crate does positive work and that acting on the truck bed does negative work.

② This problem shows that a kinetic friction force can do positive work when the surface which supports the object and generates the friction force is in motion. If the supporting surface is at rest, then the kinetic friction force acting on the moving part always does negative work.

Sample Problem 3/13

The 50-kg block at A is mounted on rollers so that it moves along the fixed horizontal rail with negligible friction under the action of the constant 300-N force in the cable. The block is released from rest at A, with the spring to which it is attached extended an initial amount $x_1 = 0.233$ m. The spring has a stiffness $k = 80$ N/m. Calculate the velocity v of the block as it reaches position B.





Solution. It will be assumed initially that the stiffness of the spring is small enough to allow the block to reach position B. The active-force diagram for the system composed of both block and cable is shown for a general position. The spring force $80x$ and the 300-N tension are the only forces external to this system which do work on the system. The force exerted on the block by the rail, the weight of the block, and the reaction of the small pulley on the cable do no work on the system and are not included on the active-force diagram.

As the block moves from $x_1 = 0.233$ m to $x_2 = 0.233 + 1.2 = 1.433$ m, the work done by the spring force acting on the block is

① $\left[U_{1\text{-}2} = \frac{1}{2}k(x_1{}^2 - x_2{}^2)\right]$ $\quad U_{1\text{-}2} = \frac{1}{2}\,80\left[0.233^2 - (0.233 + 1.2)^2\right]$

$$= -80.0 \text{ J}$$

The work done on the system by the constant 300-N force in the cable is the force times the net horizontal movement of the cable over pulley C, which is $\sqrt{(1.2)^2 + (0.9)^2} - 0.9 = 0.6$ m. Thus, the work done is $300(0.6) = 180$ J. We now apply the work-energy equation to the system and get

$$[T_1 + U_{1\text{-}2} = T_2] \qquad 0 - 80.0 + 180 = \tfrac{1}{2}(50)v^2 \qquad v = 2.00 \text{ m/s} \qquad \textit{Ans.}$$

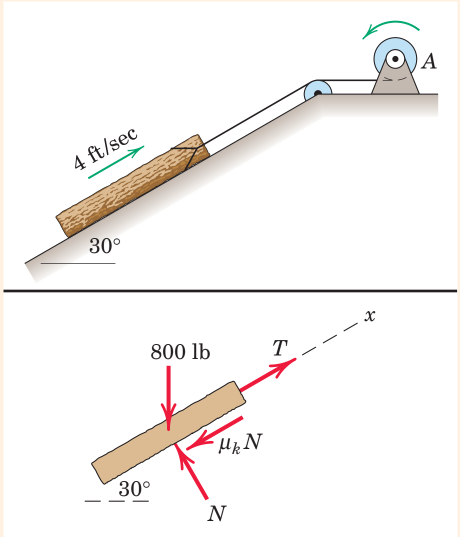
We take special note of the advantage to our choice of system. If the block alone had constituted the system, the horizontal component of the 300-N cable tension on the block would have to be integrated over the 1.2-m displacement. This step would require considerably more effort than was needed in the solution as presented. If there had been appreciable friction between the block and its guiding rail, we would have found it necessary to isolate the block alone in order to compute the variable normal force and, hence, the variable friction force. Integration of the friction force over the displacement would then be required to evaluate the negative work which it would do.

Helpful Hint

① Recall that this general formula is valid for any initial and final spring deflections x_1 and x_2, positive (spring in tension) or negative (spring in compression). In deriving the spring-work formula, we assumed the spring to be linear, which is the case here.

Sample Problem 3/14

The power winch A hoists the 800-lb log up the 30° incline at a constant speed of 4 ft/sec. If the power output of the winch is 6 hp, compute the coefficient of kinetic friction μ_k between the log and the incline. If the power is suddenly increased to 8 hp, what is the corresponding instantaneous acceleration a of the log?



Solution. From the free-body diagram of the log, we get $N = 800 \cos 30° = 693$ lb, and the kinetic friction force becomes $693\mu_k$. For constant speed, the forces are in equilibrium so that

$$[\Sigma F_x = 0] \qquad T - 693\mu_k - 800 \sin 30° = 0 \qquad T = 693\mu_k + 400$$

The power output of the winch gives the tension in the cable

① $$[P = Tv] \qquad T = P/v = 6(550)/4 = 825 \text{ lb}$$

Substituting T gives

$$825 = 693\mu_k + 400 \qquad \mu_k = 0.613 \qquad \textit{Ans.}$$

When the power is increased, the tension momentarily becomes

$$[P = Tv] \qquad T = P/v = 8(550)/4 = 1100 \text{ lb}$$

and the corresponding acceleration is given by

$$[\Sigma F_x = ma_x] \qquad 1100 - 693(0.613) - 800 \sin 30° = \frac{800}{32.2}a$$

② $$a = 11.07 \text{ ft/sec}^2 \qquad \textit{Ans.}$$

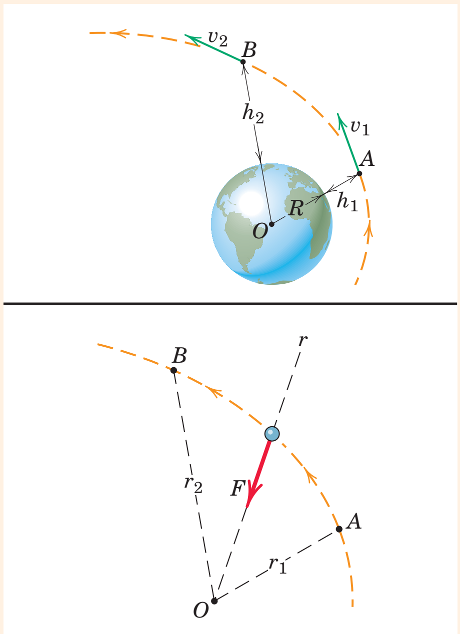
Helpful Hints

① Note the conversion from horsepower to ft-lb/sec.

② As the speed increases, the acceleration will drop until the speed stabilizes at a value higher than 4 ft/sec.

Sample Problem 3/15

A satellite of mass m is put into an elliptical orbit around the earth. At point A, its distance from the earth is $h_1 = 500$ km and it has a velocity $v_1 = 30\,000$ km/h. Determine the velocity v_2 of the satellite as it reaches point B, a distance $h_2 = 1200$ km from the earth.



Solution. The satellite is moving outside of the earth's atmosphere so that the only force acting on it is the gravitational attraction of the earth. For the large change in altitude of this problem, we cannot assume that the acceleration due to gravity is constant. Rather, we must use the work expression, derived in this article, which accounts for variation in the gravitational acceleration with altitude. Put another way, the work expression accounts for the variation of the weight $F = \dfrac{Gmm_e}{r^2}$ with altitude. This work expression is

$$U_{1\text{-}2} = mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

The work-energy equation $T_1 + U_{1\text{-}2} = T_2$ gives

① ② $$\tfrac{1}{2}mv_1^2 + mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \tfrac{1}{2}mv_2^2 \qquad v_2^2 = v_1^2 + 2gR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Substituting the numerical values gives

$$v_2^2 = \left(\frac{30\,000}{3.6}\right)^2 + 2(9.81)[(6371)(10^3)]^2\left(\frac{10^{-3}}{6371 + 1200} - \frac{10^{-3}}{6371 + 500}\right)$$

$$= 69.44(10^6) - 10.72(10^6) = 58.73(10^6)(\text{m/s})^2$$

$$v_2 = 7663 \text{ m/s} \qquad \text{or} \qquad v_2 = 7663(3.6) = 27\,590 \text{ km/h} \qquad \textit{Ans.}$$

Helpful Hints

① Note that the result is independent of the mass m of the satellite.

② Consult Table D/2, Appendix D, to find the radius R of the earth.