Physics for Computer Science

Introduction to Vectors and Scalars

Contents

Vector Algebra (1)

Two quantities are used in Mechanics

Scalars

Only a magnitude is associated

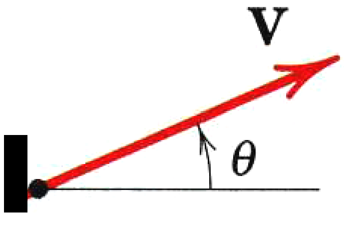
Vectors

Possess both magnitude and direction

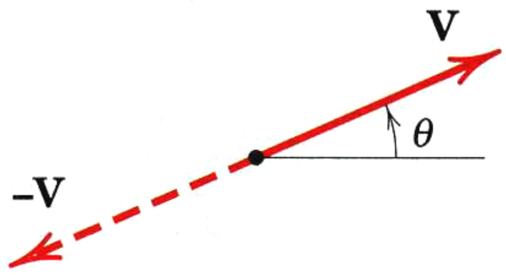
Speed, V

Velocity, **V** or $\vec{V}$

Magnitude of vector **V** is **|V|** = *V*



Vector Algebra (2)



Negative of a vector

Unit vector

The length of **v** is one

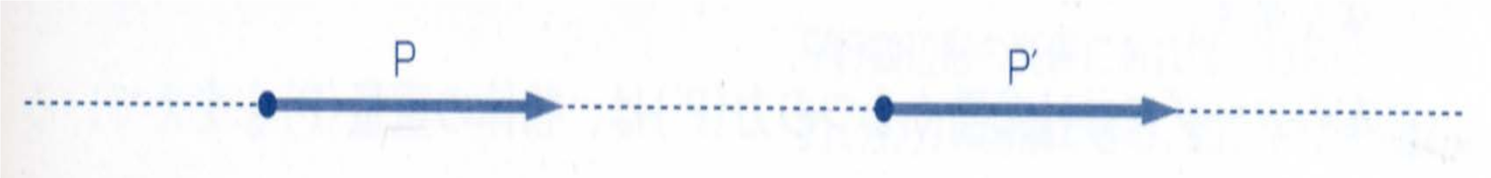
Null vector

The length of **v** is zero

Vector Types

Free vector

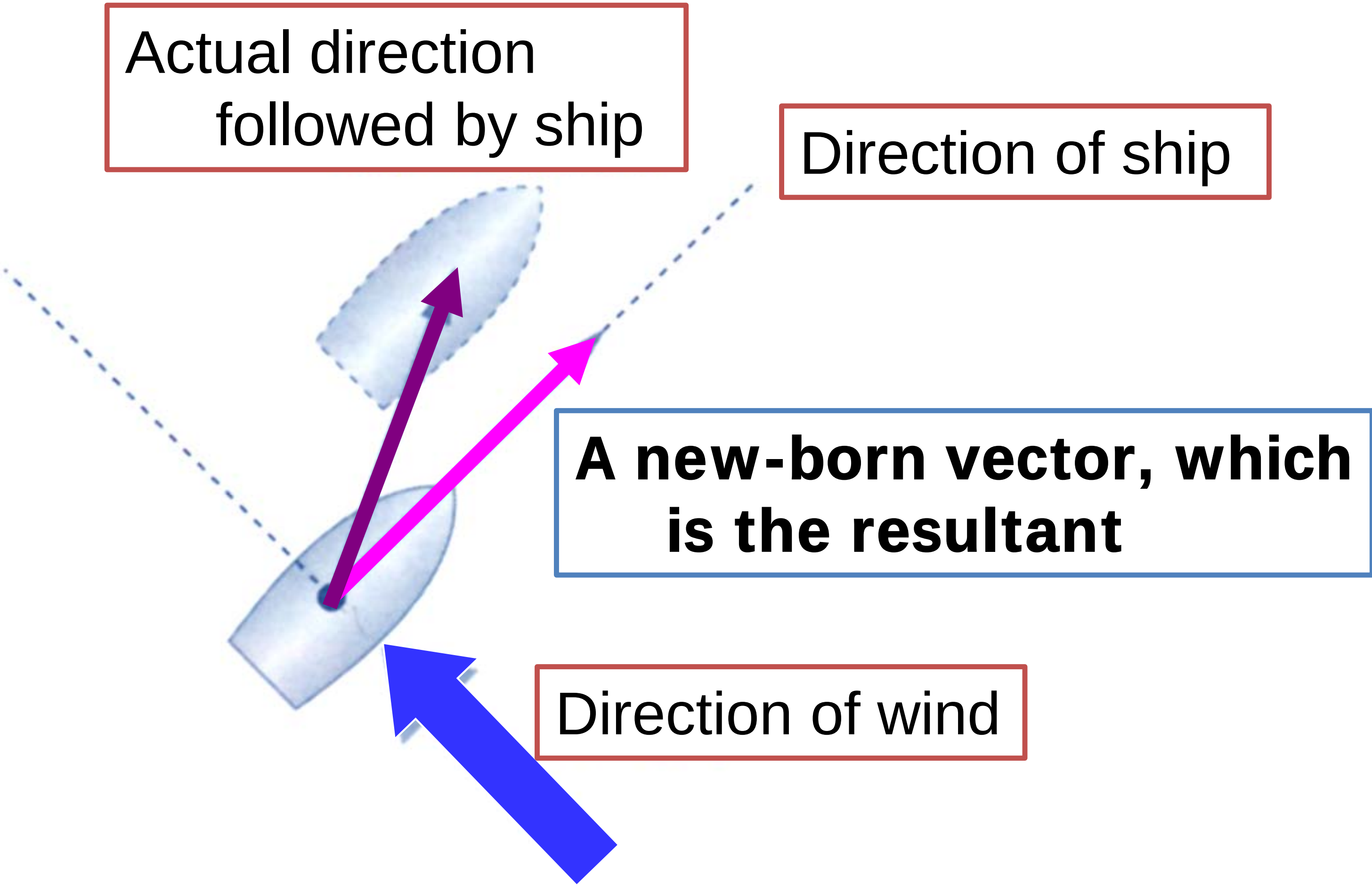
Sliding vector



Principle of transmissibility

Fixed (Bound) vector

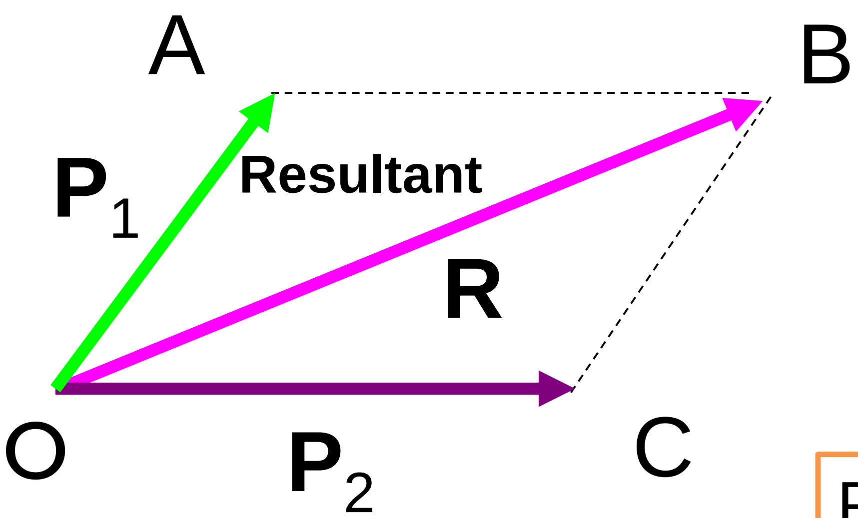
Resultant of Vectors



Condition: If they are in different lines in space

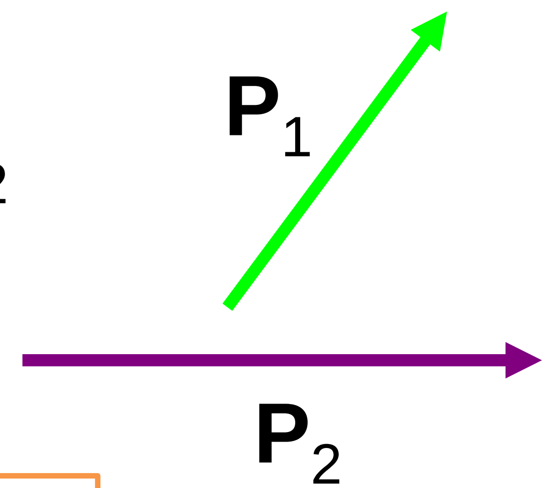
Resultant and Components

- Two free vectors ($\mathbf{P}_1$ & $\mathbf{P}_2$) can be replaced by their equivalent vector ($\mathbf{R}$), called the resultant, which is the diagonal of the parallelogram formed by $\mathbf{P}_1$ & $\mathbf{P}_2$ as its two sides



AB // $\mathbf{P}_2$

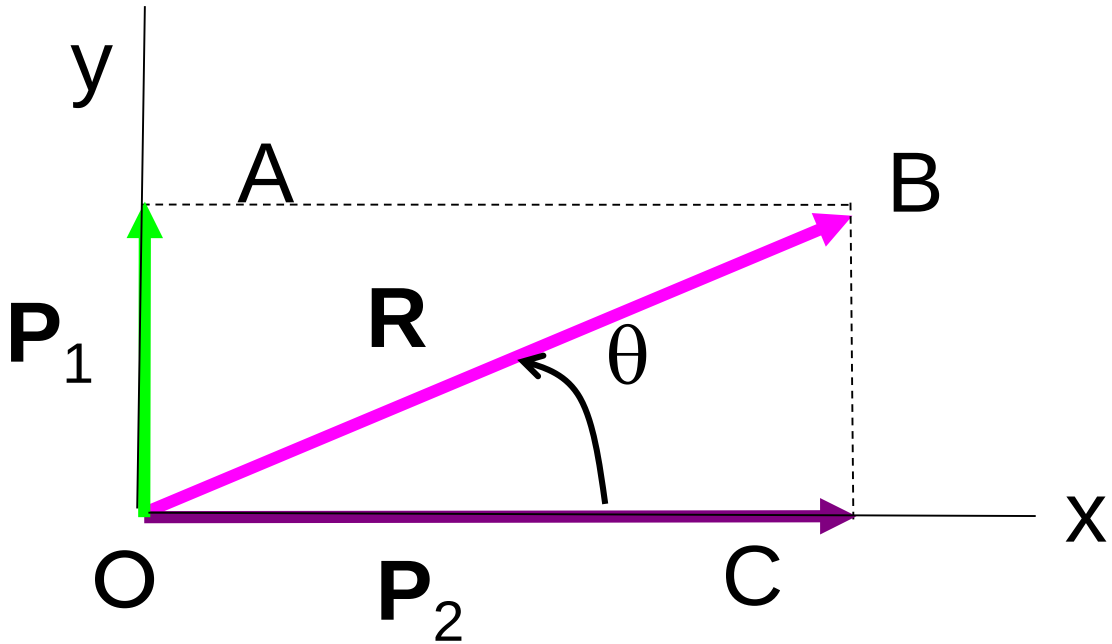
CB // $\mathbf{P}_1$



Parallelogram law

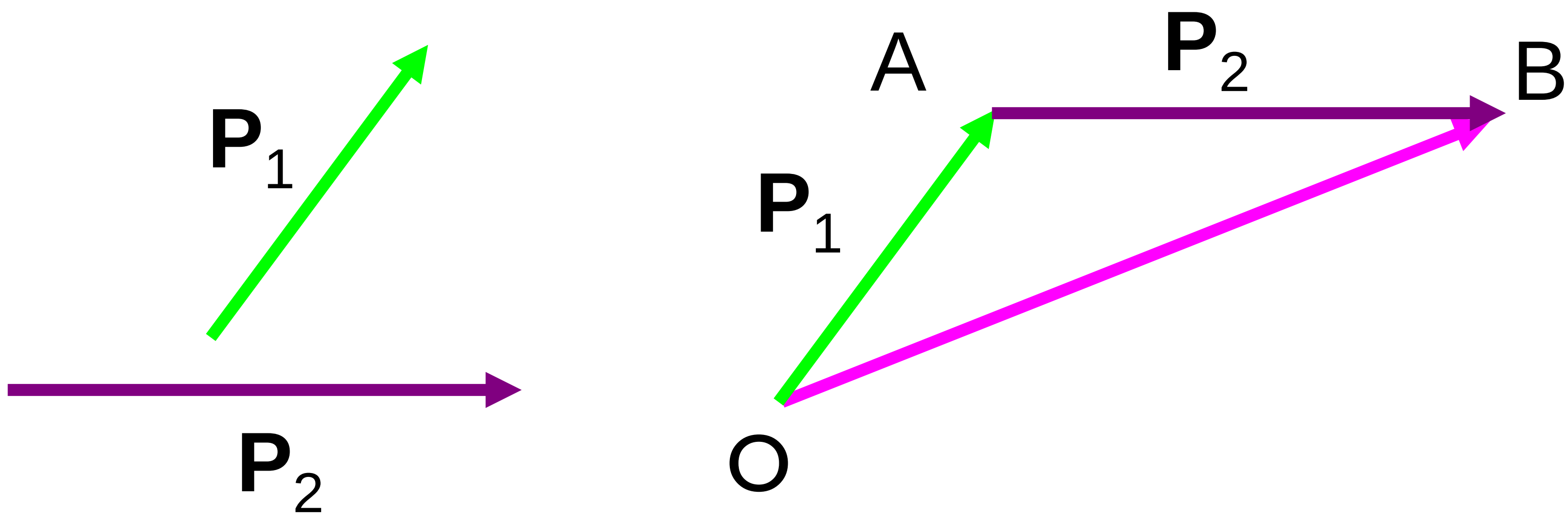
Rectangular Component

- If the sides of the parallelogram are perpendicular, the vectors (**P_1** & **P_2**) are called the rectangular component of vector (**R**)

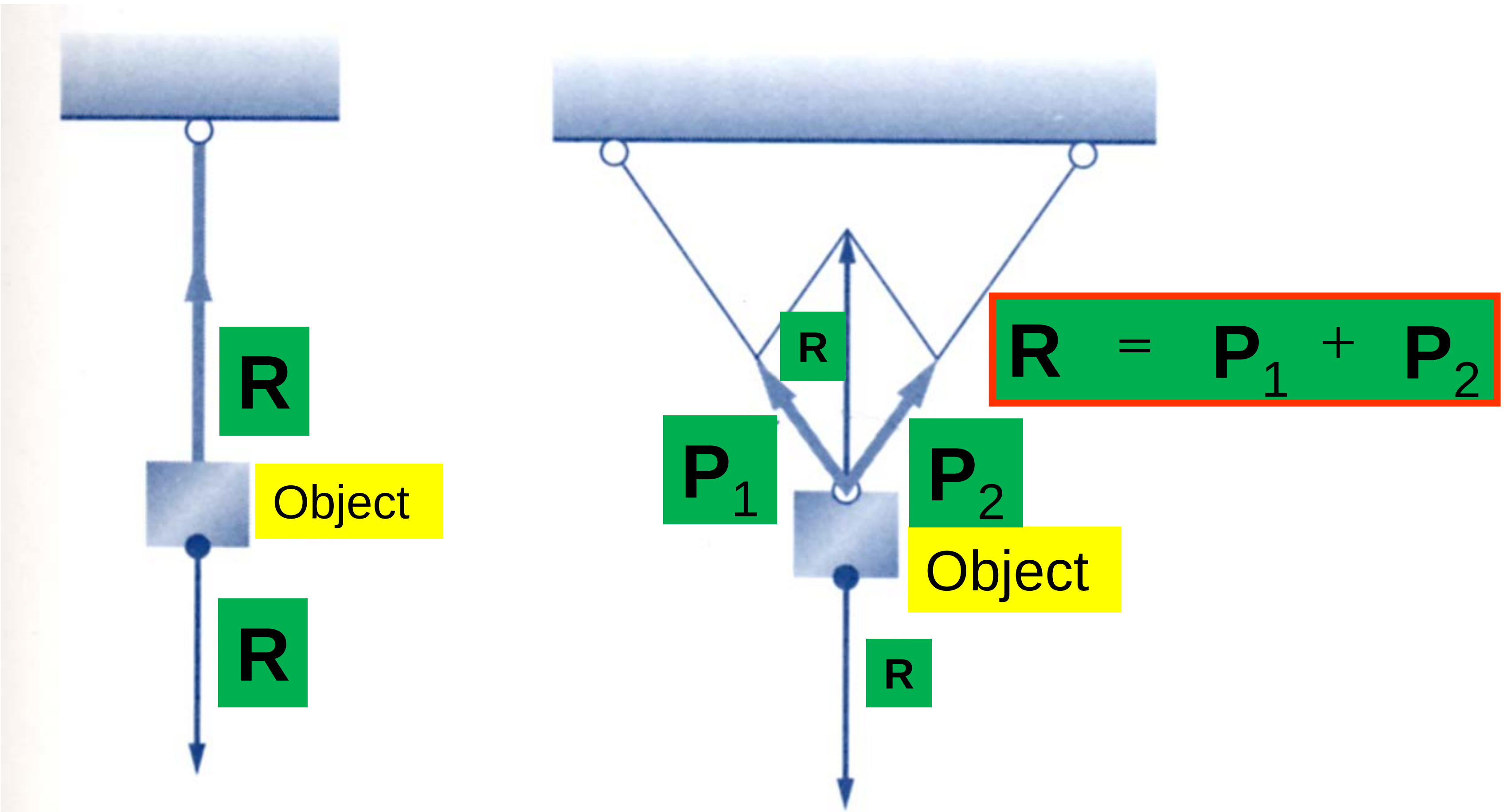


Triangle Law

- Two free vectors ($\mathbf{P}_1$ & $\mathbf{P}_2$) can also be added head-to-tail to obtain their equivalent vector **(R)**



Composition and Decomposition



Addition and Subtraction

- Vector addition is commutative

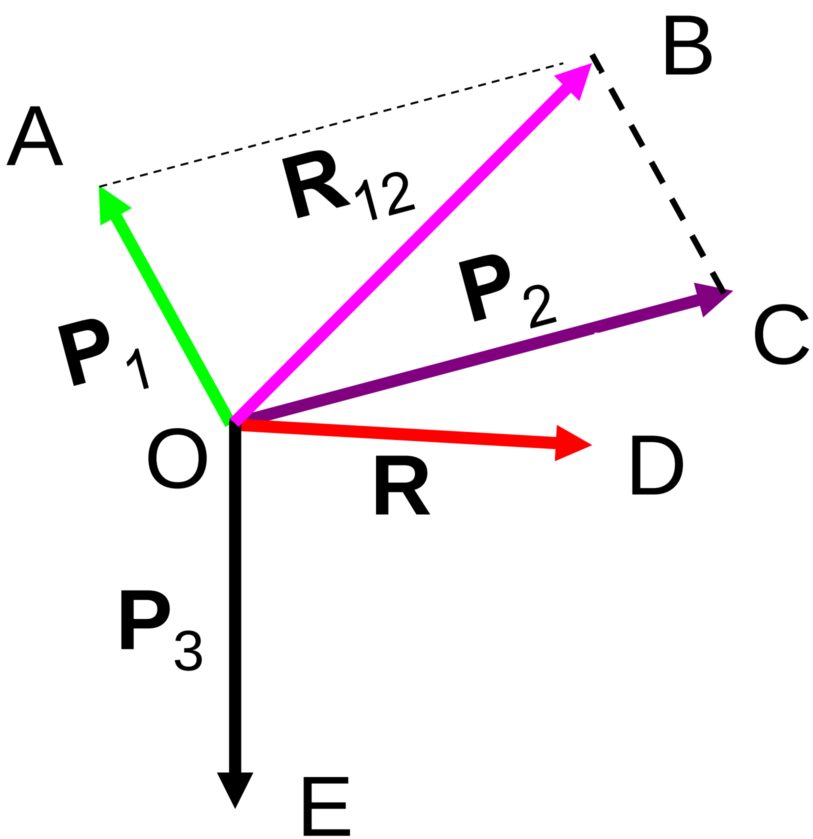
$$\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{Q} + \boldsymbol{P}$$

- Subtraction of a vector is accomplished by adding the negative of the vector

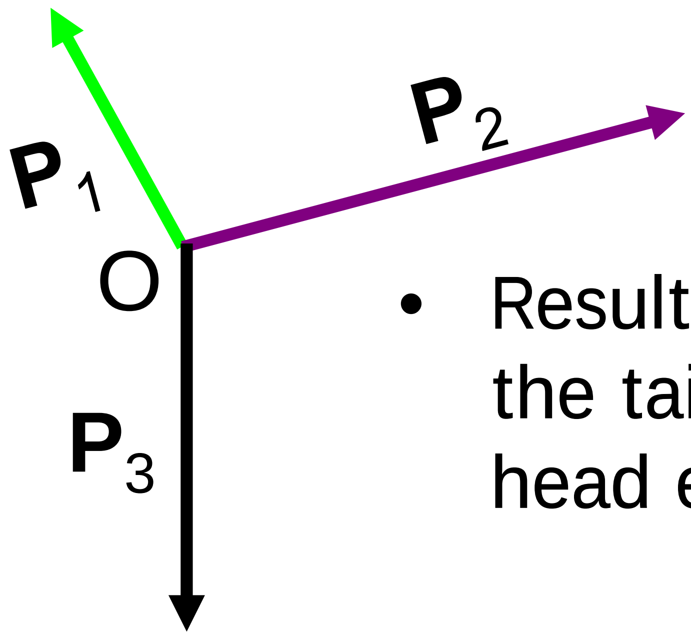
$$\boldsymbol{P} - \boldsymbol{Q} = \boldsymbol{P} + (-\boldsymbol{Q})$$

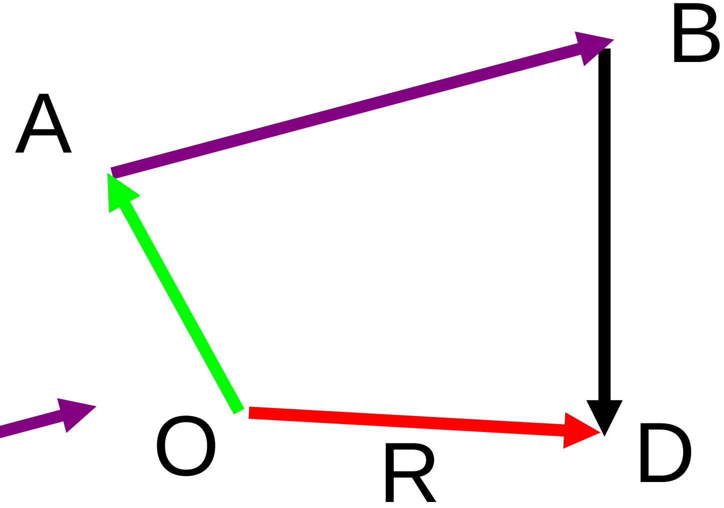
Composition of Vectors

- It is the process of determining the resultant of a system of vectors



Vector Polygon





OA // $\mathbf{P}_1$

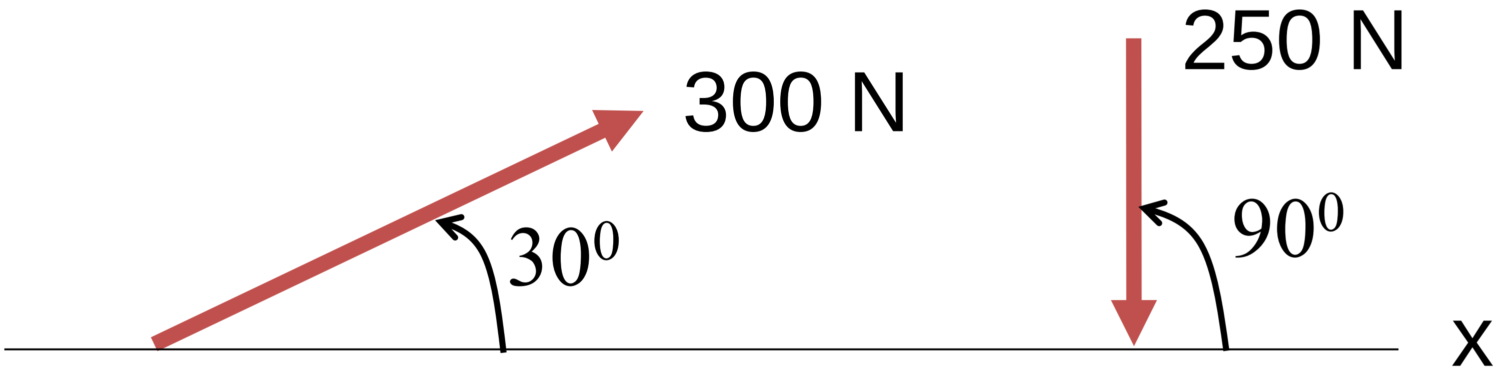
AB // $\mathbf{P}_2$

BD // $\mathbf{P}_3$

- Resultant can be obtained by joining the tail end of the first vector to the head end of the last vector

Example 1

Find the resultant of the two forces 300 N and -250 N shown in Figure.

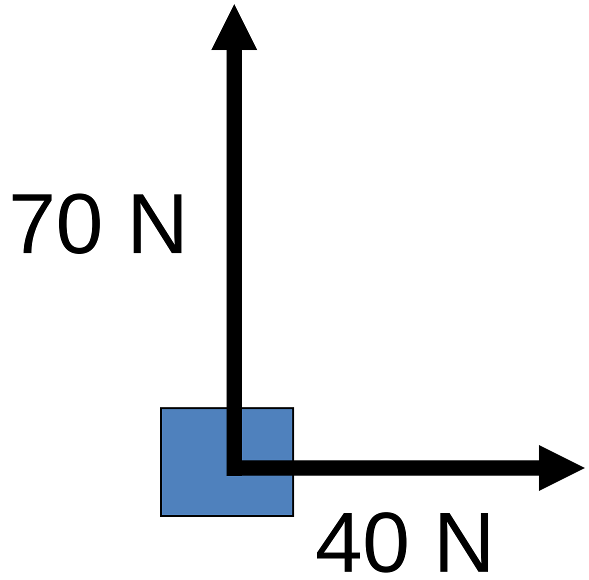


$$(300\text{N}\cos 30)\,\boldsymbol{i} + (300\text{N}\sin 30)\,\boldsymbol{j} + (-250\text{N})\,\boldsymbol{j} =$$

$$(150\sqrt{3}\,\text{N})\,\boldsymbol{i} - (100\,\text{N})\,\boldsymbol{j}$$

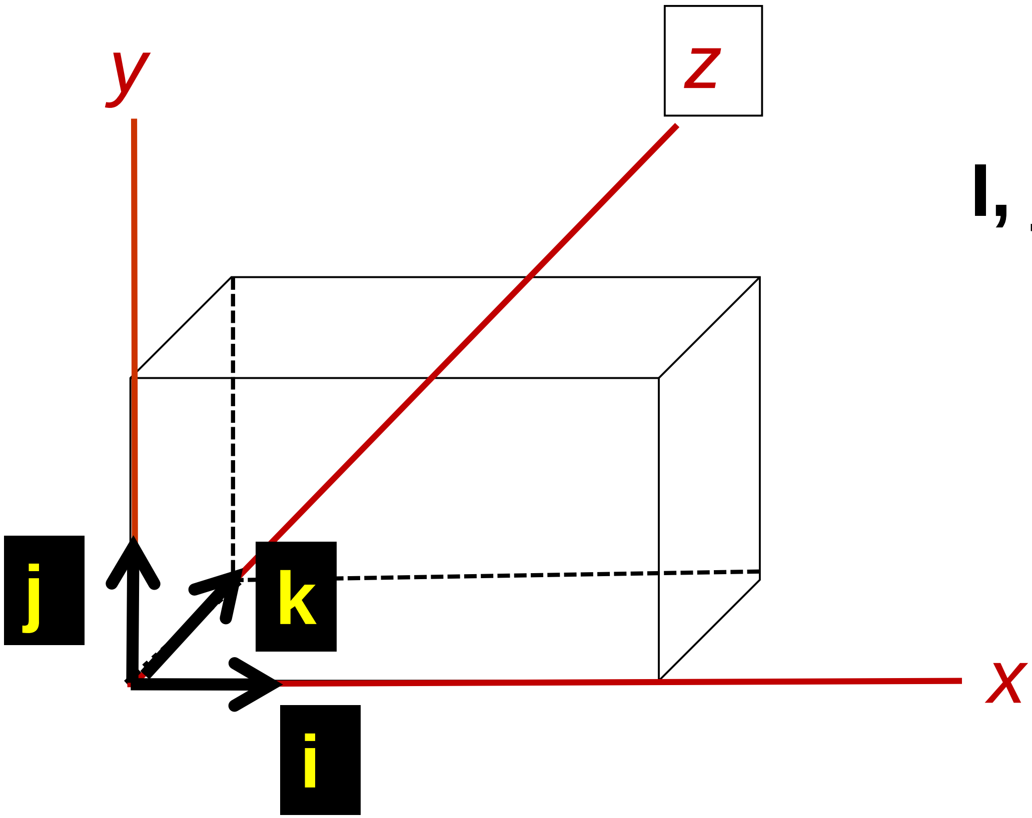
Example 2

Find the resultant of horizontal force 40 N and vertical force 70 N acting on a body.



$$(40\text{N})\,\boldsymbol{i} + (70\text{N})\,\boldsymbol{j}$$

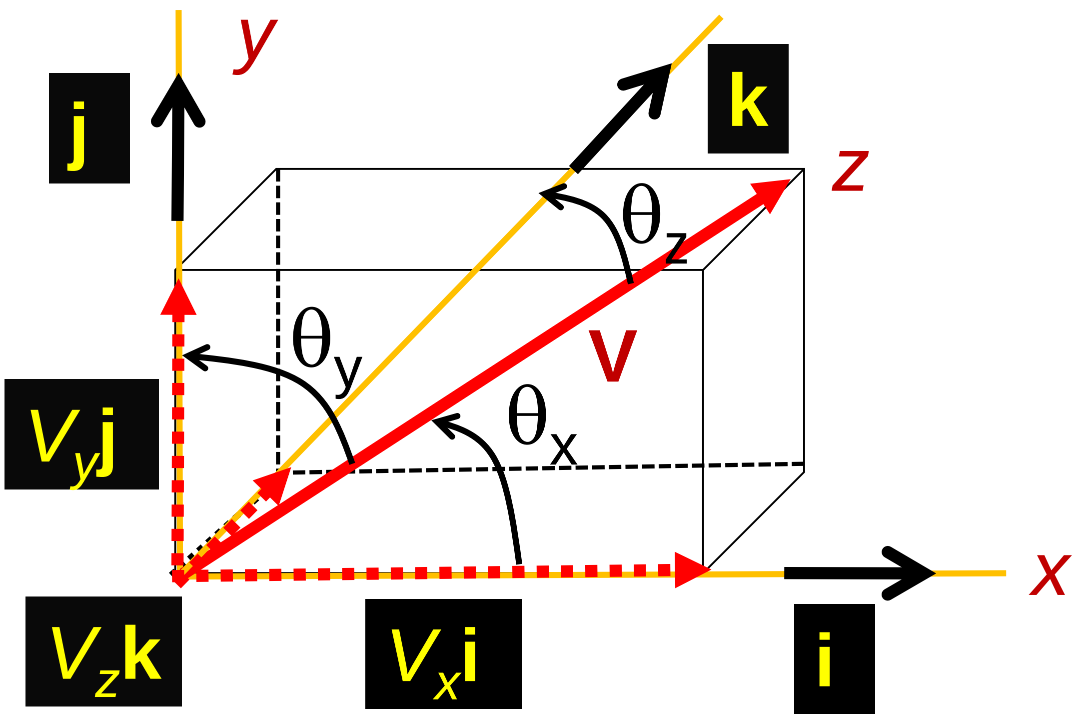
Expression of a Vector using Unit Vector



I, j, k are vectors in the x, y and z directions respectively with unit magnitudes

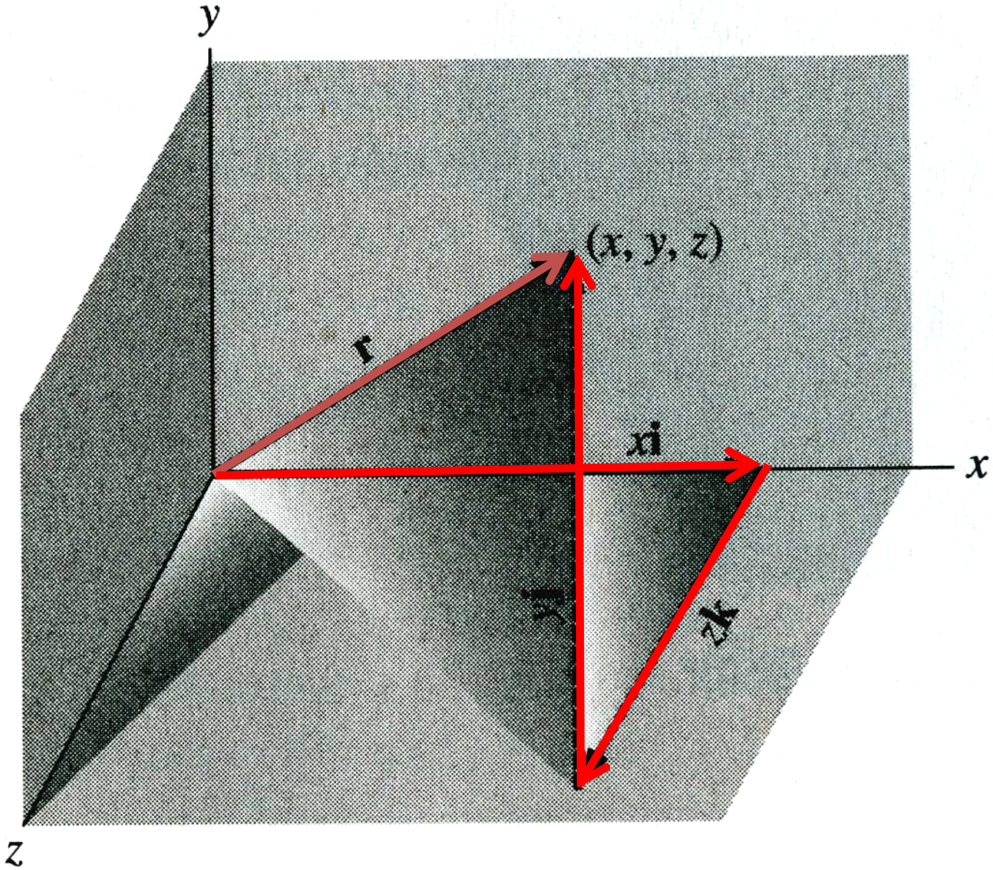
Expression of a Vector using Unit Vector

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$$



Position Vector

$$\mathbf{r} = \underline{x\mathbf{i}} + \underline{y\mathbf{j}} + \underline{z\mathbf{k}}$$



Scalar Product of Vectors

- Scalar product of two vectors **P** and **Q** is a scalar which is expressed as **P· Q** **(P dot Q)**

$$\mathbf{P}\cdot\mathbf{Q} = PQ\cos\theta$$

- Laws of dot products

$$\boldsymbol{P}\cdot\boldsymbol{Q}=\boldsymbol{Q}\cdot\boldsymbol{P}, \quad \boldsymbol{P}\cdot(\boldsymbol{Q}+\boldsymbol{R})=\boldsymbol{P}\cdot\boldsymbol{Q}+\boldsymbol{P}\cdot\boldsymbol{R}$$

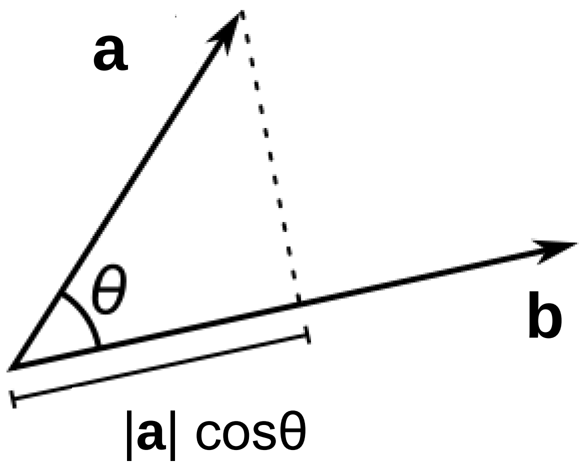
$$(\alpha\boldsymbol{P})\cdot(\beta\boldsymbol{Q})=\alpha\beta(\boldsymbol{Q}\cdot\boldsymbol{P})$$

- Dot products of base unit vectors

$$\boldsymbol{i}\cdot\boldsymbol{i}=1,\ \boldsymbol{j}\cdot\boldsymbol{j}=1,\ \boldsymbol{k}\cdot\boldsymbol{k}=1$$

$$\boldsymbol{i}\cdot\boldsymbol{j}=0,\ \boldsymbol{j}\cdot\boldsymbol{k}=0,\ \boldsymbol{k}\cdot\boldsymbol{i}=0$$

Geometric Interpretation of Dot Product



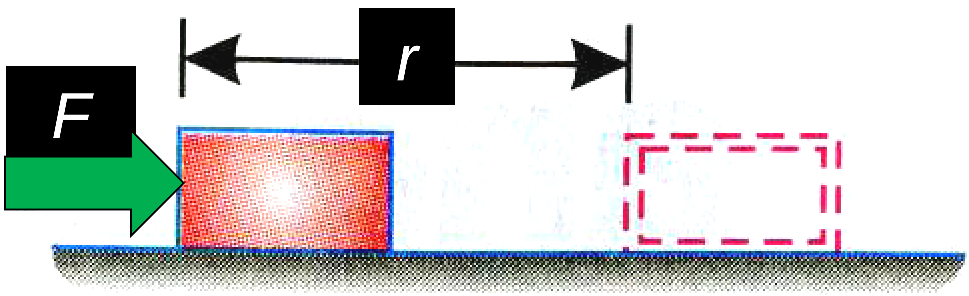
$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos(\theta)$

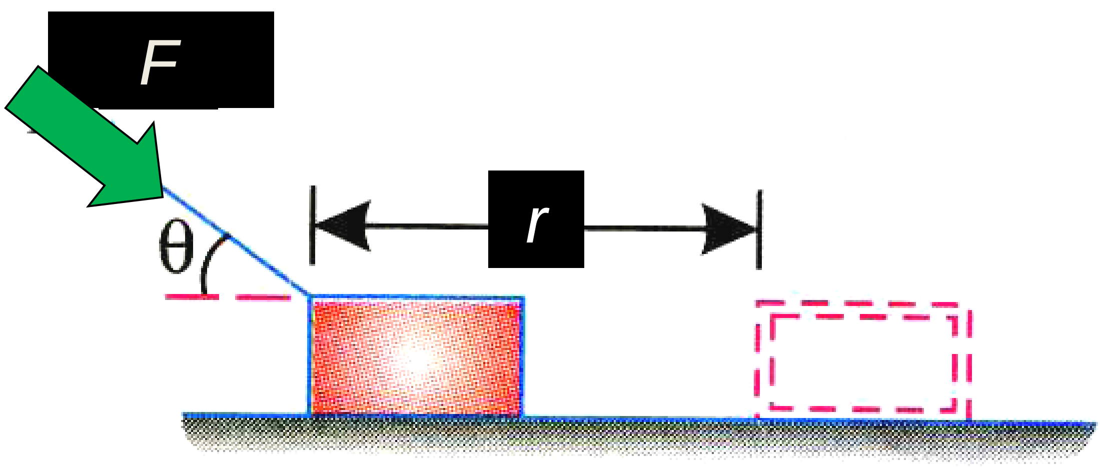
Example:

Mechanical work $W = \mathbf{F} \bullet \mathbf{r}$

Physical Interpretation of Dot Product

Work done by force **F** on the displacement **r** is the dot product of **F** and **r**





Sample Problem 1

Prove that the magnitude of the vector components of **V** along the rectangular axes can be written as.

$V_x = \mathbf{V} \cdot \mathbf{i}$ $\qquad$ $V_y = \mathbf{V} \cdot \mathbf{j}$ $\qquad$ $V_z = \mathbf{V} \cdot \mathbf{k}$

By direct computation, one has

$$\boldsymbol{V} \bullet \boldsymbol{i} = (V_x \boldsymbol{i} + V_y \boldsymbol{j} + V_z \boldsymbol{k}) \bullet \boldsymbol{i} = V_x$$

$$\boldsymbol{V} \bullet \boldsymbol{j} = (V_x \boldsymbol{i} + V_y \boldsymbol{j} + V_z \boldsymbol{k}) \bullet \boldsymbol{i} = V_y$$

$$\boldsymbol{V} \bullet \boldsymbol{k} = (V_x \boldsymbol{i} + V_y \boldsymbol{j} + V_z \boldsymbol{k}) \bullet \boldsymbol{i} = V_z$$

Sample Problem 2

Find the angle between the following two vectors

$$\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$$

$$\vec{b} = 4\vec{i} + \vec{j} - 3\vec{k}$$

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}\sqrt{\boldsymbol{b}\cdot\boldsymbol{b}}} = \frac{2}{\sqrt{14}\sqrt{26}}$$

Vector (Cross) Product of Vectors

- Vector product of two vectors **a** and **b** is a vector **c** which is expressed as **a × b** **(a cross b)**

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = |a|\,|b| \sin\theta\, \mathbf{n}$$

What is the range of θ?

Unit vector that gives the direction of c

- Vector **c** is normal to the plane of **a** and **b**

$$\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$$

⇨ Not commutative

Vector Product Rules

- Laws of cross products

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \;\Rightarrow \boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c}$$

$$\boldsymbol{a} \times (\lambda \boldsymbol{b}) = (\lambda \boldsymbol{a}) \times \boldsymbol{b} = \lambda(\boldsymbol{a} \times \boldsymbol{b}),\; \lambda \textit{ is a scalar}$$

- Cross products of Orthogonal vectors

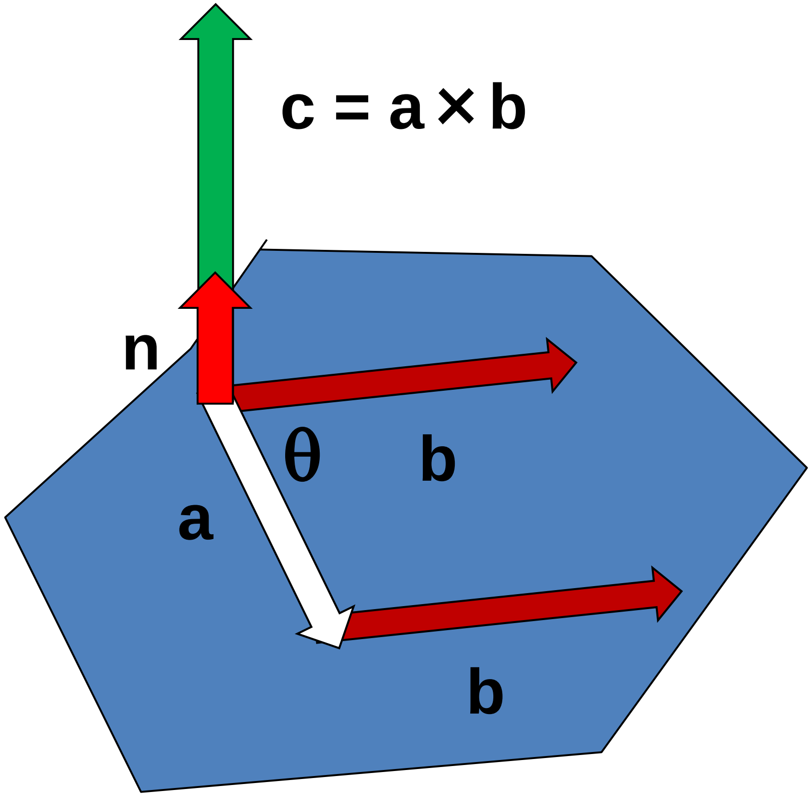
$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{0},\; \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{0},\; \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$$

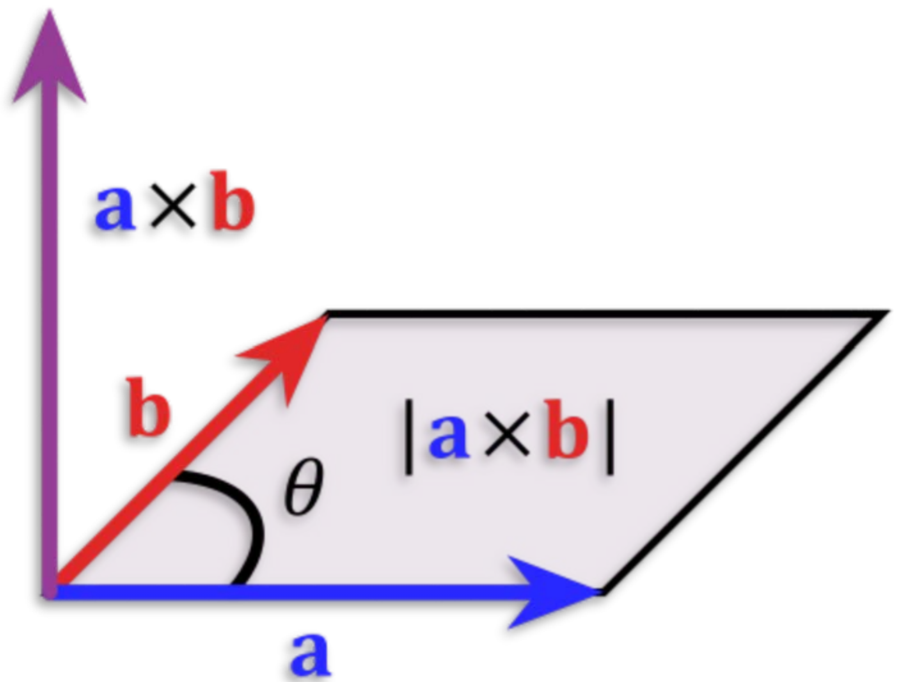
$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k},\; \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i},\; \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

$$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k},\; \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i},\; \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$$

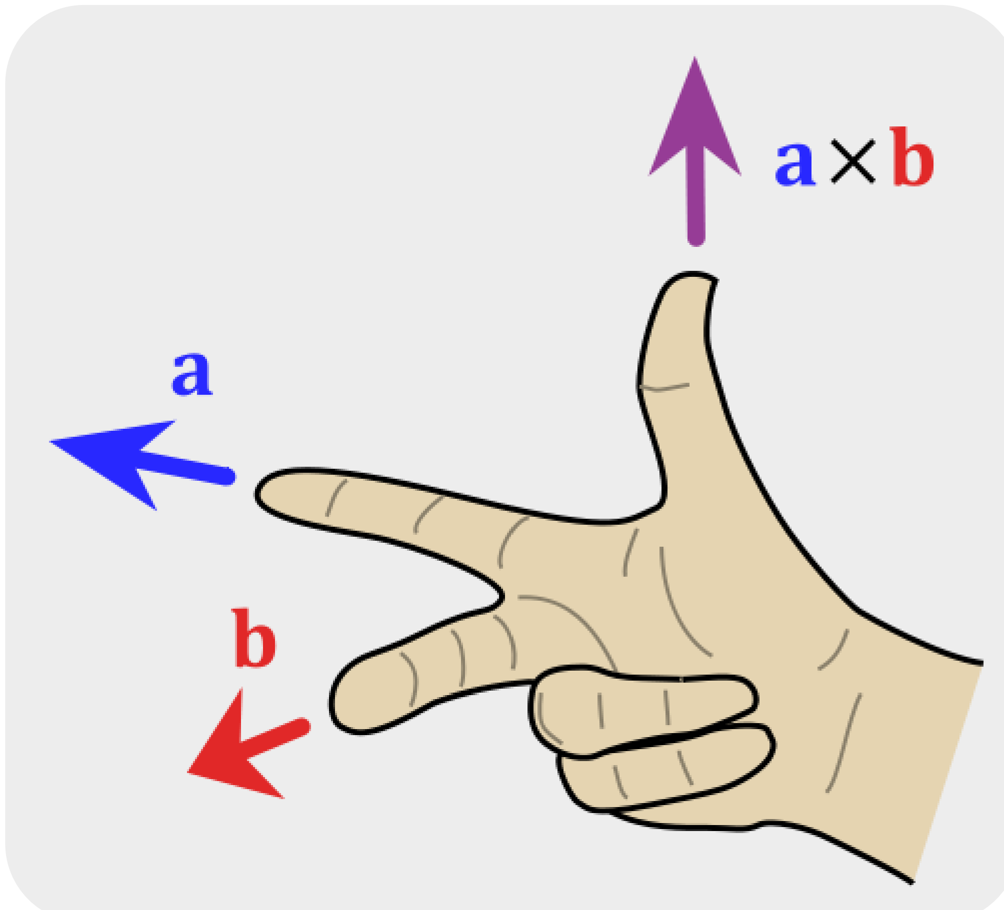
Geometric Interpretation of Cross Product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$



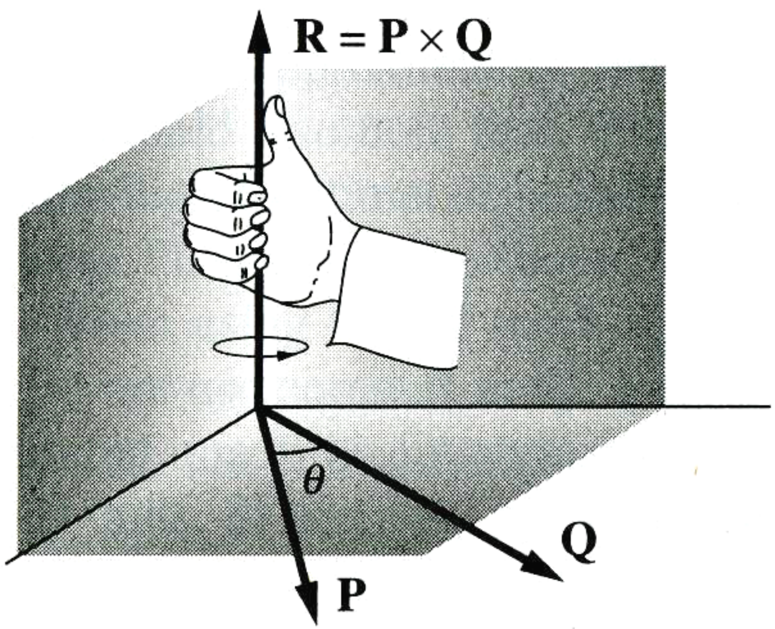


Right Hand Rule of Cross Products

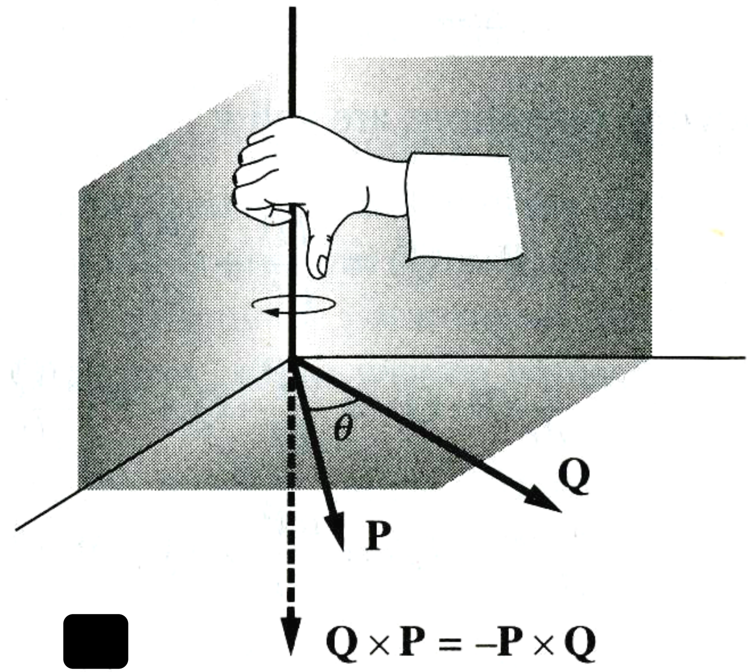


What is the expression for **a × b**
of in terms of their respective components

Importance of Order in Cross Products

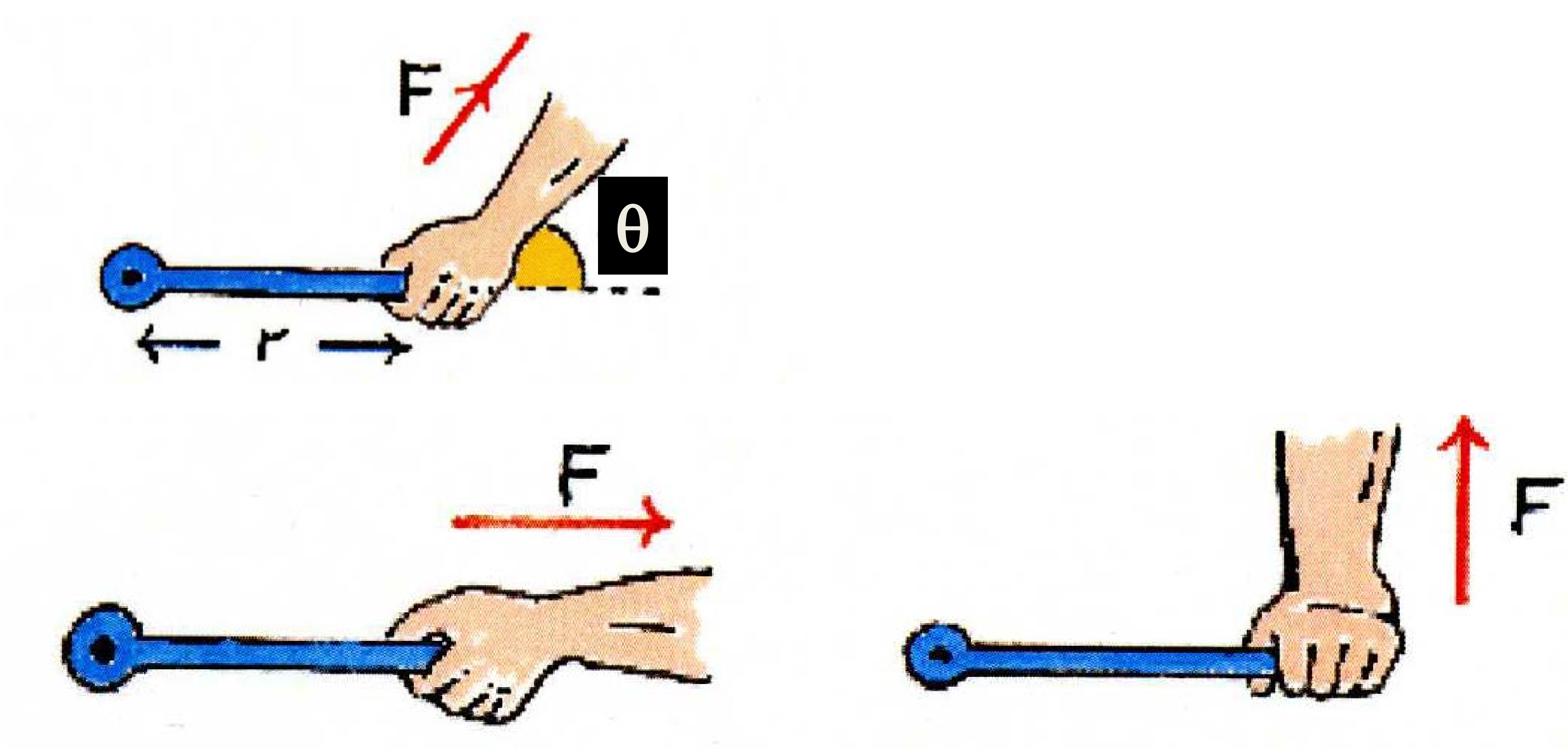


P × Q



Q × P

Physical Interpretation of Cross Product



Moment of force (also referred as torque)

Sample Problem 3

A force **F** acts through the point with position vector **r** which are given by the following.

$$\vec{F} = 3\vec{i} + 2\vec{j} + 4\vec{k}$$

$$\vec{r} = 2\vec{i} + \vec{j} + 3\vec{k}$$

What is its torque about a perpendicular axis through O? By direct computation, one has

$$\boldsymbol{F} \times \boldsymbol{r} = 3\boldsymbol{i} \times \boldsymbol{j} + 9\boldsymbol{i} \times \boldsymbol{k} + 4\boldsymbol{j} \times \boldsymbol{i} + 6\boldsymbol{j} \times \boldsymbol{k} + 8\boldsymbol{k} \times \boldsymbol{i} + 4\boldsymbol{k} \times \boldsymbol{j}$$

$$\boldsymbol{F} \times \boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

Interpretation of Dot and Cross Product Combination

- Dot product is a measure of “parallelness”
- Cross product is a measure of "perpendicularness”