Exercises in Physics

Lecture 5
Kinematics of Curvilinear Motion
(Polar Coordinates)

Sample Problem 1

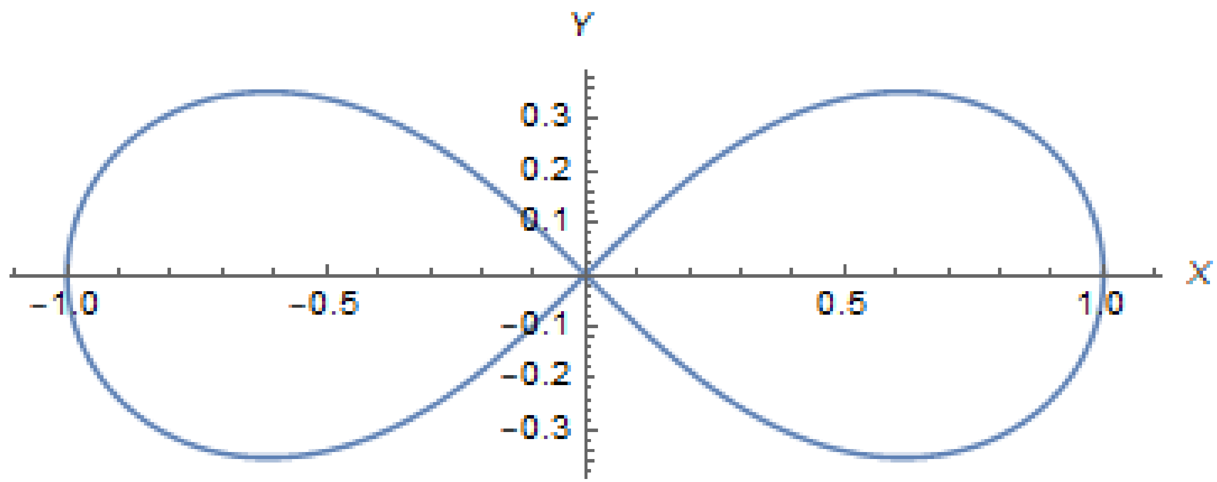
For the following expressions, find an equation in polar coordinates that has the same graph as the given equation in rectangular coordinates

- $(x^2 + y^2)^2 = x^2 - y^2$

Solution

By plugging $x = r\cos\theta$ and $y = r\sin\theta$, after simplification one obtains

- $r^2 = \cos 2\theta$



Sample Problem 2

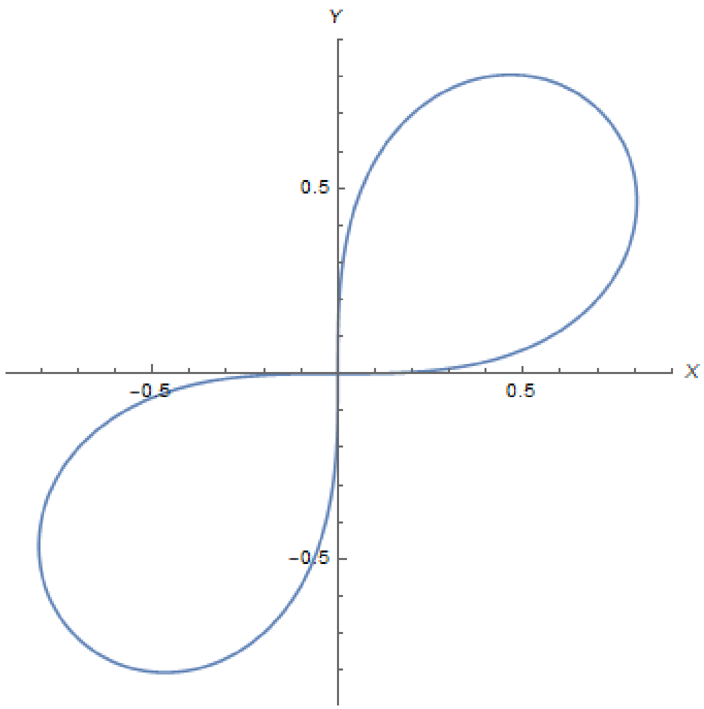
For the following expressions, find an equation in polar coordinates that has the same graph as the given equation in rectangular coordinates

- $(x^2 + y^2)^2 = 2xy$

Solution

By plugging $x = r\cos\theta$ and $y = r\sin\theta$, after simplification one obtains

- $r^2 = \sin 2\theta$



Sample Problem 3

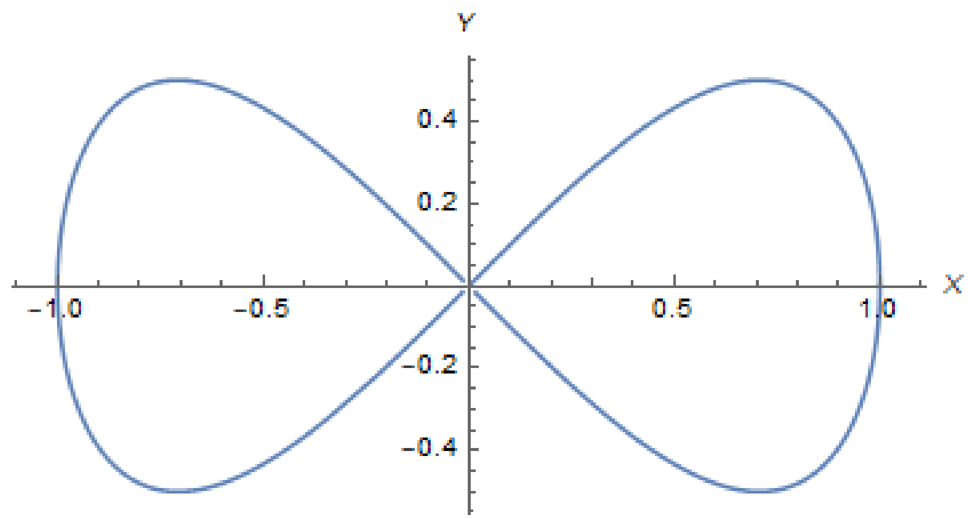
For the following expressions, find an equation in polar coordinates that has the same graph as the given equation in rectangular coordinates

- $x^4 = (x^2 - y^2)$

Solution

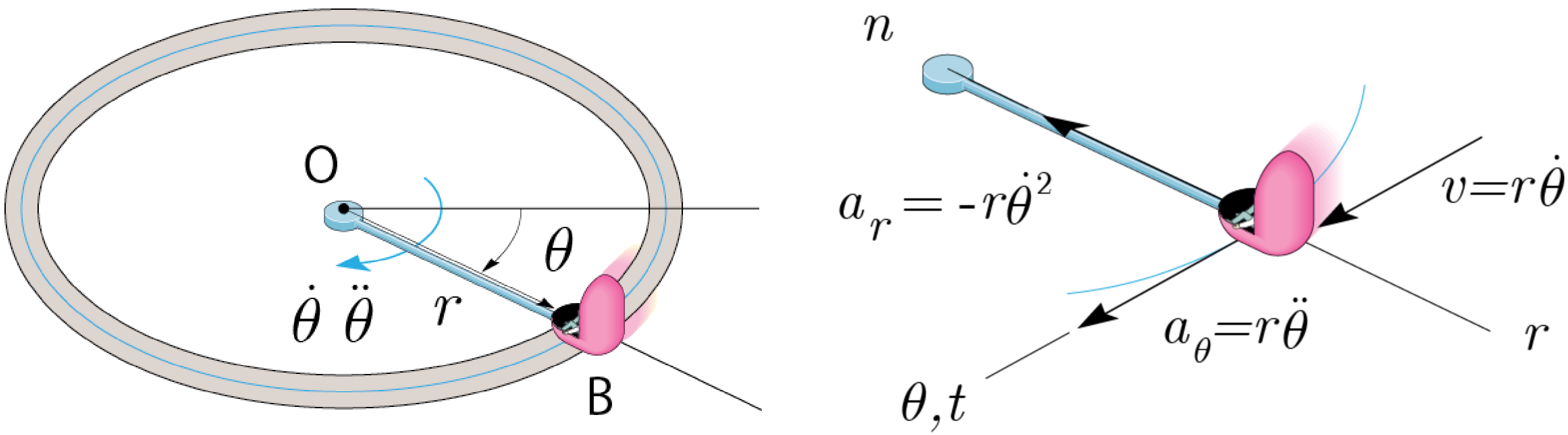
By plugging $x = r\cos\theta$ and $y = r\sin\theta$, after simplification one obtains

- $r^2 = (\cos 2\theta)/(\cos^4\theta)$



Sample Problem 4

The amusement park ride consists of a chair that is rotating in a horizontal circular path of radius r such that the arm OB has an angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$. Determine the radial and transverse components of velocity and acceleration of the passenger.



Solution

Coordinate System. Since the angular motion of the arm is reported, polar coordinates are chosen for the solution. Here θ is not related to r, since the radius is constant for all θ.

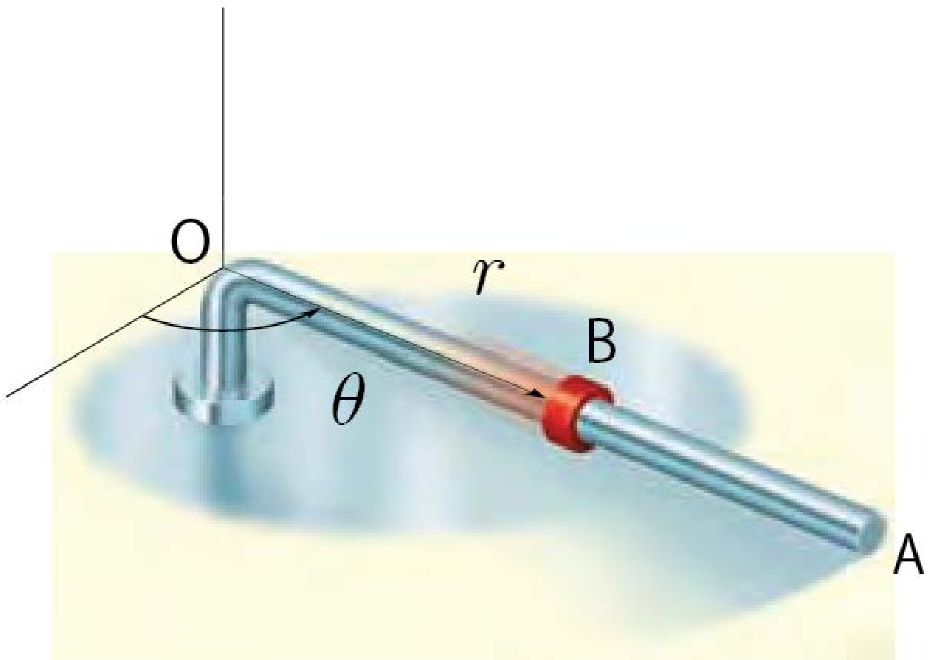
Velocity and acceleration. It is first necessary to specify the first and second time derivatives of r and θ. Since r is constant, we have $\dot{r} = 0,\ \ddot{r} = 0$. Thus $v_r = \dot{r} = 0,\ v_\theta = r\dot{\theta}$ and $a_r = \ddot{r} - r\dot{\theta}^2 = -r\dot{\theta}^2$
$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = r\ddot{\theta}$

Note: It is instructive to compare the results with n, t components (last class). The n, t axes are shown in the right figure, which in this special case of circular motion happen to be *collinear* with the r and θ axes, respectively. Since $v = v_\theta = v_t = r\dot{\theta}$, then by comparison,

$$-a_r = a_n = \frac{v^2}{\rho} = \frac{(r\dot{\theta})^2}{r} = r\dot{\theta}^2 \qquad a_\theta = a_t = \frac{dv}{dt} = \frac{d(r\dot{\theta})}{dt} = \dot{r}\dot{\theta} + r\ddot{\theta} = r\ddot{\theta}$$

Sample Problem 5

The rod OA rotates in the horizontal plane such that $\theta(t) = t^3$ rad. At the same time, the collar B is sliding outward along OA so that $r(t) = 100t^2$ mm In both cases t is in seconds, determine the velocity and acceleration of the collar when $t = 1$ s .



Solution

Coordinate System. Since time-parametric equations of the path are given, it is not necessary to relate r to θ.

Velocity. Determining the time derivatives and evaluating them when $t = 1$ s , we have

$$r = 100t^2\Big|_{t=1\text{s}} = 100\,\text{mm}, \qquad \theta = t^3\Big|_{t=1\text{s}} = 1\,\text{rad} = 57.3^\circ$$

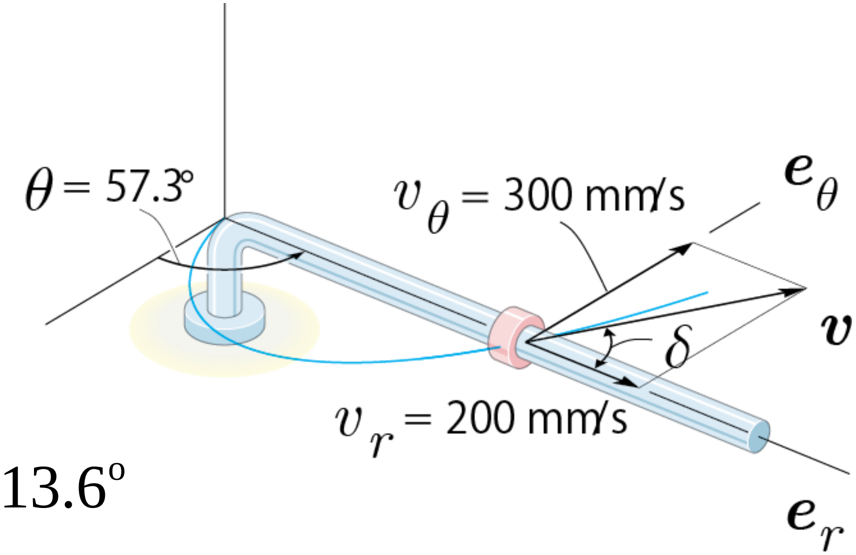
$$\dot{r} = 200t\Big|_{t=1\text{s}} = 200\,\text{mm/s}, \qquad \dot{\theta} = 3t^2\Big|_{t=1\text{s}} = 3\,\text{rad/s}$$

$$\ddot{r} = 200\Big|_{t=1\text{s}} = 200\,\text{mm/s}^2, \qquad \ddot{\theta} = 6t\Big|_{t=1\text{s}} = 6\,\text{rad/s}^2$$

Then

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta = (200\boldsymbol{e}_r + 300\boldsymbol{e}_\theta)\,\text{mm/s}$$

The magnitude of $\boldsymbol{v}$ and the orientation angle δ are



$$v = \sqrt{200^2 + 300^2} = 361\,\text{mm/s}$$

$$\delta = \tan^{-1}\left(\frac{300}{200}\right) = 56.3^\circ, \quad \delta + 57.3^\circ = 113.6^\circ$$

Solution

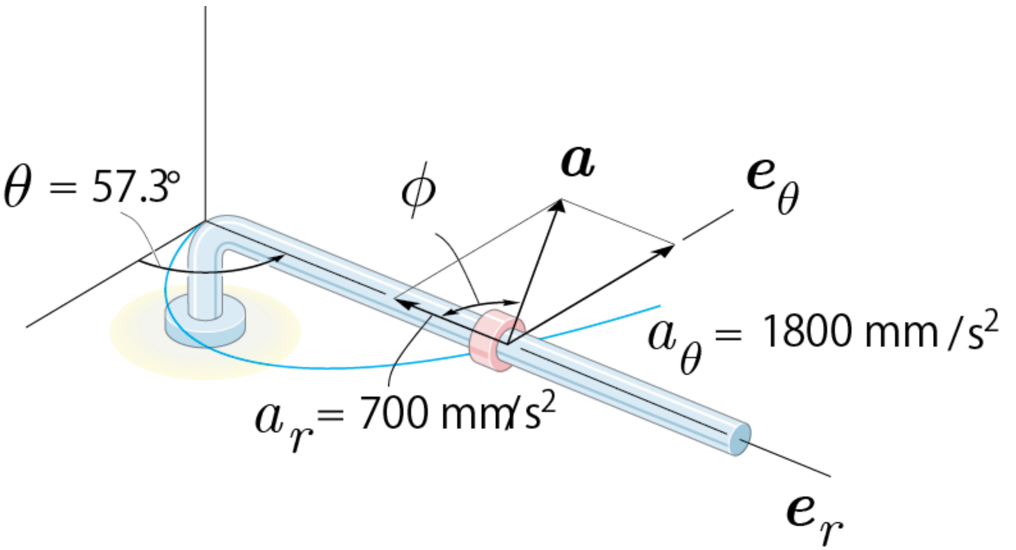
Acceleration.

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta = (-700\boldsymbol{e}_r + 1800\boldsymbol{e}_\theta)\,\mathrm{mm/s^2}$$

The magnitude of $\boldsymbol{a}$ and the orientation angle ϕ are

$$a = \sqrt{(-700)^2 + 1800^2} = 1930\ \mathrm{mm/s^2}$$

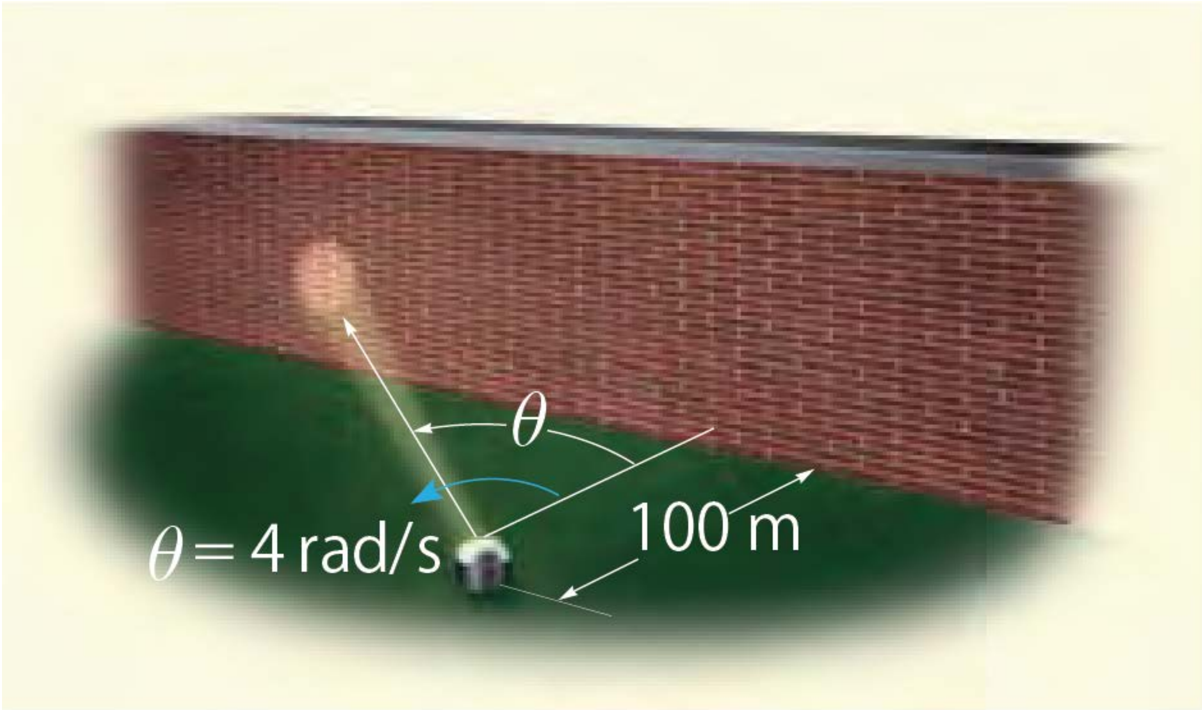
$$\phi = \tan^{-1}\left(\frac{1800}{700}\right) = 68.7^\circ, \quad (180^\circ - \phi) + 57.3^\circ = 168.6^\circ$$



Note: The velocity is tangent to the path (shown in blue); however, the acceleration is directed within the curvature of the path, as expected.

Sample Problem 6

The searchlight casts a spot of light along the face of a wall that is located 100 m from the searchlight. Determine the magnitudes of the velocity and acceleration at which the spot appears to travel across the wall at the instant $\theta = 45^{o}$. The searchlight rotates at a constant rate of $\dot{\theta} = 4\,\mathrm{rad/s}$.



Solution

Coordinate System. Polar coordinates will be used to solve this problem since the angular rate of the r searchlight is given. To find the necessary time derivatives, it is first necessary to relate r to θ

$$r = 100/\cos\theta = 100\sec\theta$$

Velocity and accelerations. Using the chain rule of calculus, noting that $\mathrm{d}(\sec\theta) = \sec\theta\tan\theta\,\mathrm{d}\theta$ and $\mathrm{d}(\tan\theta) = \sec^2\theta\,\mathrm{d}\theta$, we have

$$\dot{r} = 100(\sec\theta\tan\theta)\dot{\theta}$$

$$\ddot{r} = 100(\sec\theta\tan\theta)\dot{\theta}(\tan\theta)\dot{\theta} + 100\sec\theta(\sec^2\theta)\dot{\theta}(\dot{\theta}) + 100\sec\theta\tan\theta(\ddot{\theta})$$

$$= 100(\sec\theta\tan^2\theta)(\dot{\theta}^2) + 100\sec^3\theta(\dot{\theta}^2) + 100\sec\theta\tan\theta(\ddot{\theta})$$

Since $\dot{\theta} = 4\,\mathrm{rad/s} = \mathrm{const}$, then $\ddot{\theta} = 0$ and for $\theta = 45^\circ$ we have

$$r = 100\sec 45^\circ = 141.4$$

$$\dot{r} = 400(\sec 45^\circ\tan 45^\circ) = 565.7$$

$$\ddot{r} = 1600(\sec 45^\circ\tan^2\sec 45^\circ + 100\sec^3 45^\circ) = 6788.2$$

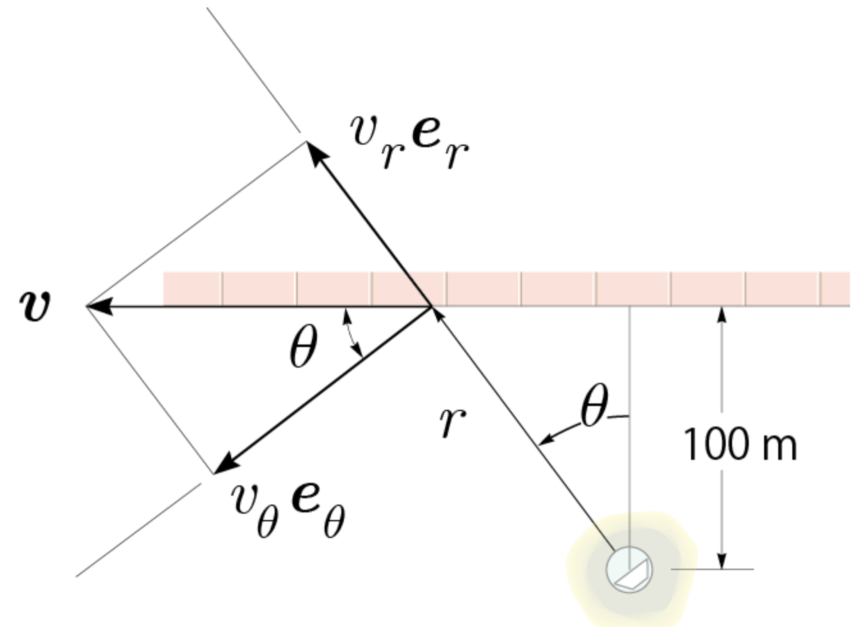
Solution

Velocity:

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$$

$$= (565.7\boldsymbol{e}_r + 565.7\boldsymbol{e}_\theta)\,\mathrm{m/s}$$

$$v = \sqrt{565.7^2 + 565.7^2} = 800\ \mathrm{m/s}$$

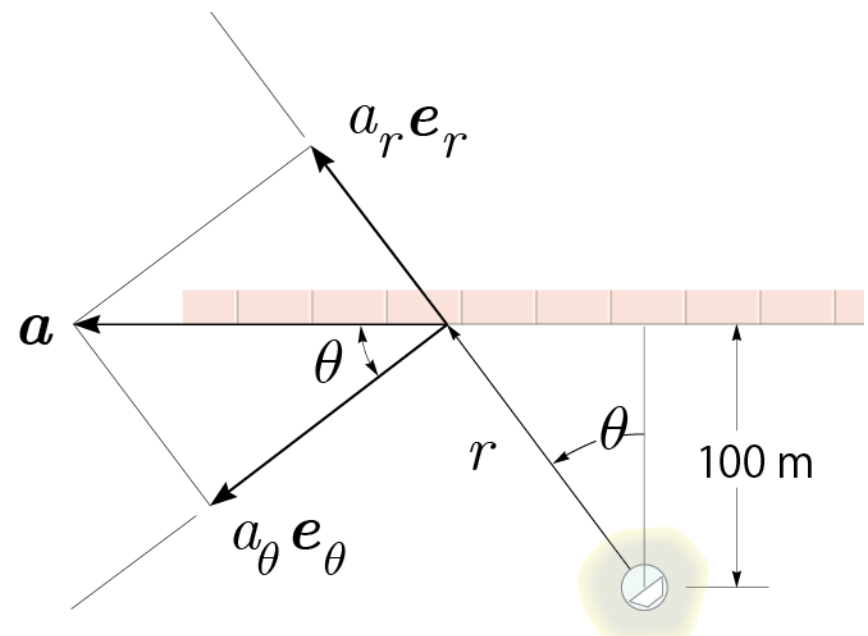


Acceleration:

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta$$

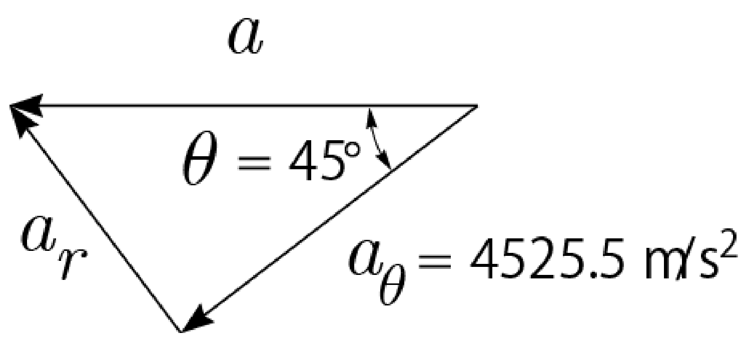
$$= (4525.5\boldsymbol{e}_r + 4525.5\boldsymbol{e}_\theta)\,\mathrm{m/s^2}$$

$$a = \sqrt{4525.5^2 + 4525.5^2} = 6400\ \mathrm{m/s^2}$$



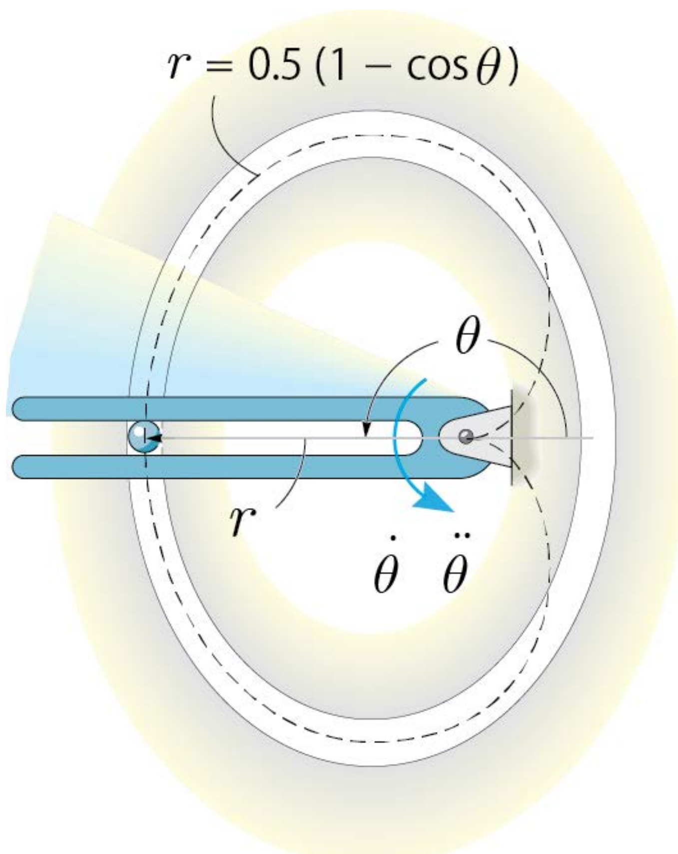
Solution

Note: It is also possible to find a without having to calculate $\ddot{r}$ (or a_r). As shown in the figure below, since $a_\theta = 4525.5\,\mathrm{m/s^2}$, then by vector resolution, $a = 4525.5/\cos 45^\circ = 6400\,\mathrm{m/s^2}$.



Sample Problem 7

Due to the rotation of the forked rod, the ball travels around the slotted path, a portion of which is in the shape of a cardioid, $r = 0.5(1 - \cos\theta)$, where u is in radians. If the ball's velocity is $v = 4\,\mathrm{m/s}$ and its acceleration is $a = 30\,\mathrm{m/s^2}$ at the instant $\theta = 180^\circ$, determine the angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ of the fork.



Solution

Coordinate System. This path is most unusual, and mathematically it is best expressed using polar coordinates, as done here, rather than rectangular coordinates. Also, since $\dot{\theta}$ and $\ddot{\theta}$ must be determined, then r and θ coordinates are an obvious choice.

Velocity and accelerations. The time derivatives of r and u can be determined using the chain rule.

$$r = 0.5(1 - \cos\theta)$$

$$\dot{r} = 0.5(\sin\theta)\,\dot{\theta}$$

$$\ddot{r} = 0.5(\cos\theta)\,\dot{\theta}(\dot{\theta}) + 0.5(\sin\theta)\,\ddot{\theta}$$

Evaluation these results at $\theta = 180^{\circ}$ we have

$$r = 1, \qquad \dot{r} = 0, \qquad \ddot{r} = -0.5\,\dot{\theta}^2$$

Solution

Since $v = 4\,\mathrm{m/s}$ we can procced as follows

$$v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2} \Rightarrow \sqrt{(0)^2 + (1\dot{\theta})^2} \Rightarrow \dot{\theta} = 4\mathrm{rad/s}$$

In a similar manner, $\ddot{\theta}$ can be found as follows.

$$a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2} \quad \Rightarrow$$

$$30 = \sqrt{(-0.5\times 4^2 - 1\times 4^2)^2 + (1\ddot{\theta} + 2\times 0\times 4^2)^2} \Rightarrow$$

$$30^2 = -24^2 + \ddot{\theta}^2 \Rightarrow$$

$$\ddot{\theta} = 18\,\mathrm{rad/s}$$

Vectors $\boldsymbol{v}$ and $\boldsymbol{a}$ are shown in the figure.