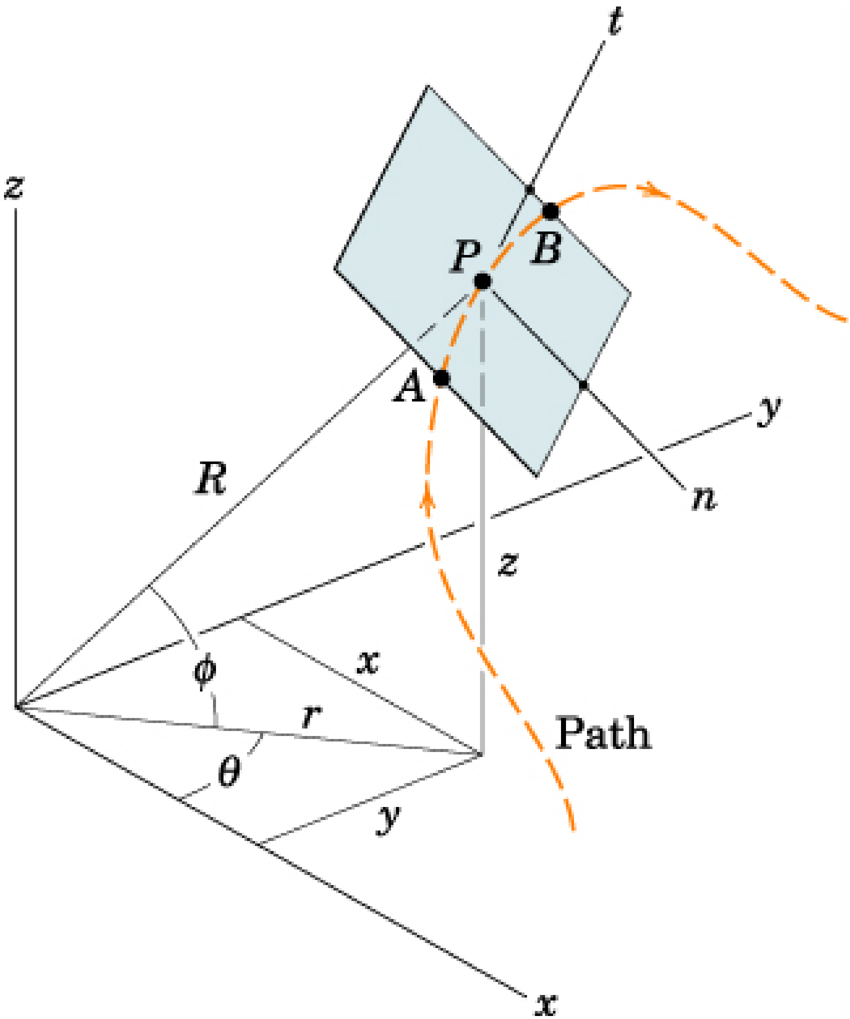
Exercises in Physics

Lecture 6
Kinematics of Curvilinear Motion (Cylindrical and Spherical Coordinates)

Different Coordinates Systems



Rectangular (x, y, z)

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$$

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}$$

Cylindrical (r, θ, z)

Spherical (R, θ, ϕ)

Cylindrical Coordinates

Position

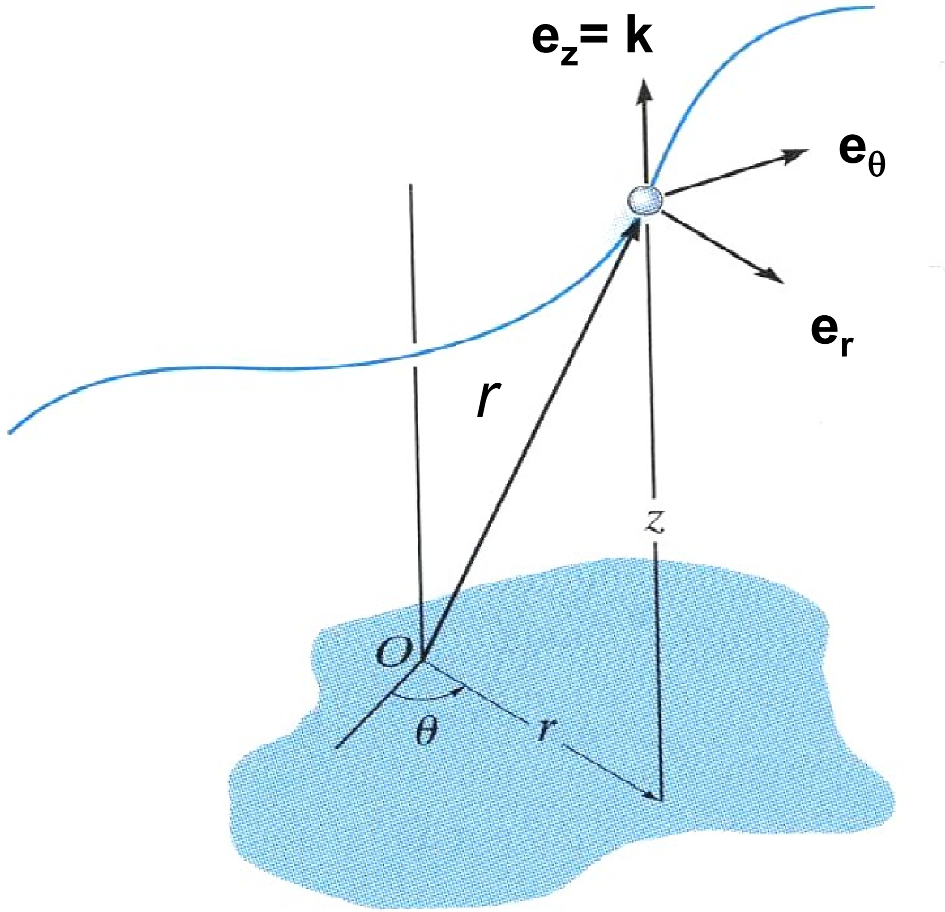
$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z$$

Velocity

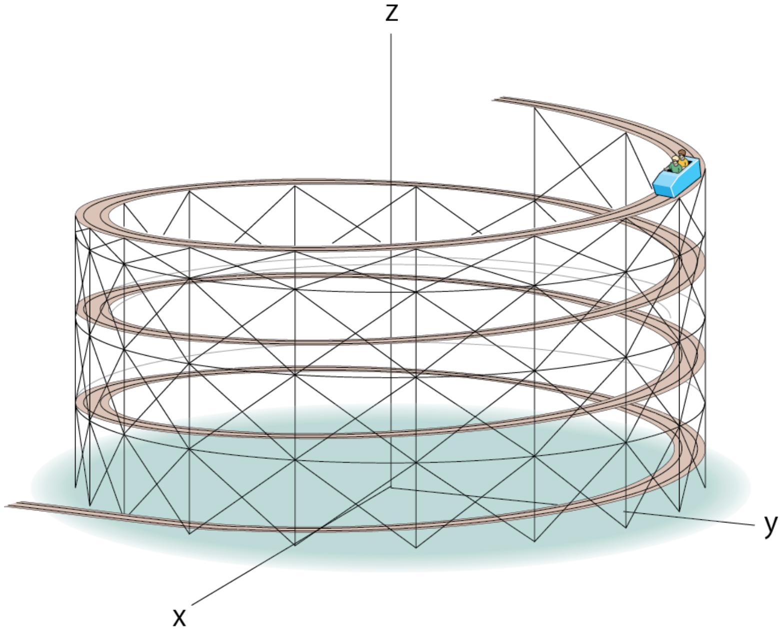
$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$$

Acceleration

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{e}_z$$

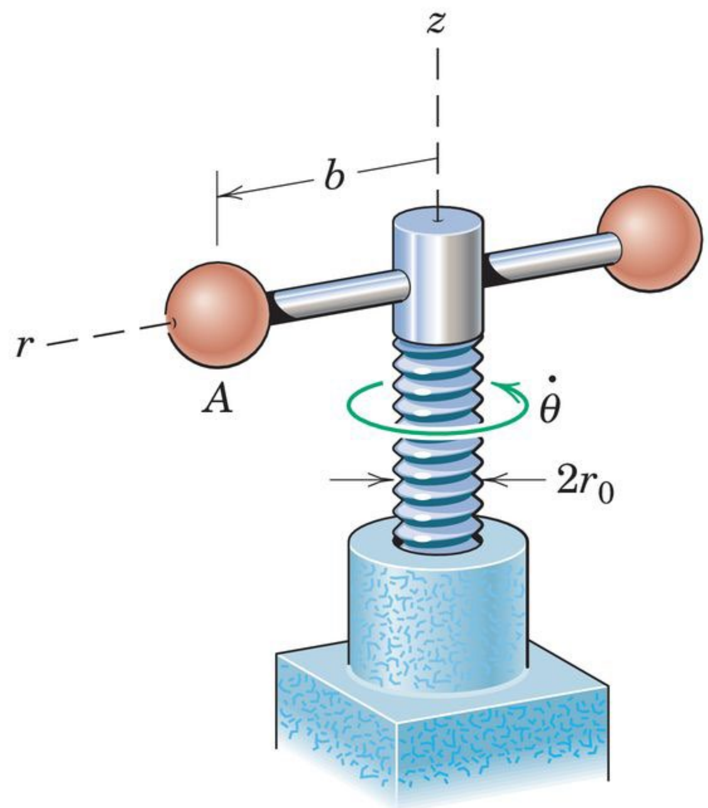


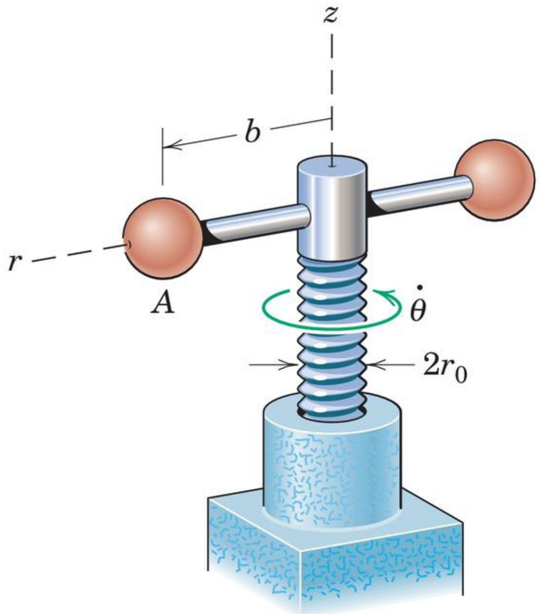
Helix



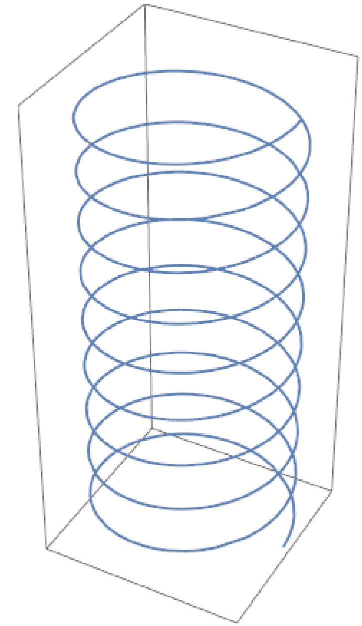
Sample Problem 1

The motion (of the power screw) starts from rest. The rotational speed $\dot{\theta}$ increases uniformly with time: $\dot{\theta} = k\,t$ where k is a constant. Find the velocity and acceleration of the ball A when the screw has turned through one complete revolution. The lead of the screw (advancement per revolution) is *L*.





Solution



The center of the ball moves in a helix on the cylindrical surface of radius b with angular velocity $\dot{\theta}(t)$

$$\dot{\theta}(t) = kt \Rightarrow$$

$$\theta(t) = \int_0^t \dot{\theta}(t)\mathrm{d}t = \int_0^t kt\mathrm{d}t = \frac{1}{2}kt^2$$

For one revolution

$$\theta(T) = 2\pi = \frac{1}{2}kT^2 \Rightarrow T = 2\sqrt{\pi / k} \Rightarrow \quad \dot{\theta}(T) = kT = 2\sqrt{\pi k}$$

Solution

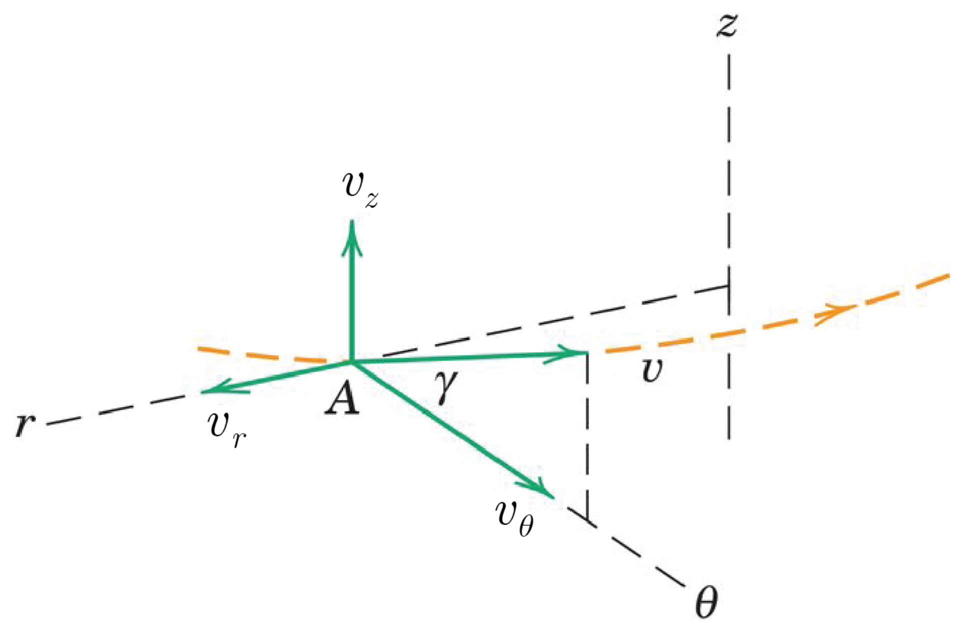
$$v_r = \dot{r}, \;\; v_\theta = r\dot{\theta}, \;\; v_z = \dot{z}$$

The helix angle

$$\frac{v_z}{v_\theta} = \tan\gamma = \frac{L}{2\pi b} \Rightarrow$$

$$\cos\gamma = \frac{2\pi b}{\sqrt{L^2 + (2\pi b)^2}} \;\; \text{and}$$

$$\sin\gamma = \frac{L}{\sqrt{L^2 + (2\pi b)^2}}$$



$$\boxed{v_r = 0, \;\; v_z = v\sin\gamma, \;\; v_\theta = v\cos\gamma}$$

The components of the velocity and the speed

$$v_\theta = b\,\dot{\theta}(T) = 2b\sqrt{\pi\,k}$$

$$v = v_\theta / \cos\gamma = 2b\sqrt{\pi k}\,\frac{\sqrt{L^2 + (2\pi b)^2}}{2\pi b} = \sqrt{\frac{k(L^2 + 4\pi^2 b^2)}{\pi}}$$

$$v_z = v\sin\gamma = v_\theta \tan\gamma = 2b\sqrt{\pi\,k}\,\frac{L}{2\pi b} = L\sqrt{k/\pi}$$

Solution

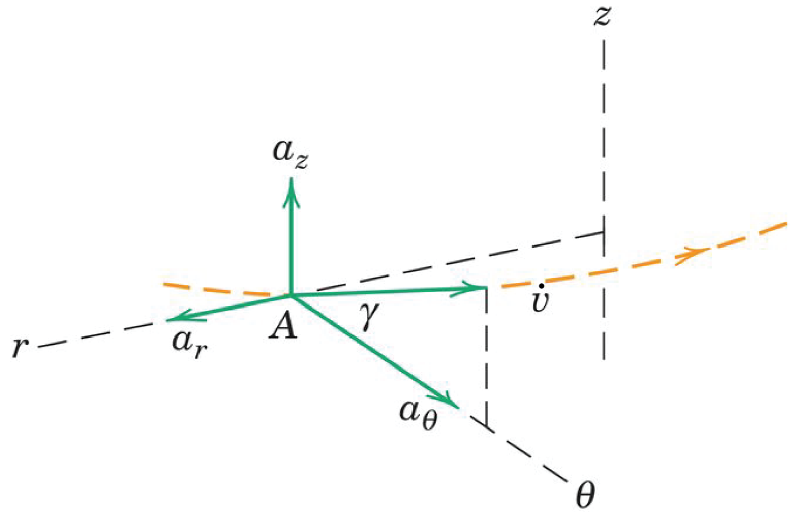
$$a_r = \ddot{r} - r\dot{\theta}^2, \;\; a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \;\; a_z = \ddot{z}$$

The vertical component

$$v_z(t) = v_\theta(t)\tan\gamma =$$

$$= b\dot{\theta}(t)\frac{L}{2\pi b} = \dot{\theta}(t)\frac{L}{2\pi}$$

$$a_z(t) = \frac{\mathrm{d}v_z(t)}{\mathrm{d}t} = \ddot{\theta}(t)\frac{L}{2\pi} = k\frac{L}{2\pi}$$



$\dot{v}$ is the rate of change of the speed

$$\boxed{a_z = \dot{v}\sin\gamma, \;\; a_\theta = \dot{v}\cos\gamma}$$

The components of the acceleration and its magnitude

$$a_r = -b\,\dot{\theta}^2\,(T) = -4b\pi\,k, \qquad a_\theta = b\ddot{\theta}(T) = bk, \qquad a_z = \frac{kL}{2\pi}$$

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2} = bk\sqrt{(1+16\pi^2) + L^2/(4\pi^2 b^2)}$$

Spherical Coordinates

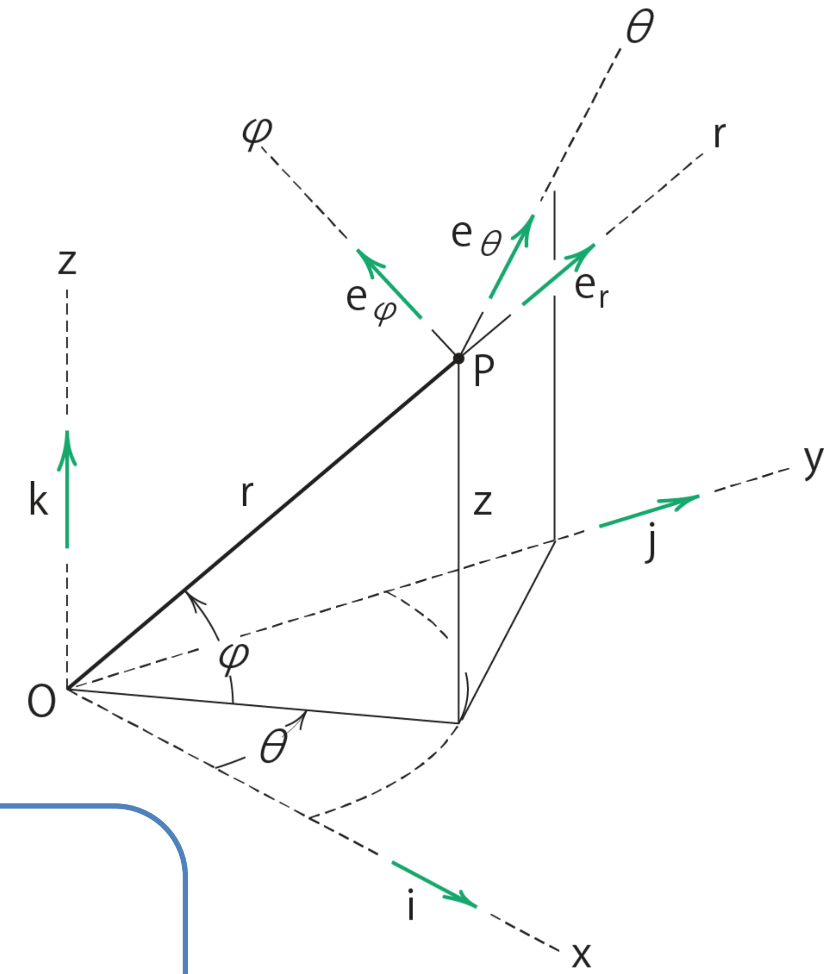
Position vector

$$\boldsymbol{r} = r\,\boldsymbol{e}_r$$

Velocity

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\cos\varphi\,\boldsymbol{e}_\theta + r\dot{\varphi}\,\boldsymbol{e}_\varphi$$

$$\begin{aligned}\boldsymbol{a} = {} & (\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2\cos^2\varphi)\ \boldsymbol{e}_r \\ & + (r\ddot{\theta}\cos\varphi + 2\dot{r}\dot{\theta}\cos\varphi - 2r\dot{\theta}\dot{\varphi}\sin\varphi)\ \boldsymbol{e}_\theta \\ & + (\,r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r\dot{\theta}^2\sin\varphi\cos\varphi)\,\boldsymbol{e}_\varphi\end{aligned}$$



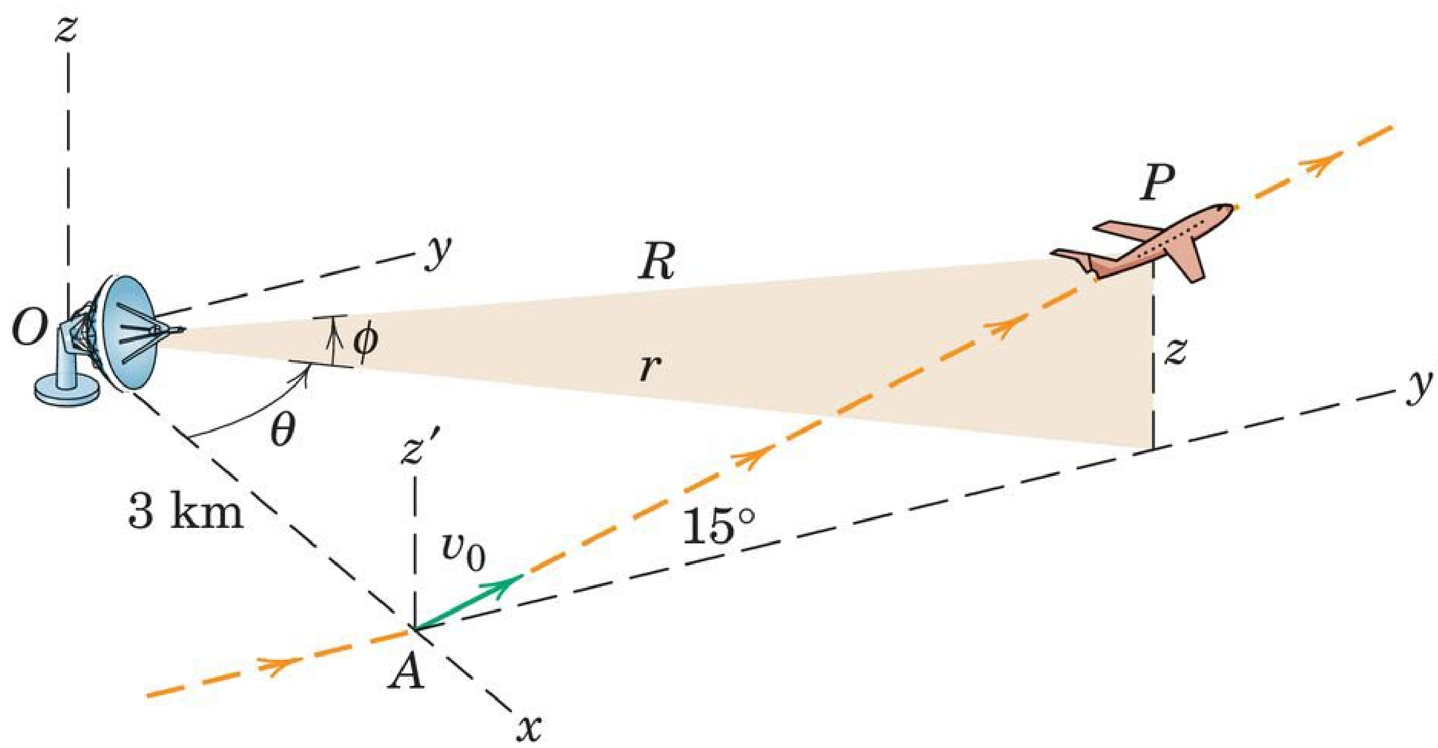
Acceleration

Sample Problem 2

An aircraft takes off at A with the velocity $v_0 = 250\text{km}/\text{h}$ and climbs in the vertical plane $y' - z'$ at the constant angle $\alpha = 15^{\circ}$ with an acceleration along its flight path of $0.8\text{m}/\text{s}^2$. Flight is monitored by radar at point O. Point P is reached after 60 seconds. Define (at point P) the time derivatives of

a) (r, θ, z) cylindrical coordinates

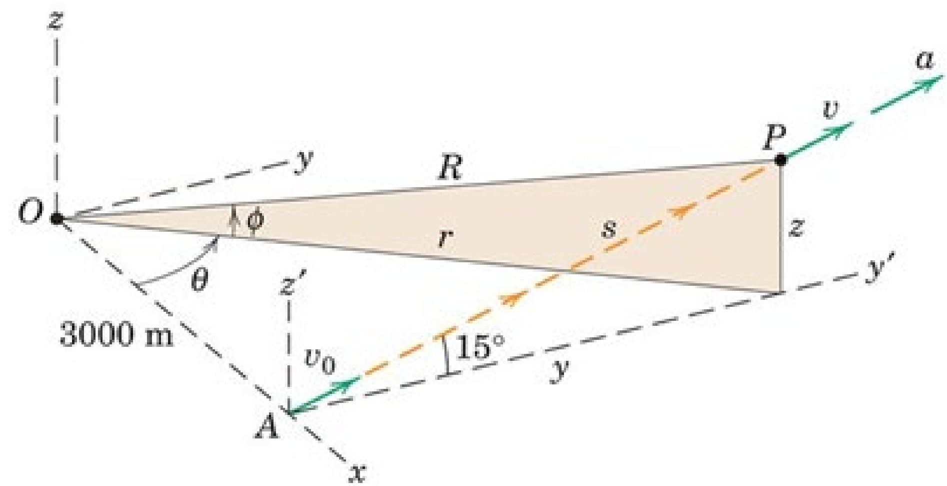
b) (R, θ, ϕ) spherical coordinates



<u>Part A.</u> The speed and the distance: for the given $v_0 = 69.4\,\mathrm{m/s}$ and $a = 0.8\mathrm{m/s^2}$ and $t = 60\,\mathrm{s}$ define

$$v = v_0 + at = 69.4 + 0.8 \times 60 = 117.4\,\mathrm{m/s}$$

$$s = |AP| = v_0 t + \frac{1}{2} a t^2 = 5610\,\mathrm{m}$$

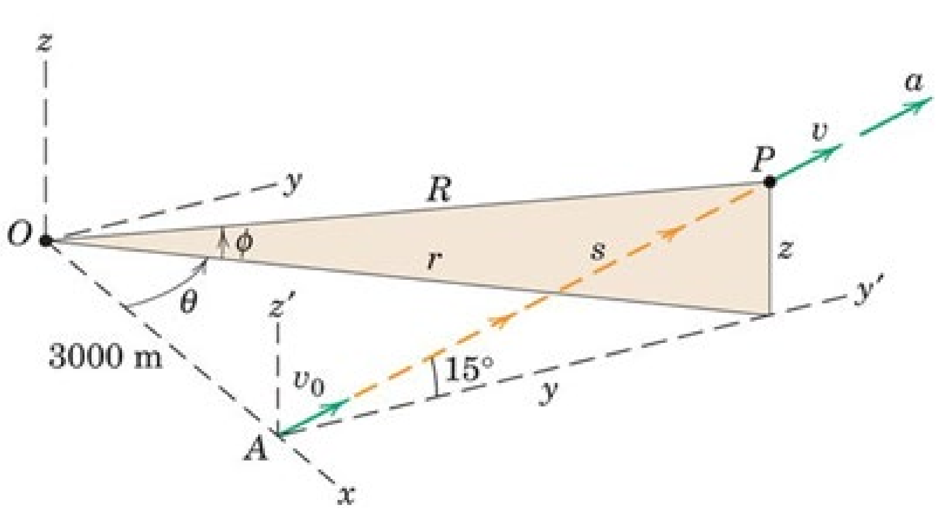


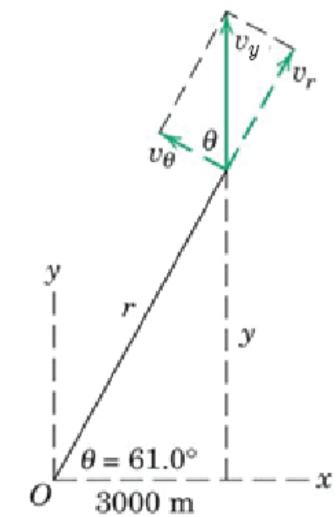
Velocity in rectangular coordinates: $\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}$

$$v_x = 0, \quad v_y = v\cos 15^\circ = 113.4\mathrm{m/s}, \quad v_z = v\sin 15^\circ = 30.4\mathrm{m/s}$$

Part B. Compute $y = s\cos 15^{\circ} = 5420\text{m},$ and establish polar coordinates:

$$r = \sqrt{3000^2 + 5420^2} = 6190\text{m}, \quad \theta = \arctan(5420/3000) = 61.0^{\circ}$$





Velocity in cylindrical coordinates: $\boldsymbol{v} = v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta + v_z \boldsymbol{k} = v_y \boldsymbol{j} + v_z \boldsymbol{k}$

Therefore, we have $v_r \boldsymbol{e}_r + v_\theta \boldsymbol{e}_\theta = v_y \boldsymbol{j}$

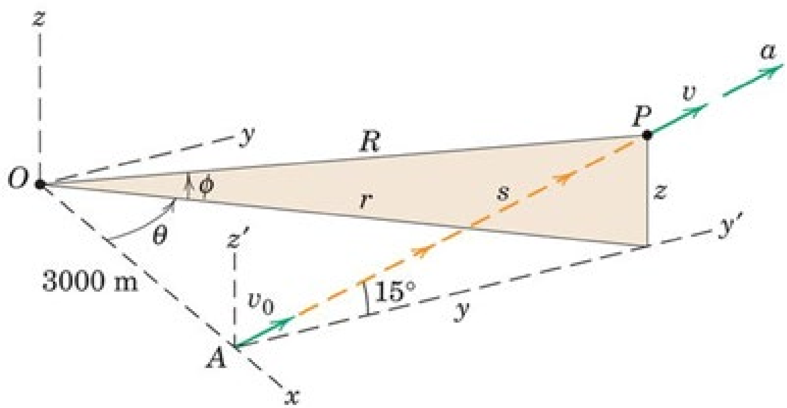
$$v_r = v_y(\boldsymbol{j} \cdot \boldsymbol{e}_r) = v_y(\boldsymbol{j} \cdot \{\cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}\}) = v_y \sin\theta = 99.2\text{m}/\text{s}$$

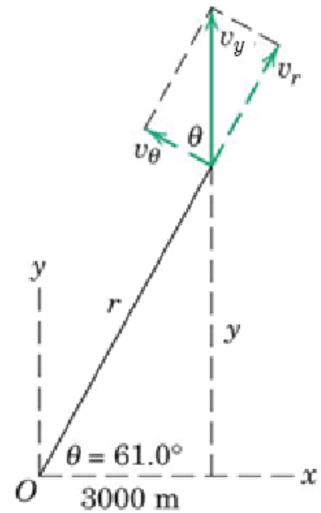
$$v_\theta = v_y(\boldsymbol{j} \cdot \boldsymbol{e}_\theta) = v_y(\boldsymbol{j} \cdot \{-\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j}\}) = v_y \cos\theta = 55.0\text{m}/\text{s}$$

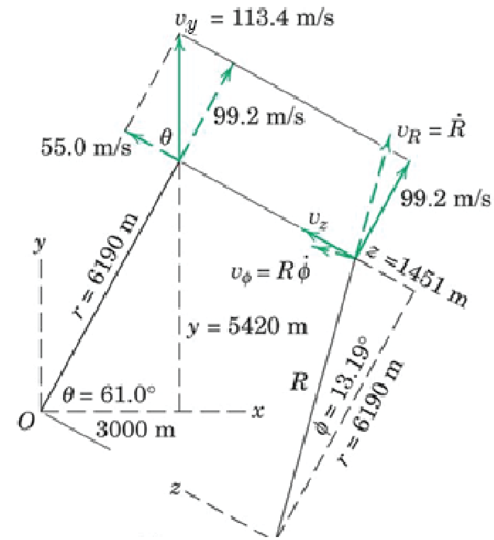
$$\boxed{\dot{r} = v_r = 99.2\text{m}/\text{s}, \quad \dot{\theta} = v_\theta / r = 8.88 \times 10^{-3}\text{rad}/\text{s}, \quad \dot{z} = v_z = 30.4\text{m}/\text{s}}$$

Part C. Compute $z = s\sin 15^\circ = 1451\text{m}$, and establish spherical coordinates:

$$R = \sqrt{r^2 + z^2} = 6360\text{m}, \quad \phi = \arctan(z/r) = 13.19^\circ$$





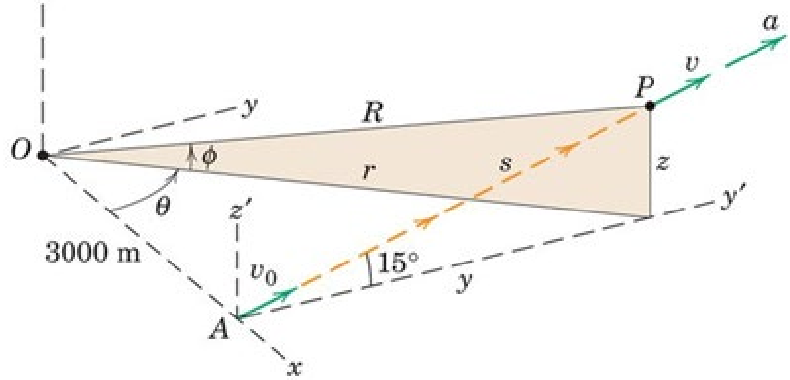


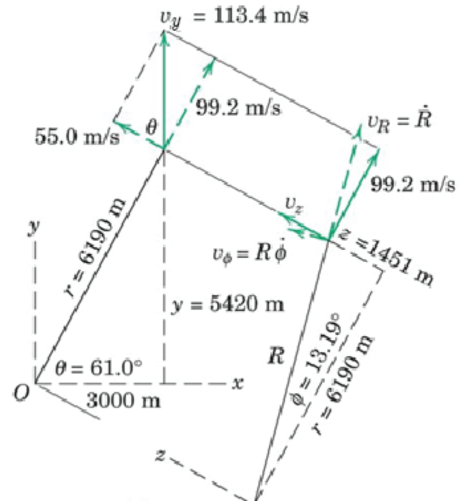
Velocity in spherical coordinates: $\boldsymbol{v} = v_R \boldsymbol{e}_R + v_\Theta \boldsymbol{e}_\theta + v_\phi \boldsymbol{e}_\phi = v_y \boldsymbol{j} + v_z \boldsymbol{k}$

Therefore, we have $v_R \boldsymbol{e}_R + v_\Theta \boldsymbol{e}_\theta + v_\phi \boldsymbol{e}_\phi = v_y \boldsymbol{j} + v_z \boldsymbol{k}$

$$v_R = (v_y \boldsymbol{j} + v_z \boldsymbol{k}) \cdot \boldsymbol{e}_R$$

$$v_\phi = (v_y \boldsymbol{j} + v_z \boldsymbol{k}) \cdot \boldsymbol{e}_\phi$$





$$\boldsymbol{e}_R = \cos\phi\cos\theta\,\boldsymbol{i} + \cos\phi\sin\theta\,\boldsymbol{j} + \sin\phi\,\boldsymbol{k}$$

$$\boldsymbol{e}_\Theta = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}$$

$$\boldsymbol{e}_\phi = -\sin\phi\cos\theta\,\boldsymbol{i} - \sin\phi\sin\theta\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k}$$

Therefore, we have

$$v_R = (v_y\boldsymbol{j} + v_z\boldsymbol{k})\cdot\boldsymbol{e}_R = v_y\sin\theta\cos\phi + v_z\sin\phi = 103.6\text{m/s}$$

$$v_\phi = (v_y\boldsymbol{j} + v_z\boldsymbol{k})\cdot\boldsymbol{e}_\phi = -v_y\sin\theta\sin\phi + v_z\cos\phi = 6.95\text{m/s}$$

$$\boxed{\dot{R} = v_R = 103.6\text{m/s}, \quad \dot{\phi} = v_\phi / R = 1.093\times10^{-3}\text{rad/s}}$$

$$\boxed{\dot{\theta} = v_\theta / r = 8.88\times10^{-3}\text{rad/s}}$$ (as in part B)