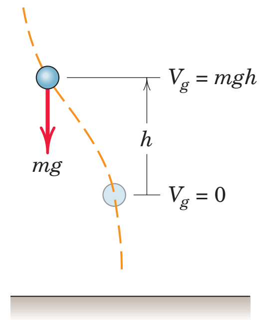
3/7 Potential Energy

In the previous article on work and kinetic energy, we isolated a particle or a combination of joined particles and determined the work done by gravity forces, spring forces, and other externally applied forces acting on the particle or system. We did this to evaluate U in the work-energy equation. In the present article we will introduce the concept of *potential energy* to treat the work done by gravity forces and by spring forces. This concept will simplify the analysis of many problems.

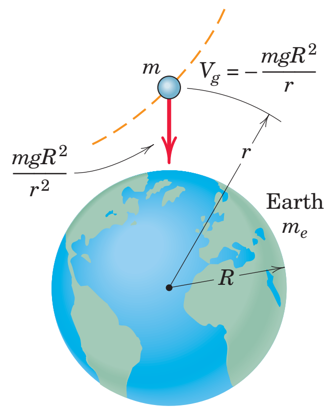
Gravitational Potential Energy

We consider first the motion of a particle of mass m in close proximity to the surface of the earth, where the gravitational attraction (weight) mg is essentially constant, Fig. 3/8a. The *gravitational potential energy* V_g of the particle is defined as the work mgh done *against* the gravitational field to elevate the particle a distance h above some arbitrary reference plane (called a *datum*), where V_g is taken to be zero. Thus, we write the potential energy as

$$\boxed{V_g = mgh} \tag{3/18}$$



(a)



(b)

Figure 3/8

This work is called potential energy because it may be converted into energy if the particle is allowed to do work on a supporting body while it returns to its lower original datum plane. In going from one level at $h = h_1$ to a higher level at $h = h_2$, the *change* in potential energy becomes

$$\Delta V_g = mg(h_2 - h_1) = mg\Delta h$$

The corresponding work done *by* the gravitational force on the particle is $-mg\Delta h$. Thus, the work done by the gravitational force is the negative of the change in potential energy.

When large changes in altitude in the field of the earth are encountered, Fig. 3/8b, the gravitational force $Gmm_e/r^2 = mgR^2/r^2$ is no longer constant. The work done *against* this force to change the radial position of the particle from r_1 to r_2 is the change $(V_g)_2 - (V_g)_1$ in gravitational potential energy, which is

$$\int_{r_1}^{r_2} mgR^2 \frac{dr}{r^2} = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = (V_g)_2 - (V_g)_1$$

It is customary to take $(V_g)_2 = 0$ when $r_2 = \infty$, so that with this datum we have

$$\boxed{V_g = -\frac{mgR^2}{r}} \tag{3/19}$$

In going from r_1 to r_2, the corresponding change in potential energy is

$$\Delta V_g = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

which, again, is the *negative* of the work done *by* the gravitational force. We note that the potential energy of a given particle depends only on its position, h or r, and not on the particular path it followed in reaching that position.

Elastic Potential Energy

The second example of potential energy occurs in the deformation of an elastic body, such as a spring. The work which is done on the spring to deform it is stored in the spring and is called its *elastic potential energy* V_e. This energy is recoverable in the form of work done by the spring on the body attached to its movable end during the release of the deformation of the spring. For the one-dimensional linear spring of stiffness k, which we discussed in Art. 3/6 and illustrated in Fig. 3/5, the force supported by the spring at any deformation x, tensile or compressive, from its undeformed position is $F = kx$. Thus, we define the elastic potential energy of the spring as the work done on it to deform it an amount x, and we have

$$\boxed{V_e = \int_0^x kx\,dx = \frac{1}{2}kx^2} \qquad \textbf{(3/20)}$$

If the deformation, either tensile or compressive, of a spring increases from x_1 to x_2 during the motion, then the change in potential energy of the spring is its final value minus its initial value or

$$\Delta V_e = \frac{1}{2}k({x_2}^2 - {x_1}^2)$$

which is positive. Conversely, if the deformation of a spring decreases during the motion interval, then the change in potential energy of the spring becomes negative. The magnitude of these changes is represented by the shaded trapezoidal area in the F-x diagram of Fig. 3/5a.

Because the force exerted *on* the spring *by* the moving body is equal and opposite to the force F exerted *by* the spring *on* the body, it follows that the work done on the spring is the negative of the work done on the body. Therefore, we may replace the work U done by the spring on the body by $-\Delta V_e$, the negative of the potential energy change for the spring, provided the spring is now included within the system.

Work-Energy Equation

With the elastic member included in the system, we now modify the work-energy equation to account for the potential-energy terms. If $U'_{1\text{-}2}$ stands for the work of all external forces *other than* gravitational forces and spring forces, we may write Eq. 3/15 as $U'_{1\text{-}2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T$ or

$$U'_{1\text{-}2} = \Delta T + \Delta V \qquad \textbf{(3/21)}$$

where ΔV is the change in total potential energy, gravitational plus elastic.

This alternative form of the work-energy equation is often far more convenient to use than Eq. 3/15, since the work of both gravity and spring forces is accounted for by focusing attention on the end-point positions of the particle and on the end-point lengths of the elastic spring. The path followed between these end-point positions is of no consequence in the evaluation of ΔV_g and ΔV_e.

Note that Eq. 3/21 may be rewritten in the equivalent form

$$\boxed{T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2} \qquad \textbf{(3/21a)}$$

To help clarify the difference between the use of Eqs. 3/15 and 3/21, Fig. 3/9 shows schematically a particle of mass m constrained to move along a fixed path under the action of forces F_1 and F_2, the gravitational force $W = mg$, the spring force F, and the normal reaction N. In Fig. 3/9b, the particle is isolated with its free-body diagram. The work done by each of the forces F_1, F_2, W, and the spring force $F = kx$ is evaluated, say, from A to B, and equated to the change ΔT in kinetic energy using Eq. 3/15. The constraint reaction N, if normal to the path, will do no work. The alternative approach is shown in Fig. 3/9c, where the spring is included as a part of the isolated system. The work done during the interval by F_1 and F_2 is the $U'_{1\text{-}2}$-term of Eq. 3/21 with the changes in elastic and gravitational potential energies included on the energy side of the equation.

We note with the first approach that the work done by $F = kx$ could require a somewhat awkward integration to account for the changes in magnitude and direction of F as the particle moves from A

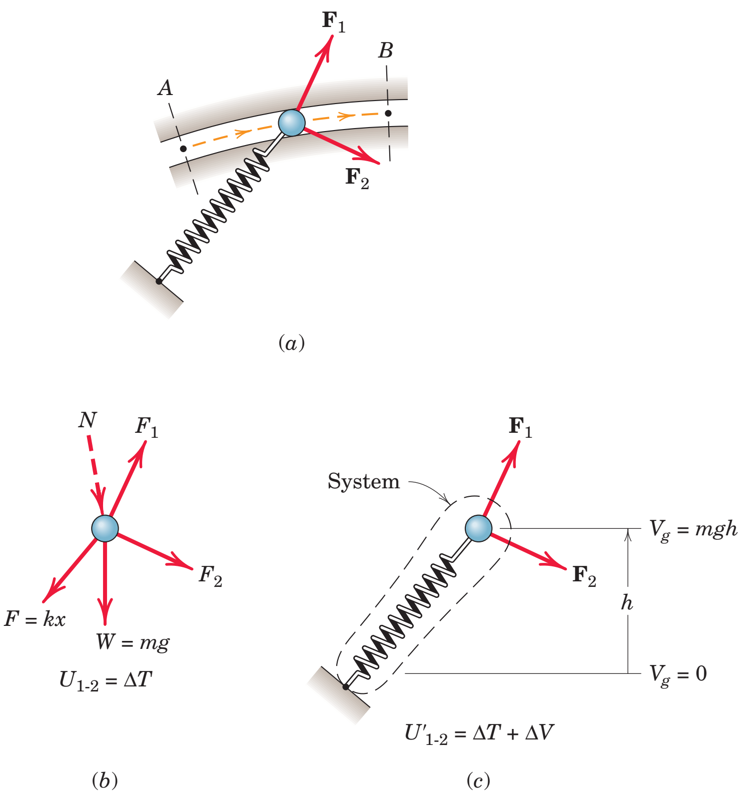


Figure 3/9

to B. With the second approach, however, only the initial and final lengths of the spring are required to evaluate ΔV_e. This greatly simplifies the calculation.

For problems where the only forces are gravitational, elastic, and nonworking constraint forces, the U'-term of Eq. 3/21a is zero, and the energy equation becomes

$$T_1 + V_1 = T_2 + V_2 \qquad \text{or} \qquad E_1 = E_2 \tag{3/22}$$

where $E = T + V$ is the total mechanical energy of the particle and its attached spring. When E is constant, we see that transfers of energy between kinetic and potential may take place as long as the total mechanical energy $T + V$ does not change. Equation 3/22 expresses the *law of conservation of dynamical energy*.

Conservative Force Fields*

We have observed that the work done against a gravitational or an elastic force depends only on the net change of position and not on the particular path followed in reaching the new position. Forces with this characteristic are associated with *conservative force fields*, which possess an important mathematical property.

Consider a force field where the force $\mathbf{F}$ is a function of the coordinates, Fig. 3/10. The work done by $\mathbf{F}$ during a displacement $d\mathbf{r}$ of its point of application is $dU = \mathbf{F}\cdot d\mathbf{r}$. The total work done along its path from 1 to 2 is

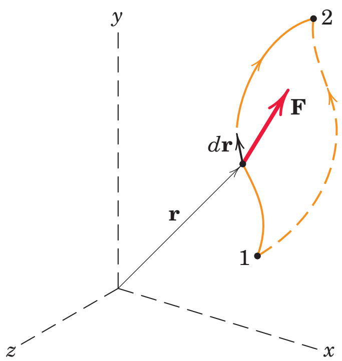


Figure 3/10

$$U = \int \mathbf{F}\cdot d\mathbf{r} = \int (F_x\,dx + F_y\,dy + F_z\,dz)$$

The integral $\int \mathbf{F}\cdot d\mathbf{r}$ is a line integral which depends, in general, on the particular path followed between any two points 1 and 2 in space. If, however, $\mathbf{F}\cdot d\mathbf{r}$ is an *exact differential*† $-dV$ of some scalar function V of the coordinates, then

$$U_{1\text{-}2} = \int_{V_1}^{V_2} -dV = -(V_2 - V_1) \tag{3/23}$$

which depends only on the end points of the motion and which is thus *independent* of the path followed. The minus sign before dV is arbitrary but is chosen to agree with the customary designation of the sign of potential energy change in the gravity field of the earth.

If V exists, the differential change in V becomes

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

*Optional.

†Recall that a function $d\Phi = P\,dx + Q\,dy + R\,dz$ is an exact differential in the coordinates x-y-z if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Comparison with $-dV = \mathbf{F}\cdot d\mathbf{r} = F_x\,dx + F_y\,dy + F_z\,dz$ gives us

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

The force may also be written as the vector

$$\mathbf{F} = -\boldsymbol{\nabla} V \tag{3/24}$$

where the symbol $\boldsymbol{\nabla}$ stands for the vector operator "del", which is

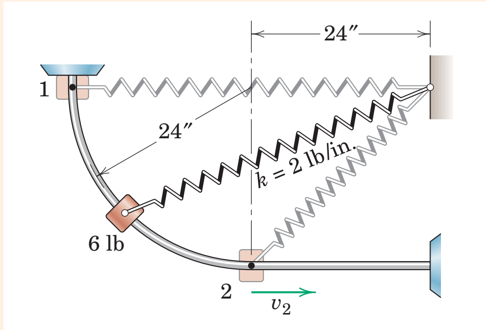
$$\boldsymbol{\nabla} = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

The quantity V is known as the *potential function*, and the expression $\boldsymbol{\nabla} V$ is known as the *gradient of the potential function*.

When force components are derivable from a potential as described, the force is said to be *conservative*, and the work done by $\mathbf{F}$ between any two points is independent of the path followed.

Sample Problem 3/16

The 6-lb slider is released from rest at position 1 and slides with negligible friction in a vertical plane along the circular rod. The attached spring has a stiffness of 2 lb/in. and has an unstretched length of 24 in. Determine the velocity of the slider as it passes position 2.



Solution. The work done by the weight and the spring force on the slider will be treated using potential-energy methods. The reaction of the rod on the slider is normal to the motion and does no work. Hence, $U'_{1\text{-}2} = 0$. We define the datum to be at the level of position 1, so that the gravitational potential energies are

① $$V_1 = 0$$

$$V_2 = -mgh = -6\left(\frac{24}{12}\right) = -12 \text{ ft-lb}$$

The initial and final elastic (spring) potential energies are

$$V_1 = \frac{1}{2}kx_1^{\,2} = \frac{1}{2}(2)(12)\left(\frac{24}{12}\right)^2 = 48 \text{ ft-lb}$$

$$V_2 = \frac{1}{2}kx_2^{\,2} = \frac{1}{2}(2)(12)\left(\frac{24\sqrt{2}}{12} - \frac{24}{12}\right)^2 = 8.24 \text{ ft-lb}$$

Substitution into the alternative work-energy equation yields

$$[T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2] \qquad 0 + 48 + 0 = \frac{1}{2}\left(\frac{6}{32.2}\right)v_2^{\,2} - 12 + 8.24$$

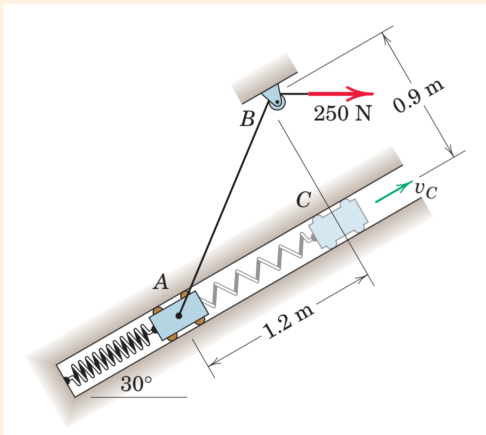
$$v_2 = 23.6 \text{ ft/sec} \qquad \textit{Ans.}$$

Helpful Hint

① Note that if we evaluated the work done by the spring force acting on the slider by means of the integral $\int \mathbf{F}\cdot d\mathbf{r}$, it would necessitate a lengthy computation to account for the change in the magnitude of the force, along with the change in the angle between the force and the tangent to the path. Note further that v_2 depends only on the end conditions of the motion and does not require knowledge of the shape of the path.

Sample Problem 3/17

The 10-kg slider moves with negligible friction up the inclined guide. The attached spring has a stiffness of 60 N/m and is stretched 0.6 m in position A, where the slider is released from rest. The 250-N force is constant and the pulley offers negligible resistance to the motion of the cord. Calculate the velocity v_C of the slider as it passes point C.



Solution. The slider and inextensible cord together with the attached spring will be analyzed as a system, which permits the use of Eq. 3/21a. The only nonpotential force doing work on this system is the 250-N tension applied to the cord. While the slider moves from A to C, the point of application of the 250-N force moves a distance of $\overline{AB} - \overline{BC}$ or $1.5 - 0.9 = 0.6$ m.

① ② $$U'_{A\text{-}C} = 250(0.6) = 150 \text{ J}$$

We define a datum at position A so that the initial and final gravitational potential energies are

$$V_A = 0 \qquad V_C = mgh = 10(9.81)(1.2 \sin 30°) = 58.9 \text{ J}$$

The initial and final elastic potential energies are

$$V_A = \frac{1}{2}kx_A^{\,2} = \frac{1}{2}(60)(0.6)^2 = 10.8 \text{ J}$$

$$V_C = \frac{1}{2}kx_B^{\,2} = \frac{1}{2}60(0.6 + 1.2)^2 = 97.2 \text{ J}$$

Substitution into the alternative work-energy equation 3/21a gives

$$[T_A + V_A + U'_{A\text{-}C} = T_C + V_C] \qquad 0 + 0 + 10.8 + 150 = \frac{1}{2}(10)v_C^{\,2} + 58.9 + 97.2$$

$$v_C = 0.974 \text{ m/s} \qquad \textit{Ans.}$$

Helpful Hints

① Do not hesitate to use subscripts tailored to the problem at hand. Here we use A and C rather than 1 and 2.

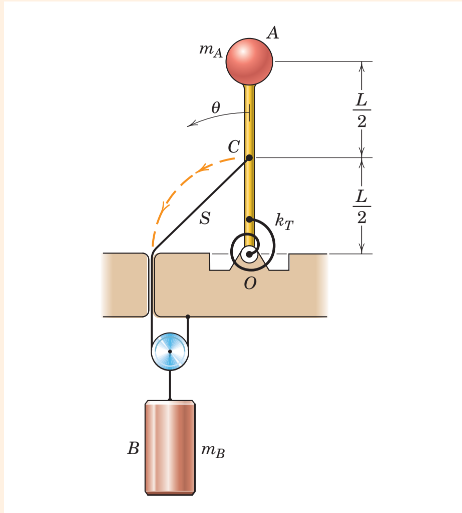
② The reactions of the guides on the slider are normal to the direction of motion and do no work.

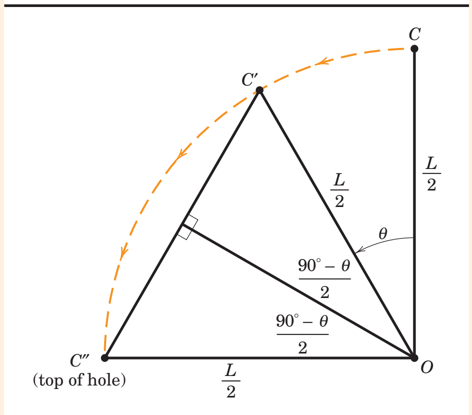
Sample Problem 3/18

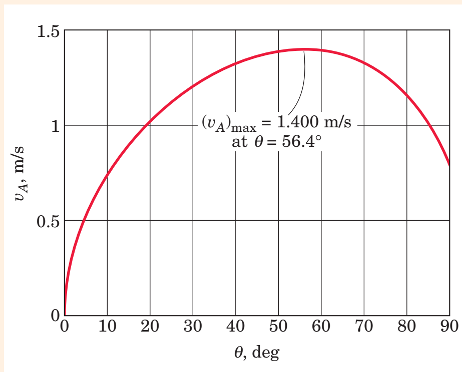
The system shown is released from rest with the lightweight slender bar OA in the vertical position shown. The torsional spring at O is undeflected in the initial position and exerts a restoring moment of magnitude $k_T\theta$ on the bar, where θ is the counterclockwise angular deflection of the bar. The string S is attached to point C of the bar and slips without friction through a vertical hole in the support surface. For the values $m_A = 2$ kg, $m_B = 4$ kg, $L = 0.5$ m, and $k_T = 13$ N·m/rad:

(*a*) Determine the speed v_A of particle A when θ reaches 90°.

(*b*) Plot v_A as a function of θ over the range $0 \le \theta \le 90°$. Identify the maximum value of v_A and the value of θ at which this maximum occurs.







Solution (a). We begin by establishing a general relationship for the potential energy associated with the deflection of a torsional spring. Recalling that the change in potential energy is the work done on the spring to deform it, we write

$$V_e = \int_0^{\theta} k_T\theta\, d\theta = \frac{1}{2}k_T\theta^2$$

We also need to establish the relationship between v_A and v_B when $\theta = 90°$. Noting that the speed of point C is always $v_A/2$, and further noting that the speed of cylinder B is one-half the speed of point C at $\theta = 90°$, we conclude that at $\theta = 90°$,

$$v_B = \frac{1}{4}v_A$$

Establishing datums at the initial altitudes of bodies A and B, and with state 1 at $\theta = 0$ and state 2 at $\theta = 90°$, we write

$$[T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2]$$

① $$0 + 0 + 0 = \frac{1}{2}m_A v_A{}^2 + \frac{1}{2}m_B v_B{}^2 - m_A gL - m_B g\left(\frac{L\sqrt{2}}{4}\right) + \frac{1}{2}k_T\left(\frac{\pi}{2}\right)^2$$

With numbers:

$$0 = \frac{1}{2}(2)v_A{}^2 + \frac{1}{2}(4)\left(\frac{v_A}{4}\right)^2 - 2(9.81)(0.5) - 4(9.81)\left(\frac{0.5\sqrt{2}}{4}\right) + \frac{1}{2}(13)\left(\frac{\pi}{2}\right)^2$$

Solving, $\quad v_A = 0.794$ m/s $\quad$ *Ans.*

(b). We leave our definition of the initial state 1 as is, but now redefine state 2 to be associated with an arbitrary value of θ. From the accompanying diagram constructed for an arbitrary value of θ, we see that the speed of cylinder B can be written as

② $$v_B = \frac{1}{2}\left|\frac{d}{dt}(\overline{C'C''})\right| = \frac{1}{2}\left|\frac{d}{dt}\left[2\frac{L}{2}\sin\left(\frac{90° - \theta}{2}\right)\right]\right|$$

$$= \frac{1}{2}\left|L\left(-\frac{\dot\theta}{2}\right)\cos\left(\frac{90° - \theta}{2}\right)\right| = \frac{L\dot\theta}{4}\cos\left(\frac{90° - \theta}{2}\right)$$

Finally, because $v_A = L\dot\theta$, $\quad v_B = \dfrac{v_A}{4}\cos\left(\dfrac{90° - \theta}{2}\right)$

$$[T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2]$$

$$0 + 0 + 0 = \frac{1}{2}m_A v_A{}^2 + \frac{1}{2}m_B\left[\frac{v_A}{4}\cos\left(\frac{90° - \theta}{2}\right)\right]^2 - m_A gL(1 - \cos\theta)$$
$$- m_B g\left(\frac{1}{2}\right)\left[\frac{L\sqrt{2}}{2} - 2\frac{L}{2}\sin\left(\frac{90° - \theta}{2}\right)\right] + \frac{1}{2}k_T\theta^2$$

Upon substitution of the given quantities, we vary θ to produce the plot of v_A versus θ. The maximum value of v_A is seen to be

$$(v_A)_{max} = 1.400 \text{ m/s at } \theta = 56.4°$$ *Ans.*

Helpful Hints

① Note that mass B will move downward by one-half of the length of string initially above the supporting surface. This downward distance is $\frac{1}{2}\left(\frac{L}{2}\sqrt{2}\right) = \frac{L\sqrt{2}}{4}$.

② The absolute-value signs reflect the fact that v_B is known to be positive.