SECTION D SPECIAL APPLICATIONS

3/11 Introduction

The basic principles and methods of particle kinetics were developed and illustrated in the first three sections of this chapter. This treatment included the direct use of Newton's second law, the equations of work and energy, and the equations of impulse and momentum. We paid special attention to the kind of problem for which each of the approaches was most appropriate.

Several topics of specialized interest in particle kinetics will be briefly treated in Section D:

1. Impact
2. Central-force motion
3. Relative motion

These topics involve further extension and application of the fundamental principles of dynamics, and their study will help to broaden your background in mechanics.

3/12 Impact

The principles of impulse and momentum have important use in describing the behavior of colliding bodies. *Impact* refers to the collision between two bodies and is characterized by the generation of relatively large contact forces which act over a very short interval of time. It is important to realize that an impact is a very complex event involving material deformation and recovery and the generation of heat and sound. Small changes in the impact conditions may cause large changes in the impact process and thus in the conditions immediately following the impact. Therefore, we must be careful not to rely heavily on the results of impact calculations.

Direct Central Impact

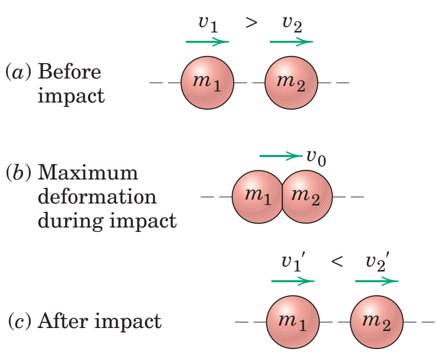

Figure 3/17

As an introduction to impact, we consider the collinear motion of two spheres of masses m_1 and m_2, Fig. 3/17*a*, traveling with velocities v_1 and v_2. If v_1 is greater than v_2, collision occurs with the contact forces directed along the line of centers. This condition is called *direct central impact*.

Following initial contact, a short period of increasing deformation takes place until the contact area between the spheres ceases to increase. At this instant, both spheres, Fig. 3/17*b*, are moving with the same velocity v_0. During the remainder of contact, a period of restoration occurs during which the contact area decreases to zero. In the final condition shown in part *c* of the figure, the spheres now have new velocities v_1' and v_2', where v_1' must be less than v_2'. All velocities are arbitrarily assumed positive to the right, so that with this scalar notation a velocity to the left would carry a negative sign. If the impact is not overly severe and if the spheres are highly elastic, they will regain their original shape following the restoration. With a more severe impact and with less elastic bodies, a permanent deformation may result.

Because the contact forces are equal and opposite during impact, the linear momentum of the system remains unchanged, as discussed in Art. 3/9. Thus, we apply the law of conservation of linear momentum and write

$$m_1 v_1 + m_2 v_2 = m_1 v_1{}' + m_2 v_2{}' \tag{3/35}$$

We assume that any forces acting on the spheres during impact, other than the large internal forces of contact, are relatively small and produce negligible impulses compared with the impulse associated with each internal impact force. In addition, we assume that no appreciable change in the positions of the mass centers occurs during the short duration of the impact.

Coefficient of Restitution

For given masses and initial conditions, the momentum equation contains two unknowns, $v_1{}'$ and $v_2{}'$. Clearly, we need an additional relationship to find the final velocities. This relationship must reflect the capacity of the contacting bodies to recover from the impact and can be expressed by the ratio e of the magnitude of the restoration impulse to the magnitude of the deformation impulse. This ratio is called the *coefficient of restitution*.

Let F_r and F_d represent the magnitudes of the contact forces during the restoration and deformation periods, respectively, as shown in Fig. 3/18. For particle 1 the definition of e together with the impulse-momentum equation give us

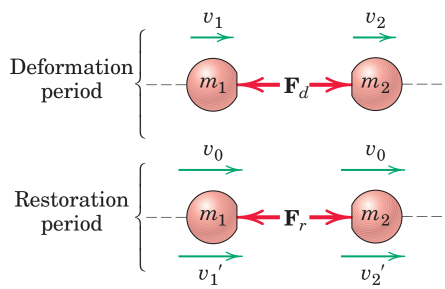

Figure 3/18

$$e = \frac{\int_{t_0}^{t} F_r\,dt}{\int_0^{t_0} F_d\,dt} = \frac{m_1[-v_1{}' - (-v_0)]}{m_1[-v_0 - (-v_1)]} = \frac{v_0 - v_1{}'}{v_1 - v_0}$$

Similarly, for particle 2 we have

$$e = \frac{\int_{t_0}^{t} F_r\,dt}{\int_0^{t_0} F_d\,dt} = \frac{m_2\,(v_2{}' - v_0)}{m_2\,(v_0 - v_2)} = \frac{v_2{}' - v_0}{v_0 - v_2}$$

We are careful in these equations to express the change of momentum (and therefore Δv) in the same direction as the impulse (and thus the force). The time for the deformation is taken as t_0 and the total time of contact is t. Eliminating v_0 between the two expressions for e gives us

$$e = \frac{v_2{}' - v_1{}'}{v_1 - v_2} = \frac{|\text{relative velocity of separation}|}{|\text{relative velocity of approach}|} \tag{3/36}$$

If the two initial velocities v_1 and v_2 and the coefficient of restitution e are known, then Eqs. 3/35 and 3/36 give us two equations in the two unknown final velocities $v_1{}'$ and $v_2{}'$.

Energy Loss During Impact

Impact phenomena are almost always accompanied by energy loss, which may be calculated by subtracting the kinetic energy of the system just after impact from that just before impact. Energy is lost through the generation of heat during the localized inelastic deformation of the material, through the generation and dissipation of elastic stress waves within the bodies, and through the generation of sound energy.

According to this classical theory of impact, the value $e = 1$ means that the capacity of the two particles to recover equals their tendency to deform. This condition is one of *elastic impact* with no energy loss. The value $e = 0$, on the other hand, describes *inelastic* or *plastic impact* where the particles cling together after collision and the loss of energy is a maximum. All impact conditions lie somewhere between these two extremes.

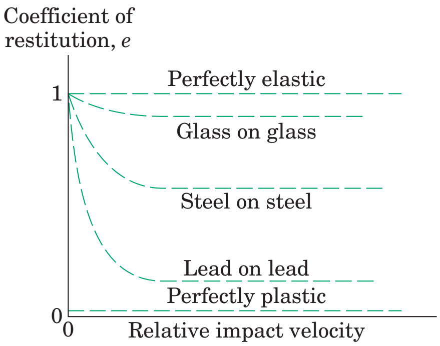

Figure 3/19

Also, it should be noted that a coefficient of restitution must be associated with a *pair* of contacting bodies. The coefficient of restitution is frequently considered a constant for given geometries and a given combination of contacting materials. Actually, it depends on the impact velocity and approaches unity as the impact velocity approaches zero as shown schematically in Fig. 3/19. A handbook value for e is generally unreliable.

Oblique Central Impact

We now extend the relationships developed for direct central impact to the case where the initial and final velocities are not parallel, Fig. 3/20. Here spherical particles of mass m_1 and m_2 have initial velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ in the same plane and approach each other on a collision course, as shown in part *a* of the figure. The directions of the velocity vectors are measured from the direction tangent to the contacting surfaces, Fig. 3/20*b*. Thus, the initial velocity components along the *t*- and *n*-axes are $(v_1)_n = -v_1 \sin\theta_1$, $(v_1)_t = v_1 \cos\theta_1$, $(v_2)_n = v_2 \sin\theta_2$, and $(v_2)_t = v_2 \cos\theta_2$. Note that

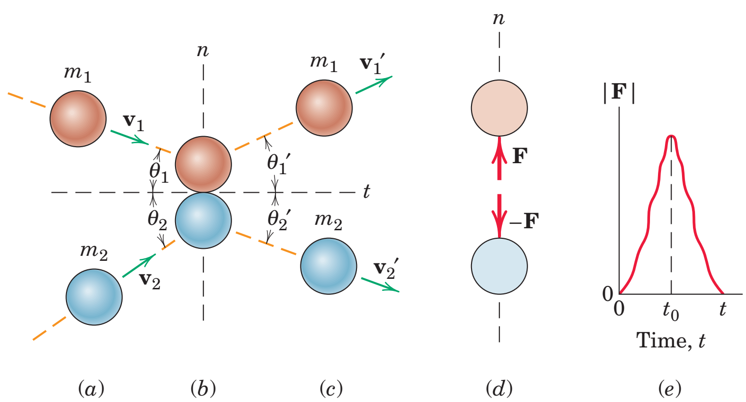

Figure 3/20

$(v_1)_n$ is a negative quantity for the particular coordinate system and initial velocities shown.

The final rebound conditions are shown in part c of the figure. The impact forces are $\mathbf{F}$ and $-\mathbf{F}$, as seen in part d of the figure. They vary from zero to their peak value during the deformation portion of the impact and back again to zero during the restoration period, as indicated in part e of the figure where t is the duration of the impact interval.

For given initial conditions of m_1, m_2, $(v_1)_n$, $(v_1)_t$, $(v_2)_n$, and $(v_2)_t$, there will be four unknowns, namely, $(v_1{}')_n$, $(v_1{}')_t$, $(v_2{}')_n$, and $(v_2{}')_t$. The four needed equations are obtained as follows:

(1) Momentum of the system is conserved in the n-direction. This gives

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1{}')_n + m_2(v_2{}')_n$$

(2) and (3) The momentum for each particle is conserved in the t-direction since there is no impulse on either particle in the t-direction. Thus,

$$m_1(v_1)_t = m_1(v_1{}')_t$$

$$m_2(v_2)_t = m_2(v_2{}')_t$$

(4) The coefficient of restitution, as in the case of direct central impact, is the positive ratio of the recovery impulse to the deformation impulse. Equation 3/36 applies, then, to the velocity components in the n-direction. For the notation adopted with Fig. 3/20, we have

$$e = \frac{(v_2{}')_n - (v_1{}')_n}{(v_1)_n - (v_2)_n}$$

Once the four final velocity components are found, the angles $\theta_1{}'$ and $\theta_2{}'$ of Fig. 3/20 may be easily determined.

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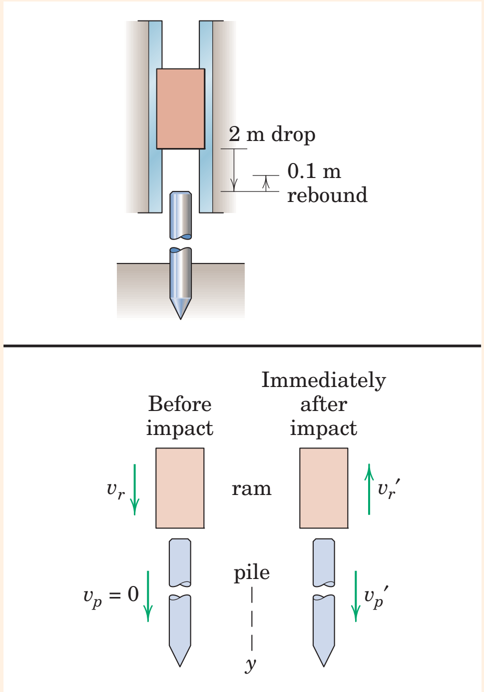
The motion of pool balls after impact is easily investigated with the principles of direct and oblique central impact.

Sample Problem 3/28

The ram of a pile driver has a mass of 800 kg and is released from rest 2 m above the top of the 2400-kg pile. If the ram rebounds to a height of 0.1 m after impact with the pile, calculate (*a*) the velocity v_p' of the pile immediately after impact, (*b*) the coefficient of restitution e, and (*c*) the percentage loss of energy due to the impact.



Solution. Conservation of energy during free fall gives the initial and final velocities of the ram from $v = \sqrt{2gh}$. Thus,

$$v_r = \sqrt{2(9.81)(2)} = 6.26 \text{ m/s} \qquad v_r' = \sqrt{2(9.81)(0.1)} = 1.40 \text{ m/s}$$

① ***(a)*** Conservation of momentum ($G_1 = G_2$) for the system of the ram and pile gives

$$800(6.26) + 0 = 800(-1.401) + 2400v_p' \qquad v_p' = 2.55 \text{ m/s} \qquad \textit{Ans.}$$

(b) The coefficient of restitution yields

$$e = \frac{|\text{rel. vel. separation}|}{|\text{rel. vel. approach}|} \qquad e = \frac{2.55 + 1.401}{6.26 + 0} = 0.631 \qquad \textit{Ans.}$$

(c) The kinetic energy of the system just before impact is the same as the potential energy of the ram above the pile and is

$$T = V_g = mgh = 800(9.81)(2) = 15\,700 \text{ J}$$

The kinetic energy T' just after impact is

$$T' = \tfrac{1}{2}(800)(1.401)^2 + \tfrac{1}{2}(2400)(2.55)^2 = 8620 \text{ J}$$

The percentage loss of energy is, therefore,

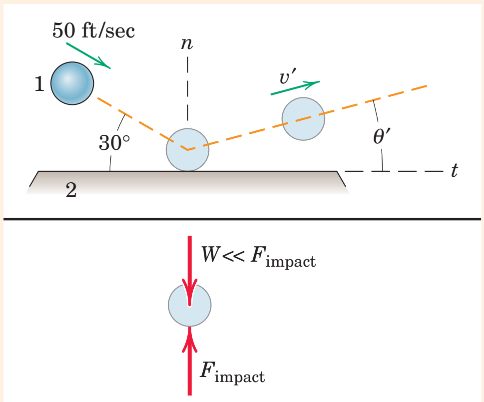
$$\frac{15\,700 - 8620}{15\,700}(100) = 45.1\% \qquad \textit{Ans.}$$

Helpful Hint

① The impulses of the weights of the ram and pile are very small compared with the impulses of the impact forces and thus are neglected during the impact.

Sample Problem 3/29

A ball is projected onto the heavy plate with a velocity of 50 ft/sec at the 30° angle shown. If the effective coefficient of restitution is 0.5, compute the rebound velocity v' and its angle θ'.



Solution. Let the ball be denoted body 1 and the plate body 2. The mass of the heavy plate may be considered infinite and its corresponding velocity zero after impact. The coefficient of restitution is applied to the velocity components normal to the plate in the direction of the impact force and gives

① $$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \qquad 0.5 = \frac{0 - (v_1')_n}{-50 \sin 30° - 0} \qquad (v_1')_n = 12.5 \text{ ft/sec}$$

Momentum of the ball in the t-direction is unchanged since, with assumed smooth surfaces, there is no force acting on the ball in that direction. Thus,

$$m(v_1)_t = m(v_1')_t \qquad (v_1')_t = (v_1)_t = 50 \cos 30° = 43.3 \text{ ft/sec}$$

The rebound velocity v' and its angle θ' are then

$$v' = \sqrt{(v_1')_n^{\,2} + (v_1')_t^{\,2}} = \sqrt{12.5^2 + 43.3^2} = 45.1 \text{ ft/sec} \qquad \textit{Ans.}$$

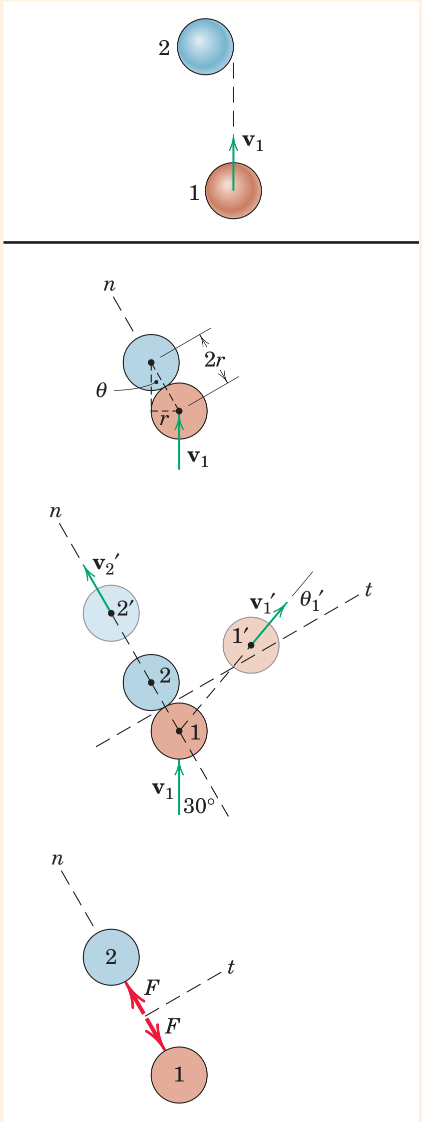
$$\theta' = \tan^{-1}\left(\frac{(v_1')_n}{(v_1')_t}\right) = \tan^{-1}\left(\frac{12.5}{43.3}\right) = 16.10° \qquad \textit{Ans.}$$

Helpful Hint

① We observe here that for infinite mass there is no way of applying the principle of conservation of momentum for the system in the n-direction. From the free-body diagram of the ball during impact, we note that the impulse of the weight W is neglected since W is very small compared with the impact force.

Sample Problem 3/30

Spherical particle 1 has a velocity $v_1 = 6$ m/s in the direction shown and collides with spherical particle 2 of equal mass and diameter and initially at rest. If the coefficient of restitution for these conditions is $e = 0.6$, determine the resulting motion of each particle following impact. Also calculate the percentage loss of energy due to the impact.



Solution. The geometry at impact indicates that the normal n to the contacting surfaces makes an angle $\theta = 30°$ with the direction of $\mathbf{v}_1$, as indicated
① in the figure. Thus, the initial velocity components are $(v_1)_n = v_1 \cos 30° = 6 \cos 30° = 5.20$ m/s, $(v_1)_t = v_1 \sin 30° = 6 \sin 30° = 3$ m/s, and $(v_2)_n = (v_2)_t = 0$.

Momentum conservation for the two-particle system in the n-direction gives

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n$$

or, with $m_1 = m_2$,

$$5.20 + 0 = (v_1')_n + (v_2')_n \qquad (a)$$

The coefficient-of-restitution relationship is

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \qquad 0.6 = \frac{(v_2')_n - (v_1')_n}{5.20 - 0} \qquad (b)$$

② Simultaneous solution of Eqs. a and b yields

$$(v_1')_n = 1.039 \text{ m/s} \qquad (v_2')_n = 4.16 \text{ m/s}$$

Conservation of momentum for each particle holds in the t-direction because, with assumed smooth surfaces, there is no force in the t-direction. Thus for particles 1 and 2, we have

$$m_1(v_1)_t = m_1(v_1')_t \qquad (v_1')_t = (v_1)_t = 3 \text{ m/s}$$

③
$$m_2(v_2)_t = m_2(v_2')_t \qquad (v_2')_t = (v_2)_t = 0$$

The final speeds of the particles are

$$v_1' = \sqrt{(v_1')_n{}^2 + (v_1')_t{}^2} = \sqrt{(1.039)^2 + 3^2} = 3.17 \text{ m/s} \qquad \textit{Ans.}$$

$$v_2' = \sqrt{(v_2')_n{}^2 + (v_2')_t{}^2} = \sqrt{(4.16)^2 + 0^2} = 4.16 \text{ m/s} \qquad \textit{Ans.}$$

The angle θ' which $\mathbf{v}_1'$ makes with the t-direction is

$$\theta' = \tan^{-1}\left(\frac{(v_1')_n}{(v_1')_t}\right) = \tan^{-1}\left(\frac{1.039}{3}\right) = 19.11° \qquad \textit{Ans.}$$

The kinetic energies just before and just after impact, with $m = m_1 = m_2$, are

$$T = \tfrac{1}{2}m_1v_1{}^2 + \tfrac{1}{2}m_2v_2{}^2 = \tfrac{1}{2}m(6)^2 + 0 = 18m$$

$$T' = \tfrac{1}{2}m_1v_1'{}^2 + \tfrac{1}{2}m_2v_2'{}^2 = \tfrac{1}{2}m(3.17)^2 + \tfrac{1}{2}m(4.16)^2 = 13.68m$$

The percentage energy loss is then

$$\frac{|\Delta E|}{E}(100) = \frac{T - T'}{T}(100) = \frac{18m - 13.68m}{18m}(100) = 24.0\% \qquad \textit{Ans.}$$

Helpful Hints

① Be sure to set up n- and t-coordinates which are, respectively, normal to and tangent to the contacting surfaces. Calculation of the 30° angle is critical to all that follows.

② Note that, even though there are four equations in four unknowns for the standard problem of oblique central impact, only one pair of the equations is coupled.

③ We note that particle 2 has no initial or final velocity component in the t-direction. Hence, its final velocity $\mathbf{v}_2'$ is restricted to the n-direction.