SECTION C IMPULSE AND MOMENTUM

3/8 Introduction

In the previous two articles, we focused attention on the equations of work and energy, which are obtained by integrating the equation of motion $\mathbf{F} = m\mathbf{a}$ with respect to the displacement of the particle. We found that the velocity changes could be expressed directly in terms of the work done or in terms of the overall changes in energy. In the next two articles, we will integrate the equation of motion with respect to time rather than displacement. This approach leads to the equations of impulse and momentum. These equations greatly facilitate the solution of many problems in which the applied forces act during extremely short periods of time (as in impact problems) or over specified intervals of time.

3/9 Linear Impulse and Linear Momentum

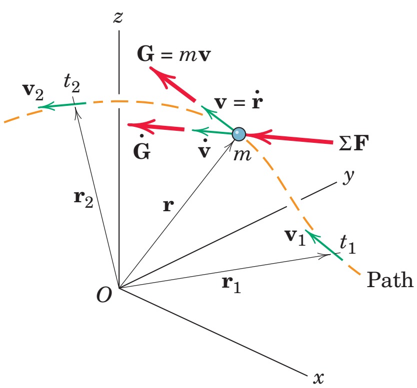


Figure 3/11

Consider again the general curvilinear motion in space of a particle of mass m, Fig. 3/11, where the particle is located by its position vector $\mathbf{r}$ measured from a fixed origin O. The velocity of the particle is $\mathbf{v} = \dot{\mathbf{r}}$ and is tangent to its path (shown as a dashed line). The resultant $\Sigma\mathbf{F}$ of all forces on m is in the direction of its acceleration $\dot{\mathbf{v}}$. We may now write the basic equation of motion for the particle, Eq. 3/3, as

$$\Sigma\mathbf{F} = m\dot{\mathbf{v}} = \frac{d}{dt}(m\mathbf{v}) \qquad \text{or} \qquad \boxed{\Sigma\mathbf{F} = \dot{\mathbf{G}}} \tag{3/25}$$

where the product of the mass and velocity is defined as the *linear momentum* $\mathbf{G} = m\mathbf{v}$ of the particle. Equation 3/25 states that *the resultant of all forces acting on a particle equals its time rate of change of linear momentum*. In SI the units of linear momentum $m\mathbf{v}$ are seen to be kg·m/s, which also equals N·s. In U.S. customary units, the units of linear momentum $m\mathbf{v}$ are [lb/(ft/sec^2)][ft/sec] = lb-sec.

Because Eq. 3/25 is a vector equation, we recognize that, in addition to the equality of the magnitudes of $\Sigma\mathbf{F}$ and $\dot{\mathbf{G}}$, the direction of the resultant force coincides with the direction of the rate of change in linear momentum, which is the direction of the rate of change in velocity. Equation 3/25 is one of the most useful and important relationships in dynamics, and it is valid as long as the mass m of the particle is not changing with time. The case where m changes with time is discussed in Art. 4/7 of Chapter 4.

We now write the three scalar components of Eq. 3/25 as

$$\Sigma F_x = \dot{G}_x \qquad \Sigma F_y = \dot{G}_y \qquad \Sigma F_z = \dot{G}_z \tag{3/26}$$

These equations may be applied independently of one another.

The Linear Impulse-Momentum Principle

All that we have done so far in this article is to rewrite Newton's second law in an alternative form in terms of momentum. But we are now able to describe the effect of the resultant force $\Sigma\mathbf{F}$ on the linear momentum of the particle over a finite period of time simply by integrating Eq. 3/25 with respect to the time t. Multiplying the equation by dt gives $\Sigma\mathbf{F}\,dt = d\mathbf{G}$, which we integrate from time t_1 to time t_2 to obtain

$$\int_{t_1}^{t_2} \Sigma\mathbf{F}\,dt = \mathbf{G}_2 - \mathbf{G}_1 = \Delta\mathbf{G} \tag{3/27}$$

Here the linear momentum at time t_2 is $\mathbf{G}_2 = m\mathbf{v}_2$ and the linear momentum at time t_1 is $\mathbf{G}_1 = m\mathbf{v}_1$. The product of force and time is defined as the *linear impulse* of the force, and Eq. 3/27 states that *the total linear impulse on m equals the corresponding change in linear momentum of m*.

Alternatively, we may write Eq. 3/27 as

$$\boxed{\mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma\mathbf{F}\,dt = \mathbf{G}_2} \tag{3/27a}$$

which says that the initial linear momentum of the body plus the linear impulse applied to it equals its final linear momentum.

The impulse integral is a vector which, in general, may involve changes in both magnitude and direction during the time interval. Under these conditions, it will be necessary to express $\Sigma\mathbf{F}$ and $\mathbf{G}$ in component form and then combine the integrated components. The components of Eq. 3/27*a* are the scalar equations

$$\begin{aligned} m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x\,dt &= m(v_2)_x \\ m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y\,dt &= m(v_2)_y \\ m(v_1)_z + \int_{t_1}^{t_2} \Sigma F_z\,dt &= m(v_2)_z \end{aligned} \tag{3/27b}$$

These three scalar impulse-momentum equations are completely independent.

Whereas Eq. 3/27 clearly stresses that the external linear impulse causes a change in the linear momentum, the order of the terms in Eqs. 3/27*a* and 3/27*b* corresponds to the natural sequence of events. While the form of Eq. 3/27 may be best for the experienced dynamicist, the form of Eqs. 3/27*a* and 3/27*b* is very effective for the beginner.

We now introduce the concept of the *impulse-momentum diagram*. Once the body to be analyzed has been clearly identified and isolated, we construct three drawings of the body as shown in Fig. 3/12. In the first drawing, we show the initial momentum $m\mathbf{v}_1$, or components thereof. In

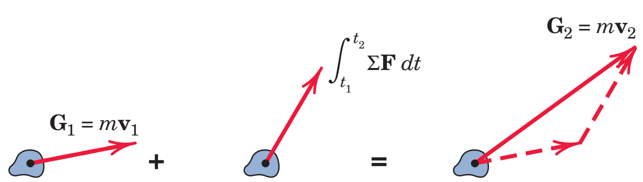


Figure 3/12

The impact force exerted by the bat on this ball will usually be much larger than the weight of the ball.

the second or middle drawing, we show all the external linear impulses (or components thereof). In the final drawing, we show the final linear momentum $m\mathbf{v}_2$ (or its components). The writing of the impulse-momentum equations 3/27*b* then follows directly from these drawings, with a clear one-to-one correspondence between diagrams and equation terms.

We note that the center diagram is very much like a free-body diagram, except that the impulses of the forces appear rather than the forces themselves. As with the free-body diagram, it is necessary to include the effects of *all* forces acting on the body, except those forces whose magnitudes are negligible.

In some cases, certain forces are very large and of short duration. Such forces are called *impulsive forces*. An example is a force of sharp impact. We frequently assume that impulsive forces are constant over their time of duration, so that they can be brought outside the linear-impulse integral. In addition, we frequently assume that *nonimpulsive forces* can be neglected in comparison with impulsive forces. An example of a nonimpulsive force is the weight of a baseball during its collision with a bat—the weight of the ball (about 5 oz) is small compared with the force (which could be several hundred pounds in magnitude) exerted on the ball by the bat.

There are cases where a force acting on a particle varies with the time in a manner determined by experimental measurements or by other approximate means. In this case a graphical or numerical integration must be performed. If, for example, a force F acting on a particle in a given direction varies with the time t as indicated in Fig. 3/13, then the impulse, $\int_{t_1}^{t_2} F\,dt$, of this force from t_1 to t_2 is the shaded area under the curve.

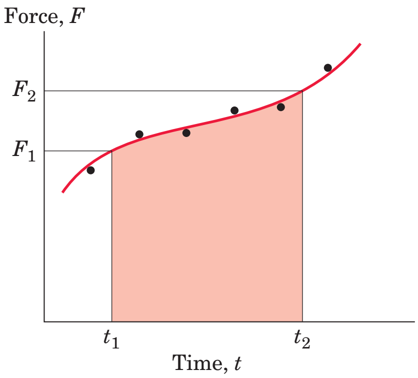


Figure 3/13

Conservation of Linear Momentum

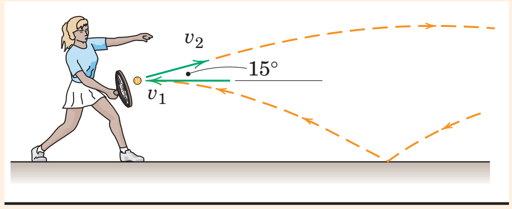
If the resultant force on a particle is zero during an interval of time, we see that Eq. 3/25 requires that its linear momentum $\mathbf{G}$ remain constant. In this case, the linear momentum of the particle is said to be *conserved*. Linear momentum may be conserved in one coordinate direction, such as x, but not necessarily in the y- or z-direction. A careful examination of the impulse-momentum diagram of the particle will disclose whether the total linear impulse on the particle in a particular direction is zero. If it is, the corresponding linear momentum is unchanged (conserved) in that direction.

Consider now the motion of two particles a and b which interact during an interval of time. If the interactive forces $\mathbf{F}$ and $-\mathbf{F}$ between them are the only unbalanced forces acting on the particles during the interval, it follows that the linear impulse on particle a is the negative of the linear impulse on particle b. Therefore, from Eq. 3/27, the change in linear momentum $\Delta\mathbf{G}_a$ of particle a is the negative of the change $\Delta\mathbf{G}_b$ in linear momentum of particle b. So we have $\Delta\mathbf{G}_a = -\Delta\mathbf{G}_b$ or $\Delta(\mathbf{G}_a + \mathbf{G}_b) = \mathbf{0}$. Thus, the total linear momentum $\mathbf{G} = \mathbf{G}_a + \mathbf{G}_b$ for the system of the two particles remains constant during the interval, and we write

$$\boxed{\Delta\mathbf{G} = \mathbf{0} \qquad \text{or} \qquad \mathbf{G}_1 = \mathbf{G}_2} \tag{3/28}$$

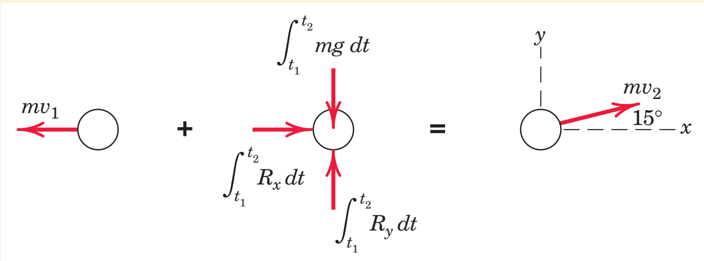
Equation 3/28 expresses the *principle of conservation of linear momentum*.

Sample Problem 3/19



A tennis player strikes the tennis ball with her racket when the ball is at the uppermost point of its trajectory as shown. The horizontal velocity of the ball just before impact with the racket is $v_1 = 50$ ft/sec, and just after impact its velocity is $v_2 = 70$ ft/sec directed at the 15° angle as shown. If the 2-oz ball is in contact with the racket for 0.02 sec, determine the magnitude of the average force **R** exerted by the racket on the ball. Also determine the angle β made by **R** with the horizontal.

Solution. We construct the impulse-momentum diagrams for the ball as follows:

① 

② $\left[m(v_x)_1 + \int_{t_1}^{t_2} \Sigma F_x dt = m(v_x)_2\right]$ $\qquad -\dfrac{2/16}{32.2}(50) + R_x(0.02) = \dfrac{2/16}{32.2}(70 \cos 15°)$

$$\left[m(v_y)_1 + \int_{t_1}^{t_2} \Sigma F_y dt = m(v_y)_2\right]$$

$$\frac{2/16}{32.2}(0) + R_y(0.02) - (2/16)(0.02) = \frac{2/16}{32.2}(70 \sin 15°)$$

We can now solve for the impact forces as

$$R_x = 22.8 \text{ lb}$$

$$R_y = 3.64 \text{ lb}$$

We note that the impact force $R_y = 3.64$ lb is considerably larger than the 0.125-lb weight of the ball. Thus, the weight mg, a nonimpulsive force, could have been neglected as small in comparison with R_y. Had we neglected the weight, the computed value of R_y would have been 3.52 lb.

We now determine the magnitude and direction of **R** as

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{22.8^2 + 3.64^2} = 23.1 \text{ lb} \qquad \textit{Ans.}$$

$$\beta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{3.64}{22.8} = 9.06° \qquad \textit{Ans.}$$

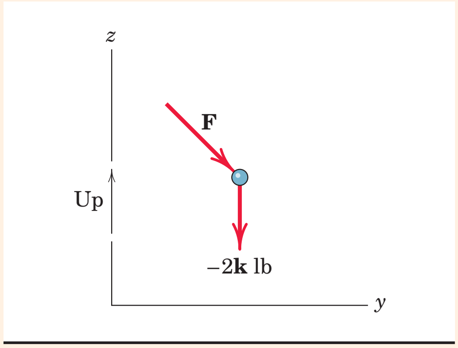
Helpful Hints

① Recall that for the impulse-momentum diagrams, initial linear momentum goes in the first diagram, all external linear impulses go in the second diagram, and final linear momentum goes in the third diagram.

② For the linear impulse $\int_{t_1}^{t_2} R_x\,dt$, the average impact force R_x is a constant, so that it can be brought outside the integral sign, resulting in $R_x \int_{t_1}^{t_2} dt = R_x(t_2 - t_1) = R_x \Delta t$. The linear impulse in the y-direction has been similarly treated.

Sample Problem 3/20

A 2-lb particle moves in the vertical y-z plane (z up, y horizontal) under the action of its weight and a force **F** which varies with time. The linear momentum of the particle in pound-seconds is given by the expression $\mathbf{G} = \frac{3}{2}(t^2 + 3)\mathbf{j} - \frac{2}{3}(t^3 - 4)\mathbf{k}$, where t is the time in seconds. Determine **F** and its magnitude for the instant when $t = 2$ sec.



Solution. The weight expressed as a vector is $-2\mathbf{k}$ lb. Thus, the force-momentum equation becomes

① $[\Sigma\mathbf{F} = \dot{\mathbf{G}}]$ $\qquad \mathbf{F} - 2\mathbf{k} = \dfrac{d}{dt}\left[\frac{3}{2}(t^2 + 3)\mathbf{j} - \frac{2}{3}(t^3 - 4)\mathbf{k}\right]$

$$= 3t\mathbf{j} - 2t^2\mathbf{k}$$

For $t = 2$ sec, $\qquad \mathbf{F} = 2\mathbf{k} + 3(2)\mathbf{j} - 2(2^2)\mathbf{k} = 6\mathbf{j} - 6\mathbf{k}$ lb $\qquad$ *Ans.*

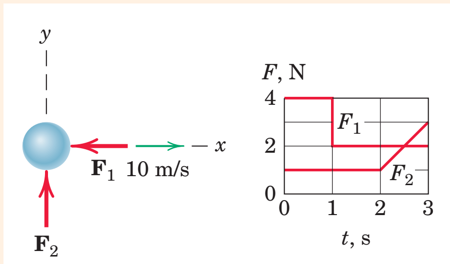
Thus, $\qquad F = \sqrt{6^2 + 6^2} = 6\sqrt{2}$ lb $\qquad$ *Ans.*

Helpful Hint

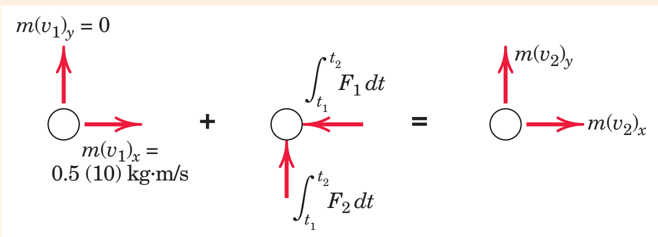
① Don't forget that $\Sigma\mathbf{F}$ includes *all* external forces acting on the particle, including the weight.

Sample Problem 3/21

A particle with a mass of 0.5 kg has a velocity of 10 m/s in the x-direction at time $t = 0$. Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the particle, and their magnitudes change with time according to the graphical schedule shown. Determine the velocity $\mathbf{v}_2$ of the particle at the end of the 3-s interval. The motion occurs in the horizontal x-y plane.



Solution. First, we construct the impulse-momentum diagrams as shown.



Then the impulse-momentum equations follow as

① $\left[m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x\, dt = m(v_2)_x\right]$ $\qquad 0.5(10) - [4(1) + 2(3 - 1)] = 0.5(v_2)_x$

$$(v_2)_x = -6 \text{ m/s}$$

$\left[m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y\, dt = m(v_2)_y\right]$ $\qquad 0.5(0) + [1(2) + 2(3 - 2)] = 0.5(v_2)_y$

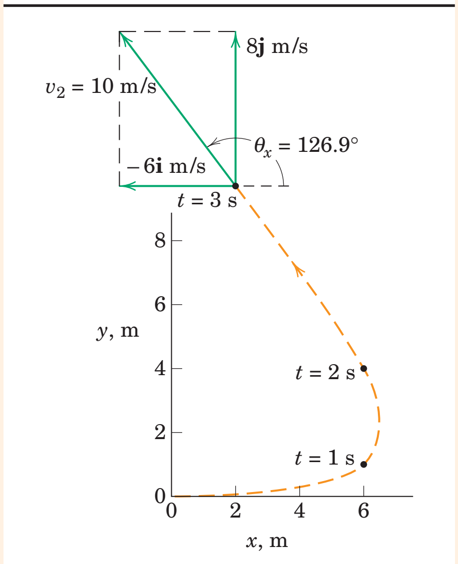
$$(v_2)_y = 8 \text{ m/s}$$

Thus,

$$\mathbf{v}_2 = -6\mathbf{i} + 8\mathbf{j} \text{ m/s} \qquad \text{and} \qquad v_2 = \sqrt{6^2 + 8^2} = 10 \text{ m/s}$$

$$\theta_x = \tan^{-1}\frac{8}{-6} = 126.9° \qquad \textit{Ans.}$$

Although not called for, the path of the particle for the first 3 seconds is plotted in the figure. The velocity at $t = 3$ s is shown together with its components.

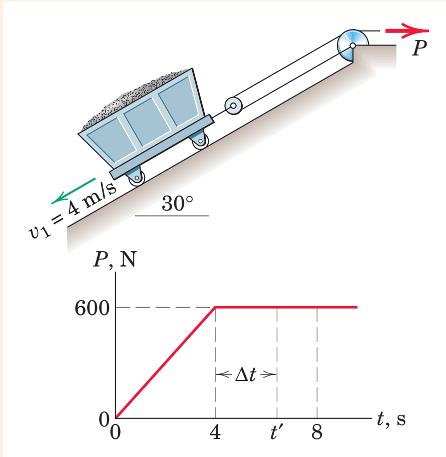


Helpful Hint

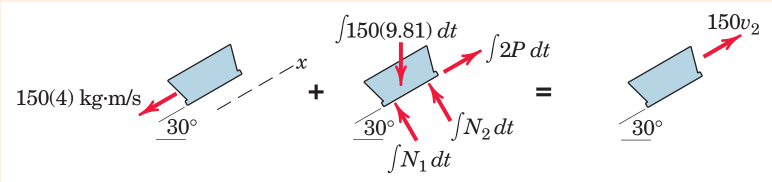
① The impulse in each direction is the corresponding area under the force-time graph. Note that F_1 is in the negative x-direction, so its impulse is negative.

Sample Problem 3/22

The loaded 150-kg skip is rolling down the incline at 4 m/s when a force P is applied to the cable as shown at time $t = 0$. The force P is increased uniformly with the time until it reaches 600 N at $t = 4$ s, after which time it remains constant at this value. Calculate (*a*) the time t' at which the skip reverses its direction and (*b*) the velocity v of the skip at $t = 8$ s. Treat the skip as a particle.



Solution. The stated variation of P with the time is plotted, and the impulse-momentum diagrams of the skip are drawn.



Part (a). The skip reverses direction when its velocity becomes zero. We will assume that this condition occurs at $t = 4 + \Delta t$ s. The impulse-momentum equation applied consistently in the positive x-direction gives

$$[m(v_1)_x + \int \Sigma F_x\,dt = m(v_2)_x]$$

① $$150(-4) + \frac{1}{2}(4)(2)(600) + 2(600)\Delta t - 150(9.81)\sin 30°(4 + \Delta t) = 150(0)$$

$$\Delta t = 2.46 \text{ s} \qquad t' = 4 + 2.46 = 6.46 \text{ s}$$ *Ans.*

Part (b). Applying the momentum equation to the entire 8-s interval gives

$$[m(v_1)_x + \int \Sigma F_x\,dt = m(v_2)_x]$$

$$150(-4) + \frac{1}{2}(4)(2)(600) + 4(2)(600) - 150(9.81)\sin 30°(8) = 150(v_2)_x$$

$$(v_2)_x = 4.76 \text{ m/s}$$ *Ans.*

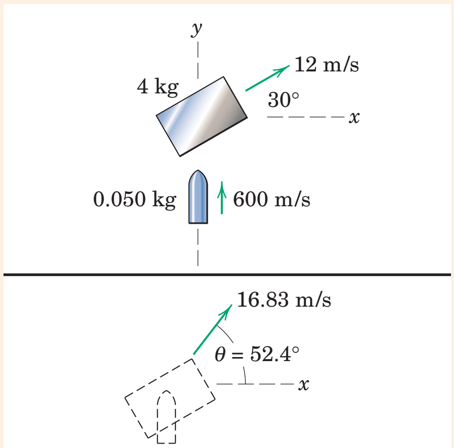
The same result is obtained by analyzing the interval from t' to 8 s.

Helpful Hint

① The impulse-momentum diagram keeps us from making the error of using the impulse of P rather than $2P$ or of forgetting the impulse of the component of the weight. The first term in the linear impulse is the triangular area of the P-t relation for the first 4 s, doubled for the force of $2P$.

Sample Problem 3/23

The 50-g bullet traveling at 600 m/s strikes the 4-kg block centrally and is embedded within it. If the block slides on a smooth horizontal plane with a velocity of 12 m/s in the direction shown prior to impact, determine the velocity $\mathbf{v}_2$ of the block and embedded bullet immediately after impact.



Solution. Since the force of impact is internal to the system composed of the block and bullet and since there are no other external forces acting on the system in the plane of motion, it follows that the linear momentum of the system is conserved. Thus,

① $$[\mathbf{G}_1 = \mathbf{G}_2] \quad 0.050(600\mathbf{j}) + 4(12)(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = (4 + 0.050)\mathbf{v}_2$$

$$\mathbf{v}_2 = 10.26\mathbf{i} + 13.33\mathbf{j} \text{ m/s}$$ *Ans.*

The final velocity and its direction are given by

$$[v = \sqrt{v_x^2 + v_y^2}] \qquad v_2 = \sqrt{(10.26)^2 + (13.33)^2} = 16.83 \text{ m/s}$$ *Ans.*

$$[\tan\theta = v_y/v_x] \qquad \tan\theta = \frac{13.33}{10.26} = 1.299 \qquad \theta = 52.4°$$ *Ans.*

Helpful Hint

① Working with the vector form of the principle of conservation of linear momentum is clearly equivalent to working with the component form.