Exercises in Physics
Assignment # 5

Date Given: May 12, 2022
Date Due: May 19, 2022

P1. (2 points) Find an equation in polar coordinates that has the same graph as the given equation in rectangular coordinates.

(a) $\left(x - \dfrac{3}{2}\right)^2 + y^2 = \dfrac{9}{4}$

(b) $\sqrt{(x^2+y^2)^3} = 3(x^2 - y^2)$

Solution:

(a) $r = 3\cos\theta$

(b) $r = 3\cos 2\theta$

P2. (2 points) Sketch the curves

(a) $r = 3\cos\theta$

(b) $r = 3\cos 2\theta$

Solution:

(a) As established in Problem P1(a), this curve in rectangular coordinates has the form $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$, which is apparently the equation for the circle of radius $3/2$ with the center in $(3/2, 0)$.

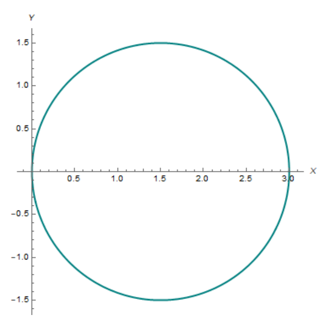

Figure 1: Illustration to Problem 2(a).

(b) The curve can be sketched by plotting points (θ, r), by setting different values for the angle θ and computing the corresponding values of the radius r. For $\theta \in [0, \pi]$ we get $(0, 3)$, $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right)$, $\left(\dfrac{\pi}{4}, 0\right)$, $\left(\dfrac{\pi}{3}, -\dfrac{3}{2}\right)$, $\left(\dfrac{\pi}{2}, -3\right)$, $\left(\dfrac{2\pi}{3}, -\dfrac{3}{2}\right)$, $\left(\dfrac{3\pi}{4}, 0\right)$, $\left(\dfrac{5\pi}{6}, \dfrac{3}{2}\right)$, $(\pi, 3)$. Plotting them will produce a sketch shown on the left side of Figure 2. The full sketch, obtained for $\theta \in [0, 2\pi]$ is shown on the right side of Figure 2.

P3. (2 points) A jet plane flying at a constant speed v at an altitude $h = 10$km is being tracked by radar located at O directly below the line of flight. If the angle θ is decreasing at the rate 0.02 rad/s when $\theta = 60^{\mathrm{o}}$, determine the value of $\ddot{r}$ at this instant and the magnitude of the velocity $\boldsymbol{v}$ of the plane.

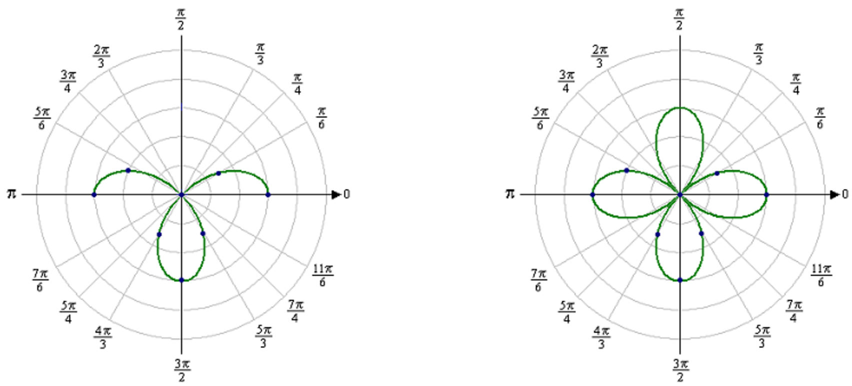

Figure 2: Illustration to Problem 2(b).

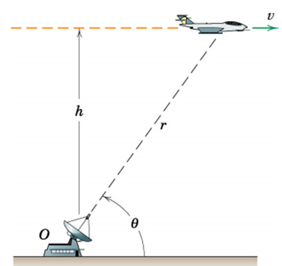

Figure 3: Illustration to Problem 3.

Solution: Since v = const, the acceleration vector $\boldsymbol{a}$ is zero (in all directions), and therefore $a_r = 0$ and $a_\theta = 0$. From $a_r = \ddot{r} - r\dot{\theta}^2 = 0$ we have $\ddot{r} = r\dot{\theta}^2$, where $\dot{\theta} = -0.02$rad/s. Next, since $r = h/\sin\theta$, we can define $r = \frac{10}{\sqrt{3}/2} \approx 11.547 \approx 11.55$km and compute

$$\boxed{\ddot{r} = 11.55(-0.020)^2 = 0.00462\text{km/s}^2 = 4.62\text{m/s}^2.}$$

Having defined r, there are several ways to define v. They can be outlined as follows.

1. From geometric considerations (see Figure 4) it follows that

$$\boxed{v = |r\dot{\theta}|/\sin\theta = |h\dot{\theta}|/\sin^2\theta = \frac{|10(-0.02)|}{(\sqrt{3}/2)^2} \approx 0.267\,\text{km/s} \approx 267\,\text{m/s} \approx 960\,\text{km/h}.}$$

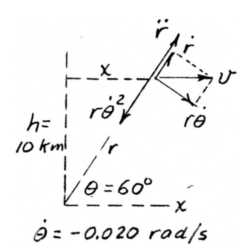

Figure 4: Illustration to Problem 3.

2. For this problem we have $v_r = \dot{r} = v\cos\theta$ and $v_\theta = r\dot{\theta} = -v\sin\theta$. Therefore $v_r/v_\theta = -\cot\theta$ and $\dot{r} = -r\dot{\theta}\cot\theta = -11.55(-0.02)/\sqrt{3} \approx 0.133$km/s, and from $v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \approx$ $0.267\,\text{km/s} \approx 267\,\text{m/s} \approx 960\,\text{km/h}$.

3. Yet another (and more general, not requiring v to be constant) way to establish v is as follows. By the problem settings $\boldsymbol{v} = v\boldsymbol{i}$. On the other hand, $\boldsymbol{v} = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta$. Recall that

$$\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} \quad \text{and} \quad \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}.$$

Therefore

$$v\boldsymbol{i} = v_r(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}) + v_\theta(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}) = (v_r\cos\theta - v_\theta\sin\theta)\boldsymbol{i} + (v_r\sin\theta + v_\theta\cos\theta)\boldsymbol{j}.$$

From this vector equation we have $v = v_r\cos\theta - v_\theta\sin\theta$ and $0 = v_r\sin\theta + v_\theta\cos\theta$. From the second equation we obtain $v_r = \dot{r} = -v_\theta\cot\theta = -r\dot{\theta}\cot\theta \approx 0.133$km/s, and from the first equation we establish $v = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \approx 0.267\,\text{km/s} \approx 267\,\text{m/s} \approx 960\,\text{km/h}$

P4. (4 points) The slider P can be moved inward by means of the string S as the bar OA rotates about the pivot O. The angular position of the bar is given by $\theta(t) = 0.4 + 0.12t + 0.06t^3$, where θ is in radians and t is in seconds. The position of the slider is given by $r(t) = 0.8 - 0.1t - 0.05t^2$, where r is in meters and t is in seconds.

(a) Determine the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ (in terms of $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$) of the slider at time $t = 2$s.

(b) Find the angles which $\boldsymbol{v}$ and $\boldsymbol{a}$ make with the positive x-axis (that is the angles between $\boldsymbol{v}$ and $\boldsymbol{i}$ and between $\boldsymbol{a}$ and $\boldsymbol{i}$).

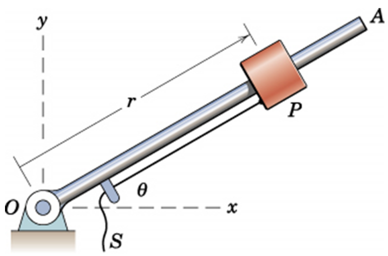

Figure 5: Illustration to Problem 4.

Solution: Here we have

$$\begin{aligned} &\theta(t) = 0.4 + 0.12t + 0.06t^3, && r(t) = 0.8 - 0.1t - 0.05t^2,\\ &\dot{\theta}(t) = 0.12 + 0.18t^2, && \dot{r}(t) = -0.1 - 0.1t,\\ &\ddot{\theta}(t) = 0.36t, && \ddot{r}(t) = -0.1. \end{aligned}$$

At $t = 2$s we have

$$\begin{aligned} &\theta = 1.12\,\text{rad} && r = 0.4\,\text{m}\\ &\dot{\theta} = 0.84\,\text{rad/s} && \dot{r} = -0.3\,\text{m/s}\\ &\ddot{\theta} = 0.72\,\text{rad/s}^2 && \ddot{r} = -0.1\,\text{m/s}^2 \end{aligned}$$

Therefore

$$\boxed{\boldsymbol{v} = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta = -0.3\boldsymbol{e}_r + 0.336\boldsymbol{e}_\theta\,\text{m/s}}$$

$$\boxed{\boldsymbol{a} = a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta = (\dot{r}r - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta = -0.382\boldsymbol{e}_r - 0.216\boldsymbol{e}_\theta\,\text{m/s}^2}$$

To find the angle between $\boldsymbol{v}$ and $\boldsymbol{i}$ one notices that $\boldsymbol{v}\cdot\boldsymbol{i} = |\boldsymbol{v}||\boldsymbol{i}|\cos(\widehat{\boldsymbol{v},\boldsymbol{i}}) = |\boldsymbol{v}|\cos(\widehat{\boldsymbol{v},\boldsymbol{i}})$. On the other hand, $\boldsymbol{v}\cdot\boldsymbol{i} = (v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta)\cdot\boldsymbol{i} = v_r(\boldsymbol{e}_r\cdot\boldsymbol{i}) + v_\theta(\boldsymbol{e}_\theta\cdot\boldsymbol{i})$. Recall that $\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$, and therefore $(\boldsymbol{e}_r\cdot\boldsymbol{i}) = \cos\theta$ and $(\boldsymbol{e}_\theta\cdot\boldsymbol{i}) = -\sin\theta$. Therefore

$$\cos(\widehat{\boldsymbol{v},\boldsymbol{i}}) = \frac{v_r\cos\theta - v_\theta\sin\theta}{\sqrt{v_r^2 + v_\theta^2}} \approx -0.96159$$

and

$$\boxed{\widehat{\boldsymbol{v}, \boldsymbol{i}} = 2.86353\,\text{rad} = 164.068^{\text{o}}.}$$

Similarly, in finding the angle between $\boldsymbol{a}$ and $\boldsymbol{i}$ we have $\boldsymbol{a} \cdot \boldsymbol{i} = |\boldsymbol{a}| \cos(\widehat{\boldsymbol{a}, \boldsymbol{i}})$, and on the other hand $\boldsymbol{a} \cdot \boldsymbol{i} = (a_r \boldsymbol{e}_r + a_\theta \boldsymbol{e}_\theta) \cdot \boldsymbol{i} = a_r(\boldsymbol{e}_r \cdot \boldsymbol{i}) + a_\theta(\boldsymbol{e}_\theta \cdot \boldsymbol{i}) = a_r \cos\theta - a_\theta \sin\theta$. Therefore

$$\cos(\widehat{\boldsymbol{a}, \boldsymbol{i}}) = \frac{a_r \cos\theta - a_\theta \sin\theta}{\sqrt{a_r^2 + a_\theta^2}} \approx 0.0635156$$

and

$$\boxed{\widehat{\boldsymbol{a}, \boldsymbol{i}} = 1.50724\,\text{rad} = 86.3584^{\text{o}}.}$$