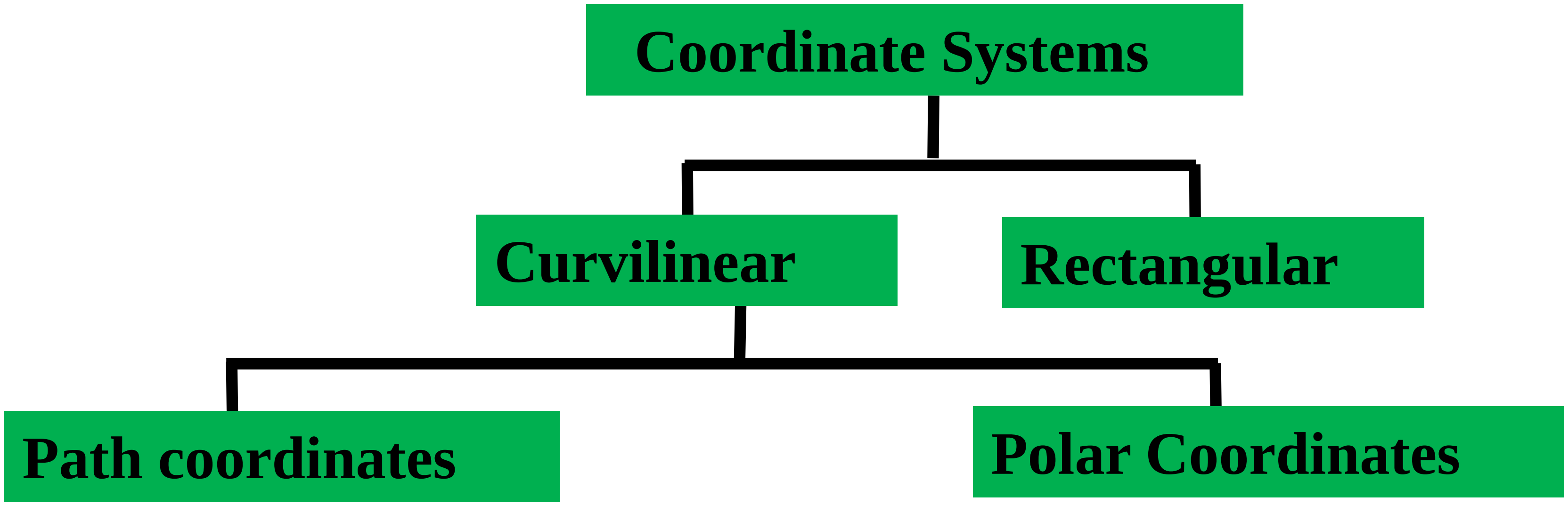
Physics

Lecture 4
Kinematics of Curvilinear Motion
(Normal & Tangential components)

Contents

- **Coordinate systems**
- **Path coordinate system**
- **Base vectors of path coordinate system**
- **Velocity and acceleration in path coordinate**
- **Components of acceleration**
- **Radius of curvature**
- **Length of curve**
- **Extension to 3D (Frenet-Serre formulae)**

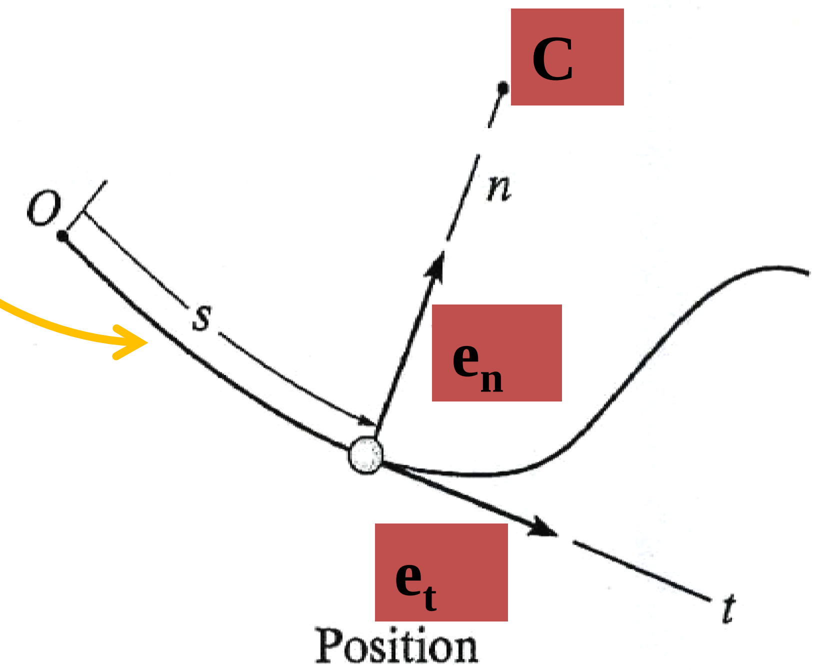
Types of Coordinate Systems



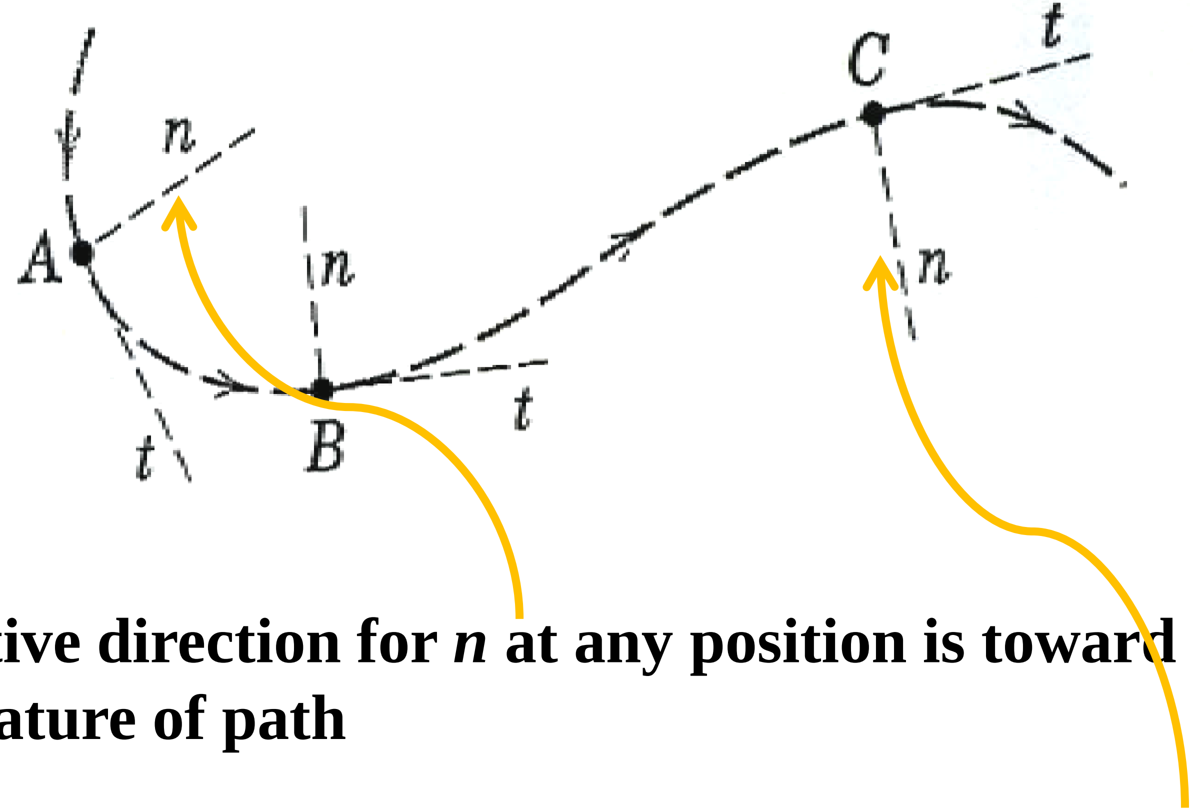
Path Coordinates

- When the path along which a particle travels is known

Characteristics



Convention Used in Path Coordinates

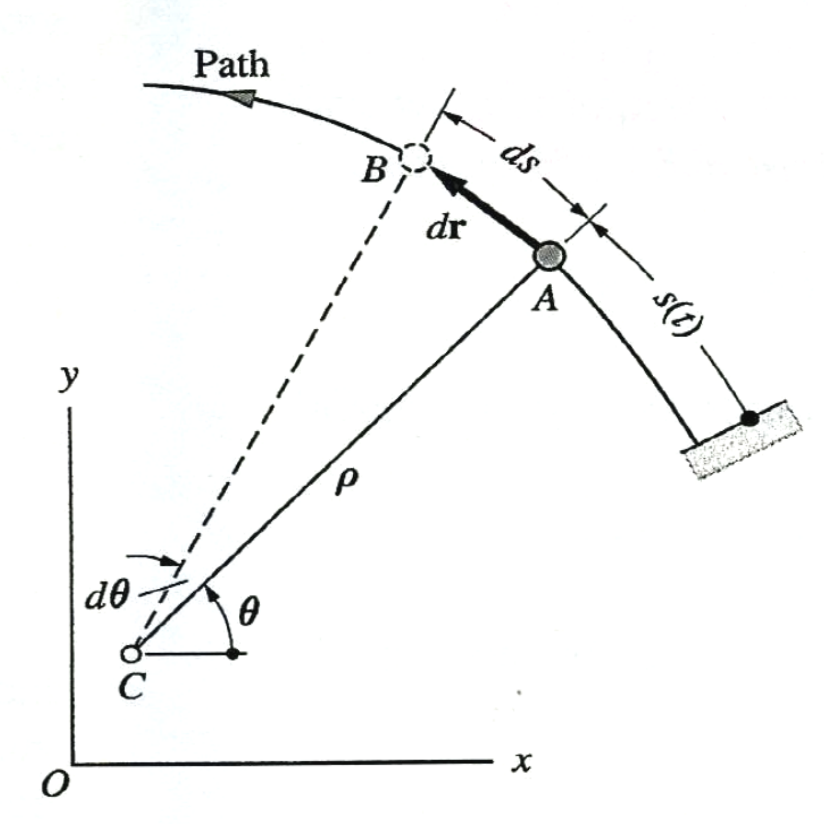


- **Positive direction for *n* at any position is toward the center of curvature of path**
- **If the curvature changes direction, the positive *n*-direction will shift from one side to another**

Geometric preliminaries

$$\left|d\mathbf{r}\right| = ds = \rho\, d\theta$$

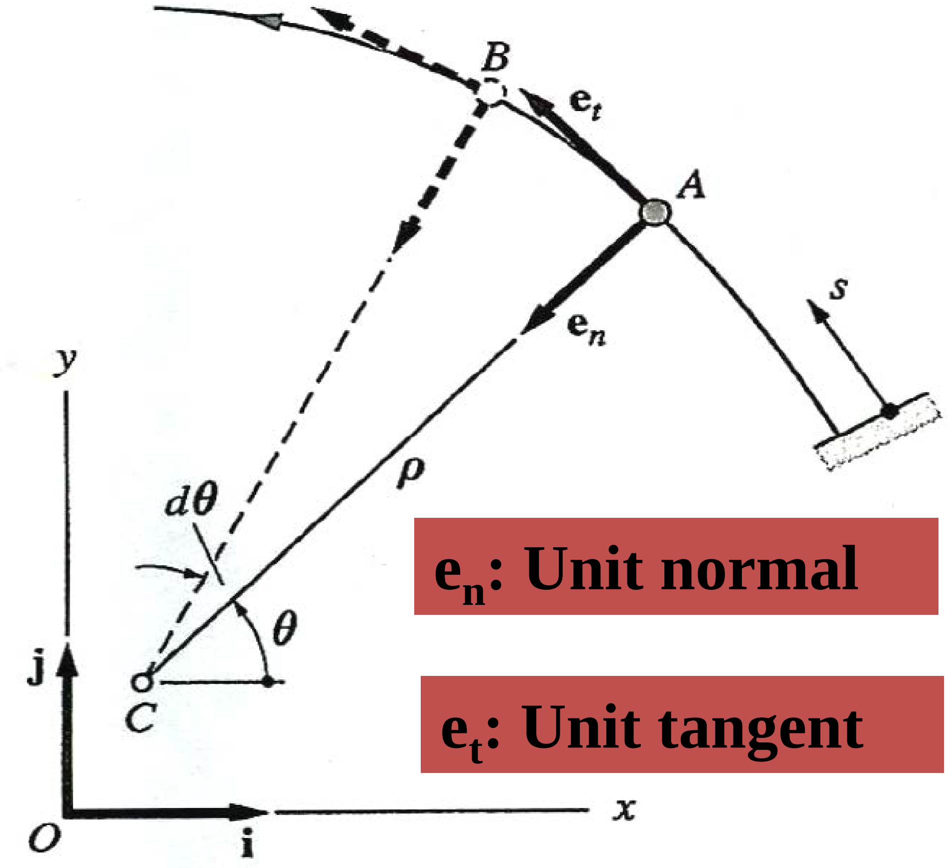
$$|\,\boldsymbol{v}\,| = v = \dot{s} = \rho\,\dot{\theta}$$



ρ is the radius of curvature of the path at point A

$v = \dot{s}$ is the speed

Base Vectors in Path Coordinates

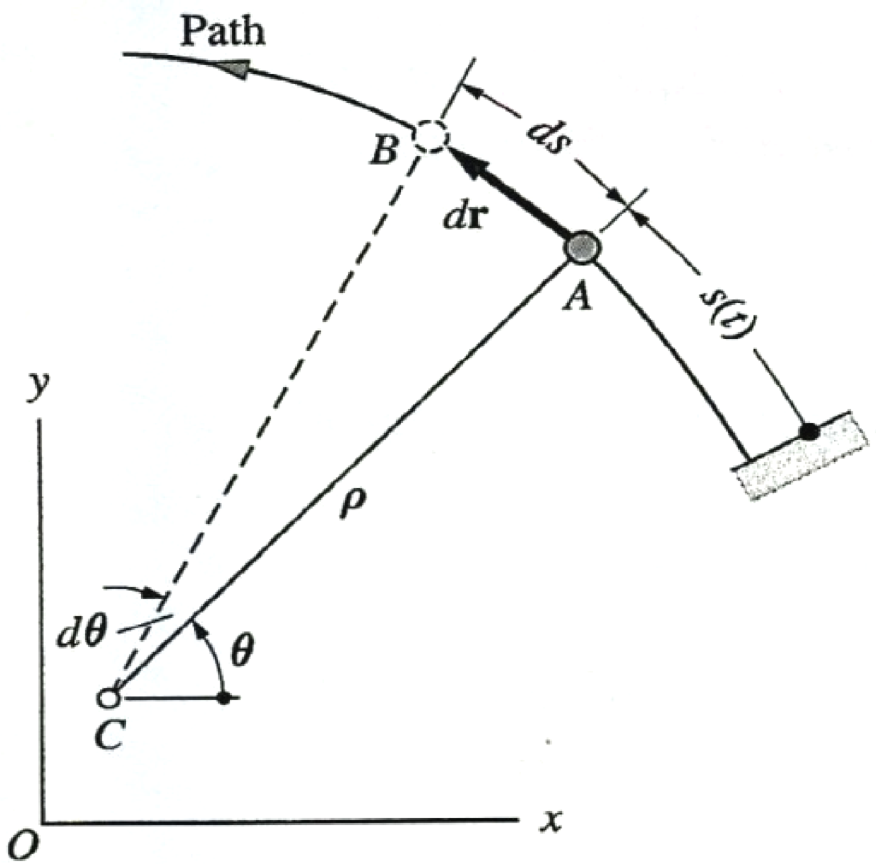


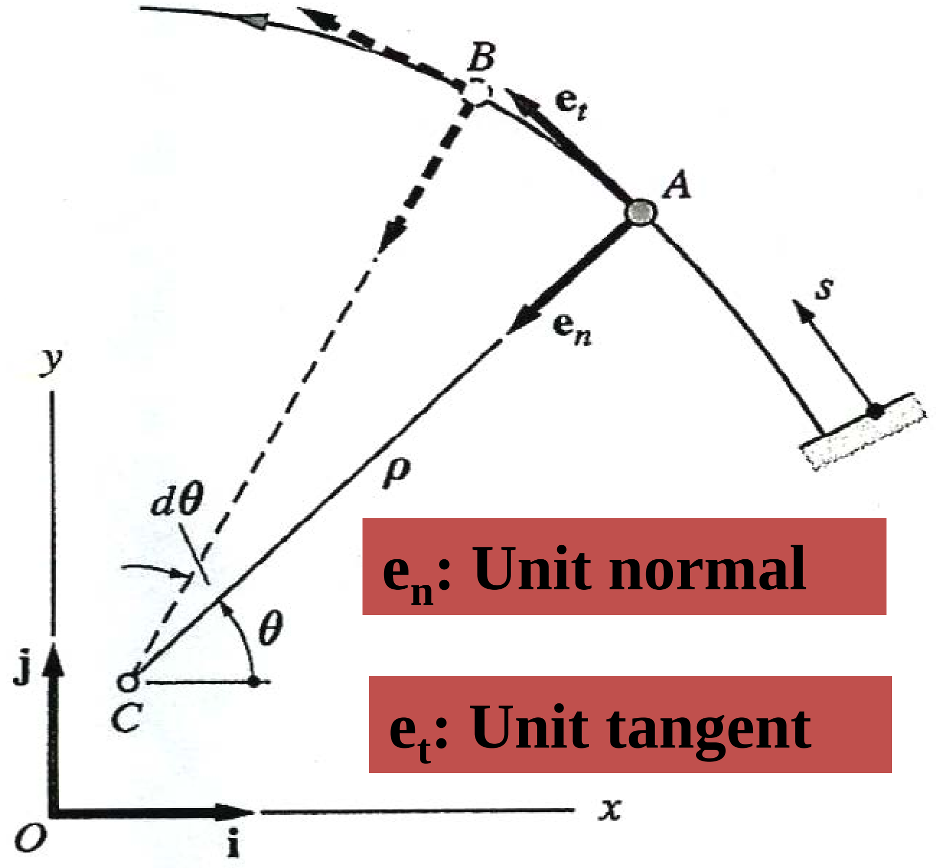
$\mathbf{e}_n$: Unit normal

$\mathbf{e}_t$: Unit tangent

$\mathbf{e}_n$ and $\mathbf{e}_t$ are called base vectors

Relationship between the Base Vectors





e_n: Unit normal

e_t: Unit tangent

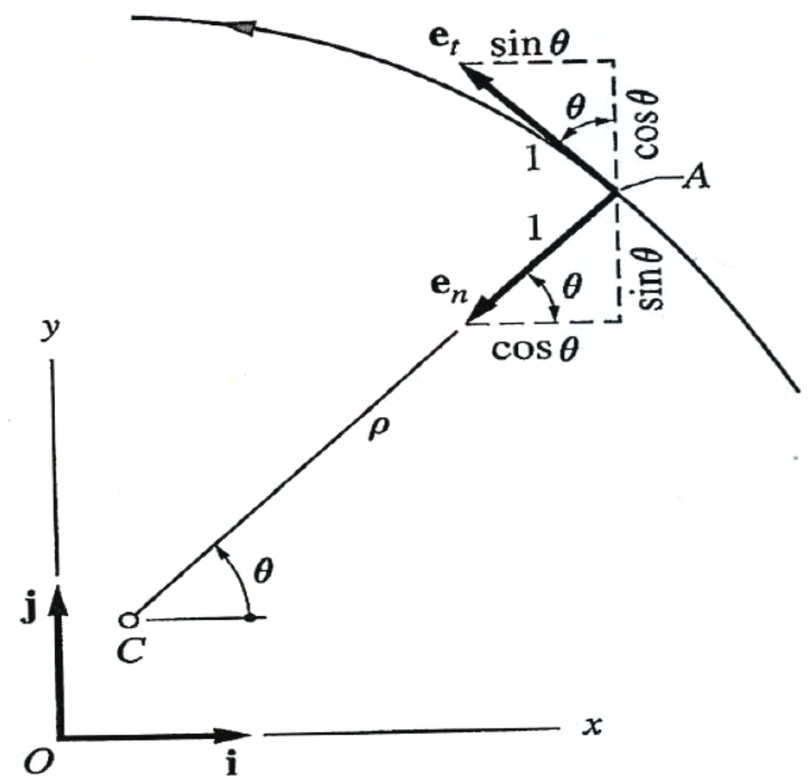
Derivative of the Base Vectors

Directions of $\mathbf{e}_n$ and $\mathbf{e}_t$ vary with position of the particle

Time derivative of $\mathbf{e}_n$ and $\mathbf{e}_t$

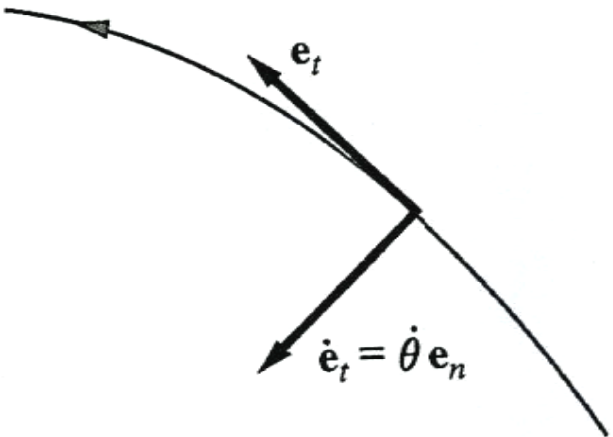
$$\frac{d\mathbf{e}_t}{dt} = \frac{v}{\rho}\mathbf{e}_n$$

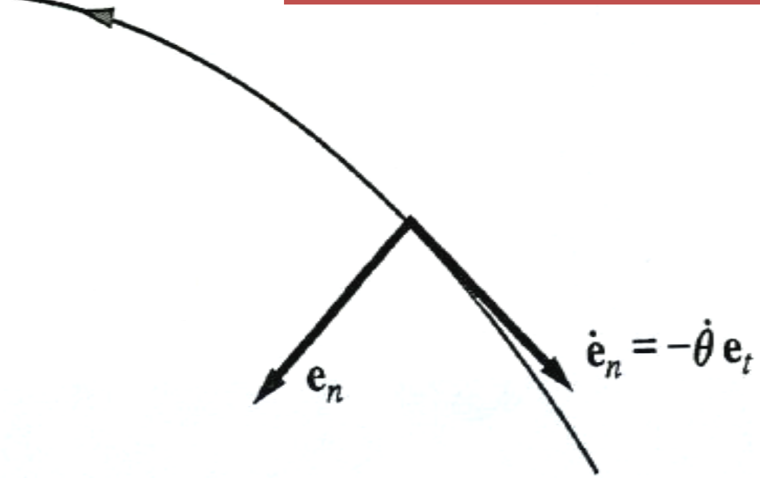
$$\frac{d\mathbf{e}_n}{dt} = -\frac{v}{\rho}\mathbf{e}_t$$



Characteristics of Derivatives of the Base Vectors

Derivative of e_n



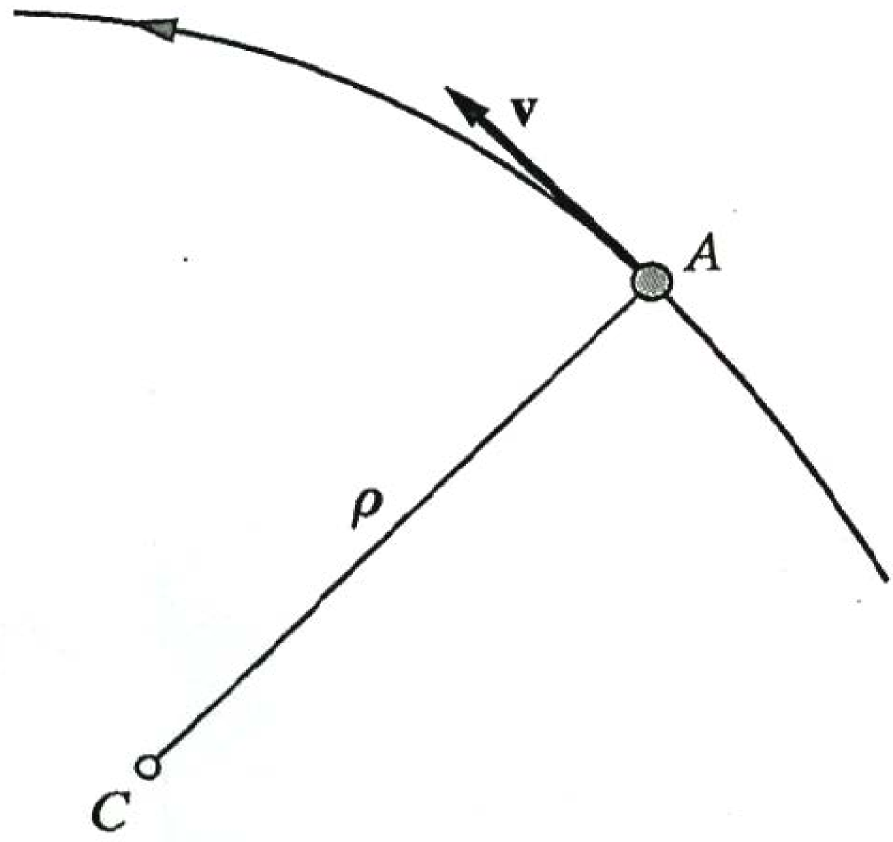


Derivative of e_t

Directions of base vectors and its derivatives are mutually perpendicular

Velocity in Path Coordinates

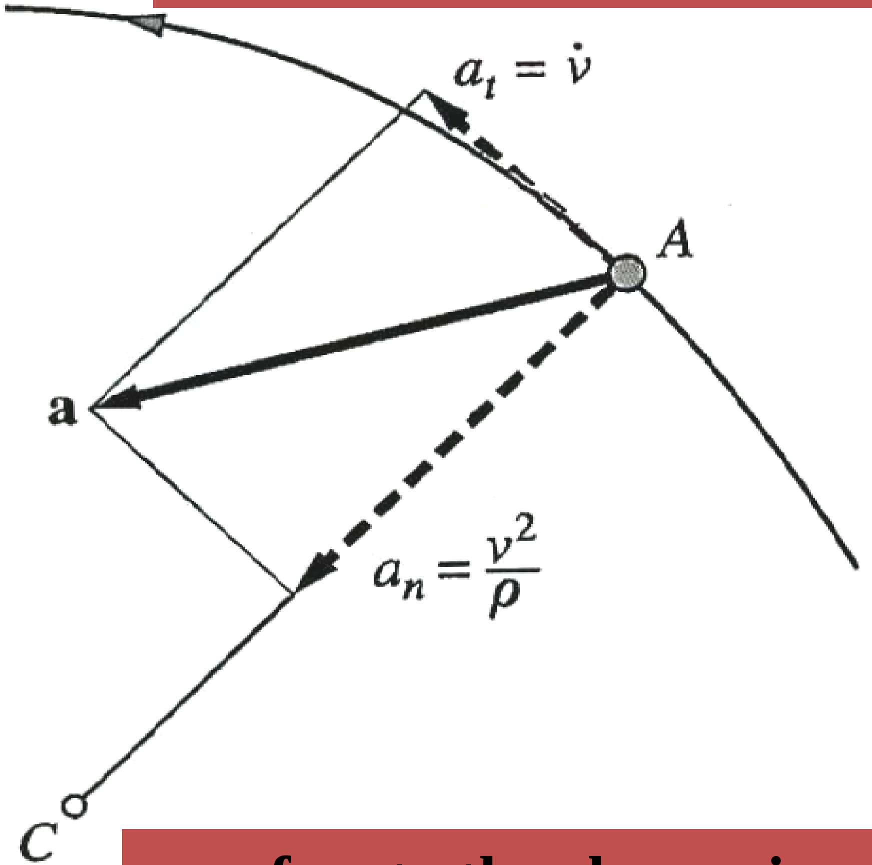
$$\mathbf{v} = v\,\mathbf{e}_t$$



Acceleration in Path Coordinates

a_t **refers to the change in speed of the particle**

$$\boldsymbol{a} = \dot{v}\,\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$



a_n **refers to the change in direction of the velocity**

Do not Confuse

Definition of Acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

Magnitude of Acceleration
In rectilinear motion

$$a = \frac{dv}{dt}$$

Tangential component of Acceleration
in plane curvilinear motion

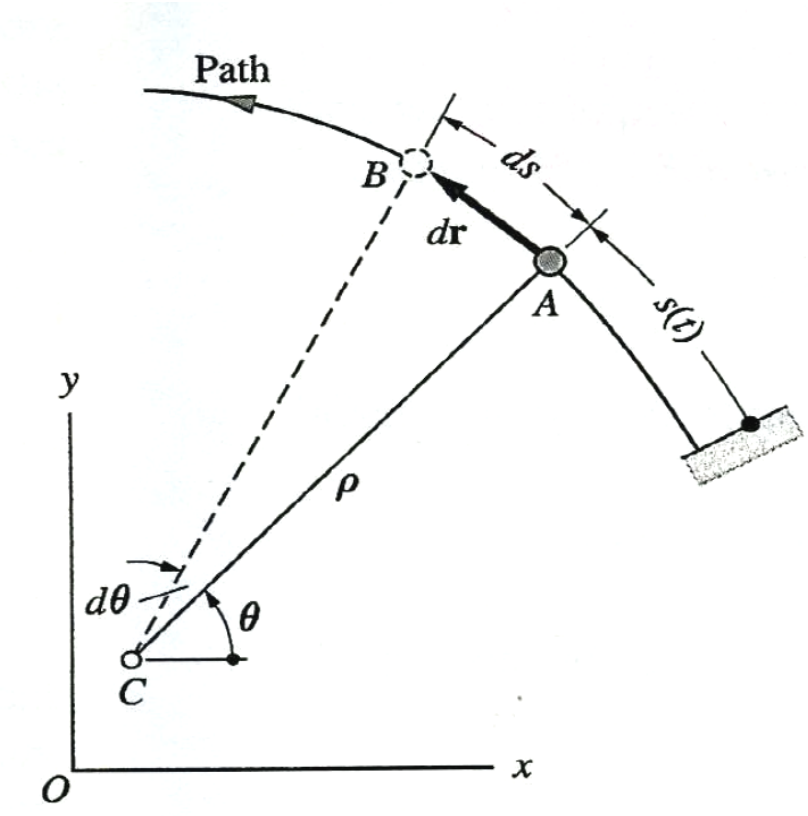
$$a_t = \frac{dv}{dt}$$

Radius of curvature

- Given velocity and acceleration vectors $\boldsymbol{v}$ and $\boldsymbol{a}$

ρ is the radius of curvature of the path at point A

$$\rho = \frac{|\boldsymbol{v}|^3}{|\boldsymbol{v} \times \boldsymbol{a}|}$$

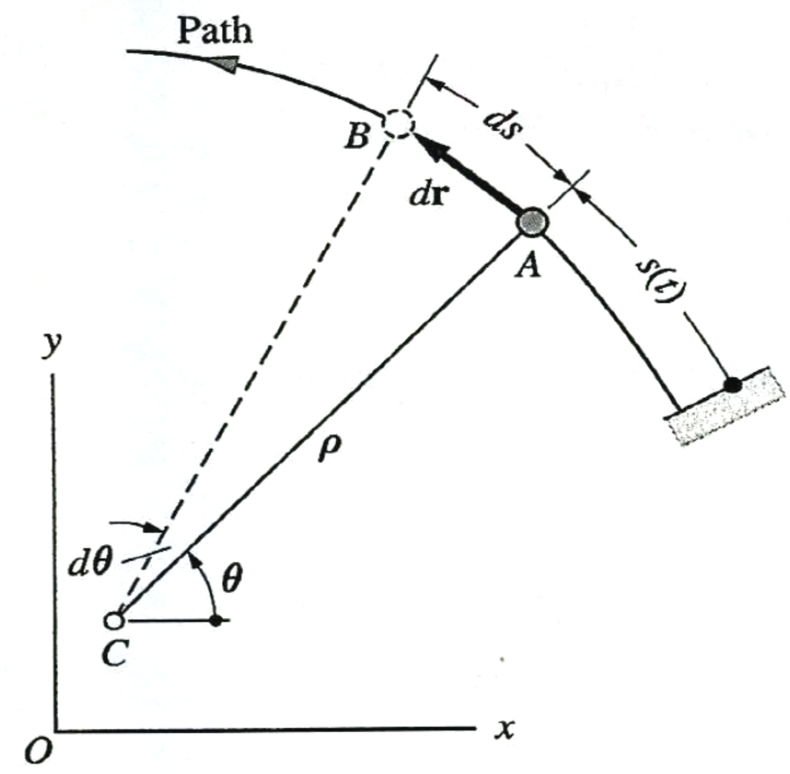


Radius of curvature

- If the curve is given parametrically

$$x = x(t), y = y(t)$$

$$\boxed{\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}}$$



- Here we use time derivatives

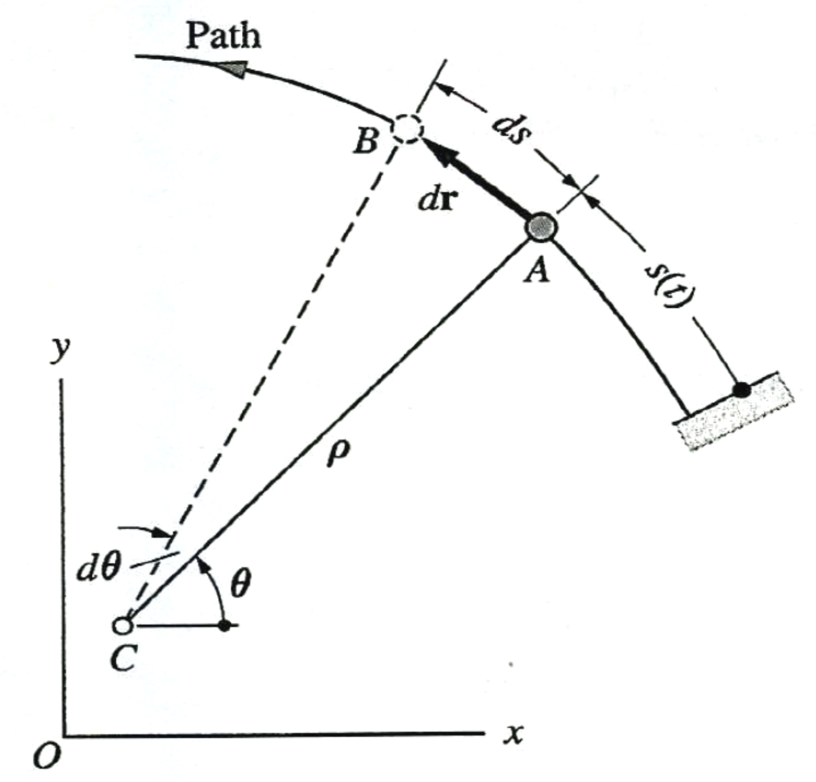
$$\dot{x} = \frac{dx}{dt}, \ \dot{y} = \frac{dy}{dt}, \ \ddot{x} = \frac{d^2x}{dt^2}, \ \ddot{y} = \frac{d^2y}{dt^2}$$

Curvature

- Curvature of a curve is defined as $\kappa = 1/\rho$
- For a straight line and for a circle of radius R
- Curvature depends on the position of the point on the curve

ρ is the radius of curvature of the path at point A

$$\boxed{\rho = \frac{|\boldsymbol{v}|^3}{|\boldsymbol{v} \times \boldsymbol{a}|}}$$



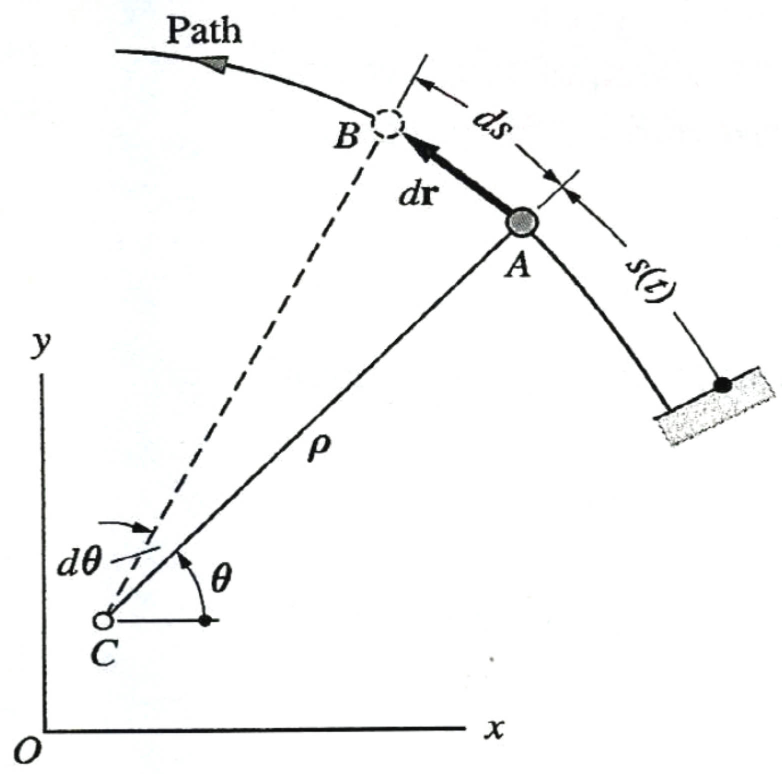
$\boldsymbol{v}$ and $\boldsymbol{a}$

Radius of curvature

- If the curve is given as a graph

$$y = y(x) \quad \text{or} \quad x = x(y)$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|}$$

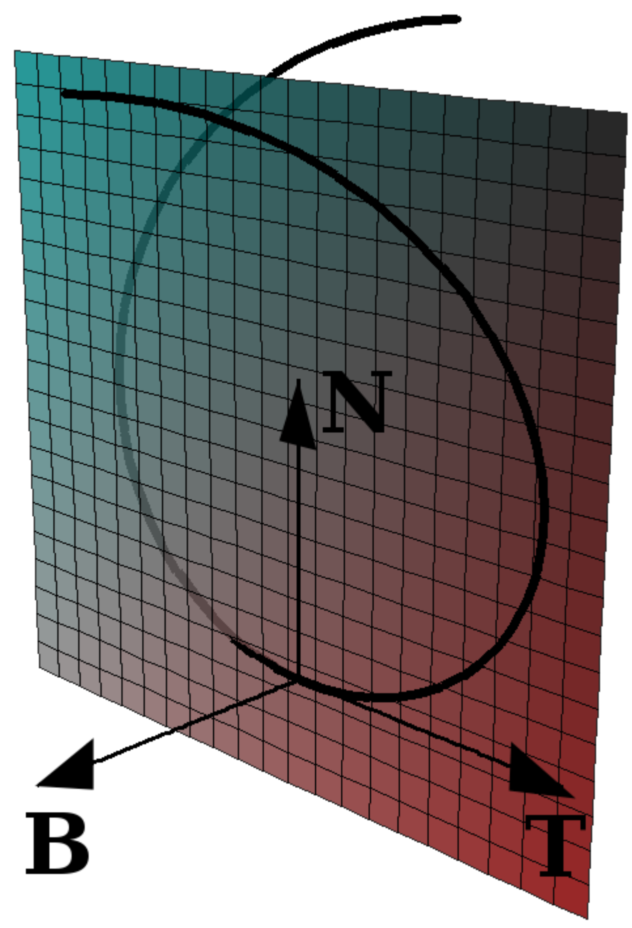


Lines in 3D

- If a curve is given parametrically

$$x = x(t), y = y(t), z = z(t) \;\; \Rightarrow \;\; \boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

- T-tangent line
- N-normal
- B-binormal



$$\boldsymbol{T} = \boldsymbol{r}'(t)$$

$$\boldsymbol{B} = \boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$$

$$\boldsymbol{N} = \{\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\} \times \boldsymbol{r}'(t)$$

$$\boldsymbol{e}_t = \boldsymbol{T} / |\boldsymbol{T}|$$

$$\boldsymbol{e}_n = \boldsymbol{N} / |\boldsymbol{N}|$$

$$\boldsymbol{e}_b = \boldsymbol{B} / |\boldsymbol{B}|$$