2/7 Space Curvilinear Motion

The general case of three-dimensional motion of a particle along a space curve was introduced in Art. 2/1 and illustrated in Fig. 2/1. Three coordinate systems, rectangular (x-y-z), cylindrical (r-θ-z), and spherical (R-θ-ϕ), are commonly used to describe this motion. These systems are indicated in Fig. 2/16, which also shows the unit vectors for the three coordinate systems.*

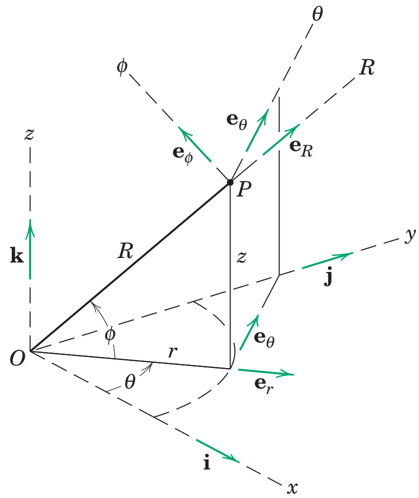


Figure 2/16

Before describing the use of these coordinate systems, we note that a path-variable description, using n- and t-coordinates, which we developed in Art. 2/5, can be applied in the osculating plane shown in Fig. 2/1. We defined this plane as the plane which contains the curve at the location in question. We see that the velocity $\mathbf{v}$, which is along the tangent t to the curve, lies in the osculating plane. The acceleration $\mathbf{a}$ also lies in the osculating plane. As in the case of plane motion, it has a component $a_t = \dot{v}$ tangent to the path due to the change in magnitude of the velocity and a component $a_n = v^2/\rho$ normal to the curve due to the change in direction of the velocity. As before, ρ is the radius of curvature of the path at the point in question and is measured in the osculating plane. This description of motion, which is natural and direct for many plane-motion problems, is awkward to use for space motion because the osculating plane continually shifts its orientation. We will confine our attention, therefore, to the three fixed coordinate systems shown in Fig. 2/16.

Rectangular Coordinates (x-y-z)

The extension from two to three dimensions offers no particular difficulty. We merely add the z-coordinate and its two time derivatives to the two-dimensional expressions of Eqs. 2/6 so that the position vector $\mathbf{R}$, the velocity $\mathbf{v}$, and the acceleration $\mathbf{a}$ become

$$\boxed{\begin{aligned} \mathbf{R} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ \mathbf{v} = \dot{\mathbf{R}} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \\ \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{R}} &= \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \end{aligned}} \qquad \textbf{(2/15)}$$

Note that in three dimensions we are using $\mathbf{R}$ in place of $\mathbf{r}$ for the position vector.

Cylindrical Coordinates (r-θ-z)

If we understand the polar-coordinate description of plane motion, then there should be no difficulty with cylindrical coordinates because all that is required is the addition of the z-coordinate and its two time derivatives. The position vector $\mathbf{R}$ to the particle for cylindrical coordinates is simply

$$\mathbf{R} = r\mathbf{e}_r + z\mathbf{k}$$

*In a variation of spherical coordinates commonly used, angle ϕ is replaced by its complement.

In place of Eq. 2/13 for plane motion, we can write the velocity as

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{k}} \tag{2/16}$$

where

$$\begin{aligned} v_r &= \dot{r} \\ v_\theta &= r\dot{\theta} \\ v_z &= \dot{z} \\ v &= \sqrt{{v_r}^2 + {v_\theta}^2 + {v_z}^2} \end{aligned}$$

Similarly, the acceleration is written by adding the z-component to Eq. 2/14, which gives us

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{k}} \tag{2/17}$$

where

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \\ a_z &= \ddot{z} \\ a &= \sqrt{{a_r}^2 + {a_\theta}^2 + {a_z}^2} \end{aligned}$$

Whereas the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ have nonzero time derivatives due to the changes in their directions, we note that the unit vector $\mathbf{k}$ in the z-direction remains fixed in direction and therefore has a zero time derivative.

Nightman 1965/Shutterstock

With base rotation and ladder elevation, spherical coordinates would be a good choice for determining the acceleration of the upper end of the extending ladder.

Spherical Coordinates (R-θ-ϕ)

Spherical coordinates R, θ, ϕ are utilized when a radial distance and two angles are utilized to specify the position of a particle, as in the case of radar measurements, for example. Derivation of the expression for the velocity $\mathbf{v}$ is easily obtained, but the expression for the acceleration $\mathbf{a}$ is more complex because of the added geometry. Consequently, only the results will be cited here.* First we designate unit vectors $\mathbf{e}_R$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ as shown in Fig. 2/16. Note that the unit vector $\mathbf{e}_R$ is in the direction in which the particle P would move if R increases but θ and ϕ are held constant. The unit vector $\mathbf{e}_\theta$ is in the direction in which P would move if θ increases while R and ϕ are held constant. Finally, the unit vector $\mathbf{e}_\phi$ is in the direction in which P would move if ϕ increases while R and θ are held constant. The resulting expressions for $\mathbf{v}$ and $\mathbf{a}$ are

$$\boxed{\mathbf{v} = v_R\mathbf{e}_R + v_\theta\mathbf{e}_\theta + v_\phi\mathbf{e}_\phi} \tag{2/18}$$

where

$$\begin{aligned} v_R &= \dot{R} \\ v_\theta &= R\dot{\theta}\cos\phi \\ v_\phi &= R\dot{\phi} \end{aligned}$$

*For a complete derivation of $\mathbf{v}$ and $\mathbf{a}$ in spherical coordinates, see the first author's book *Dynamics*, 2nd edition, 1971, or SI Version, 1975 (John Wiley & Sons, Inc.).

and

$$\boxed{\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi} \tag{2/19}$$

where

$$a_R = \ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi$$

$$a_\theta = \frac{\cos\phi}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\theta}\dot{\phi}\sin\phi$$

$$a_\phi = \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) + R\dot{\theta}^2\sin\phi\cos\phi$$

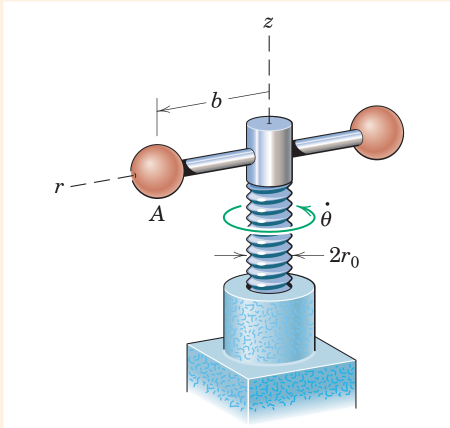
Linear algebraic transformations between any two of the three coordinate-system expressions for velocity or acceleration can be developed. These transformations make it possible to express the motion component in rectangular coordinates, for example, if the components are known in spherical coordinates, or vice versa.* These transformations are easily handled with the aid of matrix algebra and a simple computer program.

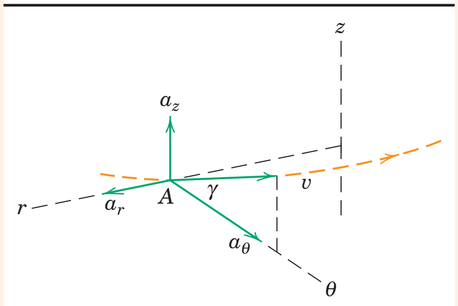
A portion of the track of this amusement-park ride is in the shape of a helix whose axis is horizontal.

*These coordinate transformations are developed and illustrated in the first author's book *Dynamics*, 2nd edition, 1971, or SI Version, 1975 (John Wiley & Sons, Inc.).

Sample Problem 2/11

The power screw starts from rest and is given a rotational speed $\dot{\theta}$ which increases uniformly with time t according to $\dot{\theta} = kt$, where k is a constant. Determine the expressions for the velocity v and acceleration a of the center of ball A when the screw has turned through one complete revolution from rest. The lead of the screw (advancement per revolution) is L.





Solution. The center of ball A moves in a helix on the cylindrical surface of radius b, and the cylindrical coordinates r, θ, z are clearly indicated.

Integrating the given relation for $\dot{\theta}$ gives $\theta = \Delta\theta = \int \dot{\theta}\, dt = \frac{1}{2}kt^2$. For one revolution from rest we have

$$2\pi = \frac{1}{2}kt^2$$

giving

$$t = 2\sqrt{\pi/k}$$

Thus, the angular rate at one revolution is

$$\dot{\theta} = kt = k(2\sqrt{\pi/k}) = 2\sqrt{\pi k}$$

① The helix angle γ of the path followed by the center of the ball governs the relation between the θ- and z-components of velocity and is given by $\tan\gamma = L/(2\pi b)$. Now from the figure we see that $v_\theta = v\cos\gamma$. Substituting $v_\theta = r\dot{\theta} = b\dot{\theta}$
② from Eq. 2/16 gives $v = v_\theta/\cos\gamma = b\dot{\theta}/\cos\gamma$. With $\cos\gamma$ obtained from $\tan\gamma$ and with $\dot{\theta} = 2\sqrt{\pi k}$, we have for the one-revolution position

$$v = 2b\sqrt{\pi k}\,\frac{\sqrt{L^2 + 4\pi^2 b^2}}{2\pi b} = \sqrt{\frac{k}{\pi}}\sqrt{L^2 + 4\pi^2 b^2} \qquad \textit{Ans.}$$

The acceleration components from Eq. 2/17 become

③ $[a_r = \ddot{r} - r\dot{\theta}^2]$ $\qquad a_r = 0 - b(2\sqrt{\pi k})^2 = -4b\pi k$

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\qquad a_\theta = bk + 2(0)(2\sqrt{\pi k}) = bk$

$[a_z = \ddot{z} = \dot{v}_z]$ $\qquad a_z = \dfrac{d}{dt}(v_z) = \dfrac{d}{dt}(v_\theta \tan\gamma) = \dfrac{d}{dt}(b\dot{\theta}\tan\gamma)$

$$= (b\tan\gamma)\ddot{\theta} = b\,\frac{L}{2\pi b}\,k = \frac{kL}{2\pi}$$

Now we combine the components to give the magnitude of the total acceleration, which becomes

$$a = \sqrt{(-4b\pi k)^2 + (bk)^2 + \left(\frac{kL}{2\pi}\right)^2}$$

$$= bk\sqrt{(1 + 16\pi^2) + L^2/(4\pi^2 b^2)} \qquad \textit{Ans.}$$

Helpful Hints

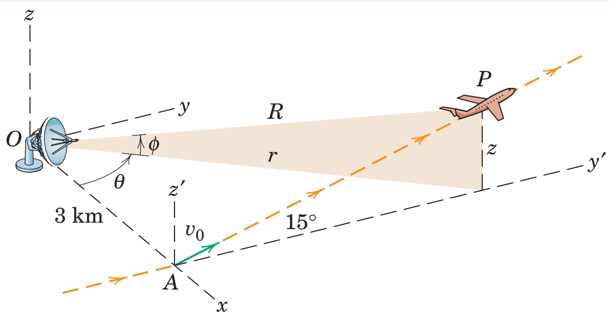
① We must be careful to divide the lead L by the circumference $2\pi b$ and not the diameter $2b$ to obtain $\tan\gamma$. If in doubt, unwrap one turn of the helix traced by the center of the ball.

② Sketch a right triangle and recall that for $\tan\beta = a/b$ the cosine of β becomes $b/\sqrt{a^2 + b^2}$.

③ The negative sign for a_r is consistent with our previous knowledge that the normal component of acceleration is directed toward the center of curvature.

Sample Problem 2/12

An aircraft P takes off at A with a velocity v_0 of 250 km/h and climbs in the vertical y'-z' plane at the constant 15° angle with an acceleration along its flight path of 0.8 m/s^2. Flight progress is monitored by radar at point O. (*a*) Resolve the velocity of P into cylindrical-coordinate components 60 seconds after takeoff and find $\dot{r}$, $\dot{\theta}$, and $\dot{z}$ for that instant. (*b*) Resolve the velocity of the aircraft P into spherical-coordinate components 60 seconds after takeoff and find $\dot{R}$, $\dot{\theta}$, and $\dot{\phi}$ for that instant.



Solution. (a) The accompanying figure shows the velocity and acceleration vectors in the y'-z' plane. The takeoff speed is

$$v_0 = \frac{250}{3.6} = 69.4 \text{ m/s}$$

and the speed after 60 seconds is

$$v = v_0 + at = 69.4 + 0.8(60) = 117.4 \text{ m/s}$$

The distance s traveled after takeoff is

$$s = s_0 + v_0 t + \frac{1}{2}at^2 = 0 + 69.4(60) + \frac{1}{2}(0.8)(60)^2 = 5610 \text{ m}$$

The y-coordinate and associated angle θ are

$$y = 5610 \cos 15° = 5420 \text{ m}$$

$$\theta = \tan^{-1}\frac{5420}{3000} = 61.0°$$

From the figure (*b*) of x-y projections, we have

$$r = \sqrt{3000^2 + 5420^2} = 6190 \text{ m}$$

$$v_{xy} = v \cos 15° = 117.4 \cos 15° = 113.4 \text{ m/s}$$

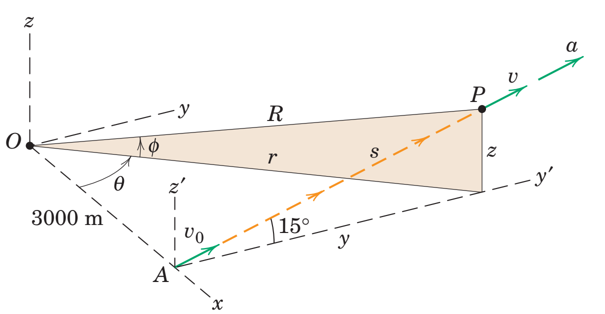
$$v_r = \dot{r} = v_{xy} \sin\theta = 113.4 \sin 61.0° = 99.2 \text{ m/s} \qquad \textit{Ans.}$$

$$v_\theta = r\dot{\theta} = v_{xy}\cos\theta = 113.4 \cos 61.0° = 55.0 \text{ m/s}$$

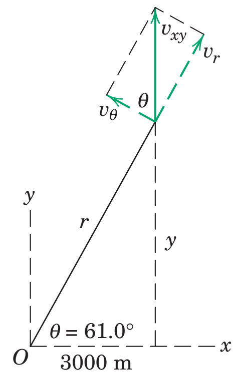
So

$$\dot{\theta} = \frac{55.0}{6190} = 8.88(10^{-3}) \text{ rad/s} \qquad \textit{Ans.}$$

Finally $\quad \dot{z} = v_z = v \sin 15° = 117.4 \sin 15° = 30.4 \text{ m/s} \qquad$ *Ans.*



(*a*)



(*b*)

(b) Refer to the accompanying figure (*c*), which shows the x-y plane and various velocity components projected into the vertical plane containing r and R. Note that

$$z = y \tan 15° = 5420 \tan 15° = 1451 \text{ m}$$

$$\phi = \tan^{-1}\frac{z}{r} = \tan^{-1}\frac{1451}{6190} = 13.19°$$

$$R = \sqrt{r^2 + z^2} = \sqrt{6190^2 + 1451^2} = 6360 \text{ m}$$

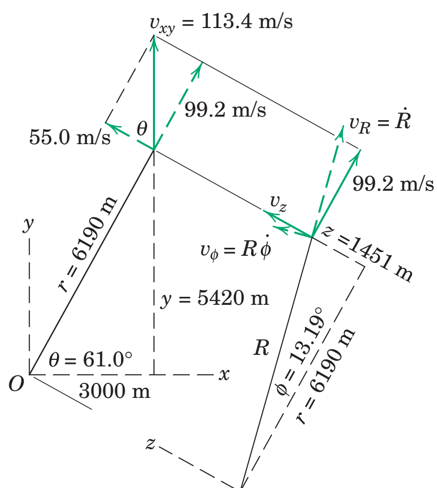
From the figure,

$$v_R = \dot{R} = 99.2 \cos 13.19° + 30.4 \sin 13.19° = 103.6 \text{ m/s} \qquad \textit{Ans.}$$

$$\dot{\theta} = 8.88(10^{-3}) \text{ rad/s, as in part } (a) \qquad \textit{Ans.}$$

$$v_\phi = R\dot{\phi} = 30.4 \cos 13.19° - 99.2 \sin 13.19° = 6.95 \text{ m/s}$$

$$\dot{\phi} = \frac{6.95}{6360} = 1.093(10^{-3}) \text{ rad/s} \qquad \textit{Ans.}$$



(*c*)