Exercises in Physics
Assignment # 11

Date Given: June 23, 2022
Date Due: June 30, 2022

P1. (2 points) For the force $\boldsymbol{F} = \boldsymbol{r}$, define the potential energy (if exists). Compute the work done by this force along the helical path $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $x(t) = a\cos t$, $y(t) = a\sin t$, $z(t) = bt$, when the parameter t is changing from $t = 0$ (start point) to $t = 2\pi$ (end point).

Solution: Here we have $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and therefore $F_x = x, F_y = y, F_z = z$. The conditions

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) = 0, \quad \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) = 0, \quad \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0.$$

are satisfied, the force is conservative and there exists a potential function V such that $F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, $F_z = -\frac{\partial V}{\partial z}$. Since

$$\mathrm{d}V = \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z = -F_x\mathrm{d}x - F_y\mathrm{d}y - F_z\mathrm{d}z = -x\mathrm{d}x - y\mathrm{d}y - z\mathrm{d}z$$

we have

$$V(x,y,z) = \int \mathrm{d}V + \text{const} = -\int x\mathrm{d}x - \int y\mathrm{d}y - \int z\mathrm{d}z + \text{const} = \boxed{-\frac{1}{2}(x^2+y^2+z^2) + \text{const}}$$

Therefore the work done along the given helical path is

$$V(P_{\text{start}}) - V(P_{\text{end}}) = V(x(0), y(0), z(0)) - V(x(2\pi), y(2\pi), z(2\pi)) =$$
$$\left(-\frac{1}{2}a^2 + \text{const}\right) - \left(-\frac{1}{2}(a^2 + 4\pi^2 b^2) + \text{const}\right) = \boxed{2\pi^2 b^2}$$

Alternatively, we can define the work by computing line integral. Here, $\boldsymbol{F} = a\cos t\boldsymbol{i} + a\sin t\boldsymbol{j} + bt\boldsymbol{k}$. Since $\boldsymbol{r}(t) = a\cos t\boldsymbol{i} + a\sin t\boldsymbol{j} + bt\boldsymbol{k}$, we get $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = -a\sin t\boldsymbol{i} + a\cos t\boldsymbol{j} + b\boldsymbol{k}$. So, the work done by the force $\boldsymbol{F}$ along this helical path can be calculated as

$$\int_{P_{\text{start}}}^{P_{\text{end}}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t}\,\mathrm{d}t = \int_0^{2\pi}\left(x(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + y(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + z(t)\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\mathrm{d}t =$$
$$\int_0^{2\pi}\left(-a^2\sin t\cos t + a^2\sin t\cos t + b^2 t\right)\mathrm{d}t = b^2\int_0^{2\pi} t\mathrm{d}t = 2\pi^2 b^2.$$

P2. (2 points) For the force $\boldsymbol{F} = \boldsymbol{k}\times\boldsymbol{r}$, define the potential energy (if exists). Compute the work done by this force along the helical path $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, where $x(t) = a\cos t$, $y(t) = a\sin t$, $z(t) = bt$, when the parameter t is changing from $t = 0$ (start point) to $t = 2\pi$ (end point).

Solution: Here we have $\boldsymbol{F} = \boldsymbol{k}\times\boldsymbol{r} = -y\boldsymbol{i} + x\boldsymbol{j}$, and therefore $F_x = -y, F_y = x, F_z = 0$. The conditions

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) = 0, \quad \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) = 0, \quad \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0.$$

are not satisfied because $\frac{\partial F_y}{\partial x} = 1 \neq \frac{\partial F_x}{\partial y} = -1$. Therefore the force $\boldsymbol{F}$ is not conservative, and potential function does not exist.

Since potential function does not exist, we can only define the work by computing line integral. Since $x(t) = a\cos t$, $y(t) = a\sin t$, we have $\boldsymbol{F} = -y(t)\boldsymbol{i} + x(t)\boldsymbol{j} = -a\sin t\boldsymbol{i} + a\cos t\boldsymbol{j}$. Next, since $\boldsymbol{r}(t) = a\cos t\boldsymbol{i} + a\sin t\boldsymbol{j} + bt\boldsymbol{k}$, we have $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = -a\sin t\boldsymbol{i} + a\cos t\boldsymbol{j} + b\boldsymbol{k}$. The work done by the force $\boldsymbol{F}$ along the given helical path is calculated as

$$\int_{P_{\text{start}}}^{P_{\text{end}}} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t}\,\mathrm{d}t = \int_0^{2\pi}\left(-y(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + x(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 0\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\mathrm{d}t =$$

$$\int_0^{2\pi}\left(a^2\sin^2 t + a^2\cos^2 t\right)\mathrm{d}t = a^2\int_0^{2\pi}\mathrm{d}t = \boxed{2\pi a^2}$$

P3. (2 points) The 10 kg collar slides on the smooth vertical rod and has a velocity $v_1 = 2\,\mathrm{m/s}$ in position A where each spring is stretched 0.1 m. Calculate the velocity v_2 of the collar as it passes point B.

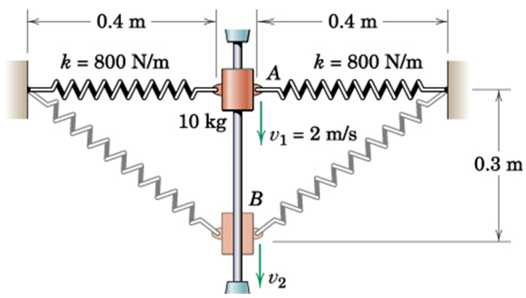

Figure 1: Illustration to Problem 3.

Solution: All the active forces in this problem are conservative (potential), and hence the total energy is conserved. Therefore we can write

$$T_A + V_A = T_B + V_B.$$

At point A we have $T_B = \frac{1}{2}mv_A^2$, where $m = 10\,\mathrm{kg}$ and $v_A = 2\,\mathrm{m/s}$. The potential energy has two sources, the gravity force and the elastic forces of the two springs, that is $V_A = V_{A,g} + V_{A,e1} + V_{A,e2}$. Let us set the reference frame (datum) at point A, with the vertical axis pointing from A to B. Then the potential energy due to gravity $V_{A,g} = 0$. The potential energy due to elasticity of the 1-st spring $V_{A,e1} = \frac{1}{2}k(l_A - l_0)^2$, where $k = 800\,\mathrm{N/m}$, $l_A = 0.4\,\mathrm{m}$ is the length of the spring at state A, and $l_0 = 0.4 - 0.1 = 0.3\,\mathrm{m}$. Similarly, due to symmetry of the springs, $V_{A,e2} = \frac{1}{2}k(l_A - l_0)^2$.

At point B we have $T_B = \frac{1}{2}mv_B^2$, where v_B is to be established. The potential energy $V_B = V_{B,g} + V_{B,e1} + V_{B,e2}$. The potential energy due to gravity $V_{B,g} = -mgh$, where $h = 0.3\,\mathrm{m}$ and $g = 9.81\,\mathrm{m/s^2}$. The potential energy due to elasticity of the 1st spring $V_{B,e1} = \frac{1}{2}k(l_B - l_0)^2$, where $l_B = \sqrt{0.4^2 + 0.3^2} = 0.5\,\mathrm{m}$ is the length of the spring at state B. Similarly, due to symmetry of the springs, $V_{B,e2} = \frac{1}{2}k(l_B - l_0)^2$.

Now, from the energy conservation equation we obtain

$$\frac{1}{2}mv_A^2 + \frac{1}{2}k(l_A - l_0)^2 + \frac{1}{2}k(l_A - l_0)^2 = \frac{1}{2}mv_B^2 + \frac{1}{2}k(l_B - l_0)^2 + \frac{1}{2}k(l_B - l_0)^2 - mgh \implies$$

$$\boxed{v_B = \sqrt{v_A^2 + 2gh + 2\tfrac{k}{m}\left\{(l_A - l_0)^2 - (l_B - l_0)^2\right\}} \approx 2.25522\,\mathrm{m/s}.}$$

P4. (2 points) The collar has a mass of 2 kg and is attached to the light spring, which has a stiffness of 30 N/m and an unstretched length of 1.5 m. The collar is released from rest at A and slides up the smooth rod under the action of constant 50 N force. Calculate the velocity v of the collar as it passes position B.

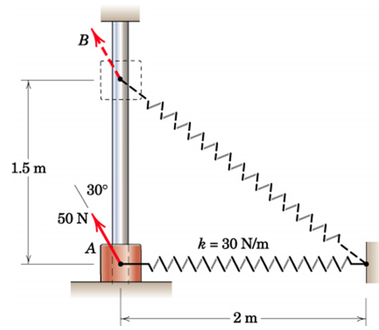

Figure 2: Illustration to Problem 4.

Solution: All the active forces in this problem are conservative[1] (potential), and hence the total energy is conserved. Therefore we can write

$$T_A + V_A = T_B + V_B.$$

At point A the system is at rest and therefore $T_A = 0$. The potential energy has three sources, the gravity force, the elastic force of the spring, and the constant force driving the collar, that is $V_A = V_{A,g} + V_{A,e} + V_{A,c}$. Let us set the reference frame (datum) at point A, with the vertical axis pointing from A to B. Then the potential energy due to gravity $V_{A,g} = 0$, and the potential of the constant force $V_{A,c} = 0$. The potential energy due to elasticity of the spring $V_{A,e} = \frac{1}{2}k(l_A - l_0)^2$, where $k = 30\,\mathrm{N/m}$, $l_A = 2\,\mathrm{m}$ is the length of the spring at state A, and $l_0 = 1.5\,\mathrm{m}$.

At point B we have $T_B = \frac{1}{2}mv_B^2$, where $m = 2\,\mathrm{kg}$ and v_B is to be established. The potential energy $V_B = V_{B,g} + V_{B,e} + V_{B,c}$. The potential energy due to gravity $V_{B,g} = mgh$, where $h = 1.5\,\mathrm{m}$, and $g = 9.81\,\mathrm{m/s^2}$. The potential energy of the constant force $V_{B,c} = -Ph\cos 30^{\mathrm{o}}$, where $P = 50\,\mathrm{N}$. The potential energy due to elasticity of the spring $V_{B,e} = \frac{1}{2}k(l_B - l_0)^2$, where $l_B = \sqrt{2^2 + 1.5^2} = 2.5\,\mathrm{m}$ is the length of the spring at state B.

Now, from the energy conservation equation we obtain

$$\frac{1}{2}k(l_A - l_0)^2 = \frac{1}{2}mv_B^2 + \frac{1}{2}k(l_B - l_0)^2 + mgh - Ph\cos 30^{\mathrm{o}} \quad \Longrightarrow$$

$$\boxed{v_B = \sqrt{2\frac{P}{m}h\cos 30^{\mathrm{o}} - 2gh + \frac{k}{m}\left\{(l_A - l_0)^2 - (l_B - l_0)^2\right\}} \approx 4.92665\,\mathrm{m/s}.}$$

[1] If we did not know that the constant driving force is potential we could write the work-energy equation as $(T_B + V_B) - (T_A + V_A) = U_{A-B}$, where the potential energy has two sources (gravity and elasticity) and then compute the work done by the driving force as $U_{A-B} = Ph\cos 30^{\mathrm{o}}$.