# Exercises in Physics
# Sample Problems # 11

Date Given: June 23, 2022

**P1.** Compute the work done by the force $\boldsymbol{F} = (z-y)\boldsymbol{i} + (x-z)\boldsymbol{j} + (y-x)\boldsymbol{k}$, given as a function of position, along the path $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k} = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$, connecting points $P_1 = (0,0,0)$ and $P_2 = (1,1,1)$ (see Figure 1), when the parameter $t$ is changing from $t=0$ to $t=1$.

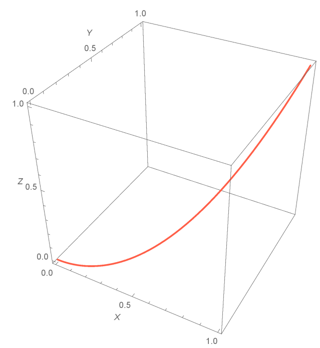


Figure 1: Illustration to Problem 1.

**Solution:** Here $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k}$, and the work done by the force $\boldsymbol{F}$ on this path is calculated as

$$\int_{P_1}^{P_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \left(F_x(t)\dot{x}(t) + F_y(t)\dot{y}(t) + F_z(t)\dot{z}(t)\right)\mathrm{d}t =$$

$$\int_0^1 \left((z(t)-y(t))\frac{\mathrm{d}x(t)}{\mathrm{d}t} + (x(t)-z(t))\frac{\mathrm{d}y(t)}{\mathrm{d}t} + (y(t)-x(t))\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\mathrm{d}t =$$

$$\int_0^1 \left\{(t^3-t^2) + (t-t^3)2t + (t^2-t)3t^2\right\}\mathrm{d}t = \int_0^1 \left\{t^4 - 2t^3 + t^2\right\}\mathrm{d}t = \left[\frac{t^5}{5} - \frac{t^4}{2} + \frac{t^3}{3}\right]_0^1 =$$

$$\frac{1}{5} - \frac{1}{4} + \frac{1}{3} = \frac{1}{30}.$$

**P2.** Compute the work done by the force $\boldsymbol{F} = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$, given as a function of position, along the segment $P_1P_2$ of the straight line[1] passing through $P_1 = (1,2,3)$ and $P_2 = (2,3,4)$.

**Solution:** The parametric equations of the straight line are $x-1=t$, $y-2=t$, $z-3=t$, and the point $P_1$ and $P_2$ correspond to $t=0$ and $t=1$, respectively. Therefore $\boldsymbol{F} = (t+2)\boldsymbol{i} + (t+3)\boldsymbol{j} + (t+1)\boldsymbol{k}$, $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (t+1)\boldsymbol{i} + (t+2)\boldsymbol{j} + (t+3)\boldsymbol{k}$, and $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, and the work done by the force $\boldsymbol{F}$ along this path is calculated as

$$\int_{P_1}^{P_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t}\,\mathrm{d}t = \int_0^1 \left(y(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + z(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + x(t)\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\mathrm{d}t =$$

$$\int_0^1 \left((t+2) + (t+3) + (t+1)\right)\mathrm{d}t = \int_0^1 (3t+6)\,\mathrm{d}t = \frac{15}{2} = 7\frac{1}{2}.$$

[1] Note that the straight line passing through $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ can be parameterized as $x - x_1 = (x_2 - x_1)t,\ \ y - y_1 = (y_2 - y_1)t,\ \ z - z_1 = (z_2 - z_1)t$.

**P3.** Compute the work done by the force $\boldsymbol{F} = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$, given as a function of position, along the the path $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $x(t) = a\cos t$, $y(t) = a\sin t$, $z(t) = bt$, and the parameter $t$ is changing from $t = 0$ to $t = 2\pi$.

**Solution:** The work done by the force on this helical path is calculated as

$$\int_{P_1}^{P_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t))\cdot \frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t}\,\mathrm{d}t = \int_0^{2\pi}\left(y(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + z(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} + x(t)\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\mathrm{d}t =$$
$$\int_0^{2\pi}\left(-a^2\sin^2 t + ab(1+t)\cos t\right)\mathrm{d}t = \left[-\frac{a^2 t}{2} + ab(\cos t + (1+t)\sin t) + \frac{a^2}{4}\sin 2t\right]_0^{2\pi} = -a^2\pi.$$

**P4.** Compute the work done by the force $\boldsymbol{F} = x^2\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$, given as a function of position, along a closed path consisting of straight line segments from $(0,0,0)$ to $(1,1,1)$ to $(1,1,0)$ to $(0,0,0)$.

**Solution:** The total work done by $\boldsymbol{F}$ over the whole path $C$ is the sum of the three integrals over the three path segments. Call these segments $C_1, C_2$, and $C_3$. The equation of these "curves", and the ranges of value of the variables along them, are:

- $C_1:\ x = y = z, \quad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$
- $C_2:\ x = y = 1, \quad 0 \le z \le 1$
- $C_3:\ z = 0, x = y, \quad 0 \le x \le 1,\ 0 \le y \le 1$

The work integral along $C_1$ is

$$W_1 = \int_{C_1} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_{C_1} x^2\mathrm{d}x + y^2\mathrm{d}y + z^2\mathrm{d}z = \int_0^1 x^2\mathrm{d}x + \int_0^1 y^2\mathrm{d}y + \int_0^1 z^2\mathrm{d}z = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

This can also be evaluated using the parametric equation of $C_1$, $x = y = z = t$, from which it follows that $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = \mathrm{d}t$ along $C_1$; then

$$W_1 = \int_{C_1} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = 3\int_0^1 t^2\mathrm{d}t = 1.$$

The other integrals are similarly evaluated. Along $C_2$ $x$ and $y$ are constant, so that $\mathrm{d}x = \mathrm{d}y = 0$:

$$W_2 = \int_{C_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_{C_2} x^2\mathrm{d}x + y^2\mathrm{d}y + z^2\mathrm{d}z = \int_{C_2} z^2\mathrm{d}z = \int_1^0 z^2\mathrm{d}z = -\frac{1}{3}.$$

Also,

$$W_3 = \int_{C_3} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_{C_3} x^2\mathrm{d}x + y^2\mathrm{d}y + z^2\mathrm{d}z = \int_{C_3} x^2\mathrm{d}x + y^2\mathrm{d}y = 2\int_1^0 x^2\mathrm{d}x = -\frac{2}{3}.$$

The total work done by $\boldsymbol{F}$ is $W_1 + W_2 + W_3 = 0$.

**P5.** Compute the work done by the force $\boldsymbol{F} = y^2\boldsymbol{i} + z^2\boldsymbol{j} + x^2\boldsymbol{k}$ along a closed path consisting of straight line segments from $(0,0,0)$ to $(1,1,1)$ to $(1,1,0)$ to $(0,0,0)$.

**Solution:** The total work done by $\boldsymbol{F}$ over the whole path $C$ is the sum of the three integrals over the three path segments. Call these segments $C_1, C_2$, and $C_3$. The equation of these "curves", and the ranges of value of the variables along them, are:

- $C_1:\ x = y = z, \quad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$
- $C_2:\ x = y = 1, \quad 0 \le z \le 1$
- $C_3:\ z = 0, x = y, \quad 0 \le x \le 1,\ 0 \le y \le 1$

For segment $C_1$, to evaluate $\int_{x=0}^{x=1} y^2 \mathrm{d}x$, we use the fact that $y = x$ along $C_1$; then $\int_0^1 y^2 \mathrm{d}x = \int_0^1 x^2 \mathrm{d}x = 1/3$. Similar substitutions lead to

$$W_1 = \int_{C_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{C_1} y^2 \mathrm{d}x + z^2 \mathrm{d}y + x^2 \mathrm{d}z = \int_{C_i} x^2 \mathrm{d}x + y^2 \mathrm{d}y + z^2 \mathrm{d}z = 3 \int_0^1 x^2 \mathrm{d}x = 1.$$

For $C_2$, since $x = y = 1$ and $\mathrm{d}x = \mathrm{d}y = 0$,

$$W_2 = \int_{C_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{C_2} y^2 \mathrm{d}x + z^2 \mathrm{d}y + x^2 \mathrm{d}z = \int_{C_2} x^2 \mathrm{d}z = \int_1^0 \mathrm{d}z = -1.$$

For $C_3$, since $x = y$ and $\mathrm{d}z = 0$,

$$W_3 = \int_{C_3} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{C_3} y^2 \mathrm{d}x + z^2 \mathrm{d}y + x^2 \mathrm{d}z = \int_{C_3} y^2 \mathrm{d}x + z^2 \mathrm{d}y = \int_{C_3} y^2 \mathrm{d}x = \int_1^0 x^2 \mathrm{d}x = -\frac{1}{3}.$$

The total work done by $\boldsymbol{F}$ is $W_1 + W_2 + W_3 = -1/3$.

**P6.** Check if the force $\boldsymbol{F} = -\dfrac{y}{(x-y)^2}\boldsymbol{i} + \dfrac{x}{(x-y)^2}\boldsymbol{j}$ is potential or not. Also, compute the work done by the force $\boldsymbol{F}$ on the path $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $x(t) = a\cos t$, $y(t) = a \sin t$, $z(t) = bt$, and the parameter $t$ is changing from $t = 0$ to $t = 2\pi$.

**Solution:** The force is conservative because

$$\mathrm{curl}\boldsymbol{F} \triangleq \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\boldsymbol{k} = \boldsymbol{0}.$$

Since the force is conservative, $F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, $F_z = -\frac{\partial V}{\partial z}$, and

$$\mathrm{d}V = \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z = -F_x \mathrm{d}x - F_y \mathrm{d}y - F_z \mathrm{d}z = \frac{y}{(x-y)^2}\mathrm{d}x - \frac{x}{(x-y)^2}\mathrm{d}y.$$

To find the potential energy

$$V(x,y,z) = \int_{M_0}^{M} \mathrm{d}V = -\int_{M}^{M_0} \mathrm{d}V = \int_{M}^{M_0} F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z = U_{MM_0}$$

we can compute the work

$$U_{MM_0} = \int_{M}^{M_0} -\frac{y}{(x-y)^2}\mathrm{d}x + \frac{x}{(x-y)^2}\mathrm{d}y$$

along the path, constructed from straight lines parallel to the coordinates axes, from $M = (x, y, z)$ to $M_0 = (x_0, y_0, z_0)$. Here, $M_0$ is the point where $V(x_0, y_0, z_0) = 0$.

Let $B = (x, y, z_0)$, and $C = (x, y_0, z_0)$.

- Moving parallel to $z$ axis along $MB$, we have $x = \text{const}$, $\mathrm{d}x = 0$, $y = \text{const}$, $\mathrm{d}y = 0$, and $z$ is changing from $z$ to $z_0$. Therefore

$$U_{MB} = \int_{z}^{z_0} F_z(x,y,z)\mathrm{d}z = \int_{z}^{z_0} 0 \ \mathrm{d}z = 0.$$

- Moving parallel to $y$ axis along $BC$, we have $z = z_0 = \text{const}$, $\mathrm{d}z = 0$, and $x = \text{const}$, $\mathrm{d}x = 0$, and $y$ is changing from $y$ to $y_0$. Therefore,

$$U_{BC} = \int_{y}^{y_0} F_y(x,y,z_0)\mathrm{d}y = \int_{y}^{y_0} \frac{x}{(x-y)^2}\mathrm{d}y = \left[\frac{x}{x-y}\right]_y^{y_0} = -\frac{x}{x-y} + \frac{x}{x-y_0}.$$

- Moving parallel to $x$ axis along $CM_0$, we have $z = z_0 = \text{const}$, $\mathrm{d}z = 0$, $y = y_0 = \text{const}$, $\mathrm{d}y = 0$, and $x$ is changing from $x$ to $x_0$. Therefore
$$U_{CM_0} = \int_x^{x_0} F_x(x, y_0, z_0)\mathrm{d}x = \int_x^{x_0} -\frac{y_0}{(x-y_0)^2}\mathrm{d}x = \left[\frac{y_0}{x-y_0}\right]_x^{x_0} = -\frac{y_0}{x-y_0} + \frac{x_0}{x_0-y_0}.$$

Therefore
$$V(x,y,z) = U_{MM_0} = U_{MB} + U_{BC} + U_{CM_0} = \frac{y}{y-x} - \underbrace{\frac{x_0}{y_0-x_0}}_{\text{constant}}$$
and $V(x_0, y_0, z_0) = 1$. Next, since $M_0 = (x_0, y_0, z_0) = \boldsymbol{r}(0) = (a, 0, 0)$ and $M_1 = \boldsymbol{r}(2\pi) = (a, 0, 2\pi b)$, the work done by $\boldsymbol{F}$ along any path from $M_0$ to $M_1$ is
$$U_{M_0M_1} = V(M_0) - V(M_1) = V(a,0,0) - V(a,0,2\pi b) = 0.$$

**P7.** Check if the force $\boldsymbol{F} = (2x+y)\boldsymbol{i} + (x+z^2)\boldsymbol{j} + (2yz+1)\boldsymbol{k}$ is potential or not. Also, compute the work done by the force $\boldsymbol{F}$ along the path $\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$, and the parameter $t$ is changing from $t = 0$ to $t = 1$.

**Solution:** The force is potential because
$$\mathrm{curl}\boldsymbol{F} \triangleq \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\boldsymbol{k} = \boldsymbol{0}.$$

Since the force is conservative, $F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, $F_z = -\frac{\partial V}{\partial z}$, and
$$\mathrm{d}V = \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z = -F_x\mathrm{d}x - F_y\mathrm{d}y - F_z\mathrm{d}z = -(2x+y)\mathrm{d}x - (x+z^2)\mathrm{d}y - (2yz+1)\mathrm{d}z.$$

To find the potential function
$$V(x,y,z) = \int_{M_0}^{M} \mathrm{d}V = -\int_M^{M_0} \mathrm{d}V = \int_M^{M_0} F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z = U_{MM_0}$$
we can compute the work
$$U_{MM_0} = \int_M^{M_0} (2x+y)\mathrm{d}x + (x+z^2)\mathrm{d}y + (2yz+1)\mathrm{d}z$$
along the path, constructed from straight lines parallel to the coordinates axes[2], from $M = (x, y, z)$ to $M_0 = (x_0, y_0, z_0)$. Here, $M_0$ is the point where $V(x_0, y_0, z_0) = 0$. Let $B = (x, y, z_0)$, and $C = (x, y_0, z_0)$.

- Moving parallel to $z$ axis along $MB$, we have $x = \text{const}$, $\mathrm{d}x = 0$, $y = \text{const}$, $\mathrm{d}y = 0$, and $z$ is changing from $z$ to $z_0$. Therefore,
$$U_{MB} = \int_z^{z_0} F_z(x,y,z)\mathrm{d}z = \int_z^{z_0} (2yz+1)\mathrm{d}z = \left[z + yz^2\right]_z^{z_0} = -z - yz^2 + z_0 + yz_0^2.$$
- Moving parallel to $y$ axis along $BC$, we have $z = z_0 = \text{const}$, $\mathrm{d}z = 0$, $x = \text{const}$, $\mathrm{d}x = 0$, and $y$ is changing from $y$ to $y_0$. Therefore,
$$U_{BC} = \int_y^{y_0} F_y(x,y,z_0)\mathrm{d}y = \int_y^{y_0} (x+z_0^2)\mathrm{d}y = \left[y(x+z_0^2)\right]_y^{y_0} = -y(x+z_0^2) + y_0(x+z_0^2).$$

[2] Note that defining $V$ by computing along the coordinate lines is only one way. Sometimes the answer can be obtained easier by direct integrating the exact differential. For example, in our case $\mathrm{d}V = \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z = -F_x\mathrm{d}x - F_y\mathrm{d}y - F_z\mathrm{d}z = -(2x+y)\mathrm{d}x - (x+z^2)\mathrm{d}y - (2yz+1)\mathrm{d}z = -2x\mathrm{d}x - y\mathrm{d}x - x\mathrm{d}y - z^2\mathrm{d}y - 2yz\mathrm{d}z - \mathrm{d}z = -\mathrm{d}(x^2) - \mathrm{d}(xy) - \mathrm{d}(yz^2) - \mathrm{d}(z) = -\mathrm{d}(x^2 + xy + yz^2 + z)$. Therefore $V = -(x^2 + xy + yz^2 + z) + C$, where $C$ is a constant. Then $U_{M_1M_0} = V(M_0) - V(M_1) = \left[x^2(t) + x(t)y(t) + y(t)z^2(t) + z(t)\right]_{t=0}^{t=1} = 4$.

- Moving parallel to $x$ axis along $CM_0$, we have $z = z_0 = \text{const}$, $\mathrm{d}z = 0$, $y = y_0 = \text{const}$, $\mathrm{d}y = 0$, and $x$ is changing from $x$ to $x_0$. Therefore

$$U_{CM_0} = \int_x^{x_0} F_x(x, y_0, z_0)\mathrm{d}x = \int_x^{x_0} (2x + y_0)\mathrm{d}x = \left[x^2 + xy_0\right]_x^{x_0} = -x^2 - xy_0 + x_0^2 + x_0y_0$$

Therefore

$$V(x, y, z) = U_{MM_0} = U_{MB} + U_{BC} + U_{CM_0} = -x(y + x) - z(1 + yz) + \underbrace{x_0(y_0 + x_0) + z_0(1 + y_0z_0)}_{\text{constant term}}$$

and $V(x_0, y_0, z_0) = 0$. Next, since $M_0 = (x_0, y_0, z_0) = \boldsymbol{r}(0) = (0, 0, 0)$ and $M_1 = \boldsymbol{r}(1) = (1, 1, 1)$, the work done by $\boldsymbol{F}$ along any path from $M_0$ to $M_1$ is

$$U_{M_0M_1} = V(M_0) - V(M_1) = V(0, 0, 0) - V(1, 1, 1) = 4.$$