Exercises in Physics
Sample Problems # 6

Date Given: May 19, 2022

P1. For a short time the arm of the robot is extending such that $\dot{r} = 0.5$m/s when $r = 1$ m, $z = t^2$ m, and $\theta = t$ rad, where t is in seconds. Determine the magnitudes of the velocity and acceleration of the grip A when $t = 3$s.

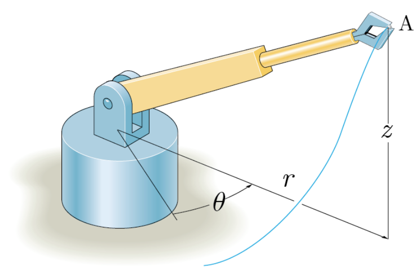

Figure 1: Illustration to Problem 1.

Solution: In the cylindrical coordinates we have

$$\begin{array}{lll} r = 1, & \theta = t & z = t^2, \\ \dot{r} = 0.5, & \dot{\theta} = 1, & \dot{z} = 2t, \\ \ddot{r} = 0, & \ddot{\theta} = 0, & \ddot{z} = 2. \end{array}$$

At the moment $t = 3$ we have

$$\begin{array}{lll} r = 1, & \theta = 3, & z = 9 \\ \dot{r} = 0.5, & \dot{\theta} = 1 & \dot{z} = 6, \\ \ddot{r} = 0, & \ddot{\theta} = 0 & \ddot{z} = 2, \end{array}$$

The components of the velocity and acceleration in the cylindrical coordinates are

$$\begin{array}{ll} v_r = \dot{r} = 0, & a_r = \ddot{r} - r\dot{\theta}^2 = -1/2, \\ v_\theta = r\dot{\theta} = 1/2, & a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0, \\ v_z = \dot{z} = 6, & a_z = \ddot{z} = 2. \end{array}$$

Therefore

$$v = \sqrt{v_r^2 + v_\theta^2 + v_z^2} = \sqrt{145}/2 \approx 6.0208 \text{ m/s},$$

and

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2} = \sqrt{17}/2 \approx 2.06155 \text{ m/s}^2.$$

P2. The rotating element in a mixing chamber is given a periodic axial movement $z = z_0 \sin 2\pi n t$ while it is rotating at the constant angular velocity $\dot{\theta} = \omega$. Determine the expression for the maximum magnitude of the acceleration of a point A on the rim of radius r. The frequency n of vertical oscillation is constant.

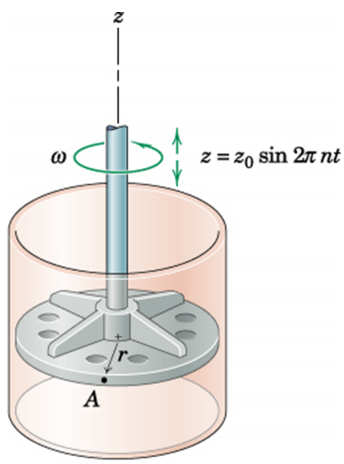

Figure 2: Illustration to Problem 2.

Solution: Here, in cylindrical coordinates we have r = const, $\dot{r} = 0$, $\ddot{r} = 0$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$. Therefore the components of the acceleration in the cylindrical coordinates are

$$a_r = \ddot{r} - r\dot{\theta}^2 = -r\omega^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0, \quad a_z = \ddot{z} = -4\pi^2 n^2 z_0 \sin 2\pi nt.$$

The acceleration does not have the transverse component ($a_\theta = 0$, see Figure 2). The acceleration magnitude is

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2} = \sqrt{a_r^2 + a_z^2} = \sqrt{r^2\omega^4 + 16\pi^4 n^4 z_0^2 \sin^2 2\pi nt}$$

It is clear that the maximum of the acceleration magnitude is attained when $\sin 2\pi nt = \pm 1$ and therefore

$$a_{\max} = \sqrt{r^2\omega^4 + 16\pi^4 n^4 z_0^2}.$$

P3. A horse on the merry-go-round moves according to the equations $r = 3$ m, $\theta = (\pi t/4)$ rad, and $z = (2\sin\theta)$ m, where t is in seconds. Determine the cylindrical components and the magnitudes of the velocity and acceleration of the horse when $t = 4$ s.

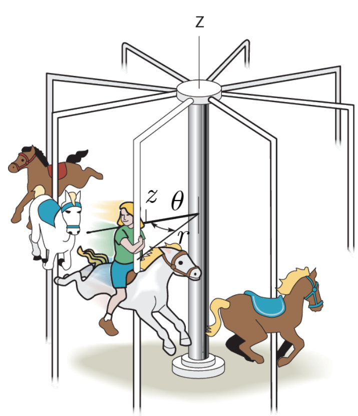

Figure 3: Illustration to Problem 3.

Solution: In the cylindrical coordinates we have[1]

$$\begin{aligned} &r = 3, && \theta = \pi t/4 && z = 2\sin\theta, \\ &\dot r = 0, && \dot\theta = \pi/4, && \dot z = 2(\cos\theta)\dot\theta, \\ &\ddot r = 0, && \ddot\theta = 0, && \ddot z = 2(\cos\theta)\ddot\theta - 2(\sin\theta)\dot\theta^2. \end{aligned}$$

At the moment $t = 4$ we have

$$\begin{aligned} &r = 3, && \theta = \pi, && z = 0 \\ &\dot r = 0, && \dot\theta = \pi/4 && \dot z = -\pi/2, \\ &\ddot r = 0, && \ddot\theta = 0 && \ddot z = 0, \end{aligned}$$

The components of the velocity and acceleration in the cylindrical coordinates are

$$\begin{aligned} &v_r = \dot r = 0, && a_r = \ddot r - r\dot\theta^2 = -3\pi^2/16 \\ &v_\theta = r\dot\theta = 3\pi/4, && a_\theta = r\ddot\theta + 2\dot r\dot\theta = 0, \\ &v_z = \dot z = -\pi/2, && a_z = \ddot z = 0. \end{aligned}$$

Therefore

$$v = \sqrt{v_r^2 + v_\theta^2 + v_z^2} = \sqrt{13}\pi/4 \approx 2.83179 \text{ m/s},$$

and

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2} = 3\pi^2/16 \approx 1.85055 \text{ m/s}^2.$$

P4. A particle moves along the cylindrical helix of radius 2 m in the positive direction of rotation. As it passes point A the magnitude of its total acceleration is 10m/s^2 and its speed along the path is increasing at the rate of 8m/s^2. For this position, find compute the velocity and acceleration vectors.

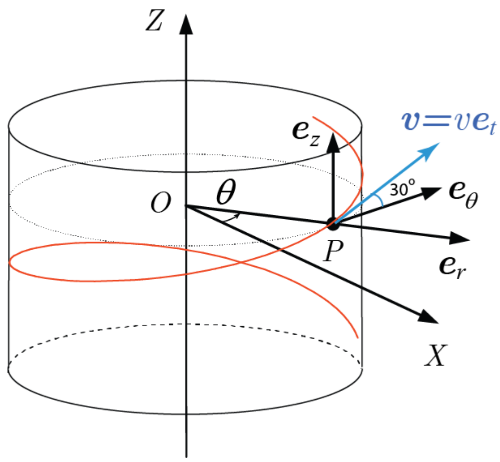

Figure 4: Illustration to Problem 4.

Solution: Here, in cylindrical coordinates we have $r = 2 = \text{const}$, $\dot r = 0$, $\ddot r = 0$, and therefore $v_r = \dot r = 0$, $a_r = \ddot r - r\dot\theta^2 = -r\dot\theta^2$. $a_\theta = r\ddot\theta + 2\dot r\dot\theta = r\ddot\theta$, The velocity $\boldsymbol{v}$ is tangent to the path (does not have component in $\boldsymbol{e}_r$ and is directed along the unit vector $\boldsymbol{e}_t$, see Figure 5) and we can write it down as

$$\boldsymbol{v} = v\boldsymbol{e}_t = v_\theta \boldsymbol{e}_\theta + v_z \boldsymbol{e}_z = r\dot\theta \boldsymbol{e}_\theta + \dot z \boldsymbol{e}_z.$$

[1]Note that when differentiating z we use the chain rule of calculus.

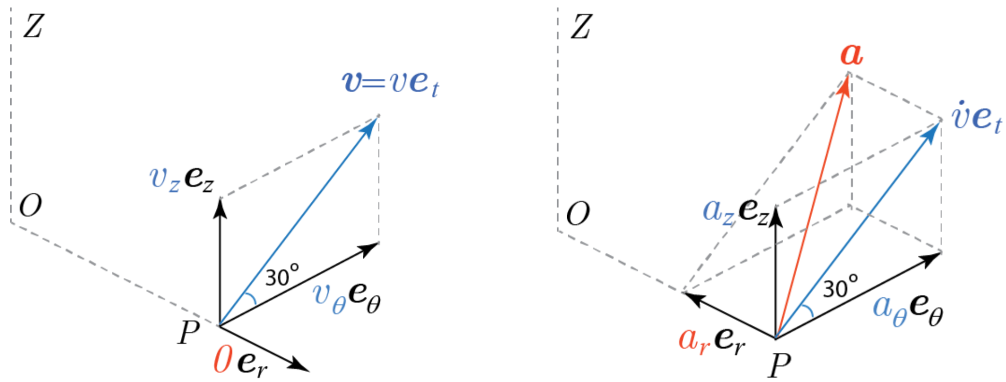

Figure 5: Illustration to Problem 4.

Similarly, we can represent the acceleration along the path

$$\dot{v}\boldsymbol{e}_t = a_\theta \boldsymbol{e}_\theta + a_z \boldsymbol{e}_z = r\ddot{\theta}\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{e}_z,$$

where $\dot{v} = 8\text{m/s}^2$.

The total acceleration is $\boldsymbol{a} = a_r\boldsymbol{e}_r + a_\theta\boldsymbol{e}_\theta + a_z\boldsymbol{e}_z$ and its magnitude is known. Therefore, from

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2} = \sqrt{a_r^2 + \dot{v}^2} \Longrightarrow 10 = \sqrt{a_r^2 + 8^2}$$

we find $a_r = -6\text{m/s}$ and from $a_r = -r\dot{\theta}^2$ we obtain $\dot{\theta} = \sqrt{3}\text{rad/s}$.

Now, we can define the components of the velocity vector,

$$v_\theta = r\dot{\theta} = 2\sqrt{3}\text{m/s}, \qquad v = v_\theta/\cos 30^\text{o} = 4\text{m/s}, \qquad v_z = \dot{z} = v\sin 30^\text{o} = 2\text{m/s},$$

and the remaining components of the acceleration vector,

$$a_\theta = r\ddot{\theta} = \dot{v}\cos 30^\text{o} \Longrightarrow \ddot{\theta} = 2\sqrt{3}\text{rad/s}^2, a_\theta = 4\sqrt{3}\text{rad/s}^2, \qquad a_z = \ddot{z} = \dot{v}\sin 30^\text{o} = 4\text{m/s}^2.$$

P5. The car A is ascending a parking-garage ramp in the form of a cylindrical helix of 7.2 m radius rising 3 m for each half turn. At the position shown the car has a speed[2] of 25 km/h, which is decreasing at the rate of 3 km/h per second. Determine the r-, θ-, and z-components of the acceleration of the car.

Solution: The helix angle (also called the lead angle) is defined as

$$\tan\gamma = \frac{\text{pitch}}{2\pi r} = \frac{2\times 3}{2\pi\times 7.2} \quad\Longrightarrow\quad \gamma \approx 0.13186\,\text{rad} \approx 7.555^\text{o}$$

The speed $v = |\boldsymbol{v}| = 25\,\text{km/h} = 6.94\,\text{m/s}$. Since the velocity does not have the radial component, from the speed and the helix angle we can define $v_z = v\sin\gamma \approx 0.915\,\text{m/s}$, and $v_\theta = v\cos\gamma \approx 6.88\,\text{m/s}$. By definition $v_z = \dot{z}$ and $v_\theta = r\dot{\theta}$. Therefore $\dot{\theta} = v_\theta/r \approx 0.956\,\text{rad/s}$.

The radial component of the acceleration vector $a_r = \ddot{r} - r\dot{\theta}^2 = -r\omega^2$. The acceleration along the path, $\boldsymbol{a}_{\text{path}} = \boldsymbol{a}_{\theta z} = a_\theta\boldsymbol{e}_\theta + a_z\boldsymbol{e}_z$ is in the tangent (to the cylinder) plane. The magnitude of $\boldsymbol{a}_{\theta z}$ is known: $a_{\theta z} = |\dot{v}| = 3\,\text{km/h} \approx 0.833\,\text{m/s}$, and its direction is negative because v is decreasing. Therefore $a_\theta = -a_{\theta z}\cos\gamma$ and $a_\theta = -a_{\theta z}\sin\gamma$. Finally, in numbers,

$$a_r \approx -6.58\text{m/s}^2, \quad a_\theta \approx -0.826\text{m/s}^2, \quad a_z \approx -0.1096\text{m/s}^2.$$

[2]The magnitude of the velocity vector.

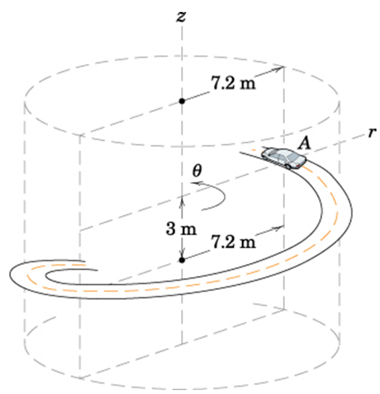

Figure 6: Illustration to Problem 5.

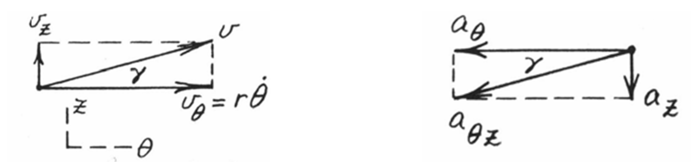

Figure 7: Illustration to Problem 5.

P6. The base structure of the fire truck ladder rotates about a vertical axis through O with a constant angular velocity $\Omega = 10$deg/s. At the same time, the ladder unit OB elevates at a constant rate $\dot{\varphi} = 7$deg/s, and section AB of the ladder extends from within section OA at the constant rate of 0.5 m/s. At the instant under consideration, $\varphi = 30^o$, $OA = 9$m, and $AB = 6$m. Determine the magnitudes of the velocity and acceleration of the end B of the ladder.

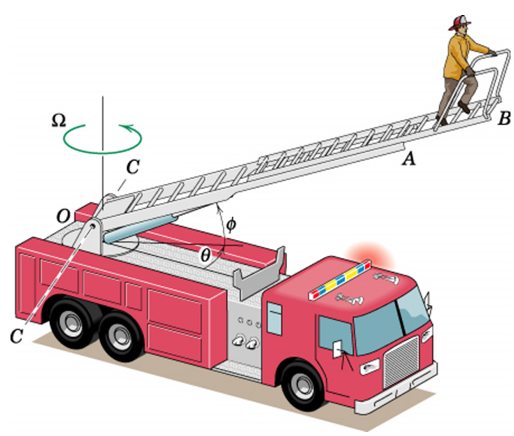

Figure 8: Illustration to Problem 6.