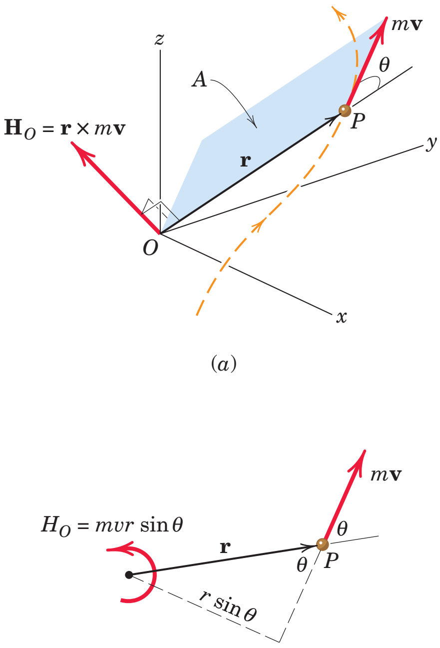


View in plane A

(b)

Figure 3/14

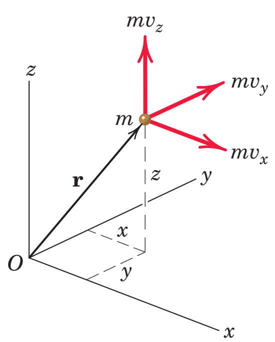


Figure 3/15

3/10 Angular Impulse and Angular Momentum

In addition to the equations of linear impulse and linear momentum, there exists a parallel set of equations for angular impulse and angular momentum. First, we define the term *angular momentum*. Figure 3/14a shows a particle P of mass m moving along a curve in space. The particle is located by its position vector $\mathbf{r}$ with respect to a convenient origin O of fixed coordinates x-y-z. The velocity of the particle is $\mathbf{v} = \dot{\mathbf{r}}$, and its linear momentum is $\mathbf{G} = m\mathbf{v}$. The *moment* of the *linear momentum vector* $m\mathbf{v}$ about the origin O is defined as the *angular momentum* $\mathbf{H}_O$ of P about O and is given by the cross-product relation for the moment of a vector

$$\boxed{\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}} \tag{3/29}$$

The angular momentum then is a vector perpendicular to the plane A defined by $\mathbf{r}$ and $\mathbf{v}$. The sense of $\mathbf{H}_O$ is clearly defined by the right-hand rule for cross products.

The scalar components of angular momentum may be obtained from the expansion

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} = m(v_z y - v_y z)\mathbf{i} + m(v_x z - v_z x)\mathbf{j} + m(v_y x - v_x y)\mathbf{k}$$

$$\mathbf{H}_O = m \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} \tag{3/30}$$

so that

$$H_x = m(v_z y - v_y z) \qquad H_y = m(v_x z - v_z x) \qquad H_z = m(v_y x - v_x y)$$

Each of these expressions for angular momentum may be checked easily from Fig. 3/15, which shows the three linear-momentum components, by taking the moments of these components about the respective axes.

To help visualize angular momentum, we show in Fig. 3/14b a two-dimensional representation in plane A of the vectors shown in part a of the figure. The motion is viewed in plane A defined by $\mathbf{r}$ and $\mathbf{v}$. The magnitude of the moment of $m\mathbf{v}$ about O is simply the linear momentum mv times the moment arm $r \sin\theta$ or $mvr \sin\theta$, which is the magnitude of the cross product $\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$.

Angular momentum is the moment of linear momentum and must not be confused with linear momentum. In SI units, angular momentum has the units $\text{kg}\cdot(\text{m/s})\cdot\text{m} = \text{kg}\cdot\text{m}^2/\text{s} = \text{N}\cdot\text{m}\cdot\text{s}$. In the U.S. customary system, angular momentum has the units $[\text{lb}/(\text{ft/sec}^2)][\text{ft/sec}][\text{ft}] = \text{lb-ft-sec}$.

Rate of Change of Angular Momentum

We are now ready to relate the moment of the forces acting on the particle P to its angular momentum. If $\Sigma\mathbf{F}$ represents the resultant of *all* forces acting on the particle P of Fig. 3/14, the moment $\mathbf{M}_O$ about the origin O is the vector cross product

$$\Sigma\mathbf{M}_O = \mathbf{r} \times \Sigma\mathbf{F} = \mathbf{r} \times m\dot{\mathbf{v}}$$

where Newton's second law $\Sigma \mathbf{F} = m\dot{\mathbf{v}}$ has been substituted. We now differentiate Eq. 3/29 with time, using the rule for the differentiation of a cross product (see item 9, Art. C/7, Appendix C) and obtain

$$\dot{\mathbf{H}}_O = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}}$$

The term $\mathbf{v} \times m\mathbf{v}$ is zero since the cross product of parallel vectors is identically zero. Substitution into the expression for $\Sigma \mathbf{M}_O$ gives

$$\boxed{\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O} \qquad \textbf{(3/31)}$$

Equation 3/31 states that the *moment about the fixed point O of all forces acting on m equals the time rate of change of angular momentum of m about O*. This relation, particularly when extended to a system of particles, rigid or nonrigid, provides one of the most powerful tools of analysis in dynamics.

Equation 3/31 is a vector equation with scalar components

$$\Sigma M_{O_x} = \dot{H}_{O_x} \qquad \Sigma M_{O_y} = \dot{H}_{O_y} \qquad \Sigma M_{O_z} = \dot{H}_{O_z} \qquad \textbf{(3/32)}$$

The Angular Impulse-Momentum Principle

Equation 3/31 gives the instantaneous relation between the moment and the time rate of change of angular momentum. To obtain the effect of the moment $\Sigma \mathbf{M}_O$ on the angular momentum of the particle over a finite period of time, we integrate Eq. 3/31 from time t_1 to time t_2. Multiplying the equation by dt gives $\Sigma \mathbf{M}_O \, dt = d\mathbf{H}_O$, which we integrate to obtain

$$\int_{t_1}^{t_2} \Sigma \mathbf{M}_O \, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1 = \Delta \mathbf{H}_O \qquad \textbf{(3/33)}$$

where $(\mathbf{H}_O)_2 = \mathbf{r}_2 \times m\mathbf{v}_2$ and $(\mathbf{H}_O)_1 = \mathbf{r}_1 \times m\mathbf{v}_1$. The product of moment and time is defined as *angular impulse*, and Eq. 3/33 states that the *total angular impulse on m about the fixed point O equals the corresponding change in angular momentum of m about O*.

Alternatively, we may write Eq. 3/33 as

$$\boxed{(\mathbf{H}_O)_1 + \int_{t_1}^{t_2} \Sigma \mathbf{M}_O \, dt = (\mathbf{H}_O)_2} \qquad \textbf{(3/33}\boldsymbol{a}\textbf{)}$$

which states that the initial angular momentum of the particle plus the angular impulse applied to it equals its final angular momentum. The units of angular impulse are clearly those of angular momentum, which are N·m·s or kg·m^2/s in SI units and lb-ft-sec in U.S. customary units.

As in the case of linear impulse and linear momentum, the equation of angular impulse and angular momentum is a vector equation where changes in direction as well as magnitude may occur during the interval of integration. Under these conditions, it is necessary to express $\Sigma \mathbf{M}_O$

and $\mathbf{H}_O$ in component form and then combine the integrated components. The x-component of Eq. 3/33a is

$$(H_{O_x})_1 + \int_{t_1}^{t_2} \Sigma M_{O_x}\, dt = (H_{O_x})_2$$

or

$$m(v_z y - v_y z)_1 + \int_{t_1}^{t_2} \Sigma M_{O_x}\, dt = m(v_z y - v_y z)_2 \qquad \textbf{(3/33}\boldsymbol{b}\textbf{)}$$

where the subscripts 1 and 2 refer to the values of the respective quantities at times t_1 and t_2. Similar expressions exist for the y- and z-components of the angular impulse-momentum equation.

Plane-Motion Applications

The foregoing angular-impulse and angular-momentum relations have been developed in their general three-dimensional forms. Most of the applications of interest to us, however, can be analyzed as plane-motion problems where moments are taken about a single axis normal to the plane of motion. In this case, the angular momentum may change magnitude and sense, but the direction of the vector remains unaltered.

Thus, for a particle of mass m moving along a curved path in the x-y plane, Fig. 3/16, the angular momenta about O at points 1 and 2 have the magnitudes $(H_O)_1 = |\mathbf{r}_1 \times m\mathbf{v}_1| = mv_1d_1$ and $(H_O)_2 = |\mathbf{r}_2 \times m\mathbf{v}_2| = mv_2d_2$, respectively. In the illustration both $(H_O)_1$ and $(H_O)_2$ are represented in the counterclockwise sense in accord with the direction of the moment of the linear momentum. The scalar form of Eq. 3/33a applied to the motion between points 1 and 2 during the time interval t_1 to t_2 becomes

$$(H_O)_1 + \int_{t_1}^{t_2} \Sigma M_O\, dt = (H_O)_2 \quad \text{or} \quad mv_1d_1 + \int_{t_1}^{t_2} \Sigma Fr \sin\theta\, dt = mv_2d_2$$

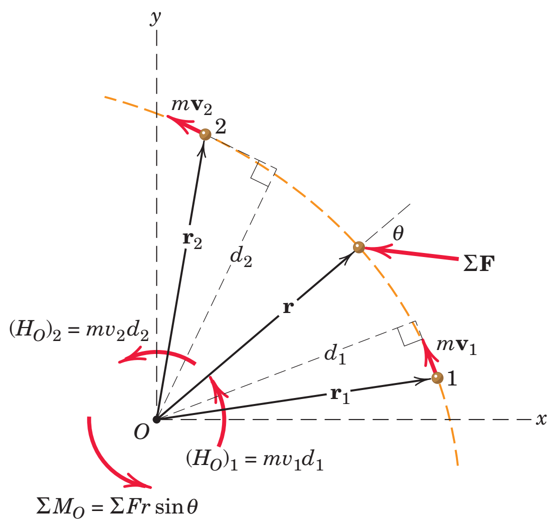


Figure 3/16

This example should help clarify the relation between the scalar and vector forms of the angular impulse-momentum relations.

Whereas Eq. 3/33 clearly stresses that the external angular impulse causes a change in the angular momentum, the order of the terms in Eqs. 3/33*a* and 3/33*b* corresponds to the natural sequence of events. Equation 3/33*a* is analogous to Eq. 3/27*a*, just as Eq. 3/31 is analogous to Eq. 3/25.

As was the case for linear-momentum problems, we encounter *impulsive* (large magnitude, short duration) and *nonimpulsive* forces in angular-momentum problems. The treatment of these forces was discussed in Art. 3/9.

Equations 3/25 and 3/31 add no new basic information since they are merely alternative forms of Newton's second law. We will discover in subsequent chapters, however, that the motion equations expressed in terms of the time rate of change of momentum are applicable to the motion of rigid and nonrigid bodies and provide a very general and powerful approach to many problems. The full generality of Eq. 3/31 is usually not required to describe the motion of a single particle or the plane motion of rigid bodies, but it does have important use in the analysis of the space motion of rigid bodies introduced in Chapter 7.

Conservation of Angular Momentum

If the resultant moment about a fixed point O of all forces acting on a particle is zero during an interval of time, Eq. 3/31 requires that its angular momentum $\mathbf{H}_O$ about that point remain constant. In this case, the angular momentum of the particle is said to be *conserved*. Angular momentum may be conserved about one axis but not about another axis. A careful examination of the free-body diagram of the particle will disclose whether the moment of the resultant force on the particle about a fixed point is zero, in which case, the angular momentum about that point is unchanged (conserved).

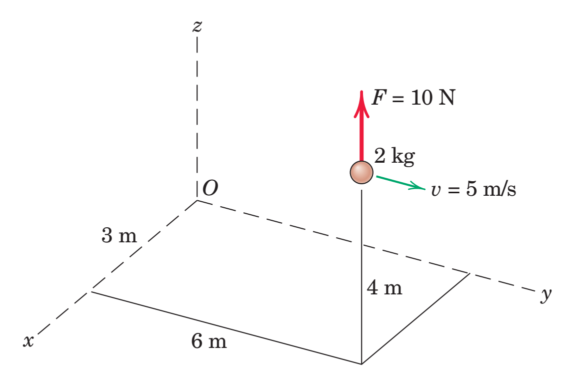
Consider now the motion of two particles a and b which interact during an interval of time. If the interactive forces $\mathbf{F}$ and $-\mathbf{F}$ between them are the only unbalanced forces acting on the particles during the interval, it follows that the moments of the equal and opposite forces about any fixed point O not on their line of action are equal and opposite. If we apply Eq. 3/33 to particle a and then to particle b and add the two equations, we obtain $\Delta\mathbf{H}_a + \Delta\mathbf{H}_b = \mathbf{0}$ (where all angular momenta are referred to point O). Thus, the total angular momentum for the system of the two particles remains constant during the interval, and we write

$$\boxed{\Delta\mathbf{H}_O = \mathbf{0} \quad \text{or} \quad (\mathbf{H}_O)_1 = (\mathbf{H}_O)_2} \qquad \textbf{(3/34)}$$

which expresses the *principle of conservation of angular momentum*.

Sample Problem 3/24

A small sphere has the position and velocity indicated in the figure and is acted upon by the force F. Determine the angular momentum $\mathbf{H}_O$ about point O and the time derivative $\dot{\mathbf{H}}_O$.



Solution. We begin with the definition of angular momentum and write

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$

$$= (3\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}) \times 2(5\mathbf{j})$$

$$= -40\mathbf{i} + 30\mathbf{k} \text{ N·m/s} \qquad \textit{Ans.}$$

From Eq. 3/31,

$$\dot{\mathbf{H}}_O = \mathbf{M}_O$$

$$= \mathbf{r} \times \mathbf{F}$$

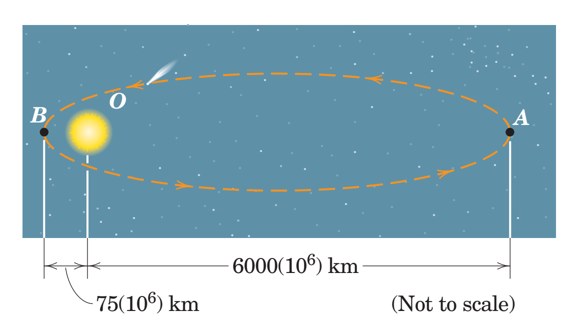
$$= (3\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}) \times 10\mathbf{k}$$

$$= 60\mathbf{i} - 30\mathbf{j} \text{ N·m} \qquad \textit{Ans.}$$

As with moments of forces, the position vector must run *from* the reference point (O in this case) *to* the line of action of the linear momentum $m\mathbf{v}$. Here $\mathbf{r}$ runs directly to the particle.

Sample Problem 3/25

A comet is in the highly eccentric orbit shown in the figure. Its speed at the most distant point A, which is at the outer edge of the solar system, is $v_A = 740$ m/s. Determine its speed at the point B of closest approach to the sun.



Solution. Because the only significant force acting on the comet, the gravitational force exerted on it by the sun, is central (points to the sun center O), angular momentum about O is conserved.

$$(H_O)_A = (H_O)_B$$

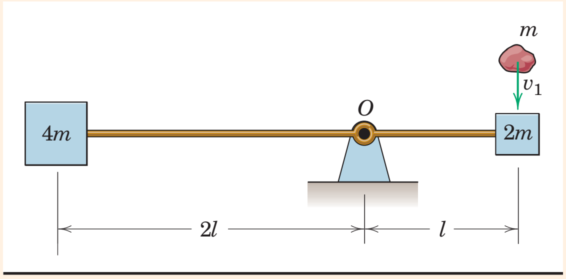
$$mr_Av_A = mr_Bv_B$$

$$v_B = \frac{r_Av_A}{r_B} = \frac{6000(10^6)740}{75(10^6)}$$

$$v_B = 59\,200 \text{ m/s} \qquad \textit{Ans.}$$

Sample Problem 3/26

The assembly of the light rod and two end masses is at rest when it is struck by the falling wad of putty traveling with speed v_1 as shown. The putty adheres to and travels with the right-hand end mass. Determine the angular velocity $\dot{\theta}_2$ of the assembly just after impact. The pivot at O is frictionless, and all three masses may be assumed to be particles.



Solution. If we ignore the angular impulses associated with the weights during the collision process, then system angular momentum about O is conserved during the impact.

$$(H_O)_1 = (H_O)_2$$

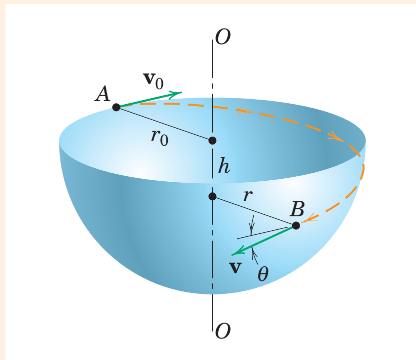
$$mv_1 l = (m + 2m)(l\dot{\theta}_2)l + 4m(2l\dot{\theta}_2)2l$$

$$\dot{\theta}_2 = \frac{v_1}{19l} \text{ CW} \qquad \textit{Ans.}$$

Note that each angular-momentum term is written in the form mvd, and the final transverse velocities are expressed as radial distances times the common final angular velocity $\dot{\theta}_2$.

Sample Problem 3/27

A small mass particle is given an initial velocity $\mathbf{v}_0$ tangent to the horizontal rim of a smooth hemispherical bowl at a radius r_0 from the vertical centerline, as shown at point A. As the particle slides past point B, a distance h below A and a distance r from the vertical centerline, its velocity $\mathbf{v}$ makes an angle θ with the horizontal tangent to the bowl through B. Determine θ.



Solution. The forces on the particle are its weight and the normal reaction exerted by the smooth surface of the bowl. Neither force exerts a moment about the axis O-O, so that angular momentum is conserved about that axis. Thus,

① $[(H_O)_1 = (H_O)_2]$ $\qquad mv_0 r_0 = mvr\cos\theta$

Also, energy is conserved so that $E_1 = E_2$. Thus

$[T_1 + V_1 = T_2 + V_2]$ $\qquad \frac{1}{2}mv_0^{\,2} + mgh = \frac{1}{2}mv^2 + 0$

$$v = \sqrt{v_0^{\,2} + 2gh}$$

Eliminating v and substituting $r^2 = r_0^{\,2} - h^2$ give

$$v_0 r_0 = \sqrt{v_0^{\,2} + 2gh}\sqrt{r_0^{\,2} - h^2}\cos\theta$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{1 + \dfrac{2gh}{v_0^{\,2}}}\sqrt{1 - \dfrac{h^2}{r_0^{\,2}}}} \qquad \textit{Ans.}$$

Helpful Hint

① The angle θ is measured in the plane tangent to the hemispherical surface at B.