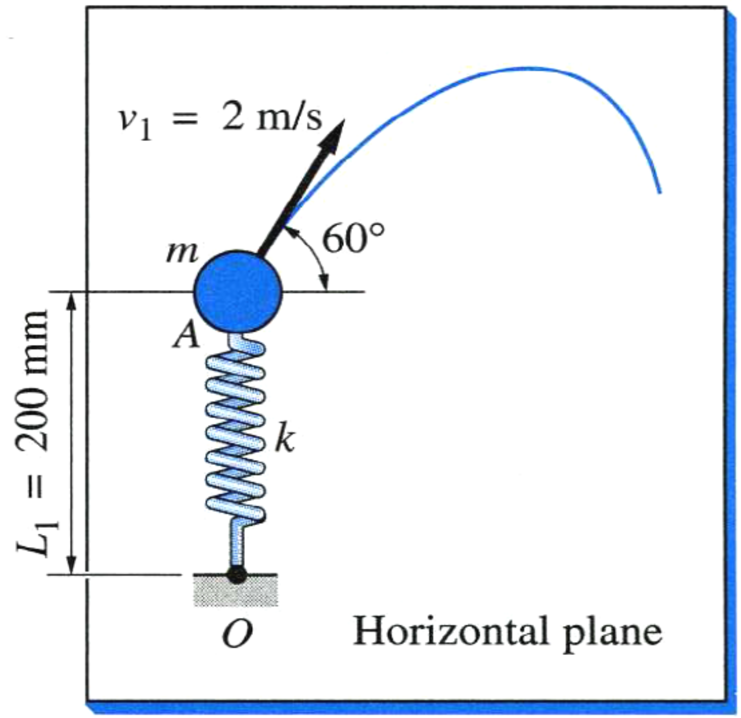
Exercises in Physics

Lecture 13
Angular Impulse & Momentum; Generalization to system of points (Linear Momentum)

Sample Problem 1

The particle of mass $m = 0.3$ kg moves on a frictionless horizontal plane. One end of the spring is attached to the particle, and the other end to the fixed point O. If the particle is launched from position A with the velocity v_1, determine the stiffness k if the maximum distance between the path of the particle and point is 400 mm. The spring is undeformed at position A.



Solution

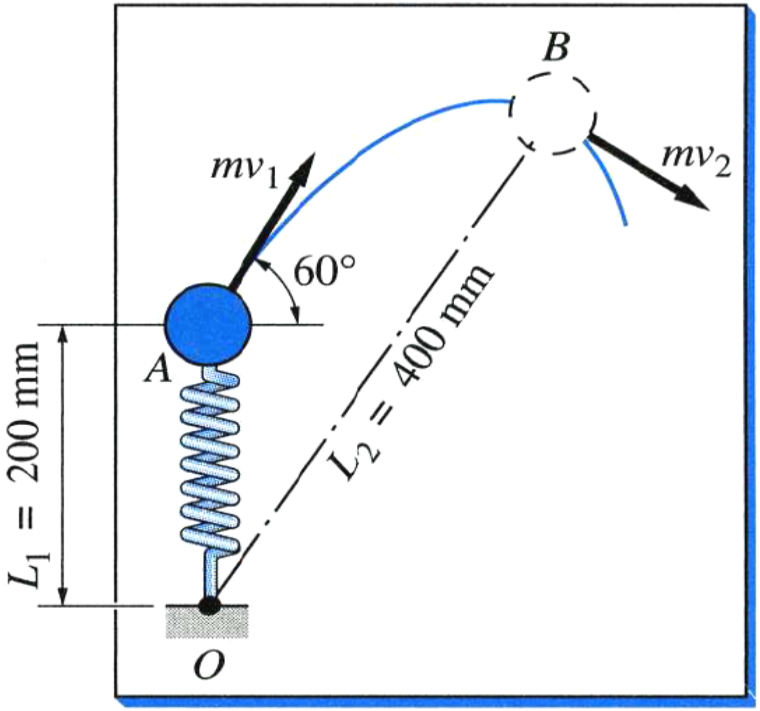
Conservation of Energy.

Let $L_2 = 0.4\text{m}$ be the maximal distance. The system is conservative (and the gravity is ignored as the motion occurs in the horizontal plane). Therefore,

$$T_A + V_A = T_B + V_B \quad \Rightarrow$$

$$\frac{1}{2}mv_1^2 + 0 = \frac{1}{2}mv_2^2 + \frac{1}{2}k(L_2 - L_1)^2 \quad \Rightarrow$$

$$k = \frac{m(v_1^2 - v_2^2)}{(L_2 - L_1)^2} = \frac{0.3(4 - v_2^2)}{(0.2)^2}\,\text{N/m}$$



It remains to stablish v_2 .

Solution

Conservation of angular momentum.

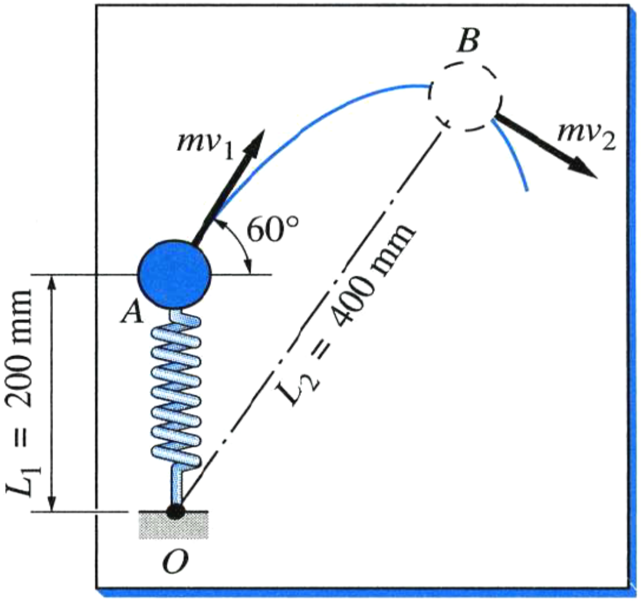
Since the spring force is central (it is always directed toward fixed point O), the angular momentum about O is conserved.

By setting the conventional $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coordinate system at point O, we get

$$(\boldsymbol{H}_O)_1 = \boldsymbol{r}_1 \times m\boldsymbol{v}_1 =$$

$$L_1 \boldsymbol{j} \times m(v_1 \cos 60^{\circ} \boldsymbol{i} + v_1 \sin 60^{\circ} \boldsymbol{j}) =$$

$$-mL_1 v_1 \cos 60^{\circ} \boldsymbol{k}$$



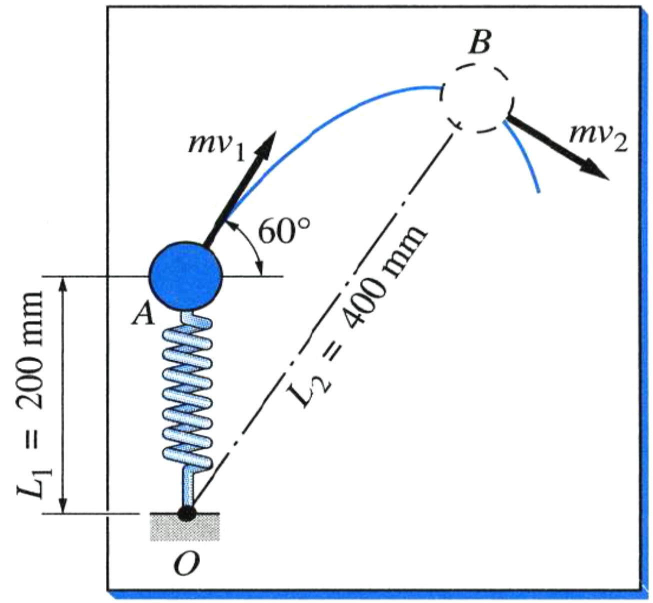
Note that $(\boldsymbol{H}_O)_1$ is a vector and (in our problem) it is perpendicular to the plane of motion.

Solution

The easiest to compute angular momentum at point B is by setting $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coordinate system in such a way so that $\boldsymbol{j}$ is directed from O to B and $\boldsymbol{i}$ is directed along the particle velocity vector $\boldsymbol{v}_2$ (since it is the maximal point path $\boldsymbol{r}_2$ is perpendicular to $\boldsymbol{v}_2$). Then

$$(\boldsymbol{H}_O)_2 = \boldsymbol{r}_2 \times m\boldsymbol{v}_2 =$$

$$L_1 \boldsymbol{j} \times m(v_2 \boldsymbol{i}) = -mL_2 v_2 \boldsymbol{k}$$



Since the angular momentum is conserved

$$(\boldsymbol{H}_O)_1 = (\boldsymbol{H}_O)_2 \quad \Rightarrow$$

$$L_1 v_1 \cos 60^{\circ} = L_2 v_2 \Rightarrow$$

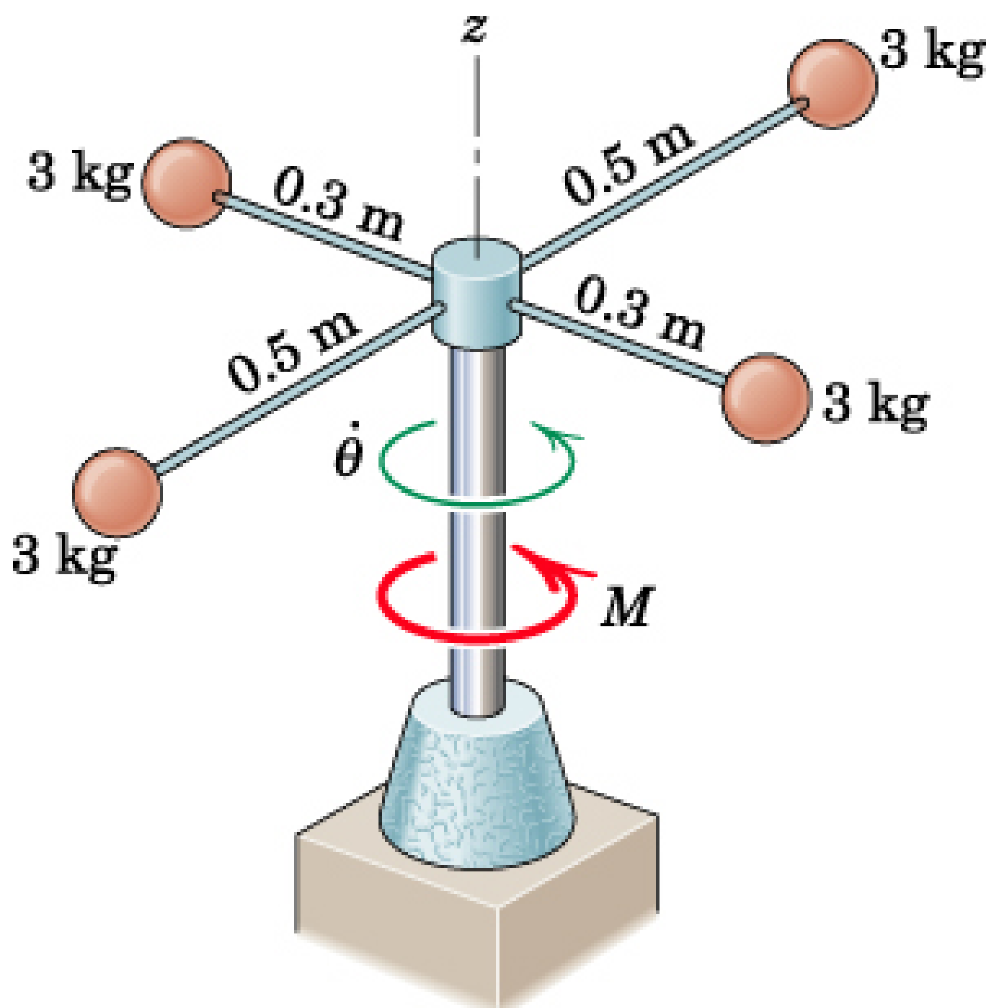
$$v_2 = L_1 v_1 \cos 60^{\circ} / L_2 = 0.5 \mathrm{m/s}$$

Finally

$$k = \frac{0.3(4 - v_2^2)}{(0.2)^2} = \frac{0.3(4 - 0.5^2)}{(0.2)^2} = 28.125 \, \mathrm{N/m}$$

Sample Problem 2

The four 3-kg balls are rigidly mounted to the rotating frame and shaft, which are initially at rest. If a constant torque M=20 N.m is applied to the shaft, calculate the time t to reach angular velocity 20 rad/s and find out the total angular displacement at the time t.



Solution

Angular impulse momentum principle. Here we have considered both cars as a single system.

$$(H_z)_1 + \int_0^t M(t)\mathrm{d}t = (H_z)_2$$

We can use polar coordinates to compute angular momentum.

$$H_z \boldsymbol{e}_z = \sum_{i=1}^{4} \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = \sum_{i=1}^{4} \rho_i \boldsymbol{e}_r \times m_i \dot{\theta}_i \boldsymbol{e}_\theta = \sum_{i=1}^{4} \rho_i m_i (\rho_i \dot{\theta}_i) \boldsymbol{e}_z = \sum_{i=1}^{4} m_i \rho_i^2 \dot{\theta} \boldsymbol{e}_z$$

Therefore

$$\int_0^t 20\,\mathrm{d}t = 3\times 0.3^2\,\dot{\theta} + 3\times 0.3^2\,\dot{\theta} + 3\times 0.5^2\,\dot{\theta} + 3\times 0.5^2\,\dot{\theta} = 2.04\,\dot{\theta}$$

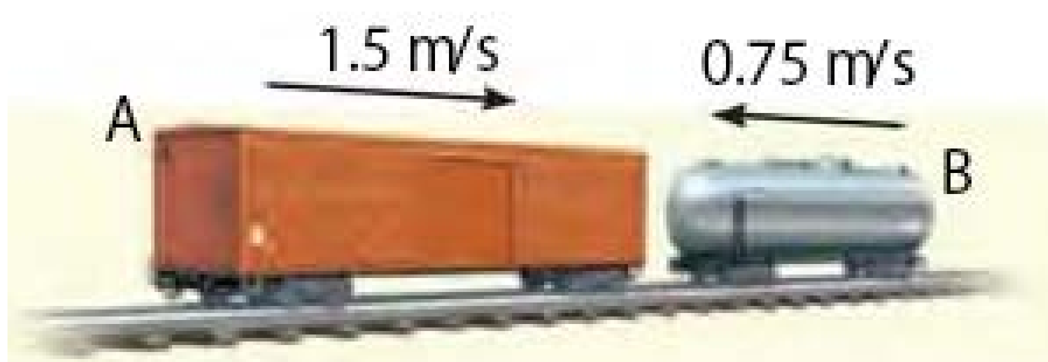
and

$$20t = 2.04\times 3\times 20 \;\Rightarrow\; t = 2.72\mathrm{s}$$

$$\theta = 20\times 2.72 = 54.4\mathrm{rad} = 3116.89^{\mathrm{o}}$$

Sample Problem 3

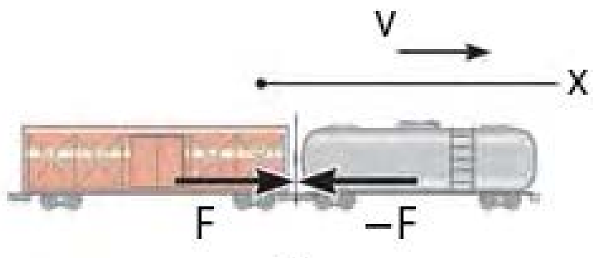
The 15000 kg boxcar A is coasting at 1.5 m/s on the horizontal track when it encounters a 12000 kg tank car B coasting at 0.75 m/s toward it. If the cars collide and couple together, determine (a) the speed of both cars just after the coupling, and (b) the average force between them if the coupling takes place in 0.8 s.



(a)

Solution

Part (a) Free-Body Diagram (only horizontal forces are shown). Here we have considered both cars as a single system. By inspection, momentum is conserved in the x direction since the coupling force F is internal to the system and will therefore cancel out. It is assumed both cars, when coupled, move at v2 in the positive x direction.



(b)

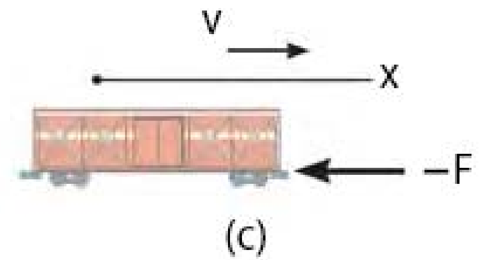
Conservation of Linear Momentum.

$$m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)v_2 \;\Rightarrow$$
$$15000 \times 1.5 - 12000 \times 0.75 = 27000\, v_2 \;\Rightarrow$$
$$v_2 = 0.5\mathrm{m/s}$$

Solution

Part (b). The average (impulsive) coupling force, Favg , can be determined by applying the principle of linear momentum to either one of the cars.

Free-Body Diagram. As shown below, by isolating the boxcar the coupling force is external to the car.



Principle of Impulse and Momentum.

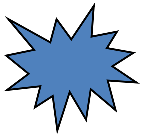
Since $\int F\mathrm{d}t = F_{avg}\Delta t = 0.8F_{avg}$ we have

$$m_A(v_A)_1 - \int F\mathrm{d}t = m_A v_2 \;\Rightarrow$$

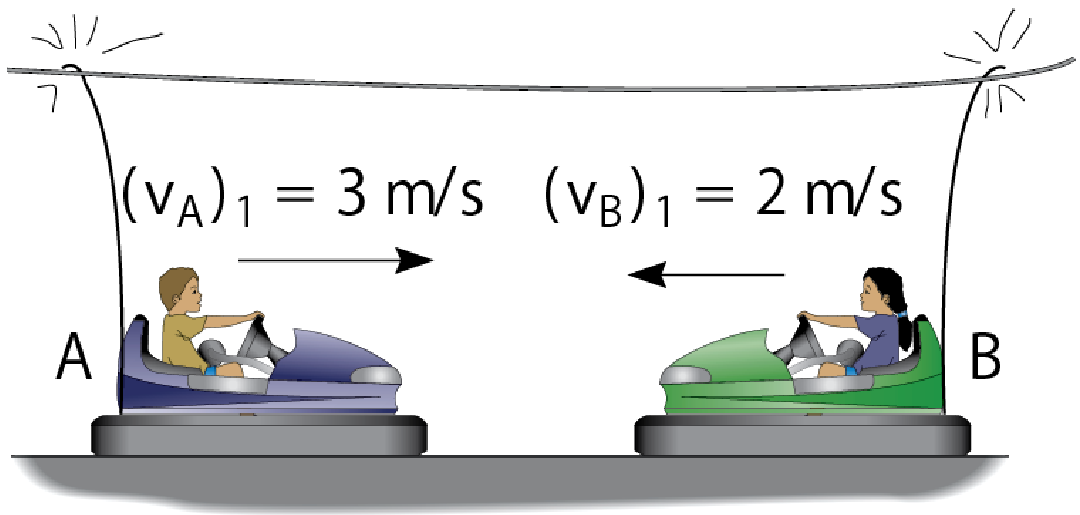
$$15000\times 1.5 - F_{avg}(0.8) = 15000\times 0.5 \;\Rightarrow$$

$$F_{avg} = 18750\,\mathrm{N}$$

Sample Problem 4

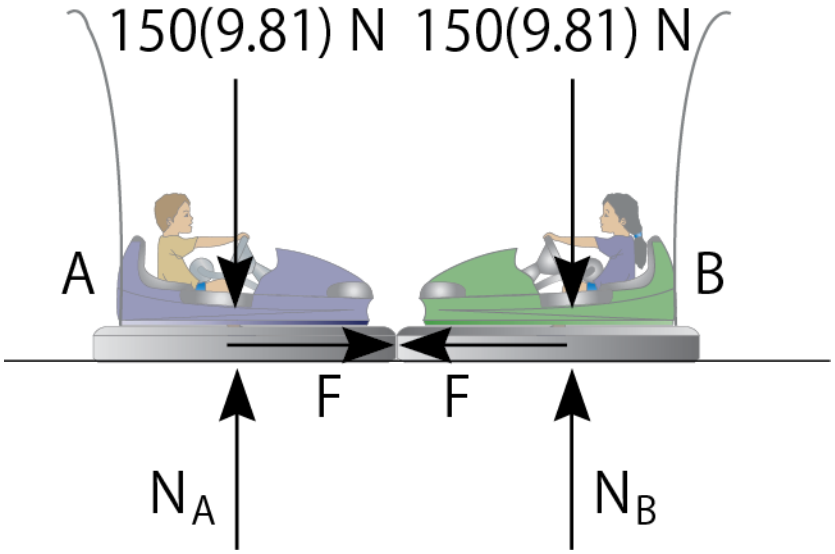


The bumper cars A and B each have a mass of 150 kg and are coasting with the velocities shown before they freely collide head on. If no energy is lost during the collision, determine their velocities after collision.



Solution

Free-Body Diagram. The cars will be considered as a single system. The free-body diagram is shown below.



Conservation of Linear Momentum.

$$m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2 \;\Rightarrow$$
$$150 \times 3 + 150 \times (-2) = 150(v_A)_2 + 150(v_B)_2 \;\Rightarrow$$
$$(v_A)_2 = 1 - (v_B)_2$$

Solution

Conservation of Energy. Since no energy is lost, the conservation of energy theorem gives

$$T_1 + V_1 = T_2 + V_2 \quad \Rightarrow$$

$$\frac{1}{2}m_A(v_A)_1^2 + \frac{1}{2}m_A(v_B)_1^2 + 0 = \frac{1}{2}m_A(v_A)_2^2 + \frac{1}{2}m_A(v_B)_2^2 + 0 \quad \Rightarrow$$

$$\frac{1}{2}150\times 3^2 + \frac{1}{2}150\times 2^2 = \frac{1}{2}150\,(v_A)_2^2 + \frac{1}{2}150\,(v_B)_2^2 \quad \Rightarrow$$

$$(v_A)_2^2 + (v_B)_2^2 = 13$$

Solving it together with $(v_A)_2 = 1 - (v_B)_2$, we get $(v_B)_2^2 - (v_B)_2 - 6 = 0$. Solving for the two roots, $(v_B)_2 = 3\mathrm{m/s}$ and $(v_B)_2 = -2\mathrm{m/s}$. Since $(v_B)_2 = -2\mathrm{m/s}$ refers to the velocity of B just before collision, then the velocity of B just after collision must be

$$(v_B)_2 = 3\mathrm{m/s}$$

Substituting this result into $(v_A)_2 = 1 - (v_B)_2$, we obtain

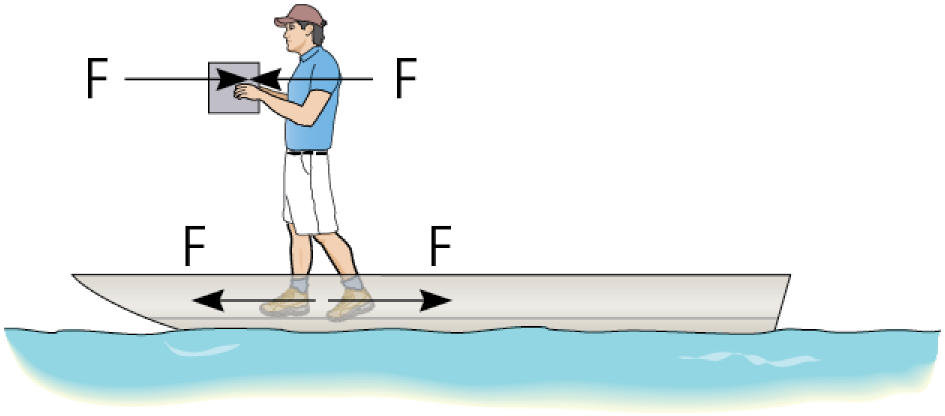
$$(v_A)_2 = -2\mathrm{m/s}$$

Sample Problem 5

The 80-kg man can throw the 20-kg box horizontally at 4 m/s when standing on the ground. If instead he firmly stands in the 120-kg boat and throws the box, as shown, determine how far the boat will move in three seconds. Neglect water resistance.

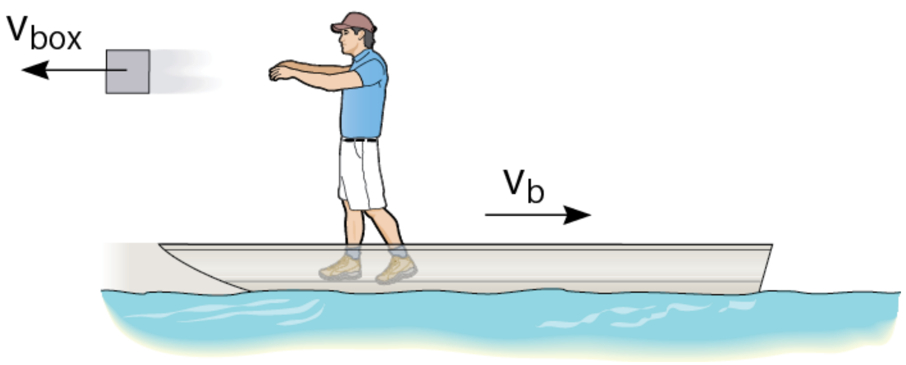
Solution

Free-Body Diagram. If the man, boat, and box are considered as a single system, the horizontal forces between the man and the boat and the man and the box become internal to the system, and so linear momentum will be conserved along the x axis.



Solution

Conservation of Linear Momentum. When writing the conservation of momentum equation, it is important that the velocities be measured from the same inertial coordinate system, assumed here to be fixed. From this coordinate system, we will assume that the boat and man go to the right while the box goes to the left, as shown in below.



Applying the conservation of linear momentum to the man, boat, box system,

$$0+0+0=(m_{man}+m_{boat})v_{boat}+m_{box}v_{box} \Rightarrow$$

$$0=(80+120)v_{boat}+20v_{box} \Rightarrow$$

$$v_{box}=-10v_{boat}$$

Solution

Kinematics. Since the velocity of the box *relative* to the man (and boat), $v_{box/boat}$ is known, then v_{boat} can also be related to v_{box} using the relative velocity equation:

$$v_{box} = v_{boat} + v_{box/boat} \Rightarrow$$
$$v_{box} = v_{boat} - 4$$

Solving this equation together with $v_{box} = -10\,v_{boat}$, we get

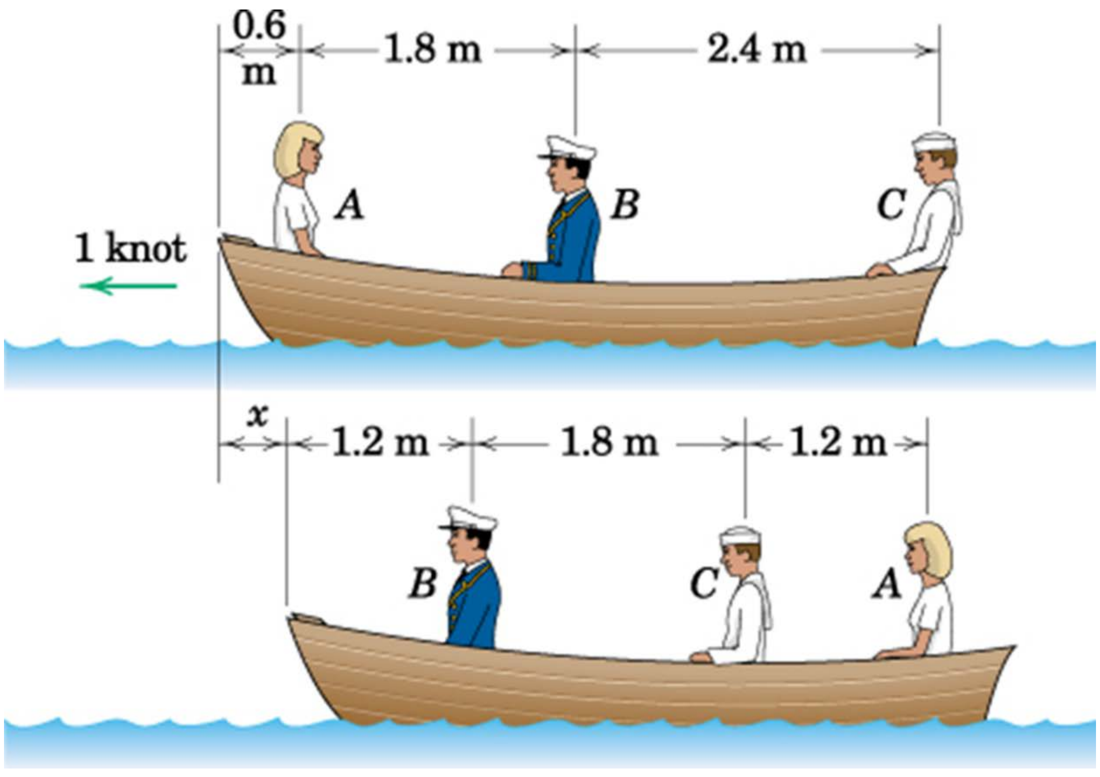
$$v_{boat} = 4/11\text{m}/\text{s} = 0.3636\,\text{m}/\text{s}$$
$$v_{box} = -40/11\text{m}/\text{s} = -3.636\,\text{m}/\text{s}$$

The displacement of the boat in three seconds is therefore

$$s_{boat} = v_{boat}\ t = 0.3636 \times 3 = 1.09\,\text{m}$$

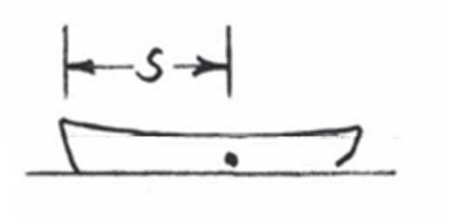
Sample Problem 6

The woman A, the captain B, and the sailor C weigh 60, 90, and 80 kg, respectively, and are sitting in the 150-kg skiff, which is gliding through the water with a speed of 1 knot. If the three people change their positions as shown in the second figure, find the distance x from the skiff to the position where it would have been if the people had not moved. Neglect any resistance to motion afforded by the water. Does the sequence or timing of the change in positions affect the final result?



Solution

With the neglect of hydraulic forces, the linear momentum of the system is conserved and therefore $v_1 = v_2 = 1\,\text{knot}$. Center of mass does not change position with respect to Newtonian system (inertial reference frame) moving with constant speed of 1 knot.



Center of mass. Let us s be the mass center of the skiff. Since the center of mass of the whole system does not change, we have

$$(m_A x_A + m_B x_B + m_C x_C + m_S\, x_S)_1 = (m_A x_A + m_B x_B + m_C x_C + m_S\, x_S)_2 \;\Rightarrow$$

$$60\times 0.6 + 90\times 2.4 + 80\times 4.8 + 150s =$$

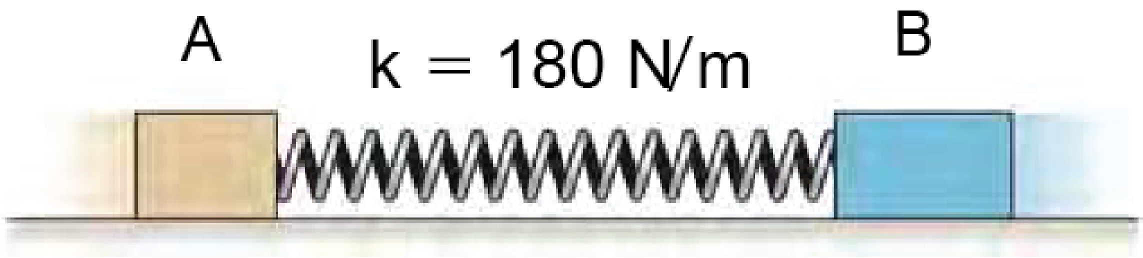
$$60\times(4.2 + x) + 90\times(1.2 + x) + 80\times(3 + x) + 150(x + s) \;\Rightarrow$$

$$x = 36/380 = 0.0947\,\text{m}$$

Timing and sequence of changed positions has no effect on final result since all forces are internal.

Sample Problem 7

Blocks A and B have masses of 40 kg and 60 kg, respectively. They are placed on a smooth surface and the spring connected between them is stretched 2 m. If they are released from rest, determine the speeds of both blocks the instant the spring becomes unstretched.



Solution

Conservation of Linear Momentum. Since the spring force is internal (and there are no external forces) , the linear momentum is conserved and for the two states under consideration (1: rest position with stretched spring, and 2: unstretched spring) we can write

$$(m_A v_A + m_B v_B)_1 = (m_A v_A + m_B v_B)_2 \;\Rightarrow$$

$$0 = (m_A v_A + m_B v_B)_2 \;\Rightarrow\; 0 = 40\,(v_A)_2 + 60\;(v_B)_2$$

Solution

Conservation of Energy. Next, this system is potential (conservative) and the total energy is conserved. Therefore,

$$T_1 + V_1 = T_2 + V_2 \quad \Rightarrow$$

$$0 + \frac{1}{2}ks^2 = \frac{1}{2}m_A(v_A)_2^2 + \frac{1}{2}m_A(v_B)_2^2 + 0 \quad \Rightarrow$$

$$\frac{1}{2}180\times 2^2 = \frac{1}{2}40(v_A)_2^2 + \frac{1}{2}60(v_B)_2^2 \quad \Rightarrow \quad 40(v_A)_2^2 + 60(v_B)_2^2 = 720$$

Solving it together with $40(v_A)_2 + 60\,(v_B)_2 = 0$, we obtain

$$(v_A)_2 = -3\sqrt{6/5} = -3.286\,\text{m/s}, \quad (v_B)_2 = 2\sqrt{6/5} = 2.191\,\text{m/s},$$

Note that the equations admit second solution: $(v_A)_2 = 3\sqrt{6/5}$, and $(v_B)_2 = -2\sqrt{6/5}$, but this second solution not physical because upon releasing the stretched system block A moves to the left and block B to the right. However, if we directed the axis of motion from right to left, then the second solution will be physical while the first will be not.