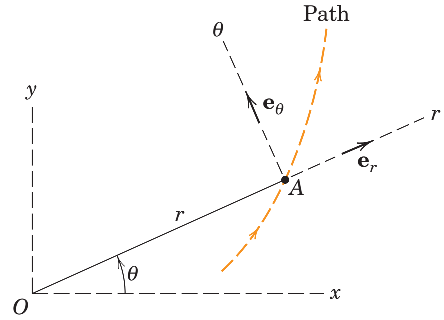
2/6 Polar Coordinates (r-θ)

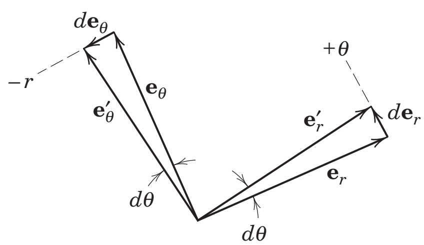
We now consider the third description of plane curvilinear motion, namely, polar coordinates where the particle is located by the radial distance r from a fixed point and by an angular measurement θ to the radial line. Polar coordinates are particularly useful when a motion is constrained through the control of a radial distance and an angular position or when an unconstrained motion is observed by measurements of a radial distance and an angular position.

Figure 2/13a shows the polar coordinates r and θ which locate a particle traveling on a curved path. An arbitrary fixed line, such as the x-axis, is used as a reference for the measurement of θ. Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are established in the positive r- and θ-directions, respectively. The position vector $\mathbf{r}$ to the particle at A has a magnitude equal to the radial distance r and a direction specified by the unit vector $\mathbf{e}_r$. Thus, we express the location of the particle at A by the vector

$$\mathbf{r} = r\mathbf{e}_r$$



(a)



(b)

Figure 2/13

Time Derivatives of the Unit Vectors

To differentiate this relation with respect to time to obtain $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \dot{\mathbf{v}}$, we need expressions for the time derivatives of both unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. We obtain $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$ in exactly the same way we derived $\dot{\mathbf{e}}_t$ in the preceding article. During time dt the coordinate directions rotate through the angle $d\theta$, and the unit vectors also rotate through the same angle from $\mathbf{e}_r$ and $\mathbf{e}_\theta$ to $\mathbf{e}'_r$ and $\mathbf{e}'_\theta$, as shown in Fig. 2/13b. We note that the vector change $d\mathbf{e}_r$ is in the plus θ-direction and that $d\mathbf{e}_\theta$ is in the minus r-direction. Because their magnitudes in the limit are equal to the unit vector as radius times the angle $d\theta$ in radians, we can write them as $d\mathbf{e}_r = \mathbf{e}_\theta\, d\theta$ and $d\mathbf{e}_\theta = -\mathbf{e}_r\, d\theta$. If we divide these equations by $d\theta$, we have

$$\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \qquad \text{and} \qquad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r$$

If, on the other hand, we divide them by dt, we have $d\mathbf{e}_r/dt = (d\theta/dt)\mathbf{e}_\theta$ and $d\mathbf{e}_\theta/dt = -(d\theta/dt)\mathbf{e}_r$, or simply

$$\boxed{\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \qquad \text{and} \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r} \tag{2/12}$$

Velocity

We are now ready to differentiate $\mathbf{r} = r\mathbf{e}_r$ with respect to time. Using the rule for differentiating the product of a scalar and a vector gives

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

With the substitution of $\dot{\mathbf{e}}_r$ from Eq. 2/12, the vector expression for the velocity becomes

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta} \tag{2/13}$$

where
$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$
$$v = \sqrt{{v_r}^2 + {v_\theta}^2}$$

The r-component of $\mathbf{v}$ is merely the rate at which the vector $\mathbf{r}$ stretches. The θ-component of $\mathbf{v}$ is due to the rotation of $\mathbf{r}$.

Acceleration

We now differentiate the expression for $\mathbf{v}$ to obtain the acceleration $\mathbf{a} = \dot{\mathbf{v}}$. Note that the derivative of $r\dot{\theta}\mathbf{e}_\theta$ will produce three terms, since all three factors are variable. Thus,

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r) + (\dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta)$$

Substitution of $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_\theta$ from Eq. 2/12 and collecting terms give

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta} \qquad \textbf{(2/14)}$$

where
$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$a = \sqrt{{a_r}^2 + {a_\theta}^2}$$

We can write the θ-component alternatively as

$$a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

which can be verified easily by carrying out the differentiation. This form for a_θ will be useful when we treat the angular momentum of particles in the next chapter.

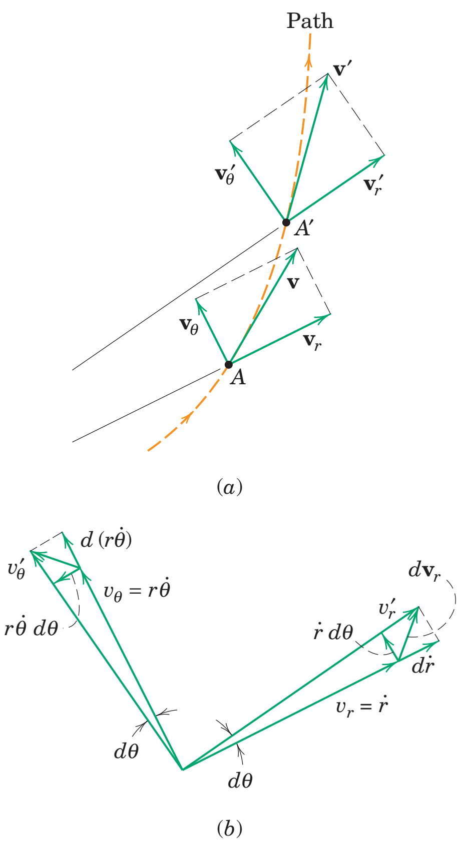


Figure 2/14

Geometric Interpretation

The terms in Eq. 2/14 can be best understood when the geometry of the physical changes can be clearly seen. For this purpose, Fig. 2/14a is developed to show the velocity vectors and their r- and θ-components at position A and at position A' after an infinitesimal movement. Each of these components undergoes a change in magnitude and direction as shown in Fig. 2/14b. In this figure we see the following changes:

(a) Magnitude Change of $\mathbf{v}_r$. This change is simply the increase in length of v_r or $dv_r = d\dot{r}$, and the corresponding acceleration term is $d\dot{r}/dt = \ddot{r}$ in the positive r-direction.

(b) Direction Change of $\mathbf{v}_r$. The magnitude of this change is seen from the figure to be $v_r\,d\theta = \dot{r}\,d\theta$, and its contribution to the acceleration becomes $\dot{r}\,d\theta/dt = \dot{r}\dot{\theta}$ which is in the positive θ-direction.

(c) Magnitude Change of $\mathbf{v}_\theta$. This term is the change in length of $\mathbf{v}_\theta$ or $d(r\dot{\theta})$, and its contribution to the acceleration is $d(r\dot{\theta})/dt = r\ddot{\theta} + \dot{r}\dot{\theta}$ and is in the positive θ-direction.

(d) Direction Change of $\mathbf{v}_\theta$. The magnitude of this change is $v_\theta\, d\theta = r\dot{\theta}\, d\theta$, and the corresponding acceleration term is observed to be $r\dot{\theta}(d\theta/dt) = r\dot{\theta}^2$ in the negative r-direction.

Collecting terms gives $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ as obtained previously. We see that the term $\ddot{r}$ is the acceleration which the particle would have along the radius in the absence of a change in θ. The term $-r\dot{\theta}^2$ is the normal component of acceleration if r were constant, as in circular motion. The term $r\ddot{\theta}$ is the tangential acceleration which the particle would have if r were constant, but is only a part of the acceleration due to the change in magnitude of $\mathbf{v}_\theta$ when r is variable. Finally, the term $2\dot{r}\dot{\theta}$ is composed of two effects. The first effect comes from that portion of the change in magnitude $d(r\dot{\theta})$ of v_θ due to the change in r, and the second effect comes from the change in direction of $\mathbf{v}_r$. The term $2\dot{r}\dot{\theta}$ represents, therefore, a combination of changes and is not so easily perceived as are the other acceleration terms.

Note the difference between the vector change $d\mathbf{v}_r$ in $\mathbf{v}_r$ and the change dv_r in the magnitude of v_r. Similarly, the vector change $d\mathbf{v}_\theta$ is not the same as the change dv_θ in the magnitude of v_θ. When we divide these changes by dt to obtain expressions for the derivatives, we see clearly that the magnitude of the derivative $|d\mathbf{v}_r/dt|$ and the derivative of the magnitude dv_r/dt are *not* the same. Note also that a_r is not $\dot{v}_r$ and that a_θ is not $\dot{v}_\theta$.

The total acceleration $\mathbf{a}$ and its components are represented in Fig. 2/15. If $\mathbf{a}$ has a component normal to the path, we know from our analysis of n- and t-components in Art. 2/5 that the sense of the n-component *must* be toward the center of curvature.

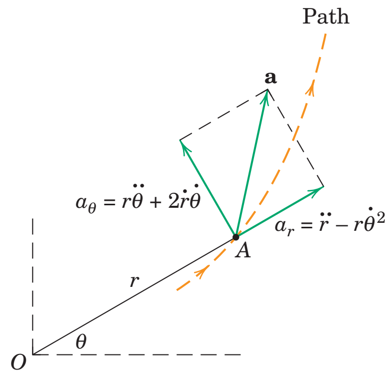


Figure 2/15

Circular Motion

For motion in a circular path with r constant, the components of Eqs. 2/13 and 2/14 become simply

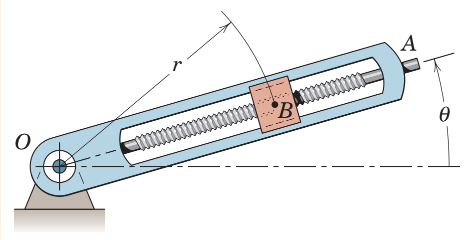
$$v_r = 0 \qquad v_\theta = r\dot{\theta}$$
$$a_r = -r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta}$$

This description is the same as that obtained with n- and t-components, where the θ- and t-directions coincide but the positive r-direction is in the negative n-direction. Thus, $a_r = -a_n$ for circular motion centered at the origin of the polar coordinates.

The expressions for a_r and a_θ in scalar form can also be obtained by direct differentiation of the coordinate relations $x = r \cos \theta$ and $y = r \sin \theta$ to obtain $a_x = \ddot{x}$ and $a_y = \ddot{y}$. Each of these rectangular components of acceleration can then be resolved into r- and θ-components which, when combined, will yield the expressions of Eq. 2/14.

Sample Problem 2/9

Rotation of the radially slotted arm is governed by $\theta = 0.2t + 0.02t^3$, where θ is in radians and t is in seconds. Simultaneously, the power screw in the arm engages the slider B and controls its distance from O according to $r = 0.2 + 0.04t^2$, where r is in meters and t is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when $t = 3$ s.



Solution. The coordinates and their time derivatives which appear in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for $t = 3$ s. ①

$$r = 0.2 + 0.04t^2 \qquad r_3 = 0.2 + 0.04(3^2) = 0.56 \text{ m}$$

$$\dot{r} = 0.08t \qquad \dot{r}_3 = 0.08(3) = 0.24 \text{ m/s}$$

$$\ddot{r} = 0.08 \qquad \ddot{r}_3 = 0.08 \text{ m/s}^2$$

$$\theta = 0.2t + 0.02t^3 \qquad \theta_3 = 0.2(3) + 0.02(3^3) = 1.14 \text{ rad}$$

$$\text{or } \theta_3 = 1.14(180/\pi) = 65.3°$$

$$\dot{\theta} = 0.2 + 0.06t^2 \qquad \dot{\theta}_3 = 0.2 + 0.06(3^2) = 0.74 \text{ rad/s}$$

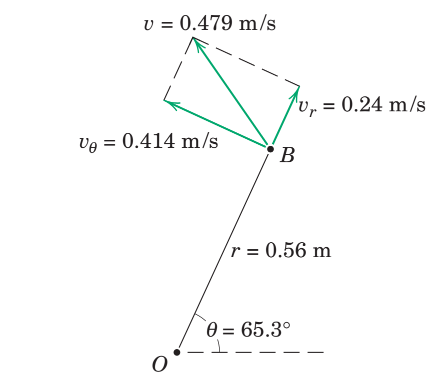
$$\ddot{\theta} = 0.12t \qquad \ddot{\theta}_3 = 0.12(3) = 0.36 \text{ rad/s}^2$$

The velocity components are obtained from Eq. 2/13 and for $t = 3$ s are

$$[v_\theta = r\dot{\theta}] \qquad v_r = 0.24 \text{ m/s}$$

$$[v_\theta = r\dot{\theta}] \qquad v_\theta = 0.56(0.74) = 0.414 \text{ m/s}$$

$$[v = \sqrt{v_r^2 + v_\theta^2}] \qquad v = \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s} \qquad \textit{Ans.}$$



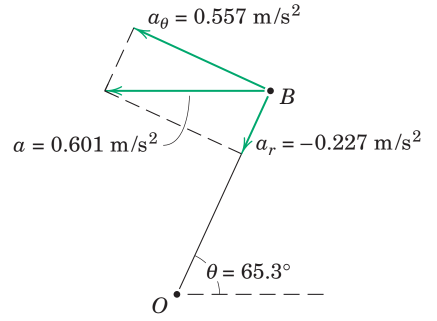
The velocity and its components are shown for the specified position of the arm.

The acceleration components are obtained from Eq. 2/14 and for $t = 3$ s are

$$[a_r = \ddot{r} - r\dot{\theta}^2] \qquad a_r = 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2$$

$$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] \qquad a_\theta = 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2$$

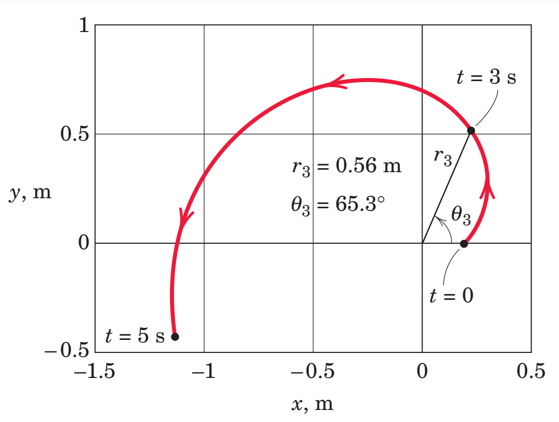
$$[a = \sqrt{a_r^2 + a_\theta^2}] \qquad a = \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2 \qquad \textit{Ans.}$$



The acceleration and its components are also shown for the 65.3° position of the arm.

Plotted in the final figure is the path of the slider B over the time interval $0 \le t \le 5$ s. This plot is generated by varying t in the given expressions for r and θ. Conversion from polar to rectangular coordinates is given by

$$x = r \cos \theta \qquad y = r \sin \theta$$

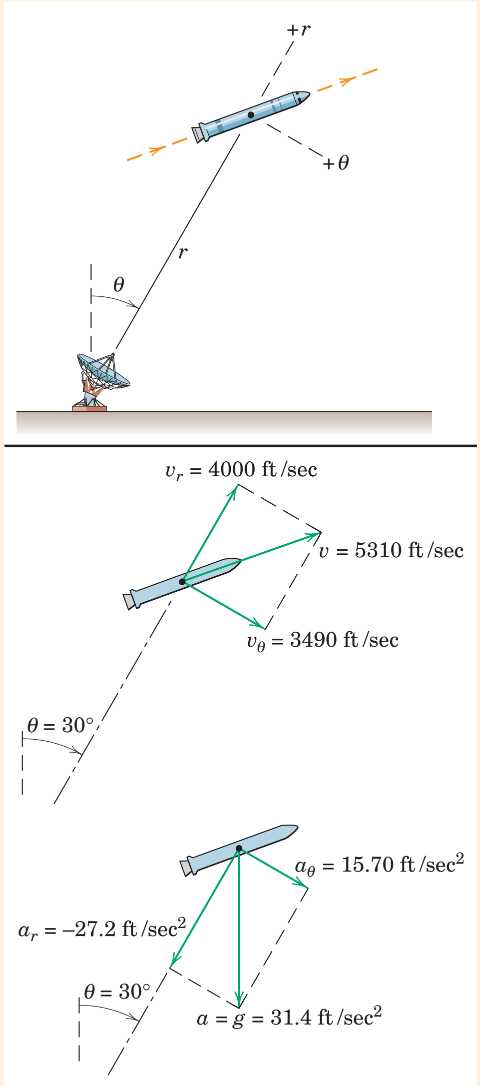


Helpful Hint

① We see that this problem is an example of constrained motion where the center B of the slider is mechanically constrained by the rotation of the slotted arm and by engagement with the turning screw.

Sample Problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when $\theta = 30°$, the tracking data give $r = 25(10^4)$ ft, $\dot{r} = 4000$ ft/sec, and $\dot{\theta} = 0.80$ deg/sec. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is 31.4 ft/sec² vertically down. For these conditions determine the velocity v of the rocket and the values of $\ddot{r}$ and $\ddot{\theta}$.



Solution. The components of velocity from Eq. 2/13 are

$[v_r = \dot{r}]$ $\qquad v_r = 4000$ ft/sec

① $[v_\theta = r\dot{\theta}]$ $\qquad v_\theta = 25(10^4)(0.80)\left(\dfrac{\pi}{180}\right) = 3490$ ft/sec

$[v = \sqrt{v_r^{\,2} + v_\theta^{\,2}}]$ $\qquad v = \sqrt{(4000)^2 + (3490)^2} = 5310$ ft/sec $\qquad$ *Ans.*

Since the total acceleration of the rocket is $g = 31.4$ ft/sec² down, we can easily find its r- and θ-components for the given position. As shown in the figure, they are

② $$a_r = -31.4 \cos 30° = -27.2 \text{ ft/sec}^2$$

$$a_\theta = 31.4 \sin 30° = 15.70 \text{ ft/sec}^2$$

We now equate these values to the polar-coordinate expressions for a_r and a_θ which contain the unknowns $\ddot{r}$ and $\ddot{\theta}$. Thus, from Eq. 2/14

③ $[a_r = \ddot{r} - r\dot{\theta}^2]$ $\qquad -27.2 = \ddot{r} - 25(10^4)\left(0.80\,\dfrac{\pi}{180}\right)^2$

$$\ddot{r} = 21.5 \text{ ft/sec}^2 \qquad \textit{Ans.}$$

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\qquad 15.70 = 25(10^4)\ddot{\theta} + 2(4000)\left(0.80\,\dfrac{\pi}{180}\right)$

$$\ddot{\theta} = -3.84(10^{-4}) \text{ rad/sec}^2 \qquad \textit{Ans.}$$

Helpful Hints

① We observe that the angle θ in polar coordinates need not always be taken positive in a counterclockwise sense.

② Note that the r-component of acceleration is in the negative r-direction, so it carries a minus sign.

③ We must be careful to convert $\dot{\theta}$ from deg/sec to rad/sec.