Vector Spaces #2

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Outline

- Vector Spaces
- Subspaces of Vector Spaces
- Linear Combination of Vectors
- Spanning Sets
- Linear Independence

สัจพจน์ (Axiom)

- หมายถึง ข้อความที่ยอมรับว่าเป็นจริงโดยไม่ต้องพิสูจน์ ซึ่งตรงข้ามกับคำว่า “ทฤษฎีบท” ซึ่งจะถูกยอมรับว่าเป็นจริงได้ก็ต่อเมื่อมีการพิสูจน์
- ดังนั้นสัจพจน์จึงถูกใช้เป็นจุดเริ่มต้นในการพิสูจน์ทางคณิตศาสตร์ และทฤษฎีบททุกอัน จะต้องอนุมาน (inference) มายังสัจพจน์ได้ เช่น
    - เราสามารถลากเส้นตรงผ่านจุดสองจุดได้
    - เราสามารถขยายส่วนของเส้นตรงไปเป็นเส้นตรงได้เส้นเดียว
    - มุมฉากทุกมุมย่อมเท่ากัน
    - สมการที่ถูกบวกด้วยค่าเท่ากันทั้งสองข้างก็ยังเป็นสมการอยู่

ที่มา: https://th.wikipedia.org/wiki/%E0%B8%AA%E0%B8%B1%E0%B8%88%E0%B8%9E%E0%B8%88%E0%B8%99%E0%B9%8C

เซต (Set)

- หมายถึง กลุ่มของสิ่งใดสิ่งหนึ่ง
- เซต = {สมาชิกของเซตตัวที่ 1, สมาชิกของเซตตัวที่ 2, ... }
- สมาชิกของเซต จะอยู่ภายใต้เครื่องหมาย ปีกกา { }

  เช่น เซต A คือของจำนวนคู่ที่เป็นบวก จะได้ว่า

  $$A = \{2, 4, 6, \dots\}$$

  การเป็นสมาชิกของเซต ใช้เครื่องหมาย ∈ เช่น
- เซตที่เป็นที่รู้จัก เช่น
  - เซตของจำนวนจริง แทนด้วย R
  - เซตของคู่ลำดับในระนาบ แทนด้วย R^2
  - เซตของจำนวนเต็มบวก แทนด้วย I^+
  - เซตของเมทริกซ์ ขนาด $m \times n$ แทนด้วย $M_{m\times n}$

Vector Spaces

- *Any* set that satisfies these properties (or **axioms**) is called a **vector space**, and the objects in the set are called **vectors**.

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space.**

Addition:	
1. $\mathbf{u} + \mathbf{v}$ is in V.	**Closure under addition**
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	**Commutative property**
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	**Associative property**
4. V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	**Additive identity**
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	**Additive inverse**
Scalar Multiplication:	
6. $c\mathbf{u}$ is in V. ปรภูมิเวกเตอร์ V	**Closure under scalar multiplication**
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	**Distributive property**
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	**Distributive property**
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	**Associative property**
10. $1(\mathbf{u}) = \mathbf{u}$	**Scalar identity**

R^2 with the Standard Operations Is a Vector Space

The set of all ordered pairs of real numbers R^2 with the standard operations is a vector space. To verify this, look back at Theorem 4.1. Vectors in this space have the form

$$\mathbf{v} = (v_1, v_2).$$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in the plane.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

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R^n with the Standard Operations Is a Vector Space

The set of all ordered n-tuples of real numbers R^n with the standard operations is a vector space. This is verified by Theorem 4.2. Vectors in this space are of the form

$$\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n).$$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

The Vector Space of All 2 × 3 Matrices

Show that the set of all 2×3 matrices with the operations of matrix addition and scalar multiplication is a vector space.

SOLUTION If A and B are 2×3 matrices and c is a scalar, then $A + B$ and cA are also 2×3 matrices. The set is, therefore, closed under matrix addition and scalar multiplication. Moreover, the other eight vector space axioms follow directly from Theorems 2.1 and 2.2 (see Section 2.2). You can conclude that the set is a vector space. Vectors in this space have the form

$$\mathbf{a} = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

$$u = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix}_{2\times 3} \qquad v = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{bmatrix}_{2\times 3}$$

$$u + v = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} & u_{13}+v_{13} \\ u_{21}+v_{21} & u_{22}+v_{22} & u_{23}+v_{23} \end{bmatrix}_{2\times 3}$$

(ยังเป็น Matrix 2×3 เหมือนเดิม)

The Vector Space of All Polynomials of Degree 2 or Less

$\{x^2+2x, x+1, x, \dots\}$

มีทั้งดีกรี 2, 1

Let P_2 be the set of all polynomials of the form

$$p(x) = a_2x^2 + a_1x + a_0,$$

where $a_0, a_1,$ and a_2 are real numbers. The *sum* of two polynomials $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0$ is defined in the usual way by

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0),$$

and the *scalar multiple* of $p(x)$ by the scalar c is defined by

$$cp(x) = ca_2x^2 + ca_1x + ca_0.$$

Show that P_2 is a vector space.

SOLUTION Verification of each of the ten vector space axioms is a straightforward application of the properties of real numbers. For instance, because the set of real numbers is closed under addition, it follows that $a_2 + b_2$, $a_1 + b_1$, and $a_0 + b_0$ are real numbers, and

$p(x)+q(x)=q(x)+p(x)$

$$p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$

is in the set P_2 because it is a polynomial of degree 2 or less. P_2 is closed under addition. Similarly, you can use the fact that the set of real numbers is closed under multiplication to show that P_2 is closed under scalar multiplication. To verify the commutative axiom of addition, write

$p(x)=x^2$

$q(x)=-x^2+x+1$

$p(x)+q(x)=x+1$

ยังอยู่ใน degree 2 or less

$$\begin{aligned} p(x) + q(x) &= (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) \\ &= (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \\ &= (b_2 + a_2)x^2 + (b_1 + a_1)x + (b_0 + a_0) \\ &= (b_2x^2 + b_1x + b_0) + (a_2x^2 + a_1x + a_0) \\ &= q(x) + p(x). \end{aligned}$$

Can you see where the commutative property of addition of real numbers was used? The zero vector in this space is the zero polynomial given by $\mathbf{0}(x) = 0x^2 + 0x + 0$, for all x. Try verifying the other vector space axioms. You may then conclude that P_2 is a vector space.

Properties of Scalar Multiplication

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

1. $0\mathbf{v} = \mathbf{0}$
2. $c\mathbf{0} = \mathbf{0}$
3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$

PROOF To prove these properties, you are restricted to using the ten vector space axioms. For instance, to prove the second property, note from axiom 4 that $\mathbf{0} = \mathbf{0} + \mathbf{0}$. This allows you to write the steps below.

$c\mathbf{0} = c(\mathbf{0} + \mathbf{0})$	Additive identity
$c\mathbf{0} = c\mathbf{0} + c\mathbf{0}$	Left distributive property
$c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$	Add $-c\mathbf{0}$ to both sides.
$c\mathbf{0} + (-c\mathbf{0}) = c\mathbf{0} + [c\mathbf{0} + (-c\mathbf{0})]$	Associative property
$\mathbf{0} = c\mathbf{0} + \mathbf{0}$	Additive inverse
$\mathbf{0} = c\mathbf{0}$	Additive identity

To prove the third property, suppose that $c\mathbf{v} = \mathbf{0}$. To show that this implies either $c = 0$ or $\mathbf{v} = \mathbf{0}$, assume that $c \neq 0$. (If $c = 0$, you have nothing more to prove.) Now, because $c \neq 0$, you can use the reciprocal $1/c$ to show that $\mathbf{v} = \mathbf{0}$, as follows.

$$\mathbf{v} = 1\mathbf{v} = \left(\frac{1}{c}\right)(c)\mathbf{v} = \frac{1}{c}(c\mathbf{v}) = \frac{1}{c}(\mathbf{0}) = \mathbf{0}$$

Note that the last step uses Property 2 (the one you just proved). The proofs of the first and fourth properties are left as exercises.

The Set of Integers Is Not a Vector Space

The set of all integers (with the standard operations) does not form a vector space because it is not closed under scalar multiplication. For example,

$$\underset{\text{Scalar}}{\tfrac{1}{2}}(\underset{\text{Integer}}{1}) = \underset{\text{Noninteger}}{\tfrac{1}{2}}.$$

ไม่ได้อยู่ในset ของ integer

vector space

cu is in V.

$cv \in V$

The Set of Second-Degree Polynomials Is Not a Vector Space

The set of all second-degree polynomials is not a vector space because it is not closed under addition. To see this, consider the second-degree polynomials

$$p(x) = x^2 \qquad \text{and} \qquad q(x) = -x^2 + x + 1,$$

$\{x^2, x^2+1, x^2+2x+3, \ldots\}$

whose sum is the first-degree polynomial

$$p(x) + q(x) = x + 1.$$

Set of P_2

Not in vector space

A Set That Is Not a Vector Space

R^2 เซ็ตของคู่อันดับ ในระนาบ

?

Let $V = R^2$, the set of all ordered pairs of real numbers, with the standard operation of addition and the *nonstandard* definition of scalar multiplication listed below.

$$c(x_1, x_2) = (cx_1, 0)$$ ถูกบังคับไว้

Show that V is not a vector space.

SOLUTION In this example, the operation of scalar multiplication is not the standard one. For instance, the product of the scalar 2 and the ordered pair (3, 4) does not equal (6, 8). Instead, the second component of the product is 0,

$$2(3, 4) = (2 \cdot 3, 0) = (6, 0).$$

This example is interesting because it actually satisfies the first nine axioms of the definition of a vector space (try showing this). The tenth axiom is where you get into trouble. In attempting to verify that axiom, the nonstandard definition of scalar multiplication gives you

$\boxed{1 \cdot u = u}$

$$1(1, 1) = (1, 0) \neq (1, 1).$$ ไม่เท่ากัน

The tenth axiom is not verified and the set (together with the two operations) is not a vector space.

Subspaces of Vector Spaces

ปริภูมิย่อย

- In most important applications in linear algebra, vector spaces occur as **subspaces** of larger spaces.
- For instance, you will see that the solution set of a homogeneous system of linear equations in n variables is a subspace of R^n.

A nonempty subset W of a vector space V is called a **subspace** of V if W is a vector space under the operations of addition and scalar multiplication defined in V.

REMARK: Note that if W is a subspace of V, it must be closed under the operations inherited from V.

Homogeneous system

$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$

$a_{21}x_1 + a_{21}x_2 + \dots a_{2n}x_n = b_2$

$b_1 = b_2 = \dots = b_m = 0$

A Subspace of R^3

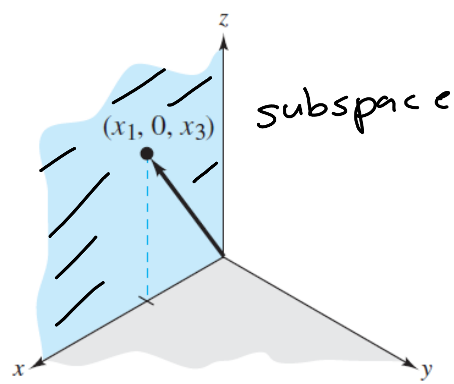
อยู่ใน W

$u = (x_1, 0, x_3)$
$v = (y_1, 0, y_3)$

Show that the set $W = \{(x_1, 0, x_3): x_1$ and x_3 are real numbers$\}$ is a subspace of R^3 with the standard operations.

SOLUTION The set W is nonempty because it contains the zero vector $(0, 0, 0)$.

Graphically, the set W can be interpreted as simply the xz-plane, as shown in Figure 4.9. The set W is closed under addition because the sum of any two vectors in the xz-plane must also lie in the xz-plane. That is, if $(x_1, 0, x_3)$ and $(y_1, 0, y_3)$ are in W, then their sum $(x_1 + y_1, 0, x_3 + y_3)$ is also in W (because the second component is zero). Similarly, to see that W is closed under scalar multiplication, let $(x_1, 0, x_3)$ be in W and let c be a scalar. Then $c(x_1, 0, x_3) = (cx_1, 0, cx_3)$ has zero as its second component and must be in W. The other eight vector space axioms can be verified as well, and these verifications are left to you.



subspace = XZ plane

Figure 4.9

Test for a Subspace

ถ้า 2 อย่างเป็นจริง เป็น subspace

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

PROOF The proof of this theorem in one direction is straightforward. That is, if W is a subspace of V, then W is a vector space and must be closed under addition and scalar multiplication.

To prove the theorem in the other direction, assume that W is closed under addition and scalar multiplication. Note that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in W, then they are also in V. Consequently, vector space axioms 2, 3, 7, 8, 9, and 10 are satisfied automatically. Because W is closed under addition and scalar multiplication, it follows that for any $\mathbf{v}$ in W and scalar $c = 0$,

$$c\mathbf{v} = \mathbf{0}$$

and

$$(-1)\mathbf{v} = -\mathbf{v}$$

both lie in W, which satisfies axioms 4 and 5.

The Subspace of $M_{2,2}$

transpose แล้วยังเท่าเดิม

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$, with the standard operations of matrix addition and scalar multiplication.

SOLUTION Recall that a matrix is called *symmetric* if it is equal to its own transpose. Because $M_{2,2}$ is a vector space, you only need to show that W (a subset of $M_{2,2}$) satisfies the conditions of Theorem 4.5. Begin by observing that W is *nonempty*. W is closed under addition because $A_1 = A_1^T$ and $A_2 = A_2^T$, which implies that

ข้อ 1. ลองบวกกัน

$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2.$$

So, if A_1 and A_2 are symmetric matrices of order 2, then so is $A_1 + A_2$. Similarly, W is closed under scalar multiplication because $A = A^T$ implies that $(cA)^T = cA^T = cA$. If A is a symmetric matrix of order 2, then so is cA.

The Set of Singular Matrices Is Not a Subspace of $M_{n,n}$

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2,2}$ with the standard operations.

SOLUTION By Theorem 4.5, you can show that a subset W is not a subspace by showing that W is empty, W is not closed under addition, or W is not closed under scalar multiplication. For this particular set, W is nonempty and closed under scalar multiplication, but it is not closed under addition. To see this, let A and B be

$\det(A)=0$
$\det(B)=0$
(ไม่มี Inverse)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\det(A)=0$

Then A and B are both singular (noninvertible), but their sum

$\det \neq 0$

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

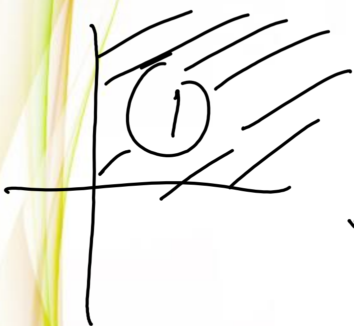
พอบวกกันแล้วเป็น nonsingular เลย เลยไม่ closed

is nonsingular (invertible). So W is not closed under addition, and by Theorem 4.5 you can conclude that it is not a subspace of $M_{2,2}$.

$$2A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \det(2A) = 0$$

The Set of First Quadrant Vectors Is Not a Subspace of R^2

$x \geq 0, y \geq 0$



Show that $W = \{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2.

multiplied by c

SOLUTION This set is nonempty and closed under addition. It is not, however, closed under scalar multiplication. To see this, note that $(1, 1)$ is in W, but the scalar multiple

$$(-1)(1, 1) = (-1, -1)$$ → ไม่อยู่ Q, ละ

is not in W. So W is not a subspace of R^2.

$(2, 3) + (4, 5) = (6, 8)$

Linear Combination of Vectors

- This section begins to develop procedures for representing each vector in a vector space as a **linear combination** of a select number of vectors in the space.

A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

ผลรวมเชิงเส้น

where $c_1, c_2, \ldots, c_k$ are scalars.

Examples of Linear Combinations

(a) For the set of vectors in R^3,

$$S = \{\overset{\mathbf{v}_1}{(1, 3, 1)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(1, 0, -5)}\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ because

$$\begin{aligned} \mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 &= 3(0, 1, 2) + (1, 0, -5) \\ &= (1, 3, 1). \end{aligned}$$

(b) For the set of vectors in $M_{2,2}$,

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}, \overset{\mathbf{v}_4}{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}} \right\},$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ because

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_2 + 2\mathbf{v}_3 - \mathbf{v}_4 \\ &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

Finding a Linear Combination

Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set S.

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 3)}, \overset{\mathbf{v}_2}{(0, 1, 2)}, \overset{\mathbf{v}_3}{(-1, 0, 1)}\}$$

SOLUTION You need to find scalars c_1, c_2, and c_3 such that

$$\begin{aligned}(1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)\\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

Equating corresponding components yields the system of linear equations below.

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 1\\ 2c_1 + \ c_2 \qquad\ \ &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

Using Gauss-Jordan elimination, you can show that this system has an infinite number of solutions, each of the form

$$c_1 = 1 + t, \qquad c_2 = -1 - 2t, \qquad c_3 = t.$$

To obtain one solution, you could let $t = 1$. Then $c_3 = 1$, $c_2 = -3$, and $c_1 = 2$, and you have

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Other choices for t would yield other ways to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Finding a Linear Combination

If possible, write the vector $\mathbf{w} = (1, -2, 2)$ as a linear combination of vectors in the set S from Example 2.

SOLUTION Following the procedure from Example 2 results in the system

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 1 \\ 2c_1 + c_2 \qquad\quad &= -2 \\ 3c_1 + 2c_2 + c_3 &= 2. \end{aligned}$$

IS Aug ?

The augmented matrix of this system row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the third row you can conclude that the system of equations is inconsistent, and that means that there is no solution. Consequently, **w** *cannot* be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Spanning Sets เซตแผ่ทั่ว

- If every vector in a vector space can be written as a linear combination of vectors in a set S, then S is called a **spanning set** of the vector space.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space V. The set S is called a **spanning set** of V if *every* vector in V can be written as a linear combination of vectors in S. In such cases it is said that S **spans** V.

Examples of Spanning Sets

(a) The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

(b) The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial function $p(x) = a + bx + cx^2$ in P_2 can be written as

พหุนาม degree 2

$$\begin{aligned} p(x) &= a(1) + b(x) + c(x^2) \\ &= a + bx + cx^2. \end{aligned}$$

A Spanning Set of R^3

Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3.

SOLUTION Let $\mathbf{u} = (u_1, u_2, u_3)$ be *any* vector in R^3. You need to find scalars c_1, c_2, and c_3 such that

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

This vector equation produces the system

$$\begin{aligned} c_1 - 2c_3 &= u_1 \\ 2c_1 + c_2 &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3. \end{aligned}$$

The coefficient matrix for this system has a nonzero determinant, and it follows from the list of equivalent conditions given in Section 3.3 that the system has a unique solution. So, any vector in R^3 can be written as a linear combination of the vectors in S, and you can conclude that the set S spans R^3.

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$

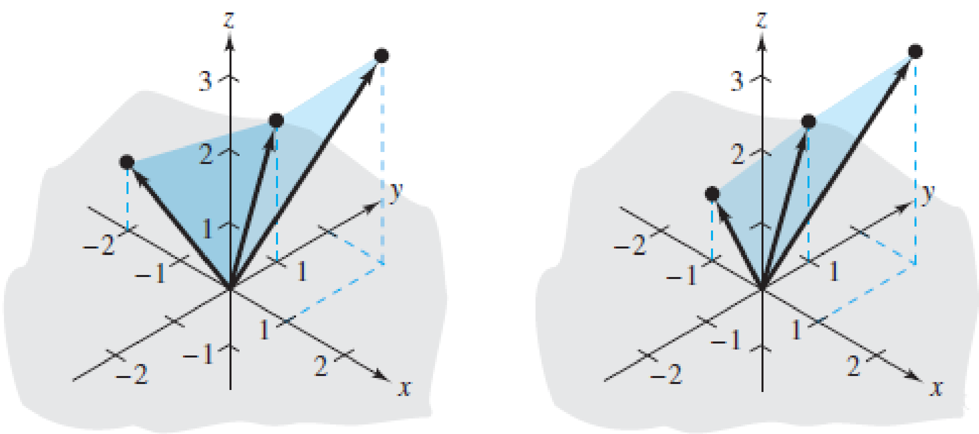
A Set That Does Not Span R^3

From Example 3 you know that the set

$$S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$$

does not span R^3 because $\mathbf{w} = (1, -2, 2)$ is in R^3 and cannot be expressed as a linear combination of the vectors in S.

coplanar



$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$
The vectors in S_1 do not lie in a common plane.

$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$
The vectors in S_2 lie in a common plane.

Figure 4.16

The Span of a Set

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are real numbers}\}.$$

The span of S is denoted by span(S) or span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. If span($S$) $= V$, it is said that V is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, or that S **spans** V.

span ไม่ออกสอบ
ผลรวมเชิงเส้น ออก

Span(S) Is a Subspace of V

If $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$ is a set of vectors in a vector space V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span(S).

PROOF To show that span(S), the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k$, is a subspace of V, show that it is closed under addition and scalar multiplication. Consider any two vectors $\mathbf{u}$ and $\mathbf{v}$ in span(S),

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_k\mathbf{v}_k$$
$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdot \cdot \cdot + d_k\mathbf{v}_k,$$

where $c_1, c_2, . . . , c_k$ and $d_1, d_2, . . . , d_k$ are scalars. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdot \cdot \cdot + (c_k + d_k)\mathbf{v}_k$$

and

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2 + \cdot \cdot \cdot + (cc_k)\mathbf{v}_k,$$

which means that $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are also in span(S) because they can be written as linear combinations of vectors in S. So, span(S) is a subspace of V. It is left to you to prove that span(S) is the smallest subspace of V that contains S.

ออกสอบ

Linear Dependence and Linear Independence

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

Result สุดท้ายเป็น 0

has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$. If there are also nontrivial solutions, then S is called **linearly dependent.**

Examples of Linearly Dependent Sets

(a) The set $S = \{(1, 2), (2, 4)\}$ in R^2 is linearly dependent because $c_1, c_2 \neq 0$

$$-2(1, 2) + (2, 4) = (0, 0).$$

(b) The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in R^2 is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

(c) The set $S = \{(0, 0), (1, 2)\}$ in R^2 is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

อิสระเชิงเส้น / ขึ้นอยู่แก่กันเชิงเส้น

Testing for Linear Independence

Determine whether the set of vectors in R^3 is linearly independent or linearly dependent.

$$S = \{\underset{\mathbf{v}_1}{(1, 2, 3)}, \underset{\mathbf{v}_2}{(0, 1, 2)}, \underset{\mathbf{v}_3}{(-2, 0, 1)}\}$$

SOLUTION To test for linear independence or linear dependence, form the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

If the only solution of this equation is

$$c_1 = c_2 = c_3 = 0,$$

then the set S is linearly independent. Otherwise, S is linearly dependent. Expanding this equation, you have

$$\begin{aligned} c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) &= (0, 0, 0) \\ (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) &= (0, 0, 0), \end{aligned}$$

which yields the homogeneous system of linear equations in c_1, c_2, and c_3 shown below.

$$\begin{aligned} c_1 \qquad - 2c_3 &= 0 \\ 2c_1 + c_2 \qquad &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gauss-Jordan elimination as follows.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This implies that the only solution is the trivial solution

$$c_1 = c_2 = c_3 = 0.$$

So, S is linearly independent.

Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or linearly dependent, perform the following steps.

1. From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_k\mathbf{v}_k = \mathbf{0}$, write a homogeneous system of linear equations in the variables $c_1, c_2, . . . ,$ and c_k.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, . . . , c_k = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

ถ้าคำตอบไม่ได้เป็น 0

Testing for Linear Independence

Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

SOLUTION Expanding the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produces

$$c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) = 0 + 0x + 0x^2$$
$$(c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 = 0 + 0x + 0x^2.$$

Equating corresponding coefficients of equal powers of x produces the homogeneous system of linear equations in c_1, c_2, and c_3 shown below.

$$\begin{aligned} c_1 + 2c_2 &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned}$$

The augmented matrix of this system reduces by Gaussian elimination as follows.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This implies that the system has an infinite number of solutions. So, the system must have nontrivial solutions, and you can conclude that the set S is linearly dependent.

One nontrivial solution is

$$c_1 = 2, \quad c_2 = -1, \quad \text{and} \quad c_3 = 3,$$

which yields the nontrivial linear combination

$$(2)(1 + x - 2x^2) + (-1)(2 + 5x - x^2) + (3)(x + x^2) = 0.$$

Testing for Linear Independence

Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

SOLUTION From the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

Vector 0

you have

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which produces the system of linear equations in c_1, c_2, and c_3 shown below.

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 \qquad &= 0 \end{aligned}$$

Using Gaussian elimination, the augmented matrix of this system reduces as follows.

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system has only the trivial solution and you can conclude that the set S is linearly independent.

A Property of Linearly Dependent Sets

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, . . . , \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in S.

PROOF To prove the theorem in one direction, assume S is a linearly dependent set. Then there exist scalars $c_1, c_2, c_3, . . . , c_k$ (not all zero) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Because one of the coefficients must be nonzero, no generality is lost by assuming $c_1 \neq 0$. Then solving for $\mathbf{v}_1$ as a linear combination of the other vectors produces

$$c_1\mathbf{v}_1 = -c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \cdots - c_k\mathbf{v}_k$$

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 - \frac{c_3}{c_1}\mathbf{v}_3 - \cdots - \frac{c_k}{c_1}\mathbf{v}_k.$$

Conversely, suppose the vector $\mathbf{v}_1$ in S is a linear combination of the other vectors. That is,

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k.$$

Then the equation $-\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has at least one coefficient, -1, that is nonzero, and you can conclude that S is linearly dependent.

Writing a Vector as a Linear Combination of Other Vectors

In Example 9, you determined that the set

$$S = \{\overset{\mathbf{v}_1}{1 + x - 2x^2}, \overset{\mathbf{v}_2}{2 + 5x - x^2}, \overset{\mathbf{v}_3}{x + x^2}\}$$

is linearly dependent. Show that one of the vectors in this set can be written as a linear combination of the other two.

SOLUTION In Example 9, the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ produced the system

$$\begin{aligned} c_1 + 2c_2 \phantom{{}+c_3} &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0. \end{aligned}$$

This system has an infinite number of solutions represented by $c_3 = 3t$, $c_2 = -t$, and $c_1 = 2t$. Letting $t = 1$ results in the equation $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$. So, $\mathbf{v}_2$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$ as follows.

$$\mathbf{v}_2 = 2\mathbf{v}_1 + 3\mathbf{v}_3$$

A check yields

$$\begin{aligned} 2 + 5x - x^2 &= 2(1 + x - 2x^2) + 3(x + x^2) \\ &= 2 + 2x - 4x^2 + 3x + 3x^2 \\ &= 2 + 5x - x^2. \end{aligned}$$

Corollary

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

REMARK: The zero vector is always a scalar multiple of another vector in a vector space.

Testing for Linear Dependence of Two Vectors

(a) The set

$$S = \{\overset{\mathbf{v}_1}{(1, 2, 0)}, \overset{\mathbf{v}_2}{(-2, 2, 1)}\}$$

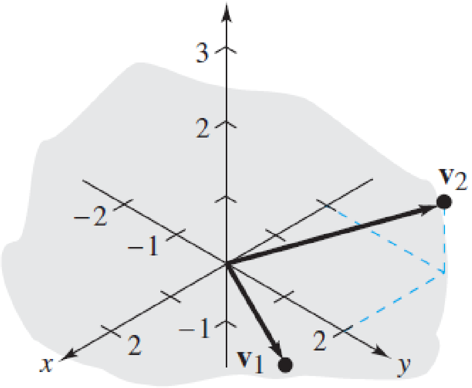
is linearly independent because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of each other, as shown in Figure 4.17(a).

(b) The set

$$S = \{\overset{\mathbf{v}_1}{(4, -4, -2)}, \overset{\mathbf{v}_2}{(-2, 2, 1)}\}$$

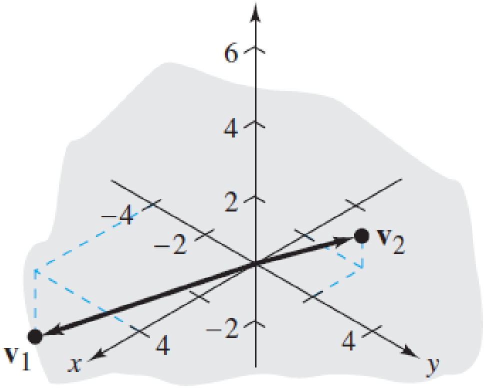
is linearly dependent because $\mathbf{v}_1 = -2\mathbf{v}_2$, as shown in Figure 4.17(b).

(a)



$S = \{(1, 2, 0), (-2, 2, 1)\}$
The set S is linearly independent.

(b)



$S = \{(4, -4, -2), (-2, 2, 1)\}$
The set S is linearly dependent because $\mathbf{v}_1 = -2\mathbf{v}_2$.

Figure 4.17

Q & A