Linear Transformations

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or as an advance appointment

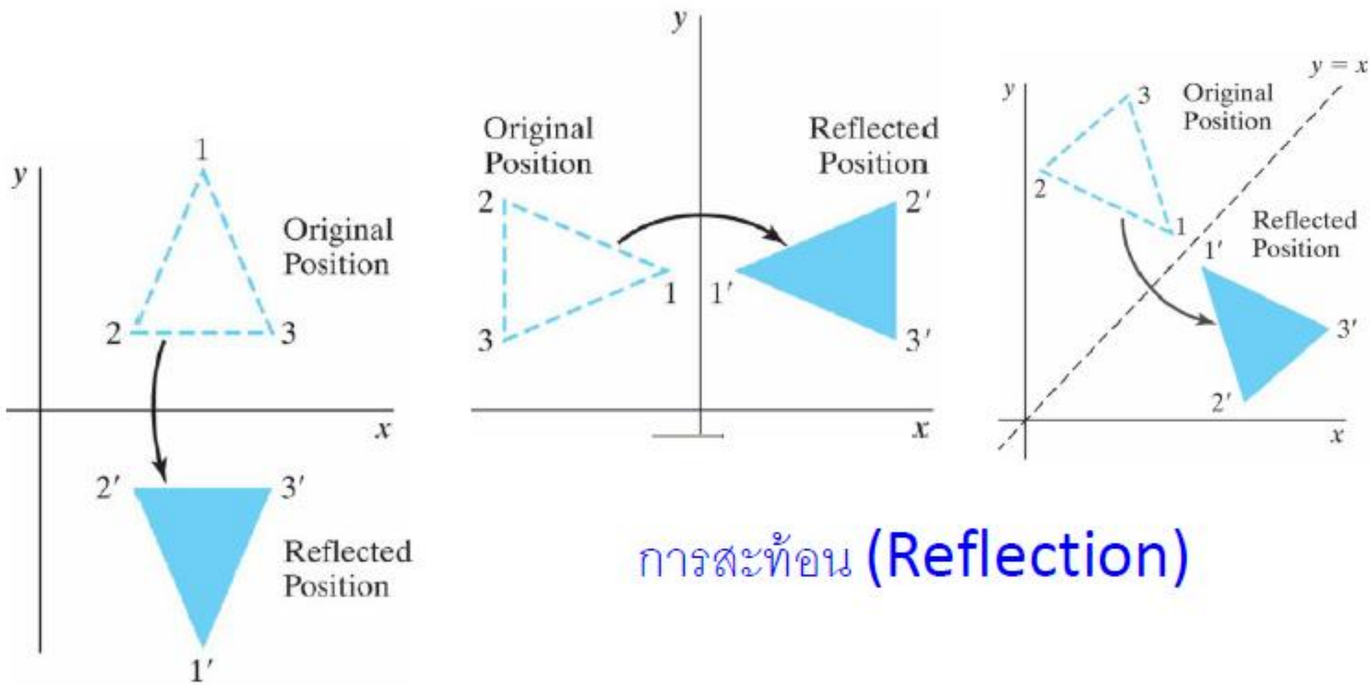
Outline

- Introduction to Linear Transformations
- Matrices for Linear Transformations
- Applications of Linear Transformations

technical terms

- image
- preimage
- standard matrices
- elementary matrices

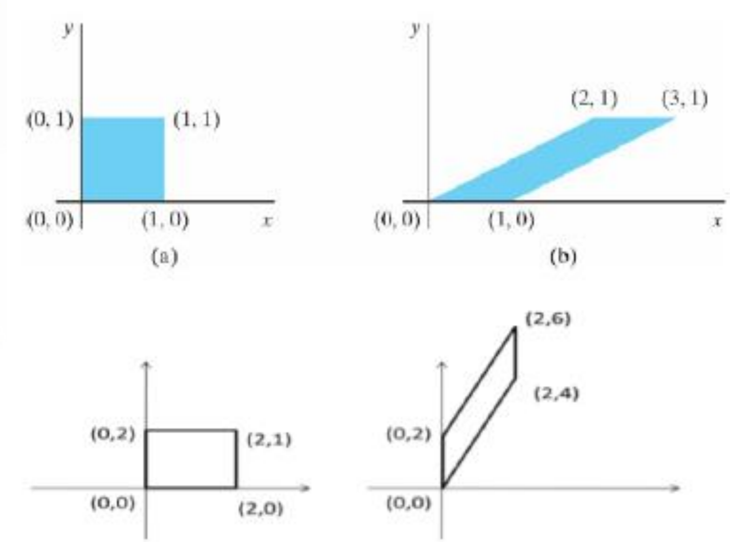
การประยุกต์การแปลงเชิงเส้น : Linear Transformation
Computer Graphics



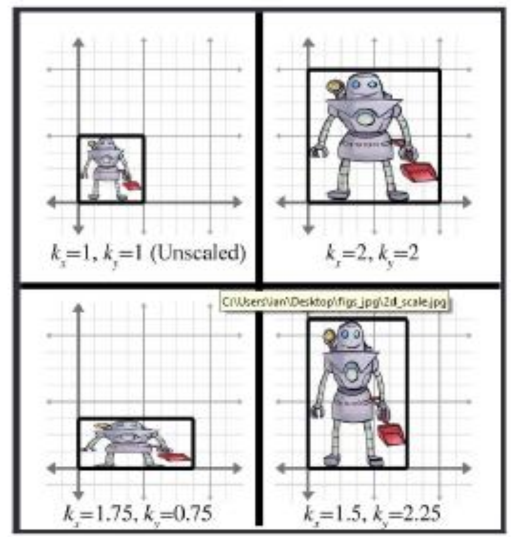
การสะท้อน (Reflection)

การประยุกต์การแปลงเชิงเส้น : Computer Graphics

Cartesian coordinate

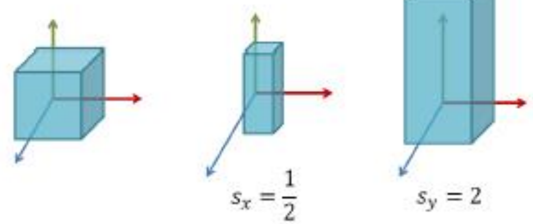


ตัดเฉือน (Shearing)

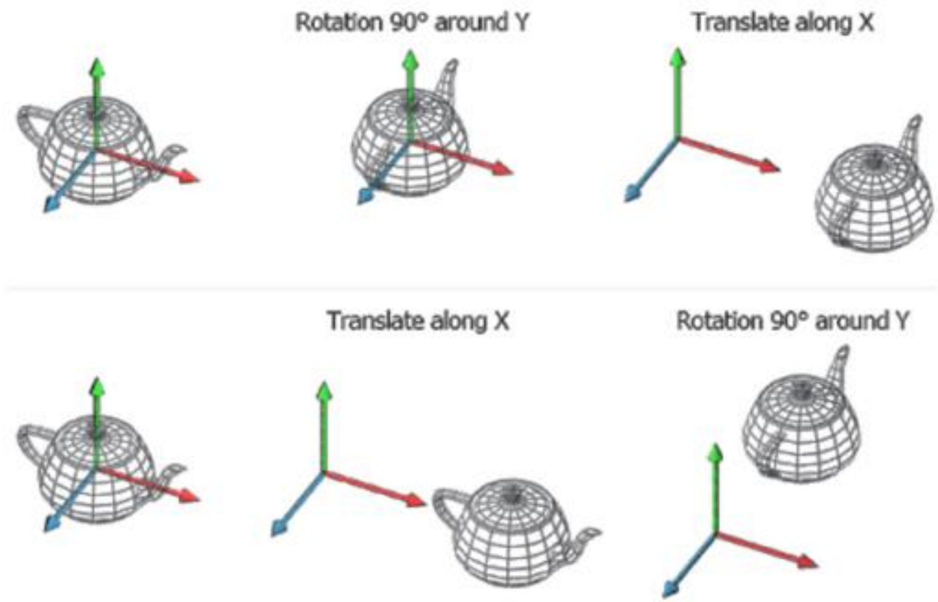


สเกล (Scaling)

การประยุกต์การแปลงเชิงเส้น : Computer Graphics



สเกล (Scaling)



หมุน (Rotation)

Introduction to Linear Transformations

ส่ง/แปลง

A function that maps a vector space V into a vector space W is denoted by

$$T: V \to W.$$

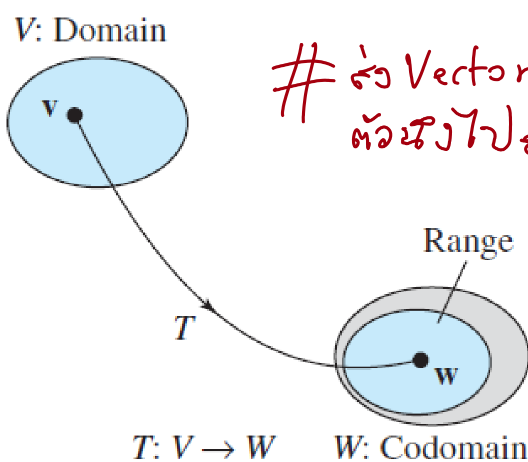
The standard function terminology is used for such functions. For instance, V is called the domain of T, and W is called the codomain of T. If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W such that

$$T(\mathbf{v}) = \mathbf{w},$$ image

then $\mathbf{w}$ is called the image of $\mathbf{v}$ under T. The set of all images of vectors in V is called the range of T, and the set of all $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$ is called the preimage of $\mathbf{w}$.

REMARK: For a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n, it would be technically correct to use double parentheses to denote $T(\mathbf{v})$ as $T(\mathbf{v}) = T((v_1, v_2, \ldots, v_n))$. For convenience, however, one set of parentheses is dropped, producing

$$T(\mathbf{v}) = T(\overbrace{v_1, v_2, \ldots, v_n}^{\text{vector}}).$$



ส่ง Vector space ตัวหนึ่งไปยังอีกตัว

A Function from R^2 into R^2

$R^2 \to R^2$ transformation Preimage

$T(v) = W$

For any vector $\mathbf{v} = (v_1, v_2)$ in R^2, let $T: R^2 \to R^2$ be defined by

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2).$$

(a) Find the image of $\mathbf{v} = (-1, 2)$.

(b) Find the preimage of $\mathbf{w} = (-1, 11)$.

a) $T(-1,2) = (-1-2, -1+2(2))$
$= (-3, 3)$

b) $T(V) = (V_1 - V_2, V_1 + 2V_2) = (-1, 11)$

$V_1 - V_2 = -1$ — 1

$V_1 + 2V_2 = 11$ — 2

$-3V_2 = -12$

$V_2 = 4$, $V_1 = 3$

Definition of a Linear Transformation

Let V and W be vector spaces. The function $T\colon V \to W$ is called a **linear transformation** of V into W if the following two properties are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and for any scalar c.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

ต้อง verify สองข้อนี้ถึงเป็น linear trans.

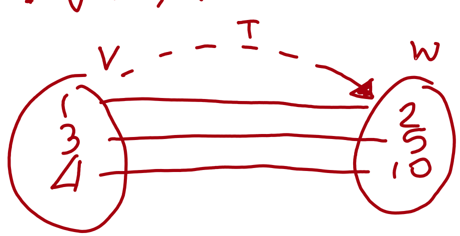
การคงสภาพการดำเนินการ

A linear transformation is said to be *operation preserving,* because the same result occurs whether the operations of addition and scalar multiplication are performed before or after the linear transformation is applied. Although the same symbols are used to denote the vector operations in both V and W, you should note that the operations may be different, as indicated in the diagram below.

from V to W

$$\underbrace{T(\mathbf{u} + \mathbf{v})}_{\text{Addition in } V} = \underbrace{T(\mathbf{u}) + T(\mathbf{v})}_{\text{Addition in } W} \qquad \underbrace{T(c\mathbf{u})}_{\text{Scalar multiplication in } V} = \underbrace{cT(\mathbf{u})}_{\text{Scalar multiplication in } W}$$

$T : V \to W$



$T(1) = 2$

$T(3) = 5$

$T(4) = 10$

$T(4) \neq T(1) + T(3)$ not linear trans.

Verifying a Linear Transformation from R^2 into R^2

Show that the function given in Example 1 is a linear transformation from R^2 into R^2.

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

$u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

$T(u+v) = T(u_1 + v_1, u_2 + v_2)$

$? \quad \ldots$

Ans. $T(u) + T(v)$

$T(cu) = T(cu_1, cu_2)$

$? \quad \ldots$

Some Functions That Are Not Linear Transformations

(a) $f(x) = \sin x$ is not a linear transformation from R into R because, in general,

$$\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2.$$

$\sin(x_1+x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2$

For instance, $\sin[(\pi/2) + (\pi/3)] \neq \sin(\pi/2) + \sin(\pi/3)$.

(b) $f(x) = x^2$ is not a linear transformation from R into R because, in general,

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2.$$

For instance, $(1 + 2)^2 \neq 1^2 + 2^2$.

(c) $f(x) = x + 1$ is not a linear transformation from R into R because

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

whereas

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2.$$

So $f(x_1 + x_2) \neq f(x_1) + f(x_2)$.

Properties of Linear Transformations

Two simple linear transformations are the **zero transformation** and the **identity transformation,** which are defined as follows.

1. $T(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v}$ **Zero transformation** ($T: V \to W$) การแปลงศูนย์
2. $T(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v}$ **Identity transformation** ($T: V \to V$) การแปลงเอกลักษณ์

Let T be a linear transformation from V into W, where $\mathbf{u}$ and $\mathbf{v}$ are in V. Then the following properties are true.

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(-\mathbf{v}) = -T(\mathbf{v})$
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$
4. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_n\mathbf{v}_n$, linear combination

   then

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdot \cdot \cdot + c_n\mathbf{v}_n)$$
$$= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdot \cdot \cdot + c_nT(\mathbf{v}_n).$$

Proof

1. $T(0) = T(0v) = 0\,T(v) = 0$
2. $T(-v) = T(-1(v)) = -1(T(v)) = -T(v)$

Linear Transformations and Bases

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$R_3 \longrightarrow R_3$

$$T(1, 0, 0) = (2, -1, 4)$$
$$T(0, 1, 0) = (1, 5, -2)$$
$$T(0, 0, 1) = (0, 3, 1).$$

Find $T(2, 3, -2)$.

$$(2,3,-2) = \overbrace{2(\underline{1,0,0}) + 3(\underline{0,1,0}) - 2(\underline{0,0,1})}^{\text{linear combination}}$$

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1)$$

Bases (Basis)
Std. spanning set

B เป็น subset ที่ไม่ใช่ $\emptyset$ ของ V

B เป็นฐานหลักของ V ก็ต่อเมื่อ

1) B span V

2) B เป็น independent

A Linear Transformation Defined by a Matrix

The function $T: R^2 \to R^3$ is defined as follows.

$$T(\mathbf{v}) = A\mathbf{v} = \underbrace{\overset{A}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}}}_{3\times 2} \overset{V}{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{2\times 1} = \overset{\to R^3}{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{3\times 1}$$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$.

(b) Show that T is a linear transformation from R^2 into R^3.

a) $T(v) = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$

b) • $T(u+v) = A(u+v) = Au + Av = T(u) + T(v)$ ✓

• $T(cu) = A(cu) = c(Au) = cT(u)$ ✓

The Linear Transformation Given by a Matrix

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m. In order to conform to matrix multiplication with an $m \times n$ matrix, the vectors in R^n are represented by $n \times 1$ matrices and the vectors in R^m are represented by $m \times 1$ matrices.

REMARK: The $m \times n$ zero matrix corresponds to the zero transformation from R^n into R^m, and the $n \times n$ identity matrix I_n corresponds to the identity transformation from R^n into R^n.

Be sure you see that an $m \times n$ matrix A defines a linear transformation from R^n into R^m.

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$m \times n$

Vector in R^n → Vector in R^m

$V_{m \times 1}$

Linear Transformation Given by Matrices

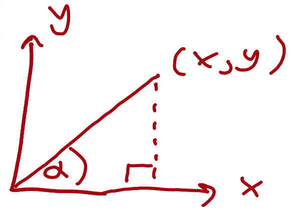
The linear transformation $T: R^n \to R^m$ is defined by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of R^n and R^m for the linear transformation represented by each matrix.

(a) $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix}_{3\times 3}$ $\quad T: R^3 \to R^3$

(b) $A = \begin{bmatrix} 2 & -3 \\ -5 & 0 \\ 0 & -2 \end{bmatrix}_{3\times 2}$ $\quad T: R^2 \to R^3$

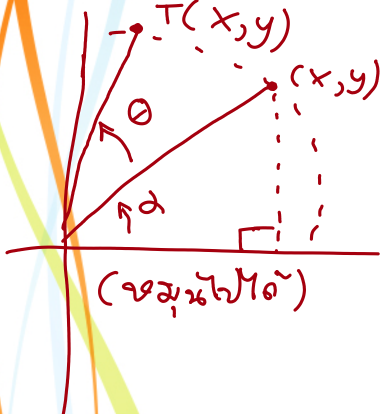
(c) $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$ $\quad T: R^4 \to R^2$

Rotation in the Plane



$x = r\cos\alpha$

$y = r\sin\alpha$



Show that the linear transformation $T: R^2 \to R^2$ represented by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

$V = (x, y) = (r\cos\alpha, r\sin\alpha)$

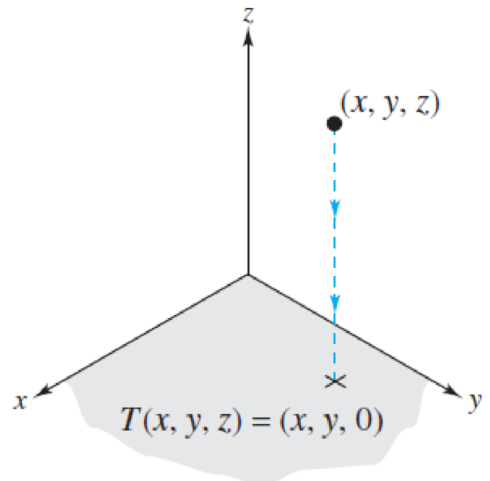
$T(v) = Av$

A Projection in R^3

The linear transformation $T: R^3 \to R^3$ represented by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a **projection** in R^3. If $\mathbf{v} = (x, y, z)$ is a vector into R^3, then $T(\mathbf{v}) = (x, y, 0)$. In other words, T maps every vector in R^3 to its orthogonal projection in the xy-plane, as shown in Figure 6.3.



Projection onto xy-plane

Figure 6.3

A Linear Transformation from $M_{m,n}$ into $M_{n,m}$

Let $T: M_{m,n} \rightarrow M_{n,m}$ be the function that maps an $m \times n$ matrix A to its transpose. That is,

$$T(A) = A^T.$$

Show that T is a linear transformation.

$$T(A+B) = (A+B)^T$$
$$= A^T + B^T$$
$$= T(A) + T(B) \checkmark$$
$$T(cA) = (cA)^T$$
$$= c(A)^T$$
$$= c\,T(A) \checkmark$$

Matrices for Linear Transformations

Which representation of $T: R^3 \rightarrow R^3$ is better,

$$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

or

แบบล่างอ่านง่ายกว่า

$$T(\mathbf{x}) = A\mathbf{x} = \underbrace{\begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix}}_{3\times 3}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times 1}? = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{3\times 1} R^3$$

The second representation is better than the first for at least three reasons: it is simpler to write, simpler to read, and more easily adapted for computer use. Later you will see that matrix representation of linear transformations also has some theoretical advantages. In this section you will see that for linear transformations involving finite-dimensional vector spaces, matrix representation is always possible.

The key to representing a linear transformation $T: V \rightarrow W$ by a matrix is to determine how it acts on a basis of V. Once you know the image of every vector in the basis, you can use the properties of linear transformations to determine $T(\mathbf{v})$ for any $\mathbf{v}$ in V.

For convenience, the first three theorems in this section are stated in terms of linear transformations from R^n into R^m, relative to the standard bases in R^n and R^m. At the end of the section these results are generalized to include nonstandard bases and general vector spaces.

Recall that the standard basis for R^n, written in column vector notation, is represented by

ฐาน

$$B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} = \left\{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right\}.$$

$(1,0,0), (0,1,0), (0,0,1)$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Standard Matrix for a Linear Transformation

Let $T: R^n \to R^m$ be a linear transformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

สอดคล้อง

Then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix},$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n. A is called the **standard matrix** for T.

Finding the Standard Matrix for a Linear Transformation

Find the standard matrix for the linear transformation $T: R^3 \rightarrow R^2$ defined by

$$T(x, y, z) = (x - 2y, 2x + y).$$

Vector Notation

$T(e_1) = T(1,0,0) = (1,2)$

$T(e_2) = T(0,1,0) = (-2,1)$

$T(e_3) = T(0,0,1) = (0,0)$

Matrix Notation

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

— etc. —

เอา Matrix ฐานมาต่อกันเลย

$$A = \left[T(e_1) : T(e_2) : T(e_3)\right] = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Finding the Standard Matrix for a Linear Transformation

ดูตรงนี้ว่า R ไหน

The linear transformation $T: R^2 \rightarrow R^2$ is given by projecting each point in R^2 onto the x-axis, as shown in Figure 6.8. Find the standard matrix for T.

$$T(e_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

$$A = \left[\, T(e_1) \,\vdots\, T(e_2) \,\right]$$

$$= \left[\begin{array}{c:c}1 & 0\\0 & 0\end{array}\right]$$

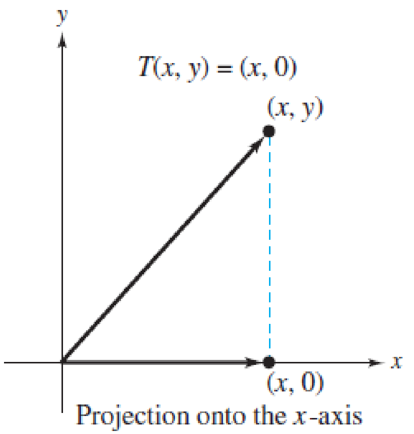


Figure 6.8

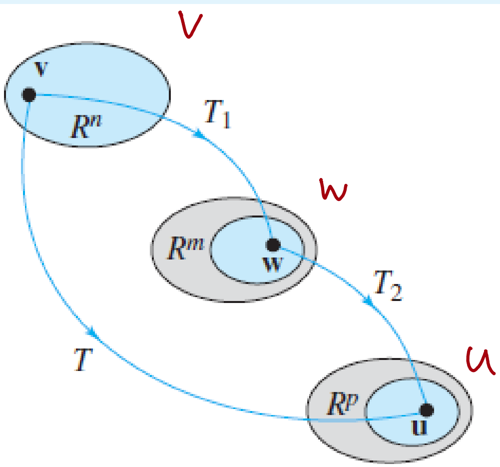
Composition of Linear Transformations

Let $T_1: R^n \rightarrow R^m$ and $T_2: R^m \rightarrow R^p$ be linear transformations with standard matrices A_1 and A_2. The **composition** $T: R^n \rightarrow R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation. Moreover, the standard matrix A for T is given by the matrix product

$$A = A_2 A_1.$$

$T_2 \circ T_1$

(T_2 ประกอบกับ T_1)



Composition of Transformations

The Standard Matrix for a Composition

Let T_1 and T_2 be linear transformations from R^3 into R^3 such that

$$T_1(x, y, z) = (2x + y, 0, x + z) \qquad \text{and} \qquad T_2(x, y, z) = (x - y, z, y).$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

composition

$$T_1(e_1) = T_1(1,0,0) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$T_1(e_2) = T_1(0,1,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T_1(e_3) = T_1(0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

std. matrix for T

$$A = A_2 A_1$$

$$= \ldots$$

$$A' = A_1 A_2$$

Definition of Inverse Linear Transformation

If $T_1: R^n \to R^n$ and $T_2: R^n \to R^n$ are linear transformations such that for every $\mathbf{v}$ in R^n

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v},$$

$T_2 \circ T_1 = T_1 \circ T_2$

then T_2 is called the **inverse** of T_1, and T_1 is said to be **invertible.**

Not every linear transformation has an inverse. If the transformation T_1 is invertible, however, then the inverse is unique and is denoted by T_1^{-1}.

Just as the inverse of a function of a real variable can be thought of as undoing what the function did, the inverse of a linear transformation T can be thought of as undoing the mapping done by T. For instance, if T is a linear transformation from R^3 onto R^3 such that

$$T(1, 4, -5) = (2, 3, 1)$$

Inverse จะ maps ย้อนกลับ

and if T^{-1} exists, then T^{-1} maps $(2, 3, 1)$ back to its preimage under T. That is,

$$T^{-1}(2, 3, 1) = (1, 4, -5).$$

Existence of an Inverse Transformation

Let $T: R^n \rightarrow R^n$ be a linear transformation with standard matrix A. Then the following conditions are equivalent.

ถ้า T เป็นฟังก์ชันแบบหนึ่งต่อหนึ่ง

$T(u) = T(v)$ injective

1. T is invertible.
2. T is an isomorphism. การแปลงถอดแบบ
3. A is invertible.

And, if T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

Matrix มาตรฐาน

ทั่วถึง

$T(v) = w$ subjective

Finding the Inverse of a Linear Transformation

The linear transformation $T: R^3 \to R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3).$$

Show that T is invertible, and find its inverse.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

- หา Inverse A
- เขียน A ให้ได้

$$V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(ทำซ้ายให้เหมือนขวา)

$$A^{-1} = \left[\begin{array}{ccc:ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

$\hookrightarrow A^{-1}$

$$T^{-1} = A^{-1}V = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3)$$

? Finding Inverse using Gaussian-elimination

Matrix มูลฐาน

Elementary Matrices for Linear Transformations in the Plane

Reflection in y-Axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in x-Axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection in Line $y = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

scaling

Horizontal Expansion ($k > 1$) or Contraction ($0 < k < 1$)

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical Expansion ($k > 1$) or Contraction ($0 < k < 1$)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

เป็น Matrix ที่คิดมาให้แล้ว

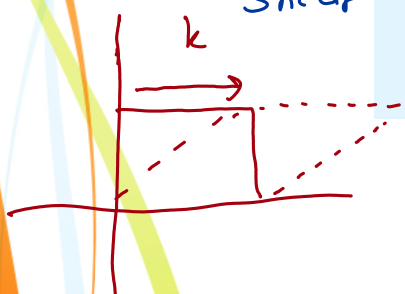
Shear

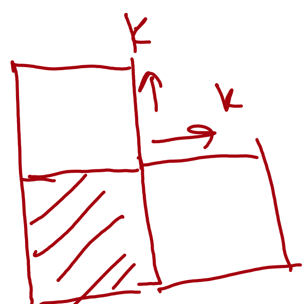
Horizontal Shear

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

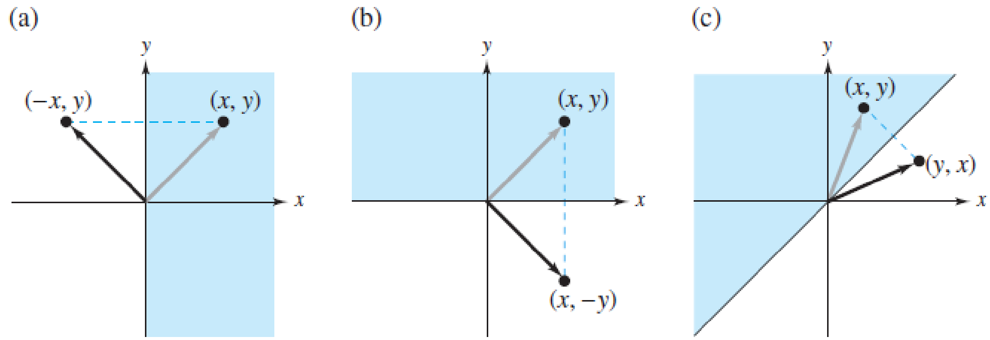
$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$





Reflections in the Plane

The transformations defined by the matrices listed below are called **reflections.** Reflections have the effect of mapping a point in the xy-plane to its "mirror image" with respect to one of the coordinate axes or the line $y = x$, as shown in Figure 6.11.



Reflections in the Plane

Figure 6.11

(a) Reflection in the y-axis:

$$T(x, y) = (-x, y)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

(b) Reflection in the x-axis:

ลอง Proof หา A ดู

$$T(x, y) = (x, -y)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

(c) Reflection in the line $y = x$:

$$T(x, y) = (y, x)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

Expansions and Contractions in the Plane

The transformations defined by the matrices below are called **expansions** or **contractions,** depending on the value of the positive scalar k.

(a) Horizontal contractions and expansions:

$$T(x, y) = (kx, y)$$

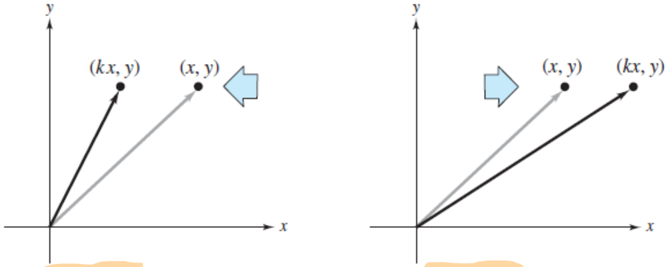
$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$$

(b) Vertical contractions and expansions:

$$T(x, y) = (x, ky)$$

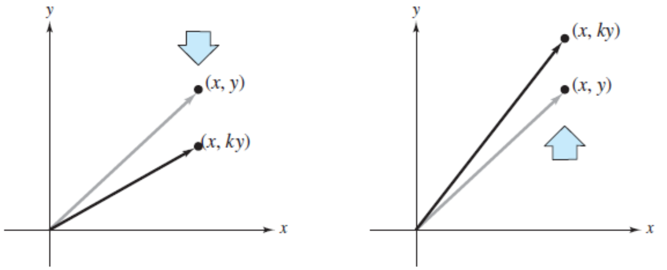
$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$$

Note that in Figures 6.12 and 6.13, the distance the point (x, y) is moved by a contraction or an expansion is proportional to its x- or y-coordinate. For instance, under the transformation represented by $T(x, y) = (2x, y)$, the point $(1, 3)$ would be moved one unit to the right, but the point $(4, 3)$ would be moved four units to the right.



Contraction $(0 < k < 1)$ Expansion $(k > 1)$

Figure 6.12



Contraction $(0 < k < 1)$ Expansion $(k > 1)$

Figure 6.13

Shears in the Plane

The transformations defined by the following matrices are shears.

$$T(x, y) = (x + ky, y) \qquad\qquad T(x, y) = (x, y + kx)$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

(a) The horizontal shear represented by $T(x, y) = (x + 2y, y)$ is shown in Figure 6.14. Under this transformation, points in the upper half-plane are "sheared" to the right by amounts proportional to their y-coordinates. Points in the lower half-plane are "sheared" to the left by amounts proportional to the absolute values of their y-coordinates. Points on the x-axis are unmoved by this transformation.

(b) The vertical shear represented by $T(x, y) = (x, y + 2x)$ is shown in Figure 6.15. Here, points in the right half-plane are "sheared" upward by amounts proportional to their x-coordinates. Points in the left half-plane are "sheared" downward by amounts proportional to the absolute values of their x-coordinates. Points on the y-axis are unmoved.

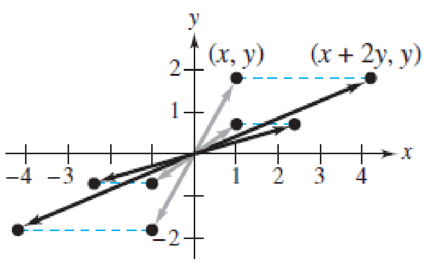


Figure 6.14

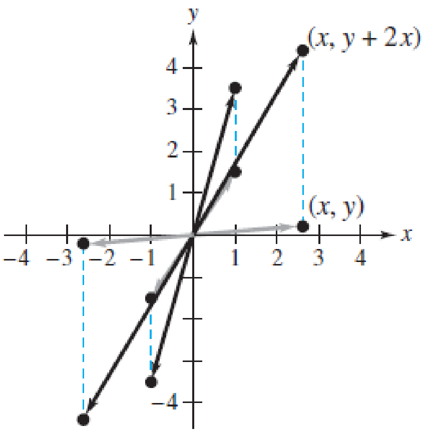


Figure 6.15

Computer Graphics

Linear transformations are useful in computer graphics. In Example 7 in Section 6.1, you saw how a linear transformation could be used to rotate figures in the plane. Here you will see how linear transformations can be used to rotate figures in three-dimensional space.

Suppose you want to rotate the point (x, y, z) counterclockwise about the z-axis through an angle θ, as shown in Figure 6.16. Letting the coordinates of the rotated point be (x', y', z'), you have

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}.$$

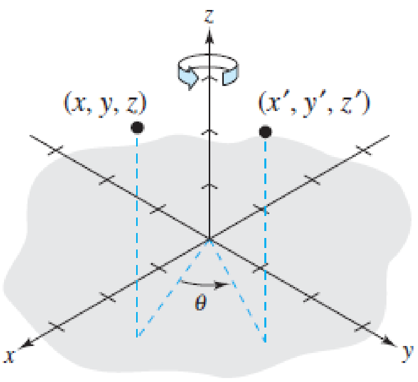


Figure 6.16

Rotation About the z-Axis

8 points around the box

The eight vertices of a rectangular box having sides of lengths 1, 2, and 3 are as follows.

$V_1 = (0, 0, 0)$, $V_2 = (1, 0, 0)$, $V_3 = (1, 2, 0)$, $V_4 = (0, 2, 0)$,
$V_5 = (0, 0, 3)$, $V_6 = (1, 0, 3)$, $V_7 = (1, 2, 3)$, $V_8 = (0, 2, 3)$

Find the coordinates of the box when it is rotated counterclockwise about the z-axis through each angle.

(a) $\theta = 60°$ (b) $\theta = 90°$ (c) $\theta = 120°$

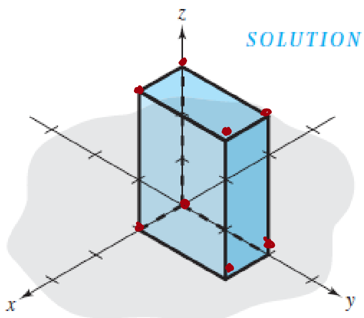


Figure 6.17

SOLUTION The original box is shown in Figure 6.17.

(a) The matrix that yields a rotation of 60° is

A for 60° rotating

$$A = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Rotation About the z-Axis (cont.)

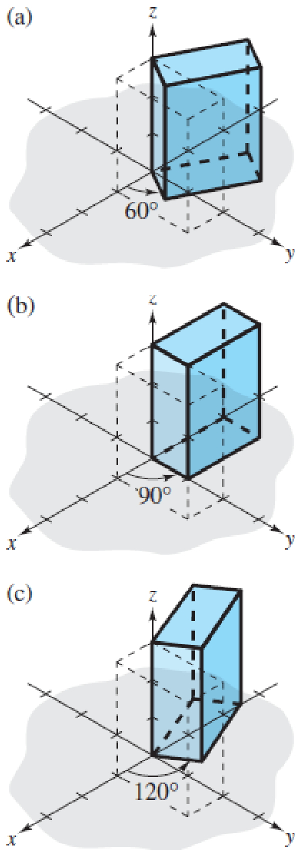


Figure 6.18

Multiplying this matrix by the eight vertices produces the rotated vertices listed below

Original Vertex	*Rotated Vertex*
$V_1 = (0, 0, 0)$	$(0, 0, 0)$
$V_2 = (1, 0, 0)$	$(0.5, 0.87, 0)$
$V_3 = (1, 2, 0)$	$(-1.23, 1.87, 0)$
$V_4 = (0, 2, 0)$	$(-1.73, 1, 0)$
$V_5 = (0, 0, 3)$	$(0, 0, 3)$
$V_6 = (1, 0, 3)$	$(0.5, 0.87, 3)$
$V_7 = (1, 2, 3)$	$(-1.23, 1.87, 3)$
$V_8 = (0, 2, 3)$	$(-1.73, 1, 3)$

A computer-generated graph of the rotated box is shown in Figure 6.18(a). Note that in this graph, line segments representing the sides of the box are drawn between images of pairs of vertices connected in the original box. For instance, because V_1 and V_2 are connected in the original box, the computer is told to connect the images of V_1 and V_2 in the rotated box.

(b) The matrix that yields a rotation of 90° is

$$A = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the graph of the rotated box is shown in Figure 6.18(b).

(c) The matrix that yields a rotation of 120° is

$$A = \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ & 0 \\ \sin 120^\circ & \cos 120^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the graph of the rotated box is shown in Figure 6.18(c).

Rotation About the x- or y-Axis

In Example 4, matrices were used to perform rotations about the z-axis. Similarly, you can use matrices to rotate figures about the x- or y-axis. All three types of rotations are summarized as follows.

Rotation About the x-Axis	*Rotation About the y-Axis*	*Rotation About the z-Axis*
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In each case the rotation is oriented counterclockwise relative to a person facing the negative direction of the indicated axis, as shown in Figure 6.19.

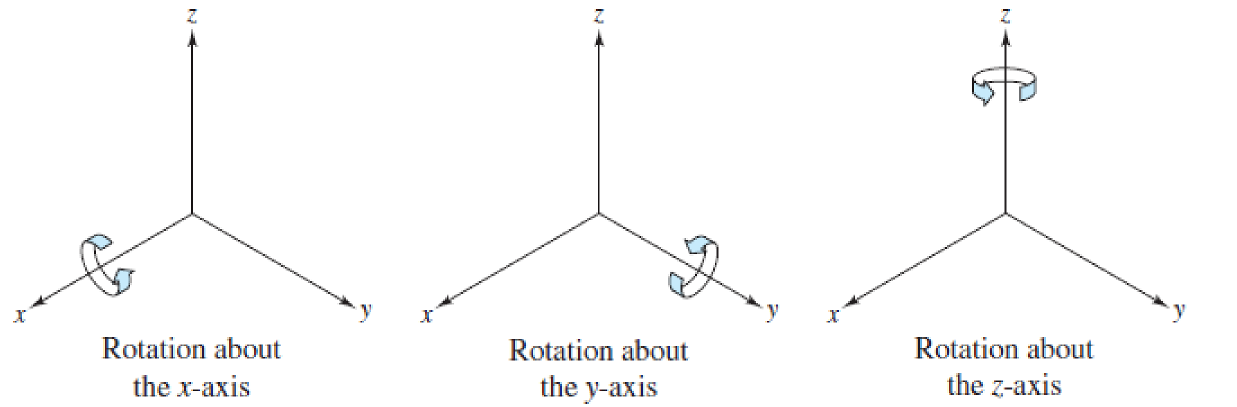


Figure 6.19

Rotation About the x-Axis and y-Axis

(a) The matrix that yields a rotation of 90° about the x-axis is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the graph of the rotated box from Example 4 is shown in Figure 6.20(a) below.

(b) The matrix that yields a rotation of 90° about the y-axis is

$$A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

and the graph of the rotated box from Example 4 is shown in Figure 6.20(b) below.

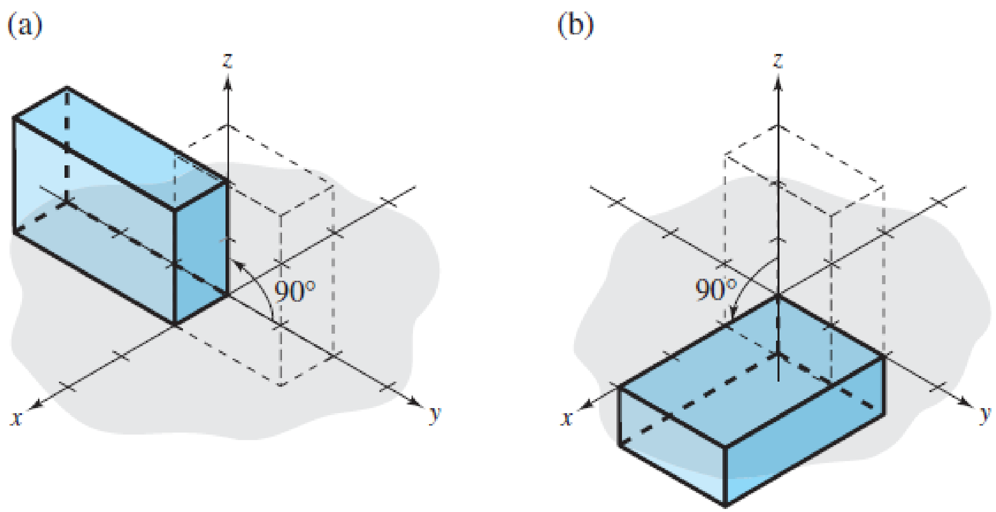


Figure 6.20

Rotations

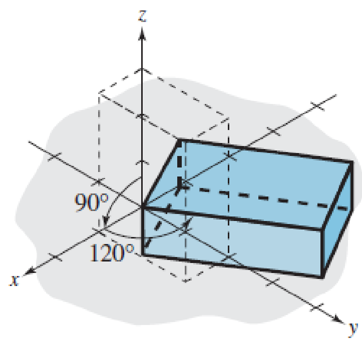


Figure 6.21

Rotations about the coordinate axes can be combined to produce any desired view of a figure. For instance, Figure 6.21 shows the rotation produced by first rotating the box (from Example 4) 90° about the *y*-axis, then further rotating the box 120° about the *z*-axis.

The use of computer graphics has become common among designers in many fields. By simply entering the coordinates that form the outline of an object into a computer, a designer can see the object before it is created. As a simple example, the images of the toy boat shown in Figure 6.22 were created using only 27 points in space. Once the points have been stored in the computer, the boat can be viewed from any perspective.

ex. เรือใบ
มี 27 pts

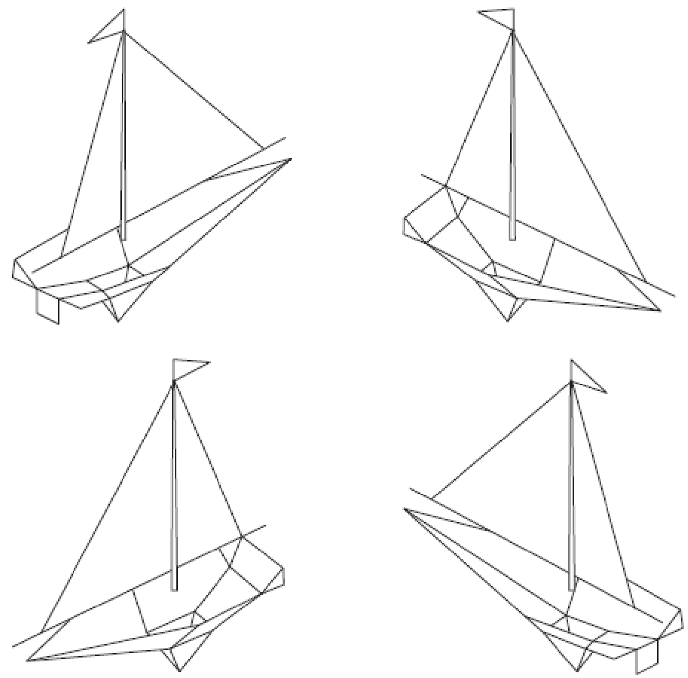


Figure 6.22

Q & A