Mathematics
for
Information Technology

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Technical Term
Derivative
Differentiation
Differentiable
Differentibility

DIFFERENTIATION (I)

Differentiation

- The most important processes of calculus---*differentiation*.
  - Learn new methods and rules for finding derivatives of functions.
  - Apply these rules to find such things as velocity, acceleration, and the rates of change of two or more related variables.

Differentiation

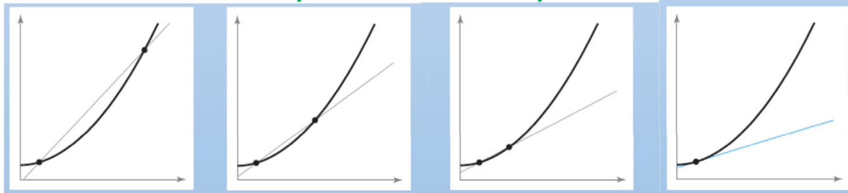
- **OBJECTIVES**

  อนุพันธ์; อัตราการเปลี่ยนแปลง

  Learn how to ...
  - find the derivative of a function using the limit definition and

    understand the relationship between differentiability and continuity.

E.g., slope at a point



To approximate the slope of a tangent line to a graph at a given point, find the slope of the secant line through the given point and a second point on the graph.
As the second point approaches the given point, the approximation tends to become more accurate.

Differentiation

- **OBJECTIVES**

  Learn how to …

  – find the derivative of a function using basic differentiation rules.

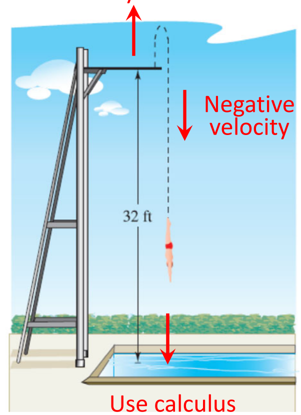
  – find the derivative of a function using the Product Rule and the Quotient Rule.

Differentiation

- **OBJECTIVES**

  Learn how to …

  – find a related rate.



The Derivative and the Tangent Line Problem

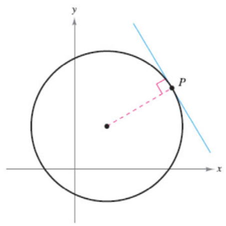
I

Objectives

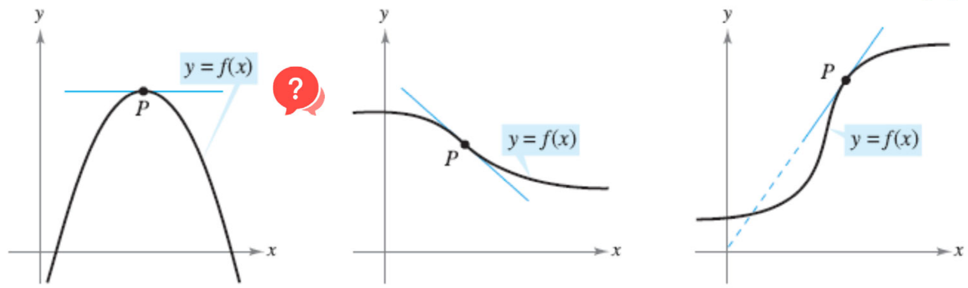
อัตราการเปลี่ยนแปลงของ y เมื่อ x เปลี่ยนไป 1 หน่วย

- Find the slope of the tangent line to a curve at a point.
- Use the limit definition to find the derivative of a function.
- Understand the relationship between differentiability and continuity.

The Tangent Line Problem



Tangent line to a circle

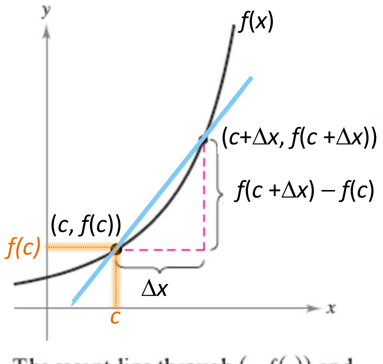


Tangent line to a curve at a point

The Tangent Line Problem

- Approximate the slope using a secant line through the point of tangency and a second point on the curve.

???



The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

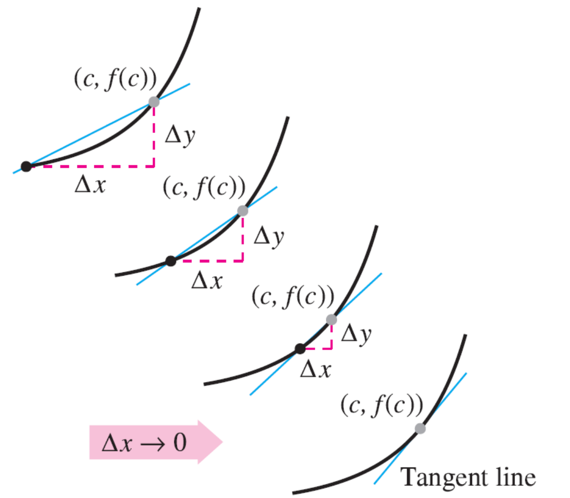
$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$m_{\text{sec}} = \boxed{\frac{f(c + \Delta x) - f(c)}{\Delta x}}. \qquad \text{Slope of secant line}$$

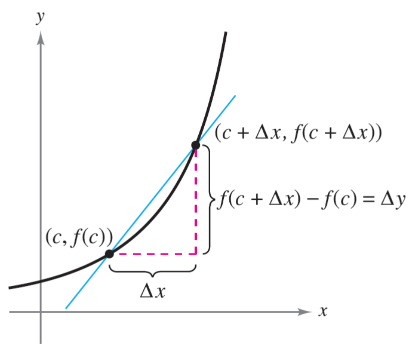
Difference quotient

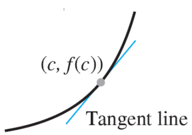
The Tangent Line Problem

- Obtain more and more accurate approximations of the slope of the tangent line by **choosing points** closer and **closer to the point of tangency**.



The Tangent Line Problem





DEFINITION OF TANGENT LINE WITH SLOPE m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$. the slope of the graph of f at $x = c$

EXAMPLE 1 The Slope of the Graph of a Linear Function

Find the slope of the graph of

$$f(x) = 2x - 3$$

at the point (2, 1). $x = 2$

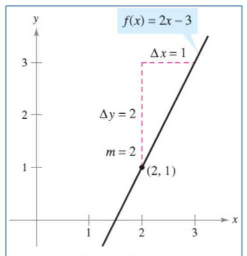
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

Solution To find the slope of the graph of f when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[2(2 + \Delta x) - 3] - [2(2) - 3]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} \\
&= \lim_{\Delta x \to 0} 2 \\
&= 2
\end{aligned}$$

หา slope (ความชัน)

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$



The slope of f at $(2, 1)$ is $m = 2$.

The slope of f at $(c, f(c)) = (2, 1)$ is $m = 2$

EXAMPLE 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slopes of the tangent lines to the graph of

$$f(x) = x^2 + 1$$

at the points $(0, 1)$ and $(-1, 2)$, as shown in Figure

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

Solution Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ is given by หาแบบติด c ไว้ก่อน

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(c + \Delta x)^2 + 1] - (c^2 + 1)}{\Delta x}$$

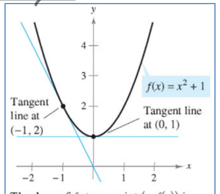
จุดใดๆ

$$= \lim_{\Delta x \to 0} \frac{\cancel{c^2} + 2c(\Delta x) + (\Delta x)^2 + \cancel{1} - \cancel{c^2} - \cancel{1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2c + \Delta x)$$

$$= 2c.$$



The slope of f at any point $(c, f(c))$ is $m = 2c$.

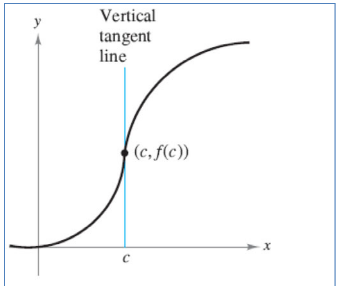
So, the slope at *any* point $(c, f(c))$ on the graph of f is $m = 2c$. At the point $(0, 1)$, the slope is $m = 2(0) = 0$, and at $(-1, 2)$, the slope is $m = 2(-1) = -2$. ■

The Tangent Line Problem

- For vertical tangent lines:
  - If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line *x* = *c* passing through (*c, f(c)*) is a **vertical tangent line** to the graph of *f*.



The graph of f has a vertical tangent line at $(c, f(c))$.

The Derivative of a Function

DEFINITION OF THE DERIVATIVE OF A FUNCTION

The **derivative** of f at x is given by

Derivative คือเรื่องเดียวกันกับ การหา Tangent Line

เป็นฟังก์ชันตัวใหม่

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Differentiation

provided the limit exists. For all x for which this limit exists, f' is a function of x.

gives the slope of the tangent line to the graph of f at the point $(x, f(x))$

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at x if its derivative exists at x and is **differentiable on an open interval (a, b)** if it is differentiable at every point in the interval.

The Derivative of a Function

"f prime of x,"

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

เทียบกับ

"the derivative of y *with respect to* x" or simply "dy, dx."

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= f'(x).$$

EXAMPLE 3 Finding the Derivative by the Limit Process

Find the derivative of $f(x) = x^3 + 2x$.

$$\boxed{(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3}$$

Solution ใช้ limit ในการหา

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3 + 2x)}{\Delta x}$$

$$= (\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 \cancel{+2x} + 2\Delta x \cancel{-x^3} \cancel{-2x})/\Delta x$$

$$= \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2\Delta x}{\Delta x}$$

$$= 3x^2 + 3x\cancel{\Delta x} + \cancel{\Delta x^2} + 2$$

$$= 3x^2 + 2$$

$\lim_{\Delta x \to 0}$ แทน Δx เป็น 0

STUDY TIP When using the definition to find a derivative of a function, the key is to rewrite the difference quotient so that Δx does not occur as a factor of the denominator.

EXAMPLE 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$. Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$. Discuss the behavior of f at $(0, 0)$.

Solution

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

แทน Δx แล้วได้ ตัวส่วน $=0$

conjugate!

$$= \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

Rationalizing

$$= \frac{\cancel{x} + \cancel{\Delta x} - \cancel{x}}{\cancel{\Delta x}(\sqrt{x+\Delta x} + \sqrt{x})} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}};\ x > 0$$

↳ แทนค่าเป็น 0

จุด 0 หา slope ไม่ได้

Differentiability and Continuity

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Alternative form of derivative

$$\boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m}$$

requires that the one-sided limits

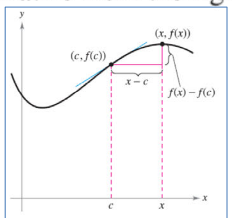
$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Differentiability

lim ซ้ายต้องเท่ากับ lim ขวา

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that f is **differentiable on the closed interval** $[a, b]$ if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.

- หาปลายซ้าย/ขวา
- หาในช่วงปิด



As x approaches c, the secant line approaches the tangent line.

If a function is not continuous at $x = c$, it is also not differentiable at $x = c$. For instance, the greatest integer function ↳ hole, jump, gap

$$f(x) = [\![x]\!]$$ ฟังก์ชันขั้นบันได

is not continuous at $x = 0$, and so it is not differentiable at $x = 0$. You can verify this by observing that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{[\![x]\!] - 0}{x} = \infty$$

Derivative from the left

and

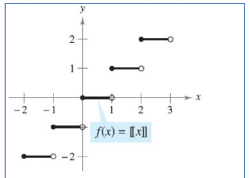
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{[\![x]\!] - 0}{x} = 0.$$

Derivative from the right

Although it is true that differentiability implies continuity (as shown in Theorem 2.1 on the next page), the converse is not true. That is, it is possible for a function to be continuous at $x = c$ and *not* differentiable at $x = c$.

ถ้า Differentiate ได้ → continuous

แต่ continuous ไม่จำเป็นต้อง Differentiate ได้



The greatest integer function is not differentiable at $x = 0$, because it is not continuous at $x = 0$.

EXAMPLE 6 A Graph with a Sharp Turn

NOT differentiable

1/2

The function

$$f(x) = |x - 2|$$

is continuous at $x = 2$. However, the one-sided limits

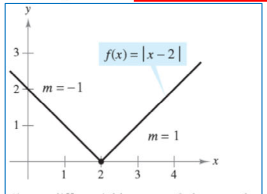
$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x - 2| - 0}{x - 2} = -1$$ Derivative from the left

and

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{|x - 2| - 0}{x - 2} = 1$$ Derivative from the right

are not equal. So, f is not differentiable at $x = 2$ and the graph of f does not have a tangent line at the point $(2, 0)$.

Continuous แต่ไม่ Differentiable



f is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.

EXAMPLE 7 A Graph with a Vertical Tangent Line

NOT differentiable

2/2

The function

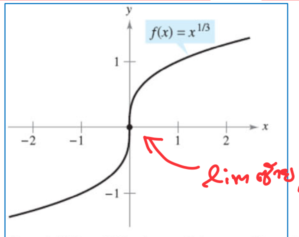
$$f(x) = x^{1/3}$$

is continuous at $x = 0$, However, because the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{1/3} - 0}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, f is not differentiable at $x = 0$. ■

a function is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line.



lim ซ้าย ≠ lim ขวา

f is not differentiable at $x = 0$, because f has a vertical tangent line at $x = 0$.

THEOREM 2.1 DIFFERENTIABILITY IMPLIES CONTINUITY ***

If f is differentiable at $x = c$, then f is continuous at $x = c$.

The following statements summarize the relationship between continuity and differentiability.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability.

II

Basic Differentiation Rules and Rates of Change

Objectives:

Find

- the derivative of a function using the Constant Rule.
- the derivative of a function using the Power Rule.
- the derivative of a function using the Constant Multiple Rule.
- the derivative of a function using the Sum and Difference Rules.
- the derivatives of the sine function and of the cosine function.

Use

- derivatives to find rates of change.

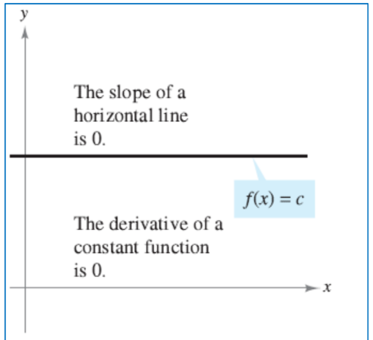
The Constant Rule

ค่าคงที่ตลอด ไม่ว่า x จะเท่าไหร่

THEOREM 2.2 THE CONSTANT RULE

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$



Notice that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.

Derivative of constant function is always zero

The Constant Rule

EXAMPLE 1 Using the Constant Rule

Function	*Derivative*
a. $y = 7$	$dy/dx = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(t) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$

$y' = 0$

The Power Rule

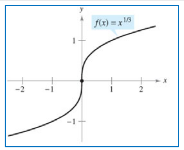
THEOREM 2.3 THE POWER RULE

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

Proved ด้วย limit process มาแล้ว

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

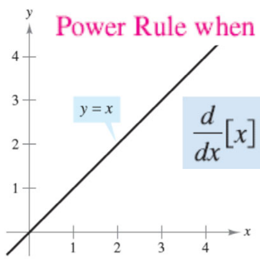
For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.



$(1/x^{2/3})$
Divide by zero
when x=0

NOTE you know that the function $f(x) = x^{1/3}$ is defined at $x = 0$, but is not differentiable at $x = 0$. This is because $x^{-2/3}$ is not defined on an interval containing 0.

Power Rule when $n = 1$



$$\frac{d}{dx}[x] = 1.$$

The slope of the line $y = x$ is 1.

The Power Rule

EXAMPLE 2 Using the Power Rule

หาค่าของ f(x) ไม่ได้เมื่อ
x=0

Function	*Derivative*
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
c. $y = \frac{1}{x^2} = x^{-2}$	$\frac{dy}{dx} = \frac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\frac{2}{x^3}$

แปลงก่อน ถึงจะใช้ Power Rule

Given:	Rewrite:	Differentiate:	Simplify:
$y = \frac{1}{x^2}$	$y = x^{-2}$	$\frac{dy}{dx} = (-2)x^{-3}$	$\frac{dy}{dx} = -\frac{2}{x^3}$

EXAMPLE 3 Finding the Slope of a Graph

used Power Rule

Find the slope of the graph of $f(x) = x^4$ when $f'(x) = 4x^3$

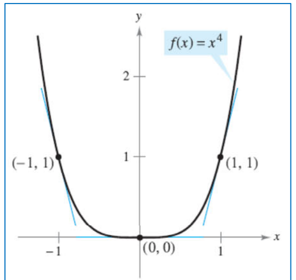
a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

Solution The slope of a graph at a point is the value of the derivative at that point. The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$. Slope is negative.

b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$. Slope is zero.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$. Slope is positive.



Note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.

EXAMPLE 4 Finding an Equation of a Tangent Line สมการของ Target Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution

$f'(x) = 2x$

$m = f'(-2) = -4$

$x = -2$

$f(x) = x^2$

$f(-2) = 4$

แสดงว่า คือจุด $(-2,4)$

$y - y_1 = m(x - x_1)$ Point-slope form

$y = m(x - x_1) + y_1$

$y = -4(x+2) + 4$

$y = -4x - 8 + 4$

$y = -4x - 4$

ได้สมการเส้นตรง (Tangent Line) ตรงจุด $(-2,4)$ เส้นจะลากผ่านจุดนั้นไป

BREAK

The Constant Multiple Rule

THEOREM 2.4 THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

ตัว constant ออกมาได้

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[\ f(x)] = cf'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{c}\right] = \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right]$$

$$= \left(\frac{1}{c}\right)\frac{d}{dx}[\ f(x)] = \left(\frac{1}{c}\right)f'(x)$$

EXAMPLE 5 Using the Constant Multiple Rule

Function	Derivative
a. $y = \dfrac{2}{x}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}\left[2x^{-1}\right] = 2\dfrac{d}{dx}\left[x^{-1}\right] = -2(-1)x^{-2} = \dfrac{-2}{x^2}$
b. $f(t) = \dfrac{4t^2}{5}$	$f'(t) = \dfrac{4}{5}\dfrac{d}{dx}\left[t^2\right] = \dfrac{4}{5}(2t) = \dfrac{8}{5}t$
c. $y = 2\sqrt{x}$	$\dfrac{dy}{dx} = 2\dfrac{d}{dx}\left[x^{\frac{1}{2}}\right] = \cancel{2}\left(\cancel{\tfrac{1}{2}}\right)\left(x^{-\frac{1}{2}}\right) = \dfrac{1}{\sqrt{x}}$
d. $y = \dfrac{1}{2\sqrt[3]{x^2}}$	$\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{d}{dx}\left[x^{-\frac{2}{3}}\right] = -\dfrac{\cancel{2}}{3}\left(\dfrac{1}{\cancel{2}}\right)x^{-\frac{5}{3}} = \dfrac{-1}{3}\cdot\dfrac{1}{\sqrt[3]{x^5}} = \dfrac{-1}{3x^{5/3}}$
e. $y = -\dfrac{3x}{2}$	$y' = \dfrac{-3}{2}\dfrac{d}{dx}[x] = \dfrac{-3}{2}(1) = \dfrac{-3}{2}$ #

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

$$\frac{d}{dx}[cx^n] = cnx^{n-1}.$$

→ รวมกันเป็นตัวเดียวได้

EXAMPLE 6 Using Parentheses When Differentiating

Original Function	Rewrite	Differentiate	Simplify
a. $y = \dfrac{5}{2x^3}$	$y = \dfrac{5}{2}(x^{-3})$	$y' = \dfrac{5}{2}(-3x^{-4})$	$y' = -\dfrac{15}{2x^4}$ $y' = \dfrac{-15}{8x^4}$
b. $y = \dfrac{5}{(2x)^3}$	$y = \dfrac{5}{8}x^{-3}$	$y' = \dfrac{5}{8}(-3)x^{-4}$	$y' = \dfrac{-15}{8}x^{-4}$
c. $y = \dfrac{7}{3x^{-2}}$	$y = \dfrac{7}{3}x^2$	$y' = \dfrac{7}{3}(2)x$	$y' = \dfrac{14}{3}x$
d. $y = \dfrac{7}{(3x)^{-2}}$	$y = \dfrac{7}{3^{-2}\cdot x^{-2}}$ $y = 63x^2$	$y' = 2(63)x$	$y' = 126x$

The Sum and Difference Rules

THEOREM 2.5 THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

EXAMPLE 7 Using the Sum and Difference Rules

Function	Derivative
a. $f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b. $g(x) = -\dfrac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -\dfrac{4}{2}x^3 + 9x^2 - 2 = -2x^3 + 9x^2 - 2$

Derivatives of the Sine and Cosine Functions

THEOREM 2.6 DERIVATIVES OF SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

EXAMPLE 8 Derivatives Involving Sines and Cosines

Function	Derivative
a. $y = 2\sin x$	$y' = 2\cos x$
b. $y = \dfrac{\sin x}{2} = \dfrac{1}{2}\sin x$	$y' = \dfrac{1}{2}\cos x = \dfrac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$

Rates of Change

- The derivative can also be used to determine the **rate of change** จุดชี้วัดอัตราการเปลี่ยนแปลงเท่าไร of one variable with respect to another.
- E.g., บ่งบอกได้ว่าอัตราการเปลี่ยนแปลงของสิ่งหนึ่ง
  - Population growth rates ว่าเปลี่ยนแปลงมาก/น้อยเท่าใด
  - Production rates
  - Water flow rates
  - Velocity
  - Acceleration

Rates of Change

- A common use for rate of change is to describe the motion of an object moving in a straight line.
- The function ***s*** that gives the position (relative to the origin) of an object as **a function of time *t*** is called a **position function**.

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}.$$

Average velocity

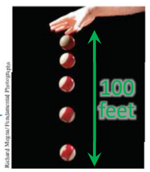
EXAMPLE 9 Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function ต้องดูความสัมพันธ์ของตัวแปร

$$s = -16t^2 + 100 \qquad \text{Position function}$$

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$



Time-lapse photograph of a free-falling billiard ball

Solution

c. $\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6$ ft/s

a. For the interval $[1, 2]$, the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \ ?$$

a. $\frac{\Delta s}{\Delta t} = \frac{s(2) - s(1)}{2 - 1} = \frac{36 - 84}{1} = -48$ ft/s

b. $\frac{\Delta s}{\Delta t} = \frac{s(1.5) - s(1)}{0.5} = -40$ ft/s

EXAMPLE 9 Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

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where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

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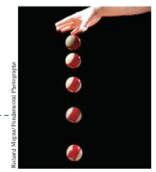
Time-lapse photograph of a free-falling billiard ball

Solution

b.

?

EXAMPLE 9 Finding Average Velocity of a Falling Object

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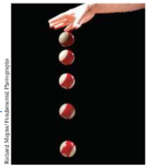


Time-lapse photograph of a free-falling billiard ball

Solution

c.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, the **velocity** of the object at time t is

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t). \qquad \text{Velocity function}$$

In other words, the velocity function is the derivative of the position function. Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0 \qquad \text{Position function}$$

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.

EXAMPLE 10 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32 \qquad \text{Position function}$$

$v(t) = -32t + 16$

$t = 0,\ s = 32$ ft

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

ความสูงมีค่า = 0

a.) $s(t) = 0$

$-16t^2 + 16t + 32 = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1)$

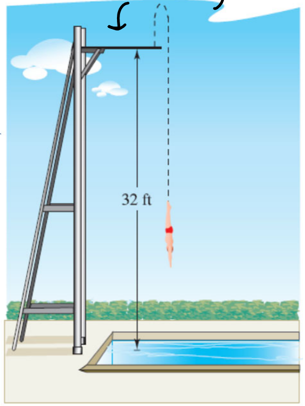
$t = 2,\ t = -1$

เพราะมันเป็นเวลาที่ย้อนถอยหลัง

$v(2) = -64 + 16$

$= -48$ ft/s

จาก Platform



Velocity is positive when an object is rising, and is negative when an object is falling. Notice that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.

Product and Quotient Rules

ผลหาร

Objectives

- Find the derivative of a function using the Product Rule.
- Find the derivative of a function using the Quotient Rule.
- Find the derivative of a trigonometric function.

The Product Rule

THEOREM 2.7 THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \qquad \boxed{uv' + u'v}$$

The Product Rule can be extended to cover products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

EXAMPLE 1 Using the Product Rule

Find the derivative of $h(x) = (3x - 2x^2)(5 + 4x)$.

Solution

$$h'(x) = \underbrace{(3x - 2x^2)}_{\text{First}}\underbrace{\frac{d}{dx}[5 + 4x]}_{\text{Derivative of second}} + \underbrace{(5 + 4x)}_{\text{Second}}\underbrace{\frac{d}{dx}[3x - 2x^2]}_{\text{Derivative of first}} \qquad \text{Apply Product Rule.}$$

$$= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$

$$= (12x - 8x^2) + (15 - 8x - 16x^2)$$

$$= -24x^2 + 4x + 15$$

In Example 1, you have the option of finding the derivative with or without the Product Rule. To find the derivative without the Product Rule, you can write

$$D_x[(3x - 2x^2)(5 + 4x)] = D_x[-8x^3 + 2x^2 + 15x]$$

$$= -24x^2 + 4x + 15.$$

หน้า Diff หลัง + หลัง Diff หน้า

EXAMPLE 2 Using the Product Rule

Find the derivative of $y = 3x^2 \sin x$.

Solution

$$\frac{d}{dx}[3x^2 \sin x] = 3x^2\frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[3x^2]$$

$$= 3x^2 \cos x + \sin x(6x)$$

$$= 3x^2 \cos x + 6x \sin x$$

EXAMPLE 3 Using the Product Rule

เป็น constant ไม่ต้องใช้ Product Rule

Find the derivative of $y = 2x \cos x - 2 \sin x$.

Solution

$$\frac{dy}{dx} = 2x\left(\frac{d}{dx}[\cos x]\right) + \cos x\left(\frac{d}{dx}[2x]\right)$$

NOTE In Example 3, notice that you use the Product Rule when both factors of the product are variable, and you use the Constant Multiple Rule when one of the factors is a constant.

The Quotient Rule

THEOREM 2.8 THE QUOTIENT RULE

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

$$\frac{\text{ล่าง Diff บน} - \text{บน Diff ล่าง}}{\text{ล่าง}^2}$$

EXAMPLE 4 Using the Quotient Rule

Find the derivative of $y = \dfrac{5x-2}{x^2+1}$.

Solution

$$\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right] = \frac{(x^2+1)\frac{d}{dx}[5x-2) - 5x-2\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$

$$= 5x^2+5x$$

ไม่จำเป็นต้องใช้ Quotient Rule

$$\boxed{\frac{u'v-uv'}{v^2}}$$

EXAMPLE 6 Using the Constant Multiple Rule

Original Function	Rewrite	Differentiate	Simplify
$\frac{2x+3}{6}$	$y = \frac{1}{6}(x^2+3x)$	$y = \frac{1}{6}(2x+3)$	$y = \frac{2x+3}{6}$
a. $y = \dfrac{x^2+3x}{6}$			
$\frac{5}{2}x^3$ b. $y = \dfrac{5x^4}{8}$	$y = \frac{5}{8}x^4$	$y' = \frac{20}{8}x^3$	$y' = \frac{5}{2}x^3$
c. $y = \dfrac{-3(3x-2x^2)}{7x}$	$y = \frac{-3\cancel{x}(3-2x)}{7\cancel{x}} = \frac{-3(3-2x)}{7} = \frac{-3}{7}(-2) =$		$\frac{6}{7}$
d. $y = \dfrac{9}{5x^2}$	$y = \frac{9}{5}x^{-2}$	$y' = \frac{-18}{5}x^{-3}$	$y' = \frac{-18}{5x^3}$

$y = -$

NOTE To see the benefit of using the Constant Multiple Rule for some quotients, try using the Quotient Rule to differentiate the functions in Example 6—you should obtain the same results, but with more work.

??b

Derivatives of Trigonometric Functions

THEOREM 2.9 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

*** ควรท่องสูตรให้ได้ไม่งั้นๆ

Product Rule

$= uv' + u'v$

Quotient Rule

$$= \frac{u'v - uv'}{v^2}$$

EXAMPLE 8 Differentiating Trigonometric Functions

Function	*Derivative*
a. $y = x - \tan x$	$\dfrac{dy}{dx} = 1 - \sec^2 x$
b. $y = x \sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$ $= (\sec x)(1 + x \tan x)$

NOTE Because of trigonometric identities, the derivative of a trigonometric function can take many forms. This presents a challenge when you are trying to match your answers to those given in the back of the text.

EXAMPLE 9 Different Forms of a Derivative

Differentiate both forms of $y = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x$.

สมการเดียวกัน

Solution

First form: $y = \dfrac{1 - \cos x}{\sin x}$

$$y' = \frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin^2 x}$$

Second form: $y = \csc x - \cot x$

$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

แยก $\sin^2 x$ เป็นสองก้อน

$$= \csc^2 x - \csc x \cot x.$$