# Mathematics
# for
# Information Technology


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# Applications of Integration (II)

Objectives

- Find centers of mass and centroids.



# Moments, Centers of Mass, and Centroids

- **Understand the definition of mass.**
- **Find the center of mass in a one-dimensional system.**
- **Find the center of mass in a two-dimensional system.**
- **Find the center of mass of a planar lamina.**



# Center of Mass in a One-Dimensional System

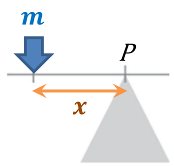
Two types of moments of mass:

moment about a point

moment about a line

If a mass $m$ is concentrated at a point, and the **length of the moment arm** $x$ is the distance between this point mass and another point $P$, the **moment of** $m$ **about the point** $P$ **is**

$$\text{Moment} = mx$$

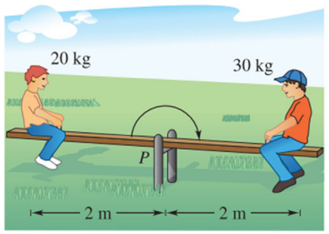


The concept of moment can be demonstrated simply by a seesaw

A child of mass 20 kilograms sits 2 meters to the left of fulcrum $P$, and an older child of mass 30 kilograms sits 2 meters to the right of $P$. From experience, you know that the seesaw will begin to rotate clockwise, moving the larger child down. This rotation occurs because the moment produced by the child on the left is less than the moment produced by the child on the right.

$$\text{Left moment} = (20)(2) = 40 \text{ kilogram-meters}$$
$$\text{Right moment} = (30)(2) = 60 \text{ kilogram-meters}$$

To balance the seesaw, the two moments must be equal. For example, if the larger child moved to a position $\frac{4}{3}$ meters from the fulcrum, the seesaw would balance, because each child would produce a moment of 40 kilogram-meters.

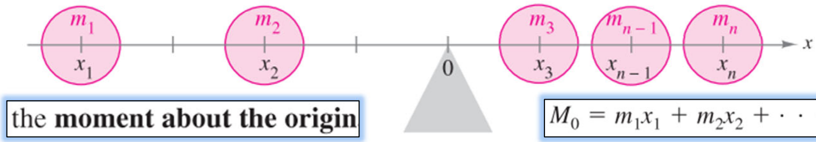


The seesaw will balance when the left and the right moments are equal.

$$(20)(2) = 40 \text{ kg-m}$$
$$(30)\left(\frac{4}{3}\right) = 40 \text{ kg-m}$$



To generalize this, you can introduce a coordinate line on which the origin corresponds to the fulcrum.




the **moment about the origin** $M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n$

If $m_1x_1 + m_2x_2 + \cdots + m_nx_n = 0$, the system is in equilibrium.

If $M_0$ is 0, the system is said to be in **equilibrium.**

For a system that is not in equilibrium, the **center of mass** is defined as the point $\bar{x}$ at which the fulcrum could be relocated to attain equilibrium. If the system were translated $\bar{x}$ units, each coordinate $x_i$ would become $(x_i - \bar{x})$, and because the moment of the translated system is 0, you have

$$\sum_{i=1}^{n} m_i(x_i - \bar{x}) = \sum_{i=1}^{n} m_ix_i - \sum_{i=1}^{n} m_i\bar{x} = 0.$$

Solving for $\bar{x}$ produces

$$\bar{x} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} = \frac{\text{moment of system about origin}}{\text{total mass of system}}.$$

**MOMENTS AND CENTER OF MASS: ONE-DIMENSIONAL SYSTEM**

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $x_1, x_2, \ldots, x_n$.

1. The **moment about the origin** is $M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
2. The **center of mass** is $\boxed{\bar{x} = \frac{M_0}{m}}$, where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

## EXAMPLE 2 The Center of Mass of a Linear System

Find the center of mass of the linear system shown in Figure


**Solution** The moment about the origin is

$$\begin{aligned} M_0 &= m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 \\ &= 10(-5) + 15(0) + 5(4) + 10(7) \\ &= -50 + 0 + 20 + 70 \\ &= 40. \end{aligned}$$

Because the total mass of the system is $m = 10 + 15 + 5 + 10 = 40$, the center of mass is

$$\bar{x} = \frac{M_0}{m} = \frac{40}{40} = 1.$$

**NOTE** In Example 2, where should you locate the fulcrum so that the point masses will be in equilibrium?

# Center of Mass in a Two-Dimensional System

You can extend the concept of moment to two dimensions by considering a system of masses located in the $xy$-plane at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Rather than defining a single moment (with respect to the origin), two moments are defined—one with respect to the $x$-axis and one with respect to the $y$-axis.

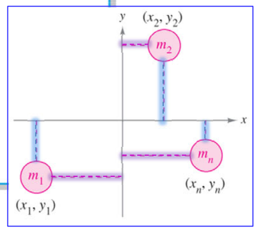
**MOMENT AND CENTER OF MASS: TWO-DIMENSIONAL SYSTEM**

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The **moment about the $y$-axis** is $M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$.
2. The **moment about the $x$-axis** is $M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n$.
3. The **center of mass** $(\bar{x}, \bar{y})$ (or **center of gravity**) is

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

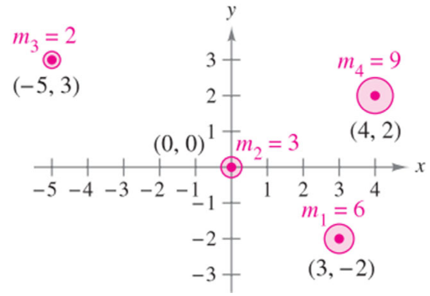





## EXAMPLE 3 The Center of Mass of a Two-Dimensional System

Find the center of mass of a system of point masses $m_1 = 6$, $m_2 = 3$, $m_3 = 2$, and $m_4 = 9$, located at

$(3, -2)$, $(0, 0)$, $(-5, 3)$, and $(4, 2)$

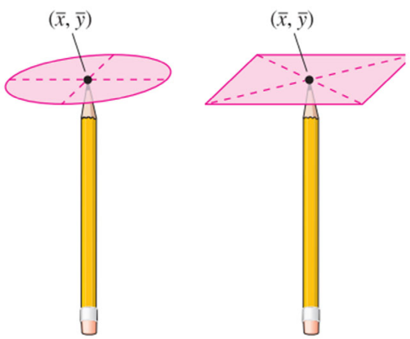





# Center of Mass of a Planar Lamina

So far ... a line or a plane $\longrightarrow$ Now ... a planar lamina

(a thin, flat plate of material of constant density ρ)




You can think of the center of mass $(\bar{x}, \bar{y})$ of a lamina as its balancing point. For a circular lamina, the center of mass is the center of the circle. For a rectangular lamina, the center of mass is the center of the rectangle.



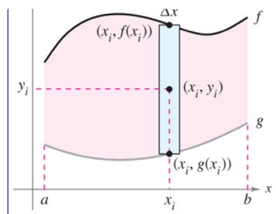
Consider an irregularly shaped planar lamina of uniform density $\rho$, bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \le x \le b$, as shown in Figure

The mass of this region is given by

$$\begin{aligned} m &= (\text{density})(\text{area}) \\ &= \rho \int_a^b [f(x) - g(x)]\,dx \\ &= \rho A \end{aligned}$$

where $A$ is the area of the region. To find the center of mass of this lamina, partition the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x$. Let $x_i$ be the center of the $i$th subinterval. You can approximate the portion of the lamina lying in the $i$th subinterval by a rectangle whose height is $h = f(x_i) - g(x_i)$. Because the density of the rectangle is $\rho$, its mass is

$$\begin{aligned} m_i &= (\text{density})(\text{area}) \\ &= \underbrace{\rho}_{\text{Density}}\underbrace{[f(x_i) - g(x_i)]}_{\text{Height}}\underbrace{\Delta x}_{\text{Width}}. \end{aligned}$$



Planar lamina of uniform density $\rho$

Now, considering this mass to be located at the center $(x_i, y_i)$ of the rectangle, the directed distance from the $x$-axis to $(x_i, y_i)$ is $y_i = [f(x_i) + g(x_i)]/2$. So, the moment of $m_i$ about the $x$-axis is

$$\begin{aligned}\text{Moment} &= (\text{mass})(\text{distance})\\ &= m_i y_i\\ &= \rho[f(x_i) - g(x_i)]\,\Delta x\left[\frac{f(x_i) + g(x_i)}{2}\right].\end{aligned}$$

The moment of $m_i$ about the $y$-axis is

$$= m_i x_i$$
$$= (\rho[f(x_i) - g(x_i)]\Delta x)(x_i)$$

Summing the moments and taking the limit as $n \to \infty$ suggest the definitions below.

**MOMENTS AND CENTER OF MASS OF A PLANAR LAMINA**

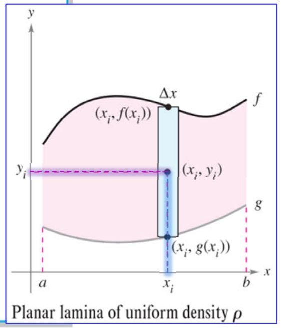
Let $f$ and $g$ be continuous functions such that $f(x) \ge g(x)$ on $[a, b]$, and consider the planar lamina of uniform density $\rho$ bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \le x \le b$.

1. The **moments about the $x$- and $y$-axes** are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2}\right][f(x) - g(x)]\,dx$$

$$M_y = \rho \int_a^b x[f(x) - g(x)]\,dx.$$

2. The **center of mass** $(\bar{x}, \bar{y})$ is given by $\bar{x} = \dfrac{M_y}{m}$ and $\bar{y} = \dfrac{M_x}{m}$, where $m = \rho \int_a^b [f(x) - g(x)]\,dx$ is the mass of the lamina.



Planar lamina of uniform density $\rho$

**EXAMPLE 4 The Center of Mass of a Planar Lamina**

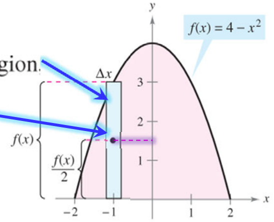
Find the center of mass of the lamina of uniform density $\rho$ bounded by the graph of $f(x) = 4 - x^2$ and the $x$-axis.

**Solution** Because the center of mass lies on the axis of symmetry, you know that $\bar{x} = 0$. Moreover, the mass of the lamina is

$$m = \rho \int_{-2}^{2} (4 - x^2)\,dx = \rho\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32\rho}{3}.$$

To find the moment about the $x$-axis, place a representative rectangle in the region. The distance from the $x$-axis to the center of this rectangle is

$$y_i = \frac{f(x)}{2} = \frac{4 - x^2}{2}.$$




Because the mass of the representative rectangle is

$$\rho f(x)\,\Delta x = \rho(4 - x^2)\,\Delta x$$

you have

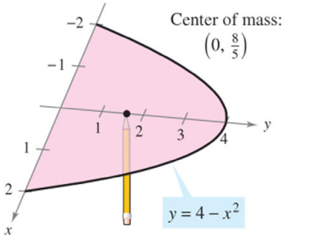
$$M_x = \rho \int_{-2}^{2} \frac{4 - x^2}{2}(4 - x^2)\,dx = \frac{\rho}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\,dx$$

$$= \frac{\rho}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{256\rho}{15}$$

and $\bar{y}$ is given by

$$\bar{y} = \frac{M_x}{m} = \frac{256\rho/15}{32\rho/3} = \frac{8}{5}.$$

So, the center of mass (the balancing point) of the lamina is $\left(0, \frac{8}{5}\right)$.



The center of mass is the balancing point.

The density $\rho$ in Example 4 is a common factor of both the moments and the mass, and as such divides out of the quotients representing the coordinates of the center of mass. So, the center of mass of a lamina of *uniform* density depends only on the shape of the lamina and not on its density. For this reason, the point

$(\bar{x}, \bar{y})$ Center of mass or centroid

is sometimes called the center of mass of a *region* in the plane, or the **centroid** of the region. In other words, to find the centroid of a region in the plane, you simply assume that the region has a constant density of $\rho = 1$ and compute the corresponding center of mass.



BREAK

# PARTIAL DERIVATIVES

Function of two or more variables



# Partial Derivatives

## Objectives

- Find and use partial derivatives of a function of two variables.
- Find and use partial derivatives of a function of three or more variables.
- Find higher-order partial derivatives of a function of two or three variables.



## Partial Derivatives of a Function of Two Variables

In applications of functions of several variables, the question often arises, "How will the value of a function be affected by a change in one of its independent variables?" You can answer this by considering the independent variables one at a time.

E.g., to determine the effect of a catalyst in an experiment, a chemist could conduct the experiment several times using varying amounts of the catalyst, while keeping constant other variables such as temperature and pressure.

similar procedure to determine the rate of change of a function $f$ with respect to one of its several independent variables. This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of $f$ with respect to the chosen independent variable.



## Partial Derivatives of a Function of Two Variables

**DEFINITION OF PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES**

If $z = f(x, y)$, then the **first partial derivatives** of $f$ with respect to $x$ and $y$ are the functions $f_x$ and $f_y$ defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

This definition indicates that if $z = f(x, y)$, then to find $f_x$ you *consider y constant* and differentiate with respect to $x$. Similarly, to find $f_y$, you *consider x constant* and differentiate with respect to $y$.

## EXAMPLE 1 Finding Partial Derivatives

Find the partial derivatives $f_x$ and $f_y$ for the function

$$f(x, y) = 3x - x^2y^2 + 2x^3y.$$

**Solution** Considering $y$ to be constant and differentiating with respect to $x$ produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad \text{Write original function.}$$

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y. \qquad \text{Partial derivative with respect to } x$$

Considering $x$ to be constant and differentiating with respect to $y$ produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad \text{Write original function.}$$

$$f_y(x, y) = -2x^2y + 2x^3. \qquad \text{Partial derivative with respect to } y$$



### NOTATION FOR FIRST PARTIAL DERIVATIVES

For $z = f(x, y)$, the partial derivatives $f_x$ and $f_y$ are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point $(a, b)$ are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a, b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a, b)} = f_y(a, b).$$



## EXAMPLE 2 Finding and Evaluating Partial Derivatives

For $f(x, y) = xe^{x^2y}$, find $f_x$ and $f_y$, and evaluate each at the point $(1, \ln 2)$.



The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation. If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure Therefore,

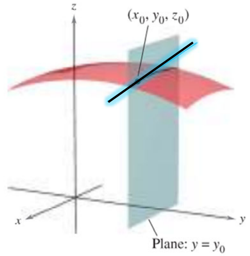
$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$. Note that both the curve and the tangent line lie in the plane $y = y_0$. Similarly,

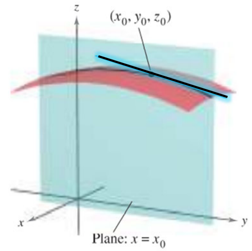
$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown

Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(x_0, y_0, z_0)$ denote the **slopes of the surface in the $x$- and $y$-directions,** respectively.




$\frac{\partial f}{\partial x}$ = slope in $x$-direction




$\frac{\partial f}{\partial y}$ = slope in $y$-direction

## EXAMPLE 3 Finding the Slopes of a Surface in the $x$- and $y$-Directions

Find the slopes in the $x$-direction and in the $y$-direction of the surface given by

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

**Solution** The partial derivatives of $f$ with respect to $x$ and $y$ are

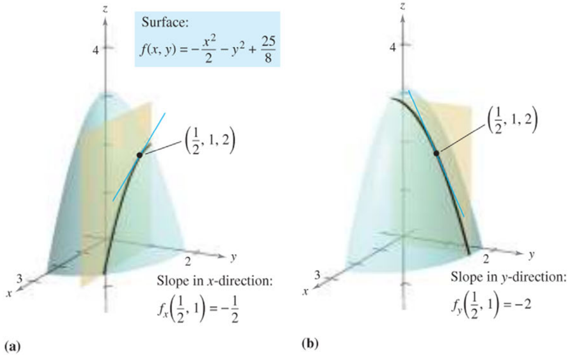
$$f_x(x, y) = -x \quad \text{and} \quad f_y(x, y) = -2y. \qquad \text{Partial derivatives}$$

So, in the $x$-direction, the slope is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2} \qquad \text{Figure (a)}$$

and in the $y$-direction, the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2. \qquad \text{Figure (b)}$$



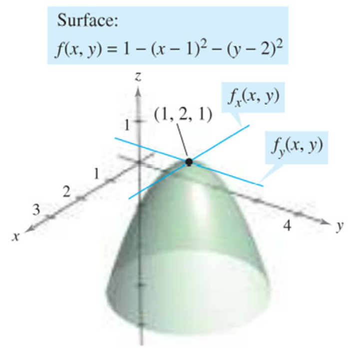



## EXAMPLE 4 Finding the Slopes of a Surface in the $x$- and $y$-Directions

Find the slopes of the surface given by

$$f(x, y) = 1 - (x - 1)^2 - (y - 2)^2$$

at the point $(1, 2, 1)$ in the $x$-direction and in the $y$-direction.





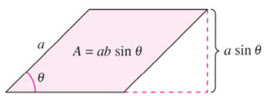

No matter how many variables are involved, partial derivatives can be interpreted as *rates of change*.

## EXAMPLE 5 Using Partial Derivatives to Find Rates of Change

The area of a parallelogram with adjacent sides $a$ and $b$ and included angle $\theta$ is given by $A = ab \sin \theta$, as shown in Figure

**a.** Find the rate of change of $A$ with respect to $a$ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

**b.** Find the rate of change of $A$ with respect to $\theta$ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.




The area of the parallelogram is $ab \sin \theta$.



# Partial Derivatives of a Function of Three or More Variables

The concept of a partial derivative can be extended naturally to functions of three or more variables.

For instance, if $w = f(x, y, z)$, there are three partial derivatives,

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are $n$ partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$

To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

## EXAMPLE 6 Finding Partial Derivatives

**a.** To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to $z$, consider $x$ and $y$ to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

**b.** To find the partial derivative of $f(x, y, z) = z\sin(xy^2 + 2z)$ with respect to $z$, consider $x$ and $y$ to be constant. Then, using the Product Rule, you obtain

**c.** To find the partial derivative of $f(x, y, z, w) = (x + y + z)/w$ with respect to $w$, consider $x$, $y$, and $z$ to be constant and obtain



# Higher-Order Partial Derivatives

Higher-order derivatives are denoted by the order in which the differentiation occurs.

For instance, the function $z = f(x, y)$ has the following second partial derivatives.

1. Differentiate twice with respect to $x$: $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$
2. Differentiate twice with respect to $y$: $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$
3. Differentiate first with respect to $x$ and then with respect to $y$: $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$
4. Differentiate first with respect to $y$ and then with respect to $x$: $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives.**

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ Right-to-left order

$(f_x)_y = f_{xy}$ Left-to-right order



## EXAMPLE 7 Finding Second Partial Derivatives

Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$, and determine the value of $f_{xy}(-1, 2)$.

**NOTE** Notice in Example 7 that the two mixed partials are equal. Sufficient conditions for this occurrence are given in Theorem 13.3. ■



**THEOREM 13.3 EQUALITY OF MIXED PARTIAL DERIVATIVES**

If $f$ is a function of $x$ and $y$ such that $f_{xy}$ and $f_{yx}$ are continuous on an open disk $R$, then, for every $(x, y)$ in $R$,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

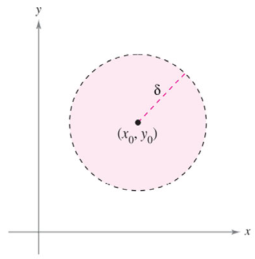
Theorem 13.3 also applies to a function $f$ of *three or more variables* so long as all second partial derivatives are continuous.

## Neighborhoods in the Plane

**$\delta$-neighborhood** about $(x_0, y_0)$

the **disk** centered at $(x_0, y_0)$ with radius $\delta > 0$

$$\left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}$$ Open disk




An open disk

## EXAMPLE 8 Finding Higher-Order Partial Derivatives

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by

$$f(x, y, z) = ye^x + x \ln z.$$