Mathematics
for
Information Technology

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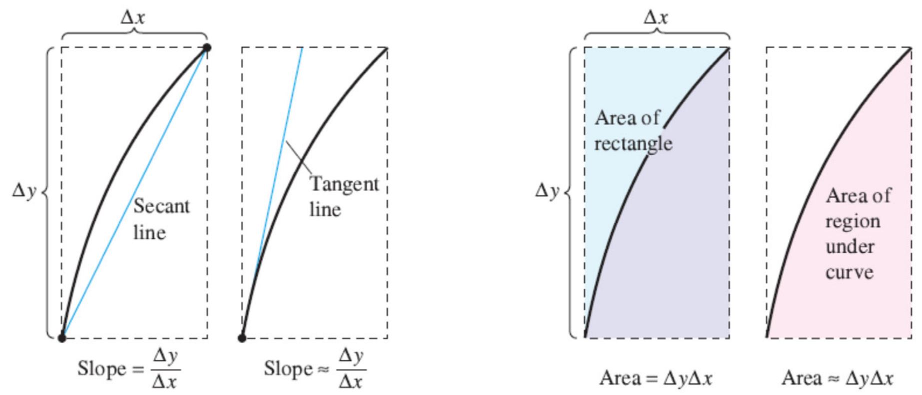
INTEGRATION (II)

The Fundamental Theorem of Calculus III

Objectives

- Evaluate a definite integral using the Fundamental Theorem of Calculus.
- Understand and use the Mean Value Theorem for Integrals.
- Find the average value of a function over a closed interval.
- Understand and use the Second Fundamental Theorem of Calculus.
- Understand and use the Net Change Theorem.

The Fundamental Theorem of Calculus



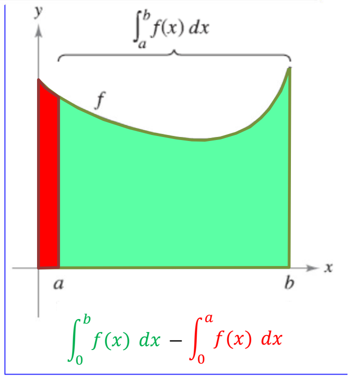
(a) Differentiation **(b)** Definite integration

Differentiation and definite integration have an "inverse" relationship.

THEOREM 4.9 THE FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$



GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\boxed{\begin{aligned}\int_a^b f(x)\,dx &= F(x)\Big]_a^b \\ &= F(b) - F(a)\end{aligned}}$$

$\Big[\ \Big]_a^b \quad \Big]_a^b$ ถ้าเทอมเดียว เขียนแบบนี้ก็ได้

เขียนได้ทั้งสอง

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

ตั้งแต่ 1 ถึง 3

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$ → พื้นที่ใต้กราฟ

3. It is not necessary to include a constant of integration C in the antiderivative because

$C - C$ ก็หายไปอยู่ดี

$$\begin{aligned}\int_a^b f(x)\,dx &= \Big[F(x) + C\Big]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a).\end{aligned}$$

วิธีลัดทางนี้ $\Big[\ \Big]_a^b$ แบบนี้

EXAMPLE 1 Evaluating a Definite Integral

Evaluate each definite integral.

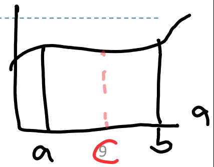
a. $\int_1^2 (x^2 - 3)\,dx$ **b.** $\int_1^4 3\sqrt{x}\,dx$ **c.** $\int_0^{\pi/4} \sec^2 x\,dx$

Solution

a. $\int_1^2 (x^2 - 3)\,dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \frac{8}{3} - 6 - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$

b. $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{\frac{1}{2}}\,dx = 3\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_1^4$

c. $\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1 - 0 = 1$



$$\int^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Integrate Abs ต้องแบ่งเป็น 2 ช่วง

EXAMPLE 2 A Definite Integral Involving Absolute Value

Evaluate $\int_0^2 |2x - 1|\,dx$. $\frac{1}{2}$

$$\int_0 (-2x+1)\,dx = \frac{-2x^2}{2} + x$$

Solution

$$= -x^2 + x$$

$$= \Big[-x^2 + x\Big]_0^{1/2}$$

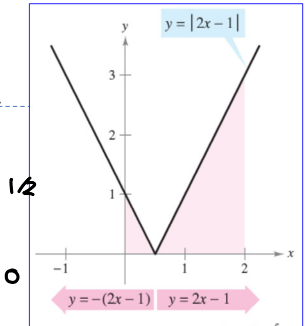
$$\int_{\frac{1}{2}}^2 (2x-1)\,dx = \Big[x^2 - x\Big]_{\frac{1}{2}}^2$$

$$\text{Area} = \Big[-x^2+x\Big]_0^{\frac{1}{2}} + \Big[x^2 - x\Big]_{\frac{1}{2}}^2 = -\frac{1}{4} + \frac{1}{2} + 4 - 2 - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{2}{4} + \frac{16}{4} - \frac{8}{4}$$

$$= -\left(\frac{1}{4} - \frac{2}{4}\right)$$

$$= \frac{9}{4} + \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$$



The definite integral of y on $[0, 2]$ is $\frac{5}{2}$.

$f_1(x) = -2x + 1$
$f_2(x) = 2x - 1$

$0 = -2x + 1$
$-1 = -2x$
$x = \frac{1}{2}$

หาจุดที่ $f(x) = 0$

EXAMPLE 3 Using the Fundamental Theorem to Find Area

Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$, as shown in Figure

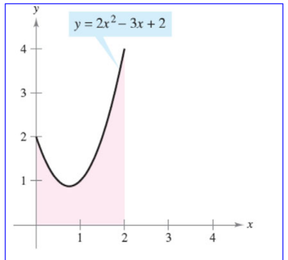
$f(x) = 2x^2 - 3x + 2$

$$\int_0^2 (2x^2 - 3x + 2)\,dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2$$

$$= \frac{2(8)}{3} - \frac{3\cancel{(4)}^{2}}{\cancel{2}} + 4$$

$$= \frac{16}{3} - 6 + 4 = \frac{16}{3} - 2$$

$$= \frac{16}{3} - \frac{6}{3} = \frac{10}{3} \ \#$$

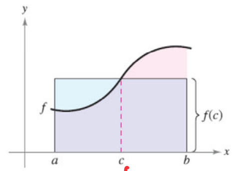


The area of the region bounded by the graph of y, the x-axis, $x = 0$, and $x = 2$ is $\frac{10}{3}$.

ทฤษฎีค่าเฉลี่ย

The Mean Value Theorem for Integrals

Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure



ความสูง Avg

Mean value rectangle:
$f(c)(b - a) = \int_a^b f(x)\,dx$

THEOREM 4.10 MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

NOTE Notice that Theorem 4.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval.

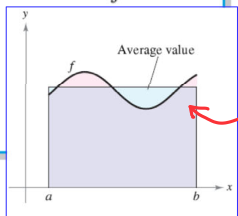
Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION ON AN INTERVAL

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$



Average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

เป็น continuous

ถ้า discrete หา Avg
ต้องใช้ Histogram
อะไรพวกนี้

NOTE Notice in Figure that the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.

EXAMPLE 4 Finding the Average Value of a Function

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution The average value is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{3}\int_1^4 \frac{\cancel{3}x^3}{\cancel{3}} - \frac{\cancel{2}x^2}{\cancel{2}}$$

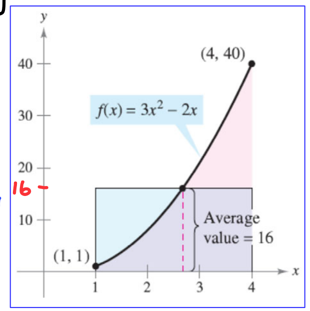
$$= \frac{1}{3}\left[x^3 - x^2\right]_1^4$$

$$= \frac{1}{3}\left[64 - 16 - 0\right]$$

$$= \frac{1}{3}(48) = 16 \ \#$$

ค่าเฉลี่ย

สรุปได้ว่า จุด x=16 พื้นที่ □
จะมีค่า พื้นที่สีชมพู



The Second Fundamental Theorem of Calculus

ทฤษฎีตัวที่สอง

The Definite Integral as a Number

Constant

$$\int_a^b f(x)\,dx$$

Constant

f is a function of x.

The Definite Integral as a Function of x

F is a function of x.

$$F(x) = \int_a^x f(t)\,dt$$

Upper Limit ไม่ได้เป็น constant

Constant

f is a function of t.

EXAMPLE 6 The Definite Integral as a Function

Evaluate the function

$$F(x) = \int_0^x \cos t\,dt$$

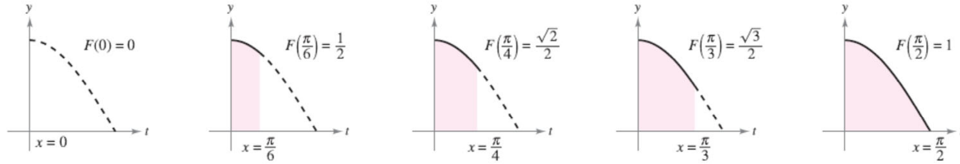
at $x = 0$, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$.

Solution You could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x (as a constant) temporarily to obtain

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x.$$

อยากแทน x เป็นอะไร ก็ได้เลย

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure



$F(x) = \int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

an **accumulation function**

accumulating the area

THEOREM 4.11 THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

* Proved มาแล้ว

บางที Integration Process มันยาว → ใช้ทฤษฎีนี้ไปเลย ว่ามันเป็น "f(x)"

It can be used to avoid the long process of integration followed by differentiation

EXAMPLE 8 Using the Second Fundamental Theorem of Calculus

Find the derivative of $F(x) = \displaystyle\int_{\pi/2}^{x^3} \cos t\,dt$.

Solution Using $u = x^3$, you can apply the Second Fundamental Theorem of Calculus with the Chain Rule as shown.

* ใช้ Chain Rule มาช่วย

$$\begin{aligned}
F'(x) &= \frac{dF}{du}\frac{du}{dx} && \text{Chain Rule} \\
&= \frac{d}{du}[F(x)]\frac{du}{dx} && \text{Definition of } \frac{dF}{du} \\
&= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } \int_{\pi/2}^{x^3} \cos t\,dt \text{ for } F(x). \\
&= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\,dt\right]\frac{du}{dx} && \text{Substitute } u \text{ for } x^3. \\
&= (\cos u)(3x^2) && \text{Apply Second Fundamental Theorem of Calculus.} \\
&= (\cos x^3)(3x^2) && \text{Rewrite as function of } x.
\end{aligned}$$

$$\boxed{\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)}$$

Alternative Because the integrand in Example 8 is easily integrated, you can verify the derivative as follows.

$$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt = \sin t\Big]_{\pi/2}^{x^3} = \sin x^3 - \sin\frac{\pi}{2} = (\sin x^3) - 1$$

อย่าลืม Chain Rule สำหรับ Composite Func.

In this form, you can apply the Power Rule to verify that the derivative is the same as that obtained in Example 8.

$$F'(x) = (\cos x^3)(3x^2)$$

Net Change Theorem

ทฤษฎีการเปลี่ยนแปลงรวม

THEOREM 4.12 THE NET CHANGE THEOREM

The definite integral of the rate of change of a quantity $F'(x)$ gives the total change, or **net change**, in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

- นำไปใช้แก้สมการ/โจทย์ของปัญหา
- หาผลรวมของอัตราการเปลี่ยนแปลง

EXAMPLE 9 Using the Net Change Theorem

Change Rate

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minute, where $0 \le t \le 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

Solution Let $c(t)$ be the amount of the chemical in the tank at time t. Then $c'(t)$ represents the rate at which the chemical flows into the tank at time t. During the first 20 minutes, the amount that flows into the tank is

$$\int_0^{20} c'(t)\,dt = \int_0^{20} (180 + 3t)\,dt$$
$$= \left[180t + \frac{3}{2}t^2\right]_0^{20}$$
$$= 3600 + 600 = 4200.$$

ปริมาณทั้งหมดที่ไหลในช่วง 20 นาทีแรก

So, the amount that flows into the tank during the first 20 minutes is 4200 liters.

ได้ $t = 1, t = 4, t = 5$

Factor แล้วหาจุดวิกฤติ ที่ $t = 0$

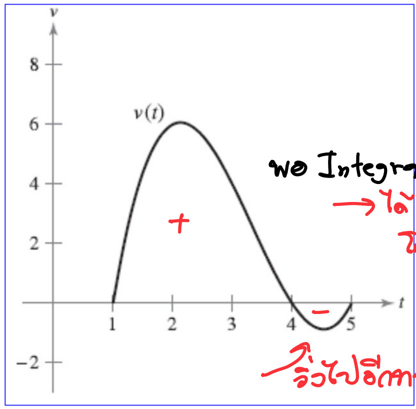
EXAMPLE 10 Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ feet per second at time t. การกระจัด; ดูแค่จุดแรก & สุดท้าย

a. What is the displacement of the particle on the time interval $1 \le t \le 5$?

b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

Solution



พอ Integrate จะหักล้างกัน → ได้ Displacement โดยปริยาย

วิ่งไปอีกทาง

EXAMPLE 10 Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ feet per second at time t.

a. What is the displacement of the particle on the time interval $1 \le t \le 5$? $= \frac{32}{3}$

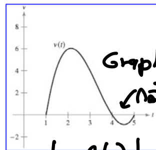
b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

Solution

→ ต้องแยกเพราะไป → หาจุดแบ่งที่ $t = 0$

$$\int_1^5 |v(t)|\,dt = \int_1^4 v(t)\,dt + \int_4^5 -v(t)\,dt$$

ความเร็วเปลี่ยนจาก + → −



Graph จะกลับหัว

$|v(t)|$

BREAK

Integration by Substitution

IV

Objectives

- Use pattern recognition to find an indefinite integral.
- Use a change of variables to find an indefinite integral.
- Use the General Power Rule for Integration to find an indefinite integral.
- Use a change of variables to evaluate a definite integral.

Pattern Recognition

In this section you will study techniques for integrating composite functions. The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***u*-substitution**. With pattern recognition you perform the substitution mentally, and with change of variables you write the substitution steps.

คิดในใจ

THEOREM 4.13 ANTIDIFFERENTIATION OF A COMPOSITE FUNCTION

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

NOTE The statement of Theorem 4.13 doesn't tell how to distinguish between $f(g(x))$ and $g'(x)$ in the integrand. As you become more experienced at integration, your skill in doing this will increase. Of course, part of the key is familiarity with derivatives.

Outside function

ถ้ามี Pattern แบบนี้

Integrate เฉพาะ ฟังก์ชันนอกได้เลย!

$$\int f(g(x))\underbrace{g'(x)}\,dx = F(g(x)) + C$$

Inside function

Derivative of inside function

EXAMPLE 1 Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2 + 1)^2(2x)\,dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Using the Power Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\,dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is the integrand of the original integral.

Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple. In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

EXAMPLE 3 Multiplying and Dividing by a Constant

EXAMPLE 1

Find $\int (x^2 + 1)^2(2x)\,dx$.

Find $\int x(x^2 + 1)^2\,dx$.

ขาด 2 ให้เป็น 2x

Solution This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2. Recognizing that $2x$ is the derivative of $x^2 + 1$, you can let $g(x) = x^2 + 1$ and supply the $2x$ as follows.

เติม $\frac{2}{2}$ มา

$$\int x(x^2 + 1)^2\,dx = \int (x^2 + 1)^2\left(\frac{1}{2}\right)(2x)\,dx \quad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx \quad \text{Constant Multiple Rule}$$

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C \quad \text{Integrate.}$$

$$= \frac{1}{6}(x^2 + 1)^3 + C \quad \text{Simplify.}$$

EXAMPLE 3 Multiplying and Dividing by a Constant

Find $\int x(x^2 + 1)^2\,dx$.

Solution

In practice, most people would not write as many steps as are shown in Example 3. For instance, you could evaluate the integral by simply writing

$$\int x(x^2 + 1)^2\,dx = \frac{1}{2}\int (x^2 + 1)^2\,2x\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C$$

$$= \frac{1}{6}(x^2 + 1)^3 + C.$$

Change of Variables

ข้อเสีย : ใช้เวลา

ข้อดี : หาคำตอบได้แน่นอน

With a formal **change of variables,** you completely rewrite the integral in terms of u and du (or any other convenient variable). Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 to 3, it is useful for complicated integrands. The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)\,dx$, and the integral in Theorem 4.13 takes the form

$\frac{du}{dx} = g'(x)$ การเปลี่ยนแปลงของ u เทียบกับ x

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

แปลงให้เป็น du, u

EXAMPLE 4 Change of Variables

Pattern Recognition

$$\boxed{\frac{1}{2}\int (2x-1)2\,dx}$$

Find $\int \sqrt{2x-1}\,dx$.

Solution First, let u be the inner function, $u = 2x - 1$. Then calculate the differential du to be $du = 2\,dx$. Now, using $\sqrt{2x-1} = \sqrt{u}$ and $dx = du/2$, substitute to obtain

$$\begin{aligned}
\int \sqrt{2x-1}\,dx &= \int \sqrt{u}\left(\frac{du}{2}\right) && \text{Integral in terms of } u \\
&= \frac{1}{2}\int u^{1/2}\,du && \text{Constant Multiple Rule} \\
&= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C && \text{Antiderivative in terms of } u \\
&= \frac{1}{3}u^{3/2} + C && \text{Simplify.} \\
&= \frac{1}{3}(2x-1)^{3/2} + C. && \text{Antiderivative in terms of } x
\end{aligned}$$

EXAMPLE 5 Change of Variables

Pattern Recognition

→ ยาก, มีส่วนเกิน x

Find $\int x\sqrt{2x-1}\,dx$. $= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.$ {ans.}

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

THEOREM 4.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

EXAMPLE 7 Substitution and the General Power Rule

a. $\int 3(3x-1)^4\,dx = \int \underbrace{(3x-1)^4}_{u^4}\underbrace{(3)\,dx}_{du} = \underbrace{\frac{(3x-1)^5}{5}}_{u^5/5} + C$

Composite Pattern

b. $\int (2x+1)(x^2+x)\,dx =$

c. $\int 3x^2\sqrt{x^3-2}\,dx =$

d. $\int \frac{-4x}{(1-2x^2)^2}\,dx = \int (-2x^2+1)^{-2}(-4x)\,dx = \int (-2x^2+1)^{-2}dx$

$= \frac{(1-2x)^{-1}}{-1} + C$

$= \frac{-1}{1-2x^2} + C$

e. $\int \cos^2 x \sin x\,dx = -\int \cos^2 x(-\sin x)\,dx$

$= \boxed{\frac{-\cos^3 x}{3} + C}$

Integration by Substitution

Some integrals whose integrands involve quantities raised to powers cannot be found by the General Power Rule. Consider the two integrals

$$\int x(x^2+1)^2\,dx \quad \text{and} \quad \int (x^2+1)^2\,dx.$$

← ขาด component ที่เป็น derivate ของฟังก์ชั่นข้างใน

The substitution $u = x^2 + 1$ works in the first integral but not in the second. In the second, the substitution fails because the integrand lacks the factor x needed for du. Fortunately, *for this particular integral*, you can expand the integrand as $(x^2+1)^2 = x^4 + 2x^2 + 1$ and use the (simple) Power Rule to integrate each term.

→ อาจจะต้องกระจาย

Integration by Substitution

EXAMPLE

$\int (2x+3)^7\,dx = \frac{1}{2}\int (2x+3)^7(2)\,dx$

$=$

Integration by Substitution

EXAMPLE

$$\int x(x^2+1)^4 dx = \frac{1}{2}\int (x^2+1)^4 (2x)dx$$

$$= \frac{(x^2+1)^5}{5(2)} = \frac{(x^2+1)^5}{10} + C$$

Integration by Substitution

EXAMPLE

$$\int 3x^2\sqrt{x^3+5}dx = \frac{(x^3+5)^{3/2}}{3/2} + C$$

$$= \frac{2}{3}(x^3+5)^{3/2} + C$$

Integration by Substitution

EXAMPLE

$$\boxed{\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0}$$

$$\boxed{\int \frac{1}{u}du = \ln|u| + C}$$

$$\int \frac{x}{3x^2+4}dx$$

$$= \int (3x^2+4)^{-1}(x)\,dx$$

$$= \frac{1}{6}\int (3x^2+4)^{-1}(6x)dx$$

$$= \frac{1}{6}\ln|3x^2+4| + C$$

Integration by Substitution

EXAMPLE

$$\int \sin(3x+2)\,dx$$

$$\frac{1}{3}\int \sin(3x+2)(3)dx$$

$$= \frac{\cos(3x+2)}{3} + C$$

Integration by Substitution

Trigonometric → ยาก

sin(2x)=2sinxcosx

EXAMPLE $\int \sin x \cos x \, dx$

$u = \sin x \Rightarrow du = \cos x \, dx$

$\int u \, du = \frac{u^2}{2} + C$

$= \frac{\sin^2 x}{2} + C$

Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

เพราะอันนี้เป็นค่า x

EXAMPLE 8 Change of Variables

Evaluate $\int_0^1 x(x^2+1)^3\,dx$.

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \Rightarrow du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit x=0 → แทนค่าเข้าไป? *Upper Limit*

When $x = 0$, $u = 0^2 + 1 = 1$. When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\int_0^1 x(x^2+1)^3\,dx = \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,dx$$

Integration limits for x

upper/lower limit ใหม่

$$= \frac{1}{2}\int_1^2 u^3\,du$$

Integration limits for u

หาตรงนี้ไปเลย

พื้นที่ใต้กราฟเท่ากัน แต่ได้ Graph คนละแบบ

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right)$$

$$= \frac{15}{8}.$$

Try rewriting the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown.

$$\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1 = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$$

Notice that you obtain the same result. ■

EXAMPLE 9 Change of Variables

Evaluate $A = \int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$.

u=2x−1

Ans. = 16/3