Mathematics for Information Technology

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Integration by Parts

- **OBJECTIVES**
  - Find an antiderivative using integration by parts.

Integration by Parts

This technique can be applied to a wide variety of functions and is particularly useful for integrands involving *products* of algebraic and transcendental functions. For instance, integration by parts works well with integrals such as

$$\int x \ln x \, dx, \quad \int x^2 e^x \, dx, \quad \text{and} \quad \int e^x \sin x \, dx.$$

Integration by parts is based on the formula for the derivative of a product

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'$$

where both u and v are differentiable functions of x. If u' and v' are continuous, you can integrate both sides of this equation to obtain

$$\boxed{\frac{dv}{dx} = v' \rightarrow dv = v'dx}$$

$$\boxed{\frac{du}{dx} = u' \rightarrow du = u'dx}$$

$$uv = \int uv'\,dx + \int vu'\,dx = \int u\,dv + \int v\,du.$$

By rewriting this equation, you obtain the following theorem.

THEOREM 8.1 INTEGRATION BY PARTS

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

This formula expresses the original integral in terms of another integral. Depending on the choices of u and dv, it may be easier to evaluate the second integral than the original one. Because the choices of u and dv are critical in the integration by parts process, the following guidelines are provided.

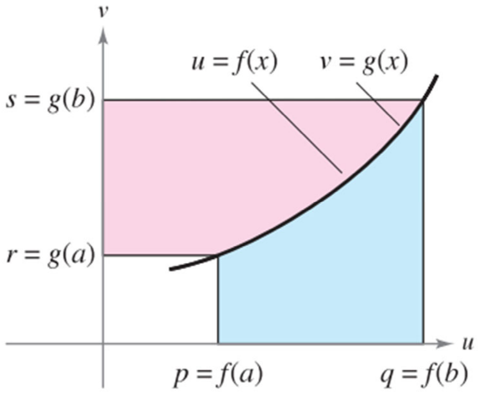
GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

$$\int u\,dv = uv - \int v\,du.$$

Proof Without Words Here is a different approach to proving the formula for integration by parts. Exercise taken from "Proof Without Words: Integration by Parts" by Roger B. Nelsen, *Mathematics Magazine*, 64, No. 2, April 1991, p. 130, by permission of the author.



Area (pink) + Area (blue) = qs − rp

$$\int_r^s u\,dv + \int_p^q v\,du = \Big[uv\Big]_{(p,r)}^{(q,s)}$$

u เป็น p
v เป็น r

$$\int_r^s u\,dv = \Big[uv\Big]_{(p,r)}^{(q,s)} - \int_p^q v\,du$$

EXAMPLE 1 Integration by Parts

$$\int u\,dv = uv - \int v\,du.$$

Find $\int xe^x\,dx$.

Solution To apply integration by parts, you need to write the integral in the form $\int u\,dv$. There are several ways to do this.

$$\boxed{\int \underbrace{(x)}_{u}\underbrace{(e^x dx)}_{dv}},\quad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv},\quad \int \underbrace{(1)}_{u}\underbrace{(xe^x\,dx)}_{dv},\quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

เลือกได้หลายแบบมาก *

the first option because the derivative of $u = x$ is simpler than x, and $dv = e^x\,dx$ is the most complicated portion of the integrand that fits a basic integration formula.

- ควรเลือก u ที่ดิฟแล้ว ง่ายกว่าเดิม
- dv ซับซ้อน, Integratable

$$dv = e^x\,dx \quad\Rightarrow\quad v = \int dv = \int e^x\,dx = e^x$$

$$u = x \quad\Rightarrow\quad \frac{du}{dx} = 1$$

Now, integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C. \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

EXAMPLE 2 Integration by Parts

$$\int u\,dv = uv - \int v\,du.$$

Find $\int x^2 \ln x\,dx$. ln x Integrate ไม่ได้

Solution In this case, x^2 is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. So, you should let $dv = x^2\,dx$.

$$dv = x^2\,dx \quad\Rightarrow\quad v = \int x^2\,dx = \frac{x^3}{3}$$

Diff สองข้าง

$$u = \ln x \quad\Rightarrow\quad du = \frac{1}{x}\,dx$$

dx ย้ายมา

Integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx \qquad \text{Substitute.}$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \qquad \text{Simplify.}$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \text{Integrate.}$$

You can check this result by differentiating.

$$\frac{d}{dx}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right] = \frac{x^3}{3}\left(\frac{1}{x}\right) + (\ln x)(x^2) - \frac{x^2}{3} = x^2 \ln x$$

EXAMPLE 4 Repeated Use of Integration by Parts

Find $\int x^2 \sin x\,dx$.

Solution The factors x^2 and $\sin x$ are equally easy to integrate. However, the derivative of x^2 becomes simpler, whereas the derivative of $\sin x$ does not. So, you should let $u = x^2$.

$$dv = \sin x\,dx \quad\Rightarrow\quad v = \int \sin x\,dx = -\cos x$$

$$u = x^2 \quad\Rightarrow\quad du = 2x\,dx$$

Now, integration by parts produces

Algebra - Transcendental
ลองทำต่อ

$$\int x^2 \sin x\,dx = \boxed{-x^2\cos x} + \boxed{\int 2x\cos x\,dx.} \qquad \text{First use of integration by parts} \qquad \int u\,dv = \boxed{uv} - \boxed{\int v\,du.}$$

This first use of integration by parts has succeeded in simplifying the original integral, but the integral on the right still doesn't fit a basic integration rule. To evaluate that integral, you can apply integration by parts again. This time, let $u = 2x$.

เลือกสลับ
↳ ท้ายสุดจะกลับไปตัวเดิม

$$\boxed{\begin{array}{l} dv = \cos x\,dx \\ u = 2x \end{array}} \quad\Rightarrow\quad \begin{array}{l} v = \int \cos x\,dx = \sin x \\ du = 2\,dx \end{array}$$

Now, integration by parts produces

$$\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx \qquad \text{Second use of integration by parts}$$

$$= 2x\sin x + 2\cos x + C.$$

Combining these two results, you can write

$$\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

not to interchange the substitutions in successive applications. the first substitution was $u = x^2$ and $dv = \sin x\,dx$. the second application, $\boxed{u = \cos x \text{ and } dv = 2x\,dx}$

$$\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx$$

$$= -x^2\cos x + x^2\cos x + \int x^2\sin x\,dx = \boxed{\int x^2\sin x\,dx}$$

undoing the previous integration and returning to the *original* integral

watch for a *constant multiple* of the original integral.

As you gain experience in using integration by parts, your skill in determining u and dv will increase. The following summary lists several common integrals with suggestions for the choices of u and dv.

SUMMARY OF COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax}\,dx, \qquad \int x^n \sin ax\,dx, \qquad \text{or} \qquad \int x^n \cos ax\,dx$$

let $u = x^n$ and let $dv = e^{ax}\,dx$, $\sin ax\,dx$, or $\cos ax\,dx$.

2. For integrals of the form

ควรเป็น u

$$\int x^n \ln x\,dx, \qquad \int x^n \arcsin ax\,dx, \qquad \text{or} \qquad \int x^n \arctan ax\,dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n\,dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx\,dx \qquad \text{or} \qquad \int e^{ax} \cos bx\,dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\,dx$.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

STUDY TIP You can use the acronym LIATE as a guideline for choosing u in integration by parts. In order, check the integrand for the following.

Is there a Logarithmic part? (log ดิฟอย่างเดียว, Integrate ไม่ได้) *

Is there an Inverse trigonometric part?

Is there an Algebraic part?

Is there a Trigonometric part?

Is there an Exponential part?

เลือก u ตามลำดับนี้

↓ 5^x, 3^{x+2}

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method, illustrated in Example 7, can help to organize the work. This method works well for integrals of the form $\int x^n \sin ax\,dx$, $\int x^n \cos ax\,dx$, and $\int x^n e^{ax}\,dx$.

EXAMPLE 7 Using the Tabular Method

Find $\int x^2 \sin 4x\,dx$.

ใช้บ่อยกับ Trig. Func. เพราะ loop ซ้ำๆ

Solution Begin as usual by letting $u = x^2$ and $dv = v'\,dx = \sin 4x\,dx$. Next, create a table consisting of three columns, as shown.

composite func. pattern recognition

Alternate Signs	*u and Its Derivatives*	*v′ and Its Antiderivatives*
+	x^2	$\sin 4x$
−	$2x$	$-\frac{1}{4}\cos 4x$
+	2	$-\frac{1}{16}\sin 4x$
−	0	$\frac{1}{64}\cos 4x$

Differentiate until you obtain 0 as a derivative.

รวมร่างตามแนวนี้ (คูณกัน)

(+C ด้วย)

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x\,dx = \boxed{-\frac{1}{4}x^2\cos 4x} + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C.$$

Tabular method

(1)

$$\int x^n e^{ax}\,dx, \qquad \int x^n \sin ax\,dx, \qquad \text{or} \qquad \int x^n \cos ax\,dx$$

①

EXAMPLE $\int x \sin x\,dx$?

$$\int u\,dv = uv - \int v\,du$$

$u = x$, $du = 1(dx)$

$dv = \sin x\,dx$

$v = \int \sin x\,dx$

$v = -\cos x$

$\int u\,dv = -x\cos x - \int -\cos x\,dx$

$= -x\cos x + \sin x + C$ ✓

(3)

$\int x^n e^{ax}\,dx, \quad \int x^n \sin ax\,dx, \quad$ or $\quad \int x^n \cos ax\,dx$ ①

EXAMPLE $\int x^2 \cos x\,dx$?

$\int u\,dv = uv - \int v\,du$

Algebra → Trig.

$u = x^2 \qquad \frac{du}{dx} = \frac{dx^2}{dx}$

$dv = \cos x\,dx \qquad du = 2x\,dx$

$v = \sin x$

$x^2 \sin x - \int \sin x\, 2x\,dx$

(5)

$\int x^n e^{ax}\,dx, \quad \int x^n \sin ax\,dx, \quad$ or $\quad \int x^n \cos ax\,dx$ ①

EXAMPLE $\int \underbrace{x^4}_{u} \underbrace{e^x\,dx}_{dv}$

$\int x^n \sin ax\,dx$, $\int x^n \cos ax\,dx$, and $\int x^n e^{ax}\,dx$

Using the Tabular Method ?

$dv = v'dx = e^x dx$

Signs	u and its derivaties	v' and its anti derivaties
+	x^4	e^x
−	$4x^3$	e^x
+	$12x^2$	e^x
−	$24x$	e^x
+	24	e^x
−	0	e^x

$\int x^4 e^x dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x$

(7)

$\int x^n \ln x\,dx, \quad \int x^n \arcsin ax\,dx, \quad$ or $\quad \int x^n \arctan ax\,dx$ ②

EXAMPLE $\int x^3 \underbrace{\ln x}_{u} \,dx$? $\quad dv$

$\int u\,dv = uv - \int v\,du$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{1}{x}(dx) \qquad v = \frac{x^4}{4}$

$\int u\,dv = \ln x\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4}\left(\frac{1}{x}\right)dx$

$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 dx$

$= \frac{x^4 \ln x}{4} - \frac{1}{4}\left(\frac{x^4}{4}\right) = \frac{4x^4 \ln x}{16} - \frac{x^4}{16}$

$= \frac{4x^4 \ln - x^4}{16} + C$ ✓

$\left(\frac{\ln x}{4} + \frac{-1}{16}\right)x^4 + C$

(9)

$\int e^{ax} \sin bx\,dx \quad$ or $\quad \int e^{ax} \cos bx\,dx$ ③

EXAMPLE $\int e^x \underbrace{\cos x}_{u}\,dx$? $\quad dv$

??

$u = \cos x \qquad dv = e^x dx$

$v = e^x$

Key ต้องย้ายไปอีกฝั่ง

แล้วบวกกัน

(9)

$\int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$ ③

EXAMPLE $\int e^{x} \cos x\, dx$

by Substitution & by part

EXAMPLE $\int \cos(\sqrt{x})\, dx$

Integration with respect to u

$$u = \sqrt{x} \rightarrow \frac{du}{dx} = \frac{1}{2}x^{-1/2} \rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \rightarrow \frac{du}{dx} = \frac{1}{2u} \rightarrow dx = 2u\, du$$

$$\int \cos(\sqrt{x})\, dx = \int \cos u\, (2u)\, du = 2\int u\,(\cos u)\, du$$

By substitution

By part

$u = x$ $\qquad \frac{du}{dx} = 1$

$$\int u\, dv = uv - \int v\, du \leftarrow du = dx$$

$dv = \cos x\, dx$ $\qquad v = \int dv = \int \cos x\, dx = \sin x$

$$\int x \cos x\, dx = (x)(\sin x) - \int (\sin x)\, dx = x \sin x - (-\cos x) + C$$
$$= x \sin x + \cos x + C$$

by Substitution & by part

EXAMPLE $\int \cos(\sqrt{x})\, dx$

Integration with respect to u

$$u = \sqrt{x} \rightarrow \frac{du}{dx} = \frac{1}{2}x^{-1/2} \rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \rightarrow \frac{du}{dx} = \frac{1}{2u} \rightarrow dx = 2u\, du$$

$$\int \cos(\sqrt{x})\, dx = \int \cos u\, (2u)\, du = 2\int u\,(\cos u)\, du$$

By substitution

By part

$$= 2(u \sin u + \cos u) + C$$

$$= 2(\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}) + C$$

$$\int x \cos x\, dx = x \sin x + \cos x + C$$

BREAK

APPLICATIONS OF INTEGRATION

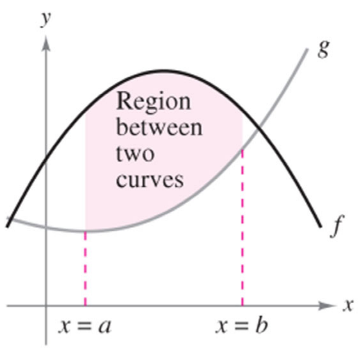
Applications of Integration (I)

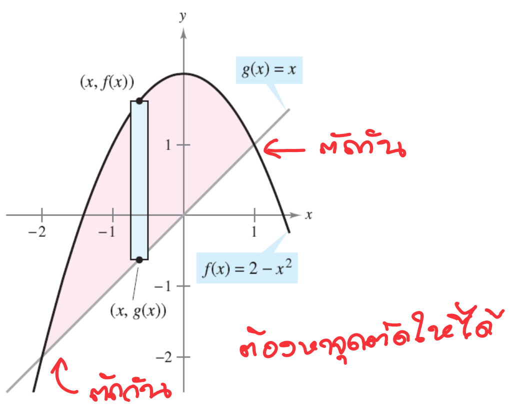
Objectives

- Use a definite integral to find the area of a region bounded by two curves.
- Find the volume of a solid of revolution.

Area of a Region Between Two Curves

- Find the area of a region between two curves using integration.
- Find the area of a region between intersecting curves using integration.

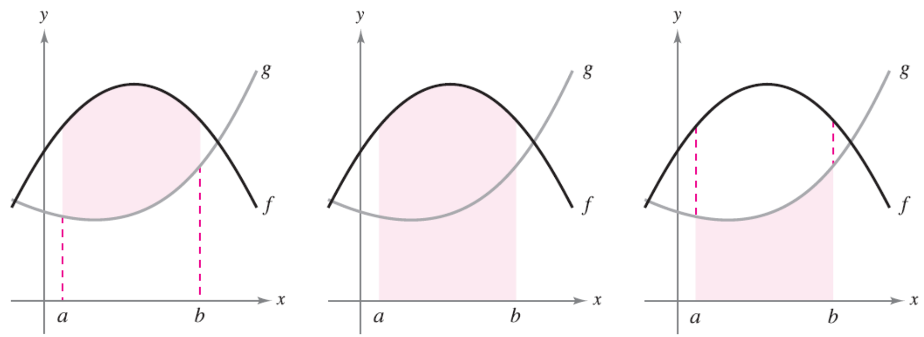




Area of a Region Between Two Curves

Previously ...

area of a region *under* a curve $\longrightarrow$ area of a region *between* two curves



Area of region between f and g	=	Area of region under f	−	Area of region under g
$\boxed{\int_a^b [f(x) - g(x)]\,dx}$	=	$\int_a^b f(x)\,dx$	−	$\int_a^b g(x)\,dx$

Area of a Region Between Two Curves

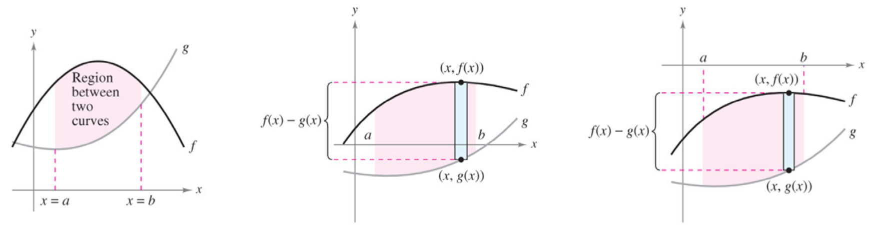
$f(x)$ สูงกว่าตลอด

ถ้าสูงบ้าง, ต่ำบ้างใช้ไม่ได้

AREA OF A REGION BETWEEN TWO CURVES

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$



continuous and $g(x) \le f(x)$ for all x in the interval $[a, b]$

EXAMPLE 1 Finding the Area of a Region Between Two Curves

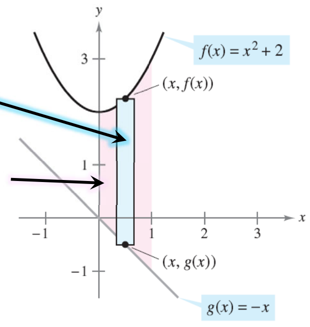
Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \le f(x)$ for all x in $[0, 1]$. So, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\,\Delta x$$
$$= [(x^2 + 2) - (-x)]\,\Delta x$$

and the area of the region is

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_0^1 [(x^2 + 2) - (-x)]\,dx$$
$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1$$
$$= \frac{1}{3} + \frac{1}{2} + 2$$
$$= \frac{17}{6}.$$



EXAMPLE 2 A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution notice that the graphs of f and g have two points of intersection. To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

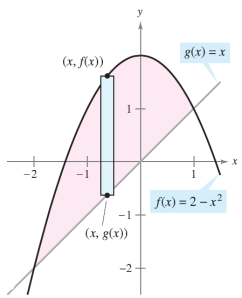
$2 - x^2 = x$	Set $f(x)$ equal to $g(x)$.
$-x^2 - x + 2 = 0$	Write in general form.
$-(x + 2)(x - 1) = 0$	Factor.
$x = -2$ or 1	Solve for x.

So, $a = -2$ and $b = 1$. Because $g(x) \le f(x)$ for all x in the interval $[-2, 1]$, the representative rectangle has an area of

$$\Delta A = [f(x) - g(x)]\,\Delta x$$
$$= [(2 - x^2) - x]\,\Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\,dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$
$$= \frac{9}{2}.$$



EXAMPLE 4 Curves That Intersect at More Than Two Points

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

$f(x) = g(x)$

$3x^3 - \cancel{x^2} - 10x = -\cancel{x^2} + 2x$

$3x^3 - 12x = 0$

$x(3x^2 - 12) = 0$

$x = 0,\quad 3x^2 = 12$

ต้องลองแทนค่า

$x^2 = 4$

$x = \pm 2$

$$\int_{-2}^{0} [f(x) - g(x)]\,dx + \int_{0}^{2} [g(x) - f(x)]\,dx$$

$$= \int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{2} (-3x^3 + 12x)\,dx$$

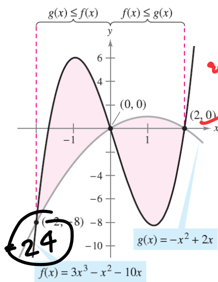
$$= \left[\frac{3x^4}{4} - \frac{12x^2}{2}\right]_{-2}^{0} + \left[\frac{-3x^4}{4} + \frac{12x^2}{2}\right]_{0}^{2}$$

$$= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[-\frac{3x^4}{4} + 6x^2\right]_{0}^{2}$$

$$= -\left(\frac{3}{\cancel{4}}\cancel{(16)} - 6(4)\right) + (-12 + 24) = 12 + 12 = 24$$

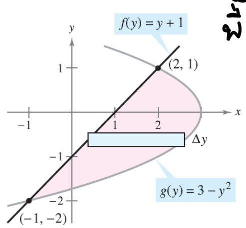
4

$12 - 24$



หลังจากนี้ปลายเปิด

function of y



พื้นที่แนวนอน

If the graph of a function of y is a boundary of a region, it is often convenient to use representative rectangles that are *horizontal* and find the area by integrating with respect to y. In general, to determine the area between two curves, you can use

เทียบกับ y

$$A = \int_{x_1}^{x_2} \underbrace{[(\text{top curve}) - (\text{bottom curve})]}_{\text{in variable } x}\, dx \qquad \text{Vertical rectangles}$$

เทียบกับ x

$$A = \int_{y_1}^{y_2} \underbrace{[(\text{right curve}) - (\text{left curve})]}_{\text{in variable } y}\, dy \qquad \text{Horizontal rectangles}$$

เทียบกับ y

where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

EXAMPLE 5 Horizontal Representative Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Solution Consider

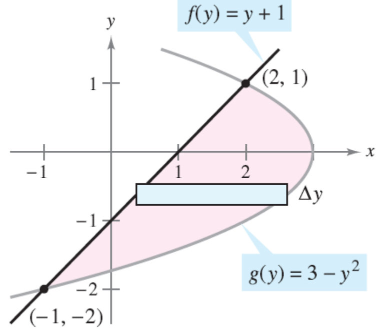
$$g(y) = 3 - y^2 \quad \text{and} \quad f(y) = y + 1. \qquad \boxed{\text{a function of } y}$$

These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure $f(y) \le g(y)$ on this interval, you have

$$\Delta A = [g(y) - f(y)]\,\Delta y = [(3 - y^2) - (y + 1)]\,\Delta y.$$

So, the area is

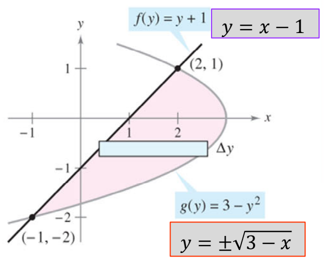
$$\begin{aligned} A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)]\, dy \\ &= \int_{-2}^{1} (-y^2 - y + 2)\, dy \\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) \\ &= \frac{9}{2}. \end{aligned}$$



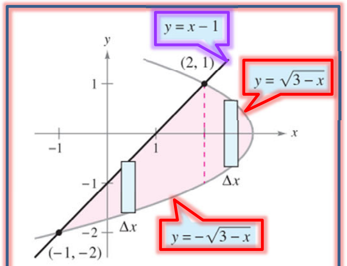
notice that by integrating with respect to y you need only one integral. If you had integrated with respect to x, you would have needed two integrals because the upper boundary would have changed at $x = 2$, as shown in Figure

$$\begin{aligned} A &= \int_{-1}^{2} (x-1) - (-\sqrt{3-x}) \quad + \int_{2}^{3} (\sqrt{3-x}) - (-\sqrt{3-x}) \\ &= \int_{-1}^{2} \left[(x - 1) + \sqrt{3 - x}\right] dx + \int_{2}^{3} \left(\sqrt{3 - x} + \sqrt{3 - x}\right) dx \\ &= \int_{-1}^{2} \left[x - 1 + (3 - x)^{1/2}\right] dx + 2\int_{2}^{3} (3 - x)^{1/2}\, dx \\ &= \left[\frac{x^2}{2} - x - \frac{(3 - x)^{3/2}}{3/2}\right]_{-1}^{2} - 2\left[\frac{(3 - x)^{3/2}}{3/2}\right]_{2}^{3} \\ &= \left(2 - 2 - \frac{2}{3}\right) \\ &\quad - \left(\frac{1}{2} + 1 - \frac{16}{3}\right) \\ &\quad - 2(0) + 2\left(\frac{2}{3}\right) \\ &= \frac{9}{2} \end{aligned}$$

More difficult



Horizontal rectangles (integration with respect to y)



Vertical rectangles (integration with respect to x)

Volume: The Disk Method

- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution using the washer method.

- 3 Dimensional

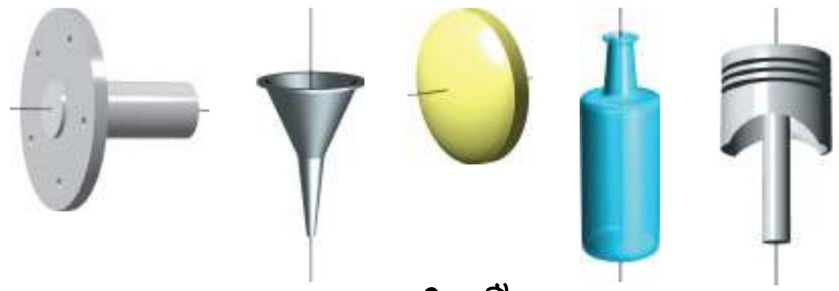
The Disk Method

Previously ...

area → volume   3-dimensional solid whose cross sections are similar

หน้าตัด

axels   funnels   pills   bottles   pistons



Solids of revolution

ถ้า เฉือนตาม แกนหมุน → cross section เป็นวงกลมหมด

A solid of revolution

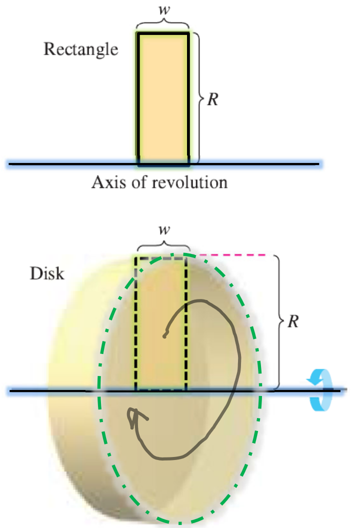
axis of revolution

The Disk Method

The simplest such solid is a right circular cylinder or **disk.**

It is formed by revolving a rectangle about an axis adjacent to one side of the rectangle.

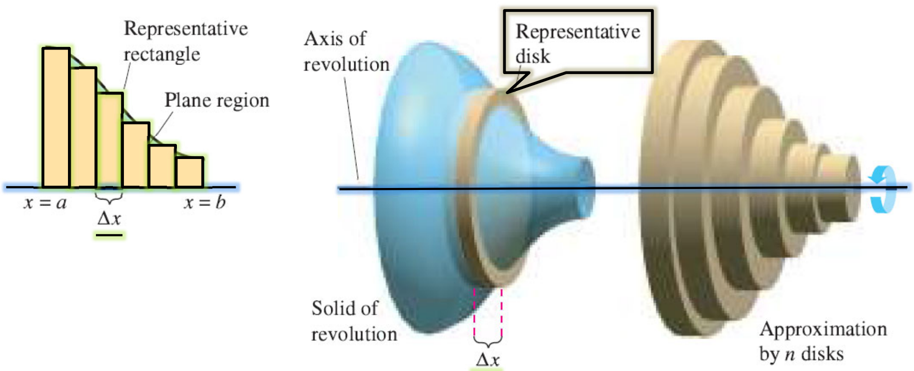
$$\text{Volume of disk} = (\text{area of disk})(\text{width of disk})$$
$$= \pi R^2 w$$



The Disk Method

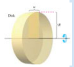
To find the volume of **a general solid of revolution** ... (see figure)
revolve a representative rectangle about the axis of revolution,
it generates a representative disk whose **volume** is ...

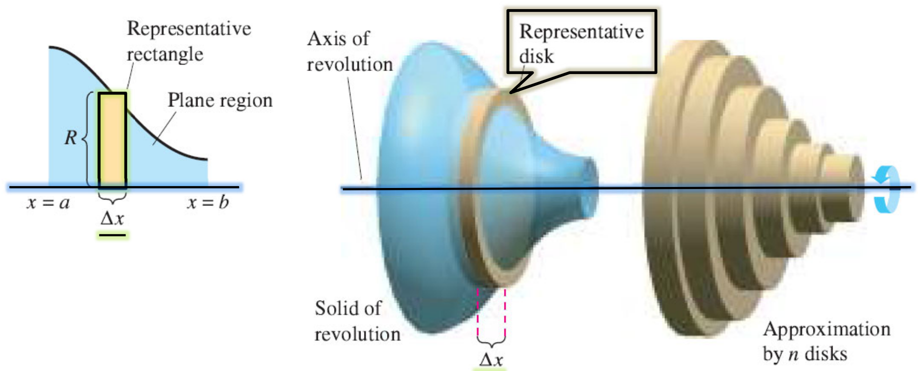
$$\Delta V = \pi R^2 \Delta x.$$



The Disk Method

Known Precalculus Formula	*Representative Element*	*New Integration Formula*
Volume of disk $V = \pi R^2 w$	$\Delta V = \pi[R(x_i)]^2\,\Delta x$	Solid of revolution $V = \pi \int_a^b [R(x)]^2\,dx$





THE DISK METHOD

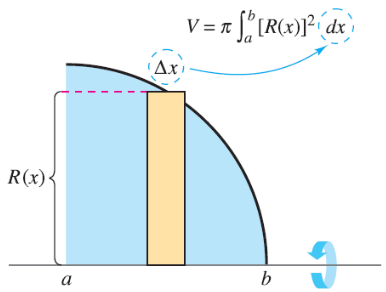
To find the volume of a solid of revolution with the **disk method,** use one of the following,

Horizontal Axis of Revolution

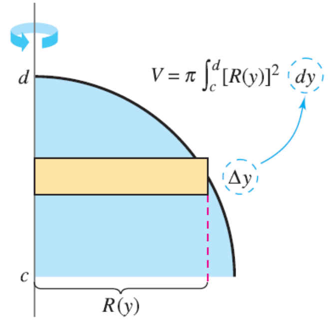
$$\text{Volume} = V = \pi \int_a^b [R(x)]^2\,dx$$

Vertical Axis of Revolution

$$\text{Volume} = V = \pi \int_c^d [R(y)]^2\,dy$$



Horizontal axis of revolution



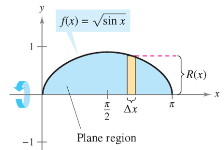
Vertical axis of revolution

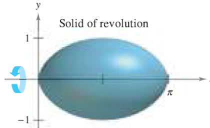
EXAMPLE 1 Using the Disk Method

Find the volume of the solid formed by revolving the region bounded by the graph of

$$f(x) = \sqrt{\sin x}$$

and the x-axis $(0 \le x \le \pi)$ about the x-axis.





Solution From the representative rectangle in the upper graph the radius of this solid is

$$R(x) = f(x) = \sqrt{\sin x}.$$

So, the volume of the solid of revolution is

$$V = \pi \int_a^b [R(x)]^2\,dx = \pi \int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx \qquad \text{Apply disk method.}$$

$$= \pi \int_0^{\pi} \sin x\,dx \qquad \text{Simplify}$$

$$= \pi \Big[-\cos x\Big]_0^{\pi} \qquad \text{Integrate.}$$

$$= \pi(1 + 1)$$

$$= 2\pi.$$

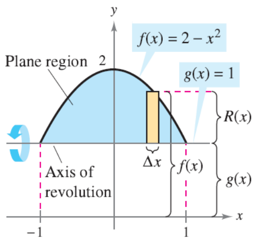
EXAMPLE 2 Revolving About a Line That Is Not a Coordinate Axis

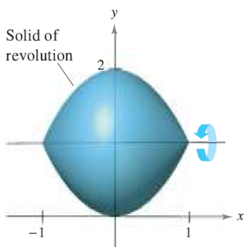
Find the volume of the solid formed by revolving the region bounded by

$$f(x) = 2 - x^2$$

and $g(x) = 1$ about the line $y = 1$, as shown in Figure

ans. $= \dfrac{16\pi}{15}$





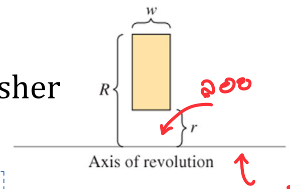
The Washer Method

วงแหวน

Solids of revolution with hole → washer

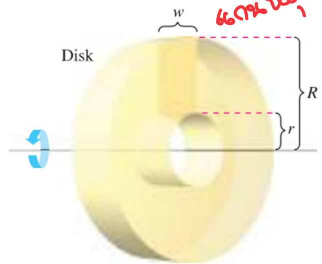
Representative disk → Representative washer

$$\text{Volume of washer} = \pi(R^2 - r^2)w$$

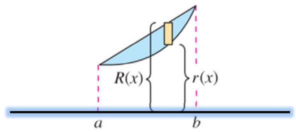


If a region bounded by an **outer radius** $R(x)$ and an **inner radius** $r(x)$, and the region is revolved about its axis of revolution, the volume of the resulting solid is given by

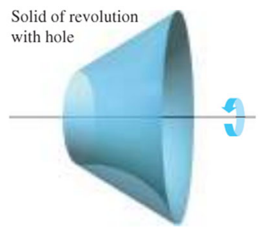
$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx.$$



Solid of revolution



Plane region



the volume of the outer radius

–

the volume of the inner radius

EXAMPLE 3 Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis, as shown in Figure

Solution the outer and inner radii are as follows.

$R(x) = \sqrt{x}$ Outer radius

$r(x) = x^2$ Inner radius

Integrating between 0 and 1 produces

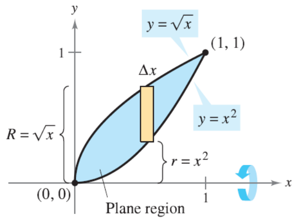
$$V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)\,dx \quad \text{Apply washer method.}$$

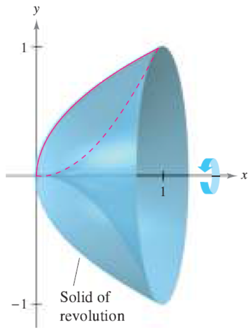
$$= \pi\int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] dx$$

$$= \pi\int_0^1 (x - x^4)\,dx \quad \text{Simplify.}$$

$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \quad \text{Integrate.}$$

$$= \frac{3\pi}{10}.$$





Solid of revolution