Systems of Linear Equations

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Outline

- Introduction to Systems of Linear Equations
- Systems of Linear Equations
- Solving a System of Linear Equations
  - Using an Inverse Matrix
  - Cramer’s Rule
  - Gaussian Elimination and Gauss-Jordan Elimination
  - The LU-*Factorization*

Introduction to Systems of Linear Equations

- Linear algebra is a branch of mathematics rich in theory and applications.
- Because linear algebra arose from the study of systems of linear equations, you shall begin with linear equations.

Linear Equations in n Variables

A **linear equation in n variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient,** and x_1 is the **leading variable.**

สัมประสิทธิ์นำ

Linear equation

ไม่มี
- ผลคูณ
- รากของตัวแปร
- ตัวแปรที่เกี่ยวกับ
  - > Trigonometric func.
  - > Exponential func.
  - > Log. func.

Examples of Linear Equations and Nonlinear Equations

Each equation is linear.

(a) $3x + 2y = 7$

(b) $\frac{1}{2}x + y - \pi z = \sqrt{2}$

constant

(c) $x_1 - 2x_2 + 10x_3 + x_4 = 0$

(d) $\left(\underbrace{\sin \frac{\pi}{2}}_{=1 \text{ (Real number)}}\right) x_1 - 4x_2 = e^2$

Each equation is not linear.

(a) $xy + z = 2$

[illegible]

(b) $e^x - 2y = 4$

exponential

(c) $\sin x_1 + 2x_2 - 3x_3 = 0$

Trigonometric

(d) $\frac{1}{x} + \frac{1}{y} = 4$

$\frac{xy}{x} + \frac{xy}{y} = 4xy$

$y + x = 4xy$

Systems of Linear Equations

A **system of *m* linear equations in *n* variables** is a set of m equations, each of which is linear in the same n variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
$$

a_{ij}

Systems of Two Equations in Two Variables

Solve each system of linear equations, and graph each system as a pair of straight lines.

เอา 2 สมการ
ไป plot graph
จะได้จุดตัด
(เป็นคำตอบของสมการ)

(a) $\left\{\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}\right.$

$2x = 2$

$x = 1, \ y = 2$

(b) $\left.\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}\right\}$ สมการเดียวกัน

คำตอบเป็นอนันต์
graph สองอันทับกัน
เลยเป็นทุกจุดเลย

(c) $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

กราฟสองเส้นไม่มีจุดตัด (ขนานกัน)

ไม่มีคำตอบ

Number of Solutions of a System of Linear Equations

For a system of linear equations in n variables, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has an infinite number of solutions (consistent system).
3. The system has no solution (inconsistent system).

Solving a System of Linear Equations

Which system is easier to solve algebraically?

A

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

B

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

เป็นขั้นบันไดโดยที่
leading coefficient = 1
(ทำให้ดูว่าแก้ง่าย)

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it follows a stair-step pattern and has leading coefficients of 1. To solve such a system, use a procedure called **back-substitution.**

Systems of Equations

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

PROOF Because A is nonsingular, the steps shown below are valid.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

If C is an invertible matrix, then

$AC = BC$, then $A = B$ right cancellation property

$CA = CB$, then $A = B$ left " "

Solving a System of Equations Using an Inverse Matrix

Use an inverse matrix to solve each system.

(a) $2x + 3y + z = -1$
$3x + 3y + z = 1$
$2x + 4y + z = -2$

(b) $2x + 3y + z = 4$
$3x + 3y + z = 8$
$2x + 4y + z = 5$

(c) $2x + 3y + z = 0$
$3x + 3y + z = 0$
$2x + 4y + z = 0$

A = เมทริกซ์สัมประสิทธิ์

$x = A^{-1}b$

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$$= \frac{1}{1}\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

The solution is trivial:

ผลเฉลยชัด

Cramer's Rule

Cramer's Rule, named after Gabriel Cramer (1704–1752), is a formula that uses determinants to solve a system of n linear equations in n variables. This rule can be applied only to systems of linear equations that have unique solutions.

To see how Cramer's Rule arises, look at the solution of a general system involving two linear equations in two variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Multiplying the first equation by $-a_{21}$ and the second by a_{11} and adding the results produces

$$\begin{aligned} \cancel{-a_{21}a_{11}x_1} - a_{21}a_{12}x_2 &= -a_{21}b_1 \\ \cancel{a_{11}a_{21}x_1} + a_{11}a_{22}x_2 &= \quad a_{11}b_2 \\ \hline (a_{11}a_{22} - a_{21}a_{12})x_2 &= \quad a_{11}b_2 - a_{21}b_1. \end{aligned}$$

สองสมการบวกกัน

Solving for x_2 (provided that $a_{11}a_{22} - a_{21}a_{12} \neq 0$) produces

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}.$$

In a similar way, you can solve for x_1 to obtain

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}}.$$

Proof คล้ายๆด้านบน

$a_{22} \times ①$

$a_{12} \times ②$

Cramer's Rule (cont.)

ที่มาของ Cramer's Rule

Finally, recognizing that the numerators and denominators of both x_1 and x_2 can be represented as determinants, you have

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad a_{11}a_{22} - a_{21}a_{12} \neq 0.$$

The denominator for both x_1 and x_2 is simply the determinant of the coefficient matrix A. The determinant forming the numerator of x_1 can be obtained from A by replacing its first column by the column representing the constants of the system. The determinant forming the numerator of x_2 can be obtained in a similar way. These two determinants are denoted by $|A_1|$ and $|A_2|$, as follows.

$$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \quad \text{and} \quad |A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

You have $x_1 = \dfrac{|A_1|}{|A|}$ and $x_2 = \dfrac{|A_2|}{|A|}$. This determinant form of the solution is called **Cramer's Rule.**

Using Cramer’s Rule

Use Cramer’s Rule to solve the system of linear equations.

$$
\begin{aligned}
4x_1 - 2x_2 &= 10 \\
3x_1 - 5x_2 &= 11
\end{aligned}
$$

$A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$

$x_1 = \dfrac{|A_1|}{|A|}$

$x_2 = \dfrac{|A_2|}{|A|}$

$\det(A) = -20 + 6$
$= -14$

อยากหา x ไหน ให้เอา constant ไปแทนตรง x นั้นแล้วทำเป็น Matrix

$x_1 = \dfrac{|A_1|}{|A|} = \dfrac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14}$

$x_2 = \dfrac{|A_2|}{|A|} = \dfrac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14}$

Cramer’s Rule (cont.)

If a system of n linear equations in n variables has a coefficient matrix with a nonzero determinant $|A|$, then the solution of the system is given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}, \qquad \ldots, \qquad x_n = \frac{\det(A_n)}{\det(A)},$$

where the ith column of A_i is the column of constants in the system of equations.

มี Proof อยู่

Using Cramer's Rule

ถ้า determinant = 0 ใช้ Cramer ไม่ได้

Use Cramer's Rule to solve the system of linear equations for x.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \phantom{{}+2y} + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

$\frac{4}{5}$

$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix}$

$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} \begin{matrix} -1 & 2 \\ 2 & 0 \\ 3 & -4 \end{matrix}$

$= 6 + 24 - 4 - 16$

$= 30 - 20$

$= 10$

$x = \dfrac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \dfrac{C_{23}}{10} = \dfrac{-\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10}$

$= \dfrac{-(-4-4)}{10}$

$= \dfrac{8}{10}$

$= \dfrac{4}{5}$

$y = \dfrac{\begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}}{10}$

$y = \dfrac{-3}{2}$

$z = \dfrac{-8}{5}$

Gaussian Elimination and Gauss-Jordan Elimination

- Gaussian elimination was introduced as a procedure for solving a system of linear equations.

Elementary Row Operations

การดำเนินการตามแถวขั้นมูลฐาน

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

- Rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination**, after the German mathematician Carl Friedrich Gauss (1777–1855).

Elementary Row Operations (cont.)

(a) Interchange the first and second rows. สลับแถว 1 กับ แถว 2

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$

(b) Multiply the first row by $\frac{1}{2}$ to produce a new first row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\left(\frac{1}{2}\right)R_1 \rightarrow R_1$ $\frac{1}{2} \times$ แถว$_1$ เพื่อได้แถว$_1$ ใหม่

(c) Add -2 times the first row to the third row to produce a new third row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{array}{lrrrrcrrrr} -2R_1 = & -2 & -4 & 8 & -6 & = & 0 & -3 & 13 & -8 \\ R_3 & 2 & 1 & 5 & -2 & & & & & \end{array}$$

Using Elementary Row Operations to Solve a System

Linear System	*Associated Augmented Matrix*
$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$
Add the first equation to the second equation.	Add the first row to the second row to produce a new second row.
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ $R_2 + R_1 \rightarrow R_2$
Add -2 times the first equation to the third equation.	Add -2 times the first row to the third row to produce a new third row.
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$ $R_3 + (-2)R_1 \rightarrow R_3$
Add the second equation to the third equation.	Add the second row to the third row to produce a new third row.
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ $R_3 + R_2 \rightarrow R_3$
Multiply the third equation by $\frac{1}{2}$.	Multiply the third row by $\frac{1}{2}$ to produce a new third row.
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $(\frac{1}{2})R_3 \rightarrow R_3$

$R_2 + 1(R_1) \rightarrow R_1$

Now you can use back-substitution to find the solution, as in the previous. The solution is $x = 1$, $y = -1$, and $z = 2$.

Row-Echelon Form of a Matrix

อยู่ในรูปขั้นบันได

A matrix in **row-echelon form** has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

แบบลดรูป

REMARK: A matrix in row-echelon form is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

Row-Echelon Form

ไม่ได้ลดรูป

The matrices below are in row-echelon form.

(a) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

0 หมดอยู่ล่างสุด

(c) $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The matrices shown in parts (b) and (d) are in *reduced* row-echelon form. The matrices listed below are not in row-echelon form.

(e) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

ติด 2 ไม่ใช่ 1 นำ

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well as an algorithmic method for solving systems of linear equations. For this algorithm, the order in which the elementary row operations are performed is important. Move from *left to right by columns*, changing all entries directly below the leading 1's to zeros.

Gaussian Elimination with Back-Substitution (cont.)

Solve the system.

$$
\begin{aligned}
x_2 + x_3 - 2x_4 &= -3 \\
x_1 + 2x_2 - x_3 \quad &= 2 \\
2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\
x_1 - 4x_2 - 7x_3 - x_4 &= -19
\end{aligned}
$$

$$
\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}
$$

อยากให้ 2 เป็น 0 (กำจัด)

$$
\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix} \quad R_1 \leftrightarrow R_2
$$

$-2R_1 + R_3 \rightarrow R_3$

$$
\begin{matrix} -2 & -4 & +2 & 0 & -4 \\ 2 & 4 & 1 & -3 & -2 \end{matrix}
$$

$$
\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ & & & & \\ & & & & \end{bmatrix}
$$

$-1R_1 + R_4 \rightarrow R_4$

Gaussian Elimination with Back-Substitution (cont.)

Gauss-Jordan Elimination

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in the next example.

ทำให้อยู่รูป reduced row-echelon ให้ได้

Gauss-Jordan Elimination (cont.)

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \quad R_1 + R_2 \to R_2$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right] \quad -2R_1 + R_3 \to R_3$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \quad \begin{array}{l} R_2 + R_3 \to R_3 \\ \frac{1}{2}R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad -3R_3 + R_2 \to R_2$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad 2R_2 + R_1 \to R_1$$

$$-3R_1 + R_1 \to R_1$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 3 & 7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x = 1,\ y = -1,\ z = 2$

✓✓

Applications of Systems of Linear Equations

- Systems of linear equations arise in a wide variety of applications and are one of the central themes in linear algebra.

Polynomial Curve Fitting

Suppose a collection of data is represented by n points in the xy-plane,

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

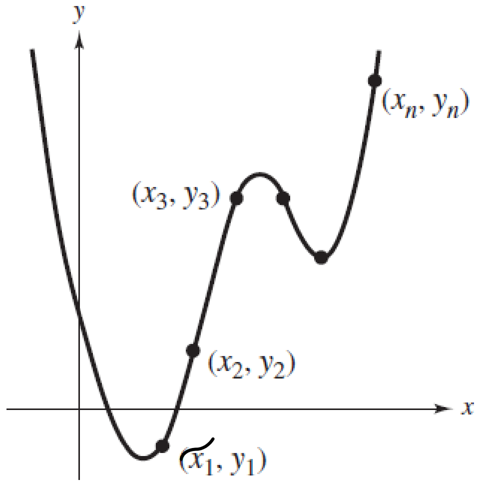
and you are asked to find a polynomial function of degree $n - 1$

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

whose graph passes through the specified points. This procedure is called **polynomial curve fitting.** If all x-coordinates of the points are distinct, then there is precisely one polynomial function of degree $n - 1$ (or less) that fits the n points, as shown in Figure 1.4.

To solve for the n coefficients of $p(x)$, substitute each of the n points into the polynomial function and obtain n linear equations in n variables $a_0, a_1, a_2, \ldots, a_{n-1}$.

$$\begin{aligned}
a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_{n-1}x_1^{n-1} &= y_1 \\
a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_{n-1}x_2^{n-1} &= y_2 \\
&\vdots \\
a_0 + a_1x_n + a_2x_n^2 + \cdots + a_{n-1}x_n^{n-1} &= y_n
\end{aligned}$$



Polynomial Curve Fitting

Polynomial Curve Fitting (cont.)

Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points $(1, 4)$, $(2, 0)$, and $(3, 12)$.

$$p(1) = a_0 + a_1 + a_2 = 4$$
$$p(2) = a_0 + 2a_1 + 4a_2 = 0$$
$$p(3) = a_0 + 3a_1 + 9a_2 = 12$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 9 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 1 & 3 & 9 & 12 \end{bmatrix} \quad R_2 + (-1)R_1 \to R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 8 & 8 \end{bmatrix} \quad R_3 + (-1)R_1 \to R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 2 & 16 \end{bmatrix} \quad R_3 - 2R_2 \to R_3$$

Polynomial Curve Fitting (cont.)

Find a polynomial that fits the points $(-2, 3)$, $(-1, 5)$, $(0, 1)$, $(1, 4)$, and $(2, 10)$.

An Application of Curve Fitting

Find a polynomial that relates the periods of the first three planets to their mean distances from the sun, as shown in Table 1.1. Then test the accuracy of the fit by using the polynomial to calculate the period of Mars. (Distance is measured in astronomical units, and period is measured in years.) (Source: *CRC Handbook of Chemistry and Physics*)

TABLE 1.1

Planet	*Mercury*	*Venus*	*Earth*	*Mars*	*Jupiter*	*Saturn*
Mean Distance	0.387	0.723	1.0	1.523	5.203	9.541
Period	0.241	0.615	1.0	1.881	11.861	29.457

The *LU*-Factorization

Solving systems of linear equations is the most important application of linear algebra. At the heart of the most efficient and modern algorithms for solving linear systems, $A\mathbf{x} = \mathbf{b}$ is the so-called LU-factorization, in which the square matrix A is expressed as a product, $A = LU$. In this product, the square matrix L is **lower triangular,** which means all the entries above the main diagonal are zero. The square matrix U is **upper triangular,** which means all the entries below the main diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

3×3 lower triangular matrix $\qquad$ 3×3 upper triangular matrix

By writing $A\mathbf{x} = LU\mathbf{x}$ and letting $U\mathbf{x} = \mathbf{y}$, you can solve for $\mathbf{x}$ in two stages. First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$; then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Each system is easy to solve because the coefficient matrices are triangular. In particular, neither system requires any row operations.

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an ***LU*-factorization** of A.

LU-Factorizations

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = LU$

is an LU-factorization of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ as the product of the lower triangular matrix $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and the upper triangular matrix $U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$

is an LU-factorization of the matrix A.

Finding the *LU*-Factorizations of a Matrix

Find the LU-factorization of the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$.

The *LU*-Factorization (cont.)

In general, if A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row, then A has an LU-factorization.

$$\begin{aligned} E_k \cdots E_2 E_1 A &= U \\ A &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} U \\ A &= LU \end{aligned}$$

Here L is the product of the inverses of the elementary matrices used in the row reduction.

Note that the multipliers in Example 6 are -2 and 4, which are the negatives of the corresponding entries in L. This is true in general. If U can be obtained from A using only the row operation of adding a multiple of one row to another row below, then the matrix L is lower triangular with 1's along the diagonal. Furthermore, the negative of each multiplier is in the same position as that of the corresponding zero in U.

Once you have obtained an LU–factorization of a matrix A, you can then solve the system of n linear equations in n variables $A\mathbf{x} = \mathbf{b}$ very efficiently in two steps.

1. Write $\mathbf{y} = U\mathbf{x}$ and solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
2. Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

The column matrix $\mathbf{x}$ is the solution of the original system because

$$A\mathbf{x} = LU\mathbf{x} = L\mathbf{y} = \mathbf{b}.$$

The second step in this algorithm is just back-substitution, because the matrix U is upper triangular. The first step is similar, except that it starts at the top of the matrix, because L is lower triangular. For this reason, the first step is often called **forward substitution.**

Solving a Linear System Using *LU*-Factorization

Solve the linear system.

$$\begin{aligned} x_1 - 3x_2 &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

Solving a Linear System Using *LU*-Factorization (cont.)

Q & A