Mathematics
for
Information Technology

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DIFFERENTIATION (II)

Differentiation

- **OBJECTIVES**

  Learn how to …
  - find the higher-order derivative of a function.
  - find the derivative of a function using the Chain Rule and …
    - the General Power Rule,
    - the Trigonometric Functions,
    - the Logarithmic and Exponential Functions.
  - find the derivative of a function using implicit differentiation.
  - find the related rates.

Higher-Order Derivatives

I

Objectives

- Find a higher-order derivative of a function.

Higher-Order Derivatives

obtain an **acceleration** function by differentiating a velocity function

$$s(t) \qquad \text{Position function}$$
$$v(t) = s'(t) \qquad \text{Velocity function}$$
$$a(t) = v'(t) = s''(t) \qquad \text{Acceleration function}$$

The function given by $a(t)$ is the **second derivative** of $s(t)$ and is denoted by $s''(t)$.

an example of a **higher-order derivative**

First derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
Second derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
Third derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
Fourth derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
⋮					
nth derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

ใส่วงเล็บจะ
ได้ไม่สับสน

EXAMPLE 10 Finding the Acceleration Due to Gravity

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by

ตกถึงพื้นพร้อมกัน

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds. What is the ratio of Earth's gravitational force to the moon's?

Solution To find the acceleration, differentiate the position function twice.

$$s(t) = -0.81t^2 + 2 \qquad \text{Position function}$$
$$s'(t) = -1.62t \qquad \text{Velocity function}$$
$$s''(t) = -1.62 \qquad \text{Acceleration function}$$

a ของวัตถุ
บนดวงจันทร์

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-9.8}{-1.62} \approx 6.0.$$

ดวงจันทร์ g น้อยกว่า

The Chain Rule

II

Objectives

- Find the derivative of a composite function using the Chain Rule.
- Find the derivative of a function using the General Power Rule.
- Simplify the derivative of a function using algebra.
- Find the derivative of a trigonometric function using the Chain Rule.

The Chain Rule

Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

This rule deals with composite functions

Without the Chain Rule		*With the Chain Rule*	
$y = x^2 + 1$	Polynomial ธรรมดา	$y = \sqrt{x^2 + 1}$	เพราะ x^2+1 เป็น 1 ฟังก์ชัน แล้วถูกส่งต่อให้ function $\sqrt{x}$ อีก
$y = \sin x$		$y = \sin 6x$	
$y = 3x + 2$		$y = (3x + 2)^5$	Composite
$y = x + \tan x$	จะมองว่าเป็น composite fx การบวกก็ได้ แต่เรามีสูตรอยู่แล้ว จึงไม่จำเป็น	$y = x + \tan x^2$	x^2 ส่ง input ให้ tan x

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.

EXAMPLE 1 The Derivative of a Composite Function

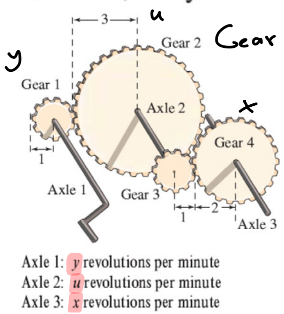
A set of gears is constructed, as shown in Figure, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles, respectively. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

$\frac{dy}{dx}$



Gear 2 ผูกกับ Gear3

หา y เทียบกับ x (แต่มันไม่ได้สัมพันธ์กันตรงๆ)
y ต้องหมุนกี่รอบ, x จะหมุน 1 รอบ
y ต้องเปลี่ยนเท่าไหร่, x จะเปลี่ยน 1 หน่วย

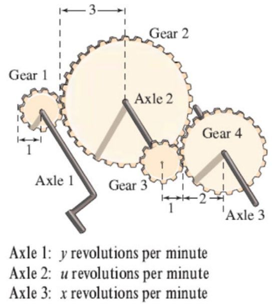
Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\frac{dy}{dx} = \boxed{\text{Rate of change of first axle with respect to second axle}} \cdot \boxed{\text{Rate of change of second axle with respect to third axle}}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6$$

$$= \boxed{\text{Rate of change of first axle with respect to third axle}}.$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x. ■



The Chain Rule

THEOREM 2.10 THE CHAIN RULE

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

มอง g(x) เป็น u
(ตัวแปร)

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

EXAMPLE 2 Decomposition of a Composite Function

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b. $y = \sin 2x$	$u = 2x$ (Func ตัวใน)	$y = \sqrt{u}$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	
d. $y = \tan^2 x$ $= (\tan x)^2$	$u = \tan x$	$y = u^2$

EXAMPLE 3 Using the Chain Rule

Find dy/dx for $y = (x^2 + 1)^3$. หรือจะใช้กำลัง 3 สมบูรณ์แล้วดิฟปกติก็ได้ ใน Study Tip

Solution For this function, you can consider the inside function to be $u = x^2 + 1$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2 + 1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2 + 1)^2.$$

STUDY TIP You could also solve the problem in Example 3 without using the Chain Rule by observing that $y = (x^2 + 1)^3$.

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}? \quad = 50(x^2+1)^{49}(2x) = 100x(x^2+1)^{49}$$

The General Power Rule

it is a special case of the Chain Rule

THEOREM 2.11 THE GENERAL POWER RULE

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n\,u^{n-1}\,u'.$$

The General Power Rule

EXAMPLE 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$.

Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$f'(x) = \underbrace{3}_{n}\underbrace{(3x - 2x^2)^2}_{u^{n-1}}\underbrace{\frac{d}{dx}[3x - 2x^2]}_{u'} \qquad \text{Apply General Power Rule.}$$

$$= 3(3x - 2x^2)^2(3 - 4x). \qquad \text{Differentiate } 3x - 2x^2.$$

The General Power Rule

$$y = (x^2 + 1)^3.$$

$$y' = \left((x^2+1)^3\right)'$$

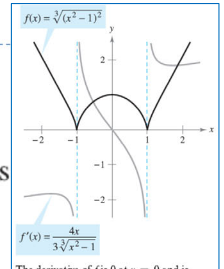
$$y' = 3(x^2+1)^2(2x)$$

$$y' = 6x(x^2+1)^2$$

is this legit?

EXAMPLE 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.

Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Simplify ได้ ! อะไร Simplify ได้, ทำก่อน

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$f'(x) = \underbrace{\frac{2}{3}}_{n}\underbrace{(x^2-1)^{-1/3}}_{u^{n-1}}\underbrace{(2x)}_{u'} \qquad \text{Apply General Power Rule.}$$

$$= \frac{4x}{3\sqrt[3]{x^2-1}}. \qquad \text{Write in radical form.}$$

So, $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$.

EXAMPLE 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t) = \dfrac{-7}{(2t-3)^2}$.

NOTE Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Solution Begin by rewriting the function as

$$g(t) = \underline{-7}(2t-3)^{-2}.$$

const ย้ายไป

Then, applying the General Power Rule produces

เหลือ const $\cdot \frac{dy}{dx}$ *constant multiple rule

$$g'(t) = 14(2t-3)^{-3}\cdot(2)$$

$$= 28(2t-3)^{-3}$$

$$= \frac{28}{(2t-3)^3}$$

จะใช้ Quotient Rule ก็ได้ แต่ยุ่งยากกว่า

EXAMPLE 9 Simplifying the Derivative of a Power

$$y = \left(\frac{3x-1}{x^2+3}\right)^2 \qquad \text{Original function}$$

Funcใน → เศษส่วนหรือก็ได้

$$y' = 2\left(\frac{3x-1}{x^2+3}\right)\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]$$

Quotient Rule

$$= \frac{2(3x-1)}{(x^2+3)}\cdot\frac{3(x^2+3)-(3x-1)(2x)}{(x^2+3)^2}$$

$$= \frac{2(3x-1)\left[3(x^2+3)-(3x-1)(2x)\right]}{(x^2+3)^3}$$

$$= \frac{6(3x-1)(x^2+3)-4x(3x-1)^2}{(x^2+3)^3}$$

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

Trigonometric Functions and the Chain Rule

Be careful !!

$\cos(x)$

$\cos(x^2)$

$\cos^2(x) = (\cos x)^2$

$\cos(2x)$

$2\cos x \neq \cos 2x$

$x\cos x \neq \cos x^2$

EXAMPLE 10 Applying the Chain Rule to Trigonometric Functions

$(\tan x)' = \sec^2 x$

a. $y = \sin \overbrace{2x}^{u}$ $\quad y' = \cos 2x \cdot 2 = 2\cos 2x$

b. $y = \cos(x-1)$ $\quad y' = -\sin(x-1)\cdot 1 = -\sin(x-1)$

c. $y = \tan 3x$ $\quad y' = \sec^2(3x)\cdot 3 = 3\sec^2 3x$

EXAMPLE 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\quad y' = -\sin(3x^2)\cdot 6x = -6x\sin(3x^2)$

b. $y = (\cos 3)x^2$ constant $\quad y' = \cos 3 \cdot 2x = 2x\cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\quad y' = -\sin(9x^2)\cdot 18x = -18x\sin(9x^2)$

d. $y = \cos^2 x = (\cos x)^2$ $\quad y' = 2\cos x \cdot -\sin x = -2\sin x\cos x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\quad y' = \frac{1}{2}(\cos x)^{-\frac{1}{2}} -\sin x = \frac{-\sin x}{2\sqrt{\cos x}}$

EXAMPLE 12 Repeated Application of the Chain Rule

$f(t) = \sin^3 4t$ Original function

$= (\sin 4t)^3$ Rewrite.

$f'(t) = 3(\sin 4t)^2(\sin 4t)'$ Apply Chain Rule once.

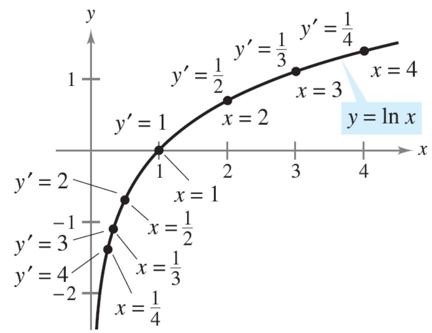
$= 3(\sin 4t)^2(\cos 4t)(4)$ Apply Chain Rule a second time.

$= 12\sin^2 4t\cos 4t$

$=$ Simplify.

BREAK

The Natural Logarithmic Function



JOHN NAPIER (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier. Napier coined the term *logarithm*, from the two Greek words *logos* (or ratio) and *arithmos* (or number).

THEOREM 5.1 PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

The Natural Logarithmic Function

THEOREM 5.2 LOGARITHMIC PROPERTIES

If a and b are positive numbers and n is rational, then the following properties are true.

$\ln(x) = \log_e x$

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

The Natural Logarithmic Function

The Derivative of the Natural Logarithmic Function

THEOREM 5.3 DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

THEOREM 5.4 DERIVATIVE INVOLVING ABSOLUTE VALUE

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u} = \frac{1}{u} \cdot \frac{d}{dx}[u]$$

$$[\ln(2x)]' = \frac{1}{2x}$$

ได้ตัวเดิม

The Natural Logarithmic Function

EXAMPLE 3 Differentiation of Logarithmic Functions

a. $\frac{d}{dx}[\ln(2x)] = \frac{u'}{u} = \frac{1}{2x} \cdot \frac{d}{dx}[2x] = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

b. $\frac{d}{dx}[\ln(x^2 + 1)] = \frac{1}{x^2+1} \cdot (x^2+1)' = \frac{2x}{x^2+1}$

c. $\frac{d}{dx}[x \ln x] = \cancel{x\left(\frac{1}{x}\right)} + (1)(\ln x)$

$= 1 + \ln x$

d. $\frac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \frac{1}{x}$

$= \frac{3}{x}(\ln x)^2$

The Natural Logarithmic Function

EXAMPLE 7 Derivative Involving Absolute Value

Find the derivative of

$f(x) = \ln|\cos x|.$ $= -\tan x$

$= ??$

The Natural Exponential Function

DEFINITION OF THE NATURAL EXPONENTIAL FUNCTION

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$f^{-1}(x) = e^x.$ $\quad f(x) = a^x,\ f^{-1} = \log_a x$

That is,

$y = e^x$ if and only if $x = \ln y.$

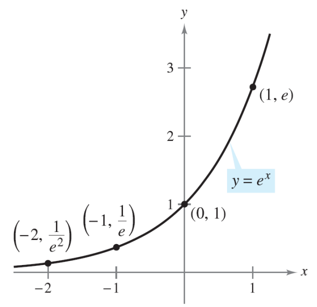
$e \approx 2.71828182846$

THEOREM 5.10 OPERATIONS WITH EXPONENTIAL FUNCTIONS

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$
2. $\frac{e^a}{e^b} = e^{a-b}$

The Natural Exponential Function



The natural exponential function is increasing, and its graph is concave upward.

PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$

ความชันจะค่อยๆเพิ่ม

The Natural Exponential Function

THEOREM 5.11 DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}[e^x] = e^x$

2. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$

composite

EXAMPLE 3 Differentiating Exponential Functions

a. $\frac{d}{dx}[e^{2x-1}] =$? $e^u \frac{du}{dx} = 2e^{2x-1}$

b. $\frac{d}{dx}[e^{-3/x}] =$ $3x^{-2}e^{-3x^{-1}} = \frac{3}{x^2}\cdot e^{\frac{-3}{x}} = \frac{3e^{-3/x}}{x^2}$

$[e^{-3x^{-1}}]$

Implicit Differentiation

III

Objectives

- Distinguish between functions written in implicit form and explicit form.
- Use implicit differentiation to find the derivative of a function.

Implicit Differentiation

to find dy/dx ➡ writing y explicitly as a function of x

Implicit Form	*Explicit Form*	*Derivative*
$xy = 1$	$y = \frac{1}{x} = x^{-1}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$
$x^2 + 2y = 1$	$y = \frac{1}{2}(1 - x^2)$	

Question: how would you find dy/dx for the equation $x^2 - 2y^3 + 4y = 2$ **?** ถ้าเป็น Explicit ได้แต่ยาก

it is very difficult to express y as a function of x explicitly

Ans: use **implicit differentiation**

Implicit Differentiation

Note!!!

Differentiating with Respect to *x*

a. $\frac{d}{dx}[x^3] = 3x^2$ Variables agree: use Simple Power Rule.

ดิฟแบบปกติ ; ถ้าตัวแปรเดียวกัน

Variables agree

b. $\frac{d}{dx}[\underbrace{y^3}_{u^n}] = \underbrace{3y^2}_{nu^{n-1}} \underbrace{\frac{dy}{dx}}_{u'}$ Variables disagree: use Chain Rule.

ต้องทำ Chain Rule ด้วย

Variables disagree

Implicit Differentiation

Note!!!

Differentiating with Respect to x

$$\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$$

Chain Rule: $\frac{d}{dx}[3y] = 3y'$

$$\frac{d}{dx}[xy^2] = x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x]$$ Product Rule

$$= x\left(2y\frac{dy}{dx}\right) + y^2(1)$$ Chain Rule

ดิฟตัวใน

$$= 2xy\frac{dy}{dx} + y^2$$ Simplify.

Implicit Differentiation

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

EXAMPLE Find dy/dx given that $x^2 + 4y^2 = 4$

Solution

$$x^2 + 4y^2 = 4$$ Write original equation.

$$2x + 8y\frac{dy}{dx} = 0$$ Differentiate with respect to x.

$$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y}$$ Solve for $\frac{dy}{dx}$.

Implicit Differentiation

EXAMPLE 2 Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Solution

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

ถ้าอยากรู้ slope
ก็แทน (x,y)

Implicit Differentiation

2. Collect the dy/dx terms on the left side of the equation and move all other terms to the right side of the equation.

3. Factor dy/dx out of the left side of the equation.

4. Solve for dy/dx by dividing by

Implicit Differentiation

Find dy/dx given that $x^4 + x^2y^2 - y^2 = 0$

$$4x^3 + (x^2)[y^2]' + [x^2]'(y^2) - [y^2]' = 0$$

$$4x^3 + x^2(2y)\frac{dy}{dx} + 2xy^2 - 2y\frac{dy}{dx} = 0$$

$$2x^2y\frac{dy}{dx} - 2y\frac{dy}{dx} = -2xy^2 - 4x^3$$

$$\frac{dy}{dx} = \frac{-2xy^2 - 4x^3}{2x^2y - 2y} = \frac{-xy^2 - 2x^3}{x^2y - y}$$

$$\frac{dy}{dx}(2x^2y - 2y)$$

Final Answer

$$\boxed{\frac{x(2x^2+y^2)}{y(1-x^2)}}$$

EXAMPLE 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$. Evaluate the first and second derivatives at the point $(-3, 4)$.

Solution Differentiating each term with respect to x produces

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y} = -xy^{-1}$$

At $(-3,4)$: $\frac{dy}{dx} = \frac{3}{4}$

At $(-3, 4)$: $\frac{dy}{dx} =$

Differentiating a second time with respect to x yields

$$\frac{d^2y}{dx^2} =$$

$$=$$

$$\frac{d^2y}{d^2x} = \left[-\frac{x}{y}\right]' = -\frac{1(y) - x\frac{dy}{dx}}{y^2}$$

แทนเลย

$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$= -\frac{y + \frac{x^2}{y}}{y^2}$$

$$= -\frac{x^2+y^2}{y}\left(\frac{1}{y^2}\right)$$

$$= -\frac{x^2+y^2}{y^3}$$

At $(3,4)$: $-\frac{9+16}{64}$

At $(-3, 4)$: $\frac{d^2y}{dx^2} =$

$$= -\frac{25}{64}$$

Related Rates

IV

Objectives

- Find a related rate.
- Use related rates to solve real-life problems.

Related Rates

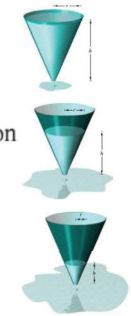
บางอย่างมีความสัมพันธ์กัน, ไม่ได้เป็นอิสระต่อกัน

Finding Related Rates

For example, when water is drained out of a conical tank, the volume V, the radius r, and the height h of the water level are all functions of time t. Knowing that these variables are related by the equation

$$V = \frac{\pi}{3}r^2h \qquad \text{Original equation}$$

หาสมการที่แสดงความสัมพันธ์มา



you can differentiate implicitly with respect to t to obtain the **related-rate** equation

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3}r^2h\right)$$

Diff เทียบกับ t ให้หมด

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] \qquad \text{Differentiate with respect to } t.$$

$$= \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right).$$

From this equation you can see that the rate of change of V is related to the rates of change of both h and r.

EXAMPLE 2 Ripples in a Pond

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

$r = 4$

Solution The variables r and A are related by $A = \pi r^2$. The rate of change of the radius r is $dr/dt = 1$.

สมการที่เชื่อม
โยงตัวแปร

Equation: ?

Given rate:

Find: when

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Total area increases as the outer radius increases.

With this information,

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2] \quad \text{Differentiate with respect to } t.$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad \text{Chain Rule}$$

$$\frac{dA}{dt} = 2\pi(4)(1) = 8\pi \quad \text{Substitute 4 for } r \text{ and 1 for } dr/dt.$$

When the radius is 4 feet, the area is changing at a rate of ?? square feet per second.

8π

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

NOTE When using these guidelines, be sure you perform Step 3 before Step 4. Substituting the known values of the variables before differentiating will produce an inappropriate derivative.

EXAMPLE 4 The Speed of an Airplane Tracked by Radar

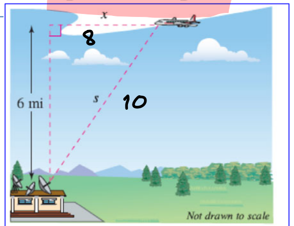
An airplane is flying on a flight path that will take it directly over a radar tracking station. If s is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?

Solution Let x be the horizontal distance from the station. Notice that when $s = 10$, $x = 8$ (Pythagoras)

?

Given rate:

Find:



An airplane is flying at an altitude of 6 miles, s miles from the station.

You can find the velocity of the plane as shown.

Equation: $x^2 + 6^2 = s^2$ Pythagorean Theorem

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt} \quad \text{Differentiate with respect to } t.$$

Solve for dx/dt.

$$\frac{dx}{dt} = \frac{10}{8}(-400) \quad \text{Substitute for } s, x, \text{ and } ds/dt.$$

$$= -500 \text{ miles/hr} \quad \text{Simplify.}$$

Because the velocity is ?? miles per hour, the *speed* is ?? miles per hour.

NOTE Note that the velocity in Example 4 is negative because x represents a distance that is decreasing.