Vector Spaces #1

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or as an advance appointment

Outline

- Vectors in the Plane
- Operations with Vectors in the Plane
- Vectors in R^n
- Length and Dot Product in R^n
- Angle Between Two Vectors
- The Cross Product of Two Vectors

Scalars and Vectors

- A scalar quantity has only magnitude.
- A vector quantity has both magnitude and direction.

Scalar Quantities

length, area, volume, speed, mass, density, pressure, temperature, energy, entropy, work, power

Vector Quantities

displacement, velocity, acceleration, momentum, force, lift, drag, thrust, weight

Scalar

24

Vector

$\begin{bmatrix} 2 & -8 & 7 \end{bmatrix}$

row or column $\begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$

Matrix

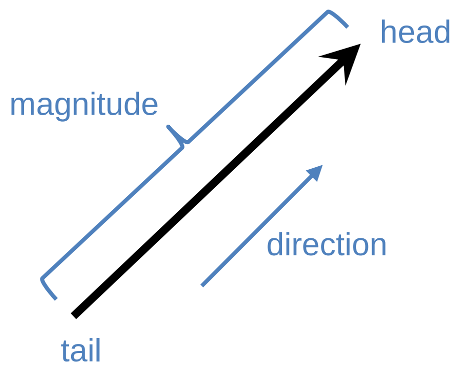
$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$

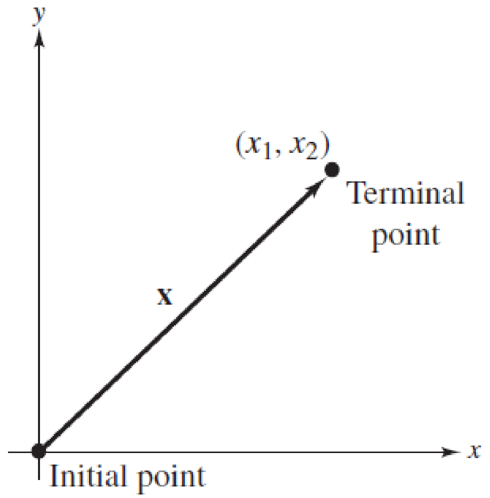
row(s) × column(s)

Vectors in the Plane

- A **vector in the plane** is represented geometrically by a **directed line segment** whose **initial point** is the origin and whose **terminal point** is the point (x_1, x_2).
- This vector is represented by the same ordered pair used to represent its terminal point. That is,

$$\mathbf{x} = (x_1, x_2)$$





Vectors in the Plane (cont.)

- The coordinates x_1 and x_2 are called the **components** of the vector $\mathbf{x}$.
- Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

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$u(u_1, u_2)$

$v(v_1, v_2)$

$u = v$ if and only if $u_1 = v_1$ and $u_2 = v_2$

Vectors in the Plane (cont.)

Use a directed line segment to represent each vector in the plane.
(a) $\mathbf{u} = (2, 3)$ (b) $\mathbf{v} = (-1, 2)$

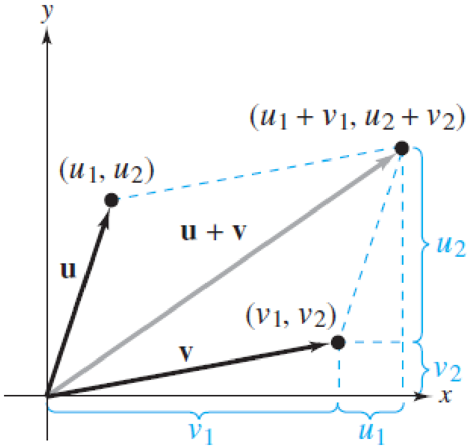
Operations with Vectors in the Plane

Vector Addition

- To add two vectors in the plane, add their corresponding components. That is, the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u}+\mathbf{v}=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2).$$

- Geometrically, the sum of two vectors in the plane is represented as the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides.



Vector Addition

Adding Two Vectors in the Plane

Find the sum of the vectors.

(a) $\mathbf{u} = (1, 4)$, $\mathbf{v} = (2, -2)$ (b) $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-3, 2)$ (c) $\mathbf{u} = (2, 1)$, $\mathbf{v} = (0, 0)$

a = (3, 2)

b = (0, 0)

c = (2, 1)

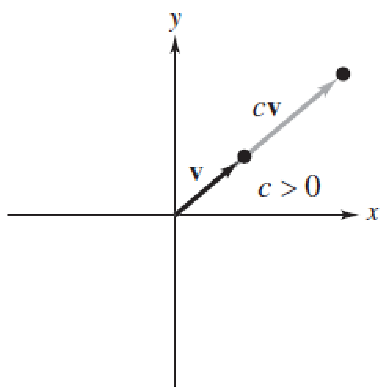
Operations with Vectors in the Plane (cont.)

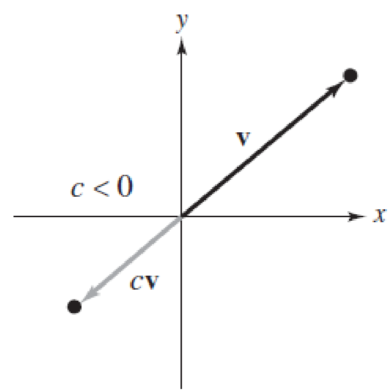
Scalar Multiplication

- To multiply a vector $\mathbf{v}$ by a scalar c, multiply each of the components of $\mathbf{v}$ by c. That is,

$$c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2).$$

- The word *scalar* is used to mean a real number.
- In general, for a scalar c, the vector $c\mathbf{v}$ will be $|c|$ times as long as $\mathbf{v}$. If c is positive, then $c\mathbf{v}$ and $\mathbf{v}$ have the same direction, and if c is negative, then $c\mathbf{v}$ and $\mathbf{v}$ have opposite directions.





Operations with Vectors in the Plane (cont.)

Provided with $\mathbf{v} = (-2, 5)$ and $\mathbf{u} = (3, 4)$, find each vector.

(a) $\frac{1}{2}\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\frac{1}{2}\mathbf{v} + \mathbf{u}$

$a = \left(-1, \frac{5}{2}\right)$ $c = \left(2, \frac{13}{2}\right)$

$b = (-5, 1)$

Properties of Vector Addition and Scalar Multiplication in the Plane

Let **u**, **v**, and **w** be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane. — Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property of addition การสลับที่การบวก
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ เอกลักษณ์การบวก — Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ ตัวผกผันการบวก — Additive inverse property
6. $c\mathbf{u}$ is a vector in the plane. — Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributive property เปลี่ยนกลุ่มการคูณ
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ — Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$ — Multiplicative identity property

Vectors in R^n

- A vector in n-space is represented by an **ordered n-tuple.**
- The set of all n-tuples is called **n-space** and is denoted by R^n.

$$
\begin{aligned}
R^1 &= \text{1-space} = \text{set of all real numbers}\\
R^2 &= \text{2-space} = \text{set of all ordered pairs of real numbers}\\
R^3 &= \text{3-space} = \text{set of all ordered triples of real numbers}\\
R^4 &= \text{4-space} = \text{set of all ordered quadruples of real numbers}\\
&\vdots\\
R^n &= n\text{-space} = \text{set of all ordered } n\text{-tuples of real numbers}
\end{aligned}
$$

- The practice of using an ordered pair to represent either a point or a vector in R^2 continues in R^n.
- That is, an n-tuple $(x_1, x_2, x_3, \dots, x_n)$ can be viewed as a **point** in R^n with the x_i's as its coordinates or as a **vector**

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n) \qquad \text{Vector in } R^n$$

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with the x_i's as its components.

- As with vectors in the plane, two vectors in R^n are **equal** if and only if corresponding components are equal.

Vector Addition and Scalar Multiplication in R^n

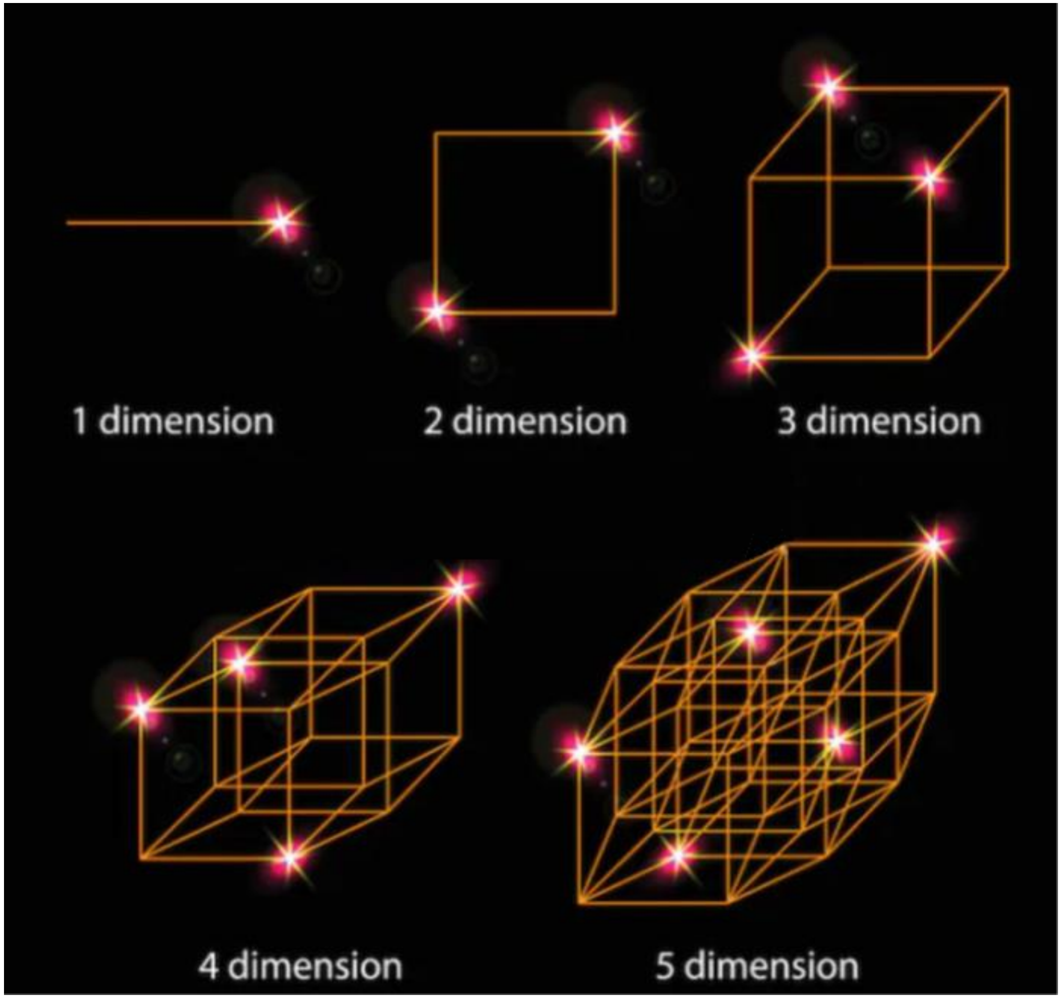
Let $\mathbf{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in R^n and let c be a real number. Then the sum of $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n),$$

and the **scalar multiple** of $\mathbf{u}$ by c is defined as the vector

$$c\mathbf{u} = (cu_1, cu_2, cu_3, \ldots, cu_n).$$

Vectors in R^n (cont.)



Vector Operations in R^3

Provided that $\mathbf{u} = (-1, 0, 1)$ and $\mathbf{v} = (2, -1, 5)$ in R^3, find each vector.

(a) $\mathbf{u} + \mathbf{v}$ (b) $2\mathbf{u}$ (c) $\mathbf{v} - 2\mathbf{u}$

$a = (1, -1, 6)$

$b = (-2, 0, 2)$

$c = (2, -1, 5) - (-2, 0, 2)$

$= (4, -1, 3)$

Properties of Vector Addition and Scalar Multiplication in R^n

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Vector Operations in R^4

$3w = (-18, 6, 0, 9)$

Let $\mathbf{u} = (2, -1, 5, 0)$, $\mathbf{v} = (4, 3, 1, -1)$, and $\mathbf{w} = (-6, 2, 0, 3)$ be vectors in R^4. Solve for $\mathbf{x}$.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$ (b) $3(\mathbf{x} + \mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

$x = (4, -2, 10, 0) - (-14, 9, 1, 8)$

$= (18, -11, 9, -8)$

b)

$3x + 3w = 2u - v + x$

$2x = 2u - v - 3w$

$x = \frac{2u - v - 3w}{2}$

$x = (9, -\frac{11}{2}, \frac{9}{2}, -4)$

Properties of Additive Identity and Additive Inverse

The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n. Similarly, the vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$. The theorem below summarizes several important properties of the additive identity and additive inverse in R^n.

Let $\mathbf{v}$ be a vector in R^n, and let c be a scalar. Then the following properties are true.

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

Length and Dot Product in R^n

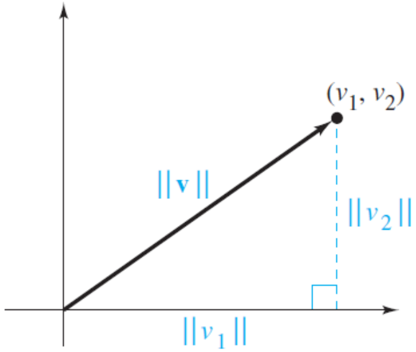
- You will begin by reviewing the definition of the length of a vector in R^2.
- If $\mathbf{v} = (v_1, v_2)$ is a vector in the plane, then the **length,** or **magnitude,** of denoted by $\|\mathbf{v}\|$, is defined as

vector norm

$$\|\mathbf{v}\| = \sqrt{\boldsymbol{v_1^2} + \boldsymbol{v_2^2}}.$$

- This definition corresponds to the usual notion of length in Euclidean geometry.
- That is, the vector is thought of as the hypotenuse of a right triangle whose sides have lengths of $|v_1|$ and $|v_2|$.
- Applying the Pythagorean Theorem produces

$$\|\mathbf{v}\|^2 = |v_1|^2 + |v_2|^2 = v_1^2 + v_2^2.$$



$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$

Length of a Vector in R^n

The **length,** or **magnitude,** of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

REMARK: The length of a vector is also called its **norm.** If $\|\mathbf{v}\| = 1$, then the vector $\mathbf{v}$ is called a **unit vector.** เวกเตอร์ 1 หน่วย

This definition shows that the length of a vector cannot be negative. That is, $\|\mathbf{v}\| \geq 0$. Moreover, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v}$ is the zero vector $\mathbf{0}$.

The Length of a Vector in R^n

(a) In R^5, the length of $\mathbf{v} = (0, -2, 1, 4, -2)$ is $\sqrt{25} = 5$

(b) In R^3, the length of $\mathbf{v} = \left(2/\sqrt{17}, -2/\sqrt{17}, 3/\sqrt{17}\right)$ is

$$= \sqrt{\frac{4}{17} + \frac{4}{17} + \frac{9}{17}}$$

$$= \sqrt{\frac{17}{17}} = 1$$

Length of a Scalar Multiple

Let $\mathbf{v}$ be a vector in R^n and let c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|,$$

where $|c|$ is the absolute value of c.

Unit Vector in the Direction of v

If **v** is a nonzero vector in R^n, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

has length 1 and has the same direction as **v**. This vector **u** is called the **unit vector in the direction of v**.

Finding a Unit Vector

Find the unit vector in the direction of $\mathbf{v} = (3, -1, 2)$, and verify that this vector has length 1.

$$\frac{v}{\|v\|} = \frac{1}{\sqrt{14}}(3, -1, 2)$$

$$= \left(\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

Distance Between Two Vectors in R^n

The **distance between two vectors u** and **v** in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

You can easily verify the three properties of distance listed below.

1. $d(\mathbf{u}, \mathbf{v}) \geq 0$
2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

Finding the Distance Between Two Vectors

The distance between $\mathbf{u} = (0, 2, 2)$ and $\mathbf{v} = (2, 0, 1)$ is

$$d(u, v) = \|u - v\|$$

$$= \sqrt{(0-2)^2 + (2-0)^2 + (2-1)^2}$$

$$= \sqrt{4 + 4 + 1}$$

$$= \sqrt{9} = 3 \ \#$$

Dot Product in R^n

The **dot product** of $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is the *scalar* quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

REMARK: Notice that the dot product of two vectors is a scalar, not another vector.

Finding the Dot Product of Two Vectors

The dot product of $\mathbf{u} = (1, 2, 0, -3)$ and $\mathbf{v} = (3, -2, 4, 2)$ is

$$u \cdot v = 1(3) + 2(-2) + 0(4) + (-3)(2)$$

$$= 3 - 4 - 6 = -7 \#$$

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar, then the following properties are true.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ distribution
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

$$u \cdot (v + w) = u \cdot v + u \cdot w$$

$$c(u \cdot v) = (cu) \cdot v$$

Finding Dot Products

$2w = (-8, 6)$

Given $\mathbf{u} = (2, -2)$, $\mathbf{v} = (5, 8)$, and $\mathbf{w} = (-4, 3)$, find

(a) $\mathbf{u} \cdot \mathbf{v}$. (b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$. (c) $\mathbf{u} \cdot (2\mathbf{v})$. (d) $\|\mathbf{w}\|^2$. (e) $\mathbf{u} \cdot (\mathbf{v} - 2\mathbf{w})$.

a) $= 2(5) - 2(8)$

$= 10 - 16$

$= -6$ #

b) $-6(-4, 3)$

$= (24, -18)$

e)

$= (2, -2) \cdot (13, 2)$

$= 26 - 4$

$= 22$ #

c) $u \cdot (10, 16)$

$= 2(10) - 2(16)$

$= 20 - 32$

$= -12$

d) $= w \cdot w$

$= 16 + 9$

$= 25$

Using Properties of the Dot Product

Provided with two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n such that $\mathbf{u} \cdot \mathbf{u} = 39$, $\mathbf{u} \cdot \mathbf{v} = -3$, and $\mathbf{v} \cdot \mathbf{v} = 79$, evaluate $(\mathbf{u} + 2\mathbf{v}) \cdot (3\mathbf{u} + \mathbf{v})$.

254 #

$u \cdot (3u+v) + 2v \cdot (3u+v)$

$u \cdot 3u + u \cdot v + 2v \cdot 3u + 2v \cdot v$

$3(u \cdot v) + u \cdot v + 6(v \cdot u) + 2(v \cdot v)$

$3(39) + (-3) + 6(\cancel{u \cdot v}) + 2(79)$

-3

$117 - 3 - 18 + 158$

$275 - 21 = 254$ #

The Angle Between Two Vectors in R^n

The **angle** θ between two nonzero vectors in R^n is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}, \quad 0 \le \theta \le \pi.$$

REMARK: The angle between the zero vector and another vector is not defined.

Finding the Angle Between Two Vectors

$u = -2v$

The angle between $\mathbf{u} = (-4, 0, 2, -2)$ and $\mathbf{v} = (2, 0, -1, 1)$ is

$$\cos\theta = \frac{-4(2) + 2(-1) - 2(1)}{\sqrt{16+8}\,(\sqrt{4+2})}$$

$$= \frac{-8-2-2}{\sqrt{24}\sqrt{6}} = \frac{-12}{\sqrt{144}} = -1$$

$\theta = \pi$ rad. #

The Angle Between Two Vectors in R^n (cont.)

Note that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos \theta$ will always have the same sign. Moreover, because the cosine is positive in the first quadrant and negative in the second quadrant, the sign of the dot product of two vectors can be used to determine whether the angle between them is acute or obtuse, as shown in Figure 5.6.

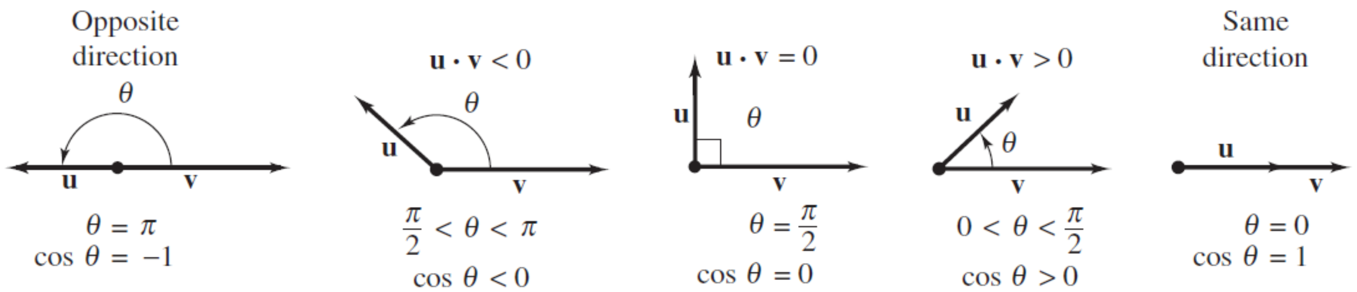


Figure 5.6

Orthogonal Vectors

Two vectors **u** and **v** in R^n are **orthogonal** if

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

REMARK: Even though the angle between the zero vector and another vector is not defined, it is convenient to extend the definition of orthogonality to include the zero vector. In other words, the vector **0** is said to be orthogonal to every vector.

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Orthogonal Vectors in R^n

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(a) The vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$ are orthogonal because

(b) The vectors $\mathbf{u} = (3, 2, -1, 4)$ and $\mathbf{v} = (1, -1, 1, 0)$ are orthogonal because

a) $u \cdot v = 0 + 0 + 0$

$= 0$

b) $u \cdot v = 3 - 2 - 1$

$= 0$

Finding Orthogonal Vectors

Determine all vectors in R^2 that are orthogonal to $\mathbf{u} = (4, 2)$.

$$V = (V_1, V_2)$$

$$U \cdot V = 4V_1 + 2V_2 = 0$$

$$2V_2 = -4V_1$$

$$V_2 = -2V_1$$

$$V = (V_1, -2V_1)$$

$$= (t, -2t) \; ; \; t \in \mathbb{R}$$

(แทนจำนวนจริงอะไรก็ได้
เพราะเป็นสมการเส้นตรง
มีจุดเป็นอนันต์)

The Cross Product of Two Vectors in Space

Many problems in linear algebra involve finding a vector orthogonal to each vector in a set. Here you will look at a vector product that yields a vector in R^3 orthogonal to two vectors. This vector product is called the **cross product,** and it is most conveniently defined and calculated with vectors written in standard unit vector form.

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in R^3. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

REMARK: The cross product is defined only for vectors in R^3. The cross product of two vectors in R^2 or of vectors in R^n, $n > 3$, is not defined here.

A convenient way to remember the formula for the cross product $\mathbf{u} \times \mathbf{v}$ is to use the determinant form shown below.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \longleftarrow \text{Components of } \mathbf{u} \\ \longleftarrow \text{Components of } \mathbf{v} \end{array}$$

Finding the Cross Product of Two Vectors

Provided that $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find

(a) $\mathbf{u} \times \mathbf{v}$. (b) $\mathbf{v} \times \mathbf{u}$. (c) $\mathbf{v} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} k$$

Algebraic Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^3 and c is a scalar, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v}$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Geometric Properties of the Cross Product

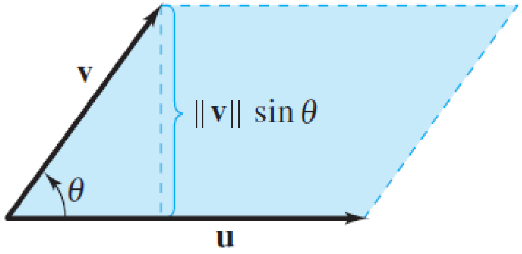
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^3, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta.$$

3. $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
4. The parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides has an area of $\|\mathbf{u} \times \mathbf{v}\|$.

$$\text{Area} = \overbrace{\|\mathbf{u}\|}^{\text{Base}}\,\overbrace{\|\mathbf{v}\| \sin \theta}^{\text{Height}} = \|\mathbf{u} \times \mathbf{v}\|.$$



Finding a Vector Orthogonal to Two Given Vectors

Find a unit vector orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

$$\vec{u} \times \vec{v} = -3\vec{i} + 2\vec{j} + 11\vec{k}$$

$$\text{unit vector} = \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|}$$

$$= \frac{-3\vec{i} + 2\vec{j} + 11\vec{k}}{\sqrt{9+4+121}}$$

$$= \frac{-3\vec{i} + 2\vec{j} + 11\vec{k}}{\sqrt{134}}$$

$$= \frac{-3}{\sqrt{134}}\vec{i} + \frac{2}{\sqrt{134}}\vec{j} + \frac{11}{\sqrt{134}}\vec{k}$$

Finding the Area of a Parallelogram

Find the area of the parallelogram that has

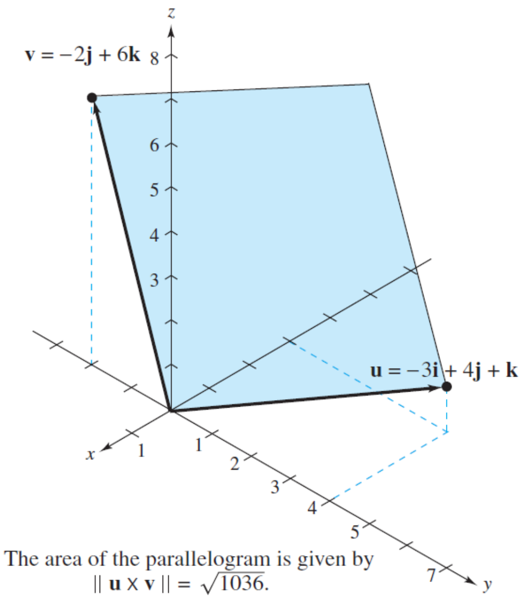
$$\mathbf{u} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{v} = -2\mathbf{j} + 6\mathbf{k}$$

as adjacent sides, as shown in Figure 5.31.

$\text{Area} = \|u\|\|v\|\sin\theta = \|\vec{u} \times \vec{v}\|$



The area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{1036}$.

Figure 5.31

Q & A