Mathematics
for
Information Technology

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APPLICATION OF DIFFERENTIATION

Application of Differentiation

- Three basic categories of application.
  - curve sketching,
  - optimization, and
  - approximation techniques.

- Limit & L'Hôpital's Rule

Extrema on an Interval

I

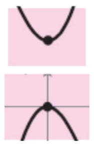
Objectives

- Understand the definition of extrema of a function on an interval.
- Understand the definition of relative extrema of a function on an open interval.
- Find extrema on a closed interval.

Extrema of a Function

DEFINITION OF EXTREMA

Let f be defined on an interval I containing c.



1. $f(c)$ is the **minimum of f on I** if $f(c) \le f(x)$ for all x in I.
2. $f(c)$ is the **maximum of f on I** if $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values,** or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum,** or the **global minimum** and **global maximum,** on the interval.

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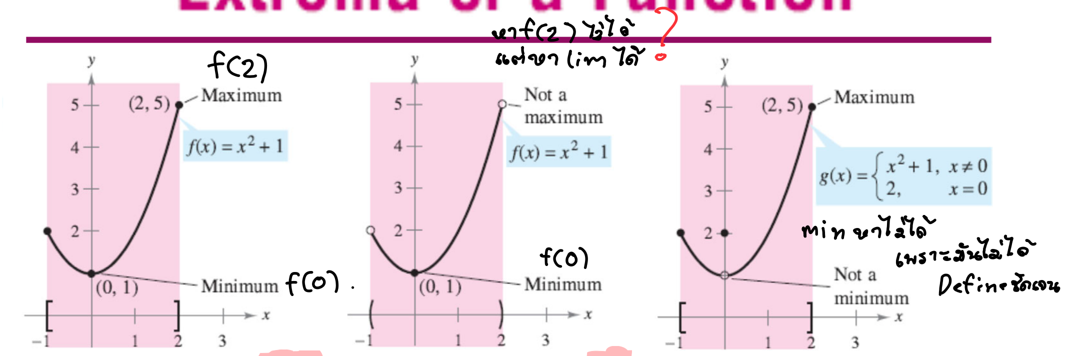
Extrema of a Function

หาf(2)ไม่ได้ แต่หา lim ได้ ?



(a) f is continuous, $[-1, 2]$ is closed. **(b)** f is continuous, $(-1, 2)$ is open. **(c)** g is not continuous, $[-1, 2]$ is closed.

Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**

THEOREM 3.1 THE EXTREME VALUE THEOREM

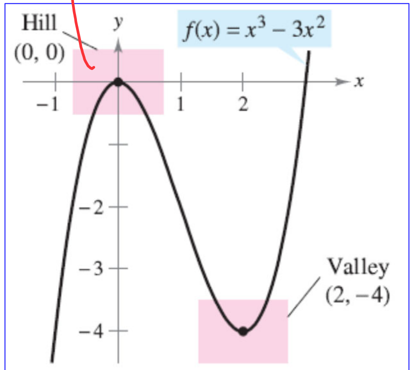
If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

global

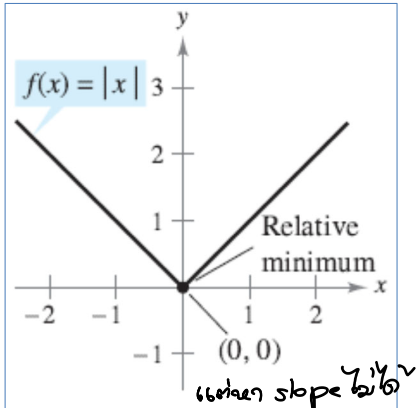
ไม่ได้เป็น absolute max. (สูงสุดทั้ง Graph)
แต่เป็น local/relative → สูงกว่าเพื่อนรอบๆ

Relative Extrema and Critical Numbers

local maximum and **local minimum**



f has a relative maximum at $(0, 0)$ and a relative minimum at $(2, -4)$.



(b) $f'(0)$ does not exist.

Such a hill and valley can occur in two ways. If the hill (or valley) is smooth and rounded, the graph has a horizontal tangent line at the high point (or low point). If the hill (or valley) is sharp and peaked, the graph represents a function that is not differentiable at the high point (or low point).

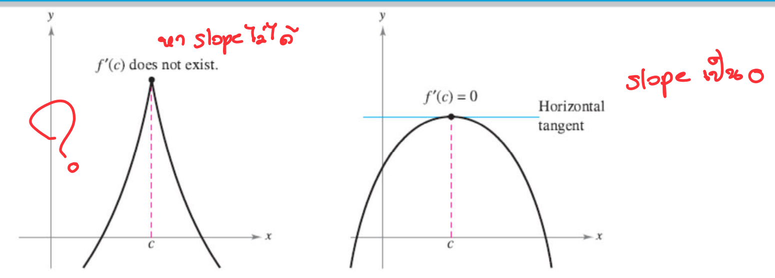
2 รูปทรง

Relative Extrema and Critical Numbers

DEFINITION OF A CRITICAL NUMBER

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

↳ หา slope ไม่ได้



c is a critical number of f.

THEOREM 3.2 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

บางจุดที่เป็น critical number แต่ไม่เป็น relative extrema

Finding Extrema on a Closed Interval

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use the following steps.

ไม่รวม endpoint

1. Find the critical numbers of f in (a, b). มีโอกาสเป็น relative extrema
2. Evaluate f at each critical number in (a, b).
3. Evaluate f at each endpoint of $[a, b]$. ตรงปลายก็มีโอกาสเป็น extrema
4. The least of these values is the minimum. The greatest is the maximum.

EXAMPLE 2 Finding Extrema on a Closed Interval

จุดที่ slope=0 / หา slope ไม่ได้

Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

Solution Begin by differentiating the function.

$$f(x) = 3x^4 - 4x^3 \qquad \text{Write original function.}$$
$$f'(x) = 12x^3 - 12x^2 \qquad \text{Differentiate.}$$

To find the critical numbers of f, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

case นี้หา slope ได้หมด, ไม่มี f'(x) ที่หาไม่ได้

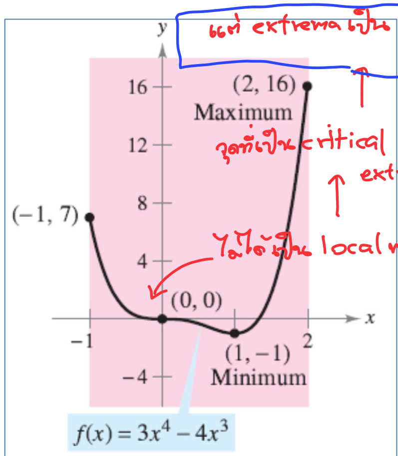
$$f'(x) = 12x^3 - 12x^2 = 0 \qquad \text{Set } f'(x) \text{ equal to 0.}$$
$$12x^2(x - 1) = 0 \qquad \text{Factor.}$$
$$x = 0, 1 \qquad \text{Critical numbers}$$

แค่ Factor

Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and at the endpoints of $[-1, 2]$, you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as shown in the table.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

Confirm with a graph.



On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.

note that the critical number $x = 0$ does not yield a relative minimum or a relative maximum. This tells you that the converse of Theorem 3.2 is not true. In other words, *the critical numbers of a function need not produce relative extrema*.

EXAMPLE 3 Finding Extrema on a Closed Interval

-2 3

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.

Solution Begin by differentiating the function.

$f'(x) = 2 - \frac{2}{3}(3)x^{-1/3}$

$0 = 2 - 2x^{-1/3}$

$-2x^{-1/3} = -2$

$x^{-1/3} = 1, \; x \neq 0 \qquad \frac{1}{\sqrt[3]{x}} = 1$

$x^{(-1/3)\cdot -3} = 1^{(-3)}$

$x = 1$

$6 - 3(3^{2/3})$

$6 - 3^{5/3}$

ต้องมี case ที่ x<0 ไหม, หา slope ตรง x=-1 ได้ไหม ?

$f(-1) = -5 \quad f(0) = 0 \quad f(1) = -1 \quad f(3) = 6 - 3^{5/3} \approx -0.24$

คำเฉลย

The Mean Value Theorem

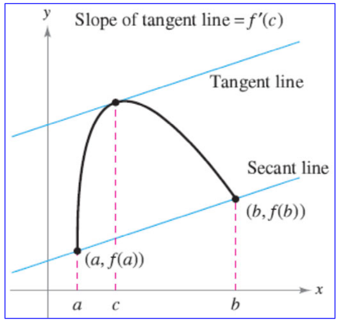
THEOREM 3.4 THE MEAN VALUE THEOREM

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$, as shown in Figure

In terms of rates of change, the Mean Value Theorem implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$. This is illustrated in Example 4.



EXAMPLE 4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.

?

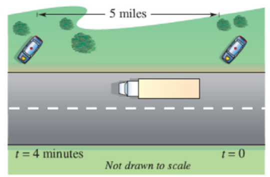
Solution Let $t = 0$ be the time (in hours) when the truck passes the first patrol car. The time when the truck passes the second patrol car is

$$t = \frac{4}{60} = \frac{1}{15} \text{ hour.}$$

By letting $s(t)$ represent the distance (in miles) traveled by the truck, you have $s(0) = 0$ and $s(\frac{1}{15}) = 5$. So, the average velocity of the truck over the five-mile stretch of highway is

$$\text{Average velocity} = \frac{s(1/15) - s(0)}{(1/15) - 0} = \frac{5}{1/15} = 75 \text{ miles per hour.}$$

Assuming that the position function is differentiable, you can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 75 miles per hour sometime during the 4 minutes. ■



At some time t, the instantaneous velocity is equal to the average velocity over 4 minutes.

First Derivative Test

Objectives

- Apply the First Derivative Test to find relative extrema of a function.

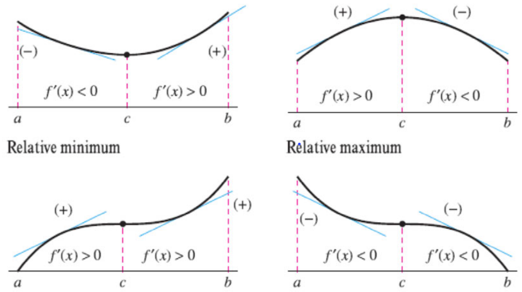
อนุพันธ์ลำดับที่ 1st

The First Derivative Test

THEOREM 3.6 THE FIRST DERIVATIVE TEST

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.

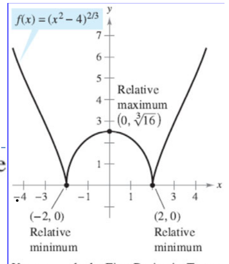


Neither relative minimum nor relative maximum

EXAMPLE 3 Applying the First Derivative Test

Find the relative extrema of

$$f(x) = (x^2 - 4)^{2/3}.$$



You can apply the First Derivative Test to find relative extrema.

Solution Begin by noting that f is continuous on the entire real number line. The derivative of f

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) \qquad \text{General Power Rule}$$

$$= \frac{4x}{3(x^2 - 4)^{1/3}} \qquad \text{Simplify.}$$

is 0 when $x = 0$ and does not exist when $x = \pm 2$. So, the critical numbers are $x = -2$, $x = 0$, and $x = 2$. The table summarizes the testing of the four intervals determined by these three critical numbers.

critical number มาเขียน

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

หาแค่ ± ก็พอ, ไม่ต้องเลขเป๊ะๆ

By applying the First Derivative Test, you can conclude that f has a relative minimum at the point $(-2, 0)$, a relative maximum at the point $(0, \sqrt[3]{16})$, and another relative minimum at the point $(2, 0)$. ■

EXAMPLE 4 Applying the First Derivative Test

- หา critical number
- 1st Derivative test
- หาว่าจุดไหน relative max/min บ้าง

Find the relative extrema of $f(x) = \dfrac{x^4 + 1}{x^2}$. หารสองตัวนี้

Domain = $\mathbb{R} - \{0\}$

Solution

$f'(x) = x^2 + \frac{1}{x^2}$

$= x^2 + x^{-2}$

$0 = 2x - 2x^{-3}$

$2x - 2x^{-3} = 0$

$2x - \frac{2}{x^3} = 0$

$\frac{2x^4 - 2}{x^3} = 0$

$\frac{2(x^4 - 1)}{x^3} = 0$

$\frac{2(x^2+1)(x^2-1)}{x^3} = 0$

$\frac{2(x^2+1)(x-1)(x+1)}{x^3} = 0$

$x = 1, x = -1, x \neq 0$

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Conclusion	Decr.	Incr.	Decr.	Incr.

Interval
Test Value
Sign of $f'(x)$
Conclusion

$$f(x) = x^2 + x^{-2}$$

$$f'(x) = \frac{2(x^4 - 1)}{x^3}$$

$$\boxed{\begin{array}{c} x = \pm 1 \\ x = 0 \end{array}}$$

BREAK

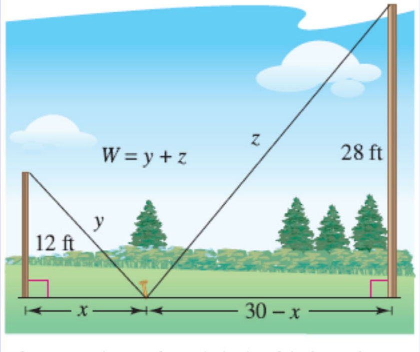
Optimization Problems

II

ออกล้อบ

Objectives

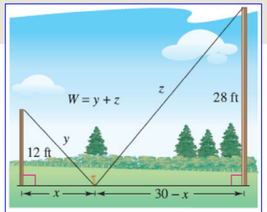
- Solve applied minimum and maximum problems.



The quantity to be minimized is length.

GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized.
3. Reduce the primary equation to one having *a single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques



The quantity to be minimized is length.

EXAMPLE 4 Finding Minimum Length

1-1

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?

1-2

Solution Let W be the wire length to be minimized. you can write

$W = y + z.$ Primary equation 2 สมการที่เชื่อมตัวแปรเหล่านั้น

In this problem, rather than solving for y in terms of z (or vice versa), you can solve for both y and z in terms of a third variable x From the Pythagorean Theorem, you obtain

$$x^2 + 12^2 = y^2$$
$$(30 - x)^2 + 28^2 = z^2$$

which implies that

$$y = \sqrt{x^2 + 144}$$
$$z = \sqrt{x^2 - 60x + 1684}.$$

So, W is given by ลดรูปให้เหลือตัวแปรเดียว

3 4

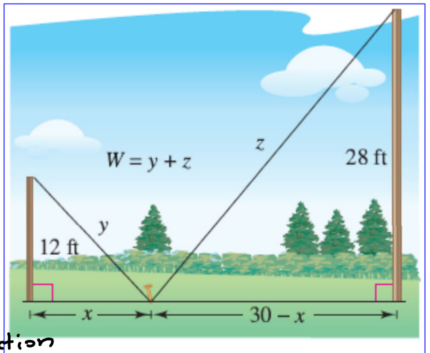
$$W = y + z$$
$$= \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}, \quad 0 \le x \le 30.$$

Domain of function

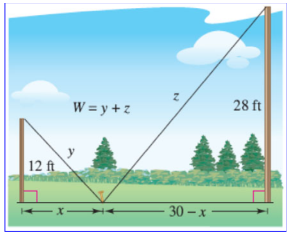
Differentiating W with respect to x yields

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}}.$$

5



The quantity to be minimized is length. From the diagram, you can see that x varies between 0 and 30.



By letting $dW/dx = 0$, you obtain

$$\frac{x}{\sqrt{x^2 + 144}} + \frac{x - 30}{\sqrt{x^2 - 60x + 1684}} = 0$$
$$x\sqrt{x^2 - 60x + 1684} = (30 - x)\sqrt{x^2 + 144}$$
$$x^2(x^2 - 60x + 1684) = (30 - x)^2(x^2 + 144)$$
$$x^4 - 60x^3 + 1684x^2 = x^4 - 60x^3 + 1044x^2 - 8640x + 129{,}600$$
$$640x^2 + 8640x - 129{,}600 = 0$$
$$320(x - 9)(2x + 45) = 0$$
$$x = 9, -22.5.$$

ไม่อยู่ใน Domain

Because $x = -22.5$ is not in the domain and

$$W(0) \approx 53.04, \quad W(9) = 50, \quad \text{and} \quad W(30) \approx 60.31$$

you can conclude that the wire should be staked at 9 feet from the 12-foot pole.

EXAMPLE 5 An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

$4x+2\pi r=4$

Domain

$0 \le x \le 1$ $x=1$ $4x=4$

$A(1)=1$

$A(0.56)\approx 0.56$

$A(0)=1.273$

$A=x^2+\pi r^2$

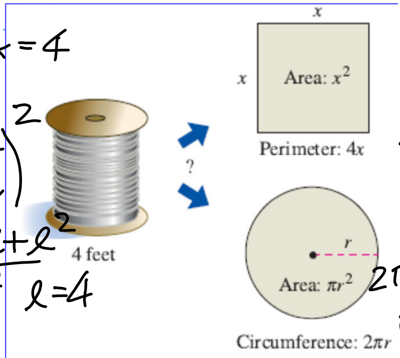
$=\left(\frac{\ell}{4}\right)^2+\pi\left(\frac{4-\ell}{2\pi}\right)^2$

$A(\ell)=\frac{\ell^2}{16}+\pi\,\frac{16-8\ell+\ell^2}{4\pi^2}$ $\ell=4$

$A(\ell)=\frac{\ell^2}{16}+\pi\left(\frac{\ell^2-8\ell+16}{4\pi^2}\right)$

$A'(\ell)=\frac{1}{16}(2)\ell+\frac{\pi}{4\pi^2}(2\ell-8)$

ทำไงต่อ



The quantity to be maximized is area: $A = x^2 + \pi r^2$.

$4x=\ell$

$x=\frac{\ell}{4}$

$2\pi r=4-\ell$

$r=\frac{4-\ell}{2\pi}$

III

Newton's Method

Objectives

Approximation

- Approximate a zero of a function using Newton's Method.

ทำให้ $f(x)=0$

Newton's Method

Approximating the zeros of a function

Newton's Method uses tangent lines to approximate the graph of the function near its x-intercepts.

To see how Newton's Method works, consider a function f that is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem, f must have at least one zero in the interval (a, b). Suppose you estimate this zero to occur at

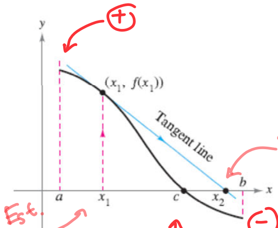
$$x = x_1 \qquad \text{First estimate}$$

Newton's Method is based on the assumption that the graph of f and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at *about* the same point. Because you can easily calculate the x-intercept for this tangent line, you can use it as a second (and, usually, better) estimate of the zero of f. The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, the equation of the tangent line is therefore

$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$y = f'(x_1)(x - x_1) + f(x_1).$$

Letting $y = 0$ and solving for x produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$



1st Est.

จุดตัด 2
2nd Est.

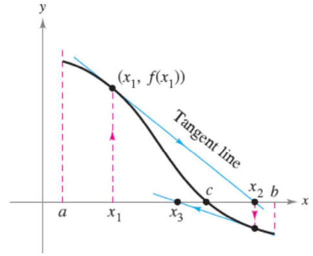
ตัดแกน X แน่ๆ

So, from the initial estimate x_1 you obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad \text{Second estimate}$$

You can improve on x_2 and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate}$$



Repeated application of this process is called Newton's Method.

NEWTON'S METHOD FOR APPROXIMATING THE ZEROS OF A FUNCTION

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

1. Make an initial estimate x_1 that is close to c. (A graph is helpful.)
2. Determine a new approximation
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration.**

EXAMPLE 1 Using Newton's Method

Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

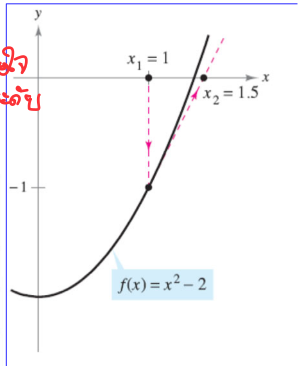
Solution Because $f(x) = x^2 - 2$, you have $f'(x) = 2x$, and the iterative process is given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

* Repeat ไปเรื่อยๆ จนกว่าจะพอใจ (ทศนิยมต่างกันระดับน้อยมากๆ)

The calculations for three iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.000000	−1.000000	2.000000	−0.500000	1.500000
2	1.500000	0.250000	3.000000	0.083333	1.416667
3	1.416667	0.006945	2.833334	0.002451	1.414216
4	1.414216				



The first iteration of Newton's Method

Of course, in this case you know that the two zeros of the function are $\pm\sqrt{2}$. To six decimal places, $\sqrt{2} = 1.414214$. So, after only three iterations of Newton's Method, you have obtained an approximation that is within 0.000002 of an actual root. The first iteration of this process is shown in Figure

L'HÔPITAL'S RULE

L'Hôpital's Rule

**

THEOREM 8.4 L'HÔPITAL'S RULE

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$. * In determinate form

L'Hôpital's Rule can also be applied to one-sided limits. For instance, if the limit of $f(x)/g(x)$ as x approaches c *from the right* produces the indeterminate form $0/0$, then

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \lim_{x\to c^+} \frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ (or $-\infty$) produces the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

L'Hôpital's Rule

EXAMPLE Evaluate $\lim_{x\to 0} \frac{e^{2x} - 1}{x}$.

Solution Because direct substitution results in the indeterminate form $0/0$

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} \quad \lim_{x\to 0} (e^{2x} - 1) = 0, \quad \lim_{x\to 0} x = 0$$

$$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}[e^u] = e^u \frac{du}{dx}$$

you can apply L'Hôpital's Rule, as shown below.

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[e^{2x} - 1]}{\frac{d}{dx}[x]}$$ Apply L'Hôpital's Rule.

$$= \lim_{x\to 0} \frac{2e^{2x}}{1}$$ Differentiate numerator and denominator.

$$= 2$$ Evaluate the limit.

L'Hôpital's Rule

EXAMPLE Evaluate $\lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \frac{\infty}{e^{\infty}} = \frac{\infty}{\infty}$

Solution Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule.

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$$

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} = \lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \frac{\infty}{\infty}$$

ใช้ L'Hôpital's Rule ใหม่

$$\lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \lim_{x \to -\infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[-e^{-x}]} = \frac{2}{e^{-x}} = \frac{2}{\infty} = 0$$