สมการเอกพันธ์ Homogenous system

Triangular Matrix

Diagonal Matrix

Orthogonal Matrix

Orthogonal Set

Orthogonal Matrix

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Outline

ค่าเฉพาะจง Vector เฉาะจง

- Eigenvalues and Eigenvectors
- Diagonalization
- Symmetric Matrices and Orthogonal Diagonalization
- Applications of Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Matrix A อาจเท่ากับ scalar 1 ตัวได้

- If A is an $n \times n$ matrix, do nonzero vectors $\mathbf{x}$ in R^n exist such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?
- The scalar, denoted by the Greek letter lambda (λ), is called an **eigenvalue** of the matrix A and the nonzero vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

สอดคล้อง

- The terms *eigenvalue* and *eigenvector* are derived from the German word *Eigenwert*, meaning “proper value.” So, you have

Eigenvalue

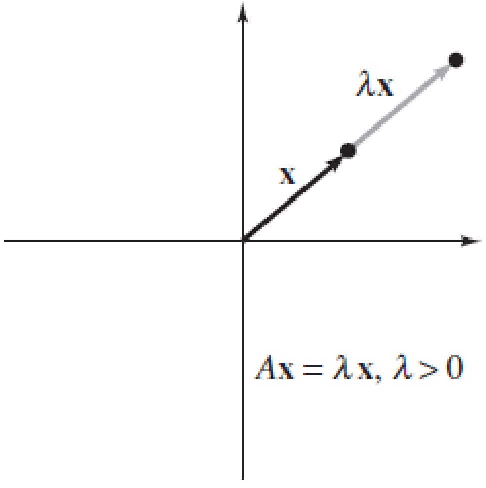
$$A\mathbf{x} = \lambda\mathbf{x}.$$

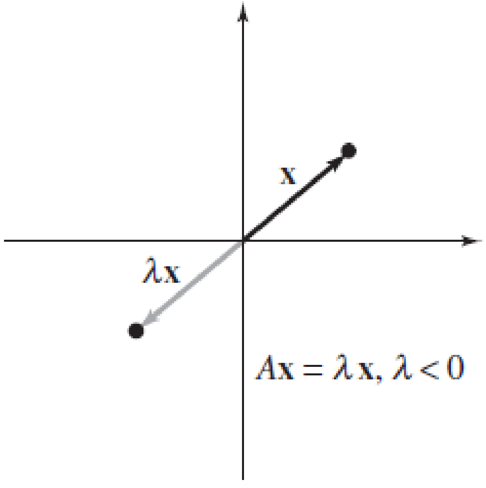
Eigenvector

$x \neq$ Vector 0

Eigenvalues and Eigenvectors (cont.)

- Eigenvalues and eigenvectors have many important applications.
- For now you will consider a geometric interpretation of the problem in R^2.
- If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is an eigenvector of A corresponding to λ, then multiplication of $\mathbf{x}$ by the matrix A produces a vector $\lambda\mathbf{x}$ that is parallel to $\mathbf{x}$.





Definitions of Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is called an **eigenvalue** of A if there is a *nonzero* vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

REMARK: Note that an eigen*vector* cannot be zero. Allowing $\mathbf{x}$ to be the zero vector would render the definition meaningless, because $A\mathbf{0} = \lambda\mathbf{0}$ is true for all real values of λ. An eigen*value* of $\lambda = 0$, however, is possible.

Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$

verify that $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and that $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

SOLUTION Multiplying $\mathbf{x}_1$ by A produces

$$\begin{aligned} A\mathbf{x}_1 &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \underbrace{2}_{\text{Eigenvalue}}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Eigenvector}}. \end{aligned}$$

So, $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$. Similarly, multiplying $\mathbf{x}_2$ by A produces

$$\begin{aligned} A\mathbf{x}_2 &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= -1\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

So, $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

verify that

$$\mathbf{x}_1 = (-3, -1, 1) \quad \text{and} \quad \mathbf{x}_2 = (1, 0, 0)$$

are eigenvectors of A and find their corresponding eigenvalues.

SOLUTION Multiplying $\mathbf{x}_1$ by A produces

λ =0 ได้

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_1 = (-3, -1, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$. Similarly, multiplying $\mathbf{x}_2$ by A produces

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (1, 0, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 1$.

Eigenspaces

ปริภูมิเจาะจง / ปริภูมิเฉพาะ

Although Examples 1 and 2 list only one eigenvector for each eigenvalue, each of the four eigenvalues in Examples 1 and 2 has an infinite number of eigenvectors. For instance, in Example 1 the vectors (2, 0) and (−3, 0) are eigenvectors of A corresponding to the eigenvalue 2. In fact, if A is an $n \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A. This may be seen by letting c be a nonzero scalar, which then produces

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x}).$$

It is also true that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to the *same* eigenvalue λ, then their sum is also an eigenvector corresponding to λ, because

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2).$$

In other words, the set of all eigenvectors of a given eigenvalue λ, together with the zero vector, is a subspace of R^n. This special subspace of R^n is called the **eigenspace** of λ.

$$A(x_1+x_2) = \lambda x_1 + \lambda x_2$$

$$= \lambda(x_1+x_2)$$

Eigenvectors of λ Form a Subspace

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector

รวม Vector 0
ไปด้วย

$$\{\mathbf{0}\} \cup \{\mathbf{x}: \mathbf{x} \text{ is an eigenvector of } \lambda\},$$

is a subspace of R^n. This subspace is called the **eigenspace** of λ.

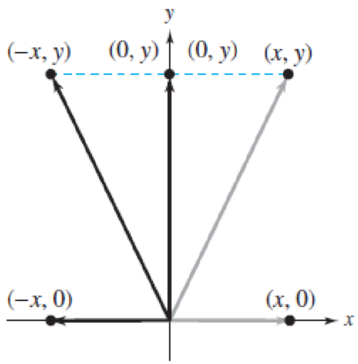
An Example of Eigenspaces in the Plane

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

SOLUTION Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y-axis. That is, if $\mathbf{v} = (x, y)$, then

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$



A reflects vectors in the *y*-axis.

Figure 7.2

Figure 7.2 illustrates that the only vectors reflected onto scalar multiples of themselves are those lying on either the x-axis or the y-axis.

For a vector on the x-axis x-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

Eigenvalue is $\lambda_1 = -1$.

For a vector on the y-axis y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$

Eigenvalue is $\lambda_2 = 1$.

So, the eigenvectors corresponding to $\lambda_1 = -1$ are the nonzero vectors on the x-axis, and the eigenvectors corresponding to $\lambda_2 = 1$ are the nonzero vectors on the y-axis. This implies that the eigenspace corresponding to $\lambda_1 = -1$ is the x-axis, and that the eigenspace corresponding to $\lambda_2 = 1$ is the y-axis.

Eigenvalues and Eigenvectors of a Matrix

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that

$$\det(\lambda I - A) = 0.$$

2. The eigenvectors of A corresponding to λ are the nonzero solutions of

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

สมการลักษณะเฉพาะ

The equation $\det(\lambda I - A) = 0$ is called the characteristic equation of A. Moreover, when expanded to polynomial form, the polynomial

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdot \cdot \cdot + c_1\lambda + c_0$$

→ characteristic polynomial
พหุนามลักษณะเฉพาะ

is called the **characteristic polynomial** of A. This definition tells you that the eigenvalues of an $n \times n$ matrix A correspond to the roots of the characteristic polynomial of A. Because the characteristic polynomial of A is of degree n, A can have at most n distinct eigenvalues.

REMARK: The Fundamental Theorem of Algebra states that an nth-degree polynomial has precisely n roots. These n roots, however, include both repeated and complex roots. In this chapter you will be concerned only with the real roots of characteristic polynomials—that is, real eigenvalues.

อาจ มีทั้งรากจริง/เชิงซ้อน

Finding Eigenvalues and Eigenvectors

$\boxed{Ax = \lambda x}$

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

$$|\lambda I - A| = \left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

SOLUTION The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = 0$$

$$= (\lambda - 2)(\lambda + 5) - (-12)$$

$$= \lambda^2 + 3\lambda - 10 + 12$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda + 1)(\lambda + 2).$$

Degree = 2
Max สุดจะมี
Eigenvalue 2 ตัว

→ แยกตัวประกอบ

So, the characteristic equation is $(\lambda + 1)(\lambda + 2) = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of A. To find the corresponding eigenvectors, use Gauss-Jordan elimination to solve the homogeneous linear system represented by $(\lambda I - A)\mathbf{x} = \mathbf{0}$ twice: first for $\lambda = \lambda_1 = -1$, and then for $\lambda = \lambda_2 = -2$. For $\lambda_1 = -1$, the coefficient matrix is

$$(-1)I - A = \begin{bmatrix} -1 - 2 & 12 \\ -1 & -1 + 5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix},$$

→ ใช้ Gaussian Elimination ลดรูป

which row reduces to

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix},$$

$x_1 - 4x_2 = 0$

$x_1 = 4x_2$

$x_1 = t$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix}$$

showing that $x_1 - 4x_2 = 0$. Letting $x_2 = t$, you can conclude that every eigenvector of λ_1 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

หา λ อีกตัว

ลดรูปเหมือนเดิม
(หลักการคือทำให้มี 0 อย่างน้อย 1 แถว)

For $\lambda_2 = -2$, you have

$$(-2)I - A = \begin{bmatrix} -2 - 2 & 12 \\ -1 & -2 + 5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

Letting $x_2 = t$, you can conclude that every eigenvector of λ_2 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

ไม่งั้นเราจะได้ 0

ทำไม $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$??

เพราะเป็น VECTOR

Try checking $A\mathbf{x} = \lambda_i\mathbf{x}$ for the eigenvalues and eigenvectors in this example.

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I - A| = 0$. It will be a polynomial equation of degree n in the variable λ.
2. Find the real roots of the characteristic equation. These are the eigenvalues of A.
3. For each eigenvalue λ_i, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. This requires row reducing of an $n \times n$ matrix. The resulting reduced row-echelon form must have at least one row of zeros.

Finding the eigenvalues of an $n \times n$ matrix can be difficult because it involves the factorization of an nth-degree polynomial. Once an eigenvalue has been found, however, finding the corresponding eigenvectors is a straightforward application of Gauss-Jordan reduction.

Finding Eigenvalues and Eigenvectors

** ออกสอบ, Pts เยอะสุด

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is the dimension of the eigenspace of each eigenvalue?

SOLUTION The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3.$$

So, the characteristic equation is $(\lambda - 2)^3 = 0$.

$\lambda = 2$ (มีค่าเดียวในcaseนี้)

So, the only eigenvalue is $\lambda = 2$. To find the eigenvectors of $\lambda = 2$, solve the homogeneous linear system represented by $(2I - A)\mathbf{x} = \mathbf{0}$.

$$2I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-x_2 = 0$

$x_2 = 0$

$x_1 = x_1 ; x_3 = x_3$

This implies that $x_2 = 0$. Using the parameters $s = x_1$ and $t = x_3$, you can find that the eigenvectors of $\lambda = 2$ are of the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s \text{ and } t \text{ not both zero.}$$

Because $\lambda = 2$ has two linearly independent eigenvectors, the dimension of its eigenspace is 2.

$\{(1,0,0), (0,0,1)\}$ Eigenspace มี dimension = 2

Finding Eigenvalues and Eigenvectors

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Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each of the corresponding eigenspaces.

SOLUTION The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 1)^2(\lambda - 2)(\lambda - 3).$$

So, the characteristic equation is $(\lambda - 1)^2(\lambda - 2)(\lambda - 3) = 0$ and the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. (Note that $\lambda_1 = 1$ has a multiplicity of 2.)

You can find a basis for the eigenspace of $\lambda_1 = 1$ as follows.

$$(1)I - A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $s = x_2$ and $t = x_4$ produces

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0s - 2t \\ s + 0t \\ 0s + 2t \\ 0s + t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

A basis for the eigenspace corresponding to $\lambda_1 = 1$ is

$B_1 = \{(0, 1, 0, 0), (-2, 0, 2, 1)\}$. **Basis for $\lambda_1 = 1$**

For $\lambda_2 = 2$ and $\lambda_3 = 3$, follow the same pattern to obtain the eigenspace bases

$B_2 = \{(0, 5, 1, 0)\}$ **Basis for $\lambda_2 = 2$**

$B_3 = \{(0, -5, 0, 1)\}$. **Basis for $\lambda_3 = 3$**

Eigenvalues of Triangular Matrices

There are a few types of matrices for which eigenvalues are easy to find. The next theorem states that the eigenvalues of an $n \times n$ triangular matrix are the entries on the main diagonal. Its proof follows from the fact that the determinant of a triangular matrix is the product of its diagonal elements.

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

Finding Eigenvalues of Diagonal and Triangular Matrices

มองเป็น △ บนหรือล่างก็ได้

Find the eigenvalues of each matrix.

(a) $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

SOLUTION (a) Without using Theorem 7.3, you can find that

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix}$$

$$= (\lambda - 2)(\lambda - 1)(\lambda + 3).$$

So, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -3$, which are simply the main diagonal entries of A.

(b) In this case, use Theorem 7.3 to conclude that the eigenvalues are the main diagonal entries $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 0$, $\lambda_4 = -4$, and $\lambda_5 = 3$.

Eigenvalues and Eigenvectors of Linear Transformations

This section began with definitions of eigenvalues and eigenvectors in terms of matrices. They can also be defined in terms of linear transformations. A number λ is called an **eigenvalue** of a linear transformation $T: V \to V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an **eigenvector** of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the **eigenspace** of λ.

Consider the linear transformation $T: R^3 \to R^3$, whose matrix relative to the standard basis is

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Standard basis:
$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

In Example 5 of Section 6.4, you found that the matrix of T relative to the basis B' is the diagonal matrix

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Nonstandard basis:
$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$

The question now is: "For a given transformation T, can you find a basis B' whose corresponding matrix is diagonal?" The next example gives an indication of the answer.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; P^{-1} = \begin{bmatrix} & & \end{bmatrix}$$

$$A' = P^{-1}AP$$

Finding Eigenvalues and Eigenspaces

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

SOLUTION Because

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} \\ &= (\lambda + 2)[(\lambda - 1)^2 - 9] \\ &= (\lambda + 2)(\lambda^2 - 2\lambda - 8) = (\lambda + 2)^2(\lambda - 4), \end{aligned}$$

the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. The eigenspaces for these two eigenvalues are as follows.

$B_1 = \{(1, 1, 0)\}$ **Basis for $\lambda_1 = 4$**

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$ **Basis for $\lambda_2 = -2$**

Diagonalization

- In this section, you will look at another classic problem in linear algebra called the **diagonalization problem**.
- Expressed in terms of matrices, the problem is this: "For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?"
- Two square matrices A and B are called **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$.
- Matrices that are similar to diagonal matrices are called **diagonalizable.**

Definition of a Diagonalizable Matrix

An $n \times n$ matrix A is **diagonalizable** if A is similar to a diagonal matrix. That is, A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Provided with this definition, the diagonalization problem can be stated as follows: "Which square matrices are diagonalizable?" Clearly, every diagonal matrix D is diagonalizable, because the identity matrix I can play the role of P to yield $D = I^{-1}DI$. Example 1 shows another example of a diagonalizable matrix.

A Diagonalizable Matrix

The matrix from Example 5 in Section 6.4,

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

is diagonalizable because

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the property

$$B = P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Diagonal Matrix

$B = P^{-1}AP$ or $D = P^{-1}AP$

Matrix filled with eigenvector from Matrix A

Similar Matrices Have the Same Eigenvalues

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

PROOF Because A and B are similar, there exists an invertible matrix P such that $B = P^{-1}AP$. By the properties of determinants, it follows that

$$\begin{aligned}
|\lambda I - B| = |\lambda I - P^{-1}AP| &= |P^{-1}\lambda IP - P^{-1}AP| \\
&= |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| \\
&= |P^{-1}||P||\lambda I - A| \\
&= |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|.
\end{aligned}$$

But this means that A and B have the same characteristic polynomial. So, they must have the same eigenvalues.

Finding Eigenvalues of Similar Matrices

The matrices A and D are similar.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Use Theorem 7.4 to find the eigenvalues of A and D.

SOLUTION Because D is a diagonal matrix, its eigenvalues are simply the entries on its main diagonal—that is,

$\lambda_1 = 1$,

$\lambda_2 = 2$, and

$\lambda_3 = 3$.

Moreover, because A is said to be similar to D, you know from Theorem 7.4 that A has the same eigenvalues. Check this by showing that the characteristic polynomial of A is

$$|\lambda I - A| = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

REMARK: Example 2 simply states that matrices A and D are similar. Try checking $D = P^{-1}AP$ using the matrices

Eigenvector
ต่อกัน

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

In fact, the columns of P are precisely the eigenvectors of A corresponding to the eigenvalues 1, 2, and 3.

Condition for Diagonalization

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Diagonalizable Matrices

(a) The matrix in Example 1 has the eigenvalues and corresponding eigenvectors listed below.

$$\lambda_1 = 4,\ \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda_2 = -2,\ \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \quad \lambda_3 = -2,\ \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix P whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, because P is row-equivalent to the identity matrix, the eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent.

(b) The matrix in Example 2 has the eigenvalues and corresponding eigenvectors listed below.

$$\lambda_1 = 1,\ \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 2,\ \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_3 = 3,\ \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The matrix P whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Again, because P is row-equivalent to the identity matrix, the eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent.

Steps for Diagonalizing an $n \times n$ Square Matrix

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.
2. If A has n linearly independent eigenvectors, let P be the $n \times n$ matrix whose columns consist of these eigenvectors. That is,

$$P = [\mathbf{p}_1 \vdots \mathbf{p}_2 \vdots \cdots \vdots \mathbf{p}_n].$$

3. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its main diagonal (and zeros elsewhere). Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

A Matrix That Is Not Diagonalizable

Show that the matrix A is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

SOLUTION Because A is triangular, the eigenvalues are simply the entries on the main diagonal. So, the only eigenvalue is $\lambda = 1$. The matrix $(I - A)$ has the reduced row-echelon form shown below.

$$I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$, and letting $x_1 = t$, you can find that every eigenvector of A has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So, A does not have two linearly independent eigenvectors, and you can conclude that A is not diagonalizable.

Diagonalizing a Matrix

Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

SOLUTION The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3).$$

So, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = 3$. From these eigenvalues you obtain the reduced row-echelon forms and corresponding eigenvectors shown below.

Eigenvector

$$2I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$-2I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$3I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Diagonalizing a Matrix

Form the matrix P whose columns are the eigenvectors just obtained.

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

This matrix is nonsingular, which implies that the eigenvectors are linearly independent and A is diagonalizable. The inverse of P is

$$P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & 1 & \frac{1}{5} \end{bmatrix},$$

and it follows that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Diagonalizing a Matrix

Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

SOLUTION In Example 6 in Section 7.1, you found that the three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$ have the eigenvectors shown below.

$$\lambda_1: \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \qquad \lambda_2: \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_3: \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

The matrix whose columns consist of these eigenvectors is

$$P = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 1 & 0 & 5 & -5 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Because P is invertible (check this), its column vectors form a linearly independent set.

$$P^{-1} = \begin{bmatrix} -\frac{5}{2} & 1 & -5 & 5 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

So, A is diagonalizable, and you have

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Sufficient Condition for Diagonalization

If an $n \times n$ matrix A has n *distinct* eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

Determining Whether a Matrix Is Diagonalizable

Determine whether the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

SOLUTION Because A is a triangular matrix, its eigenvalues are the main diagonal entries

$$\lambda_1 = 1, \qquad \lambda_2 = 0, \qquad \lambda_3 = -3.$$ สามารถหา diagonal matrix ได้

Moreover, because these three values are distinct, you can conclude from Theorem 7.6 that A is diagonalizable.

Diagonalization and Linear Transformations

So far in this section, the diagonalization problem has been considered in terms of matrices. In terms of linear transformations, the diagonalization problem can be stated as follows. For a linear transformation

$$T: V \to V,$$

does there exist a basis B for V such that the matrix for T relative to B is diagonal? The answer is "yes," provided the standard matrix for T is diagonalizable.

Finding a Diagonal Matrix for a Linear Transformation

Let $T: R^3 \to R^3$ be the linear transformation represented by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3).$$

If possible, find a basis B for R^3 such that the matrix for T relative to B is diagonal.

?

SOLUTION The standard matrix for T is represented by

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}.$$

From Example 5, you know that A is diagonalizable. So, the three linearly independent eigenvectors found in Example 5 can be used to form the basis B. That is,

$$B = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}.$$

The matrix for T relative to this basis is

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Symmetric Matrices and Orthogonal Diagonalization

- For most matrices you must go through much of the diagonalization process before you can finally determine whether diagonalization is possible.
- One exception is a triangular matrix with distinct entries on the main diagonal.
- Such a matrix can be recognized as diagonalizable by simple inspection.
- In this section you will study another type of matrix that is guaranteed to be diagonalizable: a **symmetric** matrix.

A square matrix A is **symmetric** if it is equal to its transpose:

$$A = A^T.$$

Symmetric Matrices and Nonsymmetric Matrices

The matrices A and B are symmetric, but the matrix C is not.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \qquad \text{Symmetric}$$

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \qquad \text{Symmetric}$$

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix} \qquad \text{Nonsymmetric}$$

Eigenvalues of Symmetric Matrices

If A is an $n \times n$ symmetric matrix, then the following properties are true.

1. A is diagonalizable.
2. All eigenvalues of A are real.
3. If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

REMARK: Theorem 7.7 is called the **Real Spectral Theorem,** and the set of eigenvalues of A is called the **spectrum** of A.

The Eigenvalues and Eigenvectors of a 2×2 Symmetric Matrix

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(D)

$b^2 - 4ac \rightarrow$ discriminant

$D > 0$ รากจริง

$D < 0$ เชิงซ้อน

$D = 0$ $-b/2a$

Prove that a symmetric matrix

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

is diagonalizable.

SOLUTION The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a + b)\lambda + ab - c^2.$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned}(a + b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= (a - b)^2 + 4c^2.\end{aligned}$$

Because this discriminant is the sum of two squares, it must be either zero or positive. If $(a - b)^2 + 4c^2 = 0$, then $a = b$ and $c = 0$, which implies that A is already diagonal. That is,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

On the other hand, if $(a - b)^2 + 4c^2 > 0$, then by the Quadratic Formula the characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. So, A is diagonalizable in this case also.

Dimensions of the Eigenspaces of a Symmetric Matrix

Find the eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

and determine the dimensions of the corresponding eigenspaces.

SOLUTION The characteristic polynomial of A is represented by

k

multiplicity ภาวะรากซ้ำ

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 2 & 0 & 0 \\ 2 & \lambda - 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 2 \\ 0 & 0 & 2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)^2(\lambda - 3)^2.$$

$\lambda_1 = -1,\ \lambda = 3$

So, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. Because each of these eigenvalues has a multiplicity of 2, you know from Theorem 7.7 that the corresponding eigenspaces also have dimension 2. Specifically, the eigenspace of $\lambda_1 = -1$ has a basis of $B_1 = \{(1, 1, 0, 0), (0, 0, 1, 1)\}$ and the eigenspace of $\lambda_2 = 3$ has a basis of $B_2 = \{(1, -1, 0, 0), (0, 0, 1, -1)\}$.

Orthogonal Matrices

- To diagonalize a square matrix A, you need to find an *invertible* matrix P such that $P^{-1}AP$ is diagonal.
- For symmetric matrices, you will see that the matrix P can be chosen to have the special property that $P^{-1} = P^T$.
- This unusual matrix property is defined as follows.

A square matrix P is called **orthogonal** if it is invertible and if

$$\boxed{P^{-1} = P^T.}$$

Orthogonal Matrices

(a) The matrix

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is orthogonal because

$$P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(b) The matrix

$$P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

is orthogonal because

$$P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}.$$

Property of Orthogonal Matrices

An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set.

PROOF Suppose the column vectors of P form an orthonormal set:

$$P = [\mathbf{p}_1 \vdots \mathbf{p}_2 \vdots \cdots \vdots \mathbf{p}_n] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}.$$

Then the product P^TP has the form

$$P^TP = \begin{bmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

$$P^TP = \begin{bmatrix} \mathbf{p}_1 \cdot \mathbf{p}_1 & \mathbf{p}_1 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_1 \cdot \mathbf{p}_n \\ \mathbf{p}_2 \cdot \mathbf{p}_1 & \mathbf{p}_2 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_2 \cdot \mathbf{p}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{p}_n \cdot \mathbf{p}_1 & \mathbf{p}_n \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_n \cdot \mathbf{p}_n \end{bmatrix}.$$

Because the set $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ is orthonormal, you have

$$\mathbf{p}_i \cdot \mathbf{p}_j = 0,\ i \neq j \quad \text{and} \quad \mathbf{p}_i \cdot \mathbf{p}_i = \|\mathbf{p}_i\|^2 = 1.$$

So, the matrix composed of dot products has the form

$$P^TP = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n.$$

This implies that $P^T = P^{-1}$, and you can conclude that P is orthogonal.

Conversely, if P is orthogonal, you can reverse the steps above to verify that the column vectors of P form an orthonormal set.

An Orthogonal Matrix

Show that

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

is orthogonal by showing that $PP^T = I$. Then show that the column vectors of P form an orthonormal set.

SOLUTION Because

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3,$$

it follows that $P^T = P^{-1}$, and you can conclude that P is orthogonal. Moreover, letting

$$\mathbf{p}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{\sqrt{5}} \\ -\frac{2}{3\sqrt{5}} \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ -\frac{4}{3\sqrt{5}} \end{bmatrix}, \text{ and } \mathbf{p}_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

produces

เอา Vector ที่ตั้งฉากกันต่อกันเป็น Matrix

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{p}_3 = 0$$

and

$$\|\mathbf{p}_1\| = \|\mathbf{p}_2\| = \|\mathbf{p}_3\| = 1.$$

So, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an orthonormal set, as guaranteed by Theorem 7.8.

Property of Symmetric Matrices

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

PROOF Let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. So,

$$A\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \quad \text{and} \quad A\mathbf{x}_2 = \lambda_2\mathbf{x}_2.$$

To prove the theorem, use the matrix form of the dot product shown below.

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} = \mathbf{x}_1^T\mathbf{x}_2$$

Now you can write

$$\begin{aligned} \lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2) &= (\lambda_1\mathbf{x}_1) \cdot \mathbf{x}_2 \\ &= (A\mathbf{x}_1) \cdot \mathbf{x}_2 \\ &= (A\mathbf{x}_1)^T\mathbf{x}_2 \\ &= (\mathbf{x}_1^T A^T)\mathbf{x}_2 \\ &= (\mathbf{x}_1^T A)\mathbf{x}_2 && \text{Because } A \text{ is symmetric, } A = A^T. \\ &= \mathbf{x}_1^T(A\mathbf{x}_2) \\ &= \mathbf{x}_1^T(\lambda_2\mathbf{x}_2) \\ &= \mathbf{x}_1 \cdot (\lambda_2\mathbf{x}_2) \\ &= \lambda_2(\mathbf{x}_1 \cdot \mathbf{x}_2). \end{aligned}$$

This implies that $(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0$, and because $\lambda_1 \neq \lambda_2$ it follows that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. So, $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Eigenvectors of a Symmetric Matrix

Show that any two eigenvectors of

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

corresponding to distinct eigenvalues are orthogonal.

SOLUTION The characteristic polynomial of A is

$\lambda = 2, \lambda = 4$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4),$$

which implies that the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$. Every eigenvector corresponding to $\lambda_1 = 2$ is of the form

$$\mathbf{x}_1 = \begin{bmatrix} s \\ -s \end{bmatrix}, \quad s \neq 0$$

and every eigenvector corresponding to $\lambda_2 = 4$ is of the form

$$\mathbf{x}_2 = \begin{bmatrix} t \\ t \end{bmatrix}, \quad t \neq 0.$$

So,

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \begin{bmatrix} s \\ -s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0,$$

and you can conclude that $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Orthogonal Diagonalization

- A matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.
- The following important theorem states that the set of orthogonally diagonalizable matrices is precisely the set of symmetric matrices.

Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalues if and only if A is symmetric.

Determining Whether a Matrix Is Orthogonally Diagonalizable

Which matrices are orthogonally diagonalizable?

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \checkmark \qquad A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \checkmark$$

SOLUTION By Theorem 7.10, the orthogonally diagonalizable matrices are the symmetric ones: A_1 and A_4.

Orthogonal Diagonalization of a Symmetric Matrix

Let A be an $n \times n$ symmetric matrix.

1. Find all eigenvalues of A and determine the multiplicity of each.
2. For *each* eigenvalue of multiplicity 1, choose a unit eigenvector. (Choose any eigenvector and then normalize it.)
3. For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. (You know from Theorem 7.7 that this is possible.) If this set is not orthonormal, apply the Gram-Schmidt orthonormalization process.
4. The composite of steps 2 and 3 produces an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^TAP = D$ will be diagonal. (The main diagonal entries of D are the eigenvalues of A.)

Orthogonal Diagonalization

Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}.$$

multiplicity เป็น 1

↳ eigenvector 1 ตัว

SOLUTION

1. The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 2 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 3)(\lambda - 2).$$

   So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

2. For each eigenvalue, find an eigenvector by converting the matrix $\lambda I - A$ to reduced row-echelon form.

$$\begin{array}{ccccc} & & & & \text{Eigenvector} \\ -3I - A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ 2I - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$$

   The eigenvectors $(-2, 1)$ and $(1, 2)$ form an *orthogonal* basis for R^2. Each of these eigenvectors is normalized to produce an *orthonormal* basis.

$$\mathbf{p}_1 = \frac{(-2, 1)}{\|(-2, 1)\|} = \frac{1}{\sqrt{5}}(-2, 1) = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$$\mathbf{p}_2 = \frac{(1, 2)}{\|(1, 2)\|} = \frac{1}{\sqrt{5}}(1, 2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

3. Because each eigenvalue has a multiplicity of 1, go directly to step 4.
4. Using $\mathbf{p}_1$ and $\mathbf{p}_2$ as column vectors, construct the matrix P.

$$P = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

   Verify that P is correct by computing $P^{-1}AP = P^TAP$.

$$P^TAP = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

Orthogonal Diagonalization

Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}.$$

SOLUTION

1. The characteristic polynomial of A, $|\lambda I - A| = (\lambda - 3)^2(\lambda + 6)$, yields the eigenvalues $\lambda_1 = -6$ and $\lambda_2 = 3$. λ_1 has a multiplicity of 1 and λ_2 has a multiplicity of 2.
2. An eigenvector for λ_1 is $\mathbf{v}_1 = (1, -2, 2)$, which normalizes to

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right).$$

3. Two eigenvectors for λ_2 are $\mathbf{v}_2 = (2, 1, 0)$ and $\mathbf{v}_3 = (-2, 0, 1)$. Note that $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{v}_3$, as guaranteed by Theorem 7.9. The eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$, however, are not orthogonal to each other. To find two orthonormal eigenvectors for λ_2, use the Gram-Schmidt process as follows.

$$\mathbf{w}_2 = \mathbf{v}_2 = (2, 1, 0)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 = \left(-\frac{2}{5}, \frac{4}{5}, 1\right)$$

These vectors normalize to

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}\right).$$

4. The matrix P has $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ as its column vectors.

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

A check shows that

$$P^{-1}AP = P^TAP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Gram-Schmidt

1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for an inner product space V.
2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, where $\mathbf{w}_i$ is given by

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$\vdots$$

$$\mathbf{w}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1} \rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1} \rangle}\mathbf{w}_{n-1}.$$

Then B' is an *orthogonal* basis for V.

3. Let $\mathbf{u}_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then the set $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an *orthonormal* basis for V. Moreover, span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ = span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for $k = 1, 2, \ldots, n$.

Population Growth

Matrices can be used to form models for population growth. The first step in this process is to group the population into age classes of equal duration. For instance, if the maximum life span of a member is L years, then the age classes are represented by the n intervals shown below.

$$\underbrace{\left[0, \frac{L}{n}\right)}_{\text{First age class}}, \quad \underbrace{\left[\frac{L}{n}, \frac{2L}{n}\right)}_{\text{Second age class}}, \ldots, \underbrace{\left[\frac{(n-1)L}{n}, L\right]}_{n\text{th age class}}$$

The number of population members in each age class is then represented by the **age distribution vector**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \qquad \begin{array}{l} \textbf{Number in first age class} \\ \textbf{Number in second age class} \\ \vdots \\ \textbf{Number in } n\textbf{th age class} \end{array}$$

Over a period of L/n years, the *probability* that a member of the ith age class will survive to become a member of the $(i + 1)$th age class is given by p_i, where

$$0 \le p_i \le 1, \quad i = 1, 2, \ldots, n - 1.$$

The *average number* of offspring produced by a member of the ith age class is given by b_i, where

$$0 \le b_i, \quad i = 1, 2, \ldots, n.$$

These numbers can be written in matrix form, as shown below.

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix}$$

Multiplying this **age transition matrix** by the age distribution vector for a specific time period produces the age distribution vector for the next time period. That is,

$$A\mathbf{x}_i = \mathbf{x}_{i+1}.$$

A Population Growth Model

A population of rabbits raised in a research laboratory has the characteristics listed below.

(a) Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years.

(b) During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year.

The laboratory population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. How many rabbits will be in each age class in 1 year?

SOLUTION The current age distribution vector is

$$\mathbf{x}_1 = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} \qquad \begin{matrix} \mathbf{0 \le age < 1} \\ \mathbf{1 \le age < 2} \\ \mathbf{2 \le age \le 3} \end{matrix}$$

and the age transition matrix is

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

After 1 year the age distribution vector will be

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix}. \qquad \begin{matrix} \mathbf{0 \le age < 1} \\ \mathbf{1 \le age < 2} \\ \mathbf{2 \le age \le 3} \end{matrix}$$

Finding a Stable Age Distribution Vector

Find a stable age distribution vector for the population in Example 1.

SOLUTION To solve this problem, find an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda & -6 & -8 \\ -0.5 & \lambda & 0 \\ 0 & -0.5 & \lambda \end{vmatrix} = \lambda^3 - 3\lambda - 2 = (\lambda + 1)^2(\lambda - 2),$$

which implies that the eigenvalues are -1 and 2. Choosing the positive value, let $\lambda = 2$. To find a corresponding eigenvector, row reduce the matrix $2I - A$ to obtain

$$\begin{bmatrix} 2 & -6 & -8 \\ -0.5 & 2 & 0 \\ 0 & -0.5 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, the eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}.$$

For instance, if $t = 2$, then the initial age distribution vector would be

$$\mathbf{x}_1 = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} \quad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

and the age distribution vector for the next year would be

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}\begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 4 \end{bmatrix}. \quad \begin{array}{l} 0 \le \text{age} < 1 \\ 1 \le \text{age} < 2 \\ 2 \le \text{age} \le 3 \end{array}$$

Notice that the ratio of the three age classes is still 16 : 4 : 1, and so the percent of the population in each age class remains the same.

Q & A