Mathematics
for
Information Technology

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INTEGRATION (I)

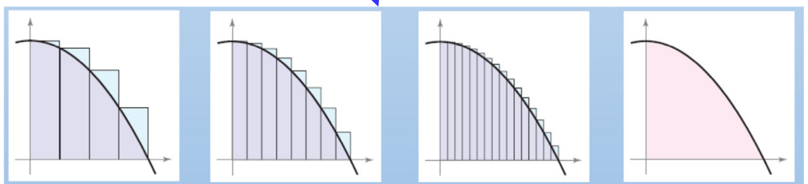
Integration

- **OBJECTIVES**

  How to

  I – evaluate indefinite integrals using basic integration rules.

  II – evaluate a sum and approximate the area of a plane region.



The area of a parabolic region can be approximated as the sum of the areas of rectangles. As you increase the number of rectangles, the approximation tends to become more and more accurate.
you will learn how the limit process can be used to find areas of a wide variety of regions.

Integration

- **OBJECTIVES**

  How to

  – evaluate a definite integral using a limit.

  III – evaluate a definite integral using the Fundamental Theorem of Calculus.

  IV – evaluate different types of definite and indefinite integrals using a variety of methods.

Antiderivatives and Indefinite Integration

I

Objectives

- Write the general solution of a differential equation.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

Antiderivatives

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note that F is called *an* antiderivative of f, rather than *the* antiderivative of f. To see why, observe that

$$F_1(x) = x^3, \quad F_2(x) = x^3 - 5, \quad \text{and} \quad F_3(x) = x^3 + 97$$

เป็น antiderivative หมดเลย

are all antiderivatives of $f(x) = 3x^2$. In fact, for any constant C, the function given by $F(x) = x^3 + C$ is an antiderivative of f.

คำตอบทั่วไปไม่เฉพาะเจาะจง

the **general solution** of the *differential equation*

THEOREM 4.1 REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

general antiderivative of f | **constant of integration**

EXAMPLE 1 Solving a Differential Equation

Find the general solution of the differential equation $y' = 2$.

Solution To begin, you need to find a function whose derivative is 2. One such function is

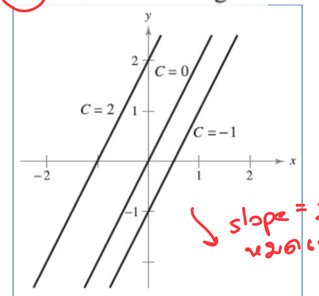
$$y = 2x. \qquad \text{2x is *an* antiderivative of 2.}$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C. \qquad \text{General solution}$$

แค่ขยับ Graph

The graphs of several functions of the form $y = 2x + C$ are shown in Figure



Functions of the form $y = 2x + C$

Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\,dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$. The general solution is denoted by

$\cancel{\int d} y = y$

(หักล้าง

Variable of integration | Constant of integration

$$y = \int f(x)\,dx = F(x) + C.$$

เป็น General sol.

Integrand | An antiderivative of $f(x)$

ไม่ได้ define ช่วงชัดเจน

The expression $\int f(x)dx$ is read as the *antiderivative of f with respect to x*. So, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

Integrate ทั้งสมการ

Integration & Differentiation หักล้างกันได้

(กระบวนการย้อนกลับของ Derivative

Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.

Basic Integration Rules

Differentiation Formula	*Integration Formula*
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$ Constant Multiple Rule
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\quad n \neq -1$ Power Rule

x^{-1} ใช้ไม่ได้

$C+1$ จะไม่กลับไปเป็นตัวเดิม

NOTE Note that the Power Rule for Integration has the restriction that $n \neq -1$.

THEOREM 5.5 LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{1}{u}\,du = \ln|u| + C$

ทำไมต้องติด abs?

Basic Integration Rules

ย้อนกลับ แล้ว + C

(ก่อนหน้าอาจมี const. อะไรก็ได้)

$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

$$\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2} + C\right) = \frac{3}{2}x^2 + 3C.$$

EXAMPLE 2 Applying the Basic Integration Rules

Describe the antiderivatives of $3x$.

Solution $\int 3x\,dx = 3\int x\,dx$ Constant Multiple Rule

ความจริง

$= 3\int x^1\,dx$ Rewrite x as x^1.

$= 3\left(\frac{x^2}{2}\right) + C$ Power Rule ($n = 1$)

$= \frac{3}{2}x^2 + C$ Simplify.

$= 3\left(\frac{x^2}{2} + C\right)$

$= \frac{3x^2}{2} + 3C$

$3C$, C ก็ const. เหมือนกัน

So, the antiderivatives of $3x$ are of the form $\frac{3}{2}x^2 + C$, where C is any constant.

the general pattern of integration

Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

Basic Integration Rules

EXAMPLE

$\boxed{\int k\,dx = kx + C}$

$\boxed{\int kf(x)\,dx = k\int f(x)\,dx}$

$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,}$

$\int 5dx = 5x + C$

$\int -8dx = -8x + C$

$\int \beta dx = \beta x + C$

มองว่าเป็น constant

$\int -2xdx = \frac{-2x^2}{2} = -x^2 + C$

$\int \pi x dx = \frac{\pi x^2}{2} = \frac{\pi}{2}x^2 + C$

Basic Integration Rules

EXAMPLE

$\int x^2dx = \frac{x^3}{3} + C$

$\int x^5dx = \frac{x^6}{6} + C$

$\int x^9dx = \frac{x^9}{9} + C$

$\int 3x^2dx = \frac{3x^3}{3} + C = x^3 + C$

$\int 10x^4dx = 10\int x^4dx$

$= 10\left(\frac{x^5}{5} + C\right)$

$= 2x^5 + C$ #

$\int 8x^7dx$

$= \frac{8x^8}{8} + C$

$= x^8 + C$

EXAMPLE 3 **Rewriting Before Integrating**

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

Remember that you can check your answer to an antidifferentiation problem by differentiating.

x^{-1} เป็นข้อยกเว้น

Basic Integration Rules

$\int \sqrt[4]{x}dx = \int x^{\frac{1}{4}}dx = \frac{4}{5}x^{\frac{5}{4}} + C$

$\int \frac{1}{x}dx = \int x^{-1}dx = \ln|x| + C$

$\int \frac{2}{x}dx = \int 2x^{-1}dx = 2\ln|x| + C$

$\int \frac{5}{x}dx = 5\ln|x| + C$

$a^x = b \quad x = \log_a b$

$\mathbb{R}$

1. $\int \frac{1}{x}\,dx = \ln|x| + C$

Basic Integration Rules

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = -4x^{-4} = \frac{-4}{x^4} + C$$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2x^{\frac{1}{2}} + C = 2\sqrt{x} + C$$

$$\int \frac{1}{\sqrt[3]{x}}\,dx \quad \int x^{-\frac{1}{3}}\,dx = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2}x^{\frac{2}{3}} + C$$

$$= \frac{3}{2}\sqrt[2]{x^3} + C$$

EXAMPLE 4 Integrating Polynomial Functions

a. $\int dx = \int 1\,dx$ — Integrand is understood to be 1.

$= x + C$ — Integrate.

b. $\int (x+2)\,dx = \int x\,dx + \int 2\,dx$

$= \frac{x^2}{2} + C_1 + 2x + C_2$ — Integrate.

$= \frac{x^2}{2} + 2x + C$ — $C = C_1 + C_2$

Integrate หลาย terms

The second line in the solution is usually omitted.

c. $\int (3x^4 - 5x^2 + x)\,dx = \frac{3x^5}{5} - \frac{5x^3}{3} + \frac{x^2}{2}$ — Integrate.

$=$ — Simplify.

BREAK

EXAMPLE 5 Rewriting Before Integrating

$\int \frac{x+1}{\sqrt{x}}\,dx = \frac{x}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}}$ — Rewrite as two fractions.

$= x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ — Rewrite with fractional exponents.

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}$ — Integrate.

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ — Simplify.

$= \frac{2}{3}\sqrt{x}(x+3) + C$

อย่าลืม

$\int \frac{x+1}{\sqrt{x}}\,dx$ is not $\cancel{\dfrac{\int (x+1)\,dx}{\int (\sqrt{x})\,dx}}$

EXAMPLE 6 Rewriting Before Integrating

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \frac{\sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx = \int \sec x \tan x \, dx$$

$$= \sec x + C$$

$$\int 3 \cos x \, dx = 3\int \cos x \, dx = 3 \sin x + C$$

$$\int \frac{1}{\sin^2 x} dx = \int (\csc x)^2 dx = -\cot x + C$$

Initial Conditions and Particular Solutions

$$y = \int (3x^2 - 1)\, dx = x^3 - x + C \qquad \text{General solutio}$$

In many applications of integration, you are given enough information to determine a **particular solution**. To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an **initial condition**. For example, in Figure, only one curve passes through the point (2, 4). To find this curve, you can use the following information.

$F(x) = x^3 - x + C$ General solution

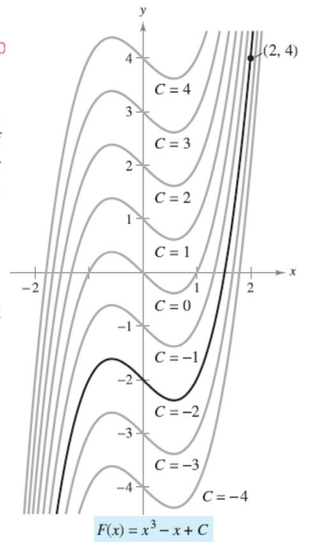
$F(2) = 4$ Initial condition

By using the initial condition in the general solution, you can determine that $F(2) = 8 - 2 + C = 4$, which implies that $C = -2$. So, you obtain

$F(x) = x^3 - x - 2.$ Particular solution

คำตอบที่แน่ชัด

จะรู้ว่าจุดนั้นอยู่ตรง C ไหน!



The particular solution that satisfies the initial condition $F(2) = 4$ is $F(x) = x^3 - x - 2$.

EXAMPLE 7 Finding a Particular Solution

Find the general solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular solution that satisfies the initial condition $F(1) = 0$.

Graph of General Sol. ที่ C มีค่าต่างกัน

Solution To find the general solution, integrate to obtain

$$F(x) = \int \frac{1}{x^2} dx \qquad F(x) = \int F'(x)dx$$

$$= \int x^{-2} dx \qquad \text{Rewrite as a power.}$$

$$= \frac{x^{-1}}{-1} + C \qquad \text{Integrate.}$$

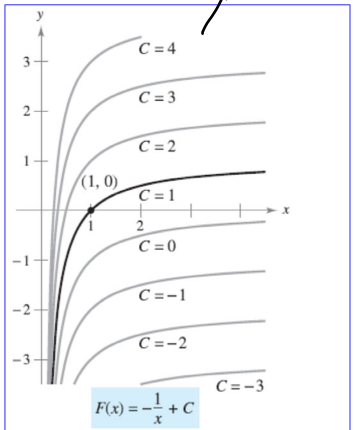
$$= -\frac{1}{x} + C, \quad x > 0. \qquad \text{General solution}$$

Using the initial condition $F(1) = 0$, you can solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \quad \Rightarrow \quad C = 1$$

So, the particular solution, as shown in Figure, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0. \qquad \text{Particular solution}$$



The particular solution that satisfies the initial condition $F(1) = 0$ is $F(x) = -(1/x) + 1$, $x > 0$.

EXAMPLE 8 Solving a Vertical Motion Problem

v ณ จุดเริ่มต้น

$s'(0) = 64$

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet. $S(0) = 80$

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?

Solution

a. Let $t = 0$ represent the initial time. The two given initial conditions can be written as follows.

$s(0) = 80$ Initial height is 80 feet.

$s'(0) = 64$ Initial velocity is 64 feet per second.

Using -32 feet per second per second as the acceleration due to gravity, you can write

$s''(t) = -32$ ความเร่ง (a)

$$s'(t) = \int s''(t)\, dt = \int -32 dt = -32t + C_1.$$ Integration ย้อนกลับ 2 รอบ

Using the initial velocity, you obtain $s'(0) = 64 = -32(0) + C_1$, which implies that $C_1 = 64$. Next, by integrating $s'(t)$, you obtain

$$s(t) = \int s'(t)\, dt = \int (-32t + 64)\, dt = -16t^2 + 64t + C_2.$$

Using the initial height, you obtain

$$s(0) = 80 = -16(0^2) + 64(0) + C_2$$

EXAMPLE 8 Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?

Solution

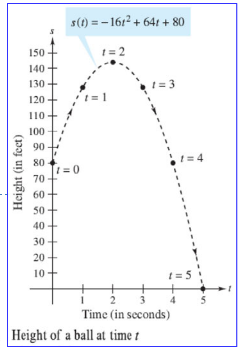
which implies that $C_2 = 80$. So, the position function is

$$s(t) = -16t^2 + 64t + 80.$$

b. Using the position function found in part (a), you can find the time at which the ball hits the ground by solving the equation $s(t) = 0$.

$$s(t) = -16t^2 + 64t + 80 = 0$$
$$-16(t+1)(t-5) = 0$$
$$t = -1, 5$$

Because t must be positive, you can conclude that the ball hits the ground 5 seconds after it was thrown.



Height of a ball at time t

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\,dx$	$\int (2x^{-\frac{1}{2}})\,dx$	$\frac{2x^{\frac{1}{2}}}{\frac{1}{2}}$	$4x^{\frac{1}{2}} = 4\sqrt{x} + C$
$\int (t^2+1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dx$	$\frac{t^5}{5} + \frac{2t^3}{3} + t$	$\frac{t^5}{5} + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3+3}{x^2}\,dx$	$\int [(x^3+3)x^{-2}]\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + \frac{3x^{-1}}{-1} + C$ $= \frac{x^2}{2} - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x-4)\,dx$	$\int x^{\frac{1}{3}}(x-4)\,dx$	$\int (x^{\frac{4}{3}} - 4x^{\frac{1}{3}})\,dx$	

$$\frac{x^{\frac{7}{3}}}{\frac{7}{3}} - \frac{4x^{\frac{4}{3}}}{\frac{4}{3}} + C$$

$$= \frac{3}{7}x^{\frac{7}{3}} - \cancel{\frac{3}{4}}(4)x^{\frac{4}{3}} + C$$

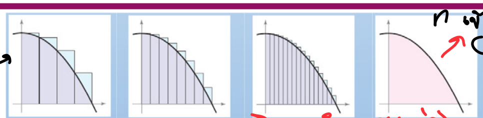
$$= \frac{3}{7}x^{7/3} - 3x^{4/3} + C$$

II

The Area of a Plane Region

n น้อย → ประมาณค่าหยาบ

n = 4



n เข้าใกล้ ∞ (แม่นยำมากสุด)

n (จำนวนแท่ง) เพิ่มเรื่อยๆ

DEFINITION OF THE AREA OF A REGION IN THE PLANE

Let f be continuous and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

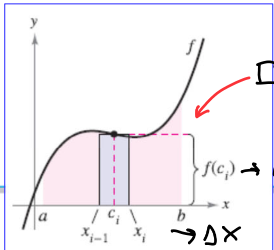
มาจาก 2 เส้นนี้

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = (b - a)/n$

นำ Limit Process มาช่วย

an *arbitrary* x-value in the ith subinterval



The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.

□ หลายๆแท่งมาบวกกัน

$f(c_i)$ → ความสูงของ c_i

→ Δx

Definite Integrals

Objectives

Root มาจากคำว่า "รวม"

- Evaluate a definite integral using limits.
- Evaluate a definite integral using properties of definite integrals.

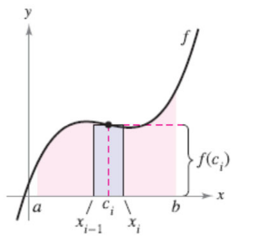
Definite Integrals

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

DEFINITION OF THE AREA OF A REGION IN THE PLANE

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

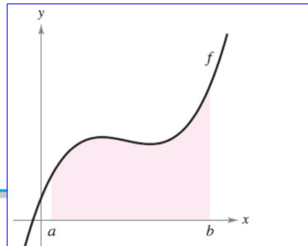


The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.

THEOREM 4.5 THE DEFINITE INTEGRAL AS THE AREA OF A REGION

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$



You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.

- A **straightforward technique** for evaluating a definite integral will be discussed in the **next lecture**.
- For now, however, you can evaluate a definite integral in two ways.
  - Use the **limit definition**. OR
  - Use the **area definition** (if the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

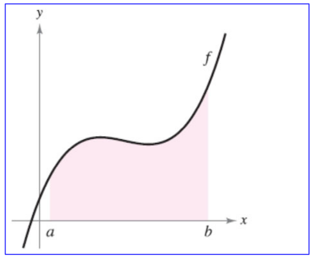
I will not demonstrate how these two methods are used.

Properties of Definite Integrals

DEFINITIONS OF TWO SPECIAL DEFINITE INTEGRALS

1. If f is defined at $x = a$, then we define $\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

$b > a$



THEOREM 4.6 ADDITIVE INTERVAL PROPERTY

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

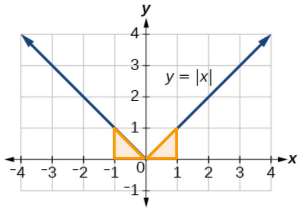
EXAMPLE 5 Using the Additive Interval Property

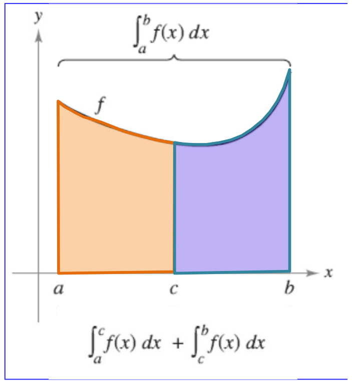
Abs แยกเป็น 2 กรณี

$$\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx \qquad \text{Theorem 4.6}$$

$$= \frac{1}{2} + \frac{1}{2} \qquad \text{Area of a triangle}$$

$$= 1$$





THEOREM 4.7 PROPERTIES OF DEFINITE INTEGRALS

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions. For example,

$$\int_a^b [f(x) + g(x) + h(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \int_a^b h(x)\,dx.$$

THEOREM 4.8 PRESERVATION OF INEQUALITY ; รักษาไว้ซึ่งความไม่เท่ากัน

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

ถ้าอยู่ใต้แกน x → Integrate แล้วจะติด (−)

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

น้อยกว่า "เสมอ"

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$