Mathematics
for
Information Technology

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What is calculus?

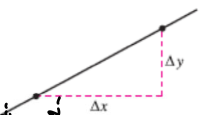
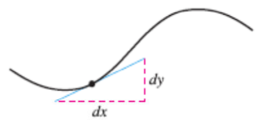
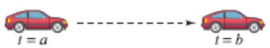
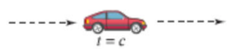
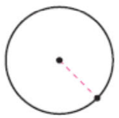
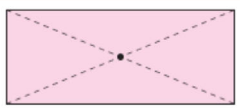
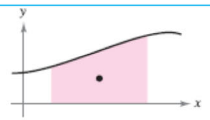
changes of specific position

- Calculus is the mathematics of changes.
  - Mathematics of velocity, accelerations, changes of velocity tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, etc.
  - To model real-life situations.

- Precalculus mathematics is more static, whereas calculus is more dynamic.

What is calculus? Example

Static Dynamic

Without Calculus		With Differential Calculus	
Slope of a line		Slope of a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	
Curvature of a circle		Curvature of a curve	
Without Calculus		**With Integral Calculus**	
Center of a rectangle		Centroid of a region	

เฉลี่ยตลอดการเคลื่อนที่

ณ จุดใดๆ

ใช้tangent line มาช่วยหา

What is calculus?

- The reformulation of precalculus mathematics through the use of a limit process.
- The calculus is a "limit machine" that involves three stages.

Precalculus mathematics ⇨ Limit process ** ⇨ Calculus

e.g., a derivative or integral

- <u>DO NOT</u> learn calculus as if it were simply a collection of new formulas.
  - Miss a great deal of understanding.

FUNCTION
LIMIT AND **CONTINUITY**

Function

- **OBJECTIVES**

  Learn …
  - Function definition
  - Function notation
  - Domain and range of a function
  - A function of x

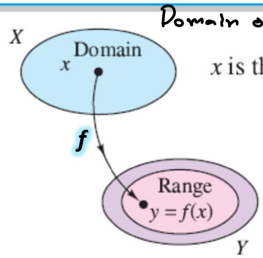
Functions Definition

DEFINITION OF A REAL-VALUED FUNCTION OF A REAL VARIABLE

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X

Domain อาจจะเป็น subset ของ $\mathbb{R}$ (บางค่าทำให้หาค่าไม่ได้)



x is the **independent variable**

Map input(x) เพื่อให้ได้ output(y)

y is the **dependent variable**

อธิบายปรากฏการณ์/changes ต่างๆในโลกด้วย Function

Function Notation

$$x^2 + 2y = 1$$ Equation in implicit form

$y \to f(x)$

$$y = \frac{1}{2}(1 - x^2)$$ Equation in explicit form ได้ค่า y เลย

ชัดเจนตรงๆ

$$f(x) = \frac{1}{2}(1 - x^2).$$ Function notation

The symbol $f(x)$ is read "f of x."

Domain of a Function

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

Explicitly

→ ไกด์ set ข้อจำกัดนี้ไว้

an explicitly defined domain given by $\{x:\ 4 \le x \le 5\}$

$\mathbb{R} - \{-2, 2\}$

$$g(x) = \frac{1}{x^2 - 4}$$

ไม่ได้บอกมาเลย

Implicitly

an implied domain that is the set $\{x:\ x \neq \pm 2\}$

ค่า x ใดๆ ที่ไม่ใช่ ±2

Finding the **Domain** and **Range** of a Function

Real Number เท่านั้นไม่เอา Complex Num

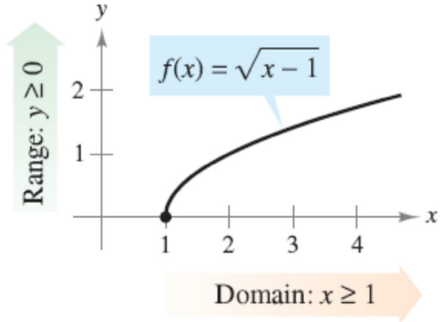
The domain of the function

$$f(x) = \sqrt{x - 1}$$

$x - 1 \ge 0$
$x \ge 1$

Domain

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative. So, the range is the interval $[0, \infty)$ → Range



A function from X to Y is จับคู่ 1 คู่ **one-to-one** X 1 ค่า → y 1 ค่า if to each y-value in the range there corresponds exactly one x-value in the domain

A function from X to Y is **onto** if its range consists of all Y.

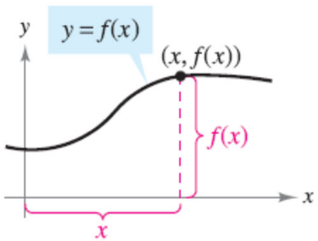
Is this function one-to-one?
Is this function onto?

(เพราะ range $\neq \mathbb{R}$)

$f(x) = \sqrt{x - 1}$

เป็น one-to-one แต่ไม่ onto

A Function of x



The graph of a function

x = the directed distance from the y-axis
$f(x)$ = the directed distance from the x-axis.

ลาก Vertical แล้วมีจุดตัดอันเดียว → เป็น Function of X

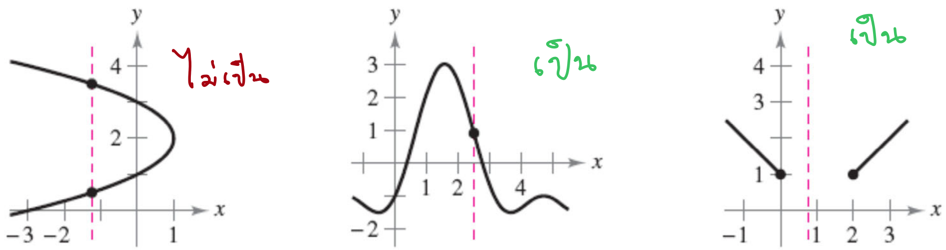
A function of x

A vertical line can intersect the graph of a function of x at most *once*.

(ไม่ตัดก็ไม่เป็นไร

↓

Vertical Line Test



ไม่เป็น

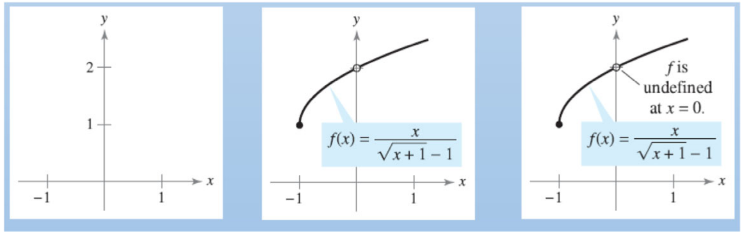
เป็น

เป็น

(a) ????????????? (b) ????????????? (c) ?????????????

Limit and Their Properties

- The limit of a function is the primary concept that distinguishes calculus from algebra and analytic geometry.
- One technique of estimating a limit is to …
  – Graph the function AND
  – Determine the behavior of the graph as the independent variable approaches a specific value.



Limit and Their Properties

According to NASA, the coldest place in the known universe is the Boomerang nebula. The nebula is five thousand light years from Earth and has a temperature of −272°C. That is only 1° warmer than absolute zero, the coldest possible temperature.

How did scientists determine that absolute zero is the "lower limit" of the temperature of matter?

plot กราฟแล้วดูว่ามันเป็นลู่เข้าไหน

Limit and Their Properties

- **OBJECTIVES**

  Learn how to …
  – Find limits graphically and numerically.
  – Evaluate limits analytically.
  – Determine continuity at a point and on an open interval.
  – Determine one-sided limits.

Finding Limits
Graphically and Numerically

Objectives

- Estimate a limit using a numerical or graphical approaches.
- Learn different way that a limit can fail to exist.

An Introduction to Limits

Numerical Approach

Suppose you are asked to sketch the graph

$$f(x) = \frac{x^3 - 1}{x - 1}, \quad \underline{x \neq 1}.$$

กำหนด Domain ไว้

หา f(x) ไม่ได้แต่หา limit ของ f(x) ได้
(ค่าที่เข้าใกล้ f(x))

x approaches 1 from the left. → ← *x* approaches 1 from the right.

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

f(x) approaches 3 from both side

การหา Limit คือการหา
ค่า f(x) ของฟังก์ชัน เมื่อ x เข้าใกล้ค่าใดค่าหนึ่ง

An Introduction to Limits

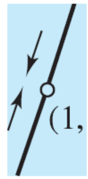
Graphical Approach

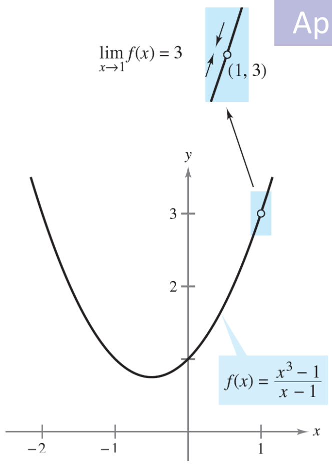
Suppose you are asked to sketch the graph

$$f(x) = \frac{x^3 - 1}{x - 1}, \quad x \neq 1.$$

This leads to a definition of limit.

$$\lim_{x \to 1} f(x) = 3$$





This is read as "the limit of $f(x)$ as x approaches 1 is 3."

An Introduction to Limits

Numerical Approach

EXAMPLE 1 Estimating a Limit Numerically

Evaluate the function $f(x) = x/\left(\sqrt{x+1} - 1\right)$ at several points near $x = 0$ and use the results to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}.$$

Why near x = 0?

Solution The table lists the values of $f(x)$ for several x-values near 0.

x approaches 0 from the left. → ← *x* approaches 0 from the right.

x	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.99995	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2. → ← $f(x)$ approaches 2.

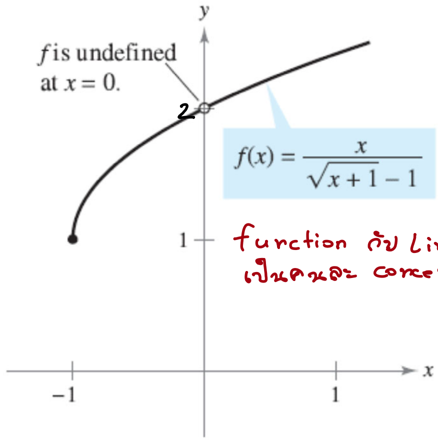
An Introduction to Limits

Graphical Approach

From the results shown in the table, you can estimate the limit to be 2. This limit is reinforced by the graph of f

- The function is undefined at $x = 0$.

The **existence or nonexistence** of $f(x)$ at $x = c$ has no bearing on the **existence** of the limit of $f(x)$ as x approaches c.



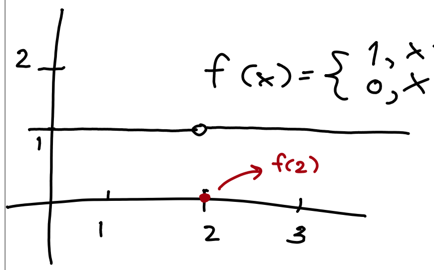
function กับ Limit
เป็นคนละ concept กัน

The limit of $f(x)$ as x approaches 0 is 2.

EXAMPLE 2 Finding a Limit

Find the limit of $f(x)$ as x approaches 2, where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$



$$f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$f(2) = 0$

$\lim_{x \to 2} f(x) = 1$

สังเกตได้ว่า limit กับ function ค่าไม่เท่ากัน

** ไม่จำเป็นต้องค่าเดียวกันเสมอ

An Introduction to Limits

นิยาม

1. $\lim_{x \to a} f(x) = b$ หมายถึง ถ้า x เข้าใกล้ a แล้ว $f(x)$ จะเข้าใกล้ค่า b
2. $\lim_{x \to a} f(x)$ จะหาค่าได้ ก็ต่อเมื่อ $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

$\lim_{x \to a^-} f(x)$ เรียกว่า Left hand Limit เข้าใกล้จากฝั่ง ซ้าย

$\lim_{x \to a^+} f(x)$ เรียกว่า Right hand Limit เข้าใกล้จากฝั่ง ขวา

Finding Limits

1. Numerical approach — Construct a table of values.
2. Graphical approach — Draw a graph by hand or using technology.
3. Analytic approach — Use algebra or calculus.

Later in this lecture …

Limits That Fail to Exist

EXAMPLE 3 Behavior That Differs from the Right and from the Left 1/3

Show that the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution Consider the graph of the function $f(x) = |x|/x$. From the definition of absolute value

แยกเป็น 2 กรณี

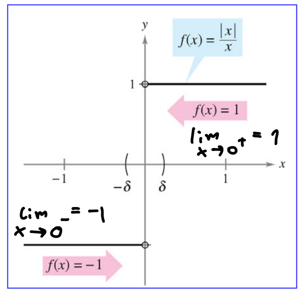
$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Definition of absolute value

you can see that

ห้าม =0, เลย >0 เฉยๆ

$$\frac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}.$$



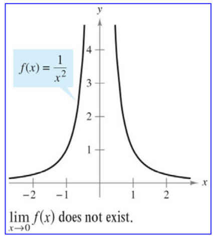
This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield $f(x) = 1$ or $f(x) = -1$.

Limits That Fail to Exist

เพิ่มไปสู่ $+\infty$ ($-\infty$ ก็ไม่ได้เหมือนกัน)

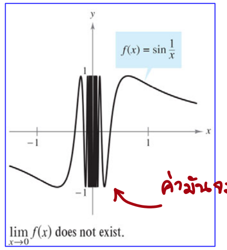
EXAMPLE 4 Unbounded Behavior 2/3

Discuss the existence of the limit $\lim_{x \to 0} \frac{1}{x^2}$.



EXAMPLE 5 Oscillating Behavior 3/3

Discuss the existence of the limit $\lim_{x \to 0} \sin \frac{1}{x}$.



ค่ามันจะกลับไปมาตลอด oscillate

Limits That Fail to Exist

COMMON TYPES OF BEHAVIOR ASSOCIATED WITH NONEXISTENCE OF A LIMIT

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

Evaluating Limits Analytically

Objectives

- Evaluate a limit using properties of limits.
- Develop and use a strategy for finding limits.
- Evaluate a limit using dividing out and rationalizing techniques.

Properties of Limits

- It may sometimes happen that the limit is precisely $f(c)$.
  - The limit is evaluated by **direct substitution**.

แทนค่าไปเลย

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at *c***.

$\lim_{x \to c} f(x) = f(c)$
หาได้ถ้าตรง c มันต่อเนื่อง

Properties of Limits

constant

1. $f(x)=b$

2. $f(x)=x$

THEOREM 1.1 SOME BASIC LIMITS

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$ **3.** $\lim_{x\to c} x^n = c^n$

EXAMPLE 1 Evaluating Basic Limits

a. $\lim_{x\to 2} 3 =$? $= 2$ **b.** $\lim_{x\to -4} x =$? $= -4$ **c.** $\lim_{x\to 2} x^2 =$? $= 4$

Properties of Limits

THEOREM 1.2 PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c} [b\,f(x)] = bL$ ตัว b ออกมาได้ (constant)

2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$ ← กระจาย limit ได้

3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

EXAMPLE 2 The Limit of a Polynomial

$$\lim_{x\to 2} (4x^2+3) = \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 \qquad \text{Property 2}$$

$$= 4\left(\lim_{x\to 2} x^2\right) + \lim_{x\to 2} 3 \qquad \text{Property 1}$$

$$= 4(2^2) + 3 \qquad \text{Example 1}$$

$$= 19 \qquad \text{Simplify.}$$

มองเป็นฟังก์ชั่น

กรณีจะดูเป็น Polynomial

แทนค่าไปเลย ไม่ต้องมาอ้างอิง Properties แล้ว

simply the value of p at $x = 2$.

$\lim_{x\to 2} p(x) = p(2) = 4(2^2) + 3 = 19$ คือ direct substitution

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators.

Function ที่เป็นเศษส่วน

THEOREM 1.3 LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \boxed{\frac{p(c)}{q(c)}}.$$

EXAMPLE 3 The Limit of a Rational Function

Find the limit: $\lim_{x\to 1} \dfrac{x^2+x+2}{x+1}$.

Solution Because the denominator is not 0 when $x = 1$, you can apply Theorem 1.3 to obtain

$$\lim_{x\to 1} \frac{x^2+x+2}{x+1} = \frac{1^2+1+2}{1+1} = \frac{4}{2} = 2.$$

BREAK

THEOREM 1.4 THE LIMIT OF A FUNCTION INVOLVING A RADICAL

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

ดูว่ารากเป็น even/odd

THEOREM 1.5 THE LIMIT OF A COMPOSITE FUNCTION

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

EXAMPLE 4 The Limit of a Composite Function

$$\lim_{x \to 0} \sqrt{x^2 + 4}$$

$$\lim_{x \to 0} (x^2 + 4) = 0^2 + 4 = 4$$

$$\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

$$\lim_{x \to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

$$\lim_{x \to 3} \sqrt[3]{2x^2 - 10}$$

$$\lim_{x \to 3} (2x^2 - 10) = 2(3^2) - 10 = 8$$

$$\lim_{x \to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$$

$$\lim_{x \to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

THEOREM 1.6 LIMITS OF TRIGONOMETRIC FUNCTIONS

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$

EXAMPLE 5 Limits of Trigonometric Functions

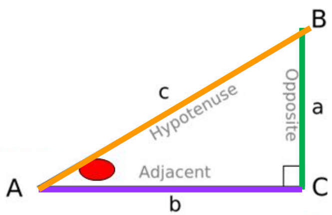
a. $\lim_{x \to 0} \tan x =$? $\tan 0 = 0$

b. $\lim_{x \to \pi} (x \cos x) =$? $\lim_{x \to \pi} x \lim_{x \to \pi} \cos x = \pi \cos \pi = -\pi$

c. $\lim_{x \to 0} \sin^2 x =$? $\lim_{x \to 0} (\sin x)^2 = 0$

Note!! Trigonometry

sin = ข้าม/ฉาก
cos = ชิด/ฉาก
tan = ข้าม/ชิด

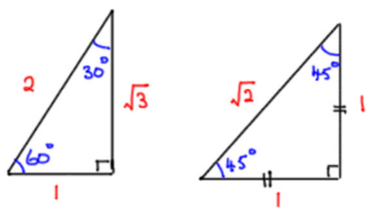


sin A = a/c cosec A = c/a

cos A = b/c sec A = c/b

tan A = a/b cot A = b/a

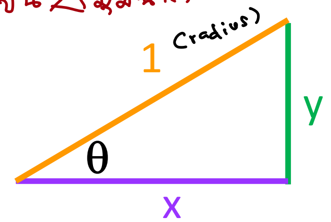
sin → cosec = $\frac{1}{\sin}$
cos → sec = $\frac{1}{\cos}$
tan → cot = $\frac{1}{\tan}$

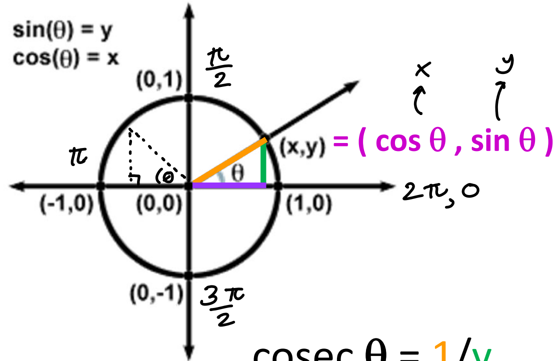


	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Note!! Trigonometry (Unit Circle)

มองเป็น △ มุมฉากรอบวงกลม





sin θ = y/1 = y

cos θ = x/1 = x

tan θ = y/x

cosec θ = 1/y

sec θ = 1/x

cot θ = x/y

A Strategy for Finding Limits

THEOREM 1.7 FUNCTIONS THAT AGREE AT ALL BUT ONE POINT

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

When the limit of $f(x)$ cannot be found, *แปลงรูปสมการ/จัดรูปใหม่

use an **equivalent** function $g(x)$ to find the limit of $f(x)$

EXAMPLE We cannot find $\lim_{x\to 1} \frac{x^3-1}{x-1}$ → แยก Factor **Why ???** ?

$$f(x) = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = x^2+x+1 = g(x), \quad x \neq 1.$$

However, we can find the limit of $g(x)$.

EXAMPLE 6 Finding the Limit of a Function

Find the limit: $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

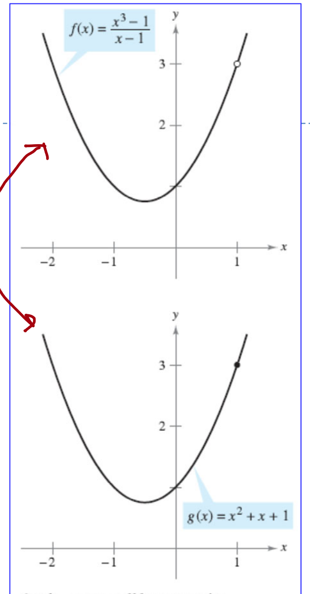
Solution you can rewrite f as

$$f(x) = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

สองสมการนี้ equivalent กัน

$$= x^2+x+1 = g(x), \quad x \neq 1.$$

So, for all x-values other than $x = 1$, the functions f and g agree



f and g agree at all but one point.

EXAMPLE 6 Finding the Limit of a Function

Find the limit: $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x-1}$.

Solution Because $\displaystyle\lim_{x\to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f and g have the same limit at $x = 1$.

$$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1} \qquad \text{Factor.}$$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} \qquad \text{Divide out like factors.}$$

$$= \lim_{x\to 1}(x^2+x+1) \qquad \text{Apply Theorem 1.7.}$$

$$= 1^2+1+1 \qquad \text{Use direct substitution.}$$

$$= 3 \qquad \text{Simplify.}$$

equivalent
g(x)

A STRATEGY FOR FINDING LIMITS

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.]
3. Apply Theorem 1.7 to conclude *analytically* that
$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$
4. Use a *graph* or *table* to reinforce your conclusion.

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$
$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$

Dividing Out and Rationalizing Techniques

dividing out common factors ตัดเศษส่วน

rationalizing the numerator of a fractional expression

EXAMPLE 7 Dividing Out Technique

Dividing Out

Find the limit: $\displaystyle\lim_{x\to -3}\frac{x^2+x-6}{x+3}$.

Solution Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

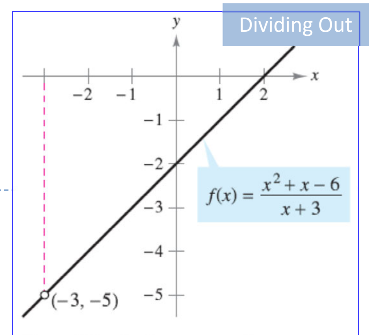
$$\lim_{x\to -3}\frac{x^2+x-6}{x+3} \quad \begin{cases} \displaystyle\lim_{x\to -3}(x^2+x-6) = 0 \\ \displaystyle\lim_{x\to -3}(x+3) = 0 \end{cases}$$

Direct substitution fails.

An expression such as 0/0 is called an **indeterminate form** you cannot (from the form alone) determine the limit.

EXAMPLE 7 Dividing Out Technique

Find the limit: $\displaystyle\lim_{x\to -3}\frac{x^2+x-6}{x+3}$.



f is undefined when $x = -3$.

Solution
you must rewrite the fraction so that the new denominator does not have 0 as its limit.

One way to do this is to *divide out like factors*

Dividing Out

$$f(x) = \frac{x^2+x-6}{x+3} = \frac{\cancel{(x+3)}(x-2)}{\cancel{x+3}} = x-2 = g(x), \quad x \neq -3.$$

Using Theorem 1.7, it follows that

$$\lim_{x\to -3}\frac{x^2+x-6}{x+3} = \lim_{x\to -3}(x-2) \qquad \text{Apply Theorem 1.7.}$$

$$= -5. \qquad \text{Use direct substitution.}$$

A second way is to *rationalize the numerator*, as shown in Example 8.

EXAMPLE 8 Rationalizing Technique

Rationalizing

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Dividing out ไม่ได้ ต้อง Rationalizing
↳ หา Conjugate

Solution By direct substitution, you obtain the indeterminate form 0/0.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \quad \begin{cases} \lim_{x\to 0}\left(\sqrt{x+1}-1\right) = 0 \\ \lim_{x\to 0} x = 0 \end{cases}$$

Direct substitution fails.

ดูก่อนว่าแยก Factor ได้ไหม ถ้าไม่ได้ → Rationalizing

$$\frac{\sqrt{x+1}-1}{x} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$$

Conjugate ต้องบวก

$$= \frac{\cancel{x}+\cancel{1}-\cancel{1}}{\cancel{x}(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

EXAMPLE 8 Rationalizing Technique

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution

you can rewrite the fraction by rationalizing the numerator

$$\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\boxed{\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)}$$

Find the conjugate

$(A+B)(A-B) = A^2 - B^2$

$$= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}$$

$$= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}$$

$$= \frac{1}{\sqrt{x+1}+1}, \quad x \neq 0$$

EXAMPLE 8 Rationalizing Technique

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution

Now, using Theorem 1.7, you can evaluate the limit as shown.

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Continuity and One-Sided Limits

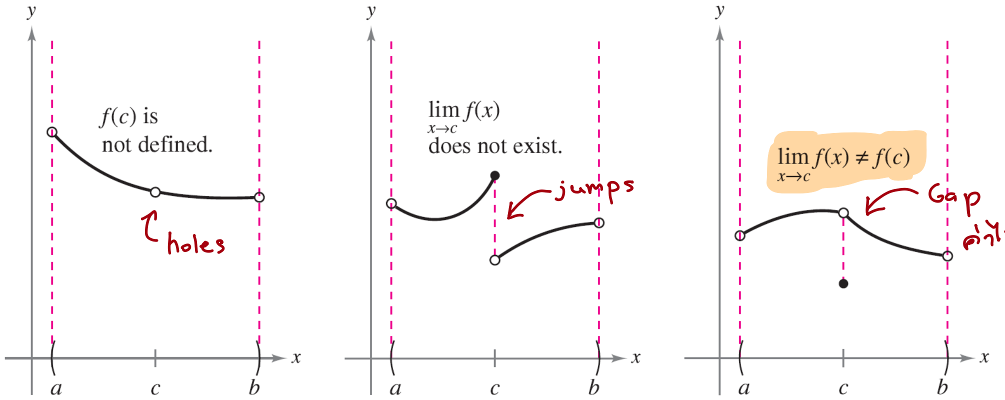
Objectives

- Determine continuity at a point and continuity on an open interval. (a, b)
- Determine one-sided limits and continuity on a closed interval. $[a, b]$

Continuity at a Point and on an Open Interval

Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c.

$\boxed{\text{no holes, jumps, or gaps.}}$ ***



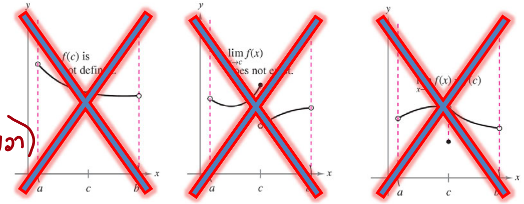
Three conditions exist for which the graph of f is not continuous at $x = c$.

Continuity at a Point and on an Open Interval

DEFINITION OF CONTINUITY

Continuity at a Point: A function f is **continuous at c** if the following three conditions are met. มองแค่จุด c

1. $f(c)$ is defined. ไม่มี holes
2. $\lim_{x\to c} f(x)$ exists. ไม่มี jumps (lim ซ้าย = ขวา)
3. $\lim_{x\to c} f(x) = f(c)$ ไม่มี gaps



Continuity on an Open Interval: A function is **continuous on an open interval (a, b)** if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**

Continuity

Determine if the following function is continuous at x=2 .

$$f(x) = \begin{cases} x^2 + 2x, & if \quad x \le 2 \\ x^3 - 6x, & if \quad x > 2 \end{cases}$$

Solution

1/3

-Function f is defined at x=2 since

$$f(2) = 2^2 + 2(2) = 8$$

2/3

-The left-hand limit $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 + 2x) = 2^2 + 2(2) = 8$

- The right-hand limit $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^3 - 6x) = 2^3 - 6(2) = -4$

limit ซ้าย ≠ ขวา

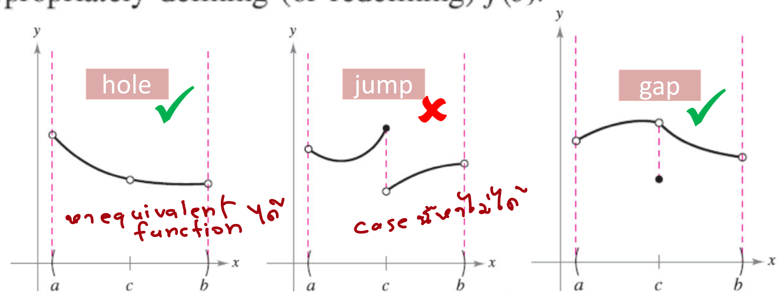
Since the left- and right-hand limits are not equal, $\lim_{x\to 2} f(x)$ does not exist, Thus, function $f(x)$ is NOT continuous at x=2 .

Continuity at a Point and on an Open Interval

f is not continuous at c → $\boxed{\textbf{discontinuity}}$ at c.

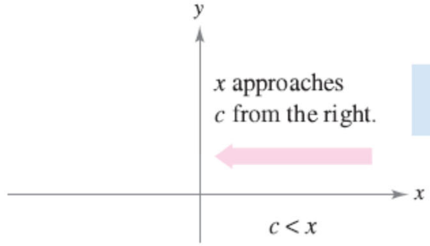
Discontinuities fall into two categories: **removable** and **nonremovable.**

A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining) $f(c)$.



(a) Removable discontinuity **(b)** Nonremovable discontinuity **(c)** Removable discontinuity

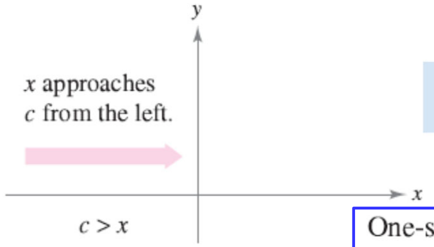
One-Sided Limits and Continuity on a Closed Interval



$$\lim_{x \to c^+} f(x) = L.$$

Limit from the right

(a) Limit from right



$$\lim_{x \to c^-} f(x) = L.$$

Limit from the left

(b) Limit from left

One-sided limits are useful in taking limits of functions involving radicals.

$$\lim_{x \to 0^+} \sqrt[n]{x} = 0. \quad n \text{ is an even integer.}$$

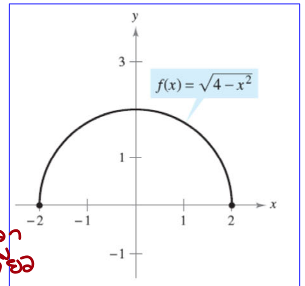
EXAMPLE 2 A One-Sided Limit

Domain = [-2, 2]

Find the limit of $f(x) = \sqrt{4 - x^2}$ as x approaches -2 from the right.

Solution

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0.$$



The limit of $f(x)$ as x approaches -2 from the right is 0.

ปลาย ซ้าย/ขวา
หา limit ได้ข้างเดียว

ex. $\lim_{x \to -2} f(x)$ ด้านขวาจะมีค่ามาอยู่ ถ้าด้านซ้ายจะไล่มาไม่ได้แล้ว

THEOREM 1.10 THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

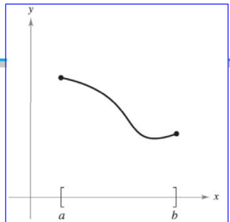
???

DEFINITION OF CONTINUITY ON A CLOSED INTERVAL

A function f is **continuous on the closed interval $[a, b]$** if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

The function f is **continuous from the right** at a and **continuous from the left** at b



Continuous function on a closed interval

EXAMPLE 4 Continuity on a Closed Interval

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

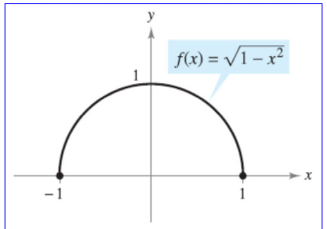
Solution The domain of f is the closed interval $[-1, 1]$. At all points in the open interval $(-1, 1)$, the continuity of f follows from Theorems 1.4 and 1.5. Moreover, because

$$\lim_{x \to -1^+} \sqrt{1 - x^2} = 0 = f(-1) \qquad \text{Continuous from the right}$$

and

$$\lim_{x \to 1^-} \sqrt{1 - x^2} = 0 = f(1) \qquad \text{Continuous from the left}$$

you can conclude that f is continuous on the closed interval $[-1, 1]$ ■



f is continuous on $[-1, 1]$.