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Introduction to Quantum Computing

Assignment 0

1) For the three subparts (a), (b), (c) of this question, let

$A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \longrightarrow$ Eigenvalues: $|A - \lambda I| = 0$

$(\lambda-5)(\lambda-2) - 12 = 0$

$\lambda^2 - 7\lambda - 2 = 0$

Roots : $\underbrace{\lambda_1 = \frac{7+\sqrt{57}}{2}, \quad \lambda_2 = \frac{7-\sqrt{57}}{2}}_{\text{2 distinct eigen-values}}$

$\Rightarrow$ diagonalizable (not neccessorily, orthogonally diagonalizable)

Corresponding Eigen vectors:

$A - \lambda_1 I = \begin{bmatrix} \frac{3-\sqrt{57}}{2} & 4 \\ 3 & \frac{-3-\sqrt{57}}{2} \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 3+\sqrt{57} \\ 6 \end{bmatrix}$

$A - \lambda_2 I = \begin{bmatrix} \frac{3+\sqrt{57}}{2} & 4 \\ 3 & \frac{-3+\sqrt{57}}{2} \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 3-\sqrt{57} \\ 6 \end{bmatrix}$

v_1 and v_2 form a basis of $\mathbb{R}^2$.

We will first do part (b).

(b) $A_2 = \begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} O & A \\ A & O \end{bmatrix}$ [using block matrices]

Let $v \in \mathbb{R}^4$ be $v = \begin{bmatrix} \alpha_1 v_1 + \alpha_2 v_2 \\ \beta_1 v_1 + \beta_2 v_2 \end{bmatrix}$ → first two elements, → last two elements

If λ is an eigen-value of A_2, $A_2 v = \lambda v$ has a non-zero solution v.

$$\begin{bmatrix} O & A \\ A & O \end{bmatrix} \begin{bmatrix} \alpha_1 v_1 + \alpha_2 v_2 \\ \beta_1 v_1 + \beta_2 v_2 \end{bmatrix} = \begin{bmatrix} \beta_1 A v_1 + \beta_2 A v_2 \\ \alpha_1 A v_1 + \alpha_2 A v_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \lambda_1 v_1 + \beta_2 \lambda_2 v_2 \\ \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda(\alpha_1 v_1 + \alpha_2 v_2) \\ \lambda(\beta_1 v_1 + \beta_2 v_2) \end{bmatrix}$$

Since v_1 and v_2 are linearly independent,

$$\left.\begin{matrix} \lambda\alpha_1 = \beta_1\lambda_1, & \lambda\beta_1 = \alpha_1\lambda_1 \\ \lambda\alpha_2 = \beta_2\lambda_2, & \lambda\beta_2 = \alpha_2\lambda_2 \end{matrix}\right\} \Rightarrow \begin{matrix} \beta_1^2\lambda_1 = \lambda\beta_1\alpha_1 = \lambda_1\alpha_1^2 \Rightarrow \alpha_1^2 = \beta_1^2 \\ \text{Similarly, } \alpha_2^2 = \beta_2^2 \end{matrix}$$

Case (i) : $\alpha_1 = \beta_1 \neq 0 \Rightarrow \lambda = \lambda_1, \quad \alpha_2 = \beta_2 = 0$

Case (ii). $\alpha_2 = \beta_2 \neq 0 \Rightarrow \lambda = \lambda_2, \quad \alpha_1 = \beta_1 = 0$

Case (iii) : $\alpha_1 = -\beta_1 \neq 0 \Rightarrow \lambda = -\lambda_1, \quad \alpha_2 = \beta_2 = 0$

Case (iv) : $\alpha_2 = -\beta_2 \neq 0 \Rightarrow \lambda = -\lambda_2, \quad \alpha_1 = \beta_1 = 0$

[$\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ is not an option, since v is non-zero.]

$\therefore$ The four eigenvalues of A_2 are $\lambda_1, \lambda_2, -\lambda_1, -\lambda_2$ with corresponding eigenvectors $\begin{bmatrix} v_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ -v_2 \end{bmatrix}$.

(a) The matrix A_1 is similar to A_2, since

$$A_1 = \begin{bmatrix} 0 & 5 & 0 & 4 \\ 5 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 5 & 4 \\ 0 & 0 & 3 & 2 \\ 5 & 4 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \underline{P A_2 P^{-1}} \qquad [\text{Here, } P^{-1} = P]$$

$\therefore$ The eigenvalues of A_1 are same as that of A_2, with the corresponding eigenvectors being those of A_2, pre-multiplied by P (i.e., 2nd and 3rd elements interchanged)

(c) $$A_3 = \begin{bmatrix} 25 & 20 & 20 & 16 \\ 15 & 10 & 12 & 8 \\ 15 & 12 & 10 & 8 \\ 9 & 6 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 5A & 4A \\ 3A & 2A \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes A$$

$$= A \otimes A$$

(Kronecker product/ tensor product of Operators)

Considering all tensor products as Kronecker products,

let $v \in \mathbb{R}^2$ be $v = u_1 \otimes v_1 + u_2 \otimes v_2$, where $u_1, u_2 \in \mathbb{R}^2$

$v_1, v_2 \rightarrow$ eigenvectors of A (in page 1)

If λ is an eigenvalue of A_3, $A_3 v = \lambda v$ has a non-zero solution.

$$A_3 v = \lambda v \Rightarrow (A \otimes A)(u_1 \otimes v_1 + u_2 \otimes v_2) = \lambda (u_1 \otimes v_1 + u_2 \otimes v_2)$$

$$\Rightarrow (Au_1) \otimes (Av_1) + (Au_2) \otimes (Av_2) = (\lambda u_1) \otimes v_1 + (\lambda u_2) \otimes v_2$$

$$\Rightarrow (Au_1) \otimes (\lambda_1 v_1) + (Au_2) \otimes (\lambda_2 v_2) = (\lambda u_1) \otimes v_1 + (\lambda u_2) \otimes v_2$$

$$\Rightarrow [\lambda_1 A u_1 - \lambda u_1] \otimes v_1 + [\lambda_2 A u_2 - \lambda u_2] \otimes v_2 = 0$$

Since v_1 and v_2 are linearly independent,

$$(\lambda_1 A - \lambda I)u_1 = 0 \quad , \quad (\lambda_2 A - \lambda I)u_2 = 0$$

Case (i) : $u_1 \neq 0 \Rightarrow \frac{\lambda}{\lambda_1} \in \{$ Set of Eigenvalues of A $\{\lambda_1, \lambda_2\}$ $\Rightarrow \lambda = \underline{\lambda_1^2}, \lambda_1\lambda_2$

$u_1 = k$ (corresponding eigenvector of A)

Case (ii) : $u_2 \neq 0 \Rightarrow \frac{\lambda}{\lambda_2} \in$ Set of eigenvalues of A $\{\lambda_1, \lambda_2\}$ $\Rightarrow \lambda = \lambda_1\lambda_2, \underline{\lambda_2^2}$

$u_2 = k$ (corresponding eigenvector of A)

$u_1 = u_2 = 0$ is not ~~po~~ valid since $v = u_1 \otimes v_1 + u_2 \otimes v_2$ is non-zero.

$\therefore$ The eigenvalues of A_3 are $\lambda_1^2, \lambda_2^2, \underbrace{\lambda_1\lambda_2, \lambda_1\lambda_2}_{\text{multiplicity} = 2}$
with corresponding eigenvectors
$v_1 \otimes v_1$, $v_2 \otimes v_2$, $v_1 \otimes v_2$ and $v_2 \otimes v_1$,
where $\lambda_1 = \frac{7+\sqrt{57}}{2}$, $\lambda_2 = \frac{7-\sqrt{57}}{2}$ and v_1 and v_2 are corresponding eigenvectors.

This method using Kronecker products can be used for the previous two subparts to get the same answers.

$$A_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

2) O is an orthogonally diagonalizable operator with eigenvalues ± 1.

$$\Rightarrow P_{+1} - P_{-1} = O \qquad [\because \sum_{\lambda} \lambda P_{\lambda} = O]$$

and $P_{+1} + P_{-1} = I \qquad [\because \sum_{\lambda} P_{\lambda} = I]$

$$\Rightarrow \quad \underline{\underline{P_{+1}}} = \frac{I+O}{2}, \quad \underline{\underline{P_{-1}}} = \frac{I-O}{2}$$

3) Given, $\|Ax\| = \|x\| \quad \forall x \in V$

$$\Rightarrow \langle Ax | Ax \rangle = \langle x | x \rangle \quad \forall x \in V$$

$$\Rightarrow \langle A^{\dagger}A x | x \rangle = \langle x | x \rangle \quad \forall x \in V$$

$$\Rightarrow \langle \underline{(A^{\dagger}A - I)} x | x \rangle = 0 \quad \forall x \in V$$

$B = (A^{\dagger}A - I)$ is Hermitian $[\because B^{\dagger} = A^{\dagger}A - I = B]$

$\rightarrow$ B is unitarily diagonalizable, by Spectral Theorem for Hermitian matrices.

For each eigenvalue λ and eigenvector v of B,

$$0 = \langle Bv | v \rangle = \langle \lambda v | v \rangle = \bar{\lambda} \|v\|^2 \Rightarrow \underline{\underline{\lambda = 0}}$$

$[\because \|v\| > 0$ for v to be an eigenvector $]$

$\therefore$ All the eigenvalues are zero, and so,

$B = P^{-1} O P = \underline{O}$ (null matrix)

$\ulcorner$

- B is Hermitian.
- $\langle Bv | v \rangle = 0 \ \forall v \in V$

$$\left.\begin{array}{l} \bullet\ B \text{ is Hermitian.} \\ \bullet\ \langle Bv|v\rangle = 0\ \forall v \in V \end{array}\right\} \Rightarrow \underline{\underline{B = O}}$$

$\lrcorner$

$\therefore \quad A^{\dagger}A = I$, Hence, A is a $\underline{\underline{\text{unitary}}}$ matrix.

Aliter (using hint):

Given, $\|Ax\| = \|x\| \quad \forall x \in V$

$$\rightarrow \langle A(x+y) | A(x+y) \rangle = \langle x+y | x+y \rangle$$

$$\Rightarrow \langle Ax|Ax \rangle + \langle Ay|Ay \rangle + \langle Ax|Ay \rangle + \langle Ay|Ax \rangle = \langle x|x \rangle + \langle y|y \rangle + \langle x|y \rangle + \langle y|x \rangle$$

$$\rightarrow \langle Ax|Ay \rangle + \overline{\langle Ax|Ay \rangle} \underline{\underline{=}} \langle x|y \rangle + \overline{\langle x|y \rangle} \quad \forall x, y \in V$$

Replacing x with ix, and cancelling i on both sides,

$$\langle Ax|Ay \rangle - \overline{\langle Ax|Ay \rangle} = \langle x|y \rangle - \overline{\langle x|y \rangle} \quad \forall x, y \in V$$

Adding both equations, $\underline{\underline{\langle Ax|Ay \rangle = \langle x|y \rangle \ \forall x, y \in V}}$

Hence, $\langle A^{\dagger}Ax|y\rangle = \langle x|y\rangle \quad \forall\, x, y \in V$

$\Rightarrow \langle (A^{\dagger}A - I)x|y\rangle = 0$

$\Rightarrow \quad (A^{\dagger}A - I)x = 0 \quad \forall\, x \in V$

$\Rightarrow \quad A^{\dagger}A = I$

$\Rightarrow \quad A$ is unitary.