MIS 381N Homework 1

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Question 1

From matrix property we know $(AB)^T = B^T A^T$, then

$$\begin{aligned} LHS \times A^T &= (A^{-1})^T A^T & (1)\\ &= (AA^{-1})^T & (2)\\ &= I^T & (3)\\ &= I & (4)\\ & & (5) \end{aligned}$$

Also,

$$\begin{aligned} RHS \times A^T = I & & (6)\\ & & (7) \end{aligned}$$

Hence, $(A^{-1})^T = (A^T)^{-1}$.

Question 2

Let dollar amount of first mortgage, second mortgage, home improvement and personal overdraf be x_1,x_2,x_3,x_4. Then

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = 250 \\ 0.45x_1 - 0.55x_2 & = 0 \\ x_2 & = 250 \times 0.25 \\ 0.14x_1 + 0.2x_2 + 0.2x_3 + 0.1x_4 & = 250 \times 0.15 \end{cases} \tag{8}$$

Hence, we have a system of equations which can be solved using matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.14 & 0.2 & 0.2 & 0.1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 250 \\ 0 \\ 62.5 \\ 37.5 \end{bmatrix} \tag{9}$$

we could use R to solve for $\mathbf{x}$,

```
b <- c(250,0,62.5,37.5)
A <- matrix(c(1,0.45,0,0.14,1,-0.55,1,0.2,1,0,0,0.2,1,0,0,0.1),4,4)
x <- solve(A)%*%b
print(x)
```

```
##          [,1]
## [1,] 76.38889
## [2,] 62.50000
## [3,] 31.94444
## [4,] 79.16667
```

Hence,

$$\mathbf{x} = \begin{bmatrix} 76.39 \\ 62.50 \\ 31.94 \\ 79.17 \end{bmatrix}$$

Question 3

Let number of units manufactured for each type of variant be x_1,x_2,x_3,x_4. Then the objective is to maximize

$$1.5x_1 + 2.5x_2 + 3x_3 + 4.5x_4$$

subject to the constraints

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + 7x_4 & \leq 100000 \\ 3x_1 + 2x_2 + 3x_3 + 4x_4 & \leq 50000 \\ 2x_1 + 3x_2 + 2x_3 + 5x_4 & \leq 60000 \\ x_1, x_2, x_3, x_4 & \geq 0 \end{cases} \tag{10}$$

We could rewrite the objective function and contraints in matrix forms $\mathbf{c}^T\mathbf{x}$ and $A\mathbf{x} \leq \mathbf{b}$ respectively, where

$$\mathbf{c} = \begin{bmatrix} 1.5 \\ 2.5 \\ 3 \\ 4.5 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.14 & 0.2 & 0.2 & 0.1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 250 \\ 0 \\ 62.5 \\ 37.5 \end{bmatrix} \tag{11}$$

we could use R to solve for $\mathbf{x}$,

```
library(lpSolve)
c <- c(1.5,2.5,3,4.5)
A <- matrix(c(2,3,2,4,2,3,3,3,2,7,4,5),3,4)
dir <- c('<=','<=','<=')
b <- c(100000,50000,60000)
```

```r
s <- lp('max',c,A,dir,b)
print(s$solution)
```

```
## [1]     0 16000  6000     0
```

Hence,

$$\mathbf{x} = \begin{bmatrix} 0 \\ 16000 \\ 6000 \\ 0 \end{bmatrix}$$

Question 4

Given b_k is the difference in points in the match k, r_i and r_j are the ratings for teams i,j and $b_k = r_i - r_j$, we have

$$\begin{cases} r_1 - r_2 = -45 \\ r_1 - r_3 = -3 \\ r_1 - r_4 = -31 \\ r_1 - r_5 = -45 \\ r_2 - r_3 = 18 \\ r_2 - r_4 = 8 \\ r_2 - r_5 = 20 \\ r_3 - r_4 = 2 \\ r_3 - r_5 = -27 \\ r_4 - r_5 = -38 \end{cases} \tag{12}$$

Then, rewrite it into a matrix form $A\mathbf{r} = \mathbf{b}$ we have,

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -45 \\ -3 \\ -31 \\ -45 \\ 18 \\ 8 \\ 20 \\ 2 \\ -27 \\ 38 \end{bmatrix} \tag{13}$$

Then we have

$$A^T A = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} -124 \\ 91 \\ -40 \\ -17 \\ 90 \end{bmatrix} \tag{14}$$

After adding the constraint that all ratings sum up to 0, we are solving for $\hat{A}\mathbf{r} = \hat{\mathbf{b}}$ where

$$\hat{A} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \hat{\mathbf{b}} = \begin{bmatrix} -124 \\ 91 \\ -40 \\ -17 \\ 90 \\ 0 \end{bmatrix} \tag{15}$$

we could use R to solve for $\mathbf{r}$,

```r
library(limSolve)
A <- matrix(c(1,1,1,1,0,0,0,0,0,0,
              -1,0,0,0,1,1,1,0,0,0,
              0,-1,0,0,-1,0,0,1,1,0,
              0,0,-1,0,0,-1,0,-1,0,1,
              0,0,0,-1,0,0,-1,0,-1,-1),
            10,5)
b <- c(-45,-3,-31,-45,18,8,20,2,-27,-38)
A_t <- t(A)
A_hat <- rbind(A_t%*%A,rep(1,5))
b_hat <- rbind(A_t%*%b,0)
r <- Solve(A_hat,b_hat)
print(r)
```

```
## [1] -24.8  18.2  -8.0  -3.4  18.0
```

Hence,

$$\mathbf{r} = \begin{bmatrix} -24.8 \\ 18.2 \\ -8 \\ -3.4 \\ 18 \end{bmatrix}$$