# Wavelets in Analysis of Climate Time Series

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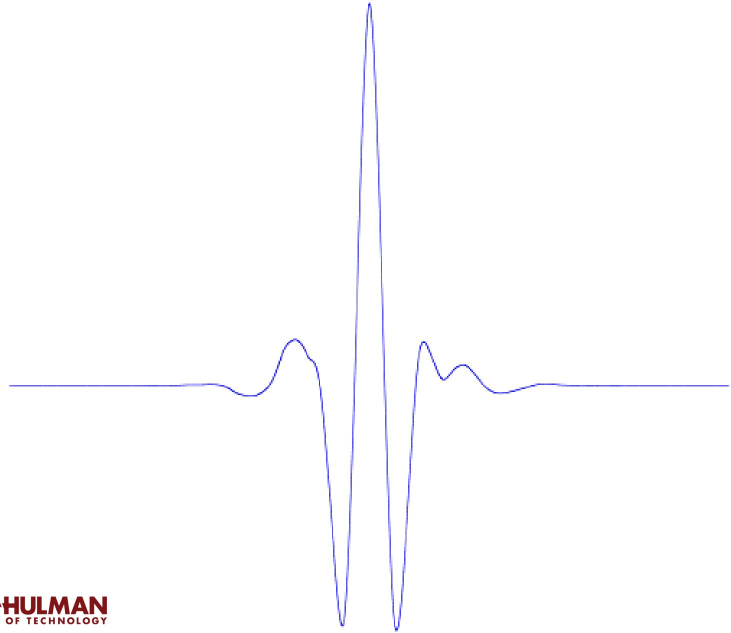
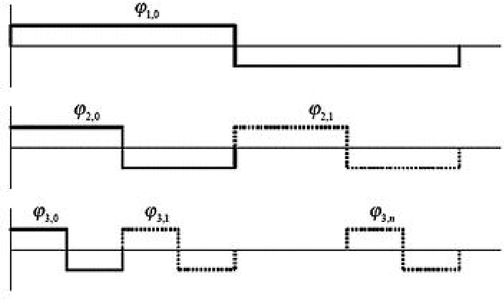
May 3, 2018

# How familiar are you with wavelets?



# Wavelet Origins – First Wavelet

1909/10 - Alfréd Haar: an example of an orthonormal system for the space of square integrable functions on [0, 1]




Images via http://wikipedia.com & http://wiki.technicalvision.ru

# Wavelet Origins – Application Based

1975 - Jean Morlet, a geophysicist, is one of the initial developers of wavelets. His motivation was to create a tool for oil prospecting.

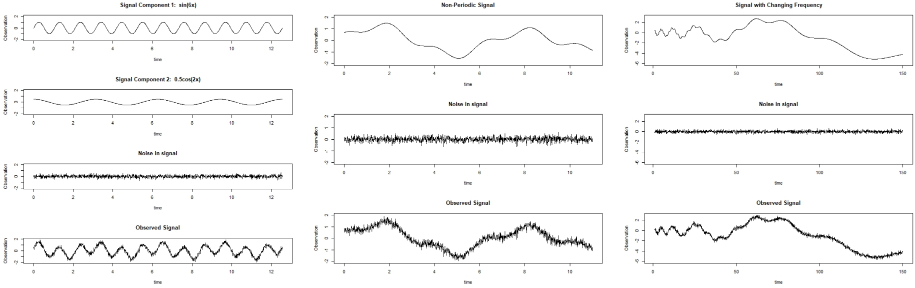
Generally, developed across multiple disciplines including:

- Robotics
- Physics
- Signal Processing

Image via https://wiki.seg.org/wiki/Jean_Morlet

# Wavelet Origins – Overcome Fourier Limitations

The multi-disciplinary development of wavelets was due to limitations of the Fourier and Windowed Fourier transform in application:

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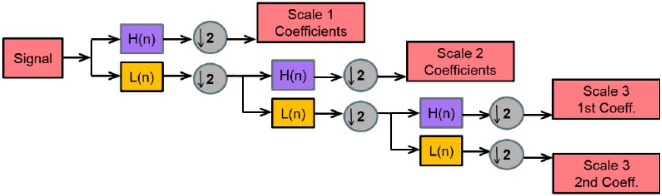
# Wavelet Origins – Fast Computing

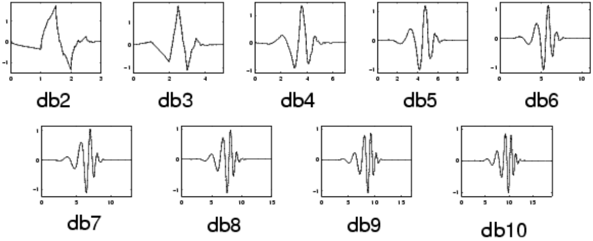
1988/89 – Stéphane Mallat proposed the “pyramid” algorithm for computing the discrete wavelet transform. It works in O(n) computations – faster than FFT with O(nlog(n)).




Images via Wikipedia & ResearchGate.net

# Wavelet Origins – Wavelet Bases

1988 - Ingrid Daubechies developed wavelet bases which are compactly supported, orthonormal, & continuous.




Images via University of Washington & Matlab

# Applications

A random smattering of wavelets in climate data analyses...

# Yearly Mean Temperature Analysis with Wavelets



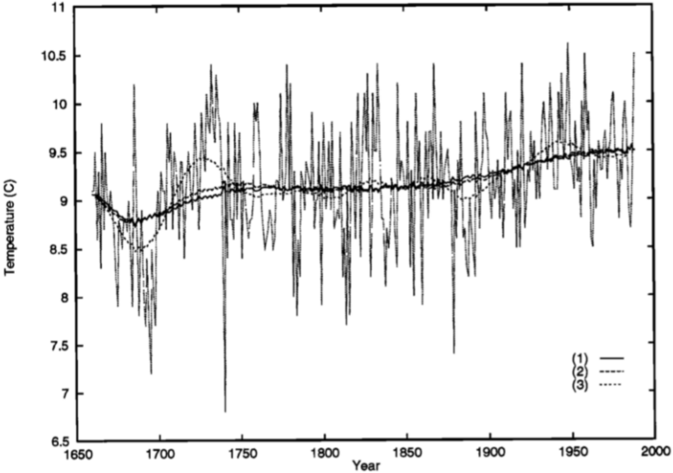
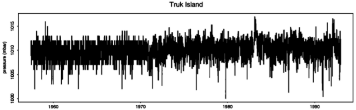



**Figure 3.** The yearly mean CET and the trend calculated as the difference between the observed data and wavelet reconstruction (based on results adopting Mexican hat wavelet) using the range of time scale between $2 \leq a \leq a_{max}$ with $a_{max}$ = 200 yr (1), 120 yr (2), and 40 yr (3).



https://doi.org/10.1029/97GL01184

# Daily Station Pressure Analysis with Wavelets




WHITCHER ET AL.: WAVELET ANALYSIS OF COVARIANCE

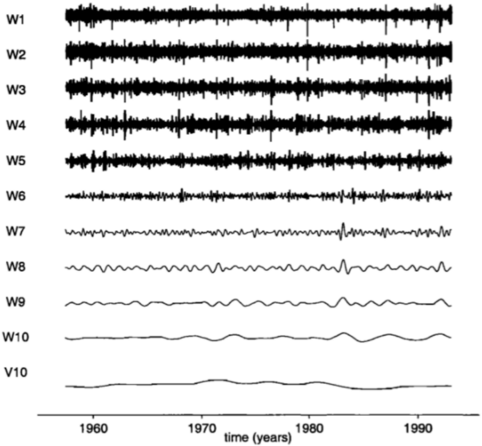



**Figure 4.** MODWT coefficients for the Truk Island SP series using the LA(8) wavelet filter. The wavelet coefficient vectors $\widetilde{\mathbf{W}}_1, \widetilde{\mathbf{W}}_2, \ldots, \widetilde{\mathbf{W}}_{10}$ are associated with variations on scales of $1, 2, \ldots, 1024$ days, and the scaling coefficient vector $\widetilde{\mathbf{V}}_{10}$ is associated with variations of 2048 days or longer.



https://doi.org/10.1029/2000JD900110

# Bluefin Tuna Trap Counts & Wavelets

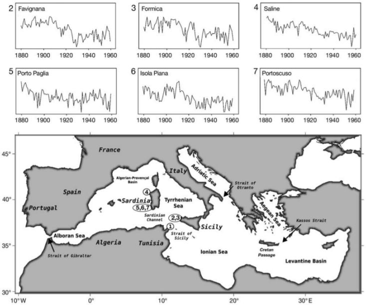



Fig. 3. Mediterranean Sea and geographic origin (1 to 7) of the Bluefin tuna trap time series. Adapted from Ravier & Fromentin (2004)

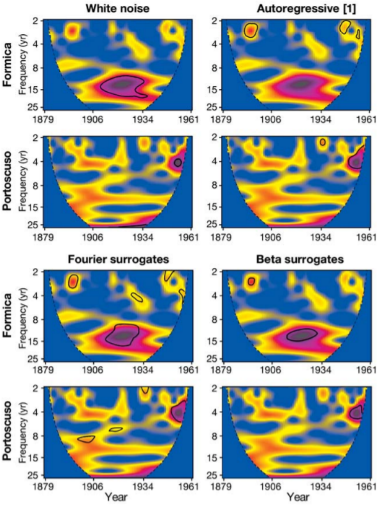



Fig. 4. Wavelet spectra of the Formica and Portoscuso time series, tested with different null hypotheses. We used a white noise process, an AR[1] process, the Fourier surrogates (Type I) and our class of surrogates (Beta surrogates). Solid black lines indicate significant areas at the 5 % level. The colour gradient, from dark blue to dark red, codes for low to high power values. Curved dashed lines: limit of the cone of influence, the area where edge effects are present

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http://www.jstor.org/stable/24872479

# Ancient Deep Sea Sediment Ice Volume

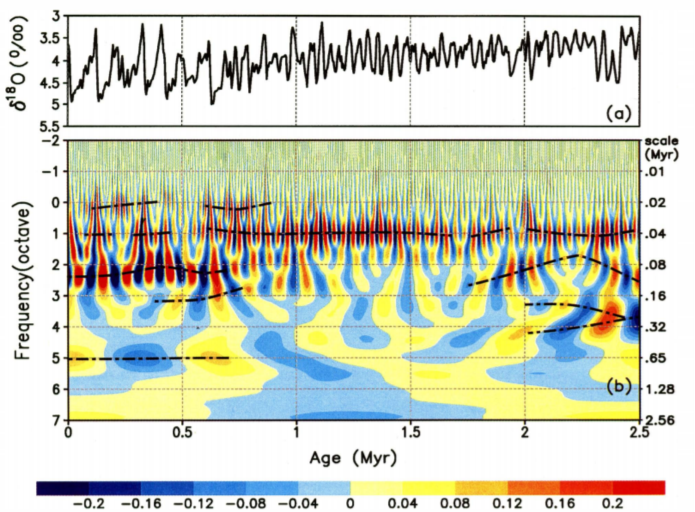


FIG. 6. (a) Time series for deep sea $\delta^{18}O$ at site 607 in the North Atlantic record for the past 2.5 Myr and (b) real part of its Morlet wavelet transform. Black dot–dashed lines in (b) indicate the frequency evolution of the dominant periodicities.

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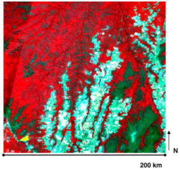
https://doi.org/10.1175/1520-0477(1995)076<2391:CSDUWT>2.0.CO;2

# Detection of Crops in Satellite Images




Download high-res image (825KB)    Download full-size image

Fig. 3. False-color infrared MODIS image (Red = 859 nm, Green = 645 nm, Blue = 555 nm) for the study area on 28 July 2005 (Upper left corner: 12° 15′ 23.61″ S, 59° 45′ 18.23″ W; Lower right corner: 13° 59′ 27.86″ S, 57°57′ 30.8″ W). Bright red areas represent dense *cerradão* woodland savanna native vegetation. Lighter reds to dark greens show the extent of *cerrado* native vegetation. Bright turquoise blues show row-crop agriculture, and very bright white areas are (bare) agricultural fields. Fazenda Santa Lordes is highlighted with a yellow polygon. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)


https://doi.org/10.1016/j.rse.2007.05.017

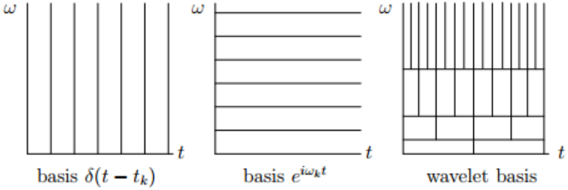
# Hopefully, your interest has been piqued

1. Basic wavelet properties & theory
2. Thresholding wavelet coefficients
    a. Definitions
    b. Demo in R
4. Visualizing wavelet models
5. Comparison of strengths & weaknesses
6. Ice core data analysis in R

# Heisenburg Uncertainty Principle

Most generally...
it is impossible to precisely measure both the position and momentum of a microscopic particle at the same time. The more precisely one is measured, the less precisely the other is known.

Similar idea: Time domain information is in the raw time series, while frequency domain information is in an orthogonal transform of the series. We can have either temporal or spectral locality regarding the information contained in the signal, but not precisely both.

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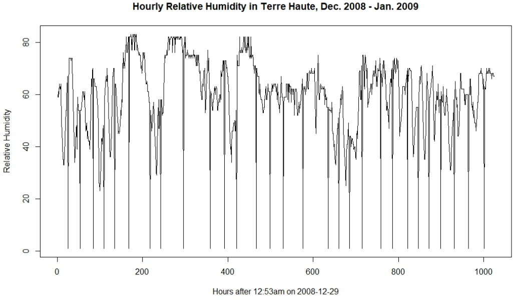
# Basic Set-up

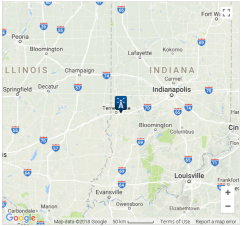
Suppose that we have a series of observations of some climate variable over time:

$$X(1), X(2), ..., X(t)$$

where $X(i)$ denotes the observation of the variable at time $i$.

Example: Hourly relative humidity in Terre Haute, IN.

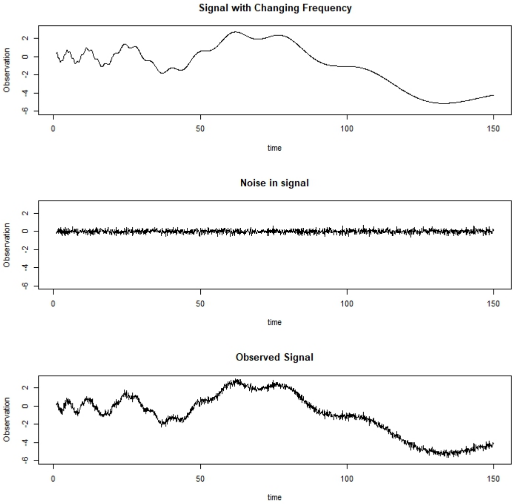



https://www.ncdc.noaa.gov/cdo-web/datasets/LCD/stations/WBAN:03868/detail

# Basic Set-up

Let's say that it is reasonable to believe that time series can be modeled as $X(t) = \mu(t) + e(t)$



# Basic Set-up

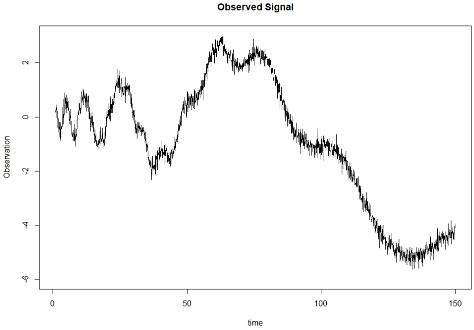
We can stack the observations over time into a vector for notation ease:

$$\boldsymbol{X} = \begin{bmatrix} X(1) \\ X(2) \\ \vdots \\ X(t) \end{bmatrix}$$

Here is the model again:

$$\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{e}$$

# Basic Set-up




How can we estimate $\mu$?

- Linear Model in time?
- Nonlinear model in time?
- Discrete fourier transform?
- **Discrete wavelet transformation**

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# Assumptions...always assumptions

Assumptions for model errors:

- $e(1), e(2), ..., e(t)$ are independent.
- $E(\boldsymbol{e}) = \mathbf{0}$
- *$Var(\boldsymbol{e}) = \sigma^2 I$
- *Errors come from a normal distribution

Other constraints:

- *$\mu : [a, b] \to \mathbb{R}$ with $a < b \in \mathbb{R}$ is a function satisfying $\int_a^b \mu^2(t)\mathrm{d}t < \infty$ and $\mu$ is continuous.
- *The time series is of length $2^J$ for $J \in \mathbb{Z}^+$.
- *The observations are equally spaced in time.

Pitfalls

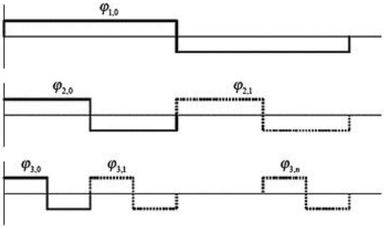
# A step back: Continuous Wavelet Function

**Wavelet**: a function, $\varphi \in \mathcal{L}_2(\mathbb{R})$, such that the translations and dyadic dilations of $\varphi$,

$$\varphi_{jk}(x) = 2^{j/2}\varphi(2^j x - k), \quad j, k \in \mathbb{Z}$$

constitute an orthonormal basis of $\mathcal{L}_2(\mathbb{R})$.

Visualize with the Haar wavelet basis:

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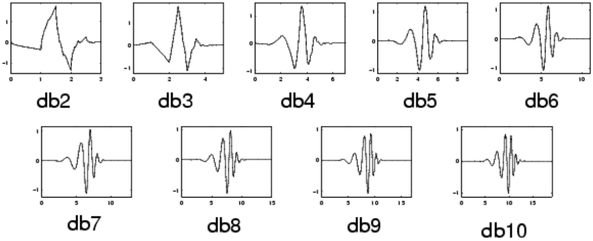
# Desirable Wavelet Properties

**Wavelets generate *local* bases.**

Fourier, Hermite, Legendre are *non-local*. What does this mean? A non-local basis has many of the basis functions contributing at any given decomposition.

Daubechies' wavelets are *compactly supported* to achieve the locality. These are also *orthogonal*.

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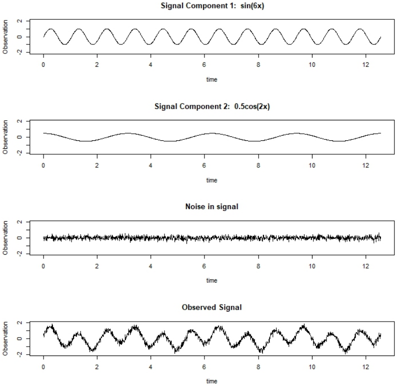
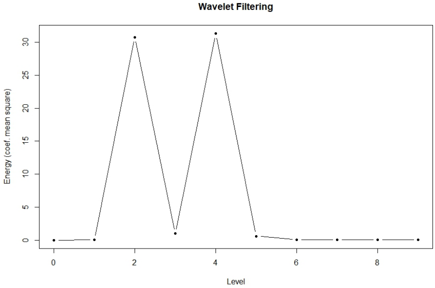
# Desirable Wavlet Properties

**Wavelets filter time series data.**

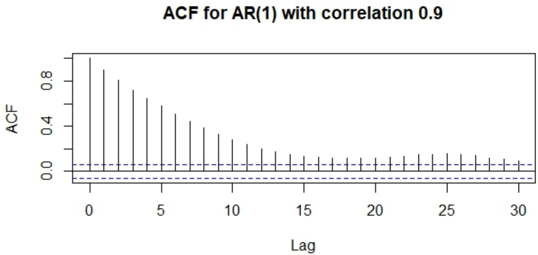
This property is similar to the Fourier transform.

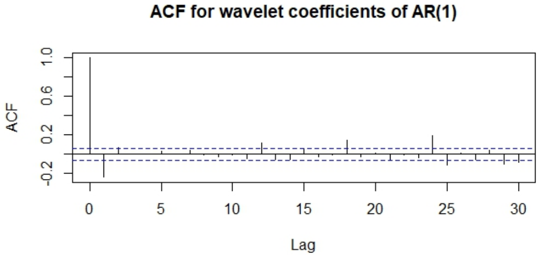


# Desirable Wavelet Properties

**Wavelet coefficients are de-correlated.**



# Discrete Wavelet Transform (DWT)

Previous slides have hinted at the discrete wavelet transform.

Formally, the discrete wavelet transform (DWT) is

$$\boldsymbol{X} = \mathcal{W}\boldsymbol{\gamma}$$

- $\mathcal{W}$ is a *fixed* wavelet basis
- $\gamma$ is the vector of wavelet coefficients.

**What do $\mathcal{W}$ and $\gamma$ look like?**

# Discrete Wavelet Matrix

$\mathcal{W}$ is an orthonormal basis

- $\mathcal{W}$ with $\mathcal{W}'\mathcal{W} = \mathcal{W}\mathcal{W}' = \mathcal{I}$

$\mathcal{W}$ has the *multiresolution* property

- Invariant under shifts of integer multiples of $2^k$.
- The l$^{th}$ and k$^{th}$ subspaces, $k > l$ are time-scaled versions of eachother with dilation factor $2^{k-l}$

Let's look at a couple $\mathcal{W}$ matrices. If you don't have R on your laptop (or are following on a smaller device) and would like to run the code: https://rdrr.io/snippets/

These slides, R code, and example data are all available in my website: https://meganheyman.github.io/

# Wavelet Coefficients

The $\mathcal{W}$ matrix helps highlight the 'level' structure present in the wavelet coefficients.

- Incorporates ideas of time and frequency together.
- Index $(j, k)$ is $k^{th}$ place in the $j^{th}$ level

For a time series of length $2^J$:

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_{0,1} \\ \gamma_{1,1} \\ \gamma_{1,2} \\ \vdots \\ \gamma_{J-1,2^{J-1}} \end{pmatrix}.$$

Larger wavelet coefficients (similar to Fourier coefficients) imply more signal at the particular time-scale.

(`wavethresh` coefficients example)

# Wavelet Model

Thus, we have motivated the idea that, given a time series $\boldsymbol{X}$, we can use the DWT to approximate the signal component.

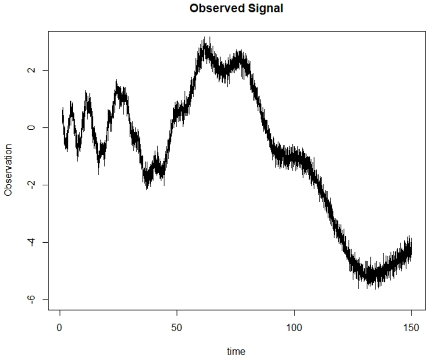
$$\begin{aligned}\boldsymbol{X} &= \boldsymbol{\mu} + \boldsymbol{e} \\ &\approx \mathcal{W}\boldsymbol{\gamma} + \boldsymbol{e}\end{aligned}$$

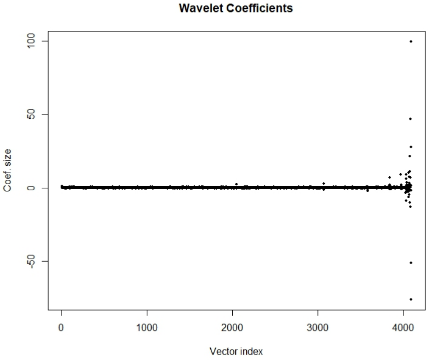
However, there is still an issue.

**The full DWT leaves no room to estimate error.**

# Thresholding Coefficients

Consider an ‘observed’ time series and its wavelet coefficients

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# Thresholding Coefficients

**Wavelet coefficients tend to be *sparse*.**

One way to take advantage of this is to 'threshold' coefficients which are small enough down to 0. The remaining coefficients estimate signal & there are also degrees of freedom to estimate error.

Wavelet thresholding in the statistical literature includes:

- Hard thresholding (Donoho and Johnstone, 1994)
- Soft thresholding (Donoho and Johnstone, 1994)
- Block thresholding (Hall et al., 1999)
- ... tons more of small variations

# Thresholding (from stackexchange.com)

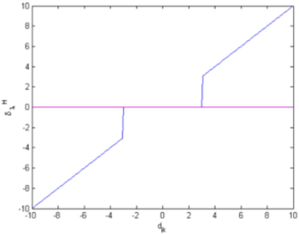
For a given threshold $\lambda$ (that can be dependent on resolution level), and value of wavelet coefficient $d$, **hard thresholding** is defined as:

$$D^H(d|\lambda) = \begin{cases} 0, & \text{for } |d| \leq \lambda \\ d, & \text{for } |d| > \lambda \end{cases}$$

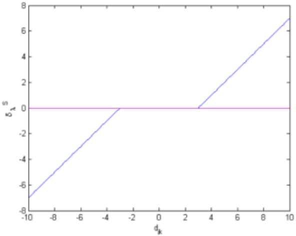
whereas **soft thresholding** is governed by following equation:

$$D^S(d|\lambda) = \begin{cases} 0, & \text{for } |d| \leq \lambda \\ d - \lambda, & \text{for } d > \lambda \\ d + \lambda, & \text{for } d < -\lambda \end{cases}$$

Figure below depicts both cases:




(a) Hard-thresholding




(b) Soft-thresholding

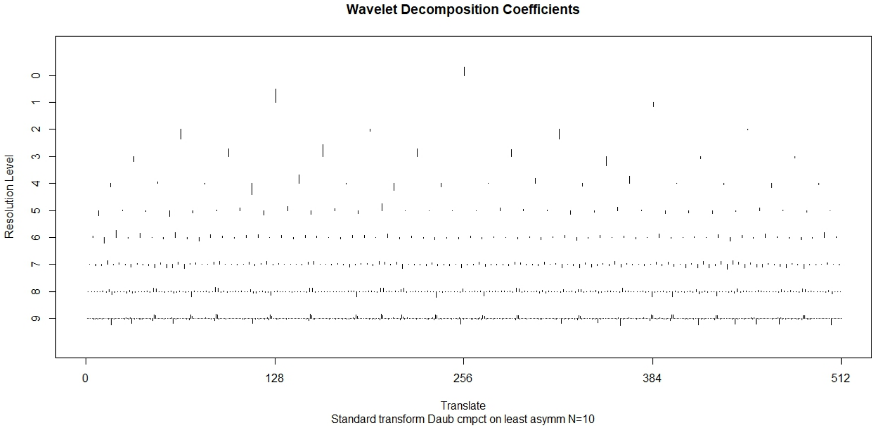
The **soft thresholding** is also called *wavelet shrinkage*, as values for both positive and negative coefficients are being "shrinked" towards zero, in contrary to **hard thresholding** which either keeps or removes values of coefficients.

# Thresholding Example

Let's give thresholding a try with a couple of examples...

# Visualizing Wavelet Model Results

Plot Individual Coefficient Values: `plot(wavObj$D)`.

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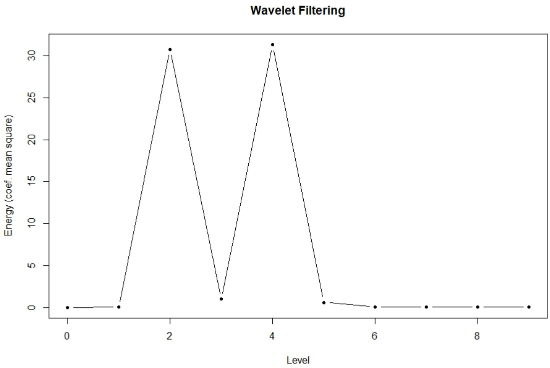
# Visualizing Wavelet Model Results

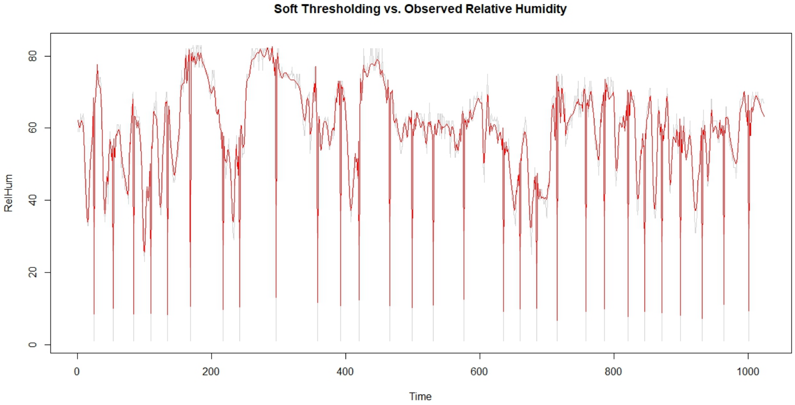
Plot 'energy' in coefficients, by level:

```
levels <- seq(0, J-1)
CoefSum <- rep(NA, J)
for(i in 1:length(levels)){
  CoefSum[i] <- mean(accessD(wavObj, level=levels[i])^2)
}
plot(levels, CoefSum)
```



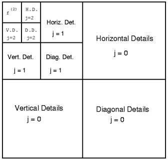
# Visualizing Wavelet Model Results

Plot thresholded inverse DWT & original observations



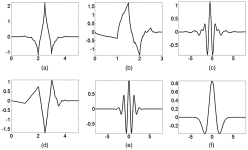
# Beyond this short course

Wavelets in 2D



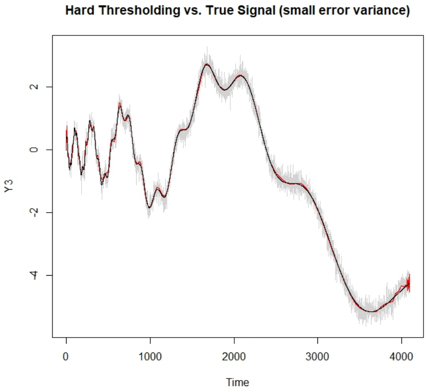

Statistical properties of wavelet coefficient estimates: $\hat{\gamma}_{j,k}$.

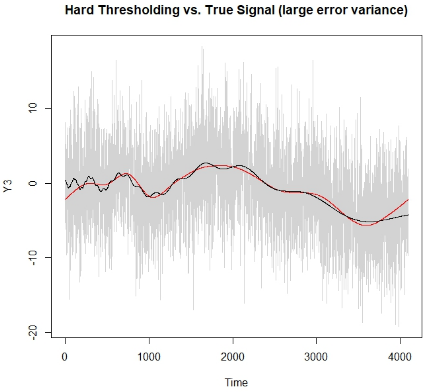
Choice of wavelet family and filter




Mallat's pyramid/cascade algorithm.

Signal-To-Noise Ratio



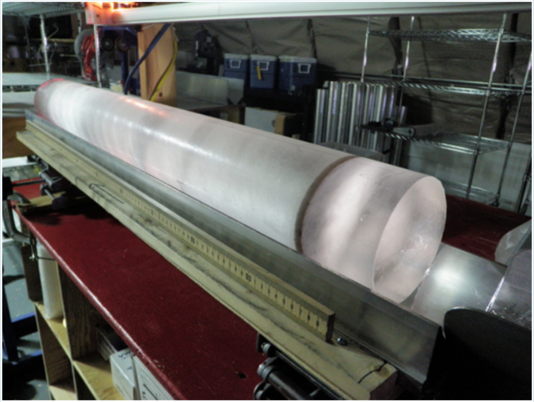



# Pitfalls of Wavelets in Statistical Modeling

- Assumptions/Constraints Assumptions
- Digging through the existing literature
- Threshold choice

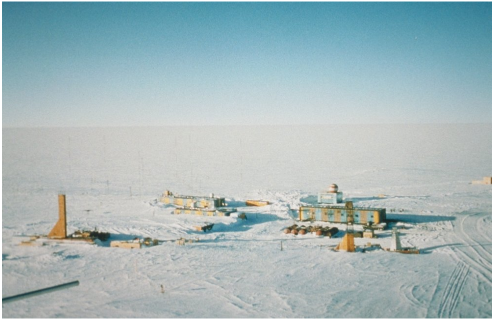
# Analyzing Ice Core Data https://icecores.org/icecores

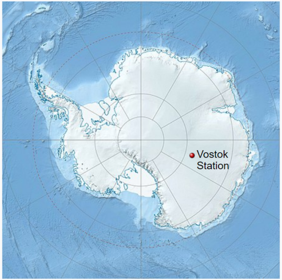
Glaciers form as layers of snow accumulate on top of each other. Each layer of snow is different in chemistry and texture, summer snow differing from winter snow. Over time, the buried snow compresses under the weight of the snow above it, forming ice. Particulates and dissolved chemicals that were captured by the falling snow become a part of the ice, as do bubbles of trapped air. Layers of ice accumulate over seasons and years, creating a record of the climate conditions at the time of formation, including snow accumulation, local temperature, the chemical composition of the atmosphere including greenhouse gas concentrations, volcanic activity, and solar activity.



The dark band in this ice core from the West Antarctic Ice Sheet Divide (WAIS Divide) is a layer of volcanic ash that settled on the ice sheet approximately 21,000 years ago.
—Credit: Heidi Roop, NSF

# Vostok Ice Core Chronology






Location of Vostok Station in Antarctica

Coordinates: 78°27′50″S 106°50′15″E

Images via Wikipedia

This particular chronology includes 1637 observations of ice age, gas age, LIDIE for depths up to 3272 m.

# What is LIDIE?

LIDIE = lock-in depth in ice equivalent

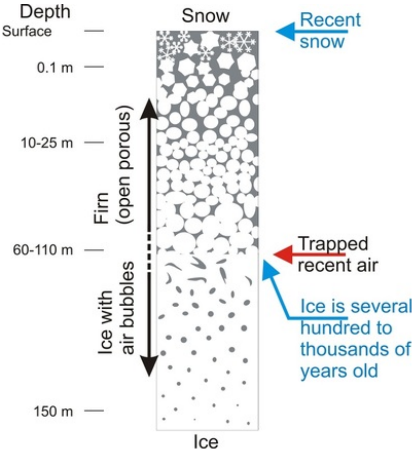
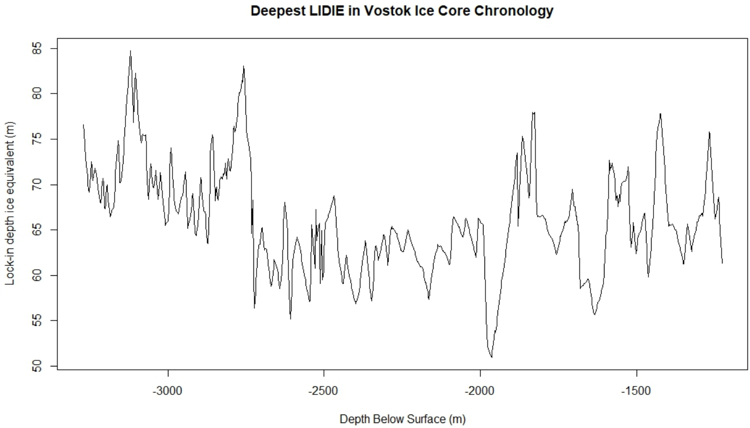



Image via https://mindynicewonger.weebly.com/blog/how-ice-cores-are-dated

# Deep Ice Core LIDIE




The Vostok Ice Core Chronology provided on my website was obtained on 2017-12-30 from `https://www.ncdc.noaa.gov/paleo-search/study/15076`

# Pop Quiz

**Learning Objectives**

Everyone who attended should be able to explain

- Why to choose wavelets for climate data analysis?
- What are desirable properties of wavelets?
- What are weaknesses of wavelets in practice?
- What are basic mechanics behind a discrete wavelet transform?

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# Research Areas with Wavelet Applications

Finding and quantifying relationships between multiple climate time series.

Extensions of wavelet methodology into 2D (and beyond) for applications in climate time series.

Properties of wavelet coefficients & estimated model with weaker assumptions.

Model selection (choice of threshold, wavelet family & filter)

# References for Getting Started with Wavelets

These are a couple of the references that I found helpful when starting to learn about wavelets.


Ogden, R.T. “Essential Wavelets for Statistical Applications and Data Analysis.” Birkhauser. 1997.

Vidakovic, B. “Statistical Modeling by Wavelets.” Wiley Series in Probability and Statistics. 1999.

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# Thank you!

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