See also 20 & 27 June 2018 at
http://phyloseminar.org/recorded.html

Bayesian Phylogenetics

Workshop on Molecular Evolution
Woods Hole, Massachusetts

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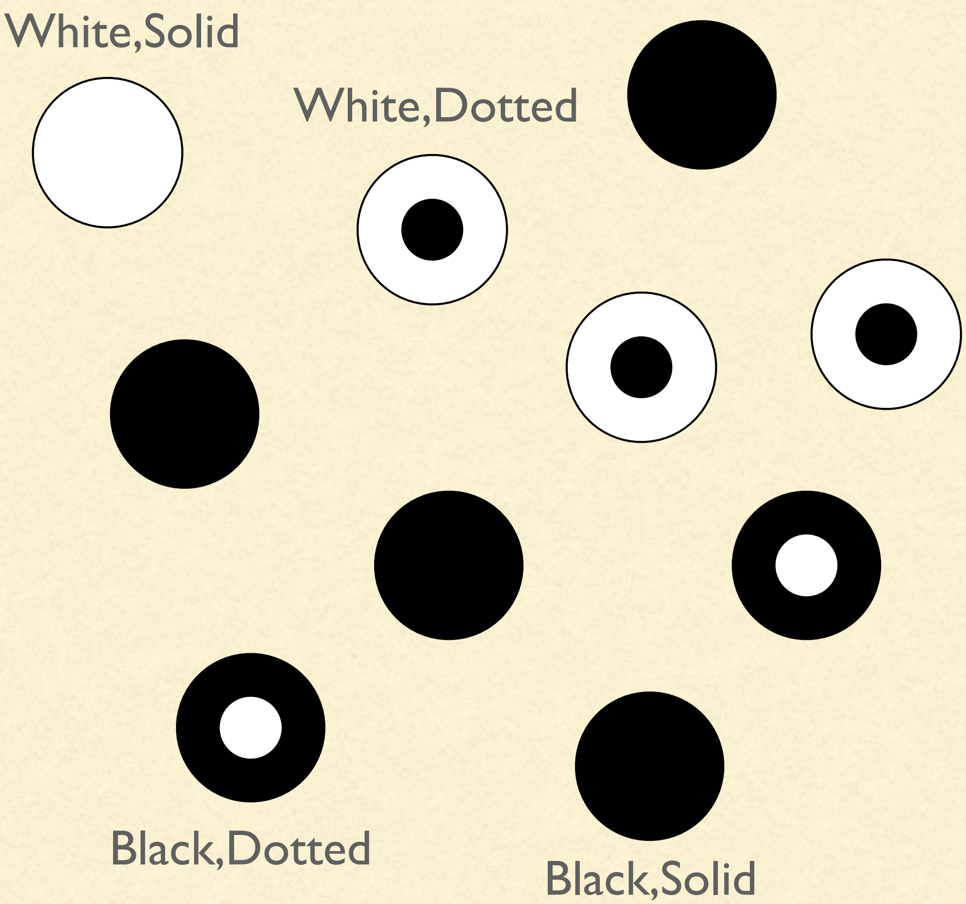
Bayesian inference

Joint probabilities



10 marbles in a bag
Sampling with replacement

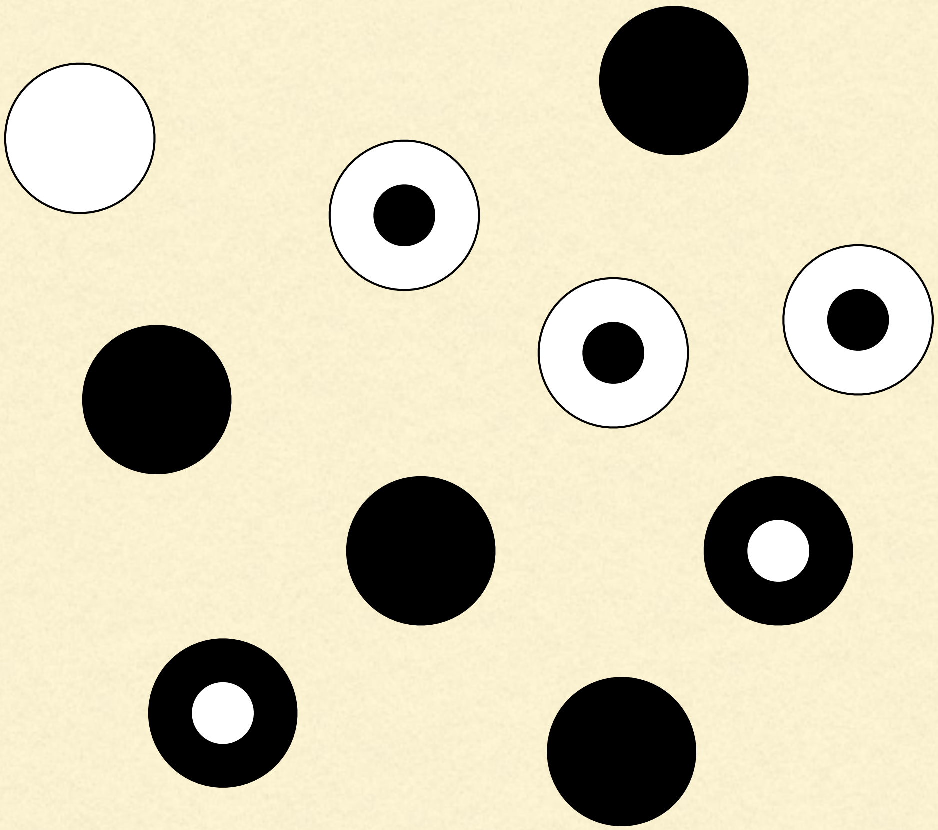
Pr(B,S) = 0.4

Pr(W,S) = 0.1

Pr(B,D) = 0.2

Pr(W,D) = 0.3

Conditional probabilities



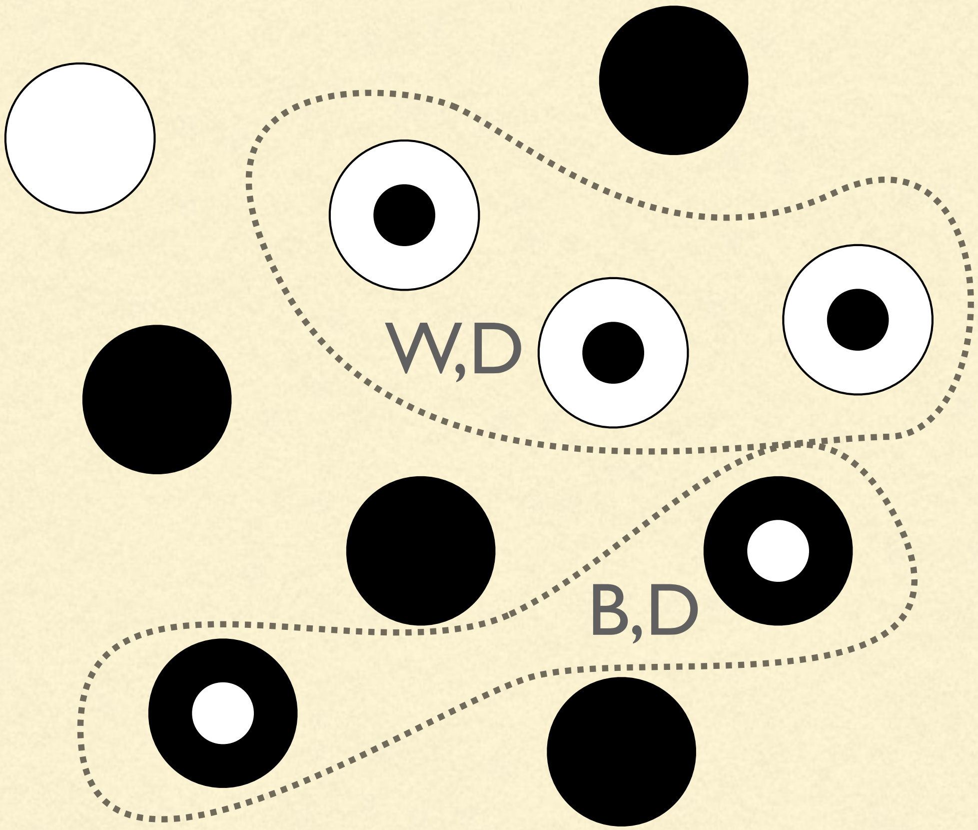
What's the probability that a marble is black given that it is dotted?

5 marbles satisfy the condition (D)

$$\Pr(B|D) = \frac{2}{5}$$

2 remaining marbles are black (B)

Marginal probabilities



Marginalizing over color yields the total probability that a marble is dotted (D)

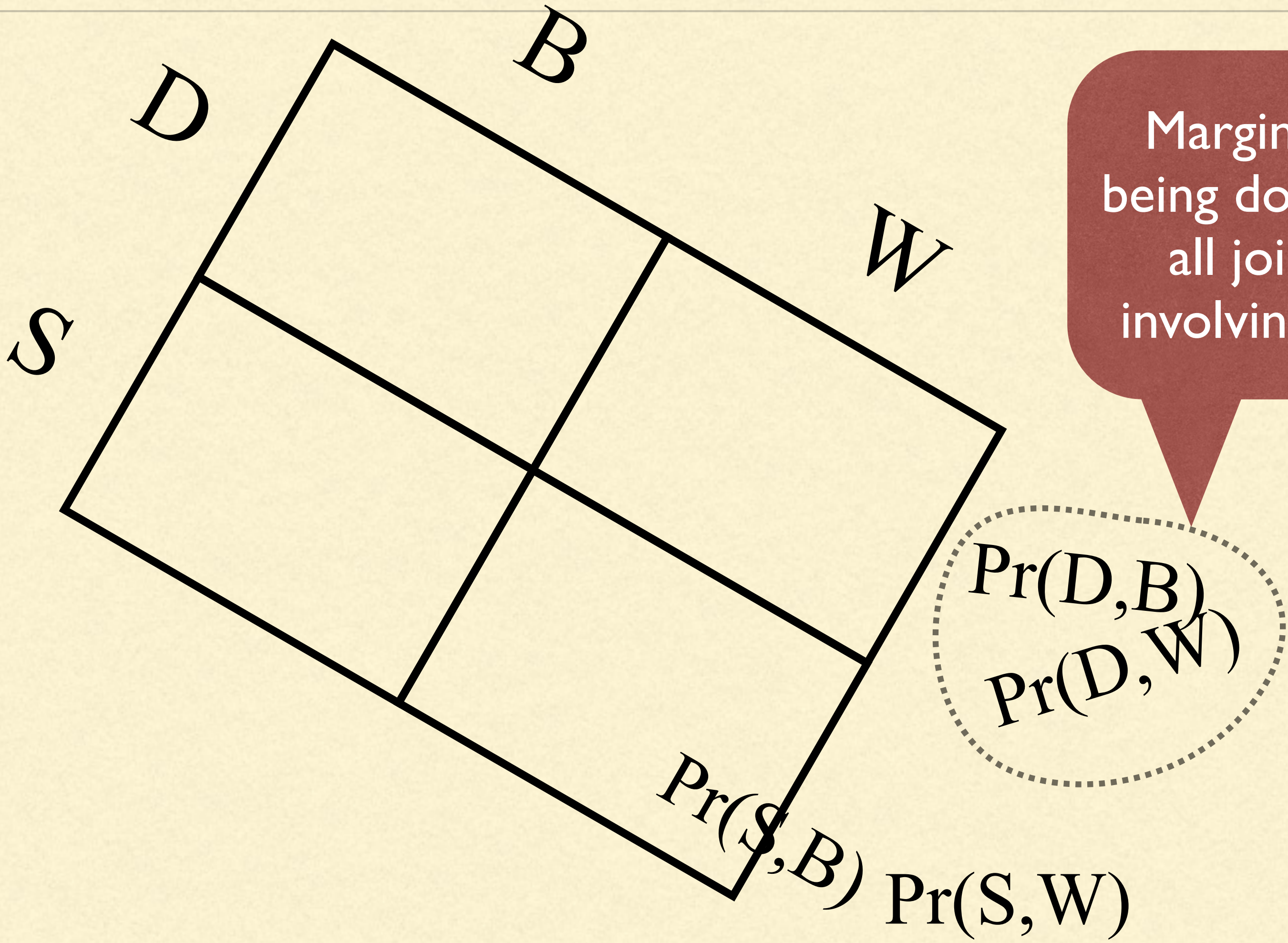
$$\Pr(\mathbf{D}) = \Pr(B,\mathbf{D}) + \Pr(W,\mathbf{D})$$
$$= 0.2 + 0.3$$
$$= 0.5$$

Marginalization involves summing all joint probabilities containing D

Marginalization

	B	W
D	Pr(D,B)	Pr(D,W)
S	Pr(S,B)	Pr(S,W)

Marginalizing over colors

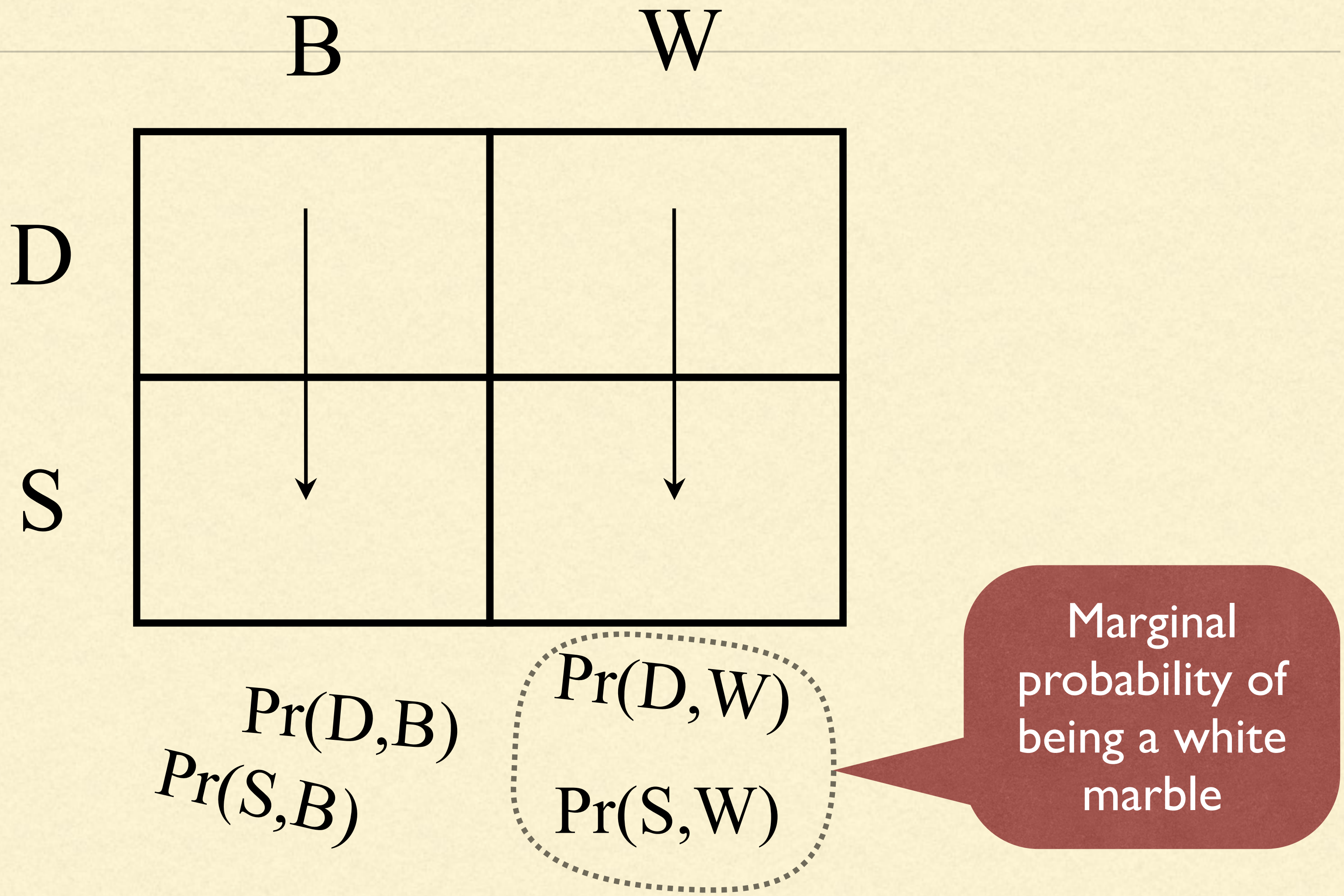


Marginal probability of being dotted is the sum of all joint probabilities involving dotted marbles

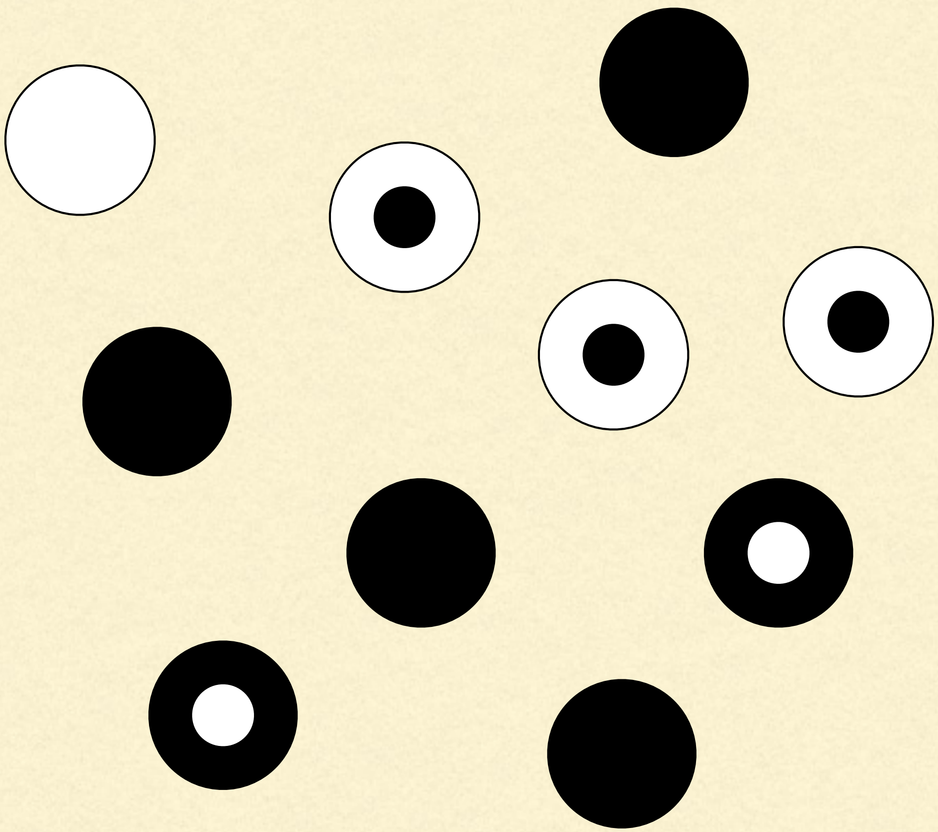
Joint probabilities

	B	W
D	Pr(D,B)	Pr(D,W)
S	Pr(S,B)	Pr(S,W)

Marginalizing over "dottedness"



Bayes' rule



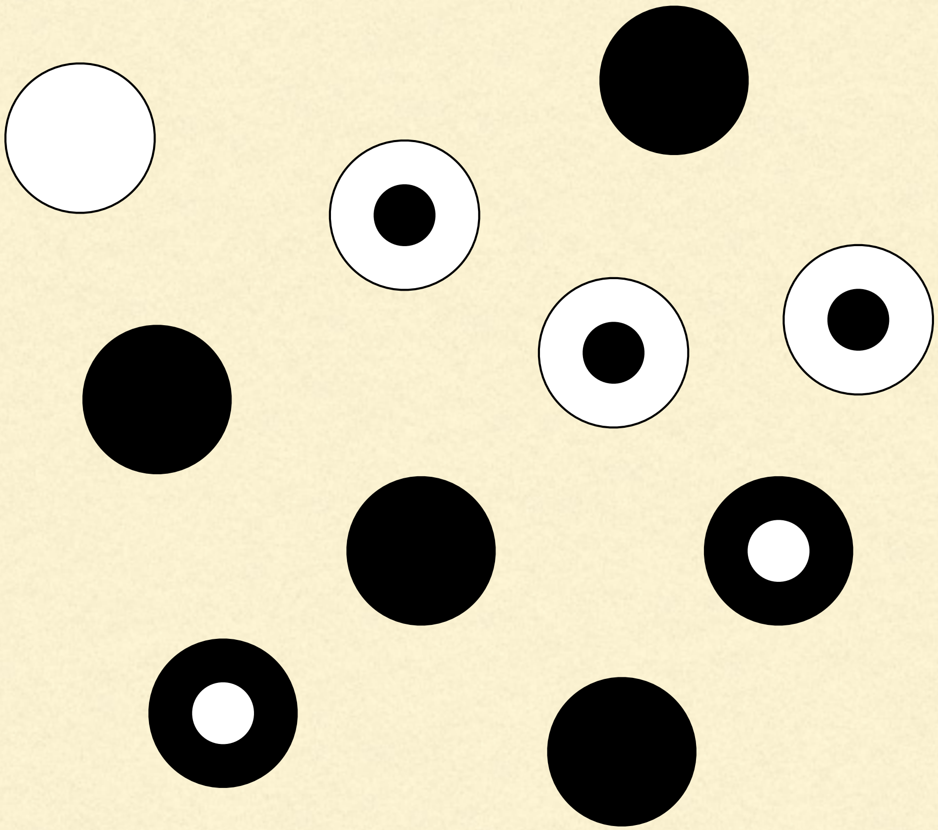
The joint probability Pr(B,D) can be written as the product of a *conditional probability* and the *probability of that condition*

$$\Pr(B,D) \rightarrow \Pr(B|D)\,\Pr(D)$$

$$\Pr(B,D) \rightarrow \Pr(D|B)\,\Pr(B)$$

Either B or D can be the condition

Bayes' rule



Equate the two ways of writing Pr(B,D)

$$\Pr(B|D)\,\Pr(D) = \Pr(D|B)\,\Pr(B)$$

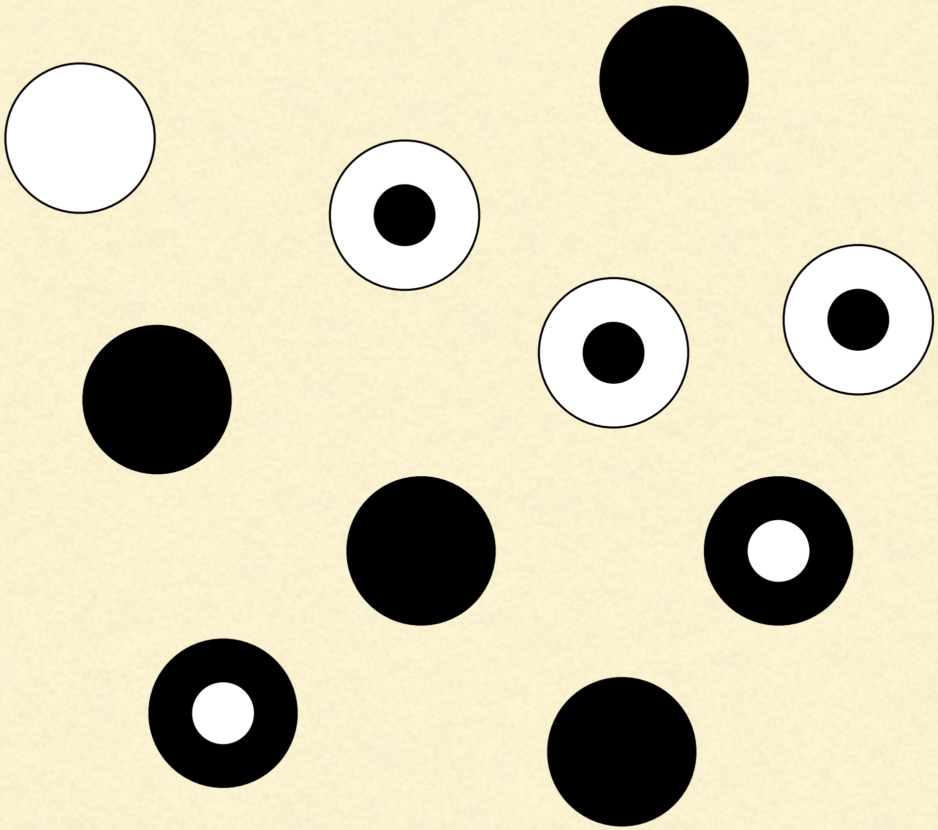
Divide both sides by Pr(D)

$$\frac{\Pr(B|D)\,\cancel{\Pr(D)}}{\cancel{\Pr(D)}} = \frac{\Pr(D|B)\,\Pr(B)}{\Pr(D)}$$

Bayes' rule

$$\Pr(B|D) = \frac{\Pr(D|B)\,\Pr(B)}{\Pr(D)}$$

Bayes' rule



$$\frac{2}{5} = \frac{\frac{1}{\cancel{3}} \times \frac{\cancel{3}}{5}}{\frac{1}{2}}$$

$$\frac{2}{5} = \frac{2}{5}$$

Bayes' rule

$$\Pr(B|D) = \frac{\Pr(D|B)\,\Pr(B)}{\Pr(D)}$$

Bayes' rule (variations)

$$\Pr(B|D) = \frac{\Pr(D|B)\Pr(B)}{\Pr(D)}$$

$$= \frac{\Pr(D|B)\Pr(B)}{\Pr(B,D) + \Pr(W,D)}$$

Pr(*D*) is the **marginal probability** of being dotted
To compute it, we **marginalize over colors**

Bayes' rule (variations)

$$\Pr(B|D) = \frac{\Pr(D|B)\Pr(B)}{\Pr(B,D) + \Pr(W,D)}$$

$$= \frac{\Pr(D|B)\Pr(B)}{\Pr(D|B)\Pr(B) + \Pr(D|W)\Pr(W)}$$

$$= \frac{\Pr(D|B)\Pr(B)}{\sum_{\theta \in \{B,W\}} \Pr(D|\theta)\Pr(\theta)}$$

Bayes' rule in statistics

Likelihood of hypothesis θ

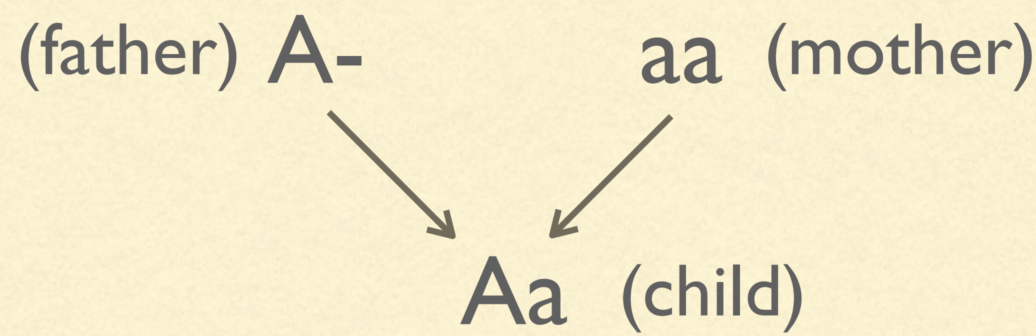
Prior probability of hypothesis θ

$$\Pr(\theta|D) = \frac{\Pr(D|\theta)\,\Pr(\theta)}{\sum_\theta \Pr(D|\theta)\,\Pr(\theta)}$$

Posterior probability of hypothesis θ

Marginal probability of the data (marginalizing over hypotheses)

Paternity example



$\Pr(\theta \mid D) = \dfrac{\Pr(D \mid \theta)\Pr(\theta)}{\sum_\theta \Pr(D \mid \theta)\Pr(\theta)}$	θ_1	θ_2	Row sum
Genotypes	AA	Aa	---
Prior	1/2	1/2	1
Likelihood	1	1/2	---
Prior X Likelihood	1/2	1/4	3/4
Posterior	2/3	1/3	1

Bayes' rule: continuous case

Likelihood

Prior probability ***density***

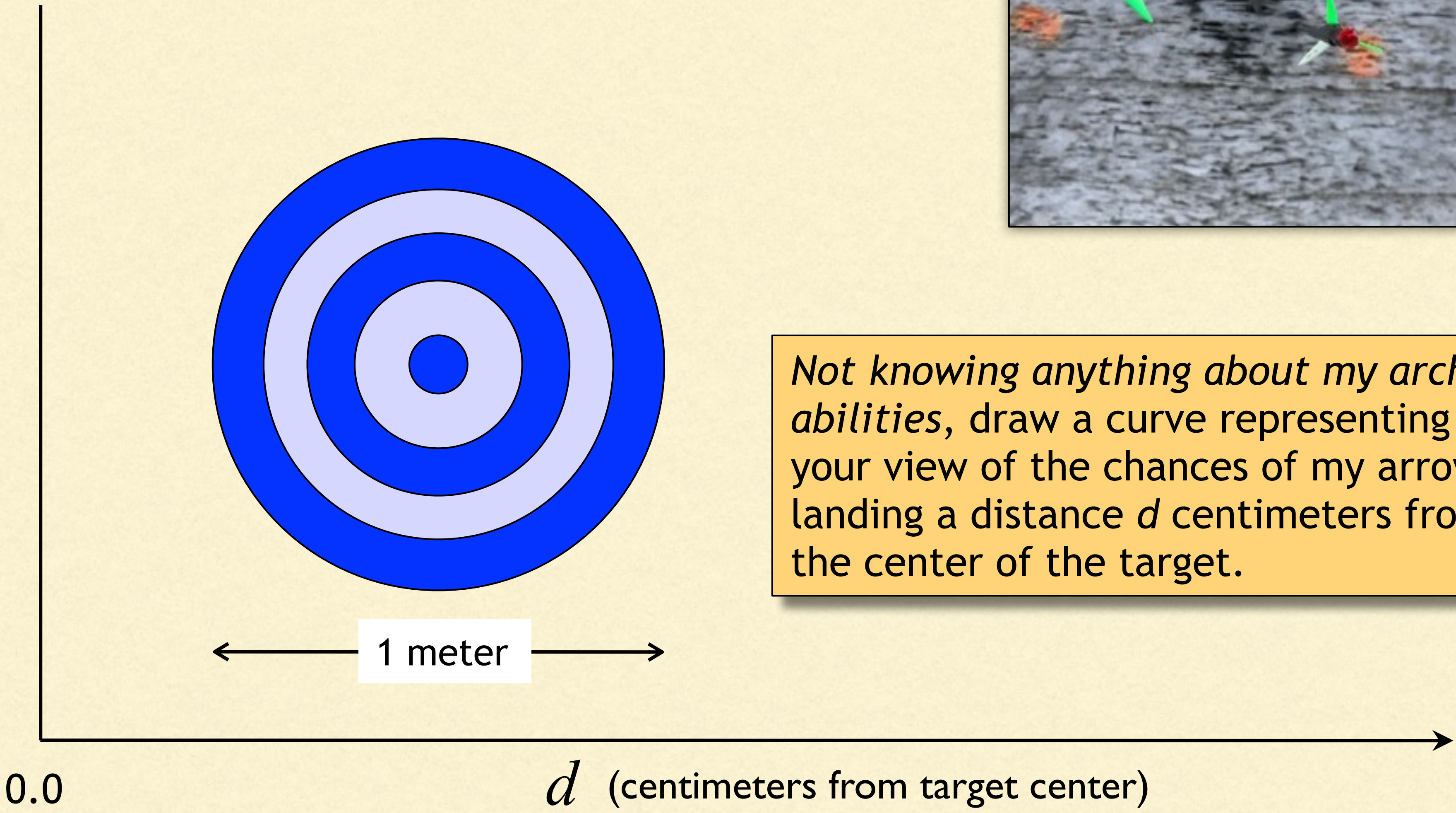
$$\boxed{p(\theta \mid D)} = \frac{\boxed{p(D \mid \theta)}\boxed{p(\theta)}}{\boxed{\int p(D \mid \theta) p(\theta) d\theta}}$$

Posterior probability ***density***

Marginal probability of the data

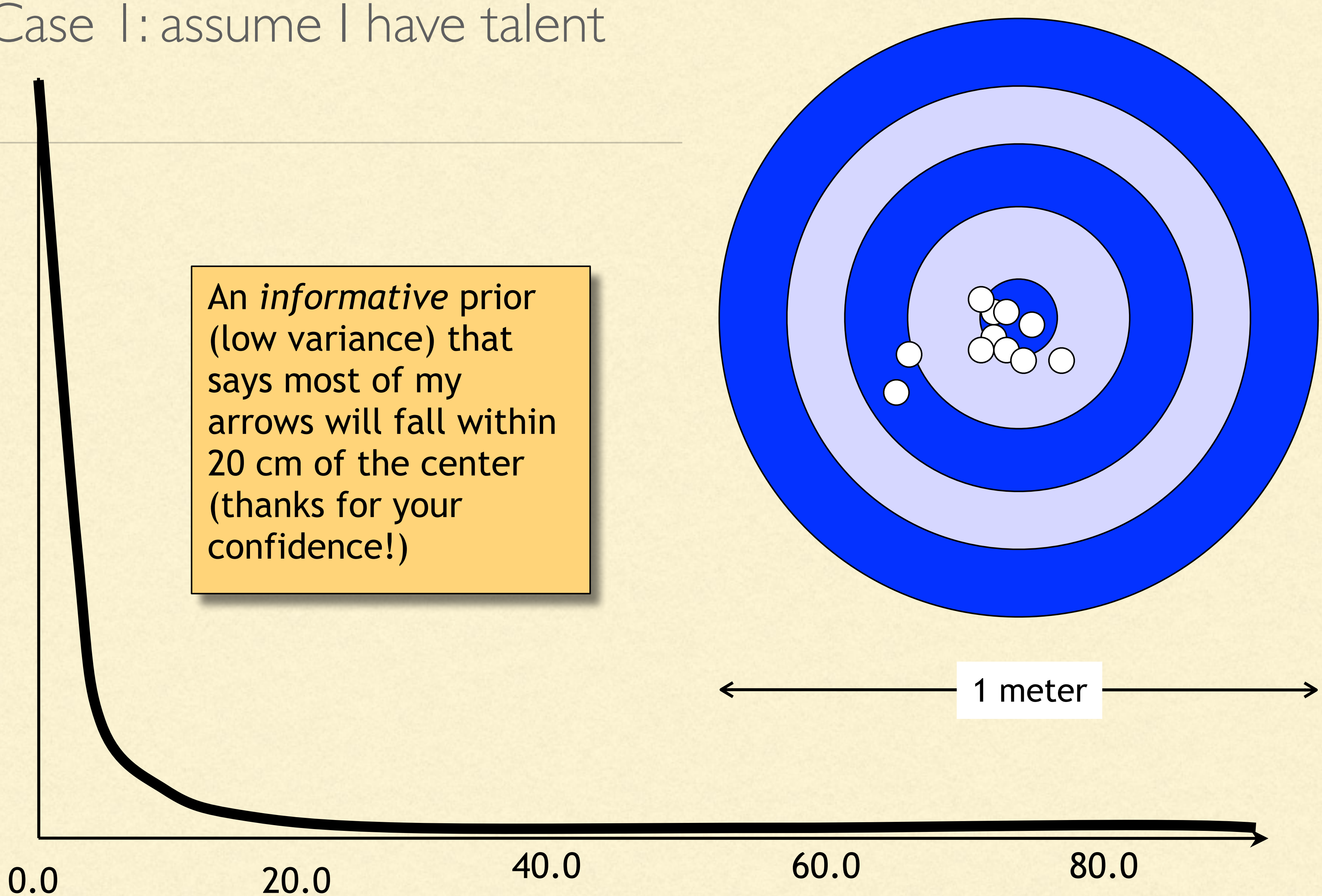
If you had to guess...

Photo by Tracy Heath



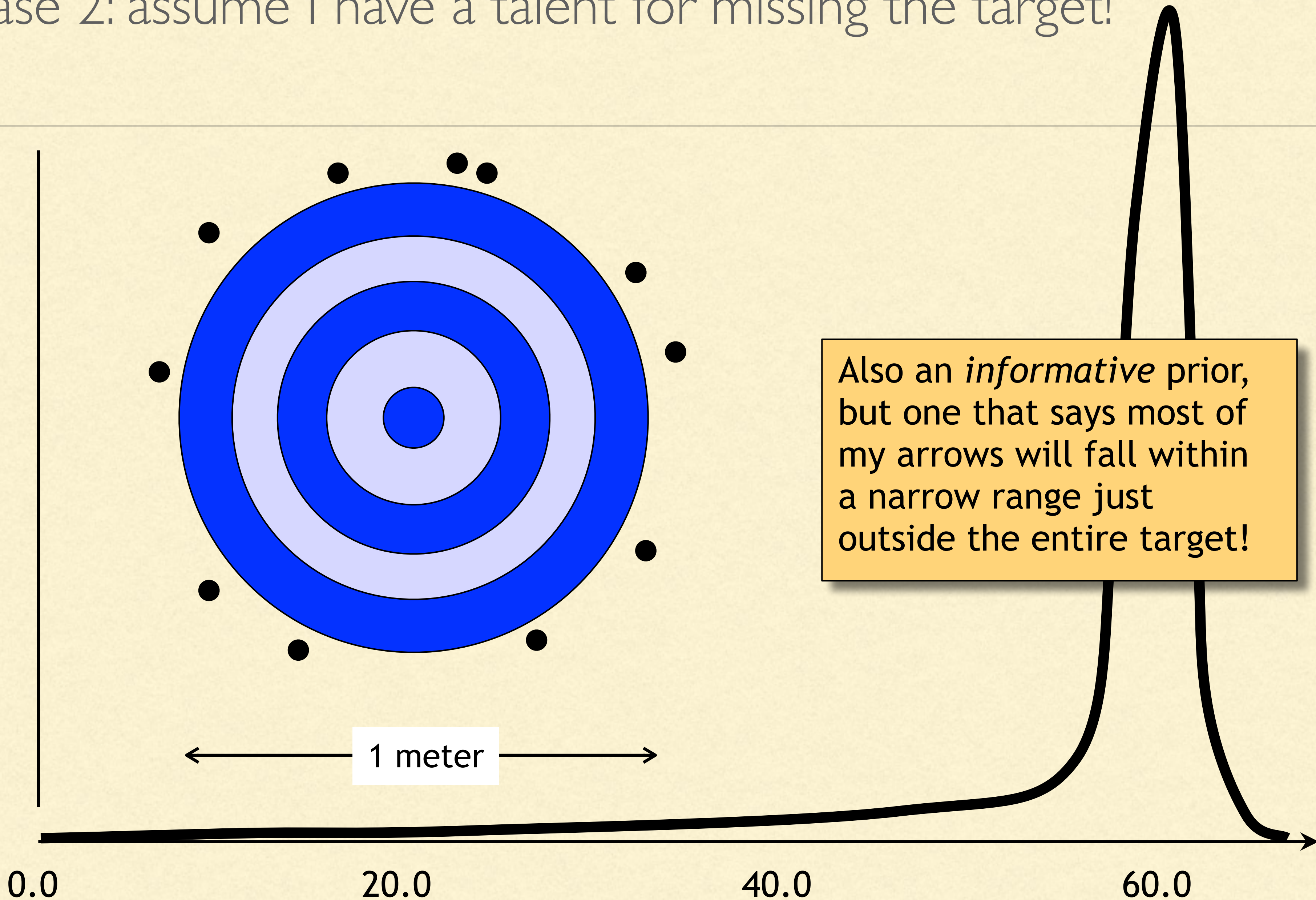
Not knowing anything about my archery abilities, draw a curve representing your view of the chances of my arrow landing a distance *d* centimeters from the center of the target.

Case 1: assume I have talent



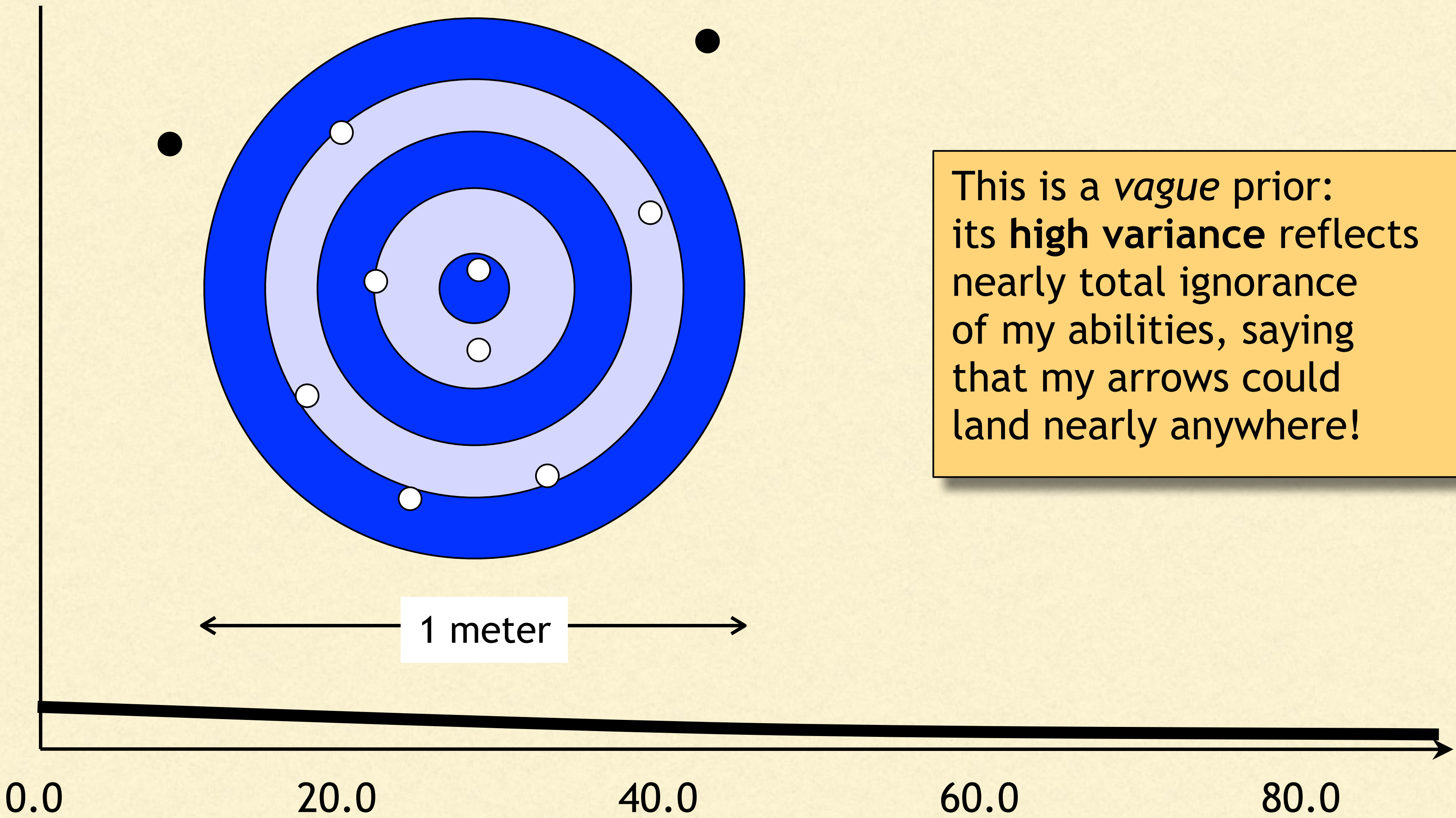
An *informative* prior (low variance) that says most of my arrows will fall within 20 cm of the center (thanks for your confidence!)

Case 2: assume I have a talent for missing the target!



Also an *informative* prior, but one that says most of my arrows will fall within a narrow range just outside the entire target!

Case 3: assume I have no talent



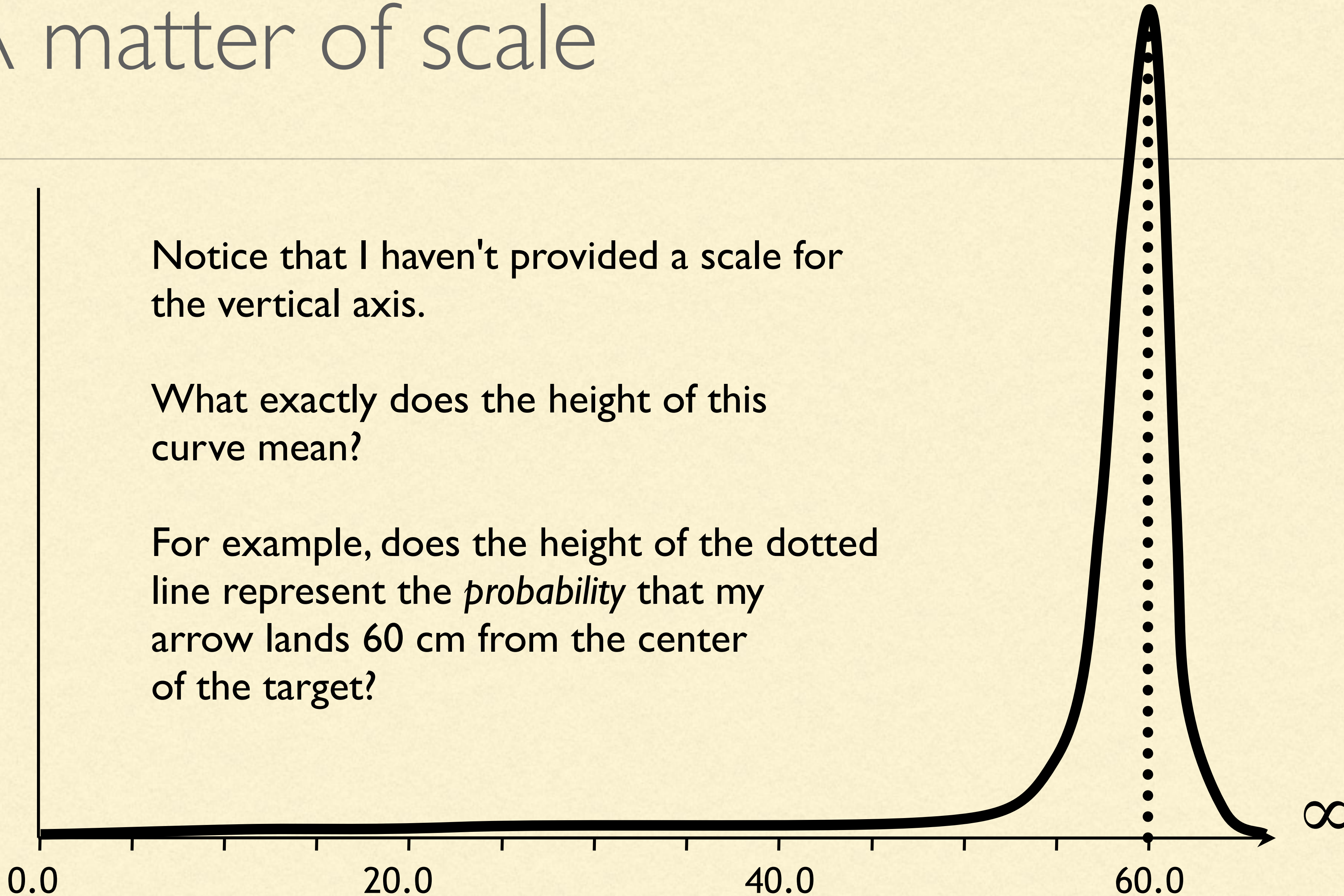
This is a *vague* prior: its **high variance** reflects nearly total ignorance of my abilities, saying that my arrows could land nearly anywhere!

A matter of scale

Notice that I haven't provided a scale for the vertical axis.

What exactly does the height of this curve mean?

For example, does the height of the dotted line represent the *probability* that my arrow lands 60 cm from the center of the target?

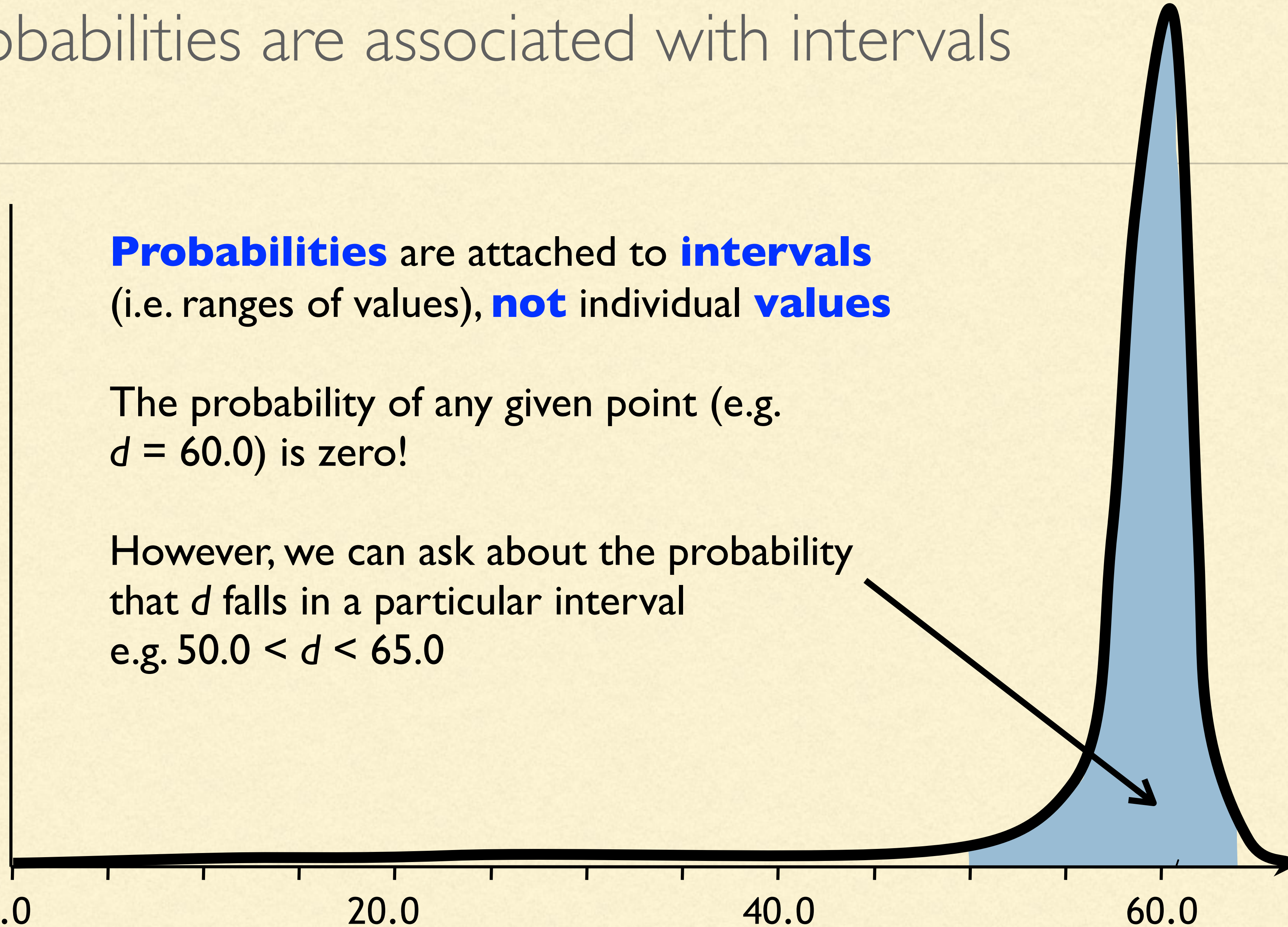


Probabilities are associated with intervals

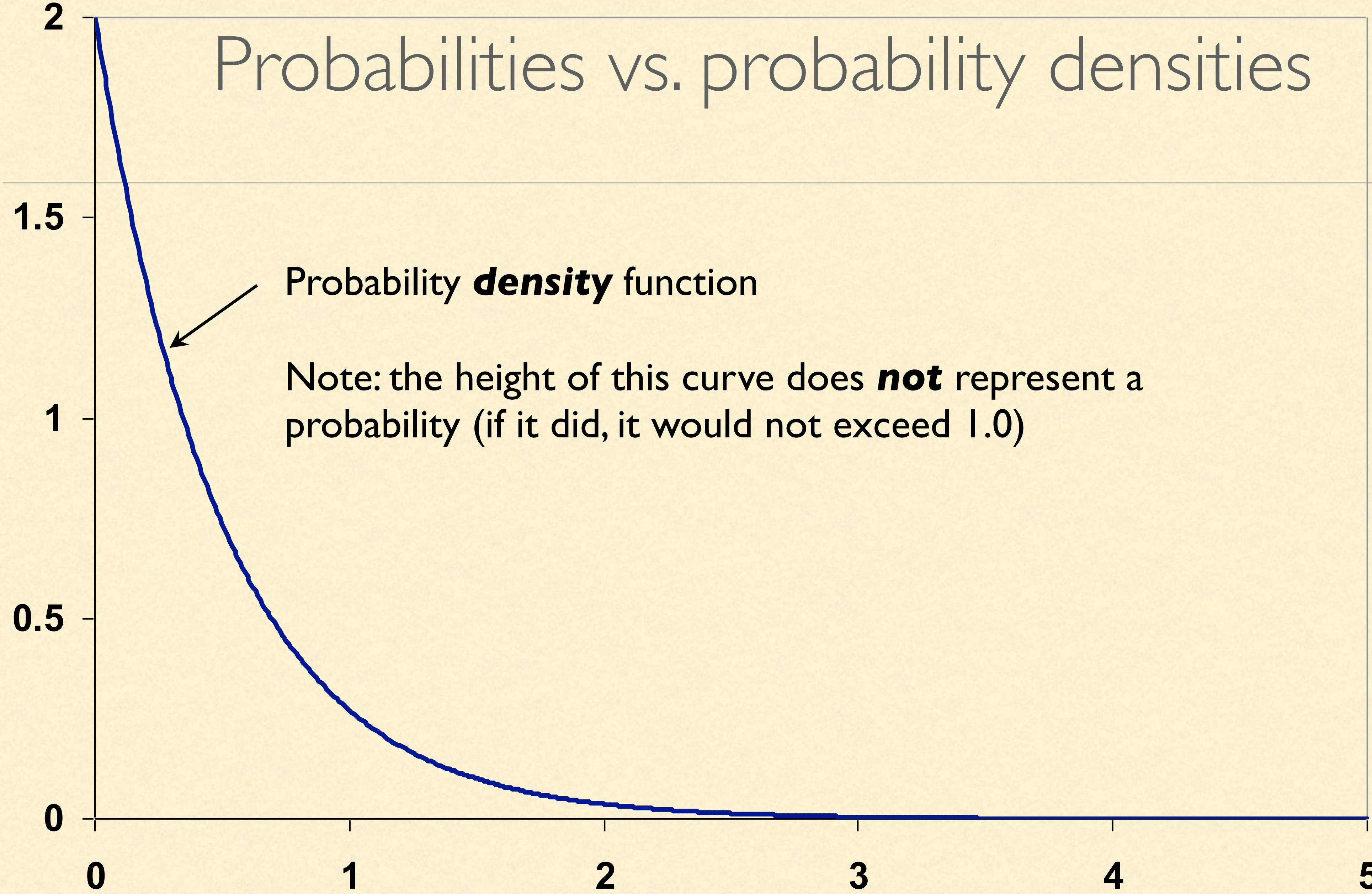
Probabilities are attached to **intervals** (i.e. ranges of values), **not** individual **values**

The probability of any given point (e.g. d = 60.0) is zero!

However, we can ask about the probability that d falls in a particular interval e.g. $50.0 < d < 65.0$



Probabilities vs. probability densities



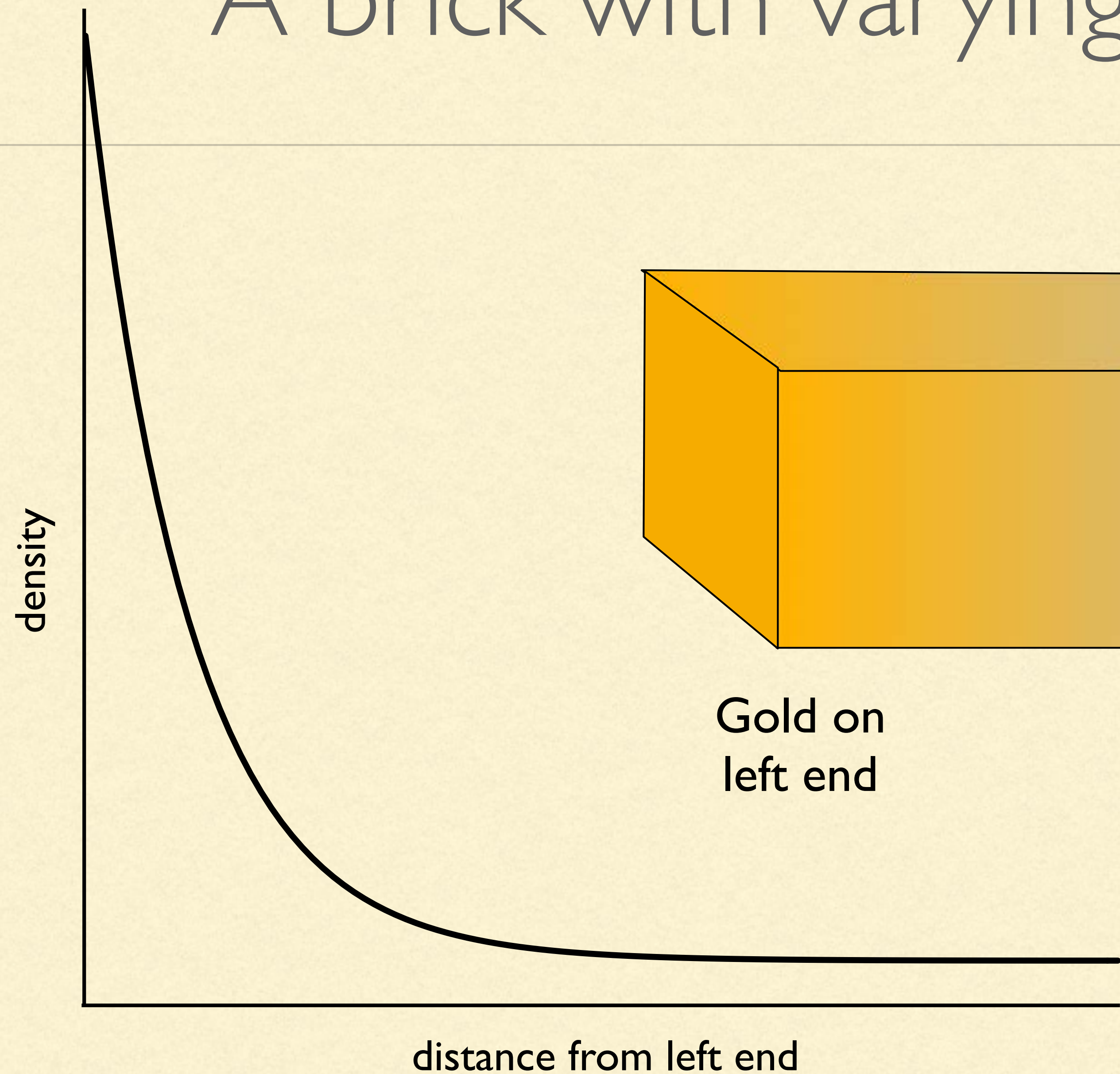
Densities of various substances

Substance	Density (g/cm^3)
Cork	0.24
Aluminum	2.7
Gold	19.3

Density does not equal mass

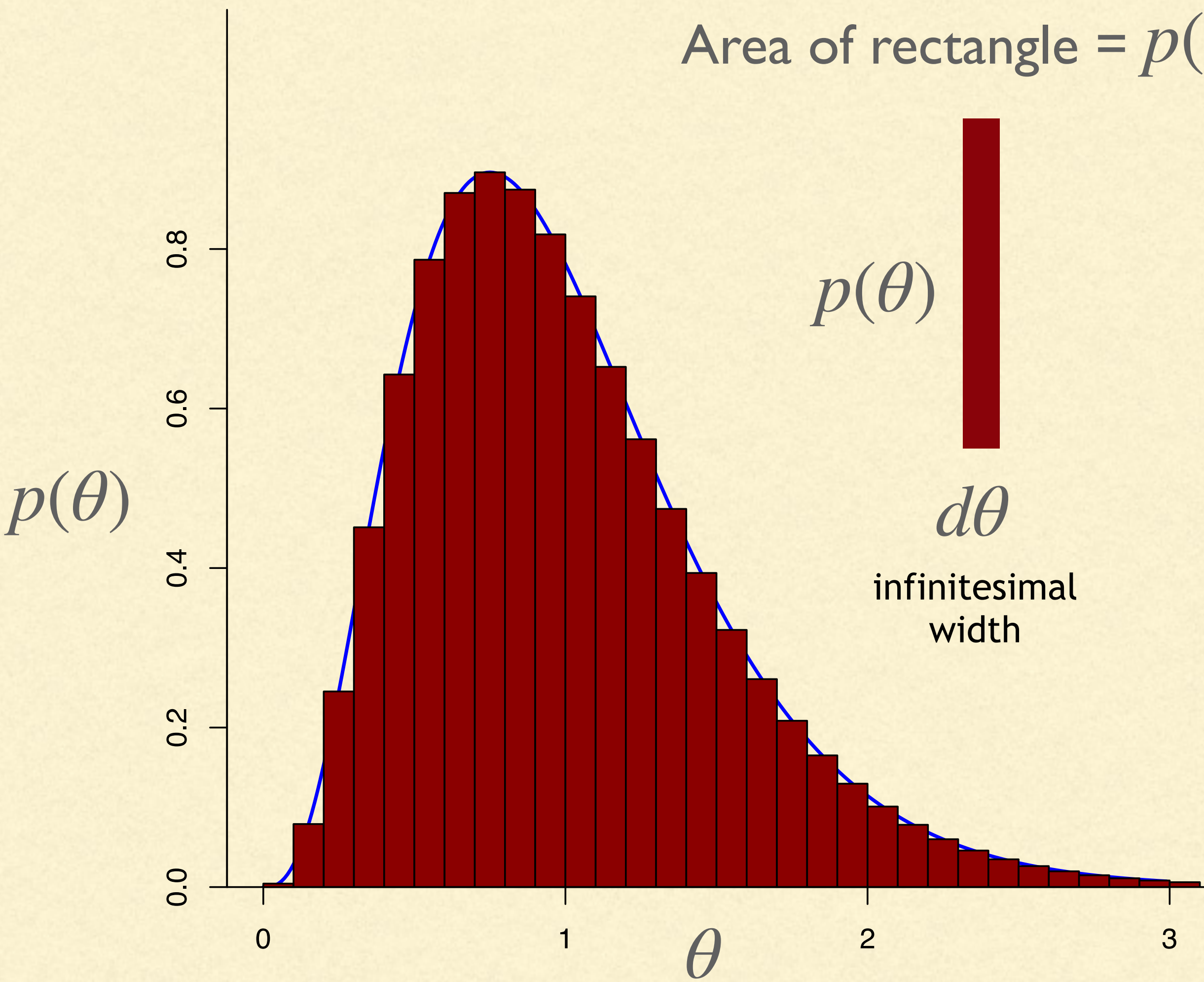
mass = density × volume

A brick with varying density



Integrating a density yields a probability

Area of rectangle = $p(\theta)d\theta$

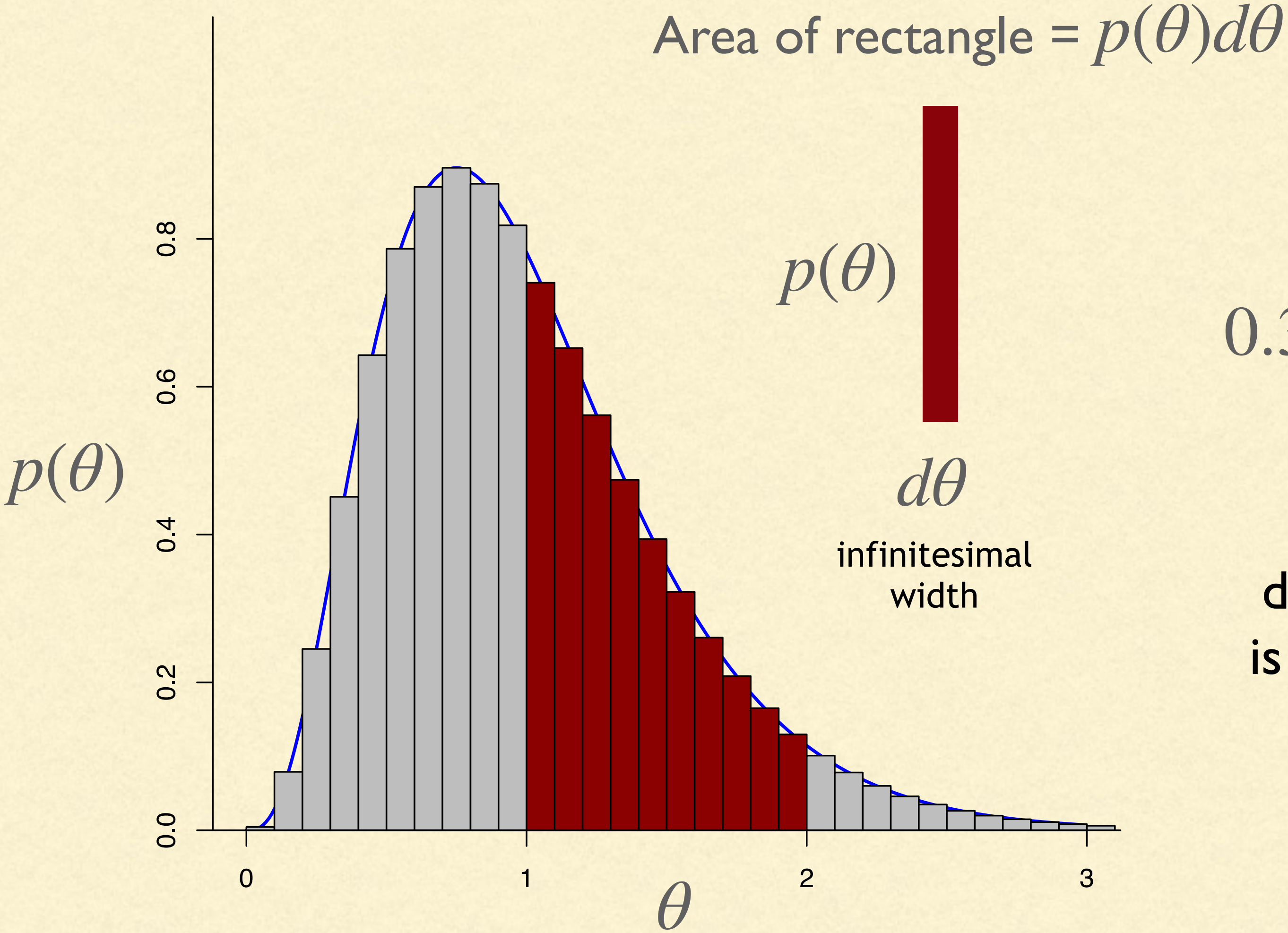


Long s from U.S. Bill of Rights

$$1.0 = \int p(\theta)d\theta$$

The density curve is scaled so that the value of this integral (i.e. the total area) equals 1.0

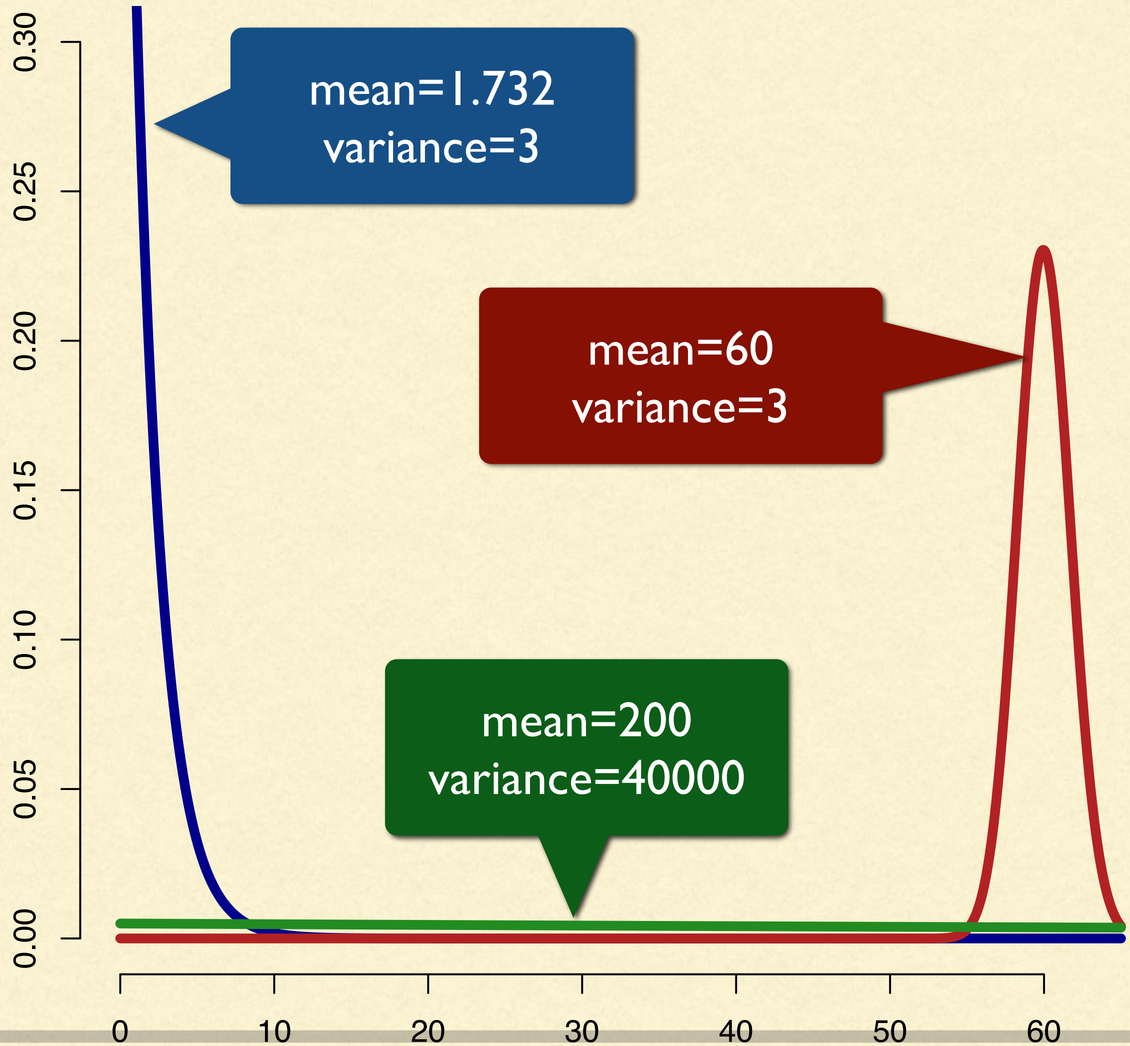
Integrating a density yields a probability



$$0.39109 = \int_1^2 p(\theta)d\theta$$

The **area** under the density curve from 1 to 2 is the **probability** that θ is between 1 and 2

Archery priors revisited



These density curves are all variations of a **gamma probability distribution**. We could have used a gamma distribution to specify each of the prior probability distributions for the archery example. Note that **higher variance** means **less informative**

Usually there are many parameters...

A 2-parameter example

Likelihood Prior density

$$p(\theta,\phi\,|\,D) = \frac{p(D\,|\,\theta,\phi)\; p(\theta)\; p(\phi)}{\int_\theta \int_\phi p(D\,|\,\theta,\phi)\; p(\theta)\; p(\phi)\; d\phi\; d\theta}$$

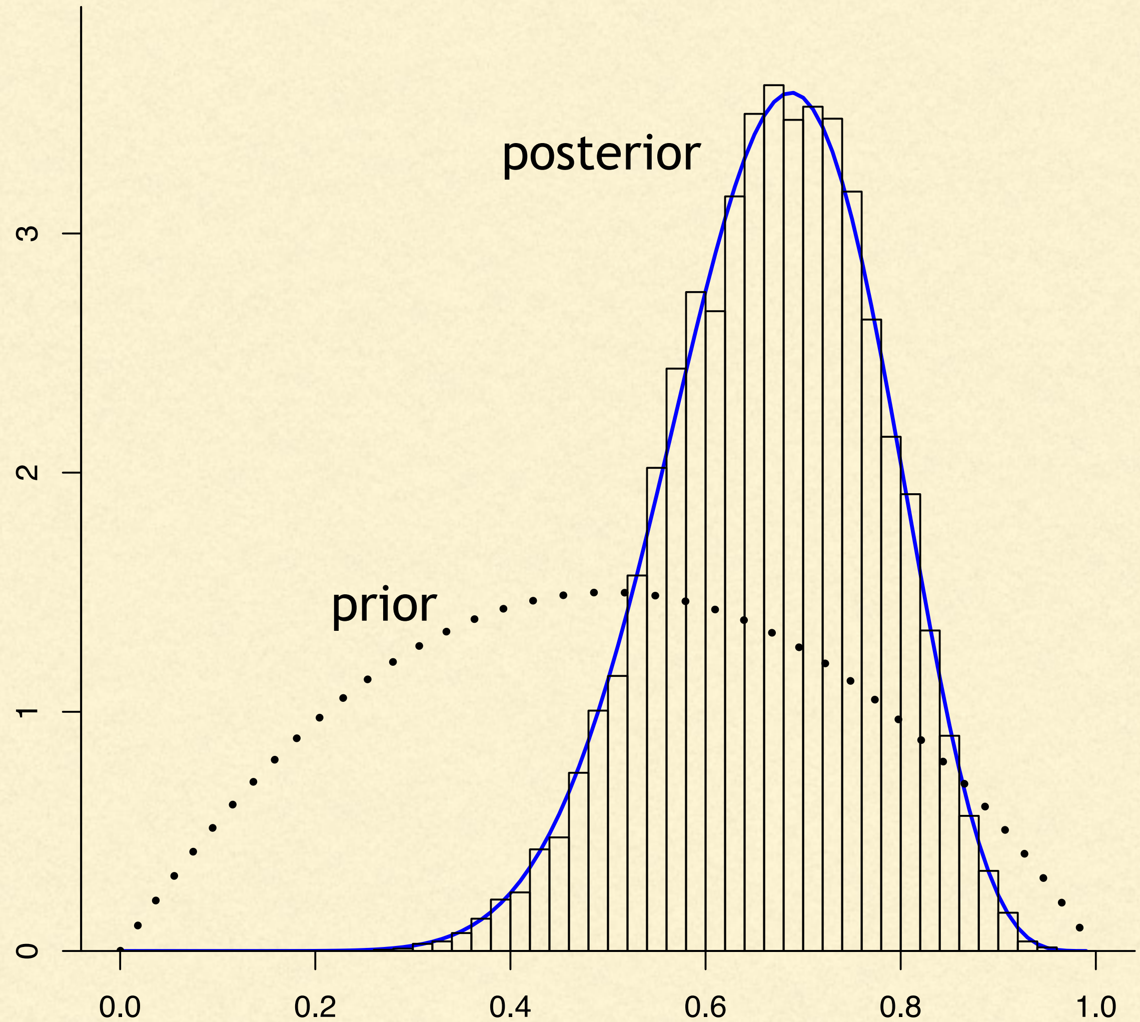
Marginal probability of data

Posterior probability density

An analysis of **100 sequences** under the simplest model (JC69) requires 197 branch length parameters.
The denominator would require a **197-fold integral** inside a sum over **all possible tree topologies**!
It would thus be nice to avoid having to calculate the marginal probability of the data...

Markov chain Monte Carlo (MCMC)

Markov chain Monte Carlo (MCMC)

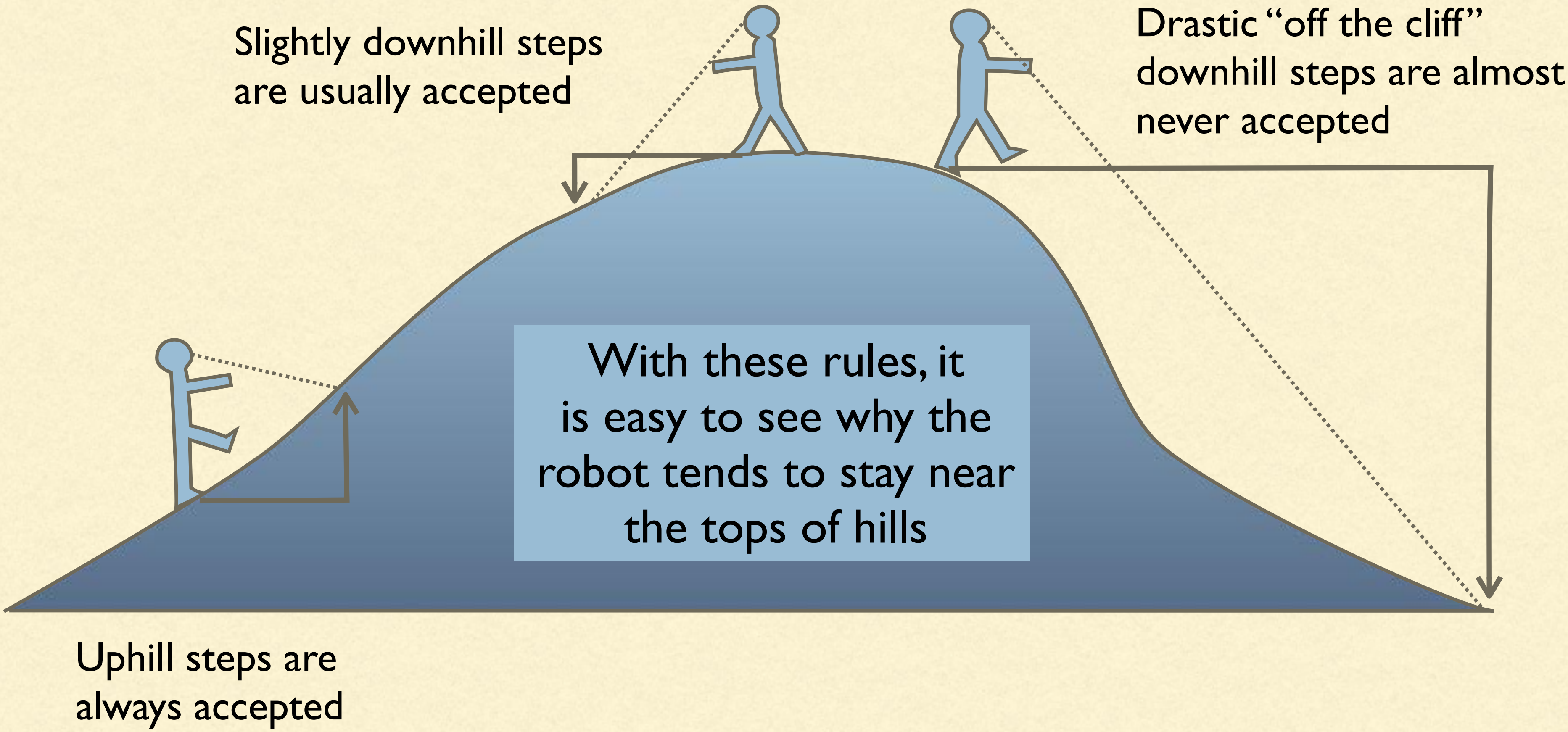


For more complex problems, we might settle for a

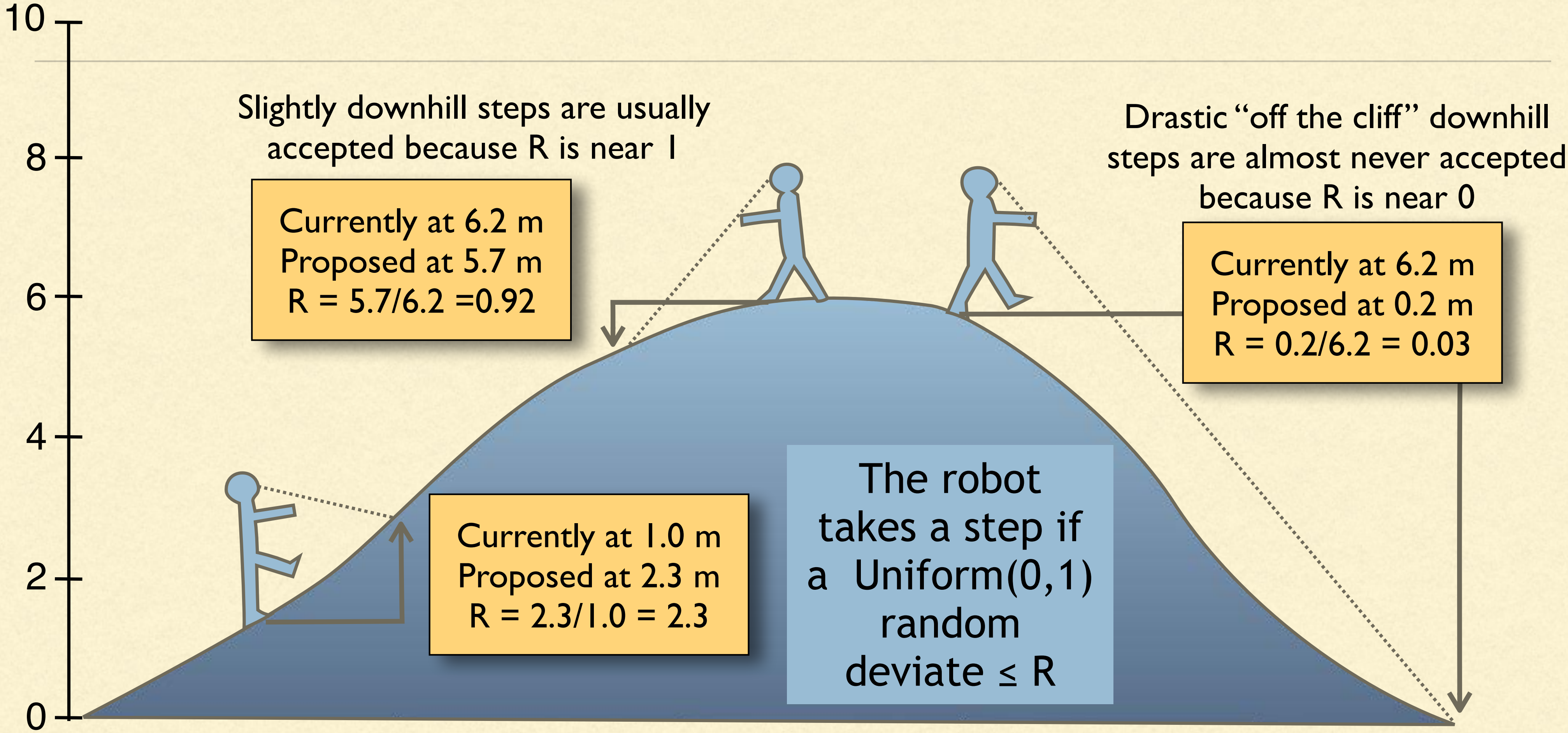
good approximation

to the posterior distribution

MCMC robot's rules



Actual rules (Metropolis algorithm)



Uphill steps are always accepted because R > 1

Metropolis et al. 1953. Equation of state calculations by fast computing machines. *J. Chem. Physics* 21(6):1087-1092.

Cancellation of marginal likelihood

When calculating the ratio (R) of posterior densities, the marginal probability of the data cancels.

$$\frac{p(\theta^* \mid D)}{p(\theta \mid D)} = \frac{\dfrac{p(D \mid \theta^*)\, p(\theta^*)}{\cancel{p(D)}}}{\dfrac{p(D \mid \theta)\, p(\theta)}{\cancel{p(D)}}} = \frac{p(D \mid \theta^*)\, p(\theta^*)}{p(D \mid \theta)\, p(\theta)}$$

Posterior odds

Apply Bayes' rule to both top and bottom

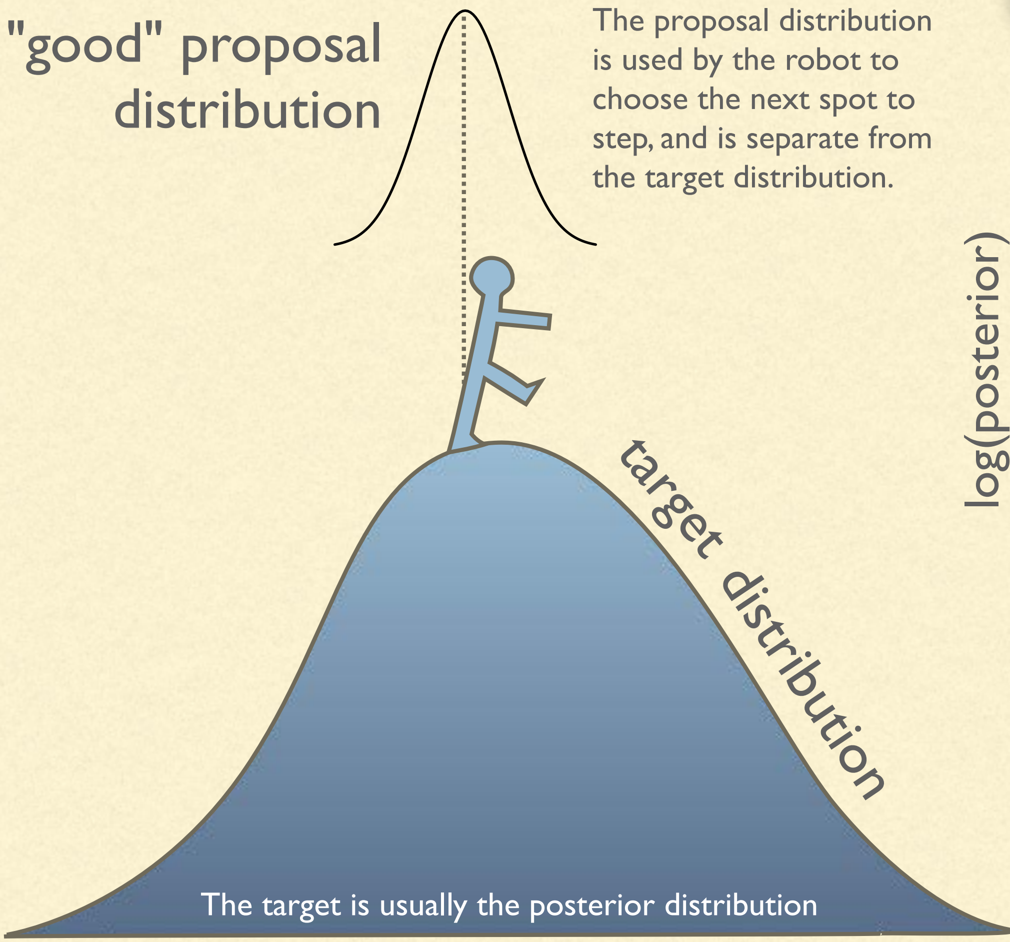
Likelihood ratio

Prior odds

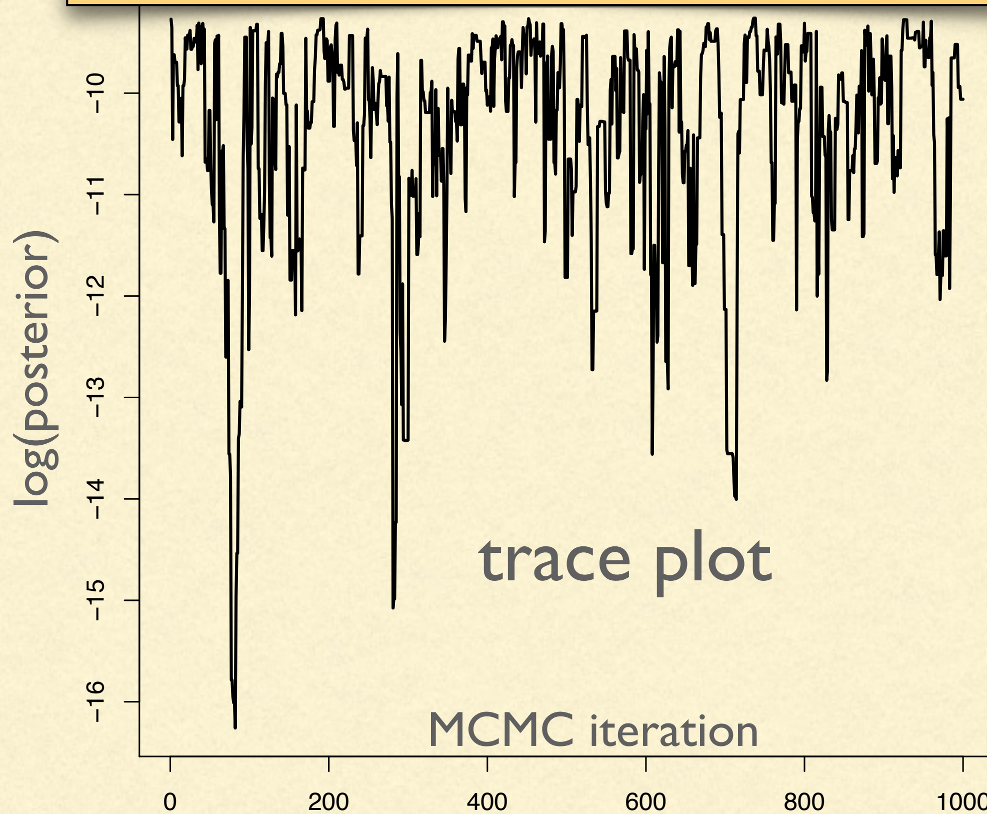
Target vs. Proposal Distributions

"good" proposal distribution

The proposal distribution is used by the robot to choose the next spot to step, and is separate from the target distribution.

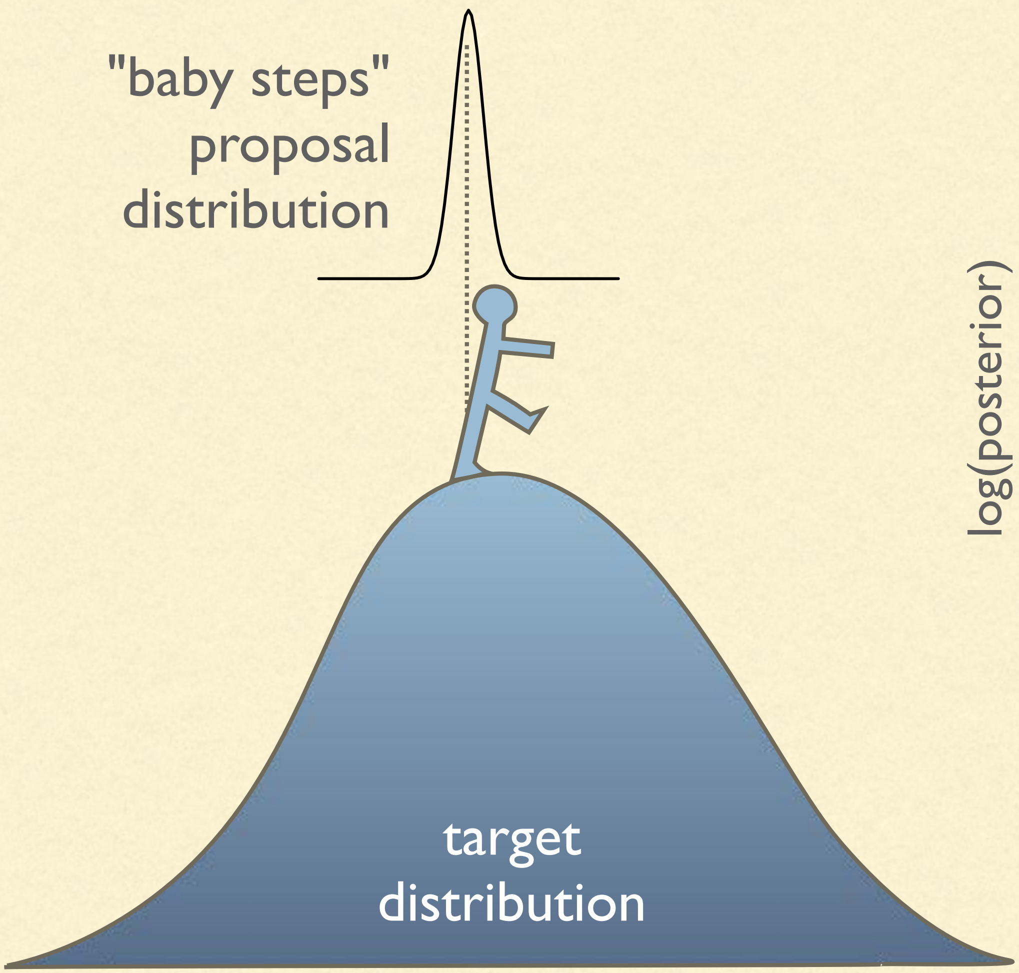


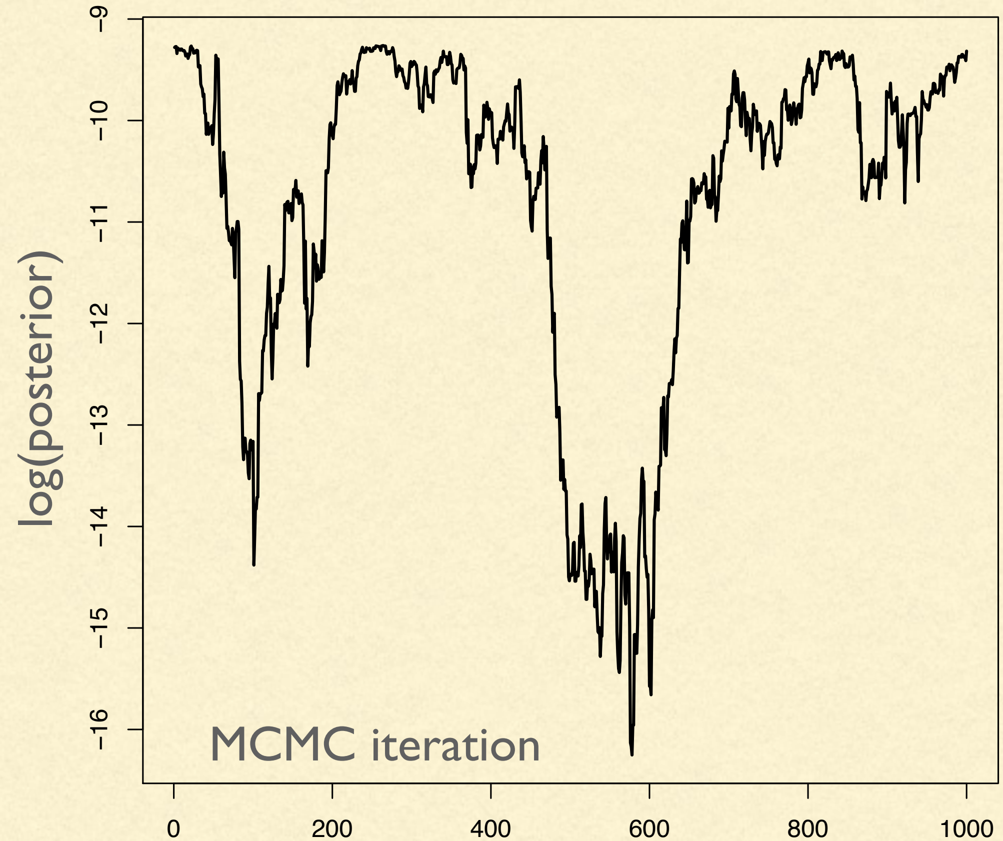
Tracer (app for generating trace plots from MCMC output):
https://github.com/beast-dev/tracer/releases/tag/v1.7.1



White noise appearance is a sign of good mixing

Target vs. Proposal Distributions

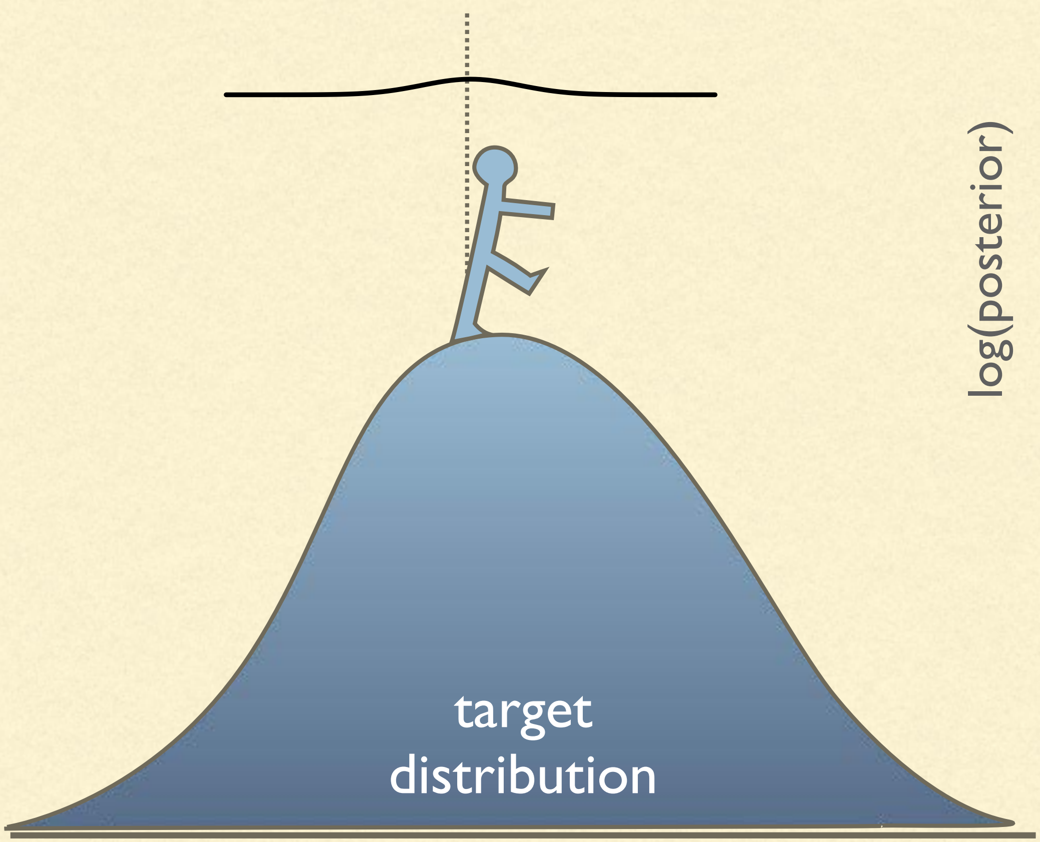


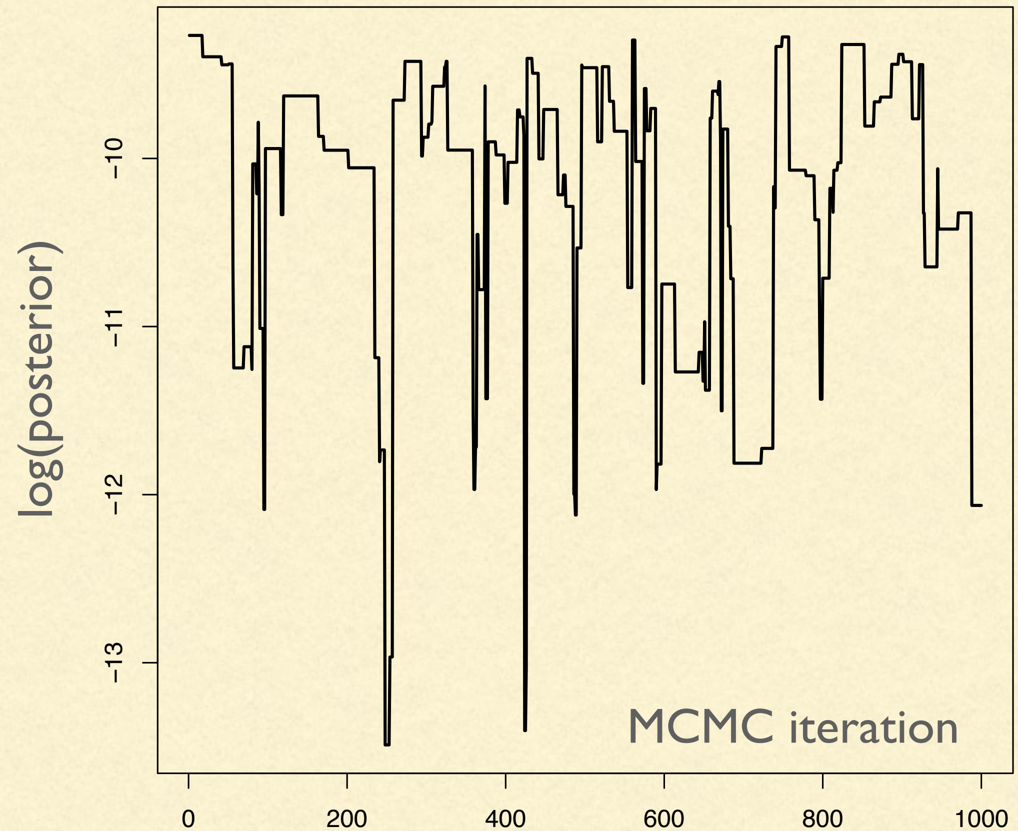


Big waves in trace plot indicate robot is crawling around

Target vs. Proposal Distributions

"overly bold" proposal distribution





Plateaus in trace plot indicate robot is often stuck in one place

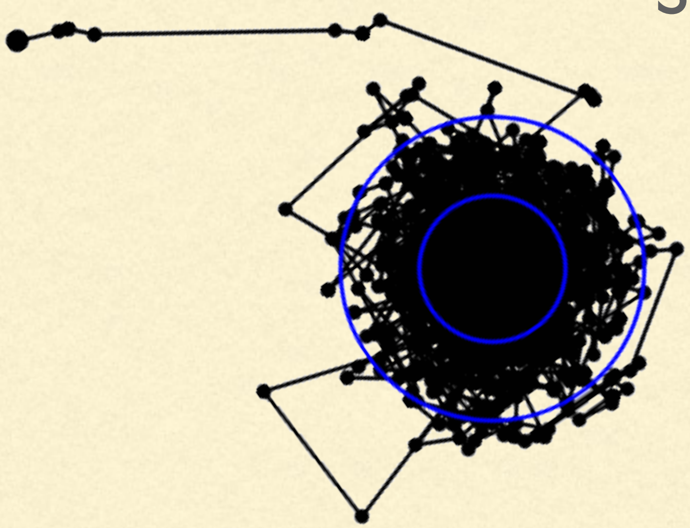
MCRobot (or "MCMC Robot")

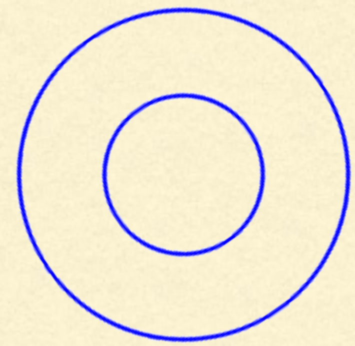
Javascript version used today will run in most web browsers and is available here:

https://plewis.github.io/applets/mcmc-robot/

Metropolis-coupled Markov chain Monte Carlo (MCMCMC)

Sometimes the robot needs some help,

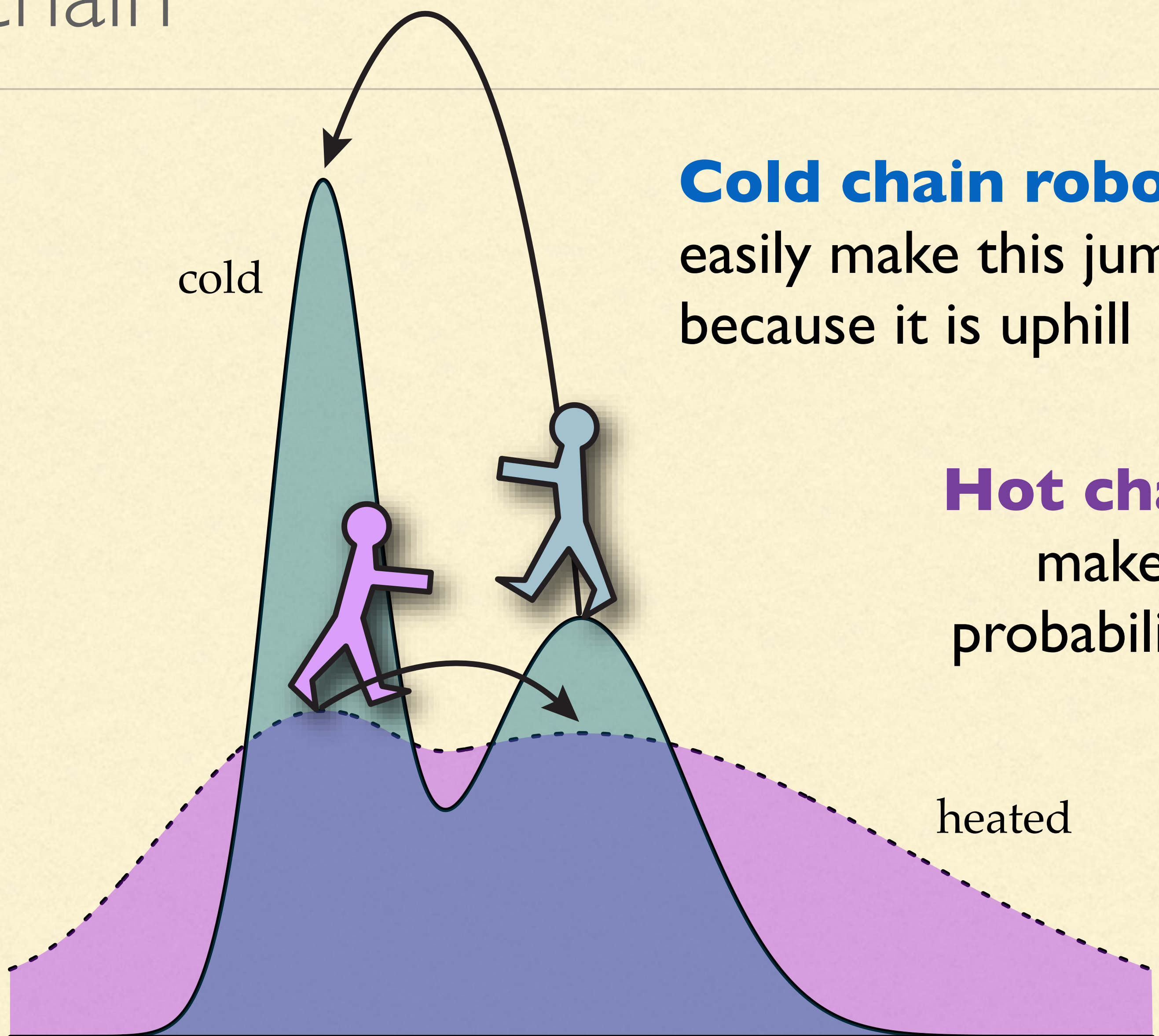




MCMCMC introduces helpers in the form of "heated chain" robots that can act as scouts.

Geyer, C. J. 1991. Markov chain Monte Carlo maximum likelihood for dependent data. Pages 156-163 *in* Computing Science and Statistics (E. Keramidas, ed.).

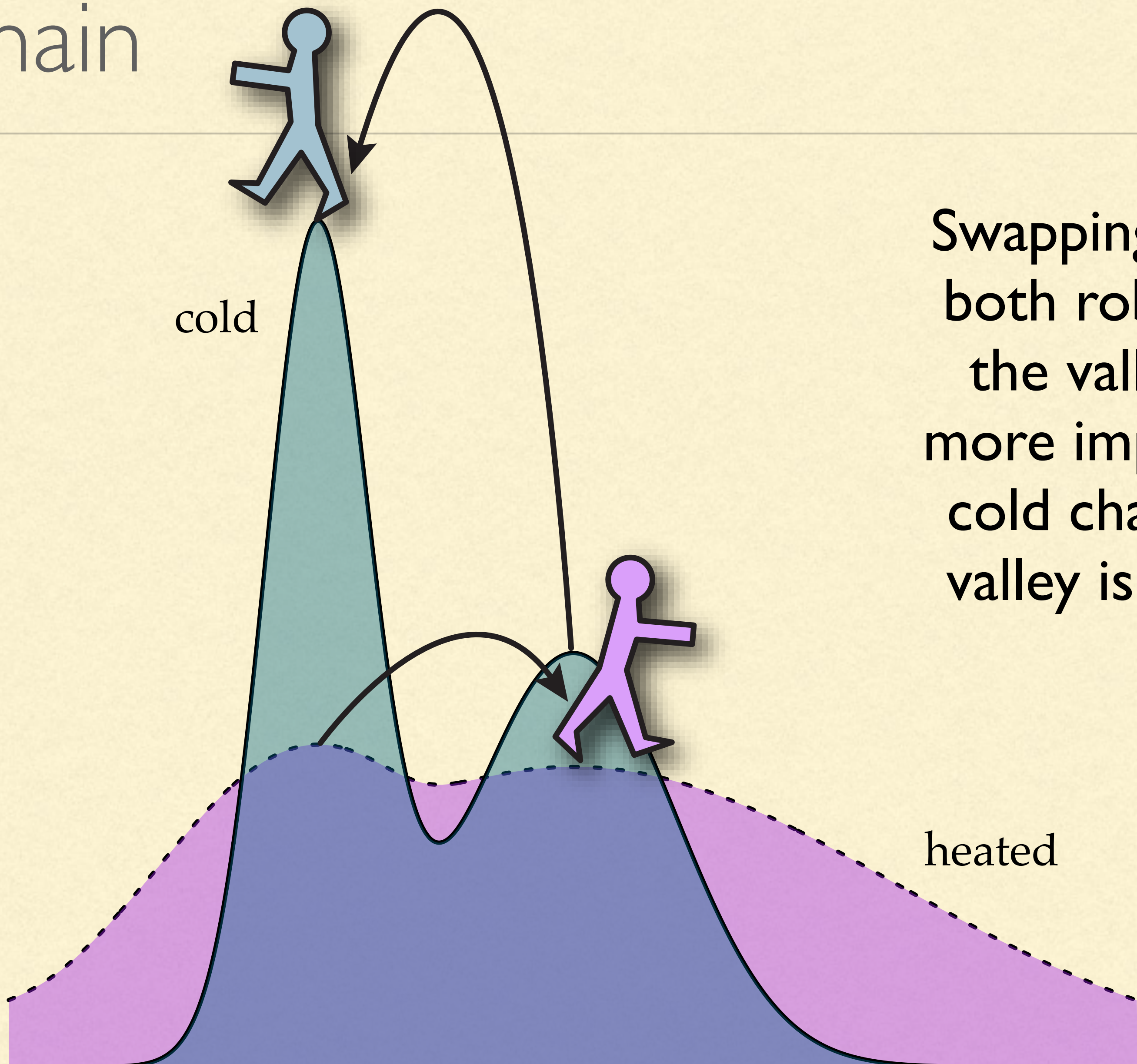
Heated chains act as scouts for the cold chain



Cold chain robot can easily make this jump because it is uphill

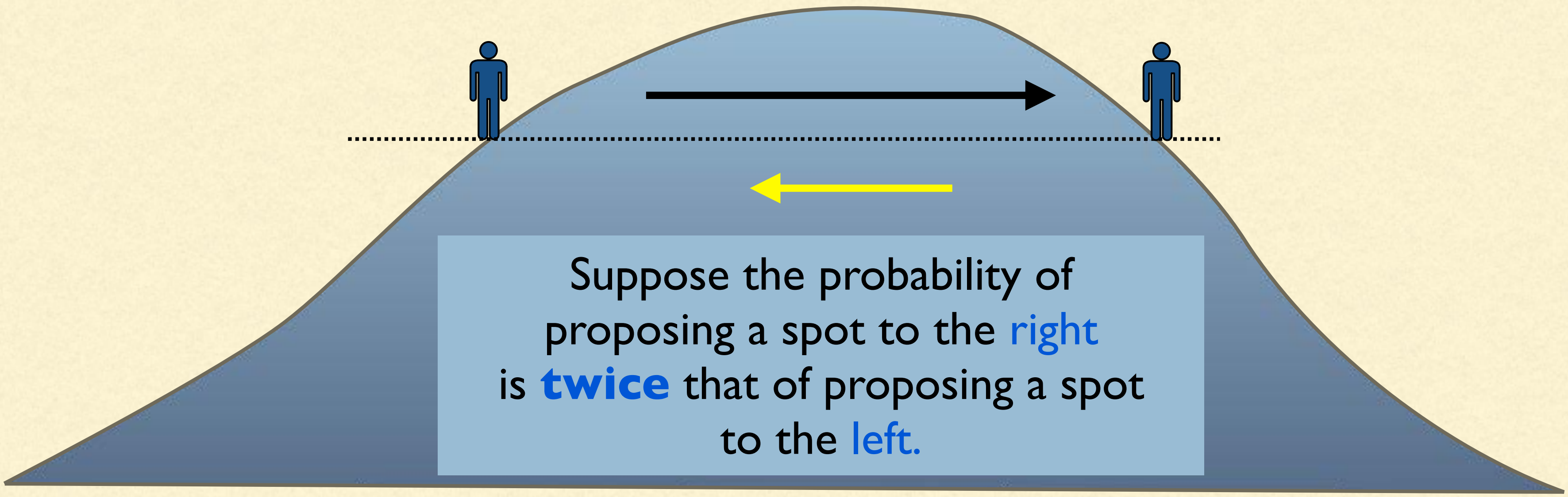
Hot chain robot can also make this jump with high probability because it is only slightly downhill

Heated chains act as scouts for the cold chain



Swapping places means both robots can cross the valley, but this is more important for the cold chain because its valley is much deeper.

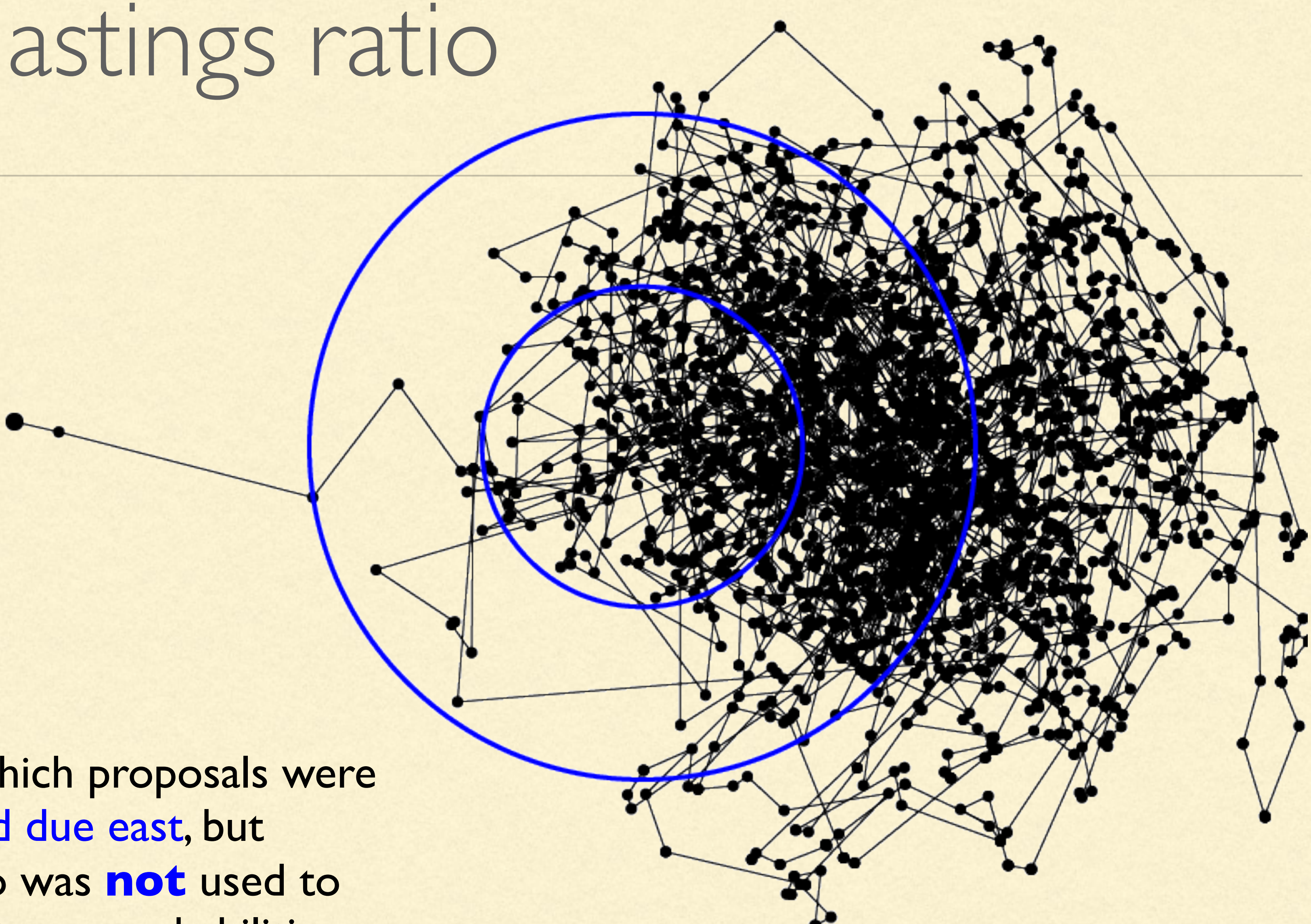
The Hastings ratio



Suppose the probability of proposing a spot to the right is **twice** that of proposing a spot to the left.

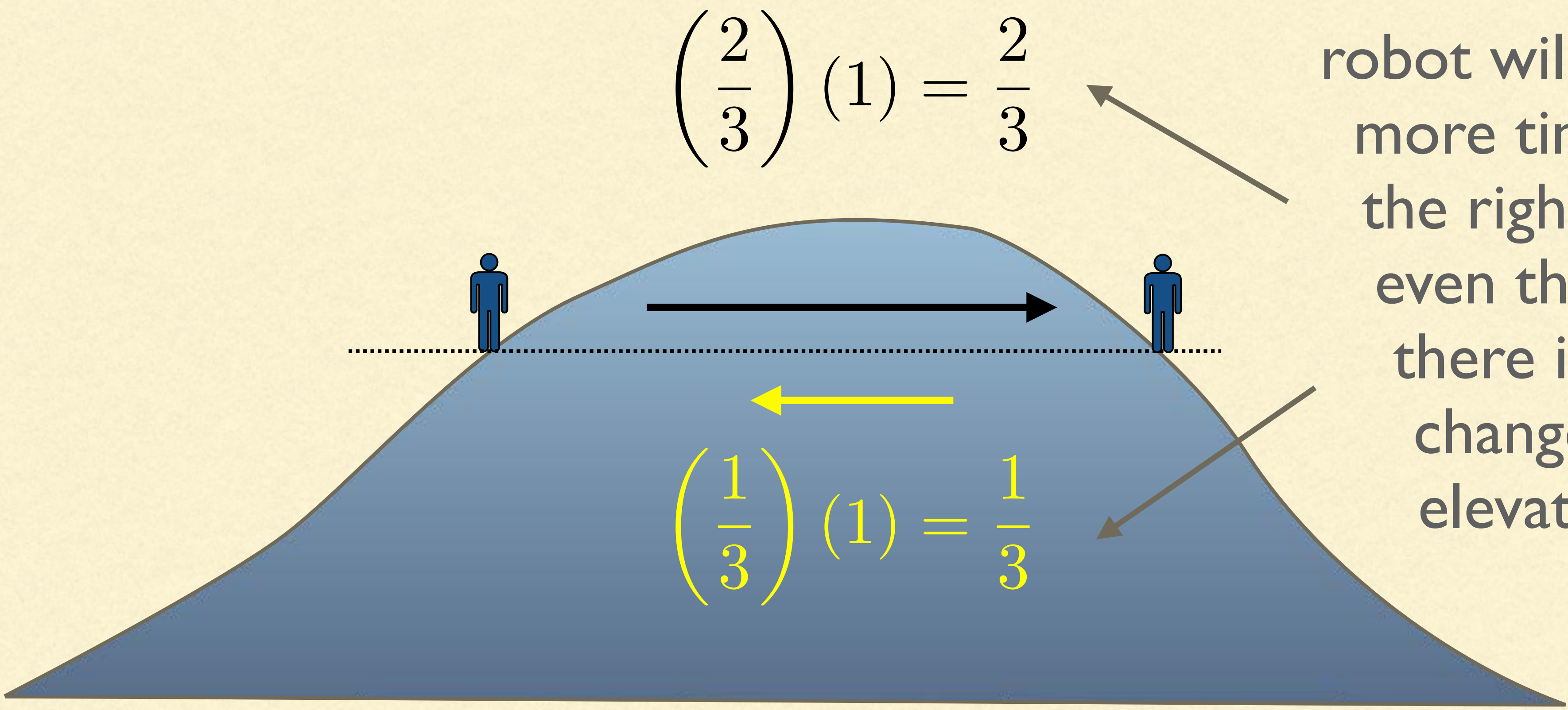
Hastings, W. K. 1970. Monte Carlo sampling methods using Markov chains and their applications. Biometrika 57:97-109.

The Hastings ratio



Example in which proposals were biased toward due east, but Hastings ratio was **not** used to modify acceptance probabilities

The Hastings ratio



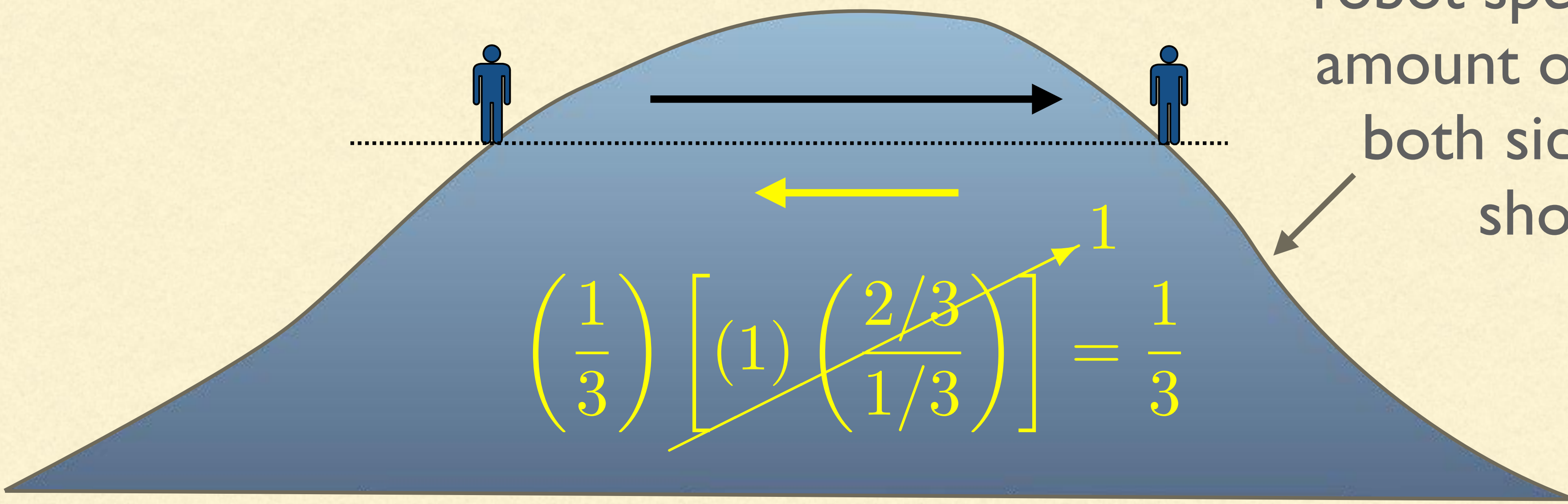
robot will spend more time on the right side even though there is no change in elevation

Hastings, W. K. 1970. Monte Carlo sampling methods using Markov chains and their applications. Biometrika 57:97-109.

The Hastings ratio

$$\left(\frac{2}{3}\right)\left[(1)\left(\frac{1/3}{2/3}\right)\right] = \frac{1}{3}$$

robot spends same amount of time on both sides, as it should



$$\left(\frac{1}{3}\right)\cancelto{1}{\left[(1)\left(\frac{2/3}{1/3}\right)\right]} = \frac{1}{3}$$

Hastings, W. K. 1970. Monte Carlo sampling methods using Markov chains and their applications. Biometrika 57:97-109.

Hastings Ratio

$$R = \min\left\{1, \underbrace{\left[\frac{p(D\,|\,\theta^*)\;p(\theta^*)}{p(D\,|\,\theta)\;p(\theta)}\right]}_{\text{posterior ratio}} \underbrace{\left[\frac{q(\theta\,|\,\theta^*)}{q(\theta^*\,|\,\theta)}\right]}_{\text{Hastings ratio}}\right\}$$

Note that the Hastings ratio is 1.0 if $q(\theta^*\,|\,\theta) = q(\theta\,|\,\theta^*)$