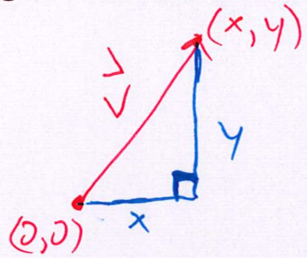
MATH 232 Section 1.2 pre-lecture comments

Today we are discussing the *geometry* of $\mathbf{R}^n$. How long is a vector? What is the angle between vectors? Can vectors in $\mathbf{R}^4$ have no direction in common? How do we break vectors down into simple directions when we have lots of dimensions?

New terminology:

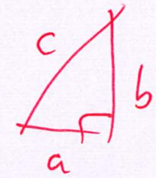
1. norm, magnitude
2. dot product
3. distance
4. unit vector
5. orthogonal
6. orthonormal

Length of a vector in $\mathbf{R}^2$



$\vec{v} = (x,y)$

length of $\vec{v} = \sqrt{x^2+y^2}$



$a^2+b^2 = c^2$

$c = \sqrt{a^2+b^2}$

Definition 1.2.1 Magnitude (or norm or length)

The norm of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n is

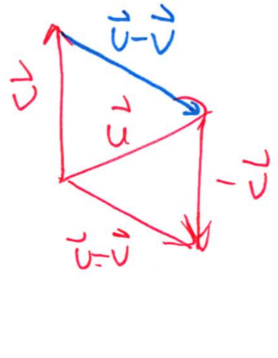
$||\mathbf{v}|| = \sqrt{v_1^2+v_2^2+v_3^2+\ldots+v_n^2}$

Examples

1. $\mathbf{u} = (1,-1,-2), ||\mathbf{u}|| = \sqrt{1^2+(-1)^2+(-2)^2} = \sqrt{1+1+4} = \sqrt{6}$
2. $\mathbf{v} = (2,-1,-1), ||\mathbf{v}|| = \sqrt{2^2+(-1)^2+(-1)^2} = \sqrt{4+1+1} = \sqrt{6}$
3. $||\mathbf{u}-\mathbf{v}|| =$

$\vec{u}-\vec{v} = (-1,0,-1)$

$||\vec{u}-\vec{v}|| = \sqrt{(-1)^2+0^2+(-1)^2} = \sqrt{1+0+1} = \sqrt{2}$



$||\vec{v}|| = \sqrt{v_1^2+v_2^2+v_3^2+\ldots+v_n^2}$ ← Norm of $\vec{v}$ (length of $\vec{v}$)

Theorem 1.2.2 Let $\mathbf{v}$ be a vector in R^n and k a scalar. Then

← length is always positive or 0

if $k>0$: $||k\vec{v}|| = k||\vec{v}||$

if $k<0$: $||k\vec{v}|| = -k||\vec{v}||$

1. $||\mathbf{v}|| \geq 0$
2. $||k\mathbf{v}|| = |k|(||\mathbf{v}||)$
3. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \vec{\mathbf{0}}$

only the $\vec{0}$ vector has a norm of 0

Definition 1.2.3 **Distance between points** $\vec{\mathbf{u}} = (u_1, u_2, \ldots, u_n)$ and $\vec{\mathbf{v}} = (v_1, v_2, \ldots, v_n)$ in $\mathbf{R}^n$, denoted by $d(\mathbf{u}, \mathbf{v})$ is the norm of the vector $\mathbf{v} - \mathbf{u}$:

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(v_1-u_1)^2+(v_2-u_2)^2+\ldots+(v_n-u_n)^2}$$

Example What is the distance from $A = (1, -2, 0, 3)$ to $B = (1, -1, 2, 1)$?

$$d(A,B) = \sqrt{(1-1)^2+(-1-(-2))^2+(2-0)^2+(1-3)^2}$$

$$= \sqrt{0^2+1^2+2^2+(-2)^2} = \sqrt{9} = \boxed{3}$$

Theorem 1.2.4 Properties of distance.

1. $d(\mathbf{u},\mathbf{v}) \geq 0$ ← length is positive or 0
2. $d(\mathbf{u},\mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
3. $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$

Problem In some cartographic system Vancouver has coordinates (31,11) and Calgary (931, 211). Your friend is flying his plane with a range of 1100. Do you go with them? Why or why not?

maximum possible distance

$$\sqrt{(931-31)^2+(211-11)^2} = \sqrt{900^2+200^2} \cancel{= \sqrt{810000+40000}}$$

$$\approx 921 \text{ (Yes)} \quad \cancel{\approx 1100 \text{ Yes (just barely)}}$$

less than 1100

Definition (p. 16) A **unit vector** is any vector $\mathbf{v}$ with $||\mathbf{v}|| = 1$. ex: $(1,0)$

Problems What unit vector(s) is/are parallel to $\mathbf{u} = (1,-1)$?

$$||\vec{u}|| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\frac{1}{\sqrt{2}}(\vec{u}) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \qquad \text{Norm: } \frac{||\vec{u}||}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

same direction ← unit vectors

$$\frac{1}{\sqrt{2}}(-\vec{u}) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

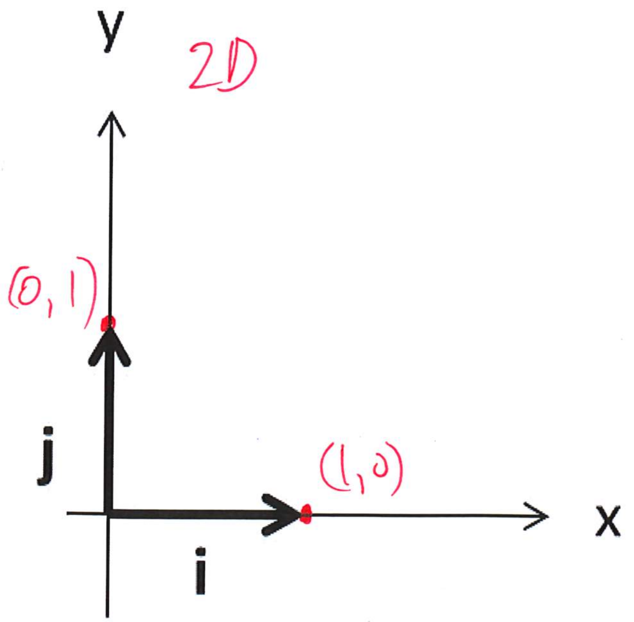
opposite direction

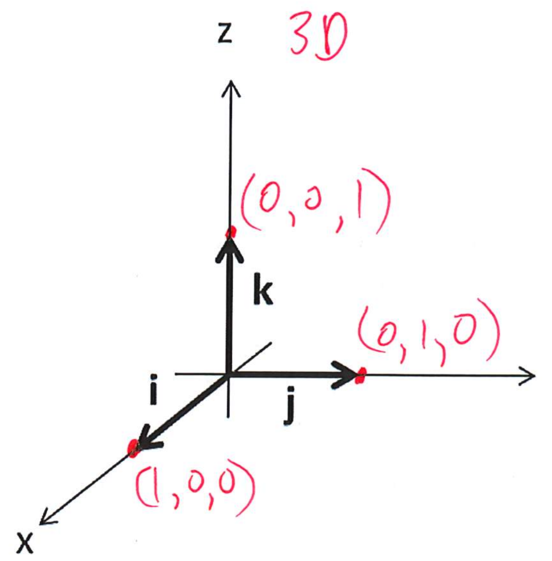
Give a unit vector parallel to nonzero $\mathbf{v}$.

$\vec{v}$

unit vector in same direction: $\dfrac{\vec{v}}{||\vec{v}||}$

The standard unit vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$.





In coordinates we have:

$\mathbf{R}^2$

$\mathbf{i} = (1,0)$

$\mathbf{j} = (0,1)$

$\mathbf{R}^3$

$\mathbf{i} = (1,0,0)$

$\mathbf{j} = (0,1,0)$

$\mathbf{k} = (0,0,1)$

Standard unit vectors in $\mathbf{R}^n$ (p. 17) In $n-$space the unit vectors are

$$\vec{\mathbf{e}}_1 = (1,0,0,\ldots,0),\ \vec{\mathbf{e}}_2 = (0,1,0,\ldots,0) \quad \ldots \quad \vec{\mathbf{e}}_n = (\overbrace{0,0,\ldots,0,1}^{n})$$

Problem Write $\mathbf{v} = (1,-\pi,0,\sqrt{2})$ as a linear combination of unit vectors.

$$\vec{v} = (1,0,0,0) + (0,-\pi,0,0) + (0,0,0,\sqrt{2})$$

$$= \boxed{\vec{e}_1 + -\pi\vec{e}_2 + \sqrt{2}\vec{e}_4}$$

Every vector in $\mathbf{R}^n$ can be written as a linear combination of standard unit vectors.

Definition 1.2.5 The **dot product** of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ is the *scalar*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Examples Given $\mathbf{u} = (1, -1, 1, 2)$, $\mathbf{v} = (0, -1, 2, 1)$ and $\mathbf{w} = (1, 1, 2, 1)$ find:

$\mathbf{u} \cdot \mathbf{v} =$ $1(0) + (-1)(-1) + 1(2) + 2(1) = 5$

$\mathbf{u} \cdot \mathbf{w} =$ $1(1) + (-1)(1) + 1(2) + 2(1) = 4$

$\mathbf{v} \cdot \mathbf{w} =$

Theorem 1.2.6 Algebraic properties

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. (a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ and (b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
4. $\mathbf{0} \cdot \mathbf{v} = 0$
5. $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

Calculate For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ calculate:

$u = (u_1, u_2, \dots, u_n)$

$\mathbf{u} \cdot \mathbf{u} =$ $u_1^2 + u_2^2 + \dots + u_n^2 = ||\vec{u}||^2$

$||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 =$

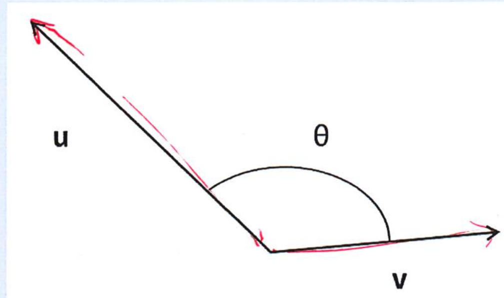
$(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) + (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$

$= \vec{u} \cdot \vec{u} + \cancel{2\vec{u} \cdot \vec{v}} + \vec{v} \cdot \vec{v} + \vec{u} \cdot \vec{u} - \cancel{2\vec{u} \cdot \vec{v}} + \vec{v} \cdot \vec{v} = 2\vec{u} \cdot \vec{u} + 2\vec{v} \cdot \vec{v}$

$= 2||\vec{u}||^2 + 2||\vec{v}||^2$

Definition

The **angle** between two non-zero vectors is the smallest non-negative angle needed to rotate one vector to other.



Theorem 1.2.8 Given non-zero vectors **u** and **v** in $\mathbf{R}^n$:

p20

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta \iff \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$|\cos\theta| = \left| \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \right| = \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$|\cos\theta| \leq 1 \rightarrow |\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$$

There are two important inequalities involving the dot product.

Theorem 1.2.12 **The Cauchy-Schwarz inequality**

$$(\mathbf{u}\cdot\mathbf{v})^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \qquad \text{or} \qquad |\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

This gives an upper bound on how big $|\mathbf{u}\cdot\mathbf{v}|$ can be without computing the dot product.

Theorem 1.2.11 **The triangle inequality**

Given non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$:

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

What does this mean in words? A picture in $\mathbf{R}^2$?

$||\vec{u}+\vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$

Theorem 1.2.8 Given non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$:
(again)

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}$$

What is the dot product of two perpendicular vectors?

$\theta = 90^\circ$ $\quad 0 = \dfrac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||} \rightarrow \vec{u}\cdot\vec{v} = 0$

Definition 1.2.9 Orthogonal

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are **orthogonal** if $\mathbf{u}\cdot\mathbf{v} = 0$.
- A ~~non-empty~~ two or more set of vectors is an **orthogonal set** if each pair of distinct vectors in the set is *orthogonal*.
- A vector~~s~~ $\mathbf{u}$ is **orthogonal to the set** S of vectors if it is *orthogonal* to every vector in S.

Examples $(1,0)$, $(0,1)$ are orthogonal $\quad (1,0)\cdot(0,1) = 0$

$(1,2)$, $(-2,1)$ are orthogonal $\quad (1,2)\cdot(-2,1) = 1(-2)+2(1) = 0$

$\{(1,0,0), (0,1,0), (0,0,1)\}$ is an orthogonal set: $\begin{cases} (1,0,0)\cdot(0,1,0) = 0 \\ (1,0,0)\cdot(0,0,1) = 0 \\ (0,1,0)\cdot(0,0,1) = 0 \end{cases}$

Pythagorean theorem for vectors

If $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* vectors in R^n, then $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$

Proof:

$$||\vec{u}+\vec{v}||^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\overbrace{\vec{u}\cdot\vec{v}}^{0 \text{ because orthogonal}} + \vec{v}\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} = ||\vec{u}||^2 + ||\vec{v}||^2$$

Definition 1.2.10 Orthonormal vectors/sets

- Two vectors **u** and **v** in R^n are **orthonormal** if they are *orthogonal* and have norm 1. $\|\vec{u}\| = \|\vec{v}\| = 1$
- A non-empty set of vectors is an **orthonormal set** if it is *orthogonal* and each vector in the set has norm 1.

Examples $(1,0), (0,1)$ are orthonormal since $(1,0)\cdot(0,1) = 0$ and $\|(1,0)\| = 1$ $\|(0,1)\| = 1$

$(1,2), (-2,1)$ are not orthonormal: $(1,2)$ is not a unit vector

~~(1,2) and (2,1) not unit vectors~~

$\{(1,0,0), (0,1,0), (0,0,1)\}$ is an orthonormal set

Standard unit vectors $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ form an orthonormal set