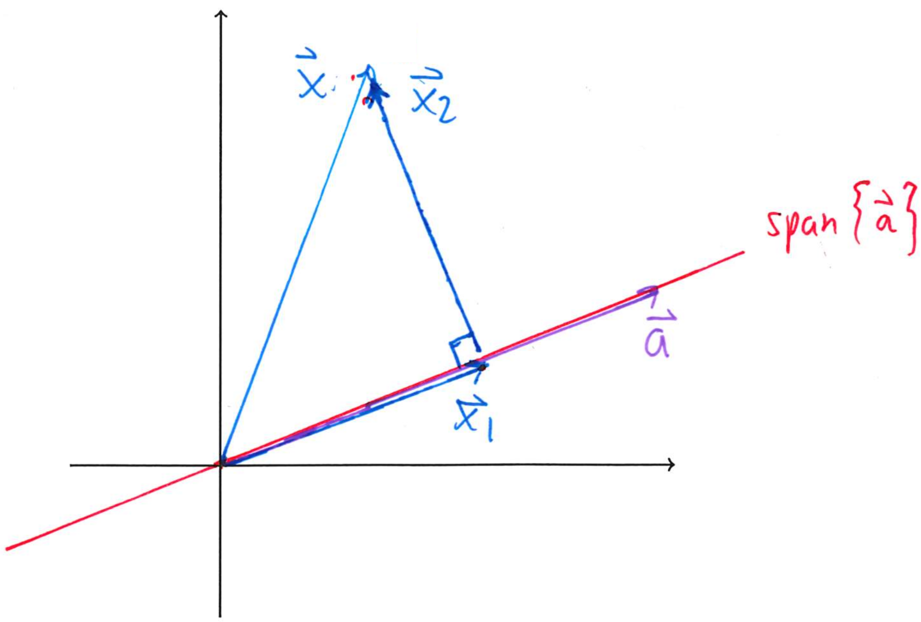
Section ~~7.3–7.6~~ 7.7 pre-lecture comments

Lecture Outline

New terminology

1. (orthogonal) projection
2. vector component along **a**
3. vector component perpendicular to **a**

Projection onto a line

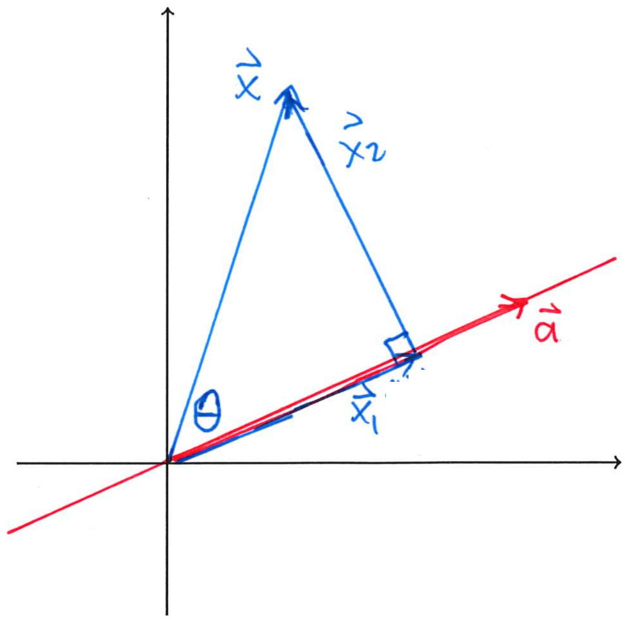


- $\vec{\mathbf{x}}_1$ is the part of $\vec{\mathbf{x}}$ parallel to $\vec{\mathbf{a}}$.
- $\vec{\mathbf{x}}_2$ is the part of $\vec{\mathbf{x}}$ perpendicular to $\vec{\mathbf{a}}$.

How do we find $\mathbf{x}_1$ and $\mathbf{x}_2$?

Assume that $\theta < 90^o$ (the case $\theta > 90^o$ is similar).

- We already know the direction of $\mathbf{x}_1$. same direction as $\vec{a}$
- So we only need to find $\|\mathbf{x}_1\|$.



From definition of cosine:

$$\cos\theta = \frac{\|\vec{x}_1\|}{\|\vec{x}\|} \quad \left(\frac{adj}{hyp}\right)$$

From the dot product $\mathbf{a} \cdot \mathbf{x}$:

$$\cos\theta = \frac{\vec{a}\cdot\vec{x}}{\|\vec{a}\|\,\|\vec{x}\|}$$

$$\|\mathbf{x}_1\| = \|\vec{x}\|\cos\theta = \cancel{\|\vec{x}\|}\,\frac{\vec{a}\cdot\vec{x}}{\|\vec{a}\|\,\cancel{\|\vec{x}\|}} = \frac{\vec{a}\cdot\vec{x}}{\|\vec{a}\|}$$ ← length of projection (scalar)

$$\mathbf{x}_1 = \left(\frac{\vec{a}\cdot\vec{x}}{\|\vec{a}\|}\right)\left(\frac{\vec{a}}{\|\vec{a}\|}\right) = \left(\frac{\vec{a}\cdot\vec{x}}{\|\vec{a}\|^2}\right)\vec{a}$$

unit vector in direction of $\vec{a}$

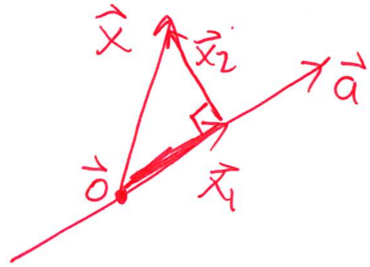
This is referred to as the (orthogonal) projection of $\mathbf{x}$ onto span($\mathbf{a}$).

Orthogonal projection Let $\mathbf{a}$ be a nonzero vector in $\mathbf{R}^n$. For any vector $\mathbf{x}$ in $\mathbf{R}^n$, the **(orthogonal) projection** of $\mathbf{x}$ onto span($\mathbf{a}$) is defined as:

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \left(\frac{\mathbf{a}\cdot\mathbf{x}}{\|\mathbf{a}\|^2}\right)\mathbf{a}$$

- $\mathbf{x}_1 = \text{proj}_{\mathbf{a}}\mathbf{x}$ is the **vector component of x along a**.
- $\mathbf{x}_2 = \mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}$ is the **vector component of x perpendicular to a**.

$\vec{x} - \vec{x}_1$



Ex: What are $\mathbf{x}_1$ and $\mathbf{x}_2$ for the following?

1) $\mathbf{a} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}2\\5\end{bmatrix}$ (projection onto $y = x$)

1) $\vec{x}_1 = \left(\frac{\vec{a}\cdot\vec{x}}{\|\vec{a}\|^2}\right)\vec{a} = \frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}7/2\\7/2\end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix}2\\5\end{bmatrix} - \begin{bmatrix}7/2\\7/2\end{bmatrix} = \begin{bmatrix}-3/2\\3/2\end{bmatrix}$

2) $\mathbf{a} = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}0\\3\\5\end{bmatrix}$

2) $\text{proj}_{\vec{a}}\vec{x} = \left(\frac{-2}{3}\right)\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}-2/3\\-2/3\\2/3\end{bmatrix} = \vec{x}_1$

$\vec{x}_2 = \vec{x} - \vec{x}_1 = \begin{bmatrix}0\\3\\5\end{bmatrix} - \begin{bmatrix}-2/3\\-2/3\\2/3\end{bmatrix} = \begin{bmatrix}2/3\\11/3\\13/3\end{bmatrix}$

Projection matrix

Standard matrix of a projection For a vector $\mathbf{a}$ in $\mathbf{R}^n$, the standard matrix of $T(\mathbf{x}) = \text{proj}_{\mathbf{a}}\mathbf{x}$ is the $n \times n$ matrix:

$$A = \left(\frac{1}{\|\mathbf{a}\|^2}\right) \mathbf{a}\mathbf{a}^T$$

where $\mathbf{a}$ is in column vector form and $\mathbf{a}^T$ is in row vector form, multiplied together using matrix multiplication.

Ex: What is the standard matrix of projection along:

1) $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (projection onto $y = x$) 2) $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3) $\mathbf{a} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ (projection onto line of angle θ)

1) $A = \frac{1}{\|a\|^2} a a^T = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

2) $A = \frac{1}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

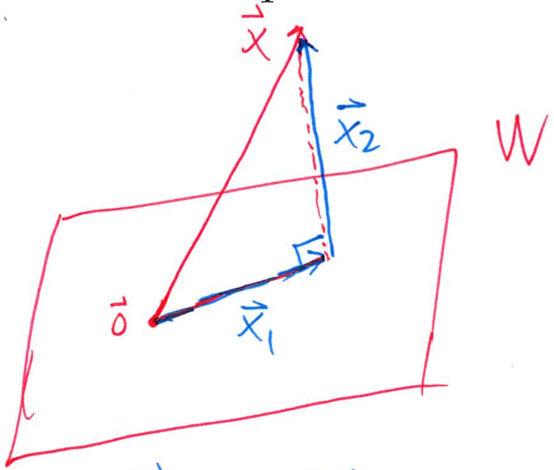
3) $A = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

$\downarrow$

1

Projections onto subspaces

Let W be a subspace of $\mathbf{R}^n$. As before, we would like to project $\mathbf{x}$ onto W.



$\vec{x}_1$ on W

$\vec{x}_2$ orthogonal to W

We can break $\mathbf{x}$ into $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1$ is in W and $\mathbf{x}_2$ is orthogonal to W (in $W^\perp$).

To calculate $\mathbf{x}$, use the following result:

> **Projection onto a subspace** Let W be a subspace of $\mathbf{R}^n$ and let M be any matrix whose column vectors form a basis for W. Then the **(orthogonal) projection** of $\mathbf{x}$ onto W is defined as:
>
> $$\text{proj}_W \mathbf{x} = \underbrace{M(M^TM)^{-1}M^T}_{A}\mathbf{x}$$

Note: The standard matrix of the transformation is $A = M(M^TM)^{-1}M^T$.

Projection onto a subspace Let W be a subspace of $\mathbf{R}^n$ and let M be any matrix whose column vectors form a basis for W. Then the **(orthogonal) projection** of $\mathbf{x}$ onto W is defined as:

$$\text{proj}_W \mathbf{x} = M(M^T M)^{-1} M^T \mathbf{x}$$

Note: The standard matrix of the transformation is $A = M(M^T M)^{-1} M^T$.

Ex: Projection of $\vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ onto the plane $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t$

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad M^T M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(M^T M)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(M^T M)^{-1} M^T = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$M(M^T M)^{-1} M^T = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = A$$

symmetric

$$A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \ldots = \frac{1}{3} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

projection mtx is always symmetric