

## MATH 232 Section 4.1 pre-lecture comments

### Lecture Outline

We will learn how to compute **determinants** for general  $n \times n$  matrices. This section gives the definition of determinants in terms of its *cofactor expansions*.

**Important:**  $A$  is invertible if and only if  $\det(A) \neq 0$ .

*All matrices in this chapter are square matrices.*

### New terminology

1. determinant
2. minor
3. cofactor

Recall the inverse of the  $2 \times 2$  matrix. Given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In particular,  $A$  has an inverse if and only if  $ad - bc \neq 0$ .

This quantity  <sup>$ad - bc$</sup>  is called the determinant. ( $2 \times 2$  mtx)

### Determinant of a $2 \times 2$ matrix

<sup>determinant</sup>  $\rightarrow$   $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc.$

Is there a similar “number test” for  $3 \times 3$  or larger matrices?

## Determinant of a $3 \times 3$ matrix

$a_{ij}$   $i$ th row  
 $j$ th column

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then  $\det(A)$  is the following:

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Diagram illustrating the Sarrus rule for a  $3 \times 3$  matrix. The matrix is written twice side-by-side. Blue arrows show the positive terms (downward diagonals) and red arrows show the negative terms (upward diagonals). Labels "negative" and "positive" are written below the respective arrows.

Ex:  $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$   $\det(A) =$

$$(-2) + (-2) + (0) - (2) - (2) - (0)$$

$$= -8$$

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = -8$$

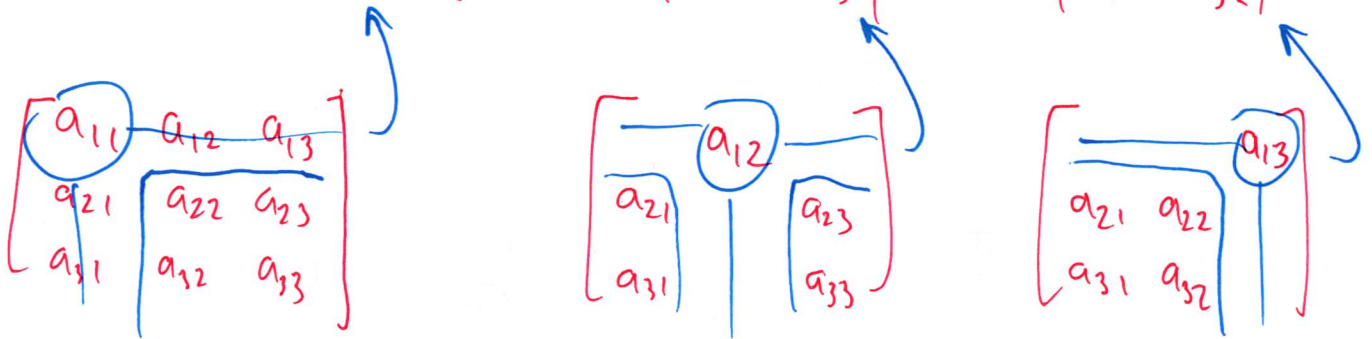
Note: This idea only works for  $2 \times 2$  and  $3 \times 3$ , not  $4 \times 4$  or larger!

~~4x4~~  $4 \times 4$  has 24 terms  
det of

There is another way to express the determinant, in terms of *smaller-sized determinants*:

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$



cofactor expansion

note the signs:  $+$   $-$   $+$



## Examples

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}$$
$$= \dots = -8$$

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + (3) \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix}$$
$$= 1(0) + 1(-1) + 3(0) = \boxed{-1}$$

**Definition: Minor (4.1.4)** For any  $n \times n$  matrix  $M_{ij}$  is the determinant of the submatrix formed by deleting  $i$ -th row and  $j$ -th column of  $A$ . This determinant is called the  $(i,j)$ -minor of  $A$  or  $M_{ij}$ .

$$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

### Examples

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = 0$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$$

$$\det(A) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

**Definition: Cofactor (4.1.4)** The  $(i,j)$ -cofactor ( $C_{ij}$ ) of  $A$  is  $C_{ij} = (-1)^{i+j}M_{ij}$ .

$$C_{ij} = \begin{cases} M_{ij} & \text{if } i+j \text{ even} \\ -M_{ij} & \text{if } i+j \text{ odd} \end{cases}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

+                  -                  +

3x3

+	-	+
-	+	-
+	-	+

checkerboard



### Theorem 4.1.5

The determinant of an  $n \times n$  matrix is given by:

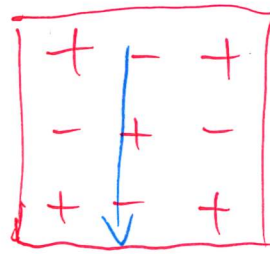
$$\begin{aligned}\det(A) &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - \dots (-1)^{n+1} a_{1n}M_{1n} \\ &= a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}\end{aligned}$$

Note that this is not the only formula. For  $3 \times 3$ , there are 5 others:

cofactor expansions	$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$	(1st row)
	$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$	(2nd row)
	$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$	(3rd row)
	$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$	(1st column)
	$= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$	(2nd column)
	$= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$	(3rd column)

We can expand on any row or column! (See Theorem 4.1.5)

Ex:  $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$



$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

$$= -(-1)M_{12} + \cancel{0M_{22}} - \cancel{0M_{32}}$$

$$= M_{12} = \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = \boxed{-1}$$

This suggests a strategy: **do the row or column with the most zeros.**



## Special cases

What about special cases like diagonal, upper triangular, lower triangular?

upper triangular

$$A = \begin{bmatrix} 1 & 3 & -4 & 7 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

1st col

$$\det(A) = a_{11} M_{11} = (1) \begin{vmatrix} -1 & 2 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & -2 \end{vmatrix} = (1)(-1) \begin{vmatrix} 2 & -5 \\ 0 & -2 \end{vmatrix}$$
$$= (1)(-1)(2)(-2) = \boxed{4}$$

if  $A$  is diagonal/upper/lower

then  $\det(A)$  is product of diagonal entries