MATH 232 Section 4.1 pre-lecture comments

Lecture Outline
We will learn how to compute **determinants** for general $n \times n$ matrices. This section gives the definition of determinants in terms of its *cofactor expansions*.

Important: A is invertible if and only if $\det(A) \neq 0$.

All matrices in this chapter are square matrices.

New terminology

1. determinant
2. minor
3. cofactor

Recall the inverse of the 2×2 matrix. Given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In particular, A has an inverse if and only if $ad - bc \neq 0$.

This quantity ($ad - bc$) is called the determinant. (2×2 mtx)

Determinant of a 2×2 matrix

determinant $\left| \begin{matrix} a & b \\ c & d \end{matrix} \right| = \det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc.$

Is there a similar "number test" for 3×3 or larger matrices?

Determinant of a 3×3 matrix

$\{a_{ij}\}$: ith row
jth column

$$\text{If} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then $\det(A)$ is the following:

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$-$ $-$ $-$ $+$ $+$ $+$

negative positive

$$\begin{array}{cc} a & b \\ c & d \end{array}$$

$-$ $+$

Ex: $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ $\det(A) = (-2) + (-2) + (0) - (2) - (2) - (0)$

$= -8$

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = -8$$

Note: This idea only works for 2×2 and 3×3, not 4×4 or larger!

det of 4×4 has 24 terms

There is another way to express the determinant, in terms of *smaller-sized determinants*:

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix} \qquad \begin{bmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix}$$

cofactor expansion

note the signs: + − +

Examples

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\det(A) = 1\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= \cdots \quad = -8$$

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\det(A) = 1\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + (3)\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix}$$

$$= 1(0) + 1(-1) + 3(0) = \boxed{-1}$$

Definition: Minor (4.1.4) For any $n \times n$ matrix M_{ij} is the determinant of the submatrix formed by deleting $i-$th row and $j-$th column of A. This determinant is called the (i,j)-minor of A or M_{ij}.

$$\begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 2$$

Examples

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$M_{11} = \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = 0$

$M_{32} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$

$\det(A) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

Definition: Cofactor (4.1.4) The (i,j)−cofactor (C_{ij}) of A is $C_{ij} = (-1)^{i+j}M_{ij}$.

$$C_{ij} = \begin{cases} M_{ij} & \text{if } i+j \text{ even} \\ -M_{ij} & \text{if } i+j \text{ odd} \end{cases}$$

$\det(A) = \underset{+}{a_{11}C_{11}} + \underset{-}{a_{12}C_{12}} + \underset{+}{a_{13}C_{13}}$

3x3

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

checkerboard

Theorem 4.1.5

The determinant of an $n \times n$ matrix is given by:

$$\det(A) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - \dots (-1)^{n+1} a_{1n}M_{1n}$$
$$= a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

Note that this is not the only formula. For 3×3, there are 5 others:

cofactor expansions

$$\begin{aligned}
\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{(1st row)} \\
&= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{(2nd row)} \\
&= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} && \text{(3rd row)} \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} && \text{(1st column)} \\
&= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} && \text{(2nd column)} \\
&= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} && \text{(3rd column)}
\end{aligned}$$

We can expand on any row or column! (See Theorem 4.1.5)

Ex: $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$

$$\boxed{\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}}$$

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

$$= -(-1)M_{12} + \cancel{0 M_{22}} - \cancel{0 M_{32}}$$

$$= M_{12} = \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = \boxed{-1}$$

This suggests a strategy: **do the row or column with the most zeros.**

Special cases

What about special cases like diagonal, upper triangular, lower triangular?

upper triangular

$$A = \begin{bmatrix} 1 & 3 & -4 & 7 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

1st col

$$\det(A) = a_{11}M_{11} = (1)\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & -2 \end{vmatrix} = (1)(-1)\begin{vmatrix} 2 & -5 \\ 0 & -2 \end{vmatrix}$$

$$= (1)(-1)(2)(-2) = \boxed{4}$$

if A is diagonal/upper/lower

then $\det(A)$ is <u>product of diagonal entries</u>