MATH 232 Section 4.4 pre-lecture comments

Lecture Outline

Given a square matrix A we wish to find a pair (or pairs) of a scalar λ and an associated vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

That is, the product $A\mathbf{v}$ is parallel to $\mathbf{v}$. We will consider the general problem as well as special cases.

New terminology

1. eigenvalue
2. eigenvector
3. eigenspace
4. characteristic polynomial

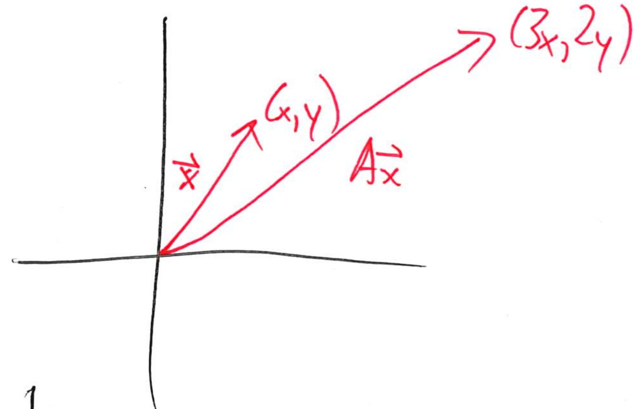
Let A be an $n \times n$ matrix. We can consider A as a "function" that transforms a vector $\mathbf{x}$ (in $\mathbf{R}^n$) to another vector $\mathbf{x}'$ (also in $\mathbf{R}^n$) as follows:

$$\underbrace{A\vec{\mathbf{x}}}_{\text{input}} = \underbrace{\vec{\mathbf{x}}'}_{\text{output}}$$

matrix multiplies a vector

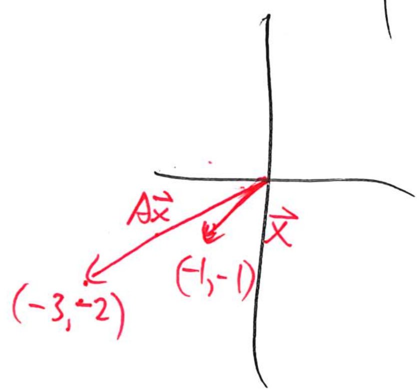
Ex: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Describe what A does to vector $\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$ in general.

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$$

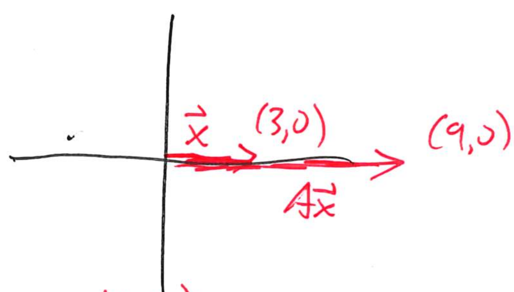


A multiplies x-value by 3
and y-value by 2

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$



$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$



$3\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ (scalar multiple)

In many applications it is important to consider when A transforms $\mathbf{x}$ to a scalar multiple of $\mathbf{x}$. That is, we look for non-trivial ($\mathbf{x} \neq \mathbf{0}$) solutions to:

$$A\vec{\mathbf{x}} = \underbrace{\lambda\vec{\mathbf{x}}}_{\text{scalar multiple of } \vec{x}},$$

This brings up the concept of eigenvalues and eigenvectors.

Eigenvalue/Eigenvector (p. 211)

Let A be a $n \times n$ matrix. If there exists a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

then λ is called an **eigenvalue** of A, and any nonzero $\mathbf{x}$ that satisfies the above is an **eigenvector** corresponding to λ.

Note: λ is allowed to be 0.

Examples

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$ not scalar multiple

2, 3 are eigenvalues

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ eigenvector for $\lambda = 3$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ———— $\lambda = 2$

Question: How do we find eigenvalues and eigenvectors?

Recall we are looking for *non-trivial* solutions to $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$. So, we write it as

n×n identity matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & & \\ 0 & & \ddots & \\ \vdots & & & 1 \end{bmatrix}$$

$$A\mathbf{x} = \lambda\mathbf{x}$$

$$A\mathbf{x} = \lambda I\mathbf{x}$$

$$(\lambda I - A)\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

If there is a non-trivial solution $\vec{\mathbf{x}}$, what does that say about $\det(\lambda I - A)$?

$\det(\lambda I - A) = 0$

non trivial solution to $B\vec{x} = \vec{0}$ $\iff$ $\det(B) = 0$

by Invertible Matrix Theorem (both equivalent to: B is not invertible)

Examples

Find λ so that $\det(\lambda I - A) = 0$:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-2 \end{bmatrix}$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 0 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-3)(\lambda-2) = 0$

$\lambda = 3$ or $\lambda = 2$

$\boxed{\lambda_1 = 3, \ \lambda_2 = 2}$ eigenvalues

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\lambda I - A = \begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-1)(\lambda-1) - (-1)(-1) = \lambda^2 - 2\lambda + 1 - 1$

$= \cancel{1 +} \lambda^2 - 2\lambda = \lambda(\lambda-2) = 0$

$\boxed{\lambda_1 = 0, \ \lambda_2 = 2}$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

cofactor expansion

$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 0 & 0 \\ 0 & \lambda-1 & 2 \\ -1 & 0 & \lambda-1 \end{vmatrix} = (\lambda-3) \begin{vmatrix} \lambda-1 & 2 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-3)(\lambda-1)^2 = 0$

$\lambda_1 = 3 \quad \lambda_2 = 1$

$\underbrace{\lambda_2 = \lambda_3 = 1}_{\text{multiple eigenvalue}}$

We call $\det(\lambda I - A)$ the *characteristic polynomial* of A. $(\lambda-3)(\lambda-1)^2$
Characteristic polynomial is a polynomial in λ.

- If A is $n \times n$, what is the degree of $\det(\lambda I - A)$? degree is n

$(\lambda-3)(\lambda-1)^2$ has deg $= 3$ A is 3×3 mtx

- How many different eigenvalues can A have? at most n (real) eigenvalues

Algebraic multiplicity

If the characteristic ~~equation~~ polynomial of $n \times n$ matrix A is $(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_k)^{m_k}$, then: $(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2} \cdots (\lambda-\lambda_k)^{m_k}$

- $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues of A.
- $m_1 + m_2 + \ldots + m_k = n$ $\leftarrow$ degree of det$(\lambda I - A)$ (characteristic polynomial)
- We say λ_i has **(algebraic) multiplicity** m_i.

$(\lambda-3)(\lambda-1)^2$ $\lambda_1 = 3$ has multiplicity 1

$\lambda_2 = 1$ has multiplicity 2

$\lambda_2 = \lambda_3 = 1$

Eigenspace

Eigenspace (p. 212)

always nontrivial

Let A be a $n \times n$ matrix. If λ is an eigenvalue of A, then the **eigenspace** corresponding to λ is the set of all possible solutions (including $\mathbf{0}$) to:

$$(\lambda I - A)\vec{\mathbf{x}} = \vec{\mathbf{0}}.$$

In other words, eigenspace corresponding to λ is the set of all eigenvectors corresponding to λ, together with $\vec{\mathbf{0}}$.

Note: Eigenspace is a subspace of $\mathbf{R}^n$.

Ex: For $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, what is the eigenspace corresponding to $\lambda = 3$?

$\lambda I - A = \begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-2 \end{bmatrix}$

$\lambda = 3$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \begin{array}{cc} x & y \end{array} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$y = 0$
$x = t$

$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t}$

Eigenspace for $\lambda = 3$

$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

What is the eigenspace corresponding to $\lambda = 2$?

$\lambda = 2$: $\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$y = t$
$x = 0$

$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t}$

Eigenspace for $\lambda = 2$

$\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

2×2 matrices

What are the eigenvalues for a 2×2 matrix?

$ax^2+bx+c=0$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-a & -b \\ -c & \lambda-d \end{vmatrix} = (\lambda-a)(\lambda-d)-(-b)(-c)$$

$$= \lambda^2 - a\lambda - d\lambda + ad - bc$$

$$= \lambda^2 - \underbrace{(a+d)}_{\substack{tr(A)\\ (\text{trace})}}\lambda + \underbrace{ad-bc}_{\det(A)} = 0$$

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

The **trace** of a square matrix A, denoted $\operatorname{tr}(A)$, is the sum of the diagonal entries of A.

sum of diagonal entries

$$\text{Ex: } B = \begin{bmatrix} 1 & 0 & 2 \\ -5 & -1 & 10 \\ -7 & 2 & 3 \end{bmatrix} \quad tr(B) = 1 + (-1) + 3 = 3$$

Trace and determinant in terms of eigenvalues

If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (*repeated according to multiplicity*), then

1. $\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$ det is product of eigenvalues
2. $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ tr is sum of eigenvalues

Ex: $(\lambda-3)(\lambda-1)^2$ $\quad \lambda_1 = 3 \quad \lambda_2 = \lambda_3 = 1$

$tr(A) = 3+1+1 = 5 \qquad \det(A) = 3 \cdot 1 \cdot 1 = 3$

More on 2×2 matrices

For a 2×2 matrix, its characteristic polynomial is quadratic in λ.

How many (distinct, real) solutions can a quadratic equation have?

$ax^2+bx+c=0$ $\qquad x = \dfrac{-b \pm \sqrt{\boxed{b^2-4ac}}}{2a}$

2 solutions if $b^2-4ac>0$ $\qquad x^2-1=0 \Rightarrow x=-1, 1$

1 solution if $b^2-4ac=0$ $\qquad x^2+2x+1=0 \Rightarrow x=-1$

0 solution if $b^2-4ac<0$ $\qquad x^2+1=0$

A 2×2 matrix can have either:

1. Two distinct (real) eigenvalues of multiplicity 1.
2. One (real) eigenvalue of multiplicity 2.
3. Two *complex* eigenvalues of multiplicity 1.