MATH 232 Section 3.6 pre-lecture comments

Lecture Outline
Matrices with special forms come up frequently in applications and are used when solving linear systems.

For instance, suppose a matrix has the form

$$A = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

$*$ = anything (incl. 0)

upper triangular matrix

then the problem $A\mathbf{x} = \mathbf{b}$ can be solved very easily using back-substitution.

All matrices in this section are square!

New terminology

1. main diagonal
2. diagonal matrix
3. triangular matrix (upper triangular and lower triangular)
4. symmetric matrix and skew-symmetric matrix

Diagonal

The main diagonal of a square matrix consists of the entries a_{ii}. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

A diagonal matrix is one where $a_{ij} = 0$ if $i \neq j$

Properties All sums and products of diagonal matrices are diagonal.

Examples

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad AB = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \qquad B^{-1} \text{ undefined}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Property If no diagonal element is zero then a diagonal matrix is invertible.

Example

($*$ means any number, including 0)

Triangular matrices come in two types:

Upper: $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ and Lower: $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$

zeros below the main diagonal   zeros above the main diagonal

Definition

A matrix $U = (u_{ij})$ is *upper triangular* if entries $u_{ij} = 0$ for $i > j$. A matrix $L = (l_{ij})$ is *lower triangular* if entries $l_{ij} = 0$ for $i < j$.

$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ $L = \begin{bmatrix} -1 & 0 & 0 \\ 1.5 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

U is invertible L is <u>not</u> invertible

Properties

1) Sums and products of upper triangular matrices are upper triangular.
2) Sums and products of lower triangular matrices are lower triangular.
3) If the diagonal entries of U are all non-zero then U is invertible (similarly for L).

Note: Diagonal matrices are both upper triangular and lower triangular

Symmetric

A is **symmetric** if $A^T = A$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 5 \\ 0 & 5 & -6 \end{bmatrix}$$

A is ~~anti~~ skew-**symmetric** if $A^T = -A$

$$\begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -6 \\ -5 & 6 & 0 \end{bmatrix}$$

opposites

diagonal is 0

Property: If A is symmetric and invertible then A^{-1} is symmetric. (Why?) if $A^T = A$

$$(A^{-1})^T = (A^T)^{-1} = A^{-1} \rightarrow (A^{-1})^T = A^{-1}$$

A^{-1} is symmetric

Remark: We will be interested later in matrices of the form A^TA and AA^T. Both of these are symmetric matrices.

symmetric

symmetric

$$\left.\begin{array}{l} A^T = A \\ A^T = -A \end{array}\right\} \rightarrow A = -A \Rightarrow A = 0 \text{ mtx}$$

Identify the following matrices by structure:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 0 & 2 & 3 \\ -2 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}$$

nothing (not skew sym)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

lower

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

diagonal / lower / upper / symmetric

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

nothing

MATH 232 Section 4.1 pre-lecture comments

Lecture Outline
We will learn how to compute **determinants** for general $n \times n$ matrices. This section gives the definition of determinants in terms of its *cofactor expansions*.

Important: A is invertible if and only if $\det(A) \neq 0$.

All matrices in this chapter are square matrices.

New terminology

1. determinant
2. minor
3. cofactor

Recall the inverse of the 2×2 matrix. Given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In particular, A has an inverse if and only if $ad - bc \neq 0$.

This quantity (ad-bc) is called the determinant. (2×2 mtx)

Determinant of a 2×2 matrix

Determinant $\rightarrow$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

Is there a similar "number test" for 3×3 or larger matrices?

Determinant of a 3×3 matrix

$\{a_{ij}\}$: ith row, jth column

$$\text{If} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then $\det(A)$ is the following:

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$\begin{matrix} a & b \\ c & d \end{matrix}$ $-$ $+$

$-$ $-$ $-$ negative $+$ $+$ $+$ positive

Ex: $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

$\det(A) = (-2) + (-2) + (0) - (2) - (2) - (0)$

$= -8$

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = -8$$

Note: This idea only works for 2×2 and 3×3, not 4×4 or larger!

det of 4×4 has <u>24</u> terms

There is another way to express the determinant, in terms of *smaller-sized determinants*:

$$\det(A) = a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})} - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix} \qquad \begin{bmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix}$$

cofactor expansion

note the signs: + − +

Examples

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= \ldots = -8$$

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$\det(A) =$

Definition: Minor (4.1.4) For any $n \times n$ matrix M_{ij} is the determinant of the submatrix formed by deleting $i-$th row and $j-$th column of A. This determinant is called the (i, j)-minor of A or M_{ij}.