MATH 232 Section 4.2 pre-lecture comments

Lecture Outline
Section 4.2 is on properties of determinants, including a way to find the determinant of a matrix using row operations.

Important: A is invertible if and only if $\det(A) \neq 0$.

All matrices in this chapter are square matrices.

Properties of determinants A and B are $n \times n$ matrices.

$\det(AB) = \det(A)\det(B)$.

If A is invertible, then $\det(A^{-1}) = 1/\det(A)$

$\det(A^k) = (\det(A))^k$

$AA^{-1} = I$

$\det(AA^{-1}) = \det(I) = 1$

$\det(A)\det(A^{-1}) = 1$

lower upper

Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & -5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -10 & 20 \\ 0 & -1 & 1.5 \\ 0 & 0 & -2 \end{bmatrix}$.

What is $\det(AB)$? $\det(A^{-1})$? $\det(B^3)$?

$\det(A) = (2)(3)(4) = 24$ $\det(B) = (1)(-1)(-2) = 2$

$\det(AB) = \det(A)\det(B) = (24)(2) = 48$

$\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{24}$

$\det(B^3) = (\det B)^3 = 2^3 = 8$

Suppose that we want to calculate $\det(A)$. Recall that we can use row operations to turn A into its row echelon form R:

$$\underbrace{E_k \cdots E_3 E_2 E_1}_{\text{elementary matrices}} A = R$$

$$\det(E_k) \cdots \det(E_3)\det(E_2)\det(E_1)\det(A) = \det(R)$$

elementary matrices

R is an upper triangular matrix ($\det(R)$ isn't too hard to calculate). But what are $\det(E_1), \det(E_2), \ldots$?

Determinant of elementary matrices (Lemma 4.2.8) Let E be an elementary matrix.

1. If E corresponds to $r_i \leftarrow kr_i$, then $\det(E) = k$. scalar mult. by $k \neq 0$
2. If E corresponds to $r_i \leftrightarrow r_j$, then $\det(E) = -1$. switch rows i and j
3. If E corresponds to $r_i \leftarrow r_i + kr_j$, then $\det(E) = 1$. add multiple of one row to another

Ex: 1) $r_1 \leftarrow 2r_1$ $\quad E_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonal $\quad \det(E_1) = 2$

2) $r_1 \leftrightarrow r_2$ $\quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \det(E_2) = -1$ by cofactor expansion

3) $r_1 \leftarrow r_1 + 5r_3$ $\quad E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ upper triangular $\quad \det(E_3) = 1$

Row reduction to compute determinants Find $\begin{vmatrix} 2 & 4 & -2 & -4 \\ 0 & -3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 3 & 2 & 1 \end{vmatrix}$

$$A = \begin{bmatrix} 2 & 4 & -2 & -4 \\ 0 & -3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow[E_1,\ \det = \frac{1}{2}]{r_1 \leftarrow \frac{1}{2} r_1} \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

E_2 $r_3 \leftarrow r_3 - r_1$ $\det = 1$

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow[E_3,\ \det = 1]{r_4 \leftarrow r_4 + r_2} \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

E_4 $r_4 \leftarrow r_4 - 2r_3$ $\det = 1$

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & -7 \end{bmatrix} = R' \quad \text{(not REF) (but good enough)}$$

$\det(R') = 42$

$E_4 E_3 E_2 E_1 A = R'$

$(1)(1)(1)\left(\frac{1}{2}\right)\det(A) = 42 \implies \det(A) = \boxed{84}$

Other properties

$\det(A) = \det(A^T)$

Because of this, "row" can be replaced with "column" below:

1. If A has a row (column) of zeros then $\det(A) = 0$. $\begin{vmatrix} & & & \\ 0 & 0 & \cdots & 0 \\ & & & \end{vmatrix} = 0$
2. If two rows (columns) of A are the same or are scalar multiples of each other, then $\det(A) = 0$. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ & & \end{vmatrix} = 0$
3. If B is obtained from A by multiplying a row (column) by k, then $\det(B) = k\det(A)$. $\begin{vmatrix} 3 & 6 \\ 0 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 3(1) = 3$
4. If B is obtained from A by switching two rows (columns) then $\det(B) = -\det(A)$. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$
5. If B is obtained from A by adding a multiple of one row (column) to another, then $\det(B) = \det(A)$. $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$

not scalar multiplication of matrix

$\det(cA) \neq c\det(A)$

if A is $n \times n$: $\underline{\det(cA) = c^n \det(A)}$

Efficient computation of determinants

Example $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 3 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ $\det(A) = ?$

$$\begin{vmatrix} 1 & 2 & -2 \\ 1 & 3 & 1 \\ 2 & 3 & -1 \end{vmatrix} \underset{(r_3 \leftarrow r_3 - 2r_1)}{\overset{(r_2 \leftarrow r_2 - r_1)}{=}} \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & -1 & 3 \end{vmatrix} \overset{(r_3 \leftarrow r_3 + r_2)}{=} \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 6(1) = 6$$

(optional step)

you can write "=" here because of determinants

in general, don't write "=" unless they are actually equal

more efficient than cofactor expansion for large matrices

Note that square matrix A is invertible if and only if REF of A has no zero rows.

A is invertible then REF $A = \begin{bmatrix} 1 & & * \\ 0 & \ddots & \\ 0 & 0 & 1 \end{bmatrix}$

- Determinant of REF is nonzero exactly when there are no zero rows.
- Determinant of elementary matrices is always nonzero.

So this gives:

Determinant test for invertibility - Theorem 4.2.4

A square matrix A is invertible if and only if $\det(A) \neq 0$.

This gives us another entry in the Invertible Matrix Theorem:

Theorem 4.2.7

If A is an $n \times n$ matrix, then the following statements are equivalent:

1. The reduced row echelon form of A is I_n.
2. A can be written as a product of elementary matrices.
3. A is invertible.
4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution: $\mathbf{x} = 0$.
5. $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.
6. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.
7. The column vectors of A are linearly independent.
8. The row vectors of A are linearly independent.
9. $\det(A) \neq 0$.

WARNING! In general, $\det(A+B) \neq \det(A) + \det(B)$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \det(A) = 1 \quad \det(B) = 1$

$\det(A+B) = 0$

$\det(A+B) \neq \det(A) + \det(B)$