MATH 232 Section 2.1 pre-lecture comments

Today we discuss linear systems, their solutions and how to solve them.

New terminology:

- linear system
- homogenous
- solution
- consistent
- elementary row operation
- augmented matrix

Linear equation A **linear equation** in n variables $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

$x_1, x_2, \ldots x_n$ power 1
not multiplied with each other

where $a_1, a_2, \ldots, a_n$ and b are constants s.t. not all a's equal to 0.
If $b = 0$, then the equation is called a **homogeneous linear equation**.

Example. Which of these are linear equations? Of those, which are homogeneous?

linear — not homogeneous

1. $2(x_1 - 0.01x_2 + 2) + x_3(1 + \frac{2}{7}) = 0 \rightarrow 2x_1 - 0.02x_2 + \frac{9}{7}x_3 = -4$
2. $(x_1 - 0.01x_2 + 2)(x_3 + \frac{2}{7}) = 0$ $\quad x_1 x_3$ not linear
3. $\sqrt{x_1} + x_2 - x_3 = 5$ $\quad \sqrt{x_1}$ not linear
4. $10y_1 + 2y_2 - 17y_3 = 0$ $\quad$ linear homogeneous

Question. What do linear equations in (a) R^2 and (b) R^3 represent?

a) $Ax + By = C$ line in $\mathbb{R}^2$

b) $Ax + By + Cz = D$ plane in $\mathbb{R}^3$

System of linear equations A finite set of linear equations is called a **system of linear equations** or a **linear system**. The variables in such a system are called **unknowns**.

In general, a linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be expressed as

$$\text{system of equations}\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

Solution set A **solution** to a linear system is a sequence of n numbers $s_1, s_2, \ldots, s_n$ that when substituted for $x_1, x_2, \ldots, x_n$, respectively, satisfies all equations. We can represent each solution as a point/vector in n-space: $(s_1, s_2, \ldots, s_n)$.
The set of all solutions is called the **solution set**.

Examples

$$\left\{\begin{array}{l} x+y=1 \rightarrow x+x=1 \rightarrow x=\frac{1}{2} \\ x-y=0 \rightarrow x=y \qquad\quad y=\frac{1}{2} \end{array}\right.$$

Solution: $\left(\frac{1}{2}, \frac{1}{2}\right)$ Solution set: $\left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ ← only one solution

2 eqs 2 unknowns

$$\begin{array}{l} 2x+3y+z=1 \\ 2x-y+2z=0 \end{array}$$

2eq 3 unknowns

$$\begin{array}{l} x+y=0 \\ x-2y=4 \\ x+14y=-\pi \end{array}$$

3 eqs 2 unknowns

Example *The* solution of $2x = -3$ is $x = -\frac{3}{2}$; $2(-\frac{3}{2}) = -3$

Some solutions of $2x_1 - x_2 = 0$ are:

1 eq 2 unknowns

$x_1 = 1, \quad x_2 = 2$

$x_1 = 2 \quad x_2 = 4 \qquad x_1 = -1 \quad x_2 = -2$

$x_1 = 0 \quad x_2 = 0$ etc.

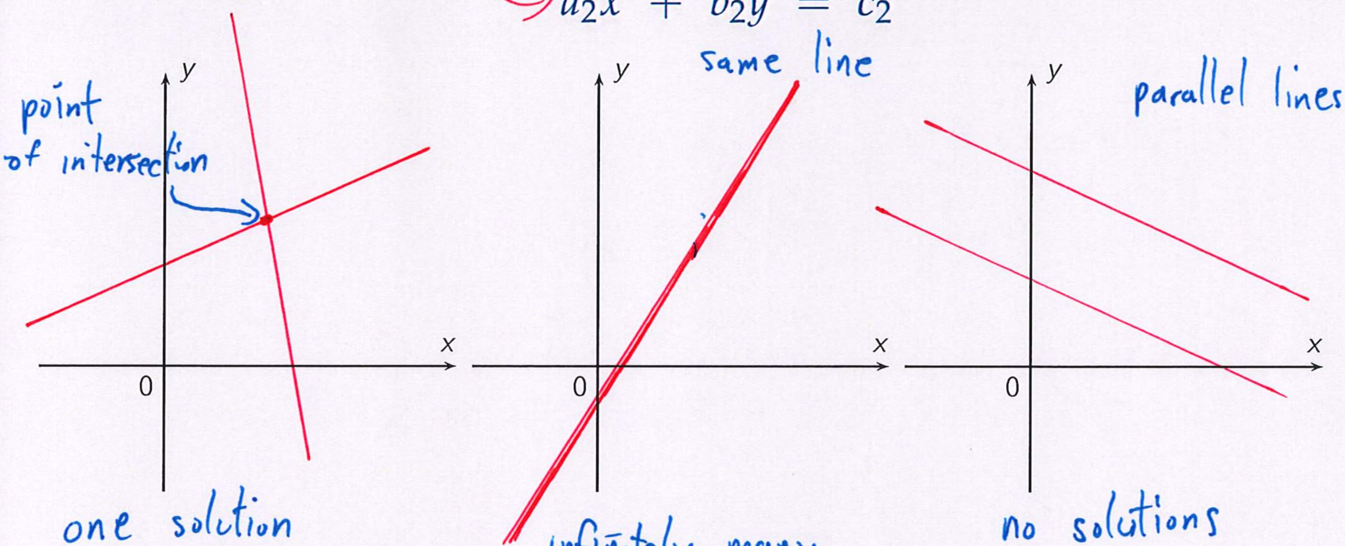
infinitely many solutions

There are *no* solutions of $\underbrace{0x_1 + 0x_2 - 0x_3}_{\text{LHS is always } 0} = 1$.

Example. Consider a linear system of two equations in two unknowns x and y. What are possible solution sets of this system?

lines

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$



one solution

infinitely many solutions

no solutions

Theorem Every linear system has zero, one or infinitely many solutions.

A linear system with ~~a~~ at least one solution is called **consistent**, else it is **inconsistent**.

at least one solution

no solution

Examples Identify each system as consistent or inconsistent.

consistent	consistent	inconsistent	consistent	subtract
$2x = -2$	$x + y = 1$ $x - y = 1$	$x + y = 1$ $2x + 2y = 1$	$x + y = 1$ $x - 2y = 1$ $2x - y = 2$	$3y = 0$ $y = 0$ $x = 1$
$x = -1$	add eqns	$2x + 2y = 2$	check	

consistent $\quad 2x = 2 \quad x = 1 \quad y = 0 \quad$ check $2(1) - 0 = 2$ ✓

Given any linear system, we can use the following operations to make a new system that has *exactly* the same solutions:

1. Multiply one of the equations by a non-zero scalar;
2. Interchange two equations;
3. Add a multiple of one equation to another.

replace 2nd equation with sum of 1st and 2nd

Demonstration: The solution remains the same

$x + y = 1 \qquad x = 1$

$x - y = 1 \qquad y = 0$

same solution

1 multiply by 2

$2x + 2y = 2 \qquad x = 1$

$x - y = 1 \qquad y = 0$

2

$x - y = 1 \qquad x = 1$

$x + y = 1 \qquad y = 0$

switch equations

3

$x + y = 1 \qquad x = 1$

$2x + 2y = 2 \qquad y = 0$

Given a linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

we can abbreviate the system by ignoring variables and $=$ symbols using the matrix notation:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

This matrix is called the **augmented matrix for the linear system**.

Example. What are the augmented matrices of the systems

$$\begin{aligned} x + y &= 1 \\ x - 2y &= 1 \\ 2x - y &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{array}\right]$$

$x = 1$

$y = 2$

$$\begin{aligned} x \quad &= 1 \\ y &= 2 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

and

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 2x_3 & + & 3x_4 & = & 4 \\ 2x_1 & + & 3x_2 & - & x_3 & + & 5x_4 & = & 5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & -1 & 5 & 5 \end{array}\right]$$

Modifications of the linear system correspond to **elementary row operations** on the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{array}\right]$$

1. Multiply one of the equations by a non-zero scalar;
2. Interchange two equations;
3. Add a multiple of one equation to another.

1. Multiply a row by a non-zero scalar;
2. Interchange two rows;
3. Add a multiple of one row to another.

e.g. starting with $\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{array}\right]$

multiply row 1 by 5

① $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{r_1 \leftarrow 5r_1} \left[\begin{array}{cc|c} 5 & 5 & 5 \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{array}\right]$

switch rows 1 and 3

② $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{cc|c} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{array}\right]$

~~add $-2r_2$ to row~~

③ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - 2r_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & -3 & 0 \end{array}\right]$

add (−2 times row 1) to row 3

Examples Solve the linear systems

first column is x, second column is y

$$\begin{aligned} x+y &= 1 \\ x-y &= 1 \end{aligned} \qquad \begin{array}{c} r_1 \\ r_2 \end{array}\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + r_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 2 \end{array}\right]$$

add row 1 to row 2

$$\xrightarrow{r_2 \leftarrow \frac{1}{2} r_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right]$$

multiply row 2 by $\frac{1}{2}$

$\cancel{x+0y=1}$

$$\cancel{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 2x_3 & + & 3x_4 & = & 4 \\ 2x_1 & + & 3x_2 & - & x_3 & + & 5x_4 & = & 5 \end{array}}$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

switch row 1 and row 2

$$\xrightarrow{r_2 \leftarrow r_2 - r_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \begin{array}{l} \to x=1 \\ \to y=0 \end{array}$$

subtract row 1 from row 2

solution:

$$\begin{aligned} x+y &= 1 \\ x-y &= 1 \\ 2x &= 1 \end{aligned}$$