MATH 232 Section 5.1 pre-lecture comments

Lecture Outline

Today we learn about Markov chains, a type of dynamical system. These model cases where a system evolves based on transitions from one state to another.

New terminology

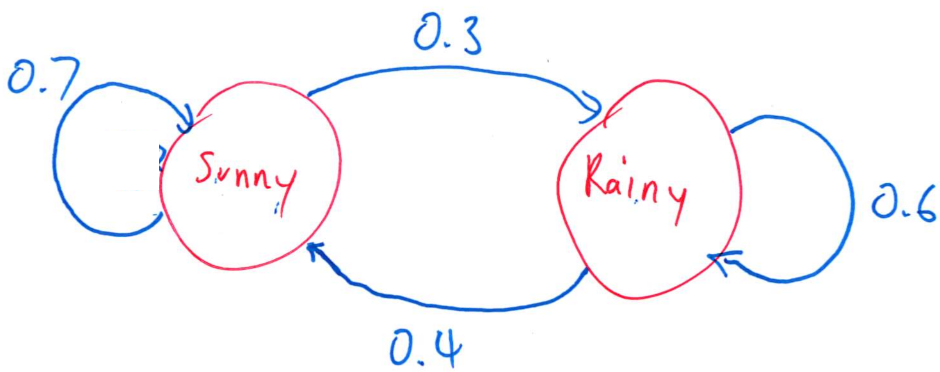
1. Markov chain
2. probability vector
3. stochastic matrix
4. transition matrix / Markov matrix
5. steady-state vector

Example

Suppose that the weather each day (either sunny or rainy) is determined as follows:

- If it is sunny today, then tomorrow there is a 70% chance that it will be sunny, and a 30% chance that it will rain.
- If it is raining today, then tomorrow there is a 60% chance that it will rain, and a 40% chance that it will be sunny.

Draw a diagram that models the probabilities of this system.



We can write it in terms of a transition matrix:

$$\begin{array}{ccc} & & \text{From} \\ & & \begin{array}{cc} S & R \end{array} \\ \text{To} & \begin{array}{c} S \\ R \end{array} & \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \end{array}$$

Note: The columns of the transition matrix add up to 1.

Probability vector and stochastic matrix

A **probability vector** $p = (p_1, p_2, \ldots, p_n)$ is a vector such that

1. Each p_i is a non-negative number.
2. $p_1 + p_2 + \ldots + p_n = 1.$

A **stochastic matrix** is a square matrix whose columns are all probability vectors. $\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$

By the book's convention, we write probability vectors as column vectors.

Examples: Probability vector

Which of the following is a probability vector? (PV)

$$\begin{bmatrix} 0.8 \\ 0.2 \\ 0.1 \end{bmatrix} \qquad \begin{bmatrix} 1.1 \\ -0.2 \\ 0.1 \end{bmatrix} \qquad \begin{bmatrix} 0.9 \\ 0 \\ 0.1 \end{bmatrix}$$

not PV (add to 1.1) | not PV | PV

Markov chain A **Markov chain** is a system which can be specified as follows:

1. n states (state 1, state 2, ..., state n). Ex: 2 states ① ②
2. A starting probability vector $\mathbf{x}_0$, where the ith entry of $\mathbf{x}_0$ represents the probability of starting in state i. Ex: $\begin{array}{cc} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} & \begin{array}{c} ① \\ ② \end{array} \end{array}$
3. A stochastic matrix P where the p_{ij} entry represents the probability of transitioning from state j to state i. Ex: $\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} ① & ② \end{array} \\ \text{To} \begin{array}{c} ① \\ ② \end{array} & \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix} \end{array}$

P is referred to as the **transition matrix** or **Markov matrix** for the Markov chain. Furthermore, we use $\mathbf{x}_1, \mathbf{x}_2, \ldots$ to denote the state vector after 1,2,... transition steps as follows:

- $\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P\mathbf{x}_1, \quad \mathbf{x}_3 = P\mathbf{x}_2, \quad \ldots$

Note that $\mathbf{x}_i$ is always a probability vector.

$$\underset{\text{start}}{\vec{x}_0} \xrightarrow{P} \vec{x}_1 \xrightarrow{P} \vec{x}_2 \xrightarrow{P} \vec{x}_3 \xrightarrow{P} \cdots$$

Example:

- On Sunday (initial day), there is a 100% chance of sun and 0% chance of rain.
- If it is sunny, then for the next day there is a 70% chance that it will be sunny, and a 30% chance that it will rain.
- If it is raining, then for the next day there is a 60% chance that it will rain, and a 40% chance that it will be sunny.

1. Write out the Markov chain in terms of its starting vector $\mathbf{x}_0$ and its transition matrix P.
2. What is the probability of sun/rain on Tuesday (two days later)?

1) state 1 = sunny (S)
state 2 = rainy (R)

$$\vec{x}_0 = \begin{array}{c|l} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{array}{l} S \\ R \end{array} \end{array} \qquad P = \begin{array}{c|cc} & S & R \leftarrow \text{From} \\ S & 0.7 & 0.4 \\ R & 0.3 & 0.6 \\ \text{To} \nearrow & & \end{array}$$

2) $$\vec{x}_1 = P\vec{x}_0 = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \quad \text{(Monday)}$$

$$\vec{x}_2 = P\vec{x}_1 = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.61 \\ 0.39 \end{bmatrix}\begin{array}{l} S \\ R \end{array} \quad \text{(Tuesday)}$$

61% chance of sun
39% chance of rain

Notice that: $\vec{x}_0 \to \vec{x}_1 \to \vec{x}_2 \to \ldots$

$\mathbf{x}_1 = P\mathbf{x}_0, \quad \mathbf{x}_2 = P\mathbf{x}_1 = P^2\vec{x}_0, \quad \mathbf{x}_3 = P\mathbf{x}_2 = P^2\vec{x}_1 = P^3\vec{x}_0 \quad \vec{x}_k = P^k\vec{x}_0$

What happens to the sequence $\mathbf{x}_0, \quad P\mathbf{x}_0, \quad P^2\mathbf{x}_0, \quad \ldots$? Does it converge?

Referred to as the *long-term behavior* of a Markov chain.

Ex: $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}.$

$\vec{x}_1 = P\vec{x}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$

$\vec{x}_3 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ doesn't converge

How do we ensure that it converges?

Regular A stochastic/transition matrix is **regular** if

- A has all positive entries, or
- There is some positive power A^k $(k>0)$ that has all positive entries.

A Markov chain is **regular** if its transition matrix is regular.

Which of the following is regular?

not regular
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
I_3

$I_3^2 = I_3$

always the same
cannot get rid of 0's

regular
$$\begin{bmatrix} 0.1 & 0.6 \\ 0.9 & 0.4 \end{bmatrix}$$
$$\begin{bmatrix} + & + \\ + & + \end{bmatrix}$$
all positive

not regular
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
switch between $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
cannot get rid of 0's

regular
$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$
$$\begin{bmatrix} + & + & 0 \\ 0 & + & + \\ + & 0 & + \end{bmatrix}\begin{bmatrix} + & + & 0 \\ 0 & + & + \\ + & 0 & + \end{bmatrix} = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$$
all positive

Convergence If P is a regular stochastic/transition matrix:

1. $\lambda_1 = 1$ is an eigenvalue of P with multiplicity 1, and
2. All other eigenvalues are smaller; that is, they satisfy $|\lambda_i| < 1$.

That means $\mathbf{x}_0, \quad P\mathbf{x}_0, \quad P^2\mathbf{x}_0, \quad \ldots$ converges!

Long-term behavior of a regular Markov chain If a regular Markov chain has transition matrix P, then:

1. There is a unique probability vector $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$.
2. For any initial probability vector $\mathbf{x}_0$ the sequence

$$\mathbf{x}_0, \quad P\mathbf{x}_0, \quad P^2\mathbf{x}_0, \quad \ldots$$

   always converges to $\mathbf{q}$.

$\mathbf{q}$ is referred to as the **steady-state vector** for the Markov chain. Note that $\mathbf{q}$ is an eigenvector corresponding to $\lambda_1 = 1$.

Example:

- If it is sunny, then for the next day there is a 70% chance that it will be sunny, and a 30% chance that it will rain.
- If it is raining, then for the next day there is a 60% chance that it will rain, and a 40% chance that it will be sunny.

What is the steady-state vector $\mathbf{q}$ for this Markov chain? (Remember that $\mathbf{q}$ is an eigenvector of P for $\lambda_1 = 1$.)

From

$$\begin{array}{cc} & \begin{array}{cc} S & R \end{array} \\ \text{To} \begin{array}{c} S \\ R \end{array} & \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \end{array} = P$$

$P\vec{q} = I\vec{q}$

$P\vec{q} = \vec{q}$

$(I-P)\vec{q} = \vec{0}$

$$I - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.3 & -0.4 \\ -0.3 & 0.4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0.3 & -0.4 & 0 \\ -0.3 & 0.4 & 0 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + r_1} \left[\begin{array}{cc|c} 0.3 & -0.4 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$y = t$

$x = \frac{4}{3}t$

$$\vec{q} = \begin{bmatrix} \frac{4}{3}t \\ t \end{bmatrix}$$

probability vector: $\frac{4}{3}t + t = 1 \rightarrow t = \frac{3}{7}$

$$\vec{q} = \begin{array}{c} \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix} \end{array} \begin{array}{c} S \\ R \end{array}$$