MATH 232 Section 2.3 pre-lecture comments

Today we will be discussing some applications of linear algebra.

A huge number of problems in science and engineering (such as GPS, electric circuits, etc.) are best modelled with linear algebra. Linear algebra is also the backbone of almost all numerical algorithms–we solve nonlinear problems by approximating them with linear ones.

Linear algebra is fundamental to many machine learning and data science algorithms.

Today we will focus on problems where you can write down and solve simple systems.

Problem 1: Food mixtures.

How do we make a meal with different ingredients that satisfies certain nutritional requirements?

quantity: per 100g

Example Suppose that, per ~~100g~~ quantity, we have

- ▶ Beans: Carbohydrates: 5g, Protein: 2 g, Fat: 1g
- ▶ Cheese: Carbohydrates: 0g, Protein: 24 g, Fat: 30g
- ▶ Tortilla flour: Carbohydrates: 30g, Protein: 6 g, Fat: 3g

Question: Can we make a tortilla with 50g Carbohydrates, 34g Protein and 35g Fat?

Let x = quantity of beans

y = ——— cheese

z = ——— tortilla flour

carbohydrates: $5x + 0y + 30z = 50$

protein: $2x + 24y + 6z = 34$

fat: $x + 30y + 3z = 35$

$$\left[\begin{array}{ccc|c} 5 & 0 & 30 & 50 \\ 2 & 24 & 6 & 34 \\ 1 & 30 & 3 & 35 \end{array}\right]$$

$\hookrightarrow$ ...

$x = 0$ no beans

$y = 1$ 1 quantity (100g) of cheese

$z = 5/3$ $5/3$ quantity of tortilla flour (166.66.. g)

Problem 2: Mixing chemicals

2 of H, 1 of S, 4 of O

Example: Balance the chemical equation

$$NaOH \quad + \quad H_2SO_4 \quad \longrightarrow \quad Na_2SO_4 \quad + \quad H_2O$$

1 of Na, 1 of O, 1 of H

$$x_1 NaOH + x_2 H_2SO_4 \longrightarrow x_3 Na_2SO_4 + x_4 H_2O$$

Balance: x_1, x_2, x_3, x_4 variables (unknowns)

Na: $x_1 = 2x_3$

O: $x_1 + 4x_2 = 4x_3 + x_4$

H: $x_1 + 2x_2 = 2x_4$

S: $x_2 = x_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -1 & 0 \\ 0 & \boxed{1} & 0 & -1/2 & 0 \\ 0 & 0 & \boxed{1} & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free

Let $x_4 = t$

$x_3 = \frac{1}{2}t$

$x_2 = \frac{1}{2}t$

$x_1 = t$

must be positive whole numbers

Choose $t = 2$: $\underbrace{x_1 = 2, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 2}$

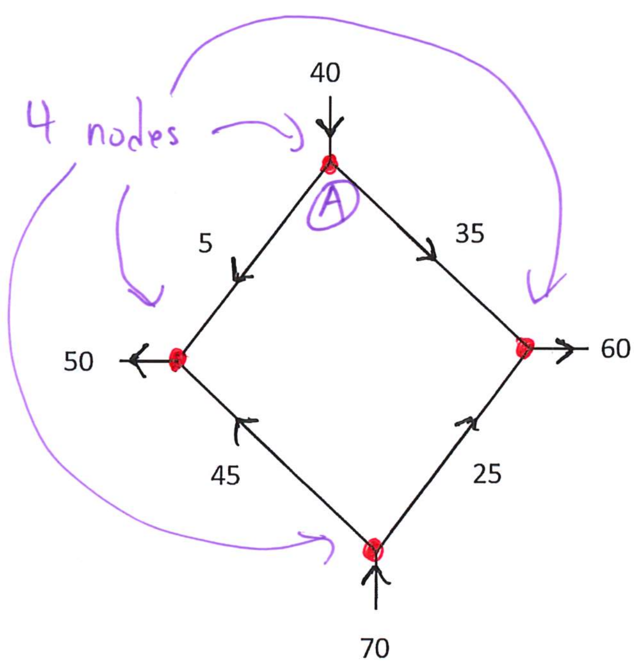
$$2NaOH + H_2SO_4 \longrightarrow Na_2SO_4 + 2H_2O$$

A **flow network** is a collection of **branches** through which something "flows". The branches meet at **nodes**.

Examples: electricity flowing through wires, liquid flowing through pipes, traffic flowing on roads, etc. Most *flow networks* satisfy the following properties:

1. **One-directional flow:** At any moment, the flow in each branch is in one direction only.
2. **Flow conservation at nodes:** The rate ~~of flow~~ of flow into each node is equal to the rate of flow out of the node. flow into node = flow out of node
3. **Flow conservation in the network:** The rate of flow in the network is equal to the rate of flow out of the network. total flow in = total flow out

In a typical problem, the rate of flow through some branches of the network is known, and we want to find the rates and directions of the flows for the remaining branches.



Suppose not all flows are known.

e.g. at node A:

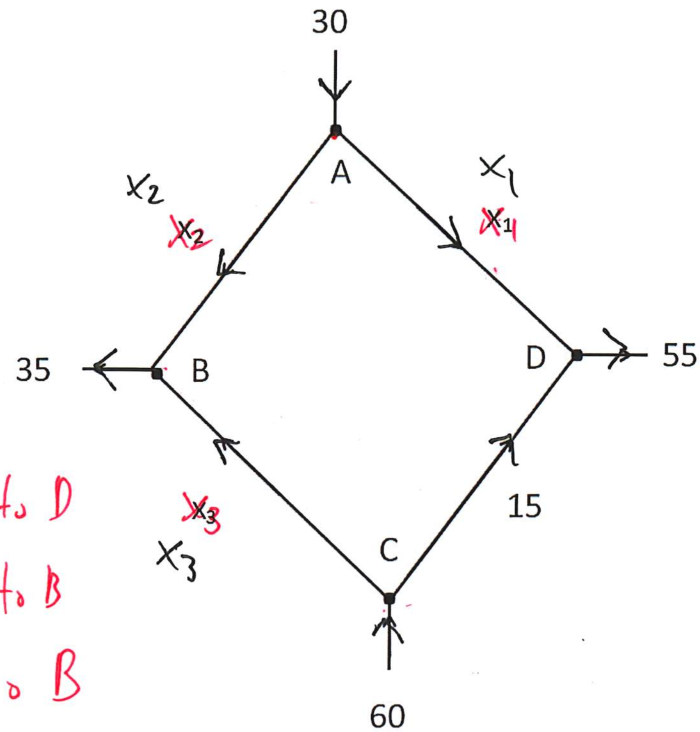
flow in = 40 same

flow out = 5+35 = 40

system:

total flow in = 40+70 = 110

total flow out = 50+60 = 110 } same



$x_1 + x_2 = 30$
$x_2 + x_3 = 35$
$x_3 + 15 = 60$
$x_1 + 15 = 55$

x_1: flow from A to D
x_2: —— A to B
x_3: —— C to B

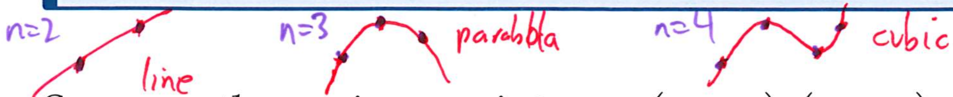
$x_1 = 40$
$x_2 = -10$
$x_3 = 45$

10 from B to A
(reversed)

Variables What are the variables?

Equations What equations describe the flow at each node?

Theorem Given any n points in R^2 that have different x-coordinates, there is a unique polynomial of degree at most $n-1$ whose graph passes through all of these points.



Suppose the n given points are $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$. We are looking for a polynomial of degree at most $n-1$, so it must have the form

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$$

If, for each given point, we substitute the values of its coordinates for x and y into the equation above, we get n **linear** equations in unknowns $a_0,a_1,\ldots,a_{n-1}$.

Example Find the system whose solution determines the polynomial passing through the points $(-1,-2),(0,2),(1,2),(2,4)$.

Cubic: $y = a_0 + a_1x + a_2x^2 + a_3x^3$

a_0, a_1, a_2, a_3 unknowns

$(-1,-2)$: $-2 = a_0 - a_1 + a_2 - a_3$

$(0,2)$: $2 = a_0$

$(1,2)$: $2 = a_0 + a_1 + a_2 + a_3$

$(2,4)$: $4 = a_0 + 2a_1 + 4a_2 + 8a_3$

$$\begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 4 \end{array}\right]$$

$\downarrow$

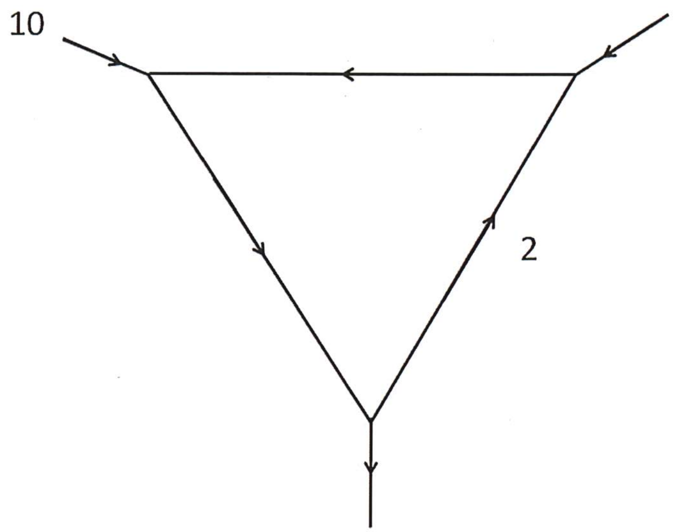
$\vdots$

$a_0 = 2, \quad a_1 = 1, \quad a_2 = -2, \quad a_3 = 1$

$$\boxed{y = 2 + x - 2x^2 + x^3}$$

Extra:

1. Find the flow in all the arcs of the following network.



- Identify variables.
- Balance equation at node A (flow in = flow out):
- Balance equation at node B:
- Balance equation at node C:
- Flow in and out of network (balanced):

Solve this system. How many feasible flows does this network have? Is the last equation (flow in and out of network) needed?

2. Find the cubic polynomial that passes through the points $(-2,2),(1,3),(0,2),(4,-2)$. Sketch these points and the curve.

   Identify variables:

   Equation for curve passing through $(-2,2)$:

   Equation for curve passing through $(1,3)$:

   Equation for curve passing through $(0,2)$:

   Equation for curve passing through $(4,-2)$:

   Solve this system.

3. Balance the following chemical equation;

$$C_3H_8 + O_2 \rightarrow CO_2 + H_2O$$

   Identify variables:

   Equation for atoms of carbon;

   Equation for atoms of hydrogen;

   Equation for atoms of oxygen;

   Solve this system.

4. A certain kind of fuel is to be prepared from 4 types of crude oil; Types 1, 2, 3 and 4. The mixture (the fuel) must satisfy the following requirements (per litre);

   octane: 8g
   carbon: 2g
   methane: 1g

   The properties of each type of crude oil are given below (per litre of oil);

grams of	Type 1	Type 2	Type 3	Type 4
octane	12	4	6	15
carbon	1	1.5	0.5	1
methane	0.4	1	0.5	0.6

   Formulate a mathematical model (system of linear equations) whose solution will specify the correct mixture of the crude oils to make exactly 10 litres of the fuel, if possible. If it is not possible to make 10 litres of the fuel, find out how much of the fuel *can* be made.