# Section 6.1 pre-lecture comments

**Lecture Outline**
We will think about matrices in terms of transformations.

Transformations are just functions $T(\mathbf{x}) = \mathbf{x}'$, where $\mathbf{x}$ (the input) is a vector in $\mathbf{R}^n$ and $\mathbf{x}'$ (the output) is a vector in $\mathbf{R}^m$.

**New/Old Terminology**

1. function/transformation/operator
2. domain
3. codomain
4. range
5. matrix transformation/operator
6. linear transformation/operator
7. standard matrix

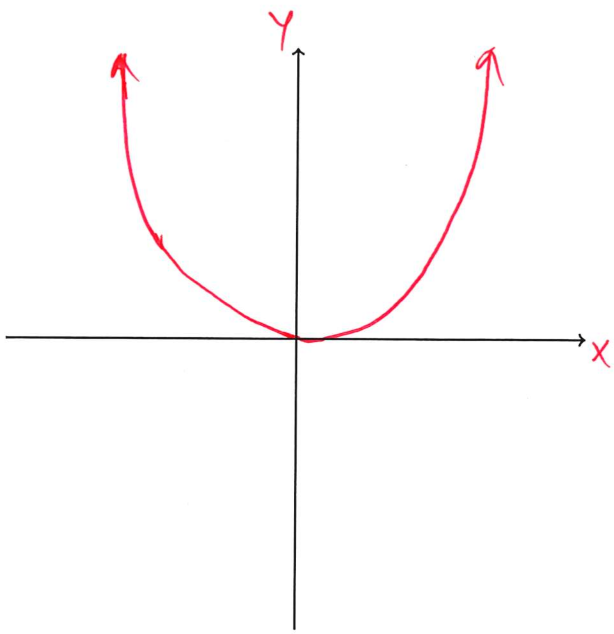
## Section 6.1 - Linear Transformations

In general, a **function** $f$ from a set $D$ to a set $E$ (denoted $f : D \to E$) is just a "rule" for taking each element of $D$ and assigning to it exactly one element of $E$.

- $D$ is called the **domain** (set of all inputs).
- $E$ is called the **codomain** (set which the outputs live in).
- The **range** is the set of all possible outputs of $f$ (not necessarily all of the codomain).

(from pre-calculus)

E.g. $f(x) = x^2$ defined on real numbers ($f : \mathbf{R} \to \mathbf{R}$).




Domain: $\mathbb{R}$

Codomain: $\mathbb{R}$

Range: $\{y \in \mathbb{R} : y \geq 0\}$

(non-negative real numbers)

Since this is linear algebra, we restrict domain/codomain to vector spaces:

**Transformation/operator**

A **transformation** is a function from $\mathbf{R}^n$ to $\mathbf{R}^m$. $(\mathbb{R}^n \to \mathbb{R}^m)$

An **operator** is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ (same domain and codomain). $\mathbb{R}^n \to \mathbb{R}^n$ (same dimension)

Note: $T$ is commonly used for transformation.

**Examples: transformations on vectors.**

$\mathbb{R}^2 \to \mathbb{R}^2$

1. Consider vectors in $\mathbf{R}^2$. $T(\vec{\mathbf{x}}) = 2\vec{\mathbf{x}}$ scales vector $\vec{x}$ by 2

Ex: $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}$ $\quad \begin{bmatrix}1\\2\end{bmatrix} \xrightarrow{T} \begin{bmatrix}2\\4\end{bmatrix}$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2x\\2y\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

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$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

2. Consider the function $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x-y\\2x+5y\\3x+4y\end{bmatrix}$.

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1-1\\2+5\\3+4\end{bmatrix} = \begin{bmatrix}0\\7\\7\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x-y\\2x+5y\\3x+4y\end{bmatrix} = \begin{bmatrix}1 & -1\\2 & 5\\3 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

matrix times $\begin{bmatrix}x\\y\end{bmatrix}$

**Matrix transformations (operators)** Let $A$ be an $m \times n$ matrix. The transformation $T_A$, defined as $T_A(\mathbf{x}) = A\mathbf{x}$, is called a **matrix transformation**. The domain is $\mathbf{R}^n$ and the codomain is $\mathbf{R}^m$ ($T_A : \mathbf{R}^n \to \mathbf{R}^m$).

- If $m = n$, the matrix transformation is called a matrix operator

Ex: $T_A$ where $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix}$. 2 rows by 3 columns

$m = 2$ $n = 3$

$T_A : \mathbb{R}^3 \to \mathbb{R}^2$

$$T_A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{in } \mathbb{R}^3} = \underbrace{\begin{bmatrix} 2x - y \\ 3x + y - 2z \end{bmatrix}}_{\text{in } \mathbb{R}^2}$$

**Linear transformation** A transformation $T : R^n \to R^m$ is called **linear**, if for all vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and scalar $k$, it satisfies

1. $T(\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2) = T(\vec{x}_1) + T(\vec{x}_2)$ (additivity)
2. $T(k\vec{\mathbf{x}}_1) = kT(\vec{x}_1)$ (homogeneity)

If $m = n$, then $T$ is called a linear operator

Ex: Is $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \end{bmatrix}$ a linear transformation? Yes.

Let $\vec{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\vec{x_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2 \end{bmatrix}$ same

$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1+x_2+y_2 \\ x_1+x_2 \end{bmatrix}$

satisfies additivity

Ex: Is $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2 \\ y \end{bmatrix}$ a linear transformation?

Note: $T(k\vec{x}_1) = kT(\vec{x}_1)$ for a linear transformation

If $k=0$: $T(\vec{0}) = 0\,T(\vec{x}_1) = \vec{0}$

$T(\vec{0}) = \vec{0}$ for a linear transformation

But above $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

So not a linear transformation

$T\left(k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix}\right)$

$= \begin{bmatrix} kx_1+ky_1 \\ kx_1 \end{bmatrix}$ same

$kT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = k\begin{bmatrix} x_1+y_1 \\ x_1 \end{bmatrix}$

satisfies homogeneity

Ex: Is $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$ a linear transformation? No, since:

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$$\underbrace{T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)}_{\begin{bmatrix} 4 \\ 4 \end{bmatrix}} \neq 2\underbrace{T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}_{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

Linear transformations:

- are linear in the variables (no powers/roots, no multiplying variables together) and

  $x^2$ $\sqrt{y}$ $xy$
- do not have added/subtracted constants. $x+1$

Eg $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

is a linear trans.

**Matrix transformations** Matrix transformations ($T(\mathbf{x}) = A\mathbf{x}$) are linear transformations.

$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2$

$A(k\vec{x}_1) = kA\vec{x}_1$

properties of matrix multiplication satisfy linear transformation conditions

**Theorem 6.1.4 (p271)**

Every linear transformation $T$ can be written as a matrix transformation for some matrix $A$ (called the **standard matrix**), where $A$ is:

$$A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \quad \text{where } \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are standard unit vectors in $\mathbb{R}^n$

Note: You can get these matrices from the *vector form*.

Ex: Suppose $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - 2y \\ -3y \\ 2x + y \end{bmatrix}$. What is the standard matrix for $T$?

$$\underbrace{\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{standard unit vectors in } \mathbb{R}^2}$$

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}$$

Note: Can also get standard mtx from:

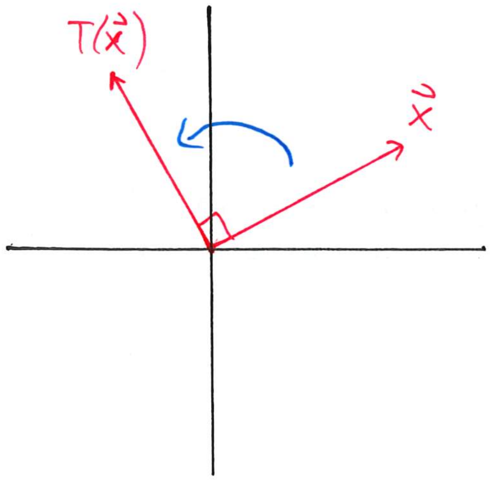
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-2y \\ -3y \\ 2x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} y$$

$$= \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad T(\vec{x}) = A\vec{x}$$

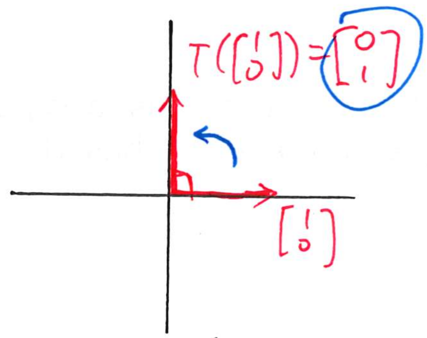
Notice that we can determine the standard matrix just by using the standard unit vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$. This will come in useful when we look at geometry of linear operators in the next section.

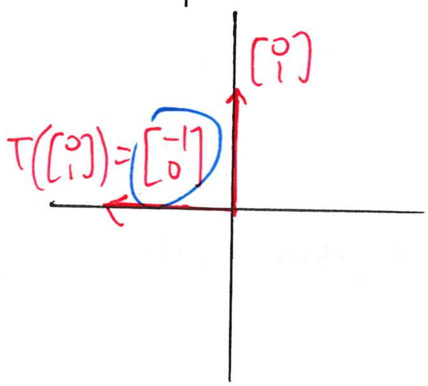
see what $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are

T: Rotation counterclockwise by $90^\circ$ in $\mathbb{R}^2$




To find standard mtx, apply T to $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:




$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

std mtx:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$