MATH 232 Section 3.5 pre-lecture comments

Lecture Outline
We discuss the relation between solutions of $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ (Same A!). We also discuss the column space and how it relates to the consistency problem.

New terminology

1. homogeneous linear system (old term, new way of thinking)
2. non-homogeneous linear system
3. general solution
4. particular solution
5. column space

Homogeneous vs. Non-Homogeneous Linear Systems

Example Let $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Compare the solution sets of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ and $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \end{array}\right]$$

$x_4 = t$

$x_3 = -3t$

$x_2 = s$

$x_1 = s - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix} t}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & 3 & 4 \end{array}\right]$$

$x_4 = t$

$x_3 = 4 - 3t$

$x_2 = s$

$x_1 = -1 + s - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix} t}$$

same

Note. We call $A\mathbf{x} = \mathbf{0}$ the homogeneous linear system **associated** with $A\mathbf{x} = \mathbf{b}$.

> **Definition**
>
> A **general solution** describes all possible solutions.
>
> A **particular solution** is one specific solution.

Even though the solution set of homogeneous system is a subspace, the solution set of a non-homogeneous system is **not a subspace**. (Why?)

$A\vec{x}=\vec{b}$ $(\vec{b}\neq\vec{0})$

$\vec{0}$ is not a solution to $A\vec{x}=\vec{b}$ $(\vec{b}\neq\vec{0})$

> **Solution to system**
>
> Every solution of a consistent non-homogeneous linear system $A\mathbf{x} = \mathbf{b}$ has the form
>
> $$\vec{\mathbf{x}} = \vec{x}_0 + \underbrace{t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k}$$
>
> where $\mathbf{x_0}$ is a **particular solution** of $A\mathbf{x} = \mathbf{b}$ and $\underbrace{t_1\mathbf{v_1} + t_2\mathbf{v_2} + \cdots + t_k\mathbf{v_k}}$ is a **general solution** of $A\mathbf{x} = \mathbf{0}$. (homogeneous)

The consistency problem revisited

> **The consistency problem** For a given matrix A, find all vectors $\mathbf{b}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution.

Rewrite $A\mathbf{x} = \mathbf{b}$ in terms of column vectors of A:

$$A = \begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Rewrite $A\vec{x} = \vec{b}$ as:

$$\underline{x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n = \vec{b}}$$

to be consistent: $\vec{b}$ is in span $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$

$A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A (span of the column vectors of A).

Span of the column vectors of A is one of the important subspaces, and it has a special name:

> **Definition.** Let A be an $m \times n$ matrix. Its **column space** $\text{col}(A)$ is the span of its column vectors:
>
> $$\text{col}(A) = \text{span}\,(\vec{\mathbf{c}}_1(A), \vec{\mathbf{c}}_2(A), \ldots, \vec{\mathbf{c}}_n(A)).$$

Column space of A is a subspace of $\mathbf{R}^m$.

This proves the following result.

> **Theorem 3.5.5** A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ belongs to the $\underbrace{\text{span of the column}\cancel{\text{s}}\text{ vectors of } A}_{\text{col}(A)}$.

So now, given a system, deciding its consistency reduces to answering if $\mathbf{b}$ is in the span of the column space, or equivalently if $\mathbf{b}$ is a linear combination of the columns of A.

> **Example** Is $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{\mathbf{x}} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ consistent?
>
> $$\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
>
> Yes, it is consistent.
>
> $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x_1\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \text{ is in } \text{col}\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

$$\uparrow$$

$$\text{span}\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Note that if $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent, it must have the same number of solutions as $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

Theorem 3.5.3 If A is an $m \times n$ matrix, then the following statements are equivalent:

1. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution ($\mathbf{x} = \mathbf{0}$).
2. For every vector $\mathbf{b} \in R^m$, the system $A\mathbf{x} = \mathbf{b}$ has either one solution, or no solutions.

inconsistent

Remarks

1) This is similar to the fundamental theorem. How?

2) If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions then $A\mathbf{x} = \mathbf{b}$ has either ∞ sol. or no sol.

Theorem 3.5.4 A non-homogeneous linear system with more unknowns than equations has either infinitely many solutions or is inconsistent.