MATH 232 Section 3.1 pre-lecture comments

Lecture Outline
Today we will define matrix multiplication and consider different geometric ways to understand it.

Make sure you review the dot product and recall the algebra of vectors.

New terminology

1. square matrix
2. matrix product
3. linearity properties
4. transpose

An $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns.

3 columns

$$\begin{array}{cc} \text{2 rows} & \begin{bmatrix} 1 & 3 & 4 \\ 1/2 & -2 & 0 \end{bmatrix} \end{array}$$

2×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

or, more compactly

$$A = [a_{ij}]_{m \times n} \quad \text{or} \quad A = [a_{ij}]$$

$$[a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

A matrix with one row ($m = 1$) is a **row vector**

$$\begin{bmatrix} 1 & 3 & 4 \end{bmatrix}$$

size 1×3

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$

whereas a matrix with one column ($n = 1$) is a **column vector**

$$\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

size 2×1

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

A **square** matrix has $m = n$.

$$\begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

3×3 square matrix

We can think of a matrix as an ordered stack of rows

$$\begin{array}{c} \vec{r}_1 \\ \vec{r}_2 \end{array} \begin{bmatrix} 1 & 3 & 4 \\ 1/2 & -2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \text{———} & \mathbf{r}_1 & \text{———} \\ \text{———} & \mathbf{r}_2 & \text{———} \\ & \vdots & \\ \text{———} & \mathbf{r}_m & \text{———} \end{bmatrix}$$

$\vec{r}_1 = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}$

$\vec{r}_2 = \begin{bmatrix} 1/2 & -2 & 0 \end{bmatrix}$

We can also think of a matrix as an ordered collection of columns

$$\begin{array}{ccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{array}$$
$$\begin{bmatrix} 1 & 3 & 4 \\ 1/2 & -2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \\ | & | & & | \end{bmatrix}$$

$\vec{c}_1 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$ $\vec{c}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $\vec{c}_3 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

Definition If matrices A and B have the same size and c is a scalar then

$$\begin{aligned} \text{Addition} \qquad & (A+B)_{ij} = (A)_{ij} + (B)_{ij} \\ \text{Subtraction} \qquad & (A-B)_{ij} = (A)_{ij} - (B)_{ij} \\ \text{Scalar multiplication} \qquad & (cA)_{ij} = c(A)_{ij} \end{aligned}$$

If they **do not** have the same size $A+B$ and $A-B$ are *undefined*.

Example. Given matrices

$$A = \begin{bmatrix} 2 & 0 \\ 7 & -5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & -3 & 0 \\ 1 & 5 & -1 \end{bmatrix}, C = \begin{bmatrix} 6 & 3 \\ 4 & 5 \\ 9 & 8 \end{bmatrix}$$

Evaluate $A+B$, $A+C$, $C-A$, and $(-2)B$

$A+B$ is undefined

$$A+C = \begin{bmatrix} 8 & 3 \\ 11 & 0 \\ 13 & 11 \end{bmatrix} \qquad C-A = \begin{bmatrix} 4 & 3 \\ -3 & 10 \\ 5 & 5 \end{bmatrix}$$

$$(-2)B = \begin{bmatrix} -8 & 6 & 0 \\ -2 & -10 & 2 \end{bmatrix}$$

Product of a matrix and a column vector:

same

The product $A\mathbf{x}$ Suppose A is an $m \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ column vector. The **product $A\mathbf{x}$** is defined as

one number

$$A\mathbf{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}}_{m \times 1}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3 \times 1} = \begin{bmatrix} 1(1)+0(2)+(-1)(3) \\ 2(1)+(-1)(2)+0(3) \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \end{bmatrix}}_{2\times 1}$$

same

Remarks: The result of multiplication is $m \times 1$ column vector. The product is only defined when the number of columns of A is the same as the number of rows of $\mathbf{x}$.
There are two geometric ways to think about matrix multiplication.

$A\mathbf{x}$ as dot products We can also view the product $A\mathbf{x}$ as a column vector composed of dot products of row vectors of A and $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

$A\mathbf{x}$ as linear combination We can also view the product $A\mathbf{x}$ as a linear combination of column vectors of A:

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \dots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n$$

$$= x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_n\vec{c}_n$$

Example

$$\begin{bmatrix} 5 & 0 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 15 + 0 + 20 \\ -3 - 8 + 10 \end{bmatrix} = \begin{bmatrix} 35 \\ -1 \end{bmatrix}$$

2×3 $\quad$ 3×1 $\quad$ 2×1

same

$$= \begin{bmatrix} 5 \\ -1 \end{bmatrix} \cdot 3 + \begin{bmatrix} 0 \\ 4 \end{bmatrix}(-2) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}(10) = \begin{bmatrix} 35 \\ -1 \end{bmatrix}$$

A has n columns, B has n rows

The product AB Given an $m \times n$ matrix A and a $n \times p$ matrix B the $m \times p$ matrix AB is defined as

$$AB = \begin{bmatrix} | & | & & | \\ A\mathbf{c}_1(B) & A\mathbf{c}_2(B) & \ldots & A\mathbf{c}_p(B) \\ | & | & & | \end{bmatrix}$$

where $\mathbf{c}_1(B)$, $\mathbf{c}_2(B)$, …, $\mathbf{c}_p(B)$ are the columns of B.

Example

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & -3 \end{bmatrix}$$

2×3 $\quad 3\times 2$ $\quad 2\times 2$

same

Entry $(AB)_{ij}$ The entry in row i and columns j of the product AB is the dot product of the ith row of A and the jth ~~row~~ column of B; that is:

$$\vec{\mathrm{r}}_{\mathrm{i}}(A) \cdot \vec{\mathrm{c}}_{\mathrm{j}}(B).$$

Example.

$$\begin{bmatrix} 4 & 0 & 5 \\ 3 & -2 & 8 \end{bmatrix} \begin{bmatrix} 1 & 10 & -2 & 0 \\ 0 & 6 & 1 & 5 \\ 2 & 2 & -7 & 9 \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & \boxed{-64} & * \end{bmatrix}$$

$3(-2)+(-2)(1)+8(-7)=-64$

Definition: Transpose If A is an $m \times n$ matrix, then the **transpose** of A, denoted as A^T, is the $n \times m$ matrix created by making the rows of A into columns. We can write this as $(A^T)_{ij} = (A)_{ji}$.

Example. Suppose $A = \begin{bmatrix} -3 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 7 & 1 \end{bmatrix}$. Find A^T, B^T and $(B^T)^T$.

2×3 $\quad$ 1×3

$A^T = \begin{bmatrix} -3 & 0 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}$ 3×2

$B^T = \begin{bmatrix} 8 \\ 7 \\ 1 \end{bmatrix}$ 3×1

$(B^T)^T = \begin{bmatrix} 8 & 7 & 1 \end{bmatrix}$

Which of the following products are defined?

	AB	A^TB	AB^T	BA	B^TA	BA^T
Defined?	no	no	yes (2×1)	no	no	yes (1×2)

Evaluate those products that are defined.

If AB is defined then $(AB)^T = B^TA^T$