6.1/6.2

Section 6.2 pre-lecture comments

Lecture Outline

Today we will look at the geometry of linear operators. We will think about both the action of geometric operators on $\mathbf{R}^2$ and geometric properties of a special class of operators.

We will be using the properties of transpose (A^T), dot product $(\mathbf{u} \cdot \mathbf{v})$, and orthonormal sets in this section.

New terminology

1. Orthogonal matrix
2. Orthogonal transformation/operator

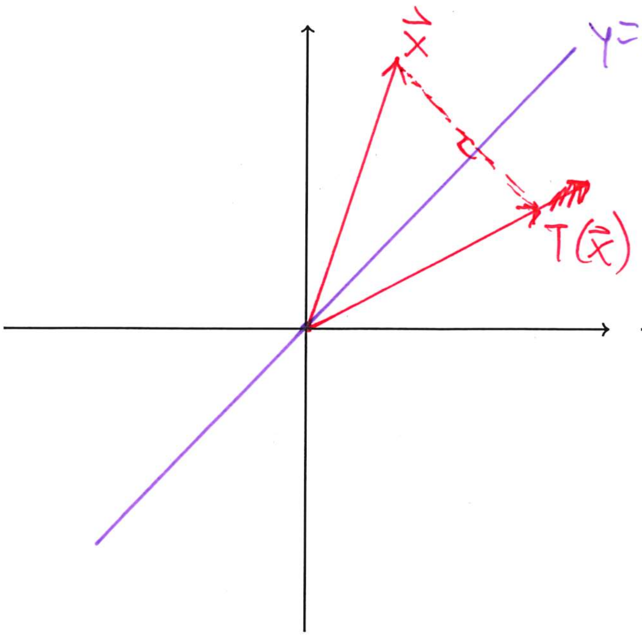
Linear operators in $\mathbb{R}^2$

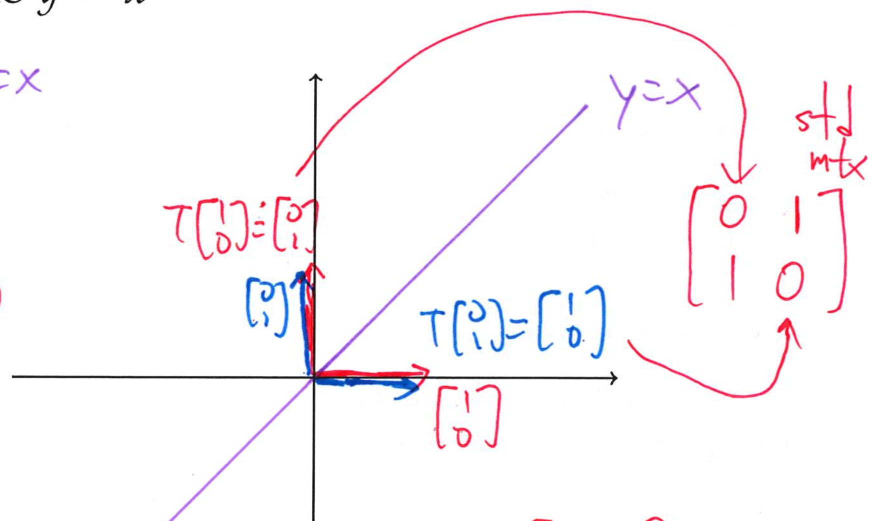
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix}$

Recall that we can find the standard matrix A of a 2D linear transformation T by seeing how T transforms $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

What is $T(\vec{e}_1)$? $T(\vec{e}_2) = ?$

Example: Reflection in the line $y = x$



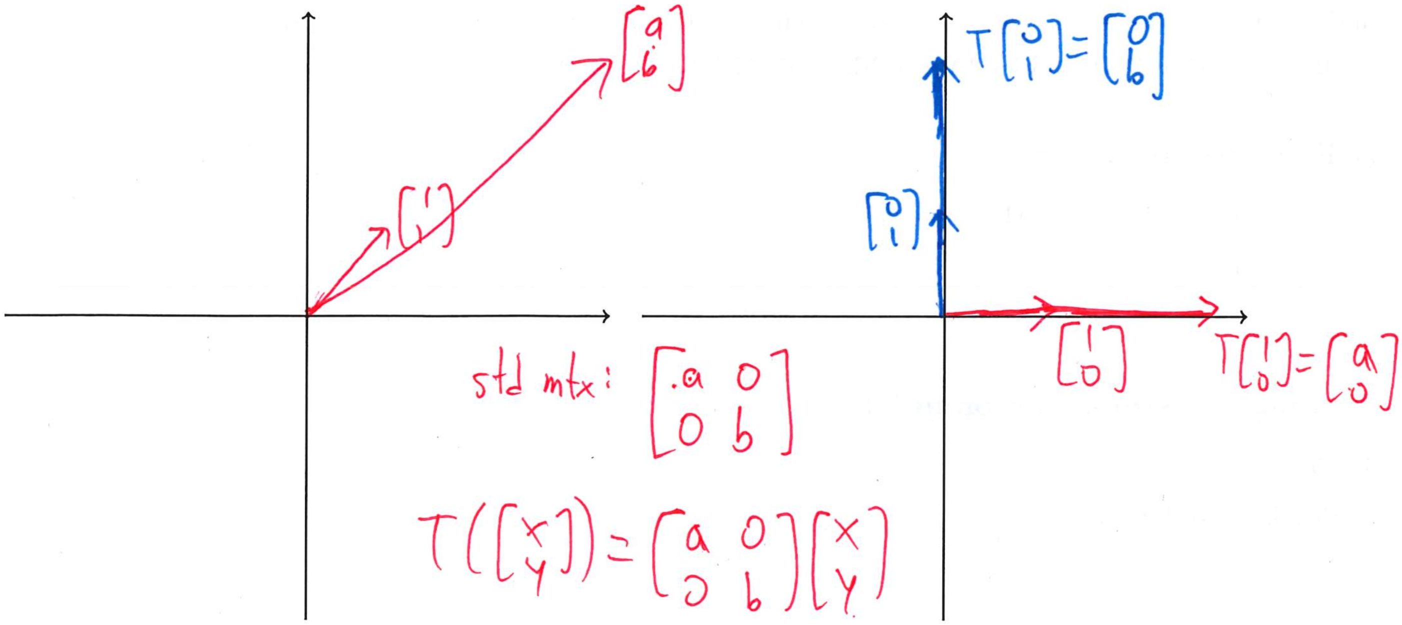


$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

$$T(\vec{x}) = A\vec{x}$$

Scaling

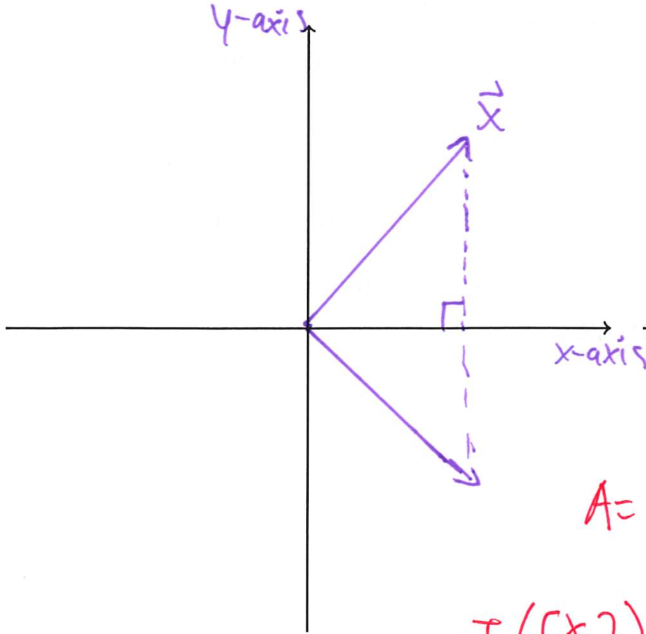
Suppose that a 2D linear transformation T scales a vector in the x direction by a and in the y direction by b. What is the standard matrix of T? Using the standard matrix, write the formula for $T(x, y)$.

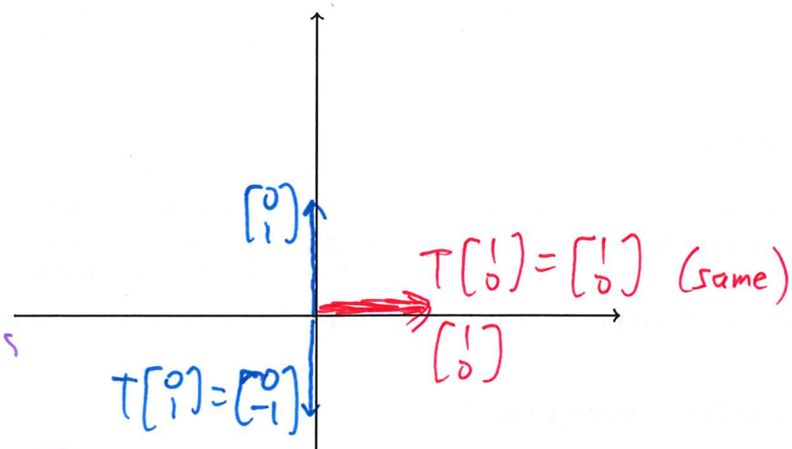


std mtx: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

Reflection across x-axis

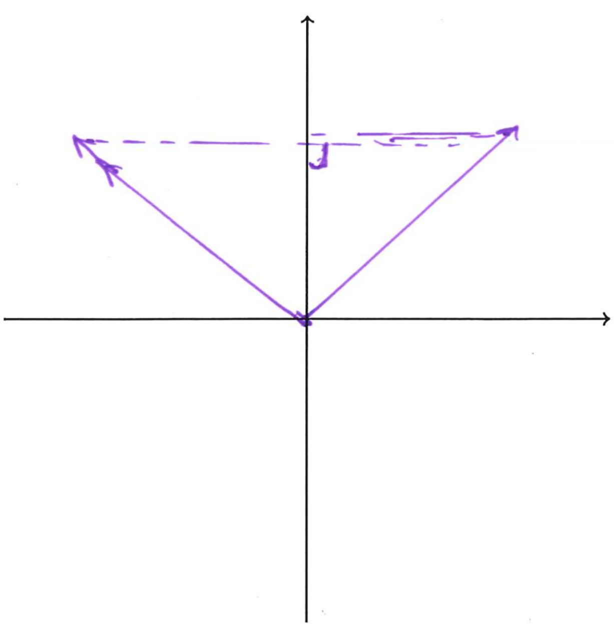


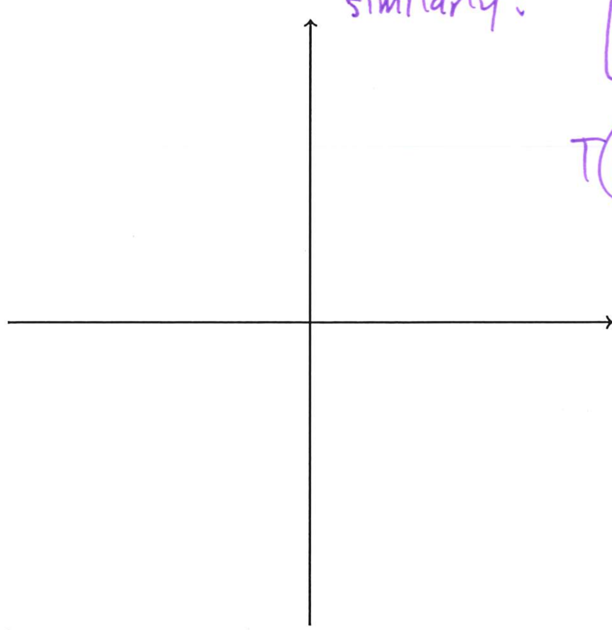


$$A=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\-y\end{bmatrix}$$

Reflection across y-axis



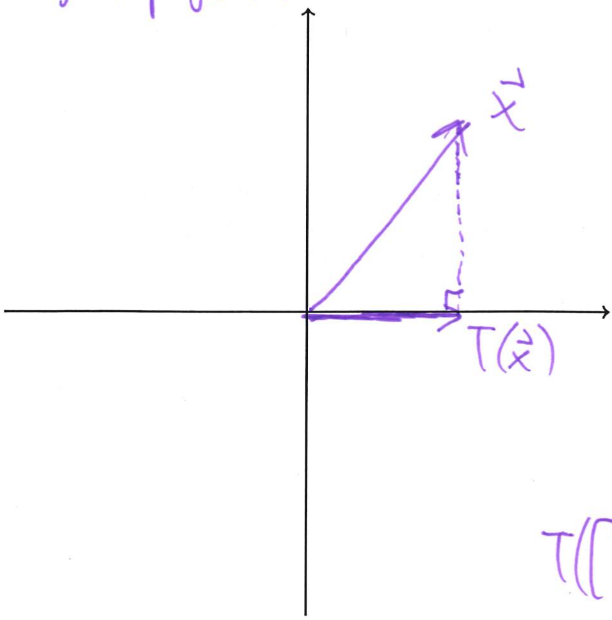


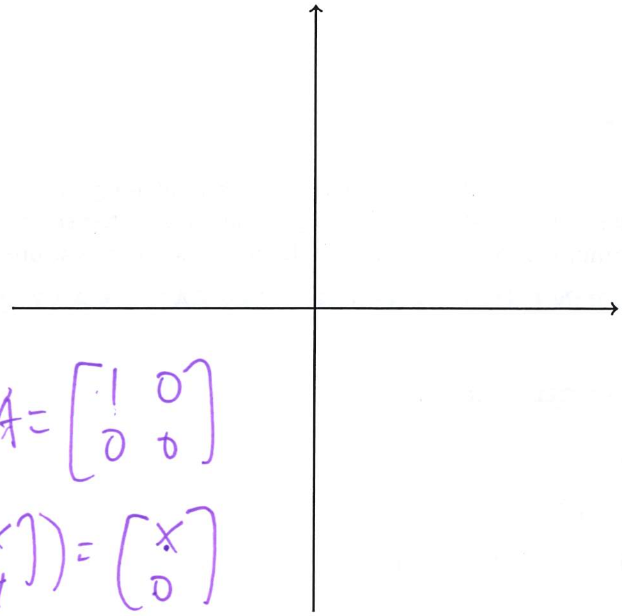
similarly: $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-x\\y\end{bmatrix}$$

Projection onto x-axis

Orthogonal projection

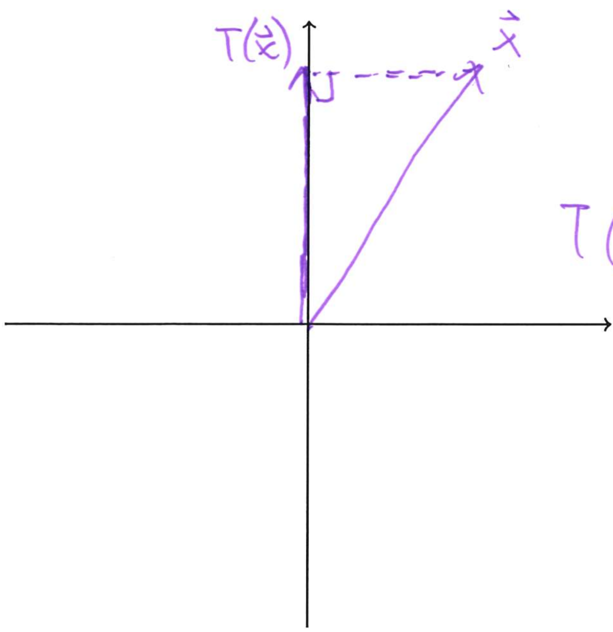




$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

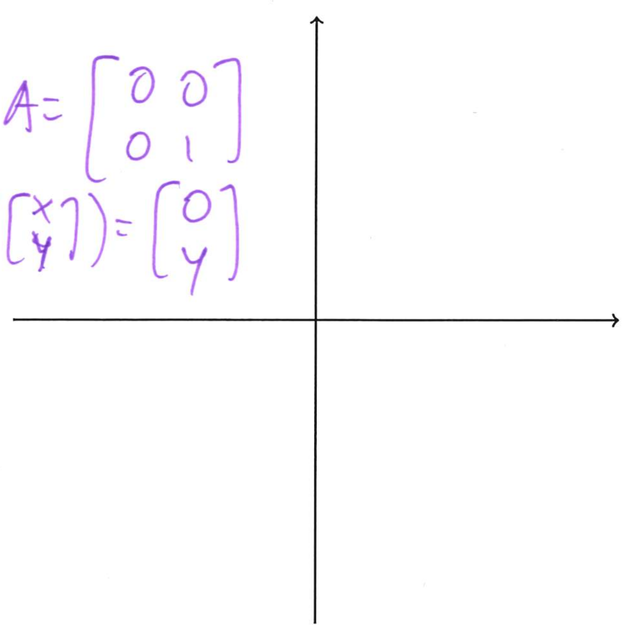
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Projection onto y-axis



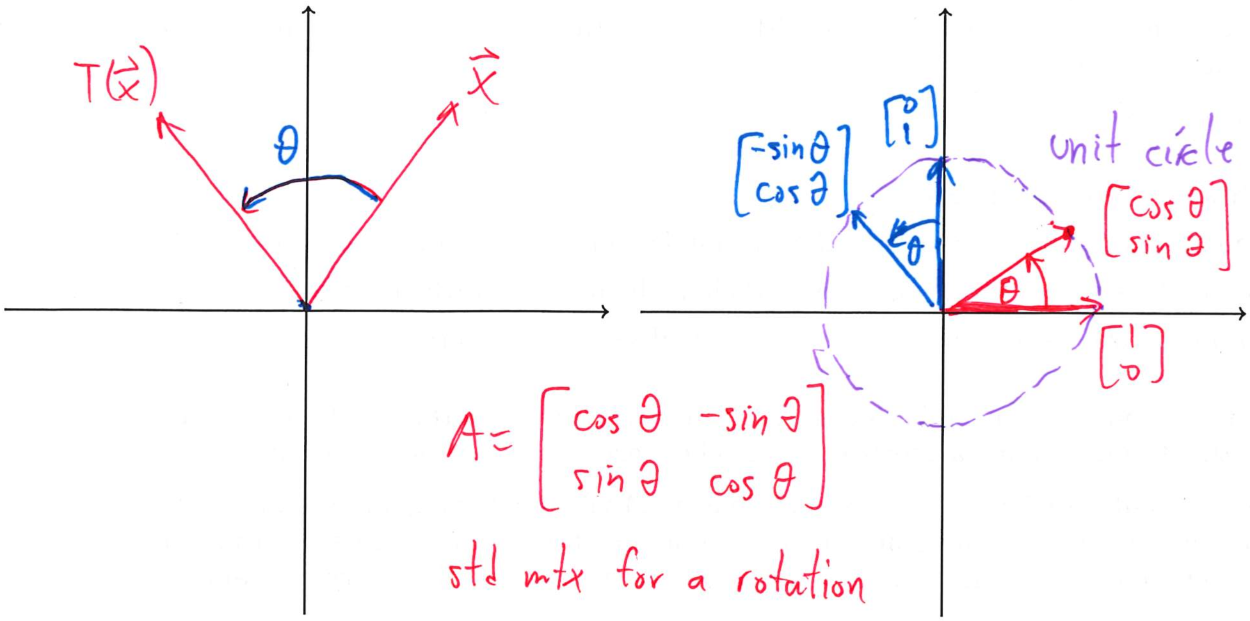
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}$$



General rotations in $\mathbb{R}^2$

Rotation by θ counterclockwise:



$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

std mtx for a rotation

Ex: What is the standard matrix of rotation by 45^o counterclockwise?

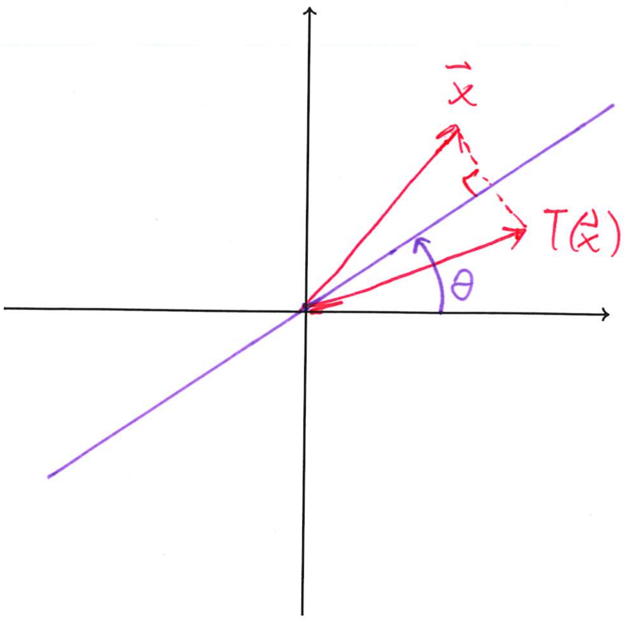
$\cos 45^\circ = 1/\sqrt{2}$ $\left(= \frac{\sqrt{2}}{2}\right)$

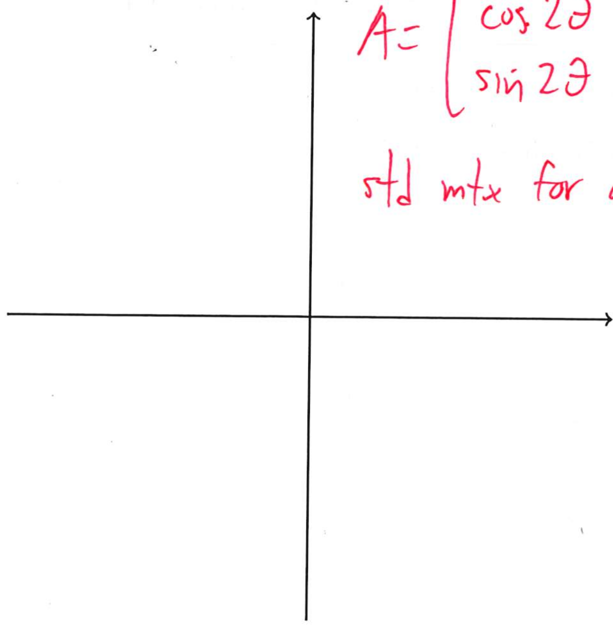
$\sin 45^\circ = 1/\sqrt{2}$

$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

General reflections in $\mathbb{R}^2$

Reflection across the line of angle θ:





$$A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

std mtx for a reflection

Ex: What is the standard matrix of reflection in a line of angle 30^o?

$\theta = 30°$

$$A = \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

$\cos 60° = \frac{1}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}$

Recall what it means to have an orthonormal set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$:

$v_i \cdot v_j = 0 \quad i \neq j$

$\|v_i\| = 1$ for all i

Orthogonal matrix

A square matrix A is called an **orthogonal matrix** if the columns of A form an orthonormal set.

Alternatively:

$A^{-1} = A^T$

A square matrix A is called an **orthogonal matrix** if $A^T A = I$.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (columns)

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ← orthogonal matrices

Examples Which of the following are orthogonal matrices?

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Yes. ↑ rotation by 90° clockwise

No. $\|\vec{v}_1\| \neq 1$

No. $\|\vec{v}_3\| \neq 1$

Yes. ↑ reflection

Note: Orthogonal matrices have $\det(A) = \pm 1$.

Orthogonal operator An **orthogonal operator** is a transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ whose standard matrix is an orthogonal matrix.

$T\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{orthogonal matrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ (Reflection in line $y=x$)

Orthogonal operators:

- preserve distances/norms ($\|T(\mathbf{u})\| = \|\mathbf{u}\|$ for all vectors $\mathbf{u}$), and
- preserve dot products and angles ($T(\mathbf{u}) \cdot T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$). Ex: (rotations, reflections)

Either of these properties may be used as an alternative definition for orthogonal operator.
(Thm. 6.2.4 p283)

Orthogonal operators in $\mathbf{R}^2$ (Thm 6.2.7 p284) If T is an orthogonal operator in $\mathbf{R}^2$, then either T is a rotation about the origin, or a reflection about a line through the origin.

std mtx: either $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some θ (rotation)

or $A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ for some θ (reflection)

For an orthogonal operator in $\mathbb{R}^2$

Note: If the standard matrix has determinant 1, then T is a rotation; if the determinant is -1, then T is a reflection.

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

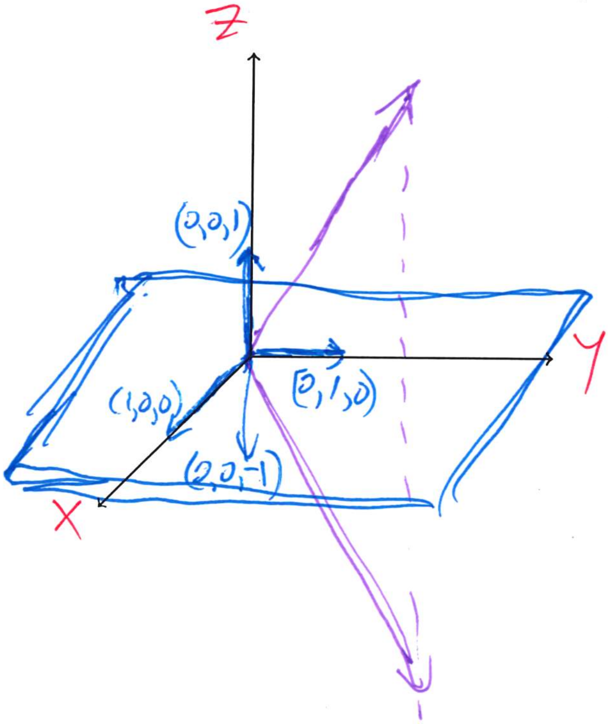
det $= -1$ reflection

Geometric transformations in 3D

Similarly to 2D, we can find the standard matrix A of a 3D linear transformation T by seeing what $T(\vec{e}_1)$, $T(\vec{e}_2)$, $T(\vec{e}_3)$ are

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Ex: Reflection across the xy-plane. What is the std mtx?



$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Read Section 6.1/6.2

for more 2D and 3D transformations