MATH 232 Section 4.3 pre-lecture comments

Lecture Outline

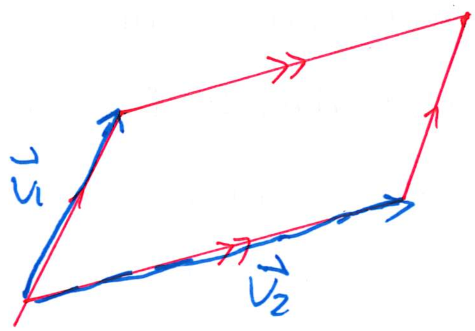
Section 4.3 is on applications of determinants. Determinants can be used to find the area of a parallelogram, or the volume of a parallelepiped.

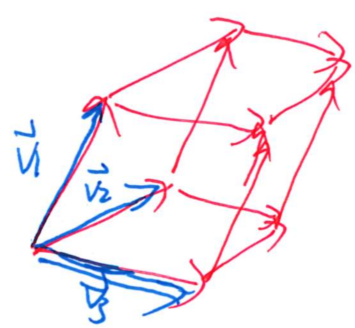
We will also look at the cross product.

All matrices in this chapter are square matrices.

New terminology

1. parallelogram
2. parallepiped
3. cross product





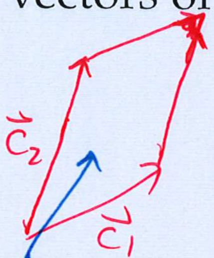
Geometric interpretation of determinants

Theorem 4.3.5

absolute value: area is positive (or 0)

1. If A is a 2×2 matrix, then $|\det(A)|$ represents the area of the parallelogram determined by the two column vectors of A, placed with their starting points together.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \vec{c}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \vec{c}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

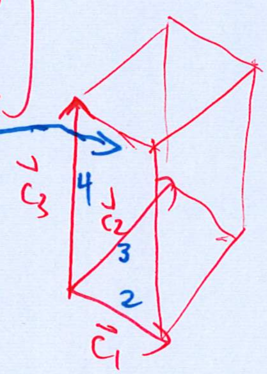


must be positive (or 0) $\boxed{\text{Area} = 2}$

$\text{Area} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

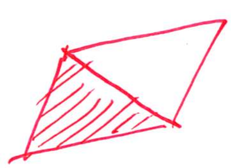
2. If A is a 3×3 matrix, then $|\det(A)|$ represents the volume of the parallelepiped determined by the three column vectors of A, placed with their starting points together.

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad \vec{c}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{c}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ $\quad \vec{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$



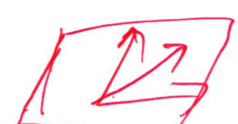
$\det(A) = (2)(3)(4) = 24$ $\quad \boxed{\text{Volume} = 24}$

Note:



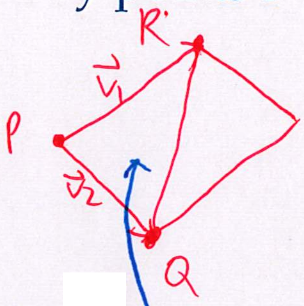
- A triangle is half a parallelogram.
- If $\det(A) = 0$, then the $\underbrace{\text{parallelogram}}_{\text{two vectors on same line}}/\underbrace{\text{parallelepiped}}_{\text{three vectors on same plane}}$ is degenerate (area/volume is 0).

Area=0



Volume=0

Example: Area of a triangle What is the area of a triangle determined by points $P = (-1,5)$, $Q = (0,3)$ and $R = (2,7)$?



$\vec{v}_1 = \overrightarrow{PR} = (3,2)$

$\vec{v}_2 = \overrightarrow{PQ} = (1,-2)$

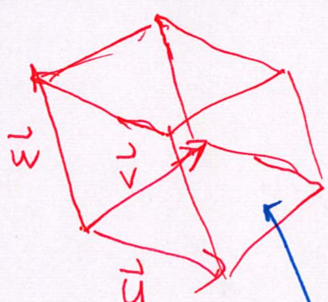
$$\det(A) = \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = -8$$

Area of parallelogram $= 8$

Area of triangle $= 4$

half of parallelogram

Example: Volume of a parallelepiped What is the volume of the parallelepiped formed by vectors $\mathbf{u} = (1,0,-1)$, $\mathbf{v} = (2,1,1)$ and $\mathbf{w} = (0,-1,1)$?



$$\det = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} \cancel{+ 0\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}}$$

$$= 1(2) - 2(-1) = 4$$

$\boxed{\text{Volume} = 4}$

The Cross Product

Definition: Cross product (4.3.7) If $\vec{\mathbf{u}} = (u_1, u_2, u_3)$ and $\vec{\mathbf{v}} = (v_1, v_2, v_3)$ are vectors in R^3, then the **cross product of u and v**, denoted by $\mathbf{u} \times \mathbf{v}$, is the vector in R^3 defined as

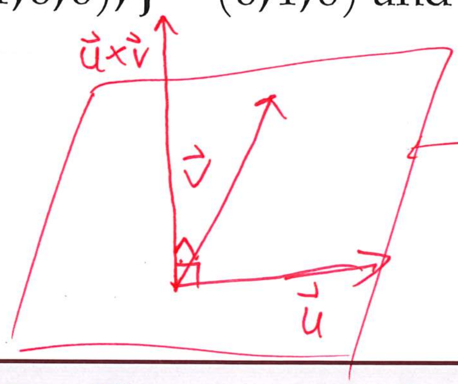
$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, \quad u_3v_1 - u_1v_3, \quad u_1v_2 - u_2v_1)$$

Equivalently: $u \times v = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$

- A way to remember this is by a "generalized determinant":

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \cdot i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

where $\hat{\mathbf{i}} = (1,0,0)$, $\hat{\mathbf{j}} = (0,1,0)$ and $\hat{\mathbf{k}} = (0,0,1)$ are standard unit vectors in R^3.



plane is span$\{\vec{u}, \vec{v}\}$

Orthogonality The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$; that is:

$$(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$$
$$(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$$

Examples Using the vectors:

$$\mathbf{u} = (1,0,-1) \quad \mathbf{v} = (2,1,1)$$

Calculate $\mathbf{u} \times \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$ and $\mathbf{u} \times \mathbf{u}$

$u \times v$: $\begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} k$

$= 1i - 3j + 1k = \boxed{(1,-3,1)}$

$v \times u = (-1, 3, -1)$ (negative of $u \times v$)

$u \times u = \vec{0} = (0,0,0)$ (parallel vectors cross product is $\vec{0}$)

Recall that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$. The magnitude of the cross product is similar:

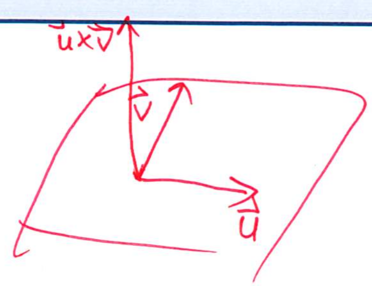
Magnitude and direction of cross product

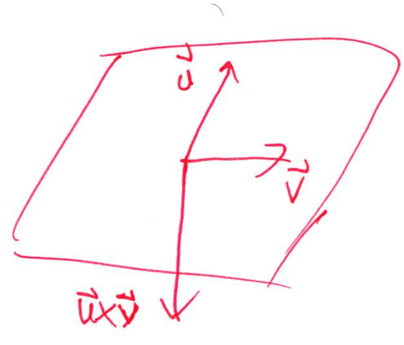
If $\mathbf{u}$ and $\mathbf{v}$ are non-parallel vectors in $\mathbf{R}^3$, then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

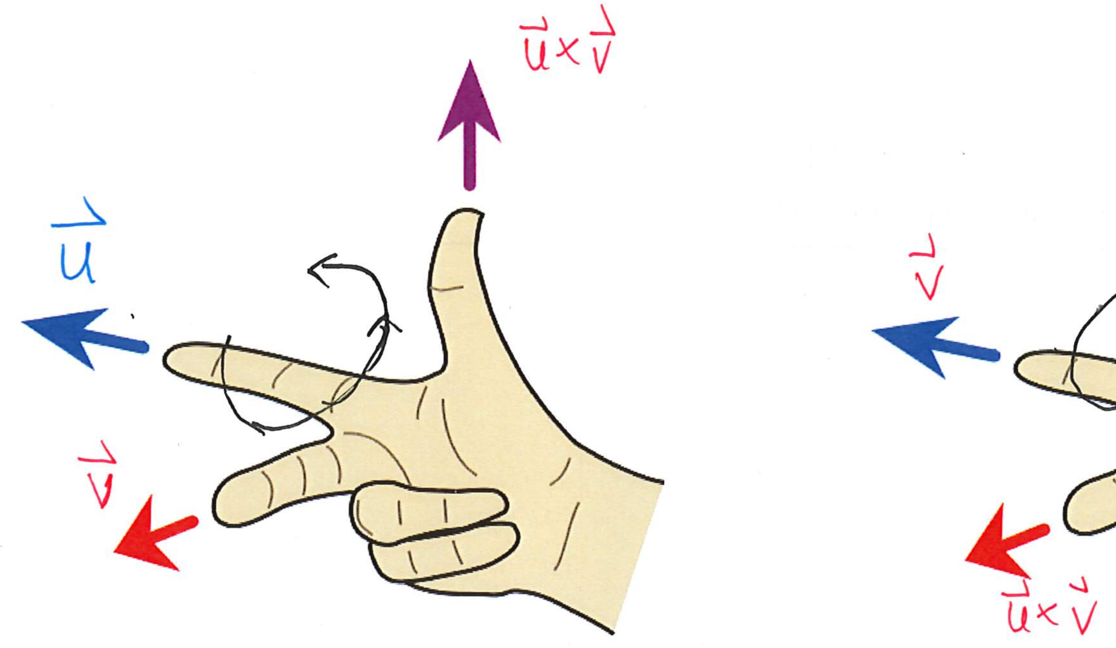
and $\mathbf{u} \times \mathbf{v}$ is

- orthogonal to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, and
- oriented so it satisfies the right-hand rule with respect to $\mathbf{u}$ and $\mathbf{v}$.



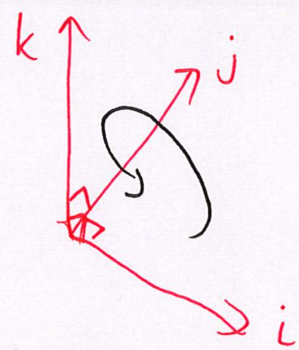


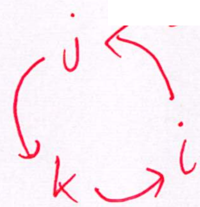
Right hand rule



Examples

- $\mathbf{i} \times \mathbf{j} =$ k
- $\mathbf{i} \times \mathbf{k} =$ $-j$
- $\mathbf{i} \times \mathbf{i} =$ $\vec{0}$
- $\mathbf{k} \times \mathbf{j} =$ $-i$





Theorem 4.3.8 - Algebraic properties

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^3 and k is a scalar, then

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(d) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Warning! In general:

- $\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}$
- $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

ambiguous

$u \times v \times w$

Normal vector to a plane

Find a normal vector to the plane $\vec{\mathbf{x}} = (0,-3,2) + (2,0,1)s + (1,3,0)t$

Normal vector $\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} k$

$(2,0,1) \times (1,3,0)$

$= -3i - (-1)j + 6k = \boxed{(-3,1,6)}$

Area of a parallelogram in 3D

Area of a parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$ and angle θ:

$$\text{Area} = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta = \|\mathbf{u} \times \mathbf{v}\|$$

in 3D

Ex: What is the area of the parallelogram formed by vectors (2,0,1) and (1,3,0)?

$\|(2,0,1) \times (1,3,0)\| = \|(-3,1,6)\|$ (same as above)

$= \sqrt{9+1+36} = \boxed{\sqrt{46}}$