Section 7.9 pre-lecture comments

Lecture Outline

We will discuss orthogonal and orthonormal bases, how they simplify projection formulas, and how to construct an orthogonal/orthonormal basis using the Gram-Schmidt Process.

New terminology

1. orthogonal basis
2. orthonormal basis

Recall what it means for a set of vectors to be orthogonal/orthonormal. For example, three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

orthogonal:

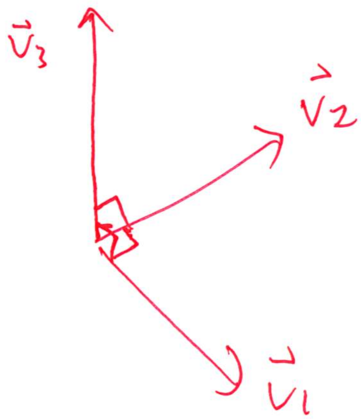
$$v_1 \cdot v_2 = 0$$

$$v_1 \cdot v_3 = 0$$

$$v_2 \cdot v_3 = 0$$

orthonormal:

orthogonal and $\|\vec{v}_1\| = 1$, $\|\vec{v}_2\| = 1$, $\|\vec{v}_3\| = 1$



Orthogonal/orthonormal basis Let W be a subspace of $\mathbf{R}^n$.
An **orthogonal basis** (similarly, **orthonormal basis**) for W is a basis for W which is an orthogonal (similarly, orthonormal) set.

Ex: W is the plane $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t.$

Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ an orthogonal basis for W? No $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \neq 0$

Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$ an orthogonal basis for W? Yes $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} = 0$

in W $\begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

How do we turn an orthogonal basis into an orthonormal basis?

Turn $\vec{v}_1$, $\vec{v}_2$ to unit vector:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \rightarrow \frac{\vec{v}_2}{\|\vec{v}_2\|} = \cdots = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Orthogonal/orthonormal basis simplifies projection

In the special case where a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ for W is orthogonal, projection onto W simplifies as:

$$\text{proj}_W(\vec{\mathbf{x}}) = \text{proj}_{\mathbf{v}_1}(\vec{\mathbf{x}}) + \text{proj}_{\mathbf{v}_2}(\vec{\mathbf{x}}) + \ldots + \text{proj}_{\mathbf{v}_k}(\vec{\mathbf{x}})$$

Projection formula for orthogonal/orthonormal basis

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for W. Then:

$$\text{proj}_W(\mathbf{x}) = \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1 + \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2 + \ldots + \left(\frac{\mathbf{v}_k \cdot \mathbf{x}}{\|\mathbf{v}_k\|^2}\right)\mathbf{v}_k$$

If the basis is orthonormal, the formula simplifies to:

$$\text{proj}_W(\mathbf{x}) = (\mathbf{v}_1 \cdot \mathbf{x})\,\mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{x})\,\mathbf{v}_2 + \ldots + (\mathbf{v}_k \cdot \mathbf{x})\,\mathbf{v}_k$$

Ex: Basis for W is $\left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \right\}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Find $\text{proj}_W(\mathbf{x})$.

orthogonal

$$\left(\frac{v_1 \cdot x}{\|v_1\|^2}\right)\vec{v}_1 = \frac{6}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

$$\left(\frac{v_2 \cdot x}{\|v_2\|^2}\right)\vec{v}_2 = \frac{4}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

$$\text{proj}_W(\vec{x}) = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 4/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 11/3 \\ 4/3 \\ 7/3 \end{bmatrix}$$

Likewise, standard matrix of projection onto W is the sum of the standard matrices of projection on each vector in an orthogonal basis.

With $[T]$ denoting standard matrix of T:

$$[\mathrm{proj}_W(\mathbf{x})] = [\mathrm{proj}_{\mathbf{v}_1}(\mathbf{x})] + [\mathrm{proj}_{\mathbf{v}_2}(\mathbf{x})] + \ldots + [\mathrm{proj}_{\mathbf{v}_k}(\mathbf{x})]$$

Standard matrix for projection (orthogonal/orthonormal)

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal basis for W. Then the standard matrix:

nxn matrices

$$[\mathrm{proj}_W(\mathbf{x})] = \left(\frac{1}{\|\mathbf{v}_1\|^2}\right)\mathbf{v}_1\mathbf{v}_1^T + \left(\frac{1}{\|\mathbf{v}_2\|^2}\right)\mathbf{v}_2\mathbf{v}_2^T + \ldots + \left(\frac{1}{\|\mathbf{v}_k\|^2}\right)\mathbf{v}_k\mathbf{v}_k^T$$

If the basis is orthonormal, the formula simplifies to:

$$[\mathrm{proj}_W(\mathbf{x})] = \mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T + \ldots + \mathbf{v}_k\mathbf{v}_k^T$$

Ex: Basis for W is $\left\{ \begin{array}{c} \vec{v}_1 \\ \begin{bmatrix}1\\0\\1\end{bmatrix} \end{array}, \begin{array}{c} \vec{v}_2 \\ \begin{bmatrix}1\\2\\-1\end{bmatrix} \end{array} \right\}$. Find the standard matrix of $\mathrm{proj}_W(\mathbf{x})$.

$$\frac{1}{\|\vec{v}_1\|^2}\vec{v}_1\vec{v}_1^T = \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}\begin{bmatrix}1&0&1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}3&0&3\\0&0&0\\3&0&3\end{bmatrix}$$

$$\frac{1}{\|\vec{v}_2\|^2}\vec{v}_2\vec{v}_2^T = \frac{1}{6}\begin{bmatrix}1\\2\\-1\end{bmatrix}\begin{bmatrix}1&2&-1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-2&1\end{bmatrix}$$

Std mtx $[\mathrm{proj}_W(\vec{x})] = \frac{1}{6}\begin{bmatrix}4&2&2\\2&4&-2\\2&-2&4\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2&1&1\\1&2&-1\\1&-1&2\end{bmatrix}$

$M(M^TM)^{-1}M^T$