MATH 232 Section 3.2 pre-lecture comments

Lecture Outline
Today we will define the inverse of a matrix. THIS IS A REALLY IMPORTANT IDEA!

We will continue on with the algebra of matrices.

New terminology

1. matrix inverse
2. zero matrix
3. identity matrix
4. invertible

Theorem 3.2.1 (p. 94) Suppose A, B and C are $m \times n$ matrices and r and s are scalars. Then

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C \qquad A(BC) = (AB)C$$

$$(rs)A = r(sA) \qquad r(BC) = (rB)C = B(rC)$$

$$(r + s)A = rA + sA \qquad (B + C)A = BA + CA$$

$$(r - s)A = rA - sA \qquad (B - C)A = BA - CA$$

$$r(A + B) = rA + rB \qquad A(B + C) = AB + AC$$

$$r(A - B) = rA - rB \qquad A(B - C) = AB - AC$$

Warning! AB is not always equal to BA!!

Matrix multiplication is different from normal multiplication.

Example. Given $A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$, calculate AB and BA.

$AB = \begin{bmatrix} -4 & 6 \\ 3 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 4 & -2 \\ 3 & -7 \end{bmatrix}$

$AB \neq BA$

AB and BA are not always equal

Given two matrices A and B, the products AB and BA may not be equal, for three possible reasons:

- AB may be defined, but BA is not; A is 2×3, B is 3×1 AB is 2×1, BA not defined
- AB and BA are both defined, but they have different sizes;
- AB and BA are both defined and have the same sizes, but the products are not equal.

Question. What conditions must A and B satisfy so that both AB and BA are defined?

A is $m \times n$ AB is $m \times m$

B is $n \times m$ BA is $n \times n$

Example

Special matrices

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Zero matrices (p. 96) A matrix with entries all zeroes is called a **zero matrix**. It is denoted by O, or by $O_{m\times n}$ if we want to specify the size. If the size of matrix A is $m \times n$ and k is a scalar, then

1. $A + O_{m\times n} = O_{m\times n} + A =$ A
2. $A - O_{m\times n} =$ A
3. $A - A = A + (-A) =$ 0 mtx
4. $0A =$ 0 mtx
5. If $kA = O_{m\times n}$, then either $k=0$ or $A = 0$ mtx

Example If $A = \begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -4 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$,

then check that $AB = AC$, and that $A(B - C) = O$.

$AB = \begin{bmatrix} 0 & 0 \\ 1 & 13 \end{bmatrix}$ $\quad AC = \begin{bmatrix} 0 & 0 \\ 1 & 13 \end{bmatrix}$

$AB - AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B - C = \begin{bmatrix} 2 & 0 \\ -3 & 0 \end{bmatrix}$

$A(B-C) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = AC$ **does not necessarily imply that** $B = C$.

$AB = O$ **does not necessarily imply that** $A = O$ **or** $B = O$.

Definition The $n \times n$ **identity** matrix, written I_n has 1s on the main diagonal and 0 in all other entries. (p, 97)

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 7 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 7 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 7 & 3 \end{bmatrix}$$

Any matrix times an identity matrix gives the same matrix back.

The inverse of a matrix (p. 98-100)

Definition If A is an $n \times n$ square matrix and there is an $n \times n$ matrix B such that $AB = BA = I_n$, then we say B is an **inverse** of A.

Example

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ are inverses of each other

Remarks

1. The notion of an inverse is only defined for a *square matrix* as we need both $AB = I_n$ and $BA = I_n$.
2. Not all matrices have an inverse.

Theorem 3.2.6 (p. 99) An invertible matrix has a unique inverse.

Remark Because the inverse is unique we denote the inverse of A by A^{-1}. Thus $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

inverse of A

Inverse of 2×2 matrices:

Theorem 3.2.7 (p. 99)

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$ in which case:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Examples If A is invertible find A^{-1}

$A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ $ad-bc=7$ $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ -5/7 & 6/7 \end{bmatrix}$

$A = \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix}$ $ad-bc=0$

not invertible

Theorem 3.2.8 If A and B are both invertible then $(AB)^{-1} = B^{-1}A^{-1}$.

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

$(AB)(B^{-1}A^{-1}) = \ldots = I$

Powers of a matrix

If A is a *square* matrix, we define the non-negative integer powers of A to be

$$A^0 = I \quad \text{and} \quad A^n = \underbrace{AA\ldots A}_{n \text{ factors}}$$

and if A is *invertible*, then we define the negative powers of A to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\ldots A^{-1}}_{n \text{ factors}}$$

Property If r and s are integers, then

$$A^r A^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}$$

Theorem 3.2.9 If A is *invertible* and n is a non-negative integer, then

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
3. If k is a non-zero scalar, then $(kA)^{-1} = \frac{1}{k}A^{-1}$

Example. Expand $(A+B)^2$

$$(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB$$

$$= \boxed{A^2 + AB + BA + B^2}$$

cannot simplify $AB+BA$ to $2AB$

because in general: $AB \neq BA$