Section 6.3/6.4 pre-lecture comments

Lecture Outline

We will go over kernel and range, and what it means for a transformation to be one-to-one or onto (from 6.3). We will also go over composition of transformations as well as inverse operators (from 6.4).

New terminology

1. composition of transformations
2. kernel
3. range
4. one-to-one
5. onto
6. inverse operator

We can combine transformations together:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathbb{R}^k \arrow[r, "T_2"] & \mathbb{R}^m \end{tikzcd}$$

Composition The composition of two ~~linear~~ transformations $T_1 : \mathbf{R}^n \to \mathbf{R}^k$ and $T_2 : \mathbf{R}^k \to \mathbf{R}^m$, denoted $T_2 \circ T_1$, is defined as:

$$T_2 \circ T_1(\vec{\mathbf{x}}) = T_2(T_1(\vec{\mathbf{x}}))$$

Ex: If $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ and $T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$, what is $T_2 \circ T_1$?

What is $T_1 \circ T_2$?

$$T_2 \circ T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T_2\left(\begin{bmatrix} 2x \\ 3y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ 2x-3y \end{bmatrix}$$

$$T_1 \circ T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T_1\left(\begin{bmatrix} x+y \\ x-y \end{bmatrix}\right) = \begin{bmatrix} 2x+2y \\ 3x-3y \end{bmatrix}$$

different

in general $T_2 \circ T_1 \neq T_1 \circ T_2$

Since both T_1 and T_2 are linear transformations, we can write them in terms of their standard matrices:

$$T_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad T_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$T_2 \circ T_1(\vec{x}) = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 2x-3y \end{bmatrix}$$

(as we have found above)

Composition and matrix multiplication If $T_1 : \mathbf{R}^n \to \mathbf{R}^k$ and $T_2 : \mathbf{R}^k \to \mathbf{R}^m$ are two linear transformations with standard matrices A_1 and A_2, respectively, then

- Their composition $T_2 \circ T_1$ is also a linear transformation, and
- the standard matrix of $T_2 \circ T_1$ is $A_2 A_1$. $T_2 \circ T_1(\vec{x}) = A_2 A_1 \vec{x}$

$T_3 \circ T_2 \circ T_1(\vec{x}) = A_3 A_2 A_1 \vec{x}$

Using this, we can break up complicated transformations into simpler ones

Ex: Suppose we have a linear transformation T in $\mathbf{R}^2$ that transforms a vector $\vec{\mathbf{x}}$ as follows:

1. First, scale $\vec{\mathbf{x}}$ by a factor of 2 (in both x and y directions). T_1
2. Then, reflect $\vec{\mathbf{x}}$ in the x-axis. T_2
3. Then, rotate $\vec{\mathbf{x}}$ by 90^0 counterclockwise. T_3

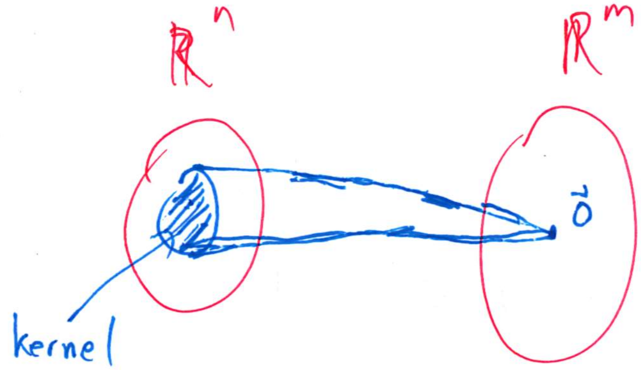
What is the standard matrix of T?

$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$A_3 A_2 A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

Kernel



Kernel of a transformation The **kernel** of a transformation $T : \mathbf{R}^n \to \mathbf{R}^m$, denoted ker($T$), is the set of all vectors $\mathbf{x}$ in $\mathbf{R}^n$ (domain) such that $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$.

1) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ (projection onto x-axis). What is the kernel?

2) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ y \end{bmatrix}$. What is the kernel?

x must be 0, but y can be anything

1) ker(T) is $\left\{\begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R}\right\}$, span $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

2) ker(T) is $\left\{\begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R}\right\}$, span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

y must be 0, but x can be anything

Ex: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x \\ 4y \end{bmatrix}$ $\ker(T) = \{\vec{0}\}$ x, y must be 0

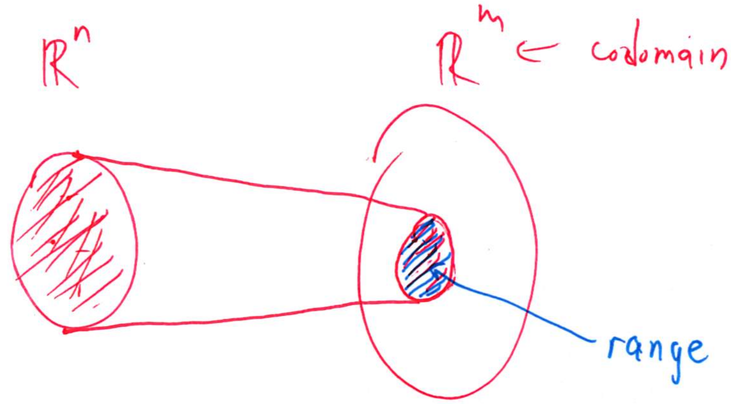
Kernel as a solution space Let T be a linear transformation with standard matrix A. Then ker(T) is the solution space of $A\mathbf{x} = \mathbf{0}$.

- ▶ Kernel of a linear transformation is always a subspace of $\mathbf{R}^n$ (the domain).

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $y = t$, $x = 0$

Solution $\begin{bmatrix} 0 \\ t \end{bmatrix}$ span $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

Range



Range of a transformation The **range** of a transformation $T : \mathbf{R}^n \to \mathbf{R}^m$, denoted $\operatorname{ran}(T)$, is the set of all possible outputs of T; that is:

- ▶ the set of all $\mathbf{b}$ in $\mathbf{R}^m$ (codomain) for which we can find a vector $\mathbf{x}$ in $\mathbf{R}^n$ satisfying $T(\mathbf{x}) = \mathbf{b}$.

1) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ (projection onto x-axis). What is the range?

2) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ y \end{bmatrix}$. What is the range?

1) ran (T) : $\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$, span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

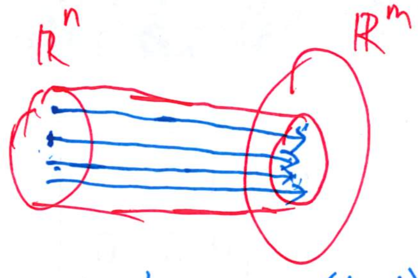
2) ran (T) : $\left\{ \begin{bmatrix} y \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$, span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Ex $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x \\ 4y \end{bmatrix}$ ran(T): all possible vectors / all of $\mathbb{R}^2$ (whole codomain)

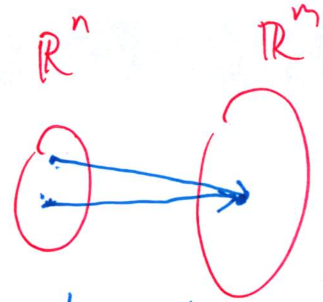
Range is a subspace

- ▶ Range of a linear transformation is always a subspace of $\mathbf{R}^m$ (the codomain).

One-to-one transformations



one-to-one (1-1)



not one-to-one

> **One-to-one** A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** if for each $\mathbf{b} \in \mathbf{R}^m$ there is *at most one* $\mathbf{x}$ in $\mathbf{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.
>
> ▶ T maps distinct vectors in $\mathbf{R}^n$ to distinct vectors in $\mathbf{R}^m$.

Ex: Which transformations below are one-to-one?

1) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ (projection onto x-axis).

different

$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

No, not 1-1.

2) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ (reflection in line $y = x$). Yes 1-1:

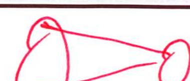
if $T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$

$\begin{bmatrix} y_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} y_2 \\ x_2 \end{bmatrix}$ $x_1 = x_2$, $y_1 = y_2$

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ (same vector)

> **One-to-one and kernel**
>
> A linear transformation T is one-to-one if and only if $\ker(T)$ is $\{\vec{\mathbf{0}}\}$.
>
> ▶ In other words, the only solution to $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$ is $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

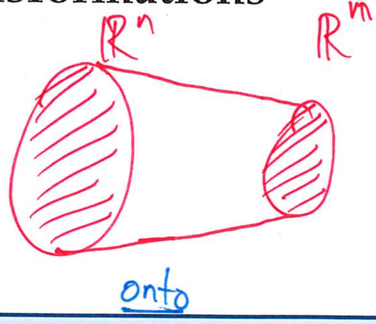
$A\vec{x} = \vec{0}$ only trivial solution

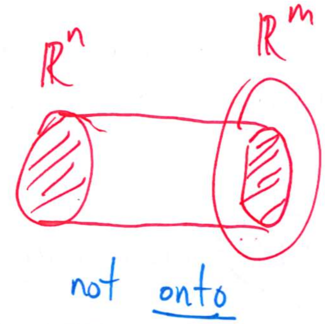


Can a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ be one-to-one? No. $A\vec{x} = \vec{0}$

A is 2×3 mtx $\Rightarrow$ free var.

Onto transformations





Onto A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is **onto** if for each $\mathbf{b} \in \mathbf{R}^m$ there is *at least one* $\mathbf{x}$ in $\mathbf{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$.

- In other words, $\operatorname{ran}(T)$ is the whole codomain $\mathbf{R}^m$.

Ex: Which transformations below are onto?

$\operatorname{ran}(T) = \operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

1) $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\0\end{bmatrix}$ (projection onto x-axis). No, not onto

There is no $\begin{bmatrix}x\\y\end{bmatrix}$ such that $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$

2) $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}y\\x\end{bmatrix}$ (reflection in line $y = x$). Yes, onto

$\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ $\quad y = b_1$, $x = b_2$

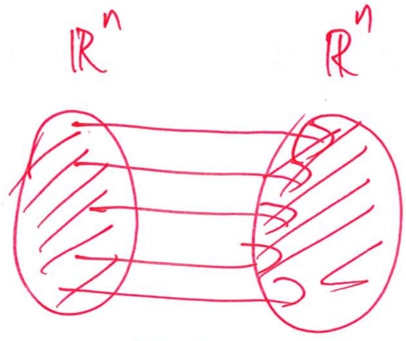
$T\left(\begin{bmatrix}b_2\\b_1\end{bmatrix}\right) = \begin{bmatrix}b_1\\b_2\end{bmatrix}$

Onto and the consistency problem ($A\vec{x} = \vec{b}$ consistent?)
A linear transformation T is onto if and only if $T(\mathbf{x}) = \mathbf{b}$ is consistent for all $\mathbf{b}$ in the codomain $\mathbf{R}^m$.

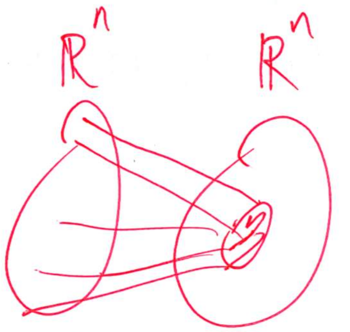
- In other words, if A is the standard matrix of T, then the column space of A is all of $\mathbf{R}^m$. (Section 3.5)

Can a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ be onto? No $\quad$ A is 3×2

Relation between one-to-one and onto when domain and codomain are the same



1-1 onto



not 1-1 not onto

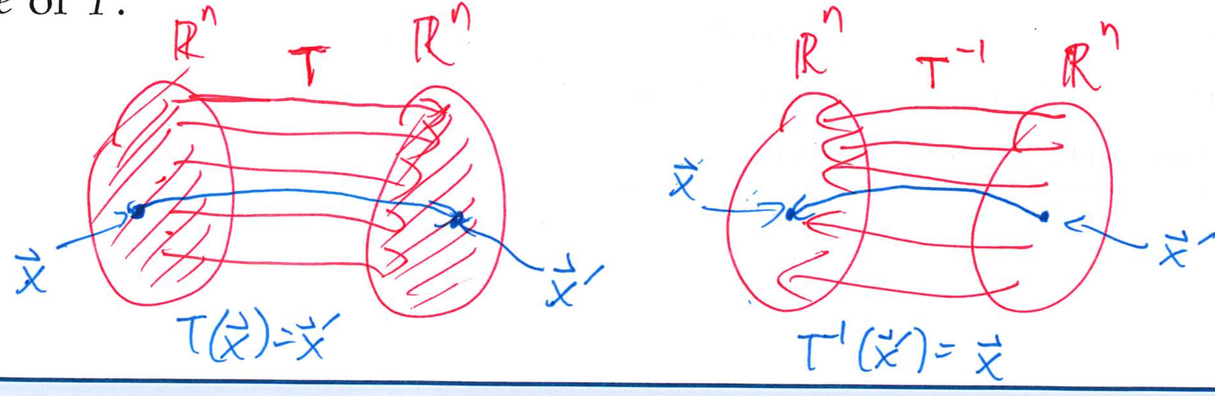
> **Thm. 6.3.14 and Invertible Mtx Thm. (p302)**
>
> If $T : \mathbf{R}^n \to \mathbf{R}^n$ is a linear operator with standard matrix A then the following are equivalent:
>
> 1. T is one-to-one.
> 2. T is onto.
> 3. A is invertible.

Ex:

1) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ (projection onto x-axis). not 1-1 not onto $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible

2) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$ (reflection in line $y = x$). 1-1 onto $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ invertible

Note that if $T : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one and onto, then for each $\mathbf{b}$ in $\mathbf{R}^n$, there is *exactly one* vector $\mathbf{x}$ in $\mathbf{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$. So we can take the *inverse* of T.



> **Inverse operator** The inverse of a one-to-one and onto operator $T : \mathbf{R}^n \to \mathbf{R}^n$, denoted $T^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ is defined as:
>
> ▸ $T^{-1}(\vec{\mathbf{x}}') = \vec{\mathbf{x}}$, where $\vec{\mathbf{x}}$ is the unique vector for which $T(\vec{\mathbf{x}}) = \vec{\mathbf{x}}'$.

If the standard matrix of T is A, what is the standard matrix of T^{-1}?

$T(\vec{x}) = \vec{x}' \longrightarrow A\vec{x} = \vec{x}' \rightarrow \vec{x} = A^{-1}\vec{x}' \rightarrow T^{-1}(\vec{x}') = \vec{x}$ $\quad A^{-1}$

Ex: What is the inverse operator of $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \end{bmatrix}$ and its standard matrix?

Standard mtx of T: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} x+y \\ x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ std mtx of T^{-1}

$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x-y \end{bmatrix}$