Constructing orthogonal bases

Example: W is the plane $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t.$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is not an orthogonal basis for W.

But we can turn it into an orthogonal basis by replacing the second vector.

If we let $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, then we can find the vector component of $\mathbf{w}_2$ that is perpendicular to $\mathbf{w}_1$:

$$\mathbf{w}_2 - \mathrm{proj}_{\mathbf{w}_1}(\mathbf{w}_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}.$$

So by replacing $\mathbf{w}_2$ with this new vector $\begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$, we get an orthogonal basis for W:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$$

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The Gram-Schmidt Process works as follows:

Given any basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ for W, we can turn it into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ for W as follows:

- $\mathbf{v}_1 = \mathbf{w}_1$
- $\mathbf{v}_2 = \mathbf{w}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{w}_2)$
- $\mathbf{v}_3 = \mathbf{w}_3 - \text{proj}_{\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}}(\mathbf{w}_3)$
- $\vdots$

That is:

- The first vector is unchanged ($\mathbf{w}_1$).
- The second vector is the component of $\mathbf{w}_2$ perpendicular to $\mathbf{v}_1$.
- The third vector is the component of $\mathbf{w}_3$ perpendicular to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
- And so on.

Notice that at each step, we are projecting a $\mathbf{w}$ vector onto an orthogonal basis (for the span of previous vectors).

This means we can use the projection formulas for orthogonal bases.

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Given any basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ for W, we can turn it into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ for W as follows:

- $\mathbf{v}_1 = \mathbf{w}_1$
- $\mathbf{v}_2 = \mathbf{w}_2 - \left(\dfrac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\|\mathbf{v}_1\|^2}\right) \mathbf{v}_1$
- $\mathbf{v}_3 = \mathbf{w}_3 - \left(\dfrac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\|\mathbf{v}_1\|^2}\right) \mathbf{v}_1 - \left(\dfrac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\|\mathbf{v}_2\|^2}\right) \mathbf{v}_2$
- $\vdots$

This process gives an orthogonal basis for W.

To get an orthonormal basis, divide each vector by its norm to turn them all into unit vectors.

See Section 7.9 Examples 9 and 10 (pages 412-413) for examples on using the Gram-Schmidt process to construct an orthogonal/orthonormal basis.