Section 7.1-7.2 pre-lecture comments

Lecture Outline

Given a subspace V, a **basis** is set of linearly independent vectors which spans V. The **dimension** of V is the number of vectors in any basis of V.

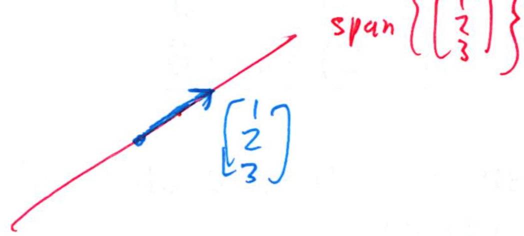
New terminology

1. basis
2. dimension

Recall lines and planes can be given as spans of vectors:

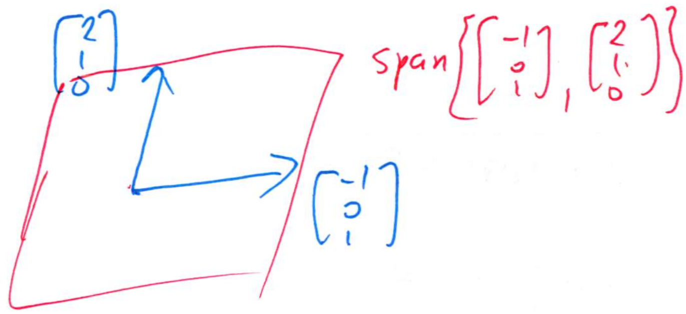
Ex: $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t$

$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ <u>line</u>



Ex: $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t$

$\text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ <u>plane</u>



Basis Let V be a nontrivial subspace of $\mathbf{R}^n$. A **basis** for V is a set of linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ such that $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

there are infinitely many bases

Ex: Give a basis for $\vec{\mathbf{x}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t.$

basis: $B = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

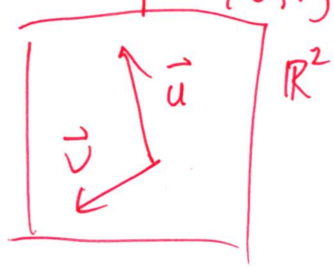
Ex: Give a basis for $\mathbf{R}^2$. How many bases for $\mathbf{R}^2$ are there?

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ standard basis for $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ basis

any $\vec{u}, \vec{v}$ independent

span$\{\vec{u}, \vec{v}\} = \mathbb{R}^2$



The **standard basis** for $\mathbf{R}^n$ is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

standard unit vectors

Size of a basis All bases for a subspace V have the same number of vectors k. This number is referred to as the **dimension** of V, denoted $\dim(V)$. (Note: We define $\dim(\{\mathbf{0}\})$ to be 0.)

Ex: What is the dimension of $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t$? 1

span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Ex: What is the dimension of $\vec{\mathbf{x}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t$? 2

span $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Ex: What is the dimension of $\mathbf{R}^n$? n

span $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

Ex: $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} s$ dimension is 1

dependent

span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Why is it called a *basis*?

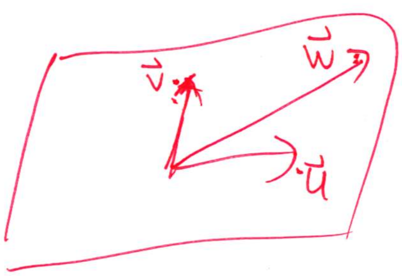
Thm. 7.2.1 (p335) If $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ is a basis for V, then every vector $\vec{\mathbf{v}}$ in V can be written as a *unique* linear combination of vectors in B.

- *Linear combination* because B spans V
- *Unique* because B is linearly independent

Ex: $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t$

Every vector in the plane can be written uniquely as $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} a + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} b$

for some $a, b \in \mathbb{R}$



$\vec{w} = a\vec{u} + b\vec{v}$

unique

Thm. 7.2.2 (p336) Suppose B is a set of vectors of a subspace V, but B is *not* a basis for V for either of the following reasons:

- B is linearly independent, but fails to span V. In this case, we can expand B using additional vectors from V to form a basis.
- B spans V, but is linearly dependent. In this case, we can remove vectors from B to form a basis.

Ex: V is the plane $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t.$

1) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ add $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$: $B = \underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{lin. indep.}}$

could add $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ instead

2) $B = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ remove $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$: $B = \underbrace{\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}}_{\text{lin. indep.}}$

could remove $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ instead