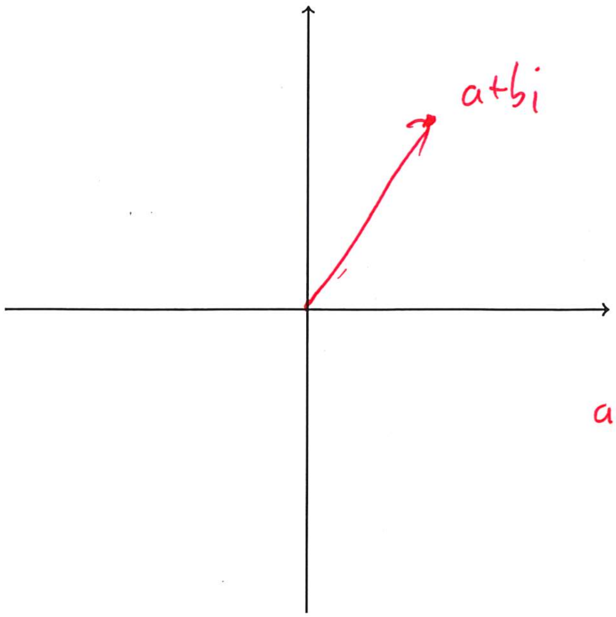
MATH 232 Additional notes on 4.4/App. B

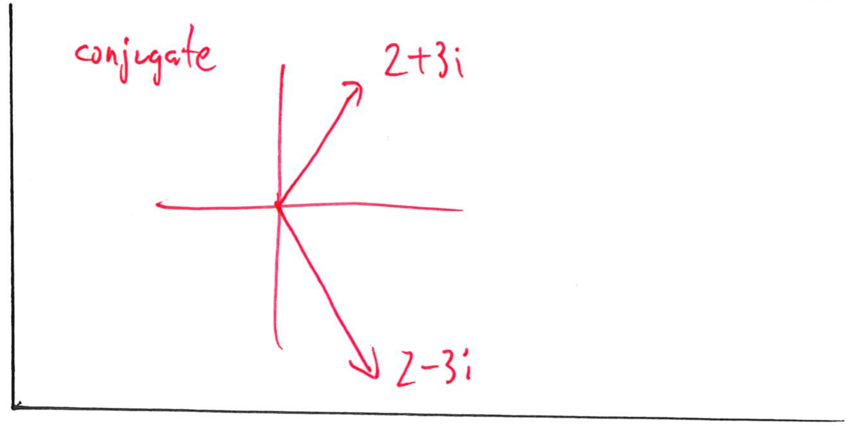
Lecture Outline
This section contains additional notes related to both complex numbers and eigenvalues/eigenvectors.

New terminology

1. absolute value (of a complex number)

Absolute value of a complex number





absolute value = norm of (a,b)

$= \sqrt{a^2+b^2}$

Absolute value The **absolute value** of $z = a + bi$, denoted $|z|$, is $|z| = \boxed{\sqrt{a^2+b^2}}$.

- Same as norm/magnitude of the 2D vector (a, b)

Ex: What is $|2+3i|$?

$\sqrt{2^2+3^2} = \sqrt{4+9} = \boxed{\sqrt{13}}$

Note that:

- $|z_1 z_2| = |z_1||z_2|$
- $|z^k| = |z|^k$
- $z\bar{z} = |z|^2$ $\quad (a+bi)(a-bi) = a^2+b^2$

as $k \to \infty$

if $|z|<1$: $|z^k| \to 0$ $(z^k \to 0)$

if $|z|>1$: $|z^k| \to \infty$ (z^k diverges)

$\downarrow$

$|z| = \sqrt{z\bar{z}}$

Other properties of eigenvalues

Note that A is invertible if and only if 0 is *not* an eigenvalue of A.
(add this to Invertible Mtx Thm.) If $\lambda = 0$ eigenvalue: $A\vec{x} = \vec{0}$ nontrivial soln

Properties If λ is an eigenvalue of A and $\mathbf{x}$ is an eigenvector of A corresponding to λ, then:

$1/\lambda$

1. λ^k is an eigenvalue of A^k for all integers $k > 0$.
2. If A is invertible, then λ^{-1} is an eigenvalue of A^{-1}.
3. $c\lambda$ is an eigenvalue of cA for all ~~nonzero~~ scalars c.
4. $\lambda - k$ is an eigenvalue of $A - kI$ for all scalars k.

Furthermore, $\mathbf{x}$ is an eigenvector for all the above matrices.

$Ax = \lambda x$ $\quad\quad A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$

Ex. If the eigenvalues of A are 2, -3, and $1/5$, what are the eigenvalues of A^2? What about A^{-1}? $5A$?

Eigenvalues of A^2: $4, 9, 1/25$

——— A^{-1}: $1/2, -1/3, 5$

——— $5A$: $10, -15, 1$

Powers of matrices and eigenvalues/eigenvectors Ex: $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

We can "apply" a matrix to a vector over and over:

$$\vec{\mathbf{v}}, \quad A\vec{\mathbf{v}}, \quad A^2\vec{\mathbf{v}}, \quad A^3\vec{\mathbf{v}}, \quad \ldots$$

$A\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$A^2\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $A^3\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

What is the long term behavior? (Does it diverge to infinity, or converge, or ...?)

switching rows

If $\mathbf{v}$ can be written as a linear combination of eigenvectors of A (which is usually the case): $\vec{\mathbf{v}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + \ldots c_n\vec{\mathbf{v}}_n$, where the $\mathbf{v}_i$ are eigenvectors corresponding to λ_i:

Then $A^k\mathbf{v} = A^k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n)$

$$= c_1 A^k \vec{v}_1 + c_2 A^k \vec{v}_2 + \ldots + c_n A^k \vec{v}_n$$

$$= c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2 + \ldots + c_n \lambda_n^k \vec{v}_n$$

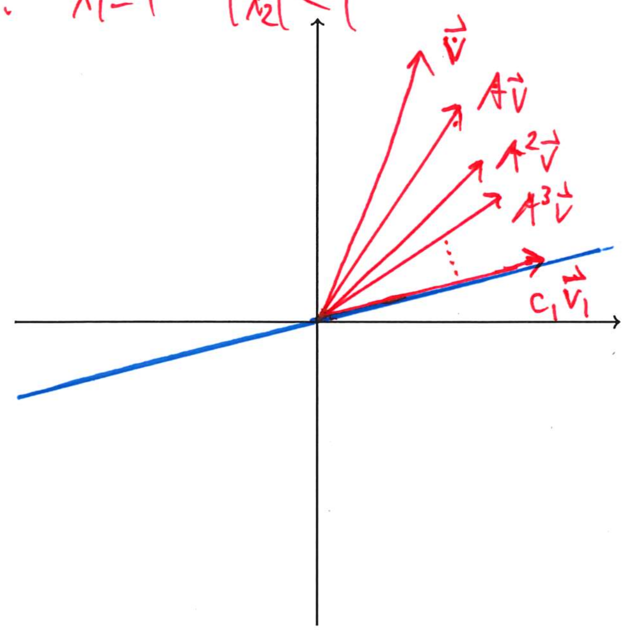
If all eigenvalues are small enough in absolute value, then $A^k\mathbf{v}$ converges as $k \to$ infinity.

"largest" "smaller" (may be complex #s)

For example, $\lambda_1 = 1$ and $|\lambda_2| < 1$, $|\lambda_3| < 1$, ..., $|\lambda_k| < 1$. What does $A^k\mathbf{v}$ converge to?

$$A^k\vec{v} \to c_1(1)\vec{v}_1 + c_2(0)\vec{v}_2 + c_3(0)\vec{v}_3 + \ldots + c_n(0)\vec{v}_n = c_1\vec{v}_1$$

2D: $\lambda_1 = 1$ $|\lambda_2| < 1$



Markov Matrices
(Section 5.1)