In the back of the textbook.

MATH 232 Appendix B pre-lecture comments

Lecture Outline
Previously, we assumed that the eigenvalues of a matrix are real numbers. However for some matrices that is not the case.

We will discuss complex numbers and how they arise when solving for eigenvalues of a matrix.

New terminology

1. complex number
2. complex plane
3. conjugate

Complex numbers

Recall that equations like $x^2 + 1 = 0$ have no solutions in real numbers. To represent solutions, we thus need *complex numbers*.

We will use the symbol "i" which has the following property:

$$i^2 = -1$$

but otherwise has the same addition/multiplication properties as the real numbers. With this, we can now solve $x^2 + a = 0$, where a is a positive real number.

$x^2+a=0 \Rightarrow x^2=-a \Rightarrow x=\pm(\sqrt{a})i$ $\quad (\sqrt{a}\,i)^2 = a(-1) = -a$

$(-\sqrt{a}\,i)^2 = a(-1) = -a$

By convention, we write $\sqrt{-a} = (\sqrt{a})i$.
In particular, we may write $i = \sqrt{-1}$.

Be careful with square roots. $\underbrace{\sqrt{(-1)\cdot(-1)}}_{1} \neq \underbrace{\sqrt{-1}}_{i}\underbrace{\sqrt{-1}}_{i} = -1$.

($\sqrt{ab} = \sqrt{a}\sqrt{b}$ only holds for real non-negative numbers!)

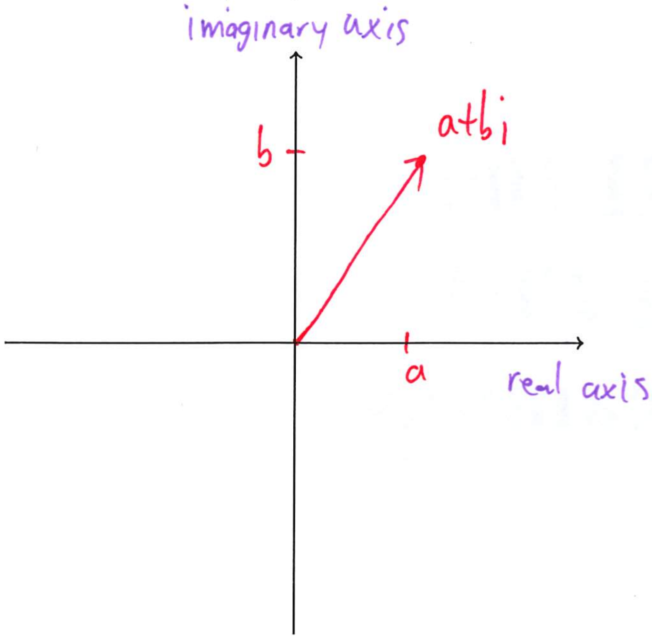
Complex number
A complex number is an expression of the form $a + bi$, where a and b are real numbers.

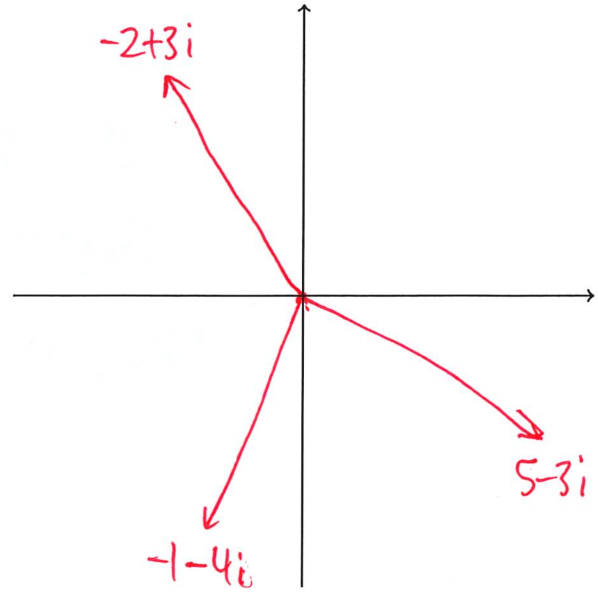
$2+3i$ $\qquad -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ $\qquad 2\pi i$ $\qquad 5$

pure imaginary number

real number

We can draw complex numbers in the **complex plane**, as if $a + bi$ were the 2D vector (a, b):





Complex numbers can even be added/subtracted just like 2D vectors:

- $(a + bi) + (c + di) = (a+c) + (b+d)i$ $\quad (2+3i) - (3+2i) = -1+i$
- $(a + bi) - (c + di) = (a-c) + (b-d)i$

Unlike vectors however, they can be multiplied together to form another complex number (replacing i^2 with -1).

- $(a + bi)(c + di) = ac + adi + bci + bd\overset{-1}{i^2} = ac - bd + (ad+bc)i$

Ex: $(1+2i)(2-3i) = 2 - 3i + 4i - 6\overset{-1}{i^2} = 8 + i$

How about division? $\frac{1}{i} = -i \quad (-i^2 = 1)$

How about $\frac{1+2i}{-2+i}$?

We can also divide complex numbers (except by 0, of course):

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\,\frac{c-di}{c-di} = \frac{ac+bd+(-ad+bc)i}{c^2+d^2+(\cancel{-cd+cd})i} = \frac{1}{c^2+d^2}\left(ac+bd+(-ad+bc)i\right)$$

$$\frac{1+2i}{-2+i} = \frac{1+2i}{-2+i}\,\frac{-2-i}{-2-i} = \frac{-2-i-4i-2i^2}{4+2i-2i-i^2} = \frac{-5i}{5} = -i$$

Note that conjugate is important enough to get its own definition:

Complex conjugate

The **(complex) conjugate** of $z = a + bi$, denoted $\bar{z}$, is $\bar{z} = a - bi$.

$z = 3+4i \qquad \bar{z} = 3-4i$

$z = i \qquad \bar{z} = -i$

Ex: Solve $x^2 - x + 6 = 0$.

$ax^2+bx+c=0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x = \frac{1 \pm \sqrt{1-24}}{2} = \frac{1 \pm \sqrt{-23}}{2} = \frac{1 \pm \sqrt{23}\,i}{2}$$

$$x = \frac{1}{2} + \frac{\sqrt{23}}{2}i, \quad \frac{1}{2} - \frac{\sqrt{23}}{2}i$$

conjugates

Back to the eigenvalue problem

Other examples of complex eigenvalues are in Section 8.8

Ex: Let $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$.

What are the eigenvalues of A?

$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda \end{bmatrix}$

$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda \end{vmatrix}$

$= \lambda^2 + 2 = 0 \Rightarrow \lambda = \pm\sqrt{2}i$

$\lambda_1 = \sqrt{2}i \qquad \lambda_2 = -\sqrt{2}i$

conjugates

Show that $\begin{bmatrix} -\frac{\sqrt{2}}{2}i \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = (\sqrt{2})i$.

$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix} = \sqrt{2}i\begin{bmatrix} -\frac{\sqrt{2}}{2}i \\ 1 \end{bmatrix}$

$\frac{1}{\sqrt{2}i} = \frac{1}{\sqrt{2}}(-i)$

$= \frac{\sqrt{2}}{2}(-i)$

$= -\frac{\sqrt{2}}{2}i$

Find the eigenspace of A corresponding to $\lambda = (\sqrt{2})i$ by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

$\lambda = \sqrt{2}i$

$\left[\begin{array}{cc|c} \lambda & -1 & 0 \\ 2 & \lambda & 0 \end{array}\right] = \left[\begin{array}{cc|c} \sqrt{2}i & -1 & 0 \\ 2 & \sqrt{2}i & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 \cdot (-\sqrt{2}i)} \left[\begin{array}{cc|c} 2 & \sqrt{2}i & 0 \\ 2 & \sqrt{2}i & 0 \end{array}\right]$

conjugate

$\xrightarrow{r_2 \leftarrow r_2 - r_1} \left[\begin{array}{cc|c} 2 & \sqrt{2}i & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 \div 2} \left[\begin{array}{cc|c} 1 & \frac{\sqrt{2}}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$y = t$

$x = -\frac{\sqrt{2}}{2}it$

(next page)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} i t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} i \\ 1 \end{bmatrix} t \qquad \boxed{\text{Span}\left\{ \begin{bmatrix} -\frac{\sqrt{2}}{2} i \\ 1 \end{bmatrix} \right\}}$$

eigenspace of $\lambda_1 = \sqrt{2}\, i$

eigenspace of $\lambda_2 = -\sqrt{2}\, i$ is span $\left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} i \\ 1 \end{bmatrix} \right\}$

conjugate

conjugate

eigenspace corresponding to conjugate eigenvalue

= conjugate of the original eigenspace

If $Ax = \lambda x$ then $A\bar{x} = \bar{\lambda}\bar{x}$