Section 7.3-7.6 pre-lecture comments

Lecture Outline

We will talk more about subspaces related to a matrix (row/column/null spaces).

New terminology

1. row space
2. column space
3. null space
4. orthogonal complement

Fundamental spaces of a matrix

m rows n columns

Fundamental spaces Let A be an $m \times n$ matrix. Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ be the columns of A and $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ be the rows of A.

$$A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \leftarrow \vec{r}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_m \rightarrow \end{bmatrix}$$

subspace of

1. **Row space** of A (row(A)) is span$\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$ $\mathbb{R}^n$
2. **Column space** of A (col(A)) is span$\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ $\mathbb{R}^m$
3. **Null space** of A (null(A)) is the solution space of $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ $\mathbb{R}^n$
4. **Null space** of A^T (null(A^T)) is the solution space of $A^T\mathbf{x} = \mathbf{0}$ $\mathbb{R}^m$

"Left nullspace of A"

Ex: $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \end{bmatrix}$. What is the row space, column space and null space of A?

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} -4 \\ 3 \end{bmatrix}} \right\} = \mathbb{R}^2$$

is a linear combination of previous vectors, can get rid of it in span

nullspace:

$$\begin{array}{ccc} x & y & z \\ \end{array}$$
$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -4 & 0 \\ 0 & \boxed{1} & 3 & 0 \end{array}\right] \rightarrow \begin{array}{l} z = t \quad \text{(free)} \\ y = -3t \\ x = 4t \end{array}$$

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \right\}$$

Rank and nullity

dim(row(A))

The **rank** of a matrix A (rank(A)) is the dimension of its row space.

The **nullity** of A (nullity(A)) is the dimension of its null space.

dim(null(A))

Ex: $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \end{bmatrix}$. What is the rank and nullity of A?

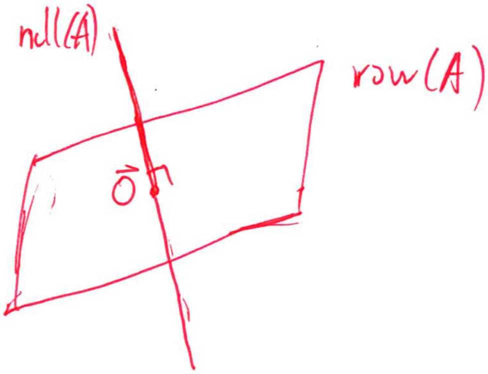
rank(A) = 2

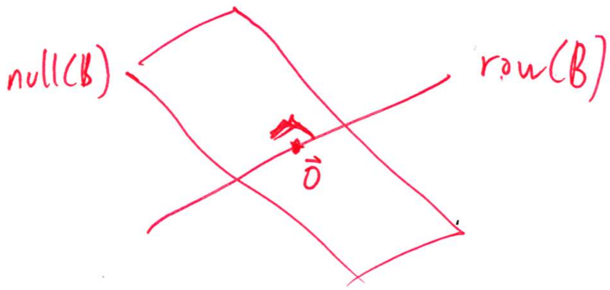
nullity (A) = 1

$$B = \begin{array}{c} \begin{array}{ccc} & \text{free} & \text{free} \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & 3 \end{bmatrix} \end{array}$$

rank(B) = 1

nullity (B) = 2





Dimension Theorem (Section 7.4) For any $m \times n$ matrix A,
rank(A) + nullity(A) = n. # of columns of A

$$A = \begin{array}{c} \begin{bmatrix} \boxed{1} & 0 & -4 \\ 0 & \boxed{1} & 3 \end{bmatrix} \\ \begin{array}{ccc} \text{leading} & \text{leading} & \text{free} \end{array} \end{array}$$

rank(A) = number of leading 1s

nullity(A) = number of free variables

rank(A) = 2

nullity(A) = 1

Finding bases for the fundamental spaces using row reduction

Note that elementary row operations do not change the row space or null space of a matrix.

> **Basis for row and null spaces** Let A be a matrix with (reduced) row echelon form R.
>
> - To get a basis for the row space of A, take all nonzero rows of R.
> - To get a basis for the null space of A, write the solution of $R\mathbf{x} = \mathbf{0}$ in vector form.

REF

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -6 & 3 \\ 1 & 3 & 4 \end{bmatrix} \qquad R = \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & \boxed{1} & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Give bases for the row space and null space of A.

basis for row(A): $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \right\}$ (row(A) = row(R))

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - 2r_2} \left[\begin{array}{ccc|c} 1 & 0 & -11 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$y = -5t$

$x = 11t$

$$\begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix} t$$

a basis for null(A): $\left\{ \begin{bmatrix} 11 \\ -5 \\ 1 \end{bmatrix} \right\}$ (null(A) = null(R))

Basis for column space (The pivot theorem, Sec. 7.6)

col(A) ≠ col(R)

Basis for column space Let A be a matrix with (reduced) row echelon form R.

- To get a basis for the column space of A, take all columns of A corresponding to the same columns as the leading ones (pivot positions) of R.

The leading ones indicate which columns are *not* the linear combinations of previous columns (or the **0** vector).

REF

$$\text{Ex: } A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & 5 \\ 2 & -6 & 3 \end{bmatrix} \qquad R = \begin{bmatrix} \boxed{1} & -3 & 2 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

Give a basis for the column space of A.

a basis for col(A): $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \right\}$

note: take the columns from A, not from R (col(A) ≠ col(R))

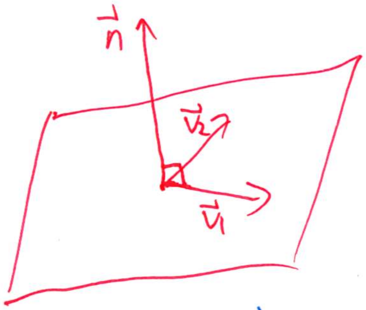
Rank Theorem (Section 7.5) Row space and column space of a matrix A have the same dimension:

dim(col(A)) = dim(row(A)) = rank(A)

Orthogonal complement

Ex: Let P be the plane $\vec{\mathbf{x}} = \begin{array}{c} \vec{v}_1 \\ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{array} s + \begin{array}{c} \vec{v}_2 \\ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{array} t.$

Find a normal vector $\mathbf{n}$ to the plane P.

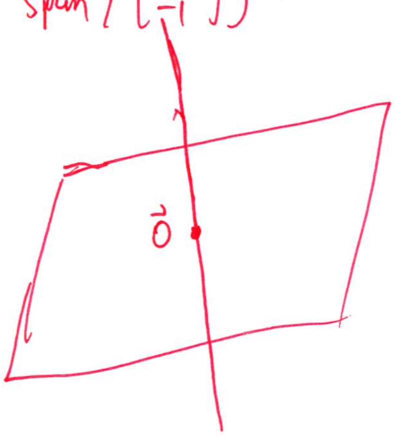


either $\begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$ $\vec{v}_1 \times \vec{v}_2$ (cross product)

or $\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{array}{c} \vec{n} \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{n} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \longleftarrow$ any scalar multiple (nonzero) is OK

$\text{span}\left\{ \begin{array}{c} \vec{n} \\ \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \end{array} \right\}$



$\text{span}\left\{ \begin{array}{c} \vec{v}_1 \\ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} \vec{v}_2 \\ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{array} \right\}$

Every vector in span($\vec{\mathbf{n}}$) is orthogonal to every vector in span($\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$).

We say that span($\mathbf{n}$) and span($\mathbf{v}_1, \mathbf{v}_2$) are orthogonal complements.

Orthogonal complement Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in $\mathbf{R}^n$. The **orthogonal complement** of S, denoted $S^{\perp}$, is the set of all vectors $\vec{\mathbf{x}}$ that are orthogonal (perpendicular) to all vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$.

Ex: $S = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}} \right\}$ $\qquad S^{\perp} = \text{span}\left\{ \underset{\vec{n}}{\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}} \right\}$ (from previous page)

Solutions to $A\mathbf{x} = \mathbf{0}$ are orthogonal to every row of A.

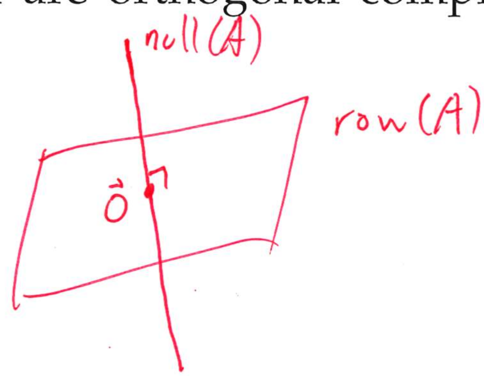
$$\begin{bmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \leftarrow \vec{r}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_m \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{x} \\ \downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{aligned} \vec{r}_1 \cdot \vec{x} &= 0 \\ \vec{r}_2 \cdot \vec{x} &= 0 \\ &\vdots \\ \vec{r}_m \cdot \vec{x} &= 0 \end{aligned}$$

if S subspace, then $(S^{\perp})^{\perp} = S$ $\qquad$ orthogonal complement is a subspace

In other words, row space and null space of A are orthogonal complements of each other.

$\text{null}(A) = \text{row}(A)^{\perp}$

$\text{row}(A) = \text{null}(A)^{\perp}$



Note: Since $\text{col}(A) = \text{row}(A^T)$

$\text{col}(A)$ and $\text{null}(A^T)$ are orthogonal complements

How to find orthogonal complement

Since row space and null space of A are orthogonal complements of each other, we can find orthogonal complement of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ as follows:

- Write $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ as the rows of a matrix A.
- Solve $A\mathbf{x} = \mathbf{0}$.

Ex: Find the orthogonal complement of $\left\{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ -3 \\ 7 \end{bmatrix}} \right\}$.

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & 7 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - r_2} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & -5 & 0 \\ 0 & \boxed{1} & -3 & 7 & 0 \end{array}\right] \end{array}$$

$x_4 = t$ (free) $\qquad x_2 = 3s - 7t$

$x_3 = s$ (free) $\qquad x_1 = -2s + 5t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 1 \end{bmatrix} t \qquad S^\perp = \text{span}\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$\dim(W) + \dim(W^\perp) = n$$

W is a subspace