MATH 232 Section 1.3 pre-lecture comments

We will be discussing lines and planes. These will give us one picture for thinking about linear systems.

You should review lines and slopes from calculus/pre-calculus.

There is no new terminology today but you will need to know the different ways of writing descriptions of lines and planes. It is important to know the names but much more important to know the geometric distinctions.

$y = mx + b$

Definition of a line in $\mathbf{R}^2$ A line in $\mathbf{R}^2$ is all points (x, y) that satisfy

$$Ax + By = C$$

where at least one of A and B is not zero. e.g. $y = 2x$ $\quad y = 4x + 1$

$y = 3 \quad x = -1$

Example Find the equation of the line passing through $(0,1)$ and $(3,2)$.

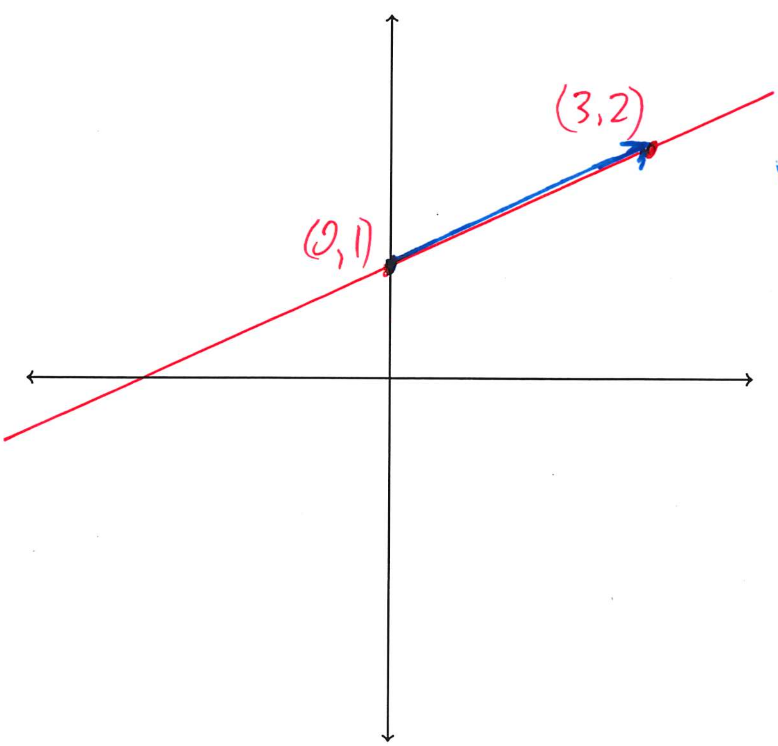
slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2-1}{3-0} = \dfrac{1}{3}$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{3}(x - 0)$$

$$\boxed{y = \frac{1}{3}x + 1}$$

What does this look like geometrically?

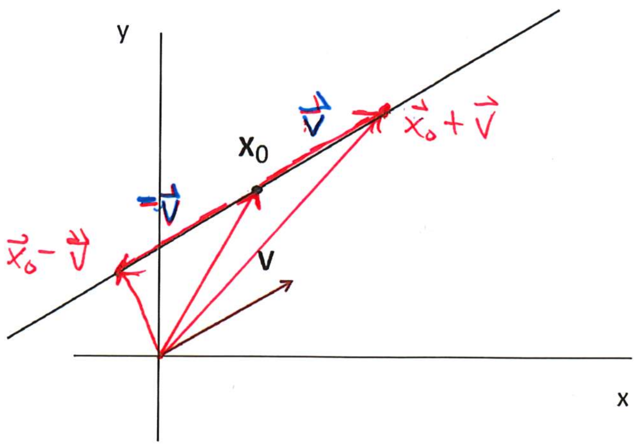


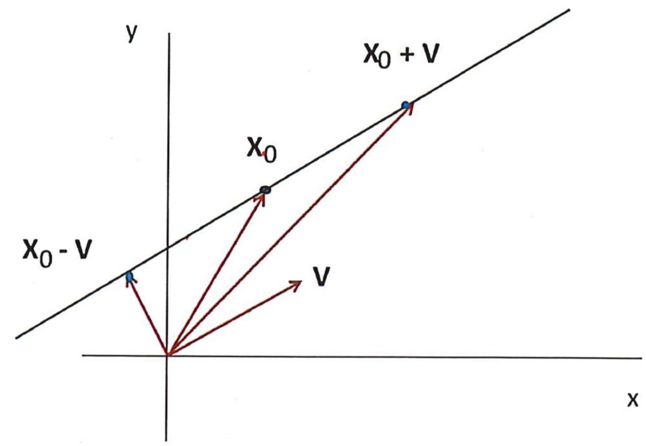
vector: $\overbrace{(3,1)}^{\text{direction}}$

parameter (any real number)

$$\underbrace{(0,1)}_{\text{starting pt}} + t\underbrace{(3,1)}_{\text{direction}}$$

Geometrically, a **line** is determined by a point $\vec{\mathbf{x}}_0$ and a direction $\vec{\mathbf{v}}$:





starting pt direction

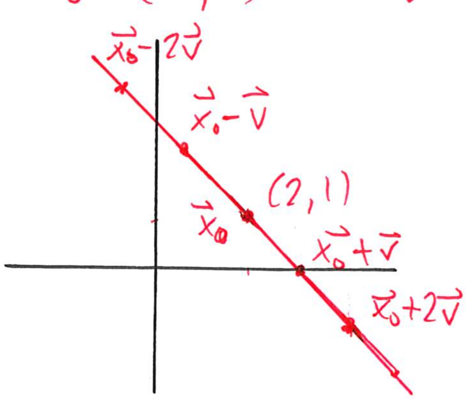
Vector form of a line With scalar parameter t:

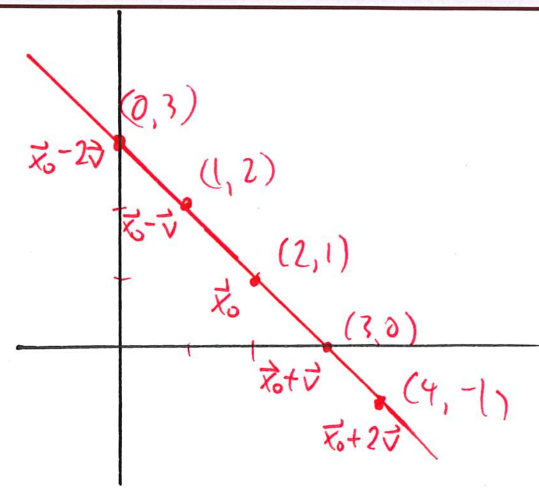
$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_0 + t\vec{\mathbf{v}}, \quad t \in \mathbf{R}$$

Question What if $\vec{\mathbf{x}}_0 = \vec{\mathbf{0}}$?

$\vec{x} = t\vec{v}$ (goes through the origin)

$\vec{x}_0 = (2,1)$ $\vec{v} = (1,-1)$





$\vec{x} = \vec{x}_0 + t\vec{v}$

Question Do the lines $(x,y,z) = (2,0,1) + t(3,-1,0)$ and $(x,y,z) = (5,1,-4) + s(1,0,2)$ intersect?

We need: $2+3t = 5+s$

$-t = 1 \longrightarrow t = -1$

$1 = -4+2s \longrightarrow s = 5/2$

$2+3(-1) \stackrel{?}{=} 5 + 5/2$

$-1 \neq 15/2$

Lines do not intersect

Parametric equation of a line in R^2 The set of all $\vec{\mathbf{x}} = (x,y)$ such that

$$x = x_0 + ta \quad \text{and} \quad y = y_0 + tb, \quad t \in R,$$

represents the line through $\mathbf{x_0} = (x_0, y_0)$ that is parallel to $\mathbf{v} = (a,b)$.

Parametric equation of a line in R^3 The set of all $\vec{\mathbf{x}} = (x,y,z)$ such that

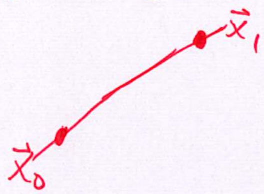
$$x = x_0 + ta \quad \text{and} \quad y = y_0 + tb \quad \text{and} \quad z = z_0 + tc, \quad t \in R,$$

represents the line through $\mathbf{x_0} = (x_0, y_0, z_0)$ that is parallel to $\mathbf{v} = (a,b,c)$.

$(x,y,z) = (2,0,1) + t(3,-1,0) \longleftarrow$ vector equation

$$\left.\begin{aligned} x &= 2+3t \\ y &= -t \\ z &= 1 \end{aligned}\right\} \text{parametric equation}$$

Example What is the vector equation of the line passing through points $\mathbf{x_0}$ and $\mathbf{x_1}$ in R^n?



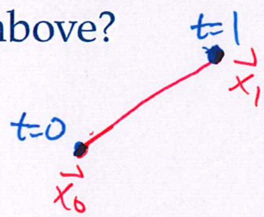
step 1: find direction $\vec{v} = \vec{x}_1 - \vec{x}_0$

step 2: find a starting pt: $\vec{x}_0$

Line: $$\boxed{\begin{aligned}\vec{x} &= \vec{x}_0 + t(\vec{x}_1 - \vec{x}_0) \\ &= (1-t)\vec{x}_0 + t\vec{x}_1\end{aligned}}$$

$t=0$: $\vec{x}_0$ $t=1$: $\vec{x}_1$

Example. What happens if we change $t \in R$ to $0 \le t \le 1$ in the equation above?



line segment from $\vec{x}_0$ to $\vec{x}_1$

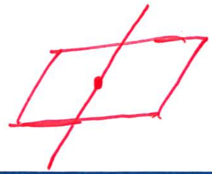
Example. What is the vector and parametric equation of the line passing through points (0,1,1) and (1,1,0)?

$\vec{x}_0 = (0,1,1)$ $\quad\quad$ $\vec{v} = \vec{x}_1 - \vec{x}_0 = (1,0,-1)$

vector equation: $\vec{x} = \vec{x}_0 + t\vec{v} = (0,1,1) + t(1,0,-1)$

parametric eqn:

$$x = t$$
$$y = 1$$
$$z = 1-t$$

We can also describe planes algebraically.  in 3D

General form of a plane in $\mathbf{R}^3$ A plane in $\mathbf{R}^3$ is all points (x, y, z) that satisfy

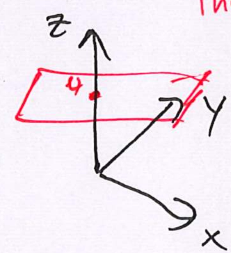
$$Ax + By + Cz = D$$

where at least one of A, B and C is not zero. e.g. $x - 2y + 3z = 5$

Examples What does it mean if $D = 0$?

$Ax + By + Cz = 0$ plane goes through origin: $(0,0,0)$ satisfies the eqn.

Describe the plane $z = 4$.



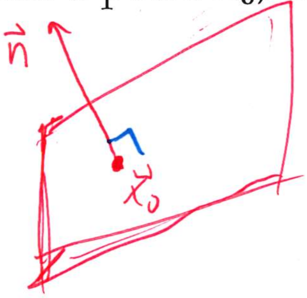
x, y anything

$z = 4$ $(x, y, 4)$

all vectors of the form $(x, y, 4)$

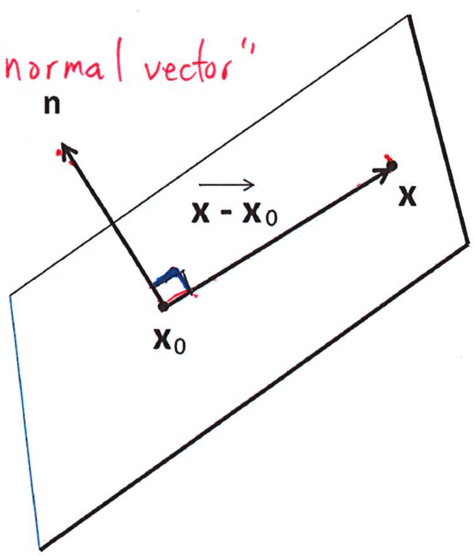
R^3 nonzero

Geometrically we can define a plane in the following way: "Given a direction $\mathbf{n}$ and a point $\mathbf{x}_0$, the plane is all $\mathbf{x}$ so that $\mathbf{x} - \mathbf{x}_0$ is perpendicular to $\mathbf{n}$"



$\vec{n}$ perpendicular to plane (orthogonal)

"normal vector" $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$



If we have $\mathbf{x_0} = (x_0, y_0, z_0)$ and $\mathbf{n} = (A, B, C)$, with the point $\vec{\mathbf{x}} = (x, y, z)$ then we can write this as

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \qquad \vec{n} = (A, B, C)$$

$$(A, B, C) \cdot \big((x, y, z) - (x_0, y_0, z_0)\big) = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

This can be simplified to

$$Ax + By + Cz = \underbrace{Ax_0 + By_0 + Cz_0}_{\text{some number}} = D$$

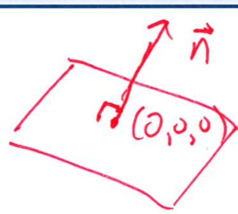
$$Ax + By + Cz = D$$

choose a normal : (A, B, C)

which is the a **general equation of a plane**.

> **Point-normal equation of a plane** The **point-normal equation of a plane** *through* $\mathbf{x_0}$ with *normal* $\mathbf{n}$ is all $\mathbf{x}$ with
>
> $$\mathbf{n} \cdot (\vec{\mathbf{x}} - \vec{\mathbf{x}}_0) = \mathbf{0}$$

What if $\vec{\mathbf{x}}_0 = \vec{0}$?



$$Ax + By + Cz = 0$$

goes through the origin

Remark Notice the similarity between the general equation of a line in 2d to the general equation of a plane in 3d.

$ax + by = c$ line in R^2

$Ax + By + Cz = D$ plane in R^3

Example. Find the point-normal and general equations of the plane through point (1,1,2) with normal (−1,2,1).

$\vec{x_0} = (1,1,2) \qquad \vec{n} = (-1,2,1)$

$\vec{n} \cdot (\vec{x} - \vec{x_0}) = 0$

$\boxed{(-1,2,1) \cdot ((x,y,z) - (1,1,2)) = 0}$ point-normal form

$-(x-1) + 2(y-1) + 1 \cdot (z-2) = 0$

$-x + 2y + z = -1 + 2 + 2 = 3$

$\boxed{-x + 2y + z = 3}$ $\leftarrow$ general form

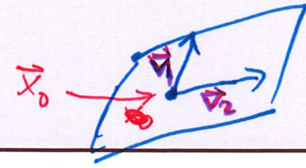
Example. Describe the plane in R^3 given by the equation $1(x-3) + 2(y+2) - 3(z-1) = 0$.

$\vec{x_0} = (3,-2,1) \qquad \vec{n} = (1,2,-3)$

Question Describe points of a plane passing through a point $\mathbf{x_0}$ and parallel to two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ (that are not parallel) in R^n.

$\vec{x} = \vec{x}_0 + s\vec{v}_1 + t\vec{v}_2$ s, t real numbers (scalars)

starting pt two directions



s, t parameters

This is called a **vector equation of a plane**.

If we have that $\mathbf{x_0} = (x_0, y_0, z_0)$, $\mathbf{v_1} = (a_1, b_1, c_1)$, $\mathbf{v_2} = (a_2, b_2, c_2)$ and $\mathbf{x} = (x, y, z)$, then we can write above as

$\vec{x} = \vec{x}_0 + s\vec{v}_1 + t\vec{v}_2$ vector eqn

$$\begin{cases} x = x_0 + sa_1 + ta_2 \\ y = y_0 + sb_1 + tb_2 \\ z = z_0 + sc_1 + tc_2 \end{cases}$$

parametric eqn

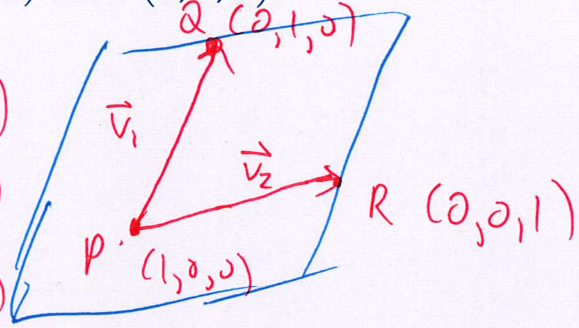
This is called a **parametric equation of a plane**.

Example. Find a description of the plane passing through points $\overset{P}{(1,0,0)}$, $\overset{Q}{(0,1,0)}$ and $\overset{R}{(0,0,1)}$.

$\vec{v}_1 = (-1,1,0)$

$\vec{v}_2 = (-1,0,1)$

$\vec{x}_0 = (1,0,0)$



$$\vec{x} = \vec{x}_0 + s\vec{v}_1 + t\vec{v}_2$$

$$\vec{x} = (1,0,0) + s(-1,1,0) + t(-1,0,1)$$ ← vector equation

$$\left.\begin{aligned} x &= 1-s-t \\ y &= s \\ z &= t \end{aligned}\right\}$$ parametric equation

general equation

$\vec{n}\cdot\vec{v}_1 = 0 \qquad \vec{n}\cdot\vec{v}_2 = 0 \qquad \vec{n} = (1,1,1) \qquad x+y+z=1$

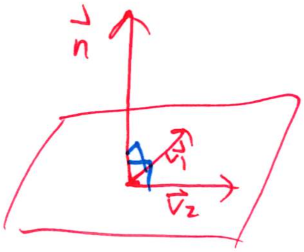
see next page for why

Summary question. Which forms of the equations of line and plane generalize to n-space?

Vector equation can be in any dimension

Can write lines and planes in 4 or more dimensional space

About finding normal vector $\vec{n}$ given direction vectors $\vec{v_1}$ and $\vec{v_2}$



"orthogonal"

$\vec{n}$ is perpendicular to both $\vec{v_1}$ and $\vec{v_2}$:

$\vec{n} \cdot \vec{v_1} = 0$ and $\vec{n} \cdot \vec{v_2} = 0$

where $\vec{v_1} = (-1, 1, 0)$ and $\vec{v_2} = (-1, 0, 1)$

Let $\vec{n} = (a, b, c)$:

We need to solve:

$$\begin{aligned} -a + b + 0c &= 0 \\ \text{and} \quad -a + 0b + c &= 0 \end{aligned} \implies \begin{aligned} a &= b \\ a &= c \end{aligned}$$

So that means $a = b = c$.

(0,0,0) is not allowed for normal vector $\vec{n}$

We can choose any value for a <u>except 0</u>

So choose $a = 1$: $a = 1, b = 1, c = 1$

$\vec{n} = (a, b, c) = (1, 1, 1)$