E.g. $\begin{aligned} x+y&=1 \\ x-y&=2 \end{aligned}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right] \xrightarrow[r_2 \leftarrow r_2 - r_1]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 \leftarrow r_2 \div (-2)} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1/2 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - r_2} \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \end{array}\right]$$

REF
row echelon form

RREF
row reduced echelon form

$$\boxed{\begin{aligned} x &= 3/2 \\ y &= -1/2 \end{aligned}}$$

or

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right] \xrightarrow[r_2 \leftarrow r_2 - r_1]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{r_1 \leftarrow 2r_1 + r_2} \left[\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & -2 & 1 \end{array}\right] \xrightarrow[r_2 \leftarrow r_2 \div (-2)]{r_1 \leftarrow r_1 \div 2} \left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \end{array}\right]$$

same

not an elementary row operation

$$\left.\begin{aligned} r_1 &\leftarrow 2r_1 \\ r_1 &\leftarrow r_1 + r_2 \end{aligned}\right\} \text{combined}$$

MATH 232 Section 2.2 pre-lecture comments

Today we are continuing to learn about solving linear systems.

Example:
Recall how to solve a system by manipulating the equations:

$$\begin{array}{rcrcl} x & + & y & = & 1 \\ x & - & y & = & 2 \end{array}$$

Subtract first equation from second one

$$\begin{array}{rcrcl} x & + & y & = & 1 \\ & & -2y & = & 1 \end{array}$$

Multiply the second question by -1/2

$$\begin{array}{rcrcl} x & + & y & = & 1 \\ & & y & = & -1/2 \end{array}$$

Subtract second equation from first one.

$$\begin{array}{rcrcl} x & & & = & 3/2 \\ & & y & = & -1/2 \end{array}$$

(Verify this solves the original equation!)

(Reduced) row echelon forms (P. 48) A matrix is said to be in **row echelon form (REF)** if it has the following properties:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & -1 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & -3 \end{bmatrix}$$

1. Each row either consists of all zeroes, or has a 1 as its first non-zero entry. (This entry is called a **leading 1**.)
2. Any rows of all zeroes are grouped together at the bottom of the matrix. $\begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix}$
3. In two successive non-zero rows, the *leading 1* in the *lower* row is to the *right* of the *leading 1* in the *upper* row. $\boxed{1}$ $\boxed{1}$ top left to bottom right

The matrix is in **reduced row echelon form (RREF)** if it additionally satisfies:

$\begin{bmatrix} 1 & \boxed{2} \\ 0 & 1 \end{bmatrix}$ not RREF but $\begin{bmatrix} 1 & \boxed{0} \\ 0 & 1 \end{bmatrix}$ is RREF

4. Each column with a *leading 1* has zeroes everywhere else.

Example. Mark each matrix as REF, RREF or NEITHER. Circle any *leading 1*.

neither
$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

$r_2 \leftrightarrow r_3$
$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

REF (not RREF)
$$\begin{bmatrix} 1 & 0 & \boxed{2} & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

RREF
$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

RREF
$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF
$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solve the linear system that has the augmented matrix

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow 0x+0y+0z=1$$

$0=1$ inconsistent (no solution)

Solve the linear system that has the augmented matrix

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2 & -3 \\ 0 & 0 & \boxed{1} & -5 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 2x_4 = -3$$
$$x_3 - 5x_4 = 8$$
$$0 = 0$$

(free variables)

Let $x_2 = s$

$x_4 = t$

x_1, x_3 leading variables

x_2, x_4 free variables

Then $x_1 = -3 - 2x_2 - 2x_4 = -3 - 2s - 2t$

$x_3 = 8 + 5x_4 = 8 + 5t$

Rewrite the solution above in vector form.

$x_1 = -3 - 2s - 2t$

$x_2 = s$

$x_3 = 8 + 5t$

$x_4 = t$.

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 8 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -2 \\ 0 \\ 5 \\ 1 \end{pmatrix} t}$$

plane in $\mathbb{R}^4$

Example. Write the following linear system as an augmented matrix, simplify the matrix using row operations, and finally solve the system:

$$\begin{aligned} 2x &- 3y &+ z &= 1 \\ 2x &- y &+ z &= 3 \\ 4x &+ 4y &- 2z &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 2 & -1 & 1 & 3 \\ 4 & 4 & -2 & 4 \end{array}\right] \xrightarrow[r_3 \leftarrow r_3 - 2r_1]{r_2 \leftarrow r_2 - r_1} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 10 & -4 & 2 \end{array}\right]$$

REF

$$\xrightarrow{r_3 \leftarrow r_3 - 5r_2} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -4 & -8 \end{array}\right] \xrightarrow[\substack{r_2 \leftarrow r_2 \div 2 \\ r_3 \leftarrow r_3 \div (-4)}]{r_1 \leftarrow r_1 \div 2} \left[\begin{array}{ccc|c} \boxed{1} & -3/2 & 1/2 & 1/2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

use <u>back substitution</u>:

$z = 2$

$y = 1$

$x - \frac{3}{2}y + \frac{1}{2}z = \frac{1}{2} \Rightarrow x = \frac{1}{2} + \frac{3}{2}y - \frac{1}{2}z = \frac{1}{2} + \frac{3}{2} - 1 = 1$

Solution: $(x, y, z) = (1, 1, 2)$

Note: You are not required to use back substitution unless it ~~says~~ says you have to use it.

You can row-reduce to RREF if you prefer.

leading variables x_1 x_2 x_3 x_4 free variable

Once a system is in row echelon form it is easily solved via **back substitution**.

Example: $\left[\begin{array}{cccc|c} \boxed{1} & \frac{2}{7} & \frac{3}{7} & \frac{2}{7} & 3 \\ 0 & \boxed{1} & -2 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \boxed{1} & 3 \end{array}\right]$

$$x_1 + \frac{2}{7}x_2 + \frac{3}{7}x_3 + \frac{2}{7}x_4 = 3$$
$$x_2 - 2x_3 - \frac{3}{2}x_4 = -\frac{1}{2}$$
$$x_4 = 3$$

Step one: Write each row in terms of the leading variables. x_1, x_2, x_4

$$x_1 = -\frac{2}{7}x_2 - \frac{3}{7}x_3 - \frac{2}{7}x_4 + 3$$
$$x_2 = 2x_3 + \frac{3}{2}x_4 - \frac{1}{2}$$
$$x_4 = 3$$

Step two: Solve by substitution *from the bottom up*.

$x_4 = 3$

let $x_3 = t \leftarrow$ free variable

$$x_2 = 2x_3 + \frac{3}{2}x_4 - \frac{1}{2} = 2t + \frac{3}{2}(3) - \frac{1}{2} = 4 + 2t$$
$$x_1 = -\frac{2}{7}x_2 - \frac{3}{7}x_3 - \frac{2}{7}x_4 + 3$$
$$= -\frac{2}{7}(4+2t) - \frac{3}{7}t - \frac{2}{7}(3) + 3 = \ldots = 1 - t$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1-t \\ 4+2t \\ t \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} t \qquad t \in \mathbb{R}$$

Gaussian elimination (P. 51) is an algorithm to converts an augmented matrix to *row echelon form* (**introduce 0's below leading 1's**).

Example Convert the following augmented matrix to *row echelon form*:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 6 & 2 & 2 \\ 3 & 6 & 18 & 9 & -6 \\ 4 & 8 & 12 & 10 & 4 \end{array}\right]$$

Step 1. Locate the leftmost column c that does not consist entirely of zeros.

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 6 & 2 & 2 \\ 3 & 6 & 18 & 9 & -6 \\ 4 & 8 & 12 & 10 & 4 \end{array}\right]$$

Step 2. Interchange the top row with another row, if necessary, to bring a non-zero entry to the top of the column c.

e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Step 3. If the entry in the top row and column c is a, multiply the top row by $1/a$ (in order to introduce a leading 1). don't have to do this right away

Step 4. Add suitable multiples of the top row to the rows below so that all entries below leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 6 & 2 & 2 \\ 3 & 6 & 18 & 9 & -6 \\ 4 & 8 & 12 & 10 & 4 \end{bmatrix}$$

turn into 0s using row operations

leading 1

$$\left[\begin{array}{ccccc} \boxed{1} & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 9 & -3 & -9 \\ 0 & 0 & 0 & -6 & 0 \end{array}\right]$$

second row: $r_2 \leftarrow r_2 - 2r_1$

third row: $r_3 \leftarrow r_3 - 3r_1$

fourth row: $r_4 \leftarrow r_4 - 4r_1$

Step 5. Cover the top row, if there are any non-zero rows left, repeat Step 1.

Step 1. $\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 9 & -3 & -9 \\ 0 & 0 & 0 & -6 & 0 \end{bmatrix}$

$r_2 \Leftrightarrow r_3$

Step 2. $\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 9 & -3 & -9 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 & 0 \end{bmatrix}$ ~~2nd and 3rd rows exchanged~~

$r_2 \Leftarrow r_2 \div 9$

Step 3. $\begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 1 \\ 0 & 0 & \boxed{1} & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 & 0 \end{bmatrix}$ ~~second row~~:

different ways to do this:

Step 4. and Step 5.

$r_4 \Leftarrow r_4 - r_3$

Step 1. - Step 3. $\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ~~row multiplied by~~

$r_3 \Leftarrow r_3 \div (-6)$

Step 4. - Step 5. $\left[\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 1 \\ 0 & 0 & \boxed{1} & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ ~~fourth row~~:

REF

Stop. Matrix is in row echelon form.

This system can be solved by back substitution

$$\begin{aligned} x_4 &= 0 \\ x_3 &= -1 \\ x_1 &= 4 - 2s, \qquad s \in \mathbf{R} \\ x_2 &= s, \leftarrow \text{free} \qquad s \in \mathbf{R} \end{aligned}$$

Question What is the solution in vector format?

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} s \qquad s \in \mathbb{R}$$

Gauss-Jordan elimination is an algorithm to converts an augmented matrix to *reduced row echelon form* (**introduce 0's above and below leading 1's**).

There are two phases:
Forward phase: Gaussian elimination.
Backward phase: Begin with the last non-zero row and working *upward*, add suitable multiples of each row to the rows above to introduce zeros above the *leading* 1's.

Example

turn into 0s

turn into 0

Starting with the final matrix from the previous Gaussian elimination.

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 1 \\ 0 & 0 & \boxed{1} & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_1 \leftarrow r_1 - 4r_3]{r_2 \leftarrow r_2 + \frac{1}{3} r_3} \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_1 \leftarrow r_1 - 3r_2} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 0 & 4 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

free

$x_2 = s$

$x_1 + 2x_2 = 4$ $\qquad$ $x_1 = 4 - 2s$

$x_3 = -1$ $\qquad$ $x_3 = -1$

$x_4 = 0$ $\qquad$ $x_4 = 0$

Theorem (P. 53)

1. Every matrix has a **unique** *reduced row echelon form*.
2. *Row echelon forms* are not unique. But all *row echelon forms* have **leading** 1's in the same positions of the matrix.

Pivot positions/columns (P. 53) The positions in a row echelon form that have the *leading* 1's are called **pivot positions**. The columns that contain the *leading* 1's are called **pivot columns**.

pivot positions

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 1 \\ 0 & 0 & \boxed{1} & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns