About linear systems:

$$x + 2y + 3z = 0$$
$$2x - y = 1$$
$$4x - 7y + 10z = 2$$

$$\underbrace{\left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 4 & -7 & 10 \end{array}\right.}_{A}\underbrace{\left|\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right]}_{\vec{b}}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 4 & -7 & 10 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

can write any linear system as $A\vec{x} = \vec{b}$

About multiple row operations:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow[\cancel{r_2 \leftarrow r_2 + r_1}]{\cancel{r_1 \leftarrow r_1 + r_2}}$$

not valid; don't do this!

r_1, r_2 depend on each other

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow[r_3 \leftarrow r_3 + r_1]{r_2 \leftarrow r_2 + r_1}$$

OK (not elementary)

r_2, r_3 don't depend on each other

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{r_2 \leftarrow 2r_2 - r_1}$$

OK (not elementary)

MATH 232 Section 3.3 pre-lecture comments

Lecture Outline
Today we continue on with understanding and constructing inverses of matrices.

You should review solving linear systems and elementary row operations.

We will also begin construction of the **invertible matrix theorem**.

New terminology

1. elementary matrix
2. row equivalence
3. trivial solution

elementary row operation

1) $r_i \leftarrow c r_i$ $(c \neq 0)$

2) $r_i \leftarrow r_i + c r_j$

3) $r_i \leftrightarrow r_j$

Row operations and elementary matrices Recall the three row operations to solve linear systems

1. Multiply a row by a **nonzero** scalar. $r_i \leftarrow c r_i$ $(c \neq 0)$
2. Exchange the order of two rows. $r_i \Leftrightarrow r_j$
3. Add a multiple of one row to another. $r_i \leftarrow r_i + c r_j$

Each operation can be done by multiplication with an **elementary matrix**

Example. Suppose that $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}$

To multiply the second row by -3 $r_2 \leftarrow r_2 \times (-3)$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -6 \\ -1 & 3 \end{bmatrix}$$

To interchange the first and third rows $r_1 \Leftrightarrow r_3$

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad FA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}$$

To add -2 times the first row to the second row $r_2 \leftarrow r_2 - 2r_1$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad GA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ -1 & 3 \end{bmatrix}$$

A EA

elementary matrix

Elementary matrices: Inverse operations $r_3 \leftarrow r_3 + 3r_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Theorem 3.3.1 Suppose A is an $m \times n$ matrix and E is the elementary matrix obtained by applying a given elementary row operation to I_m. Then the product EA is the matrix that results when the same row operation is applied to A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 + 3r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Example Find an elementary matrix that converts $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 5 \end{bmatrix}$ into RREF.

turn this into 0

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{r_1 \leftarrow r_1 + 3r_2} \begin{bmatrix} 1 & 0 & 17 \\ 0 & 1 & 5 \end{bmatrix}$$

Elementary mtx: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$$\underset{2\times 2}{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 5 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 17 \\ 0 & 1 & 5 \end{bmatrix}}$$

Elementary row operations can be *inverted* by other elementary operations.

Row operation on I to produce E	**Row operation on E to restore I**
$R_i \leftrightarrow R_j$	$r_i \leftrightarrow r_j$
For $c \neq 0$, $R_i \leftarrow cR_i$	$r_i \leftarrow \frac{1}{c} r_i$
$R_i \leftarrow R_i + cR_j$	$r_i \leftarrow r_i - cr_j$

If E corresponds to a certain elementary row operation, and if E_0 corresponds to the inverse elementary row operation, then

$$EE_0 = I \qquad \text{and} \qquad E_0E = I$$

For all A:

$$(\underbrace{E_0 E}_{I})A = A \qquad (\underbrace{EE_0}_{I})A = A$$

To do these three steps in order (first scale the second row by -3, then swap the first and third rows, and finally add -2 times the first row to the second) do

Example

$$A \xrightarrow{r_2 \leftarrow r_2 \times (-3)} EA \xrightarrow{r_1 \leftrightarrow r_3} FEA \xrightarrow{r_2 \leftarrow r_2 - 2r_1} GFEA$$

$$GF(EA) = GF\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & 3 \end{bmatrix}\right) = GF\begin{bmatrix} 1 & -1 \\ 0 & -6 \\ -1 & 3 \end{bmatrix}$$

$$= G\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -6 \\ -1 & 3 \end{bmatrix} = G\begin{bmatrix} -1 & 3 \\ 0 & -6 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 0 & -6 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -12 \\ 1 & 1 \end{bmatrix}$$

$$E_k \dots E_3 E_2 E_1 A$$

Note the order! The first operation performed is the **rightmost** matrix.

Why?

$$GFEA = GF(EA)$$

first action performed, rightmost matrix

Theorem 3.3.2 All elementary matrices are invertible.
The inverse of an elementary matrix is also an elementary matrix.

Examples

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$F^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad G = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question Why is this helpful?

We have a way to build A^{-1}, A from elementary matrices

Invertible Matrix Thm (1)

Theorem 3.3.3 If A is an $n \times n$ matrix, then the following statements are equivalent:

0. A is invertible
1. The reduced row echelon form of A is I_n.
2. A can be written as a product of elementary matrices.
3. A is invertible.

Suppose an $n \times n$ matrix A is invertible. By Theorem (3.3.3), RREF of A is I_n, and hence there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$\underbrace{E_k \cdots E_3 E_2 E_1}_{A^{-1}} A = I_n$, hence, $A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$

Taking the inverse on both sides:

$$A^{-1} = E_k \cdots E_3 E_2 E_1 \cancel{I_n}$$

Observation: The same sequence of row operations that transforms A to I_n will also transform I_n to A^{-1}. because $A^{-1} = E_k \cdots E_3 E_2 E_1 I_n$

The inversion algorithm To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to I, and then perform the same sequence of operations on I to produce A^{-1}

Note: In practice this is done simultaneously using the "compound matrix" $\left[A \mid I\right]$

$$[A \mid I] \to [\ \mid\ ] \to \cdots \to [I \mid A^{-1}]$$

Example

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad A^{-1}$:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + r_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \xrightarrow{r_2 \leftarrow \frac{1}{2} r_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{r_1 \leftarrow r_1 - r_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad A^{-1}:$$

RREF

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -1 & 4 \end{bmatrix}$$

Facts about inverses A is an $n \times n$ matrix.

1. If $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is a linear system, and the coefficient matrix A is invertible, then the system has the unique solution $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$;

   no sol..

   • If A is not invertible, then $A\mathbf{x} = \mathbf{b}$ is either inconsistent, or has infinitely many solutions.

   homogeneous system $\quad \vec{x} = A^{-1}\vec{0} = \vec{0}$

2. If A is invertible, then $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ only has the trivial solution ($\vec{\mathbf{x}} = \vec{\mathbf{0}}$).

   • If A is not invertible, then $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

We can add these two (and another one) to our theorem:

Invertible Mtx Thm (2)

Theorem 3.3.9

If A is an $n \times n$ matrix, then the following statements are equivalent:

0) A is invertible

1. The reduced row echelon form of A is I_n.
2. A can be written as a product of elementary matrices.
3. A is invertible.
4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution: $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
5. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n: $\mathbf{x} = A^{-1}\mathbf{b}$.
6. $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.

NEW

$A\vec{x} = \vec{b}$

$$\left[\quad\right]\left[\quad\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

consistent

$\rightarrow \vec{x} = A^{-1}\vec{b}$

These facts give us a strategy for solving $A\mathbf{x} = \mathbf{b}$. One method is to construct A^{-1} and then apply it to $\mathbf{b}$. The other is to apply the row operations reducing A to A^{-1} to A and $\mathbf{b}$ *simultaneously* with an augmented matrix. RREF

To solve $A\mathbf{x} = \mathbf{b}$ form $[A|\mathbf{b}]$ and reduce to $[I_n|A^{-1}\mathbf{b}]$.

Example

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad + 8x_3 &= 17 \end{aligned}$$

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

if we know A^{-1}:

$$\vec{x} = A^{-1}\vec{b} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

Note: Just because a matrix A is not invertible doesn't mean we cannot solve $A\mathbf{x} = \mathbf{b}$! It may have no solutions or *it might have infinitely many.* The consistency problem is finding those $\mathbf{b}$ so that there are infinitely many solutions.

The consistency problem For a given matrix A, find all vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution.

revisit in Section 3.5 (column space)