MATH 232 Section 3.4 pre-lecture comments

Lecture Outline
Today we look at some geometric objects in $\mathbf{R}^n$ and their connections to linear systems.

We introduce two of the big ideas in this class:

1. subspace
2. linear independence

We will use these ideas to add two more items to the big theorem. These are two of the most (maybe the most!) important theoretical ideas in this class.

New terminology

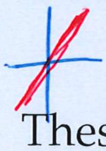
1. subspace
2. closed
3. solution space
4. general solution
5. linearly independent
6. linear dependence

Introduction to subspaces

Informally, subspaces are linear combinations of vectors.

Subspaces in $\mathbf{R}^2$ and $\mathbf{R}^3$ 1-dimensional

Lines through the origin: In $\mathbf{R}^2$ and $\mathbf{R}^3$, lines through the origin take the form



$$\vec{\mathbf{x}} = s\vec{\mathbf{v}}_1.$$

These are subspaces of dimension 1–we need one vector to define them.

$$\vec{x} = s\begin{bmatrix}1\\2\\-1\end{bmatrix}$$ line in $\mathbb{R}^3$ going through origin

Planes through the origin: In $\mathbf{R}^3$, planes through the origin take the form

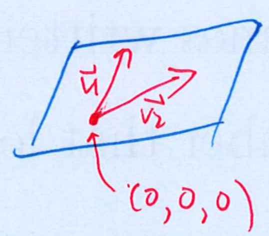
2-dimensional

$$\vec{\mathbf{x}} = s\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2.$$

These are subspaces of dimension 2–we need two non-collinear vectors to define them. not parallel

$$\vec{x} = s\overset{v_1}{\begin{bmatrix}1\\2\\-1\end{bmatrix}} + t\overset{v_2}{\begin{bmatrix}3\\4\\5\end{bmatrix}}$$

plane in $\mathbb{R}^3$



Hyperplanes in $\mathbf{R}^n$: Informally we can extend our idea to $\mathbf{R}^n$ with

$$\vec{\mathbf{x}} = t_1\vec{\mathbf{v}}_1 + t_2\vec{\mathbf{v}}_2 + \ldots + t_k\vec{\mathbf{v}}_\mathbf{k}.$$ (k-dimensional)

Definition A non-empty subset S of vectors of $\mathbf{R}^n$ is called a subspace if

belongs to the set

1. If $\vec{u} \in S$ and $\vec{v} \in S$, then $\vec{u}+\vec{v} \in S$ closed under addition

2. If $\vec{u} \in S$ and $c \in \mathbb{R}$ then $c\vec{u} \in S$ closed under scalar multiplication

Example A line through the origin.

$\vec{x} = t\vec{v} \quad t \in \mathbb{R}$

Closed under add.? $\vec{x}_1 = t_1\vec{v}$, $\vec{x}_2 = t_2\vec{v}$

$\vec{x}_1 + \vec{x}_2 = t_1\vec{v} + t_2\vec{v} = (t_1+t_2)\vec{v}$ ✓

Scalar mult.? $c\vec{x}_1 = ct_1\vec{v} = (ct_1)\vec{v}$ ✓ subspace of $\mathbb{R}^n$

Question: How about a line not through the origin?

$\vec{0}$ is not in S

not closed under scalar multiplication

But if $\vec{u} \in S$ then $c\vec{u} \in S$ for all scalars c.

if $c=0$ then $0\vec{u} \in S \Rightarrow \vec{0} \in S$

$\vec{0}$ must always be in subspace

set of 2 vectors

Example Let $S_1 = \{(1,0,0),(0,1,0)\}$. Is S_1 closed under addition?

No: $(1,0,0)+(0,1,0)=(1,1,0) \notin S_1$ $\quad$ S_1 not closed under addition

all vectors of the form $(a,b,0)$: a,b real numbers

Let $S_2 = \{(a,b,0);\ a,b \in \mathbf{R}\}$. Is S_2 closed under addition?

Yes: let $(c_1,d_1,0)$, $(c_2,d_2,0)$ in S_2:

$(c_1,d_1,0)+(c_2,d_2,0) = (c_1+c_2, d_1+d_2, 0) \in S_2$

Is S_2 closed under scalar multiplication?

Yes: For $(c_1,d_1,0) \in S_2$ and $e \in \mathbb{R}$:

$e(c_1,d_1,0) = (ec_1, ed_1, 0) \in S_2$

S_2 is a subspace

Let $S_3 = \{(x,y,0);\ x,y \in \mathbf{R}_+\}$. Is S_3 closed under scalar multiplication?

positive #s

No: For example: $c=-1$ $\quad \vec{u}=(1,0,0) \in S_3$

$c\vec{u} = -1(1,0,0) = (-1,0,0) \notin S_3$

Which of S_1, S_2 and S_3 are subspaces? Only S_2 is a subspace.

Examples

only the $\vec{0}$ vector

The Zero subspace

The zero subspace of $\mathbf{R}^n$ is the set $\{\mathbf{0}\}$ (trivial subspace).

$\vec{0}+\vec{0}=\vec{0}$

$c\vec{0}=\vec{0}$

The full space $\mathbf{R}^n$

The full space $\mathbf{R}^n$ is, according to the definition, a subspace of $\mathbf{R}^n$

"everything" (every vector in $\mathbb{R}^n$)

Ex $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$

A plane in $\mathbf{R}^3$

A plane in $\mathbf{R}^3$ is, according to the definition, a subspace of $\mathbf{R}^3$

through $(0,0,0)$

A line not through the origin.

According to the definition is a line in $\mathbf{R}^n$ not through the origin a subspace? No.

$\vec{0}$ is not in S

not closed under scalar mult.

Definition: Span If $\vec{\mathbf{v}}_1, \mathbf{v}_2, \ldots, \vec{\mathbf{v}}_s$ are vectors in $\mathbf{R}^n$, the set of all linear combinations of these vectors

scalars

Set of $$\{\vec{\mathbf{x}} = t_1\vec{\mathbf{v}}_1 + t_2\vec{\mathbf{v}}_2 + \cdots + t_s\vec{\mathbf{v}}_s,\ t_1, t_2, \ldots, t_s \in \mathbf{R}\}$$

is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ and denoted by span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$.

Examples

span $\{(1,2,-1)\}$: $\vec{x} = (1,2,-1)\,t \quad (t \in \mathbb{R})$

line in $\mathbb{R}^3$ through origin

span $\{(1,2,-1),(1,-1,0)\}$: $\vec{x} = t(1,2,-1) + s(1,-1,0) \quad (s,t \in \mathbb{R})$

plane in $\mathbb{R}^3$ through origin

Lines

In general: $\vec{x} = \vec{x}_0 + \vec{v}t$

but here, $\vec{x}_0 = \vec{0}$.

Theorem 3.4.2 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are vectors in $\mathbf{R}^n$, then span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ is a subspace of $\mathbf{R}^n$.

Theorem 3.4.3 If $A\vec{x} = \vec{\mathbf{0}}$ is a homogeneous linear system with n unknowns, then its solution set is a subspace of $\mathbf{R}^n$.

Solve (general solution)

$$x + y + z = 0$$
$$x + 2y - 3z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 - r_1} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{1} & -4 & 0 \end{array}\right]$$

Let $z = t$

$y = 4t$

$x + y + z = 0 \Rightarrow x = -4t - t = -5t$

In vector form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5t \\ 4t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix} t \leftarrow \text{span}\left\{ \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix} \right\}$ $(t \in \mathbb{R})$

Note:

- This solution set is also called the **solution space** of the system.
- The solution space of a homogeneous linear system is a span of some vectors.

Question: Is every subspace of R^n a span of some vectors? Yes!

($\{\vec{0}\}$ is an exception)

Definition: Linear Independence 1. A collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ of vectors is said to be **linearly dependent** if there are some scalars $a_1, a_2, \ldots, a_m$, at least one of which is nonzero, such that

$$a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + \ldots a_m\vec{\mathbf{v}}_m = \vec{\mathbf{0}} \tag{1}$$

2. The collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is **linearly independent** if it is not linearly dependent, i.e., if the only choice of scalars $a_1, a_2, \ldots, a_m$ that satisfies (1) is the trivial one $a_1 = a_2 = \cdots = a_m = 0$.

Observation 1. Any collection that includes the zero vector, say $\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_m$, must be linearly dependent.

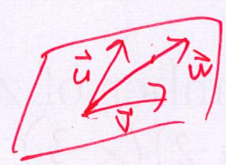
$1 \cdot \vec{0} + 0\vec{v}_2 + \ldots 0\vec{v}_m = \vec{0}$

Observation 2. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if they are parallel.

$\vec{v} = c\vec{u} \longrightarrow c\vec{u} - 1\vec{v} = \vec{0}$

Observation 3. Three vectors in $\mathbf{R}^3$ are linearly dependent if and only if they lie in the same plane.

$\vec{w} = c\vec{u} + d\vec{v} \longrightarrow c\vec{u} + d\vec{v} - 1\vec{w} = \vec{0}$



$\vec{u}, \vec{v}, \vec{w}$ in same plane

Ex: $\vec{u} = (1,0,0)$
$\vec{v} = (0,1,0)$
$\vec{w} = (1,1,0)$

$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$ let's say $c_1 \neq 0$

$\vec{u} = -\frac{c_2}{c_1}\vec{v} - \frac{c_3}{c_1}\vec{w} \longleftarrow \vec{u}$ linear combination of $\vec{v}$ and $\vec{w}$

Theorem 3.4.6 A set S with two or more vectors in $\mathbf{R}^n$ is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

Examples Which of the following are linearly independent?

a) $\left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

a) $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ are not parallel $\rightarrow$ linearly independent

c) has $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\rightarrow$ lin. dep. (linearly dependent)

d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \longrightarrow$ lin. dep.

b) $\overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{bmatrix}}^{A} \rightarrow \cdots \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$

Thm 3.4.7

$A\vec{x} = \vec{0}$ has only trivial solution $\rightarrow$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

lin. indep.

Theorem 3.4.7 A homogeneous linear system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution ($\vec{\mathbf{x}} = \vec{\mathbf{0}}$) if and only if the columns of A are linearly independent.

$$A = \begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Rewrite $A\vec{x} = \vec{0}$ as: $x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n = \vec{0}$

System has only trivial sol. $\Longleftrightarrow$ $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ lin indep.

——— nontrivial sols. $\Longleftrightarrow$ $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ lin dep.

free variable

This gives us a procedure to determine if vectors are linearly independent!

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k\}$ is linearly dependent if and only if the system $[\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_k | \mathbf{0}]$ has a nontrivial solution.

Observation 4. Any set of vectors in $\mathbf{R}^n$ with more than n vectors will always be linearly dependent.

By above procedure, will get a homogeneous system with <u>more variables than equations.</u>

At least one free variable, so infinitely many solutions → nontrivial solutions

and consistent

Invertible Matrix Theorem (3)

Theorem 3.4.9 If A is an $n \times n$ matrix, then the following statements are equivalent:

0. A is invertible
1. The reduced row echelon form of A is I_n.
2. A can be written as a product of elementary matrices.
3. A is invertible.
4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution: $\mathbf{x} = \mathbf{0}$.
5. $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.
6. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.
7. The ~~columns~~ column vectors of A are linearly independent.
8. The row vectors of A are linearly independent.

NEW

Thm 3.4.7

How do these new items relate to the earlier ones?