Notes on Finite State Machines
and Regular Languages

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Abstract

As the major concerns of computer science are *computation* and the nature and variety of the *machines* that support it, in all their glorious but unwieldy variety, a particularly productive strategy in the study of computer science is to focus on special cases of computation (and machine) that exhibit two opposing qualities: they are special enough to make a detailed study feasible, yet they are general enough to provide useful insight into many broader questions. In the whole of computer science, there is no better example of a subject that meets these two requirements than the one involving Finite State Machines and Regular Languages.

These notes present the standard definitions, techniques, and results of this subject, although in occasionally new ways. They are currently in draft form and contain parts in various stages of completion, from bare outlines to polished text, so please do not distribute them without permission of the author.

1 Introduction

Suppose we are given a string of characters, say `twilson@csufresno.edu`, and need to determine whether this string represents a valid email address. Or we are given a string and need to determine whether it represents a valid date, or zip code, or system log entry, or programming-language statement. These are all examples of *string recognition problems*, and these problems form the context of our study.

We will be studying simple machines that do nothing but read strings, a character at a time, from left to right, indicating after reading each character whether or not they *accept* (or *recognize*) the string read so far. Each such machine will have a fixed, finite amount of "memory" it can use to remember information about the characters it has read, or "reference" information it needs to use, to help it decide whether or not to accept the string it has read up to that point. It is not allowed to look ahead at the characters it has yet to read, nor look back at or re-read the characters it has already read. And it is completely *deterministic*, in that it always produces the same result when given the same string as input. Such machines will be called *Finite State Machines*, or FSMs for short.

If you'd like to have a picture in mind, consider an FSM to be a black box that has access to a *tape*, onto which is written, in distinct cells from left to right, the characters of an *input string*. The machine accesses this tape by means of a *read head*, which is always positioned either just to the left of one of the characters on the tape or at the end. The box also has a *light*, its only means of output, which is either on or off at any given time. Finally, the box has two buttons: a "reset" button that moves the read head back to the beginning of the tape and puts the machine in its *initial state*, which includes turning its light on or off; and a "read" button that causes the machine's read head to pass over and read the next character to its right (the only way the machine gets information about this character), and adjust the status of its light. The machine *accepts* a particular string when, after we write the string on the tape, push the "reset" button, and then push the "read" button for each character in the string (if any), the machine has its light on at the end of the process.

Note that, as I've described it, it makes sense to consider whether or not our machine accepts the *empty string*, the string with no characters and thus length 0: it does so if its light goes on when the machine is reset. It also makes sense to consider whether our machine accepts the various *prefixes* of the input string, since its light is on or off at the end of reading each prefix, and the rest of the characters have no influence on what the machine has done up to that point.

Although I've described the acceptance process of an FSM in physical terms, its input/output behavior, being deterministic, is completely characterized by the set of strings it accepts. Taking advantage of a linguistic metaphor, I will use the term *language* to mean any sets of strings; thus, every machine accepts a particular language. One important component of our study will be to understand the class of languages accepted by FSMs. We will call these languages *regular*.

Examples of the kinds of questions we would like to answer about FSMs and regular languages are:

- Given a set of strings, is it regular? In other words, is there an FSM that accepts exactly these strings?
- Are there other ways of representing regular languages besides using FSMs that might be more convenient for other purposes?
- What are the closure properties of regular languages? In other words, what operations, when performed on regular languages, always result in regular languages? (I'm thinking of such operations as union, intersection,

and difference, as well as many more complicated ones we'll define later.) How can the machines associated with these languages be constructed?

- Are there extensions to the notion of FSM we can make that do not change the class of languages they accept? This would allow us to investigate "higher-level machines" that might be easier to construct or work with than FSMs but still accept the same languages, the same way that higher-level programming languages make it easier to write programs than assembly languages do, even though they don't allow us to write any new programs.
- How can we show that a particular language is *not* regular? It's one thing to show that a language is regular—you just need to construct a FSM that accepts it—but it's something altogether different to show that a language is not regular: you have to show somehow that, no matter how clever you are, you will never be able to construct an accepting FSM.
- How much of the theory of FSMs and regular languages can be implemented on a computer (for example in Haskell)?
- What are the applications of FSMs to other areas of computer science? In particular, how can we use FSMs and regular languages to solve string-recognition problems, which come up all over the place?
- What other directions are there that can be explored further?

2 Basic Definitions

We now begin the formal treatment of our subject. The initial goal is to capture the informal notions introduced in the previous section as precise, mathematical defintions, most of which are inductive, and to prove our results precisely, mostly by induction.

2.1 Characters and Strings

We start with a fixed, finite set Σ of *characters*, which we will use to build our strings. We call Σ the *alphabet*.

Definition 2.1 (Strings) We define the set Σ^* of *strings* over Σ by the following rules:

$$\frac{}{\varepsilon \in \Sigma^*}\ \textsc{Empty} \qquad \frac{a \in \Sigma \qquad w \in \Sigma^*}{aw \in \Sigma^*}\ \textsc{Cons}$$

Here, ε stands for the empty string, and aw is the result of adding the character a to the beginning of the string w to get a string whose length is one greater.

DEFINITION 2.2 (Length and Concatenation of Strings) The length, $|w|$, of a string w and the concatenation, $w \cdot u$, of two strings w and u are defined by recursion on w:

$$
\begin{aligned}
|\varepsilon| &= 0, && \text{(LenEmpty)} \\
|aw| &= 1 + |w|; && \text{(LenCons)} \\
\varepsilon \cdot u &= u, && \text{(CatEmpty)} \\
(aw) \cdot u &= a(w \cdot u). && \text{(CatCons)}
\end{aligned}
$$

THEOREM 2.3 *(Length of concatenation) For all* $w, u \in \Sigma^*$, $|w \cdot u| = |w| + |u|$.

PROOF. By induction on w.

Case EMPTY: $w = \varepsilon$. For all $u \in \Sigma^*$, we have

$$
\begin{aligned}
|\varepsilon \cdot u| &= |u| && \text{(by CatEmpty)} \\
&= 0 + |u| \\
&= |\varepsilon| + |u| && \text{(by LenEmpty)}
\end{aligned}
$$

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. For all $u \in \Sigma^*$, we have

$$
\begin{aligned}
|(aw') \cdot u| &= |a(w' \cdot u)| && \text{(by CatCons)} \\
&= 1 + |w' \cdot u| && \text{(by LenCons)} \\
&= 1 + |w'| + |u| && \text{(by the IH)} \\
&= |aw'| + |u| && \text{(by LenCons)}
\end{aligned}
$$

$\dashv$

THEOREM 2.4 *(Identity and associativity of concatenation) For all* $w, u, v \in \Sigma^*$,

1. $w \cdot \varepsilon = w$
2. $w \cdot (u \cdot v) = (w \cdot u) \cdot v$

PROOF. I'll prove these together by induction on w, although they could just as easily be proven individually the same way.

Case EMPTY: $w = \varepsilon$. Then $\varepsilon \cdot \varepsilon = \varepsilon$ by (CATEMPTY), and for every $u, v \in \Sigma^*$, we have $\varepsilon \cdot (u \cdot v) = u \cdot v = (\varepsilon \cdot u) \cdot v$ by two instances of (CATCONS).

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. Then $(aw') \cdot \varepsilon = a(w' \cdot \varepsilon) = aw'$ by (CATCONS) and the IH, and for all $u, v \in \Sigma^*$, we have

$$
\begin{aligned}
(aw') \cdot (u \cdot v) &= a(w' \cdot (u \cdot v)) && \text{(by CatCons)} \\
&= a((w' \cdot u) \cdot v) && \text{(by the IH)} \\
&= (a(w' \cdot u)) \cdot v && \text{(by CatCons)} \\
&= ((aw') \cdot u) \cdot v && \text{(by CatCons)}
\end{aligned}
$$

$\dashv$

2.2 Languages and Language Operations

As I mentioned in the introduction, I'll use the term *language* to mean any set of strings. Since Σ^* is the set of all strings (over our fixed alphabet Σ), saying that L is a language is the same as saying $L \subseteq \Sigma^*$, which makes Σ^* the maximum, or largest, language. On the other end of the spectrum, the empty set, $\emptyset$, is the minimum, or smallest, language. Next up in size are the languages that contain one string, that is $\{w\}$ where $w \in \Sigma^*$, which includes the language $\{\varepsilon\}$. Note that the size of the strings in a language (their lengths) has nothing to do with the size of the language itself (its cardinality as a set).

Because languages are sets, all of the usual set operations can be performed on languages and result in languages:

- $L_1 \cup L_2 = \{w \mid w \in L_1 \vee w \in L_2\}$ (union),
- $L_1 \cap L_2 = \{w \mid w \in L_1 \wedge w \in L_2\}$ (intersection),
- $L_1 \setminus L_2 = \{w \mid w \in L_1 \wedge w \notin L_2\}$ (set difference).

More generally, if $\phi(w)$ is any property of strings, then $\{w \mid \phi(w)\}$ is a language, so we can define languages such as $\{w \mid w \text{ is the password of an NSA executive}\}$ without even knowing what strings are included! However, there are two specific operations on languages that are especially important for our study—called *language concatenation* and *Kleene star* and defined shortly—which arise because the components of our languages, namely strings, themselves have an internal structure involving characters and concatenation.

Definition 2.5 (Concatenation of languages) If $L_1, L_2 \subseteq \Sigma^*$, then we define the *concatenation* of L_1 and L_2 to be the language

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}.$$

That is,

$$w \in L_1 \cdot L_2 \quad \text{if and only if} \quad \exists w_1\, \exists w_2\; w_1 \in L_1 \wedge w_2 \in L_2 \wedge w = w_1 \cdot w_2,$$

or, in other words, $w \in L_1 \cdot L_2$ precisely when we can split w up into two parts (in at least one but possibly more than one way), so that the first part is in L_1 and the second part is in L_2.

The following theorem lists some properties of this operation:

Theorem 2.6 *(Properties of language concatenation)*

1. $\emptyset \cdot L = L \cdot \emptyset = \emptyset$
2. $\{\varepsilon\} \cdot L = L \cdot \{\varepsilon\} = L$
3. $L_1 \cdot (L_2 \cdot L_3) = (L_1 \cdot L_2) \cdot L_3$
4. $L_1 \subseteq L_2$ *implies* $L \cdot L_1 \subseteq L \cdot L_2$ *and* $L_1 \cdot L \subseteq L_2 \cdot L$

5. *$L \cdot (L_1 \cup L_2) = (L \cdot L_1) \cup (L \cdot L_2)$ and $(L_1 \cup L_2) \cdot L = (L_1 \cdot L) \cup (L_2 \cdot L)$*

6. *$L \cdot (L_1 \cap L_2) \subseteq (L \cdot L_1) \cap (L \cdot L_2)$ and $(L_1 \cap L_2) \cdot L \subseteq (L_1 \cdot L) \cap (L_2 \cdot L)$, but not necessarily the reverse.*

Proof. I'll prove the first part of (6); the proofs of the rest are left as exercises for the reader.

Let $L, L_1, L_2 \subseteq \Sigma^*$ be arbitrary, and assume $w \in L \cdot (L_1 \cap L_2)$. Then by Definition 2.5 and the comment afterward, $w = w_1 \cdot w_2$ for some $w_1, w_2 \in \Sigma^*$ with $w_1 \in L$ and $w_2 \in L_1 \cap L_2$. By the definition of intersection, this means $w_2 \in L_1$ and $w_2 \in L_2$. Therefore $w_1 \cdot w_2 \in L \cdot L_1$ and $w_1 \cdot w_2 \in L \cdot L_2$, showing that $w = w_1 \cdot w_2 \in (L \cdot L_1) \cap (L \cdot L_2)$.

Why isn't the reverse necessarily true? Intuitively, if $w \in (L \cdot L_1) \cap (L \cdot L_2)$, then $w \in L \cdot L_1$ and $w \in L \cdot L_2$, and so we can write $w = w' \cdot w_1$ with $w' \in L$ and $w_1 \in L_1$, and write $w = w'' \cdot w_2$ with $w'' \in L$ and $w_2 \in L_2$, but there is no guarantee that $w' = w''$ or $w_1 = w_2$, which we would need in order to claim $w \in L \cdot (L_1 \cap L_2)$. However, to really disprove this, we would need to find a *counterexample*, so here is one that realizes this intuitive idea: let $L = \{\mathtt{a}, \mathtt{aa}\}$, $L_1 = \{\mathtt{ab}\}$, and $L_2 = \{\mathtt{b}\}$; then the string `aab` is in both $L \cdot L_1$ and $L \cdot L_2$ (as $\mathtt{a} \cdot \mathtt{ab}$ and $\mathtt{aa} \cdot \mathtt{b}$, respectively) but is not in $L \cdot (L_1 \cap L_2)$, since $L_1 \cap L_2 = \emptyset$. $\dashv$

Note that, according to Theorem 2.6.1–2, $\emptyset$ is a *zero element* for concatenation, and $\{\varepsilon\}$ is a *unit element* for concatenation. That is, in the numerical analogy where concatenation is multiplication, $\emptyset$ acts like 0 and $\{\varepsilon\}$ acts like 1. For this reason, we will sometimes denote these two sets **0** and **1**, respectively. However, be careful with this analogy: although concatenation is associative (Theorem 2.6.3), is not commutative ($L_1 \cdot L_2$ is not necessarily equal to $L_2 \cdot L_1$).

Definition 2.7 (Kleene star) If $L \subseteq \Sigma^*$, then the *Kleene star* of L is the language L^* defined inductively by these rules:

$$\frac{}{\varepsilon \in L^*}\ {*}\textsc{Empty} \qquad\qquad \frac{w \in L \qquad u \in L^*}{w \cdot u \in L^*}\ {*}\textsc{Cat}$$

The following key property of the Kleene star is an easy consequence of this definition.

Theorem 2.8 *For any language L, we have $L^* = \{\varepsilon\} \cup L \cdot L^*$.*

Proof. We show both $L^* \subseteq \{\varepsilon\} \cup L \cdot L^*$ and $\{\varepsilon\} \cup L \cdot L^* \subseteq L^*$.

First, suppose that $v \in L^*$. By inversion on the derivation of $v \in L^*$, we have two cases:

Case *Empty: $v = \varepsilon$. Then $v \in \{\varepsilon\}$ and thus $v \in \{\varepsilon\} \cup L \cdot L^*$.

Case *Cat: $v = w \cdot u$ where $w \in L$ and $u \in L^*$. Then $v \in L \cdot L^*$ by the definition of concatenation and thus $v \in \{\varepsilon\} \cup L \cdot L^*$.

In either case, $v \in \{\varepsilon\} \cup L \cdot L^*$, and so we've shown $L^* \subseteq \{\varepsilon\} \cup L \cdot L^*$.

In the other direction, suppose $v \in \{\varepsilon\} \cup L \cdot L^*$. Then either $v = \varepsilon$, in which case $v \in L^*$ by (*Empty) or $v = w \cdot u$ where $w \in L$ and $u \in L^*$, in which case $v \in L^*$ by (*Cat). It follows that $\{\varepsilon\} \cup L \cdot L^* \subseteq L^*$. ⊣

In fact, we can extend this theorem further. Continuing the multiplication analogy, let $L^n = L \cdot L \cdots L$ (n times). That is,

$$\begin{aligned} L^0 &= \mathbf{1} \quad \text{(i.e., } \{\varepsilon\}\text{)} \\ L^{n+1} &= L \cdot L^n. \end{aligned}$$

Then, for any n, we have

$$L^* = \{\varepsilon\} \cup L \cup L^2 \cup \cdots \cup L^n \cdot L^*,$$

as well as the infinite version:

$$L^* = \{\varepsilon\} \cup L \cup L^2 \cup L^3 \cup \cdots = \bigcup_{i=0}^{\infty} L^i.$$

We won't use these characterizations of L^* in what follows, but you can try proving them as exercises.

The following theorem lists some other properites of the Kleene star:

Theorem 2.9 *(Properties of the Kleene star)*

1. $\emptyset^* = \{\varepsilon\}$ *(i.e.,* $\mathbf{0}^* = \mathbf{1}$*)*
2. $\{\varepsilon\}^* = \{\varepsilon\}$ *(i.e.,* $\mathbf{1}^* = \mathbf{1}$*)*
3. $(\Sigma)^* = \Sigma^*$*, justifying our use of* Σ^* *for the set of all strings on* Σ
4. $L \subseteq L^*$
5. $L_1 \subseteq L_2$ *implies* $L_1^* \subseteq L_2^*$
6. $L^* \cdot L^* \subseteq L^*$*, i.e.,* L^* *is closed under concatenation*
7. $(L^*)^* \subseteq L^*$*, i.e.,* L^* *is closed under Kleene star*

Moreover, the subset relationships in (6) and (7) can be replaced by equalities.

Proof. I'll prove (6) and leave the proofs of the rest as exercises.

By the observation following Definition 2.5 above, it will suffice to show that $w_1 \in L^*$ and $w_2 \in L^*$ imply $w_1 \cdot w_2 \in L^*$, which I'll do for an arbitrary $w_2 \in L^*$ by induction on the derivation of $w_1 \in L^*$.

Case *Empty: $w_1 = \varepsilon$. Then $w_1 \cdot w_2 = w_2$ by (CatEmpty), which is in L^* by assumption.

Case *Cat: $w_1 = w_1' \cdot u$ where $w_1' \in L$ and $u \in L^*$. Then

$$w_1 \cdot w_2 = (w_1' \cdot u) \cdot w_2 = w_1' \cdot (u \cdot w_2)$$

by Theorem 2.4.2. But by the IH on $u \in L^*$, we have $u \cdot w_2 \in L^*$, and so $w_1' \cdot (u \cdot w_2) \in L^*$ by *Cat, and therefore $w_1' \cdot (u \cdot w_2) = (w_1' \cdot u) \cdot w_2 = w_1 \cdot w_2$ again by Theorem 2.4.2.

To justify the statement that the subset relationship in (6) can be replaced by an equality, I need to prove the converse inclusion: $L^* \subseteq L^* \cdot L^*$. But if $w \in L^*$, then since $\varepsilon \in L^*$ by *Empty, we have $w = \varepsilon \cdot w \in L^* \cdot L^*$. ⊣

The conditions $\varepsilon \in L^*$, $L \subseteq L^*$, and $L^* \cdot L^* \subseteq L^*$ from *Empty and Theorem 2.9 can be used to give an alternative inductive definition of L^*:

Theorem 2.10 *(Alternative definition of Kleene star) L^* can be equivalently defined by the rules*

$$\frac{}{\varepsilon \in L^*} \qquad \frac{w \in L}{w \in L^*} \qquad \frac{w \in L^* \qquad u \in L^*}{w \cdot u \in L^*}$$

i.e., it is the smallest set containing ε (as an element) and L (as a subset) and closed under concatenation.

Proof. Let's number these rules 1, 2, and 3, and for the purposes of this proof use L^{*1} and L^{*2} to stand for the sets defined by the rules in Definition 2.7 and the present theorem, respectively. I'll first show that $s \in L^{*1}$ implies $s \in L^{*2}$ for every $s \in \Sigma^*$ by induction on the derivation of $s \in L^{*1}$:

Case *Empty: $s = \varepsilon$. Then $s \in L^{*2}$ by rule 1.

Case *Cat: $s = w \cdot u$ where $w \in L$ and $u \in L^{*1}$. By the IH, $u \in L^{*2}$, and since $w \in L^{*2}$ by rule 2, we have $w \cdot u \in L^{*2}$ by rule 3.

Second, I'll show the converse, that $t \in L^{*2}$ implies $t \in L^{*1}$ for every $t \in \Sigma^*$, by induction on the derivation of $t \in L^{*2}$:

Case 1: $t = \varepsilon$. Then $t \in L^{*1}$ by *Empty.

Case 2: $t \in L$. Since $\varepsilon \in L^{*1}$ by *Empty and $t = t \cdot \varepsilon$ by Theorem 2.4, we have $t \in L^{*1}$ by *Cat.

Case 3: $t = w \cdot u$ where $w \in L^{*2}$ and $u \in L^{*2}$. By the IH, both $w \in L^{*1}$ and $u \in L^{*1}$, and so $t \in L^{*1}$ by Theorem 2.9.6. ⊣

Let me finish this section with yet another way of defining the Kleene star operation that makes it clear that L^* is the collection of all possible concatenations of elements of L. My notation is inspired by Haskell.

Given a set L, the set $[L]$ of *lists over* L can be defined inductively by the rules

$$\frac{}{[\,] \in [L]} \qquad \frac{w \in L \qquad l \in [L]}{w : l \in [L]} \,.$$

If L is a language, we can define an operation $\mathsf{concat} : [L] \to \Sigma^*$ by

$$\mathsf{concat}\ [\,] = \varepsilon \qquad \text{(ConcatEmpty)}$$
$$\mathsf{concat}\ (w : l) = w \cdot \mathsf{concat}\ l \qquad \text{(ConcatCons)}$$

THEOREM 2.11 *(Yet another definition of Kleene star) L^* can be equivalently defined by the equation*

$$L^* = \{\mathsf{concat}\ l \mid l \in [L]\}.$$

PROOF. Let's number the rules defining $[L]$ as 1 and 2. I'll first show by induction on the derivation of $s \in L^*$ that $s = \mathsf{concat}\ l$ for some $l \in [L]$. So, if $s = \varepsilon$, then we can take $l = [\,]$, since $l \in [L]$ by 1 and $\mathsf{concat}\ l = \varepsilon$ by (CONCATEMPTY). And if $s = w \cdot u$ where $w \in L$ and $u \in L^*$, then by the IH, $u = \mathsf{concat}\ l'$ for some $l' \in [L]$, so we can take $l = w : l'$, and get $l \in [L]$ by 2 and $\mathsf{concat}\ l = s$ by (CONCATCONS).

In the reverse direction, I'll show by induction on the derivation of $l \in [L]$ that $\mathsf{concat}\ l \in L^*$. So, if $l = [\,]$, then $\mathsf{concat}\ l = \varepsilon$ by (CONCATEMPTY), which is in L^* by (*EMPTY). And if $l = w : l'$ where $w \in L$ and $l' \in [L]$, then $\mathsf{concat}\ l = w \cdot \mathsf{concat}\ l'$ by (CONCATCONS), and since $\mathsf{concat}\ l' \in L^*$ by the IH, we have $\mathsf{concat}\ l \in L^*$ by (*CAT). $\dashv$

3 Regular languages and regular expressions

In the introduction, I defined regular languages as those accepted by FSMs. Here, I am going to take an indirect approach: I am going to give an inductive definition of what it means to be a "regular language" and then *prove*, in later sections, that these so-called regular languages are precisely the languages accepted by FSMs and thus are deserving of the name. The end point is the same, but this will allow me to do some preliminary study of regular languages before formally introducing FSMs and developing the tools necessary for the proof.

3.1 Regular languages

DEFINITION 3.1 (REG) We define the class REG of *regular languages* by the following rules:

$$\frac{}{\emptyset \in \mathsf{REG}}\ \textsc{RegEmpty} \qquad \frac{a \in \Sigma}{\{a\} \in \mathsf{REG}}\ \textsc{RegLetter}$$

$$\frac{L_1 \in \mathsf{REG} \qquad L_2 \in \mathsf{REG}}{L_1 \cup L_2 \in \mathsf{REG}}\ \textsc{RegUnion}$$

$$\frac{L_1 \in \mathsf{REG} \qquad L_2 \in \mathsf{REG}}{L_1 \cdot L_2 \in \mathsf{REG}}\ \textsc{RegCat} \qquad \frac{L \in \mathsf{REG}}{L^* \in \mathsf{REG}}\ \textsc{RegStar}$$

Note that these rules define a *set* of languages, i.e., $\mathsf{REG} \subseteq \mathsf{Pow}(\Sigma^*)$ or, equivalently, $\mathsf{REG} \in \mathsf{Pow}(\mathsf{Pow}(\Sigma^*))$. It is the smallest set of languages that contains the empty and single-letter languages and is closed under union, concatenation, and Kleene star.

Let's play around with this definition a bit to get a sense of what's here. As a start, we can prove that every finite language is regular:

THEOREM 3.2 *(Finite languages are regular) If $L \subseteq \Sigma^*$ is finite, then $L \in \mathsf{REG}$.*

The proof is to observe that, since the one-letter languages $\{a\}$ $(a \in \Sigma)$ are all regular by (REGLETTER) and regular languages are closed under concatenation by (REGCAT), it follows that the *one-string* languages $\{w\}$ $(w \in \Sigma^*)$ are also regular. And since every finite set is a union of one-element sets, we get that every finite language $F \subseteq \Sigma^*$ is regular.

Of course, regular languages can be infinite, too: if $\mathtt{a} \in \Sigma$, then by using (REGLETTER) and (REGSTAR), we see that

$$\{\mathtt{a}\}^* = \{\varepsilon, \mathtt{a}, \mathtt{aa}, \mathtt{aaa}, \ldots\}$$

is regular. Finally, let's observe that, because of Theorem 3.2 and (REGUNION), any *finite modification* of a regular language is regular, where by finite modification I mean adding an arbitrary finite set of strings. (We'll see later that we can also *subtract* a finite set of strings and still have a regular language.)

3.2 Regular expressions

Regular expressions are just *names* of regular sets. Since, by Definition 3.1, regular sets are constructed using certain fixed operations, we can create a term language with constructors for each of these operations, and the terms will then correspond to the regular sets. Let's formalize this idea.

Let RE be the set of terms generated by this syntactic specification:

$$r \in \mathsf{RE} \quad ::= \quad 0 \mid \mathtt{a} \mid r_1 + r_2 \mid r_1 r_2 \mid r^* \qquad (\mathtt{a} \in \Sigma)$$

In writing such terms, it will be convenient to omit parentheses by declaring that the star operation has the highest precedence, followed by concatenation, and finally plus, and that both plus and concatenation associate to the right. Thus, the regular expression

$$\mathtt{b} + \mathtt{abc} + \mathtt{bc}^* \quad \text{is really} \quad (\mathtt{b} + (((\mathtt{a})((\mathtt{b})(\mathtt{c}))) + ((\mathtt{b})((\mathtt{c})^*)))),$$

when a pair of parentheses is added around the result of every constructor.

If r is a regular expression, let us write $[\![r]\!]$ for the regular set named by r. This can be defined recursively as follows:

$$\begin{aligned} [\![0]\!] &= \emptyset \\ [\![\mathtt{a}]\!] &= \{\mathtt{a}\} \\ [\![r_1 + r_2]\!] &= [\![r_1]\!] \cup [\![r_2]\!] \\ [\![r_1 r_2]\!] &= [\![r_1]\!] \cdot [\![r_2]\!] \\ [\![r^*]\!] &= [\![r]\!]^*. \end{aligned}$$

Of course, because of the nature of the regular operators, many different regular expressions can denote the same regular set. For example, both $\mathtt{a}+\mathtt{b}$ and $\mathtt{b}+\mathtt{a}$ denote the regular set $\{\mathtt{a},\mathtt{b}\}$; both $\mathtt{a}+(\mathtt{b}+\mathtt{c})$ and $(\mathtt{a}+\mathtt{b})+\mathtt{c}$ denote the regular set $\{\mathtt{a},\mathtt{b},\mathtt{c}\}$; and, for any regular expression r, the regular expressions r^*, $(r^*)^*$, and r^*r^*, all denote the regular set $[\![r]\!]^*$, as a consequence of Theorem 2.9.

If w is a string and r is a regular expression, then we say that w *matches* r if $w \in [\![r]\!]$.

Examples. I'm going to give a few easier examples of regular expressions over the two-letter alphabet $\Sigma = \{\mathtt{a},\mathtt{b}\}$ and then leave the harder ones as exercises for you!

1. The regular expression $r = (\mathtt{a}+\mathtt{b})^*$ matches all strings on Σ, i.e., every string of `a`'s and `b`'s, of any length, including the empty string.

2. Both regular expressions $r_1 = (\mathtt{a}+\mathtt{b})(\mathtt{a}+\mathtt{b})^*$ and $r_2 = (\mathtt{a}+\mathtt{b})^*(\mathtt{a}+\mathtt{b})$ match all non-empty strings, since the $(\mathtt{a}+\mathtt{b})$ term matches exactly one character.

3. All three regular expressions $r_1 = \mathtt{b}^*\mathtt{a}(\mathtt{a}+\mathtt{b})^*$, $r_2 = (\mathtt{a}+\mathtt{b})^*\mathtt{a}(\mathtt{a}+\mathtt{b})^*$ and $r_3 = (\mathtt{a}+\mathtt{b})^*\mathtt{a}\mathtt{b}^*$ match all strings containing at least one `a`: r_1 singles out the first `a`, r_2 singles out any `a`, and r_3 singles out the last `a`.

4. The regular expression $r = \mathtt{a}^* + \mathtt{a}^*\mathtt{b}\mathtt{a}^*$ matches all strings with at most one `b`: $\mathtt{a}^*$ matches strings with no `b`'s, and $\mathtt{a}^*\mathtt{b}\mathtt{a}^*$ matches strings with exactly one `b`.

Exercises. Now, try your hand at constructing regular expressions for each of the following, progressively more difficult languages, still over the alphabet $\Sigma = \{\mathtt{a},\mathtt{b}\}$:

1. all strings in which every `a` is immediately followed by `bb`
2. all strings with an even number of `a`'s
3. all strings with at least one `a` and at least one `b`
4. all strings with no instance of `bbb` (as a substring)
5. all strings with no instance of `aba`
6. all strings with every instance of `aa` coming before every instance of `bb`
7. all strings with an even number of `a`'s and an even number of `b`'s.

3.3 Recursive predicates of REs

Many properties of the languages denoted by regular expressions can be computed recursively from the regular expressions themselves.

Emptiness. A regular expression r is *empty* if $[\![r]\!] = \emptyset$. Let's denote this condition by $0(r)$; it can be defined as follows:

- $0(0)$ is true;
- $0(a)$ is false;
- $0(r_1 + r_2)$ iff $0(r_1)$ and $0(r_2)$;
- $0(r_1 \cdot r_2)$ iff $0(r_1)$ or $0(r_2)$;
- $0(r_1^*)$ is false.

Unitarity. A regular expression r is *unitary* if $[\![r]\!] = \mathbf{1}$. Let's denote this condition by $1(r)$; it can be defined as follows:

- $1(0)$ and $1(a)$ are false;
- $1(r_1 + r_2)$ iff either $1(r_1)$ and $0(r_2)$, or $0(r_1)$ and $1(r_2)$, or $1(r_1)$ and $1(r_2)$;
- $1(r_1 \cdot r_2)$ iff $1(r_1)$ and $1(r_2)$;
- $1(r_1^*)$ iff $0(r_1)$ or $1(r_1)$.

Bypassability. A regular expression r is *bypassable* if $\varepsilon \in [\![r]\!]$ (or, equivalently, $\mathbf{1} \subseteq [\![r]\!]$). This condition, denoted $b(r)$, can be defined as follows:

- $b(0)$ and $b(a)$ are false;
- $b(r_1 + r_2)$ iff $b(r_1)$ or $b(r_2)$;
- $b(r_1 \cdot r_2)$ iff $b(r_1)$ and $b(r_2)$;
- $b(r_1^*)$ is true.

Infiniteness. A regular expression r is *infinite* if $[\![r]\!]$ is an infinite set. This condition, denoted $\infty(r)$, can be defined as follows:

- $\infty(0)$ and $\infty(a)$ are false;
- $\infty(r_1 + r_2)$ iff $\infty(r_1)$ or $\infty(r_2)$;
- $\infty(r_1 \cdot r_2)$ iff $\infty(r_1)$ and not $0(r_2)$, or $\infty(r_2)$ and not $0(r_1)$;
- $\infty(r_1^*)$ iff not $0(r_1)$ and not $1(r_1)$.

We'll see some more examples later.

4 Finite State Machines

We now come to the problem of turning our previous informal description of an FSM into a formal definition. Recall from the Introduction that an FSM

- reads characters one at a time from left to right, indicating after each character whether it has accepted the string read so far,
- can only be in one of a fixed, finite number of states, and
- is deterministic (i.e., always starts in the same state and behaves the same way given the same input).

We can represent the states of an FSM as a finite set Q, and let $s \in Q$ be the state in which the machine starts (its *start state*). Since the machine is deterministic, and the only "memory" available to it is its set of states, all that matters in determining the machine's next state is the state that it is currently in and the character it just read. It follows that the machine's transition behavior is completely described by a function $\delta : Q \times \Sigma \to Q$, so that if the machine is in state $q \in Q$ and reads the character $x \in \Sigma$, it moves to the state $\delta(q, x)$.

That leaves the question of acceptance: how does the machine indicate it has accepted the string read so far? Its decision to accept a string or not must be completely determined by the state it's in after reading the last character, so that leaves us really only one choice: we designate some subset $F \subseteq Q$ of the states as *accepting states* and say that if the machine enters an accepting state then it has accepted the string it has read so far. With these preliminaries, we can now give the formal definition.

4.1 Definition of an FSM

Definition 4.1 (Finite State Machine) A *Finite State Machine* is a tuple (Q, s, F, δ), where

- Q is a finite set (the *states*)
- $s \in Q$ (the *initial state*)
- $F \subseteq Q$ (the *final states*)
- $\delta : Q \times \Sigma \to \Sigma$ (the *transition function*)

We denote by FSM the set of all finite state machines.

We want to have no limitation on what Q can be, other than that it is finite, so that we have maximum flexibility in constructing machines. A much more restrictive alternative would have been to replace Q with a single number N and call the states $\{0, 1, 2, \ldots, N-1\}$. Although this would work, and would standardize machines more, it would make it much harder to construct new machines from old ones.

4.2 Language Accepted by an FSM

Of course, the main purpose of an FSM is to accept a particular language, so we have to give a formal definition of the language accepted by an FSM $M = (Q, s, F, \delta)$ in terms of its components Q, s, F, and δ. We will denote this language by $L(M)$, which you can read as "the language of M." Actually, I'm going to give two such definitions, each of which proceeds by recursion on the input string, and then prove (by induction on the input string) that they are equivalent.

Definition 1. For this definition we extend the function δ, which tells us what M does on each input symbol, to a function δ^*, which tells us what M does on each input *string*. Since a string is just a sequence of input symbols, this can be achieved by recursion on the input string:

$$\begin{aligned} \delta^*(q, \varepsilon) &= q \\ \delta^*(q, aw) &= \delta^*(\delta(q, a), w) \end{aligned}$$

That is, after "reading" the empty string, we are still in the same state, and after reading aw, we are in the state we get to by first making the transition from q to $\delta(q, a)$ on a, and then reading the rest of the string w from there.

Now, using δ^* we can define $L(M)$:

$$w \in L(M) \quad \text{iff} \quad \delta^*(s, w) \in F.$$

In English: a string w is accepted by the FSM M if, when we start the machine in state s and read the string w, we end up in a final state.

Definition 2. For this defintion, we generalize $L(M)$ to $L_q(M)$, which is the language accepted by M when we start the machine in state q rather than s. The judgment $w \in L_q(M)$ can be defined, simultaneously for every $q \in Q$, by these rules:

$$\frac{q \in F}{\varepsilon \in L_q(M)}\ \textsc{AcceptEmpty} \qquad\qquad \frac{w \in L_{\delta(q,a)}(M)}{aw \in L_q(M)}\ \textsc{AcceptCons}$$

In English: we accept the empty string starting in state q if q is already a final state, and we accept the string aw starting in state q if we accept the string w starting in state $\delta(q, a)$, which is where we are after making the transition from q on symbol a.

Using this judgment, we can give our second definition of $L(M)$:

$$w \in L(M) \quad \text{iff} \quad w \in L_s(M).$$

The equivalence. How do we prove the equivalence of these two definitions? Clearly, given the two ways we've defined $L(M)$, this will amount to showing

$$\delta^*(s, w) \in F \quad \text{iff} \quad w \in L_s(M),$$

for all w, and we would be tempted to prove this by induction on w, since both of the definitions proceeded by recursion on w. However, this won't work, because both definitions involve s being replaced by other states as we recurse down the input string w, and so this equivalence is too weak to use as an IH. What we need is a stronger IH that quantifies over *all* states:

THEOREM 4.2 *Suppose $M = (Q, s, F, \delta)$ is a FSM, and δ^* and $L_q(M)$ are defined as above. Then for any string $w \in \Sigma^*$, we have*

$$\text{for all } q \in Q,\ \ \delta^*(q, w) \in F \text{ iff } w \in L_q(M).$$

PROOF. By induction on w (where we note that the quantification over states is now included in the IH).

Case EMPTY: $w = \varepsilon$. Let q be arbitrary. Then, by the definitions of δ^* and $L_q(M)$, we have $\delta^*(q, w) \in F$ iff $q \in F$ iff $w \in L_q(M)$, as required.

Case CONS: $w = aw'$, where $a \in \Sigma$ and $w' \in \Sigma^*$. Let q be arbitrary. Then,

$$\begin{aligned}
\delta^*(q, aw') \in F \quad &\text{iff} \quad \delta^*(\delta(q, a), w') \in F && \text{(by def of } \delta^*) \\
&\text{iff} \quad w' \in L_{\delta(q,a)}(M) && \text{(by the IH on state } \delta(q, a)) \\
&\text{iff} \quad aw' \in L_q(M), && \text{(by def of } L_q(M))
\end{aligned}$$

as required (and note how the stronger IH lets us use the equivalence at any state, in this case $\delta(q, a)$, even though we have to prove it for q). $\dashv$

The equivalence of the two definitions of $L(M)$ now follows easily from this Theorem by choosing q to be s. The practical implication of this equivalence is that we are free to use either definition where it is convenient, although it actually seems in what follows that I always use Definition 1, and so I will take this definition as the official one.

The construction of the language $L(M)$ from an FSM M is of course functional, so that we actually have $L : \mathsf{FSM} \to \mathsf{Pow}(\Sigma^*)$. But will see in Section 5 below that $L(M)$ is, in fact, regular for every $M \in \mathsf{FSM}$, meaning that our function is actually $L : \mathsf{FSM} \to \mathsf{REG}$.

4.3 Representing FSMs Visually

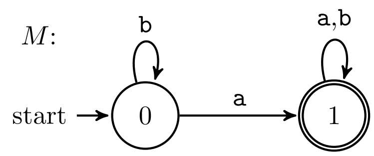
In Section 3, we represented (or named) regular languages using regular expressions, and we would like to do something similar with FSMs, but we quickly realize that the amount of information that goes into a description of a FSM is too much for a useful textual description. Fortunately, there is a nice graphical description that will do the job.

We will represent an FSM $M = (Q, s, F, \delta)$ graphically by drawing a circle for each state; drawing an arrow connecting two circles, labeled with a list (or set) of input symbols, if there are transitions from the source state to the destination state for each of input symbols in the label; drawing an arrow from the word "start" into the initial state; and making the final state(s) double circles.

For example, consider the FSM over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ with

- $Q = \{0, 1\}$
- $s = 0$
- $F = \{1\}$
- $\delta = \{(0, \texttt{a}, 1), (0, \texttt{b}, 0), (1, \texttt{a}, 1), (1, \texttt{b}, 1)\}$,

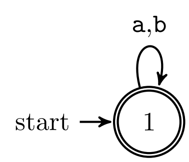
where I've given δ by its "graph;" thus, $\delta(0, \texttt{a}) = 1$, $\delta(0, \texttt{b}) = 0$, $\delta(1, \texttt{a}) = 1$, and $\delta(1, \texttt{b}) = 1$. This FSM can be represented graphically as follows:



4.4 Examples and Exercises

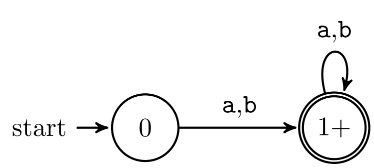
We will now revisit the examples and exercises from Section 3, this time constructing FSMs that accept the given languages. As before, I will do the easier ones and leave the more difficult ones for you! (Part of the fun of learning how to design FSMs is figuring out the tricks and techniques yourself, but I'll give you a few hints to get you started.) Again, we are keeping to a two-letter alphabet $\Sigma = \{\texttt{a}, \texttt{b}\}$.

1. The language consisting of all strings can be recognized by a one-state FSM:

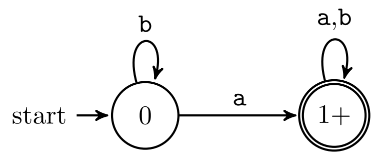
   Note, by the way, that if the single state in this machine were not a final state, then the machine would accept the empty language—consider that a bonus example!

2. The language consisting of all nonempty strings can be recognized by a machine that adds a new initial state to the previous machine:

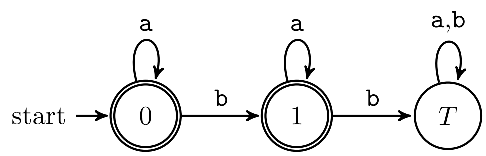
   Here, the names I have chosen for the states are suggestive: 0 is the state where we have (so far) read 0 characters, and 1+ is the state where we have read 1 or more characters.

3. The language consisting of all strings with at least one `a` is recognized by the example machine from the previous subsection, which I'll repeat here for convenience (with a new name for state 1):

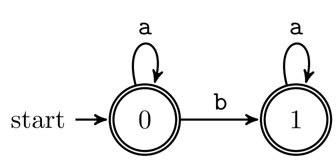
   State 0 of this machine is the state where we have read 0 `a`'s, and state 1+ is the state where we have read one or more `a`'s; note that reading a `b` doesn't change the number of `a`'s we have read, so these transitions just loop back to the same state.

4. Finally, the language consisting of all strings with at most one `b` can be recognized by this three-state FSM:

   Here, state 0 is the state where we have read 0 `b`'s, state 1 is the state where we have read 1 `b`, and state T is a "trap state," i.e., a state that is not final but to which all transitions lead back. Both 0 and 1 are final states, because we want to accept strings that have 0 `b`'s or exactly 1 `b`. However, once we read a second `b`, we will no longer accept the string, whatever the remaining characters may be, and so a trap state is appropriate.

Trap states are both useful and common, and so we will adopt a convention when drawing machines with trap states that will keep our diagrams simpler: if, in an FSM diagram, there is at least one state and input symbol for which no transition is indicated, then the machine is assumed to have a single trap state, and every missing transition is taken to be a transition to that state. Thus, we could simplify the previous machine as follows:



Now, with these examples in hand, go back to the exercises in Section 3 and construct FSMs for each of those languages. In some cases, you will be able to use the trap-state convension to keep your machines simpler.

5 Regular Languages are Accepted by FSMs

The goal of this section is to prove one half of the equivalence of FSMs and regular languages:

THEOREM 5.1 *For every regular language R, there exists a finite state machine M such that $R = L(M)$.*

Our strategy is simple, even if the individual steps are more complicated. The class of regular languages has an inductive definition (3.1), according to which the empty language and, for every $a \in \Sigma$, the single-letter language, $\{a\}$, are regular; and if L_1 and L_2 are regular, then $L_1 \cup L_2$, $L_1 \cdot L_2$, and L_1^* are regular. Moreover, we have a system of names for these regular languages, which we called regular expressions (the set of which was denoted RE), with a constructor for each of these base languages and regular operators; if $r \in \mathsf{RE}$, then $[\![r]\!] \in \mathsf{REG}$ was the regular language named by r. Therefore, to show that every regular language is accepted by some FSM, it will be enough to

- construct a machine $M(0)$ that accepts the (empty) language, $\emptyset$;
- construct, for every $a \in \Sigma$, a machine $M(a)$ that accepts the (single-letter) language, $\{a\}$;
- given machines M_1 and M_2 that accept the languages L_1 and L_2, construct a machine $M_1 \cup M_2$ that accepts the language $L_1 \cup L_2$;
- given machines M_1 and M_2 that accept the languages L_1 and L_2, construct a machine $M_1 \cdot M_2$ that accepts the language $L_1 \cdot L_2$; and
- given a machine M that accepts the language L, construct a machine M^* that accepts the language L^*.

This collection of constructions can then be used to give a recursive definition of a function $M : \mathsf{RE} \to \mathsf{FSM}$ that satisfies the condition $\forall r \in \mathsf{RE}\ L(M(r))) = [\![r]\!]$. Since every regular language, R, satisfies $[\![r]\!] = R$ for its name, r, the machine $M(r)$ will be therefore be the required machine recognizing R.

Each one of the above bulleted constructions is the subject of its own subsection below.

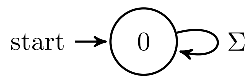
5.1 Empty

I've already given a machine accepting the empty language—it was the "bonus" machine from Example 4.4.1: let $M(0)$ be the machine with

- $Q = \{0\}$
- $s = 0$
- $F = \emptyset$

- $\delta(0, a) = 0$.

This machine can be pictured as follows:



Let me record the obvious:

THEOREM 5.2 $L(M(0)) = \emptyset$.

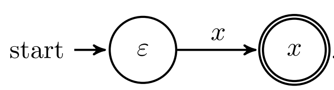
PROOF. By the definition of $L(M)$, $w \in L(M(0))$ iff $\delta^*(0, w) \in \emptyset$. But the latter is false for every w, and so $L(M(0))$ is empty. ⊣

5.2 Letter

For any $x \in \Sigma$, let $M(x)$ be the machine with

- $Q = \{\varepsilon, x, T\}$
- $s = \varepsilon$
- $F = \{x\}$
- $\delta(q, a) = \begin{cases} x, & \text{if } q = \varepsilon \text{ and } a = x; \\ T, & \text{otherwise.} \end{cases}$

This machine can be pictured as follows, where I am using the trap-state convention to avoid having to draw the trap state, T:



I've chosen the names of the (non-trap) states to indicate the strings that must have already been read by the machine if it ends up in that particular state.

Now, it's probably also obvious to you that this machine accepts exactly the language $\{x\}$. Nevertheless, I will give a formal proof of this, so that you can see how it is done. I'll start, as I will for the other constructions to come, with a characterization of δ^* for this machine:

LEMMA 5.3 *For all $w \in \Sigma^*$,*

1. $\delta^*(T, w) = T$
2. $\delta^*(x, w) = \begin{cases} x, & \textit{if } w = \varepsilon; \\ T, & \textit{otherwise} \end{cases}$

3. $\delta^*(\varepsilon, w) = \begin{cases} \varepsilon, & \textit{if } w = \varepsilon; \\ x, & \textit{if } w = x; \\ T, & \textit{otherwise} \end{cases}$

Proof. I'll prove 1 by induction on w. For $w = \varepsilon$, we have $\delta^*(T, \varepsilon) = T$ by definition of δ^*. For $w = aw'$, we have, by the definitions of δ^* and δ and the IH,

$$\delta^*(T, aw') = \delta^*(\delta(T, a), w') = \delta^*(T, w') = T.$$

I'll prove 2 by inversion on w, with two cases. For $w = \varepsilon$, we have $\delta^*(x, \varepsilon) = x$ by definition of δ^*. For $w = aw'$, we have, by the definitions of δ^* and δ and part 1,

$$\delta^*(x, aw') = \delta^*(\delta(x, a), w') = \delta^*(T, w') = T.$$

I'll prove 3 also by inversion on w, with four cases. For $w = \varepsilon$, we have $\delta^*(\varepsilon, w) = \varepsilon$ by definition of δ^*. For $w = x$, we have, by the definitions of δ^* and δ,

$$\delta^*(\varepsilon, x) = \delta^*(\delta(\varepsilon, x), \varepsilon) = \delta^*(x, \varepsilon) = x.$$

For $w = xw'$ with $w' \neq \varepsilon$, we have, by the definitions of δ^* and δ and part 2,

$$\delta^*(\varepsilon, xw') = \delta^*(\delta(\varepsilon, x), w') = \delta^*(x, w') = T.$$

Finally, for $w = aw'$ with $a \neq x$, we have, by the definitions of δ^* and δ and part 1,

$$\delta^*(\varepsilon, aw') = \delta^*(\delta(\varepsilon, a), w') = \delta^*(T, w') = T.$$

$\dashv$

Theorem 5.4 $L(M(a)) = \{a\}$.

Proof. By the definition of $L(M)$, $w \in L(M(a))$ iff $\delta^*(\varepsilon, w) \in \{a\}$. But part 3 of the Lemma shows that the latter occurs exactly when $w = a$. $\dashv$

5.3 Union

In this case, we are given two machines

$$M_1 = (Q_1, s_1, F_1, \delta_1) \qquad \text{and} \qquad M_2 = (Q_2, s_2, F_2, \delta_2),$$

and need to construct a machine $M_1 \cup M_2$ with $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$. Here is the idea. Given an input string w, we are going to start both machines M_1 and M_2 in their respective start states simultaneously reading w. As they each make their respective transitions, we will record the simultaneous state of both machines as a pair (q_1, q_2), where q_1 is the state of machine M_1 and q_2 is the state of machine M_2. Since we want $M_1 \cup M_2$ to accept a string if it is accepted by *either* M_1 or M_2 (or both), we can simply declare the final states of $M_1 \cup M_2$ to be the states (q_1, q_2) where $q_1 \in F_1$ or $q_2 \in F_2$.

Here, then, is the formal construction. Let $M_1 \cup M_2$ be the machine with

- $Q = Q_1 \times Q_2$
- $s = (s_1, s_2)$
- $F = \{(q_1, q_2) \mid q_1 \in F_1 \vee q_2 \in F_2\} = (F_1 \times Q_2) \cup (Q_1 \times F_2)$
- $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$.

To prove that this construction works, we again give the appropriate characterization of δ^* for this machine:

LEMMA 5.5 *For all* $(q_1, q_2) \in Q$ *and* $w \in \Sigma^*$,

$$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)).$$

PROOF. By induction on w. For $w = \varepsilon$, we have

$$\begin{aligned} \delta^*((q_1, q_2), \varepsilon) &= (q_1, q_2) && \text{(def of } \delta^*) \\ &= (\delta_1^*(q_1, \varepsilon), \delta_2^*(q_2, \varepsilon)). && \text{(defs of } \delta_1^* \text{ and } \delta_2^*) \end{aligned}$$

For $w = aw'$, we have

$$\begin{aligned} \delta^*((q_1, q_2), aw') &= \delta^*\big(\delta\big((q_1, q_2), a\big), w'\big) && \text{(def of } \delta^*) \\ &= \delta^*\big(\big(\delta_1(q_1, a), \delta_2(q_2, a)\big), w'\big) && \text{(def of } \delta) \\ &= \big(\delta_1^*(\delta_1(q_1, a), w'), \delta_2^*(\delta_2(q_2, a), w')\big) && \text{(IH)} \\ &= \big(\delta_1^*(q_1, aw'), \delta_2^*(q_2, aw')\big). && \text{(defs of } \delta_1^* \text{ and } \delta_2^*) \end{aligned}$$

as required. ⊣

THEOREM 5.6 $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$.

PROOF. Let $w \in \Sigma^*$ be arbitrary. Then

$$\begin{aligned} w \in L(M_1 \cup M_2) \quad &\text{iff} \quad \delta^*((s_1, s_2), w) \in \{(q_1, q_2) \mid q_1 \in F_1 \vee q_2 \in F_2\} \\ & \qquad\qquad\qquad \text{(def of machine and } L) \\ &\text{iff} \quad (\delta_1^*(s_1, w), \delta_2^*(s_2, w)) \in \{(q_1, q_2) \mid q_1 \in F_1 \vee q_2 \in F_2\} \\ & \qquad\qquad\qquad \text{(Lemma)} \\ &\text{iff} \quad \delta_1^*(s_1, w) \in F_1 \vee \delta_2^*(s_2, w) \in F_2 \\ &\text{iff} \quad w \in L(M_1) \vee w \in L(M_2) \qquad \text{(def of } L) \\ &\text{iff} \quad w \in L(M_1) \cup L(M_2), \end{aligned}$$

as required. ⊣

Exercise. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Draw the machine $M(\mathtt{a}) \cup M(\mathtt{b})$, first without using the trap-state convention, and then again using it. (Hint: the first machine will have 9 states, 5 of which are final, and 14 arrows; the second machine will have 8 states and 6 arrows.) What's the smallest machine you can draw that accepts the same language, i.e., $\{\mathtt{a}, \mathtt{b}\}$? What's going on with the extra states in $M(\mathtt{a}) \cup M(\mathtt{b})$? (The answer will be given in Section 5.6.)

5.4 Concatenation

Similarly to the union, we are given two machines M_1 and M_2 and need to construct a machine $M_1 \cdot M_2$ such that $L(M_1 \cdot M_2) = L(M_1) \cdot L(M_2)$. This construction and its analysis are a bit tricky, however, so let's first get a better idea of what's required. By definition of concatenation, $w \in L(M_1) \cdot L(M_2)$ iff we can write $w = w_1 \cdot w_2$, where $w_1 \in L(M_1)$ and $w_2 \in L(M_2)$. Now if there were just one possible way to divide w into w_1 and w_2 such that $w_1 \in L(M_1)$, then we could proceed as follows:

- Start M_1 reading w.
- If M_1 enters a final state after reading w_1 (with w_2 still to read), then abandon machine M_1 and start M_2 in its start state, s_2, reading w_2.
- If M_2 is in a final state after reading w_2, then accept, otherwise reject.

The problem is that there might be several ways to divide w as $w_1 \cdot w_2$ with $w_1 \in L(M_1)$, only one of which has $w_2 \in L(M_2)$. If we commit to running M_2 on w_2 as soon as we find the *first* w_1 such that $w = w_1 \cdot w_2$ and $w_1 \in L(M_1)$, it may be that M_2 rejects *that* w_2, whereas a different division of w into $w_1 \cdot w_2$ would have succeeded.

What to do? The solution is a combination of two insights. First, in order to keep all of our options open, we won't abandon machine M_1 when it gets into a final state, but keep it running simultaneously with M_2, which we have just started in its start state. Then, if M_1 gets into a final state again later on, we'll start *another copy* of M_2 in its start state, so that we'll have three machines running simultaneously. And, as M_1 might continue returning to a final state many more times while reading w, we might end up having many copies of M_2 running simultaneously, each in its own state. Finally, when we get to the end of the string w, we will say that this system of machines accepts w if *at least one* of the copies of M_2 is in a final state.

The problem with this idea, besides it being an accounting nightmare, is that there would be no limit to the number of copies of M_2 we'd have to keep track of simultaneously, and we only have available a fixed, finite amount of state that was independent of the length of the input string. This is where the second insight comes in. Since all we care about this set of copies of M_2 is that at least one of them gets into a final state after finishing w, all we really need to keep track of is the particular *subset* of states of M_2 that the system of machines is in, as an aggregate, at any one time—having more than one copy of M_2 in the same state doesn't make it any easier or harder to accept the string w, since they will all accept or reject together. And if Q_2 is a finite set, then $\mathsf{Pow}(Q_2)$ is also finite.

Thus, the states of the machine $M_1 \cdot M_2$ will be pairs (q_1, X_2), where $q_1 \in Q_1$ and $X_2 \subseteq Q_2$, with X_2 representing the aggregate of the states the copies of the machine M_2 are in at the moment. There is one catch, however: if $q_1 \in F_1$ then we should have $s_2 \in X_2$ as well, since we will be starting another copy of M_2 in its start state at the same time. This condition on states can insured by the following device: if X_2 occurs as the second component of a state (q_1, X_2), then we define

$$X_2^c = X_2 \cup \{s_2 \mid q_1 \in F_1\},$$

which "corrects" X_2 by adding s_2 to X_2 if the first component of the state is final and leaving it unchanged if not. Note that correction is *idempotent*, in the sense that correcting a pair that's already corrected doesn't result in any further changes: $(X_2^c)^c = X_2^c$.

In defining the final states, I will use the notation $X_2 \sqcap F_2$ to mean that the sets X_2 and F_2 *overlap*, i.e., that there exists $q_2 \in X_2 \cap F_2$ (or, written another way, $X_2 \cap F_2 \neq \emptyset$), meaning that at least one of the copies of M_2 is in a final state.

Finally, in defining the transition function, I will make use of the function $\hat{\delta}_2 : \mathsf{Pow}(Q_2) \times \Sigma \to \mathsf{Pow}(Q_2)$ defined by

$$\hat{\delta}_2(X_2, a) = \{\delta_2(q_2, a) \mid q_2 \in X_2\},$$

which are the states to which the various copies of M_2 transition on symbol a.

With these preliminaries out of the way, I can now give the formal construction: let $M_1 \cdot M_2$ be the machine with

- $Q = \{(q_1, X_2^c) \mid q_1 \in Q_1, X_2 \subseteq Q_2\}$
- $s = (s_1, \emptyset^c)$
- $F = \{(q_1, X_2) \in Q \mid X_2 \sqcap F_2\}$
- $\delta\big((q_1, X_2), a\big) = (\delta_1(q_1, a), \hat{\delta}_2(X_2, a)^c)$.

Note that, from the definition of s, we don't start any copies of M_2 at the beginning, *unless* s_1 is also a final state of M_1, in which case we are also starting one copy of M_2 at the same time. Also, we always correct the result of a transition, since $\delta_1(q_1, a)$ might be a final state of M_1. These corrections insure that $s \in Q$ and $\delta : Q \times \Sigma \to Q$, as required in the definition of an FSM.

Now, to prove the correctness of this construction, I'll again state and prove the appropriate characterization of δ^* for this machine, which in this case will make use of an auxiliary notion, called a *trace*, that will be useful also in the analysis of the construction of M^* in the next section. To motivate this notion, consider what happens when our machine is in state $(q_1, X_2) \in Q$ and we read the string w, arriving in state (q_1', X_2'): where do the elements of X_2' come from?

There are two sources: (1) a copy of M_2 that was in state $q_2 \in X_2$ will transition on w to the state $\delta_2^*(q_2, w) \in X_2'$, and (2) M_1, reading w, will transition to the state $\delta_1^*(q_1, w)$, but along the way, it may enter a final state of M_1 after reading w_1, causing a copy of M_2 to start up on the string w_2, where $w = w_1 \cdot w_2$, which will then transition to $\delta_2^*(s_2, w_2) \in X_2'$.

Here, then, is the formal definition of a trace and the corresponding characterization of δ^* for our concatenation machine:

DEFINITION 5.7 Suppose $(q_1, X_2) \in Q$ and $w \in \Sigma^*$. Then a *trace from* (q_1, X_2) *on* w is an object of one of the following two forms:

- $(q_2 \,|\, w)$, where $q_2 \in X_2$, or
- $(q_1 \,|\, w_1, w_2)$, where $w_1 \neq \varepsilon$ and $\delta_1^*(q_1, w_1) \in F_1$ and $w = w_1 \cdot w_2$.

We let $\mathsf{Tr}(q_1, X_2 \,|\, w)$ be the set of all traces from (q_1, X_2) on w and define a function $\mathsf{end} : \mathsf{Tr}(q_1, X_2 \,|\, w) \to Q_2$ by

$$\begin{aligned} \mathsf{end}(q_2 \,|\, w) &= \delta_2^*(q_2, w) \\ \mathsf{end}(q_1 \,|\, w_1, w_2) &= \delta_2^*(s_2, w_2), \end{aligned}$$

which gives the *end-point* of a trace. Finally, we define the set

$$E((q_1, X_2), w) = \{\mathsf{end}(t) \mid t \in \mathsf{Tr}(q_1, X_2 \,|\, w)\}$$

of all end-points of traces from (q_1, X_2) on w.

LEMMA 5.8 *For all* $(q_1, X_2) \in Q$ *and* $w \in \Sigma^*$,

$$\delta^*\big((q_1, X_2), w\big) = \big(\delta_1^*(q_1, w), E((q_1, X_2), w)\big).$$

PROOF. Let $(q_1, X_2) \in Q$. I'll first establish the following recursion equations for the sets $E((q_1, X_2), w) \subseteq Q_2$:

$$E((q_1, X_2), \varepsilon) = X_2 \tag{1}$$

$$E((q_1, X_2), aw) = E\big((\delta_1(q_1, a), \hat{\delta}_2(X_2, a)^c), w\big). \tag{2}$$

The proof of (1) is summarized by the table,

$t \in \mathsf{Tr}(q_1, X_2 \,\|\, \varepsilon)$	$q_2 \in X_2$
$(q_2 \,\|\, \varepsilon)$	q_2

which I claim defines a total and onto relation (in fact, bijection) between elements $t \in \mathsf{Tr}(q_1, X_2, \varepsilon)$ and elements $q_2 \in X_2$, the connection being $\mathsf{end}(t) = q_2$. The existence of such a relation establishes (1).

Similarly, the proof of (2) is summarized by the table,

$t \in \mathsf{Tr}(q_1, X_2 \,\|\, aw)$	$t' \in \mathsf{Tr}(\delta_1(q_1, a), \hat{\delta}_2(X_2, a)^c \,\|\, w)$	condition
$(q_2 \,\|\, aw)$	$(\delta_2(q_2, a) \,\|\, w)$	
$(q_1 \,\|\, aw_1', w_2)$	$(\delta_1(q_1, a) \,\|\, w_1', w_2)$	$w_1' \neq \varepsilon$
$(q_1 \,\|\, a, w)$	$(s_2 \,\|\, w)$	$\delta_1(q_1, a) \in F_1$

which I claim defines three parts of a total and onto relation between elements $t \in \mathsf{Tr}(q_1, X_2 \mid aw)$ and elements $t' \in \mathsf{Tr}(\delta_1(q_1,a), \hat{\delta}_2(X_2,a)^c \mid w)$, under the indicated conditions, such that $\mathsf{end}(t) = \mathsf{end}(t')$. The existence of such a relation establishes (2). In both cases, I leave the straightforward checks to the reader.

With (1) and (2) established, the proof of the main lemma is now a straightforward induction on w. For $w = \varepsilon$, we have

$$\begin{aligned}
\delta^*((q_1, X_2), \varepsilon) &= (q_1, X_2) && \text{(def of } \delta^*\text{)}\\
&= \big(\delta_1^*(q_1, \varepsilon), E((q_1, X_2), \varepsilon)\big). && \text{(def of } \delta_1^* \text{ and (1))}
\end{aligned}$$

For $w = aw'$, we have

$$\begin{aligned}
\delta^*((q_1, X_2), aw') &= \delta^*(\delta((q_1, X_2), a), w') && \text{(def of } \delta^*\text{)}\\
&= \delta^*((\delta_1(q_1,a), \hat{\delta}_2(X_2,a)^c), w') && \text{(def of } \delta\text{)}\\
&= \big(\delta_1^*(\delta_1(q_1,a), w'), E((\delta_1(q_1,a), \hat{\delta}_2(X_2,a)^c), w')\big) && \text{(IH)}\\
&= (\delta_1^*(q_1, aw'), E((q_1, X_2), aw')). && \text{(def of } \delta_1^* \text{ and (2))}
\end{aligned}$$

completing the proof of the Lemma. ⊣

Theorem 5.9 $L(M_1 \cdot M_2) = L(M_1) \cdot L(M_2)$.

Proof. Let $w \in \Sigma^*$ be arbitrary. Then

$$\begin{aligned}
w \in L(M_1 \cdot M_2) \quad &\text{iff} \quad \delta^*((s_1, \emptyset^c), w) \in \{(q_1, X_2) \in Q \mid X_2 \sqcap F_2\}\\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(def of machine and } L\text{)}\\
&\text{iff} \quad E((s_1, \emptyset^c), w) \sqcap F_2 \qquad\qquad \text{(Lemma)}\\
&\text{iff} \quad \exists t \in \mathsf{Tr}(s_1, \emptyset^c \mid w)\ \mathsf{end}(t) \in F_2. \qquad \text{(def of } E\text{)}
\end{aligned}$$

Now, any such t is either $t = (s_2 \mid w)$ with $s_1 \in F_1$ and $\delta_2^*(s_2, w) \in F_2$, in which case the decomposition $w = \varepsilon \cdot w$ shows that $w \in L(M_1) \cdot L(M_2)$, or $t = (s_1 \mid w_1, w_2)$ with $\delta_1^*(s_1, w_1) \in F_1$ and $\delta_2^*(s_2, w_2) \in F_2$, in which case the decomposition $w = w_1 \cdot w_2$ shows that $w \in L(M_1) \cdot L(M_2)$. Conversely, if $w \in L(M_1) \cdot L(M_2)$, then $w = w_1 \cdot w_2$ for $w_1 \in L(M_1)$ and $w_2 \in L(M_2)$, i.e., $\delta_1^*(s_1, w_1) \in F_1$ and $\delta_2^*(s_2, w_2) \in F_2$, and so the trace $t = (s_1 \mid w_1, w_2)$ satisfies $t \in \mathsf{Tr}(s_1, \emptyset^c \mid w)$ and $\mathsf{end}(t) \in F_2$, showing, as above, that $w \in L(M_1 \cdot M_2)$ and completing the proof. ⊣

Exercise. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Draw the machine $M(\mathtt{a}) \cdot M(\mathtt{b})$. (Hint: it will have 20 states, 10 of which are final.) What's the smallest machine you can draw that accepts the same language, i.e., $\{\mathtt{ab}\}$? Again, what's going on with all these extra states? (Again, the answer will be given in Section 5.6.)

5.5 Star

For the final construction, we are given a machine $M_1 = (Q_1, s_1, F_1, \delta_1)$ and we need to construct a machine M_1^* such that $L(M_1^*) = L(M_1)^*$. The idea is similar to the concatenation machine, in that we will need to run multiple copies of M_1 simultaneously, so we will use subsets $X \subseteq Q_1$ to keep track of the aggregate set of states a collection of copies of M_1 are in at any given time, with the catch that, if $X \sqcap F_1$, then we require $s_1 \in X$; that is, when at least one copy of M_1 is in a final state, then a new copy of M_1 will be started simultaneously in its initial state, which we will enforce by defining, for $X \subseteq Q_1$,

$$X^c = X \cup \{s_1 \mid X \sqcap F_1\}.$$

A set of states is final if it includes at least one final state. One final twist: since we always have $\varepsilon \in M_1^*$, the start state of M_1^* must also be final, even if the start state of M_1 is not. We will achieve this by using $\emptyset$ as the start state of M_1^*, which is not otherwise used in the construction, and set the transitions out of this state to mirror the transitions of M_1 out of s_1.

Here, then is the formal construction. Let M_1^* be the machine with

- $Q = \{X^c \mid X \subseteq Q_1\}$
- $s = \emptyset$
- $F = \{\emptyset\} \cup \{X \in Q \mid X \sqcap F_1\}$
- $\delta(X, a) = \begin{cases} \{\delta_1(s_1, a)\}^c, & \text{if } X = \emptyset; \\ \hat{\delta}_1(X, a)^c, & \text{otherwise.} \end{cases}$

Note that, since $\emptyset^c = \emptyset$ and the results of each transition are corrected, we have $s \in Q$ and $\delta : Q \times \Sigma \to Q$, as required in the definition of an FSM.

Here is the appropriate notion of trace for this machine, and the lemma characterising δ^*:

Definition 5.10 Suppose $X \in Q$ and $w \in \Sigma^*$. Then a *trace from* X *on* w is an object of one of the following two forms:

- $(\emptyset \mid w_1, \ldots, w_n)$, where $X = \emptyset$, $n \geq 1$, $w_i \neq \varepsilon$ and $\delta_1^*(s_1, w_i) \in F_1$ for all $i < n$, and $w = \mathsf{concat}\,[w_1, \ldots, w_n]$,
- $(q_1 \mid w_1, \ldots, w_n)$, where $q_1 \in X$, $n \geq 1$, $w_i \neq \varepsilon$ and $\delta_1^*(s_1, w_i) \in F_1$ for all $i < n$, and $w = \mathsf{concat}\,[w_1, \ldots, w_n]$.

$\mathsf{Tr}(X \mid w)$ is the set of all traces from X on w, and $\mathsf{end} : \mathsf{Tr}(X \mid w) \to Q_1$ is the end-point function defined by

$$\begin{aligned} \mathsf{end}(\emptyset \mid w_1, \ldots, w_n) &= \delta_1^*(s_1, w_n) \\ \mathsf{end}(q_1 \mid w) &= \delta_1^*(q_1, w) \\ \mathsf{end}(q_1 \mid w_1, \ldots, w_n) &= \delta_1^*(s_1, w_n) \qquad (n > 1). \end{aligned}$$

Finally, we define

$$E(X, w) = \{\mathsf{end}(t) \mid t \in \mathsf{Tr}(X \,|\, w)\}.$$

LEMMA 5.11 *For all $X \in Q$ and $w \in \Sigma^*$,*

$$\delta^*(X, w) = E(X, w).$$

PROOF. Again, we first establish recursion equations for the $E(X, w) \subseteq Q_1$,

$$E(X, \varepsilon) = X \tag{3}$$
$$E(\emptyset, aw) = E(\{\delta_1(s_1, a)\}^c, w) \tag{4}$$
$$E(X, aw) = E(\hat{\delta}_1(X, a)^c, w) \quad (X \neq \emptyset), \tag{5}$$

leaving the reader to check the details. The proof of (3) is given by the table

$t \in \mathsf{Tr}(X \,\vert\, \varepsilon)$	$q_1 \in X$
$(q_1 \,\vert\, \varepsilon)$	q_1

which I claim defines a total and onto relation (in fact, bijection) between elements $t \in \mathsf{Tr}(X \,|\, \varepsilon)$ and elements $q_1 \in X$, the connection being $\mathsf{end}(t) = q_1$. The existence of such a relation establishes (3).

The proof of (4) is given by the following table, where each n satisfies $n \geq 2$:

$t \in \mathsf{Tr}(\emptyset \,\vert\, aw)$	$t' \in \mathsf{Tr}(\{\delta_1(s_1, a)\}^c \,\vert\, w)$	condition
$(\emptyset \,\vert\, aw)$	$(\delta_1(s_1, a) \,\vert\, w)$	
$(\emptyset \,\vert\, aw'_1, w_2, \ldots, w_n)$	$(\delta_1(s_1, a) \,\vert\, w'_1, w_2, \ldots, w_n)$	$w'_1 \neq \varepsilon$
$(\emptyset \,\vert\, a, w_2, \ldots, w_n)$	$(s_1 \,\vert\, w_2, \ldots, w_n)$	$\delta_1(s_1, a) \in F_1$

which I claim defines three parts of a total and onto relation between elements $t \in \mathsf{Tr}(\emptyset \,|\, aw)$ and elements $t' \in \mathsf{Tr}(\{\delta_1(s_1, a)\}^c \,|\, w)$ under the indicated conditions, such that $\mathsf{end}(t) = \mathsf{end}(t')$. The existence of such a relation establishes (4).

The proof of (5) is given by the following table, where each n satisfies $n \geq 2$:

$t \in \mathsf{Tr}(X \,\vert\, aw)$	$t' \in \mathsf{Tr}(\hat{\delta}_1(X, a)^c \,\vert\, w)$	condition
$(q_1 \,\vert\, aw)$	$(\delta_1(q_1, a) \,\vert\, w)$	
$(q_1 \,\vert\, aw'_1, w_2, \ldots, w_n)$	$(\delta_1(q_1, a) \,\vert\, w'_1, w_2, \ldots, w_n)$	$w'_1 \neq \varepsilon$
$(q_1 \,\vert\, a, w_2, \ldots, w_n)$	$(s_1 \,\vert\, w_2, \ldots, w_n)$	$\delta_1(q_1, a) \in F_1$

which I claim defines three parts of a total and onto relation between elements $t \in \mathsf{Tr}(X \,|\, aw)$ and elements $t' \in \mathsf{Tr}(\hat{\delta}_1(X, a)^c \,|\, w)$ under the indicated conditions (including $X \neq \emptyset$), such that $\mathsf{end}(t) = \mathsf{end}(t')$. The existence of such a relation establishes (5).

With (3), (4), and (5) established, the proof of the main lemma is now a straightforward induction on w. For $w = \varepsilon$, we have $\delta^*(X, \varepsilon) = X = E(X, \varepsilon)$ by the definition of δ^* and by (3). For $w = aw'$ and $X = \emptyset$, we have

$$\begin{aligned} \delta^*(\emptyset, aw') = \delta^*(\delta(\emptyset, a), w') &= \delta^*(\{\delta_1(s_1, a)\}^c, w') && \text{(defs of } \delta^* \text{ and } \delta) \\ &= E(\{\delta_1(s_1, a)\}^c, w') = E(\emptyset, aw'). && \text{(IH and (4))} \end{aligned}$$

And for $w = aw'$ and $X \neq \emptyset$, we have

$$\begin{aligned}\delta^*(X, aw') = \delta^*(\delta(X, a), w') &= \delta^*(\hat{\delta}_1(X, a)^c, w') && \text{(defs of } \delta^* \text{ and } \delta)\\ &= E(\hat{\delta}_1(X, a)^c, w') = E(X, aw'), && \text{(IH and (5))}\end{aligned}$$

completing the proof of the Lemma. ⊣

THEOREM 5.12 $L(M_1^*) = L(M_1)^*$.

PROOF. Let $w \in \Sigma^*$ be arbitrary. Then

$$\begin{aligned} w \in L(M_1^*) \quad &\text{iff} \quad \delta^*(\emptyset, w) \in \{\emptyset\} \cup \{X \in Q \mid X \sqcap F_1\} && \text{(def of machine and } L)\\ &\text{iff} \quad E(\emptyset, w) = \emptyset \vee E(\emptyset, w) \sqcap F_1 && \text{(Lemma)}\\ &\text{iff} \quad w = \varepsilon \vee \exists t \in \mathsf{Tr}(\emptyset \mid w)\ \mathsf{end}(t) \in F_1. && \text{(def of } E)\end{aligned}$$

Now, if $t \in \mathsf{Tr}(\emptyset \mid w)$, then $t = (\emptyset \mid w_1, \ldots, w_n)$, where $n \geq 1$, $w_i \neq \varepsilon$ and $\delta_1^*(s_1, w_i) \in F_1$ for all $i < n$, and $w = \mathsf{concat}\ [w_1, \ldots, w_n]$. It follows that $w_i \in L(M_1)$ for all $i < n$, and $\mathsf{end}(t) \in F_1$ means that $w_n \in L(M_1)$ as well, so $w \in L(M_1)^*$ by Theorem 2.11. Conversely, using the same Theorem, if $w \in L(M_1)^*$ then $w = \mathsf{concat}\ l$ for some $l \in [L(M_1)]$. So either $l = [\,]$ and $w = \varepsilon$, or we can assume that $l = [w_1, \ldots, w_n]$, for some $n \geq 1$, where $w_i \in L(M_1)$ and $w_i \neq \varepsilon$ for all i (since any empty string in l can be eliminated without changing the value of the concatenation). Thus, for each i, $\delta_1^*(s_i, w_i) \in F_1$, and so $t = (\emptyset | w_1, \ldots, w_n) \in \mathsf{Tr}(\emptyset | w)$ and $\mathsf{end}(t) \in F_1$, showing, as above, that $w \in L(M_1^*)$ and completing the proof. ⊣

Exercise. Let $\Sigma = \{\mathsf{a}, \mathsf{b}\}$. Draw the machine $M(\mathsf{a})^*$ (Hint: it will have 6 states, 3 of which are final.) What's the smallest machine you can draw that accepts the same language, i.e., $\{\mathsf{a}\}^*$. For the last time, what's going on with these extra states? (For the answer, read on!)

5.6 An aside: Reachability

If you tried the exercises in the earlier part of this section, you realized that the constructions for union, concatenation, and star tend to produce machines that are much (and often staggeringly) bigger than they have to be. For example, if the machine M_1 has n_1 states and the machine M_2 has n_2 states, then machine $M_1 \cdot M_2$ can have up to $n_1 \cdot 2^{n_2}$ states, meaning that a machine as "simple" as $M(\mathsf{a}) \cdot (M(\mathsf{b}) \cdot M(\mathsf{c}))$, which accepts the singleton language $\{\mathsf{abc}\}$, could have up to $3 \cdot 2^{3 \cdot 2^3}$ or 50,331,648 states, even though it is easy to construct a machine that accepts this language with just 5 states—talk about overkill!

The reason that our constructions are this "wasteful" is that they are completely *general*: since they don't know anything about the constituent machines, they have to include every possibility of interaction between them. In any particular case, however, the constructions will include many states that never get used, in the sense that the machine, staring in its start state and reading an arbitrary string, can never reach those states. For example, the colossal machine mentioned above has only 6 states that can be reached from its start state, and thus 50,331,642 states that might as well not even have been included in the construction!

Let's now formalize the idea of the "reachable" portion of a machine and show that we can simultaneously throw away all of the states that are not reachable and still have a machine that accepts the same language.

Let $M = (Q, s, F, \delta)$ be a FSM. Then we can define a subset $Q_r \subseteq Q$, called the *reachable* states of M, by the following rules:

$$\frac{}{s \in Q_r} \qquad \frac{q \in Q_r \qquad a \in \Sigma}{\delta(q,a) \in Q_r}\,.$$

Why is Q_r a subset of Q? It follows by a straightforward induction on these rules, since $s \in Q$ and $\delta(q,a) \in Q$ for every $q \in Q$ and $a \in \Sigma$. Now let M_r be the machine $(Q_r, s_r, F_r, \delta_r)$, where

- $s_r = s$
- $F_r = F \cap Q_r$
- $\delta_r(q,a) = \delta(q,a)$, for $q \in Q_r$ and $a \in \Sigma$.

Clearly, $s_r \in Q_r$ and $F_r \subseteq Q_r$, and the proposition that $q \in Q_r$ and $a \in \Sigma$ imply $\delta_r(q,a) \in Q_r$ is just the second rule above, showing that $\delta_r : Q_r \times \Sigma \to Q_r$ and thus that M_r is a FSM.

LEMMA 5.13 *For all $q \in Q_r$ and $w \in \Sigma^*$,*

$$\delta_r^*(q,w) = \delta^*(q,w) \in Q_r$$

PROOF. A straightforward induction on w, which we omit. $\dashv$

THEOREM 5.14 $L(M_r) = L(M)$.

PROOF. Let $w \in L(M_r)$ be arbitrary. Then, by the definitions of the machine and acceptance, $\delta_r^*(s,w) \in F \cap Q_r$, so by the Lemma, $\delta^*(s,w) = \delta_r^*(s,w) \in F$ and thus $w \in L(M)$.

Conversely, let $w \in L(M)$ be arbitrary. The $\delta^*(s,w) \in F$, and so by the Lemma and the definitions, $\delta_r^*(s,w) = \delta^*(s,w) \in F \cap Q_r$, and so $w \in L(M_r)$, as required. $\dashv$

Exercise. Go back over the exercises in this section and determine which of the states in the machines you constructed were reachable. How does that number compare with the minimal machines you were able to find? Is there still some room for improvement? (This question will be settled completely in Section 10.2 below.)

6 FSMs Accept Regular Languages

We established in the previous section (with a brief detour into reachability) that every regular language is accepted by some FSM, so now let's move on in this section to the problem of proving the converse:

THEOREM 6.1 *For every finite state machine, M, the language $L(M)$ is regular.*

To do this, I'll first introduce the notion of a *system of proper linear equations* (SPLE) in a finite list of variables and define what it means for a list of languages to satisfy such a system. I'll then show that every SPLE has a unique solution and that every component of that solution is regular. Finally I'll show how we can construct, given an FSM M, a SPLE using the states of M as the variables so that the components of the solution are exactly the $L_q(M)$.

6.1 Systems of Proper Linear Equations

A *linear equation* between languages is an equation of the form

$$X = L_1 \cdot X \cup L_2, \tag{1}$$

where L_1 and L_2 are languages and, by convention, the right-hand side is grouped $(L_1 \cdot X) \cup L_2$, as $\cdot$ has higher precedence than $\cup$. A language L is a *solution* to (1) if, naturally, $L = L_1 \cdot L \cup L_2$.

If $\varepsilon \in L_1$, then the solutions to (1) are numerous and not very interesting: for example, $L = \Sigma^*$ is always a solution, and it is just the largest of a generally infinite (even *uncountable*, if that means anything to you) number of solutions. For example, at the extreme, when $L_1 = \{\varepsilon\}$ and $L_2 = \emptyset$ (i.e., when $L_1 = \mathbf{1}$ and $L_2 = \mathbf{0}$), *every* language L is a solution to (1). On the other hand, if we call a language L_1 *proper* when $|w| \geq 1$ for all $w \in L_1$ (which is just another way of saying $\varepsilon \notin L_1$, since ε is the only string of length 0), then we have the following remarkable result:

LEMMA 6.2 *(Arden's Lemma) For any two languages L_1 and L_2, the language $L = L_1^* \cdot L_2$ is the smallest solution to (1); i.e., L is a solution, and if L' is also a solution then $L \subseteq L'$. Moreover, if L_1 is proper, then L is the unique solution.*

PROOF. Let L_1 and L_2 be arbitrary languages, and let $L = L_1^* \cdot L_2$. Then we can use the theorems of Section 2.2 to show that L is a solution to (1):

$$\begin{aligned}
L_1 \cdot (L_1^* \cdot L_2) \cup L_2 &= (L_1 \cdot L_1^*) \cdot L_2 \cup L_2 && \text{(by 2.6.3)}\\
&= (L_1 \cdot L_1^*) \cdot L_2 \cup \{\varepsilon\} \cdot L_2 && \text{(by 2.6.2)}\\
&= (L_1 \cdot L_1^* \cup \{\varepsilon\}) \cdot L_2 && \text{(by 2.6.5)}\\
&= L_1^* \cdot L_2. && \text{(by 2.8)}
\end{aligned}$$

To show that it's the smallest solution, suppose $L' = L_1 \cdot L' \cup L_2$, and let's show by induction on $w_1 \in L_1^*$ that, for every $w_2 \in L_2$, we have $w_1 \cdot w_2 \in L'$. If $w_1 = \varepsilon$ and $w_2 \in L_2$, then

$$w_1 \cdot w_2 = w_2 \in L_2 \subseteq L_1 \cdot L' \cup L_2 = L'.$$

And if $w_1 = w \cdot w_1'$, with $w \in L_1$ and $w_1' \in L_1^*$ and $w_2 \in L_2$, then we have, using associativity of $\cdot$ and the IH,

$$w_1 \cdot w_2 = (w \cdot w_1') \cdot w_2 = w \cdot (w_1' \cdot w_2) \in L_1 \cdot L' \subseteq L_1 \cdot L' \cup L_2 = L'.$$

Finally, let's show that if L_1 is proper, then L is also the largest—and thus *only*—solution to (1), in that if L' is also a solution, then $L' \subseteq L$. So, assume L_1 is proper and $L' = L_1 \cdot L' \cup L_2$. By Theorem 2.6, we have

$$L_1 \cdot L = L_1 \cdot (L_1^* \cdot L_2) = (L_1 \cdot L_1^*) \cdot L_2 \subseteq L_1^* \cdot L_2 = L,$$

and so $L_1 \cdot L \subseteq L$. I will now prove by strong induction on $|w|$ that

$$w \in L' \text{ implies } w \in L. \tag{2}$$

So let n be a natural number, assume (2) holds for all strings w' with $|w'| < n$, let w be such that $|w| = n$, and assume $w \in L'$. Since $L' = L_1 \cdot L' \cup L_2$, either $w \in L_1 \cdot L'$ or $w \in L_2$. In the first case, there exist $w_1 \in L_1$ and $w' \in L'$ such that $w = w_1 \cdot w'$. Since L_1 is proper, we have $|w_1| \geq 1$, and so $|w'| < n$. Therefore, by the IH, $w' \in L$, and so $w = w_1 \cdot w' \in L_1 \cdot L \subseteq L$, showing $w \in L$. In the second case, we use $w = \varepsilon \cdot w$ and $\varepsilon \in L_1^*$ to conclude $w \in L_1^* \cdot L_2 = L$. $\dashv$

We will say that the equation (1) is *proper* if L_1 is proper. Thus, Arden's Lemma gives us a unique solution to any proper linear equation. Notice, moreover, the crucial point that if L_1 and L_2 are regular, then the unique solution to (1) is also regular, since it is constructed from L_1 and L_2 using Kleene star and concatenation.

Now that we have a way to solve proper linear equations in one variable, we will extend the method to systems of proper linear equations of the form

$$\begin{array}{ccccccccccc}
X_1 & = & L_{1,1} \cdot X_1 & \cup & L_{1,2} \cdot X_2 & \cup & \cdots & \cup & L_{1,n} \cdot X_n & \cup & L_1' \\
X_2 & = & L_{2,1} \cdot X_1 & \cup & L_{2,2} \cdot X_2 & \cup & \cdots & \cup & L_{2,n} \cdot X_n & \cup & L_2' \\
 & & \multicolumn{9}{c}{\dotfill} \\
X_n & = & L_{n,1} \cdot X_1 & \cup & L_{n,2} \cdot X_2 & \cup & \cdots & \cup & L_{n,n} \cdot X_n & \cup & L_n',
\end{array} \tag{3}$$

with n equations in n variables $X_1, X_2, \ldots, X_n$, where every $L_{i,j}$ is proper. Here is the formal definition:

Definition 6.3 A *system of n proper linear equations* (or n-SPLE), where $n \geq 0$, is a pair $(\{L_{i,j}\}_{ij}, \{L_i'\}_i)$, where

- $\{L_{i,j}\}_{ij}$ is an n-by-n matrix of proper languages, called the *coefficients* of the system, and

- $\{L'_i\}_i$ is an n-vector of languages, called the *constants* of the system.

This system of equations is called *regular* if every coefficient $L_{i,j}$ and every constant L'_i is a regular language. An n-vector of languages, $\{L_i\}_i$, is a *solution* to this system if, for every i,

$$L_i = \big(\bigcup_{j=1}^{n} L_{i,j} \cdot L_j\big) \cup L'_i. \tag{4}$$

Now, Arden's Lemma gives us a recursive method to solve any n-SPLE: if $n = 0$, then the system is empty, and has an empty solution; if $n > 1$, then we

- rewrite the first equation in (3) to the form

  $$X_1 = L_{1,1} \cdot X_1 \cup (L_{1,2} \cdot X_2 \cup \cdots \cup L_{1,n} \cdot X_n \cup L'_1),$$

  and use Arden's Lemma to get a solution for X_1 in terms of the other variables:
  $$X_1 = L^*_{1,1} \cdot (L_{1,2} \cdot X_2 \cup \cdots \cup L_{1,n} \cdot X_n \cup L'_1); \tag{5}$$

- substitute this solution in for the other instances of X_1 in (3), and then use the properties of $\cdot$ and $\cup$ to rearrange the result to get an $(n-1)$-SPLE, $(\{\bar{L}_{i,j}\}_{ij}, \{\bar{L}'_i\}_i)$, where we find that

  $$\bar{L}_{i,j} = L_{i,1} \cdot L^*_{1,1} \cdot L_{1,j} \,\cup\, L_{i,j} \qquad (i, j \in \{2, \ldots, n\}), \tag{6}$$
  $$\bar{L}'_i = L_{i,1} \cdot L^*_{1,1} \cdot L'_1 \,\cup\, L'_i \qquad (i \in \{2, \ldots, n\}); \tag{7}$$

- recursively solve this $(n-1)$-SPLE to get solutions $L_2, \ldots, L_n$ for the variables $X_2, \ldots, X_n$; and finally

- extend this solution to include L_1, which we get as the value of X_1 in (5) after substituting $L_2, \ldots, L_n$ in for the variables $X_2, \ldots, X_n$.

The following theorem establishes that this method produces unique regular solutions to regular systems of equations:

Theorem 6.4 *Every regular n-SPLE $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$, for $n \geq 0$, has a unique solution, all of the components of which are regular.*

Proof. By induction on n. The case $n = 0$ is trivial, so suppose $n > 0$ and $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ is regular. Let $\{\bar{L}_{i,j}\}_{ij}$ and $\{\bar{L}'_i\}_i$ be given by (6) and (7). For every $i, j \geq 2$, the language $\bar{L}_{i,j}$ is proper, since $L_{i,1}$ and $L_{i,j}$ are proper, and every $\bar{L}_{i,j}$ and $\bar{L}'_i$ is regular, since these languages are constructed from regular languages using the regular operations of union, concatenation, and Kleene star. By the IH, this $(n-1)$-SPLE has a unique solution $\{L_i\}_i$, where for every $i \geq 2$, L_i is regular and satisfies

$$L_i = \big(\bigcup_{j=2}^{n} \bar{L}_{i,j} \cdot L_j\big) \cup \bar{L}'_i \qquad (i \in \{2, \ldots, n\}), \tag{8}$$

Define

$$L_1 = L_{1,1}^* \cdot \Big((\bigcup_{j=2}^{n} L_{1,j} \cdot L_j) \cup L'_1 \Big),$$

which we again note is regular for the same reason noted above. We need to show that $\{L_i\}_i$ $(1 \le i \le n)$ is the unique solution to the original system, i.e., satisfies (4) for every $i \ge 1$. The case $i = 1$ (including the uniqueness of L_1) follows directly from Arden's Lemma and the definition of L_1. The cases where $i > 1$ follow from the definition of L_1, the properties of $\cdot$ and $\cup$, and (6), (7), and (8), as follows:

$$\begin{aligned}
&(\bigcup_{j=1}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = L_{i,1} \cdot L_1 \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = L_{i,1} \cdot \Big(L_{1,1}^* \cdot \Big((\bigcup_{j=2}^{n} L_{1,j} \cdot L_j) \cup L'_1 \Big) \Big) \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = (\bigcup_{j=2}^{n} L_{i,1} \cdot L_{1,1}^* \cdot L_{1,j} \cdot L_j) \cup L_{i,1} \cdot L_{1,1}^* \cdot L'_1 \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup L'_i \\
&\qquad = (\bigcup_{j=2}^{n} L_{i,1} \cdot L_{1,1}^* \cdot L_{1,j} \cdot L_j) \cup (\bigcup_{j=2}^{n} L_{i,j} \cdot L_j) \cup \bar{L}'_i \\
&\qquad = (\bigcup_{j=2}^{n} L_{i,1} \cdot L_{1,1}^* \cdot L_{1,j} \cdot L_j \cup L_{i,j} \cdot L_j) \cup \bar{L}'_i \\
&\qquad = (\bigcup_{j=2}^{n} (L_{i,1} \cdot L_{1,1}^* \cdot L_{1,j} \cup L_{i,j}) \cdot L_j) \cup \bar{L}'_i \\
&\qquad = (\bigcup_{j=2}^{n} \bar{L}_{i,j} \cdot L_j) \cup \bar{L}'_i, \\
&\qquad = L_i
\end{aligned}$$

⊣

In what follows, all of our SPLEs will be regular, so it will often be convenient to use regular expressions, rather than the regular languages they name, to specify these systems and their solutions.

Exercise. Use the algorithm above to find the unique solution to this regular 3-SPLE in the variables X_0, X_1, X_2, which we write using regular expressions:

$$\begin{aligned}
X_0 &= b \cdot X_0 + a \cdot X_1 \\
X_1 &= b \cdot X_1 + a \cdot X_2 + 1 \\
X_2 &= a \cdot X_0 + b \cdot X_2
\end{aligned}$$

Note that several of the coefficients and constants in this SPLE are 0.

6.2 From FSMs to SPLEs

Suppose we are given an FSM $M = (Q, s, F, \delta)$, where $Q = \{q_1, q_2, \ldots, q_n\}$, i.e., where we have numbered the states from 1 to n. We will now construct a regular n-SPLE $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ whose unique solution is $\{L_{q_i}(M)\}_i$, i.e., where the ith component of the solution is the language accepted by M if the machine is started in state q_i. Since every component of this unique solution is regular by Theorem 6.4, it will follow that $L(M)$, which is just $L_{q_i}(M)$ for the i such that $s = q_i$, is regular, achieving our goal for this section.

THEOREM 6.5 *Let $M = (Q, s, F, \delta)$ be an FSM, where $Q = \{q_1, q_2, \ldots, q_n\}$, and define, for all $i, j \in \{1, \ldots, n\}$,*

$$\begin{aligned} L_{i,j} &= \{a\varepsilon \mid a \in \Sigma \text{ and } \delta(q_i, a) = q_j\} \\ L'_i &= \{\varepsilon \mid q_i \in F\}. \end{aligned}$$

Then $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ is a regular n-SPLE whose unique solution is $\{L_{q_i}(M)\}_i$.

PROOF. Note first that each $L_{i,j}$ is a finite set of strings of length 1, and thus regular and proper, and each L'_i is either $\mathbf{1}$ or $\mathbf{0}$, and thus regular, so that $(\{L_{i,j}\}_{ij}, \{L'_i\}_i)$ is indeed a regular n-SPLE. By Theorem 6.4, there is a unique vector of languages, $\{L_i\}_i$, satisfying (4) for every $i \in \{1, \ldots, n\}$. Since each $L_{i,j}$ is proper, it follows from (4) that, for every i,

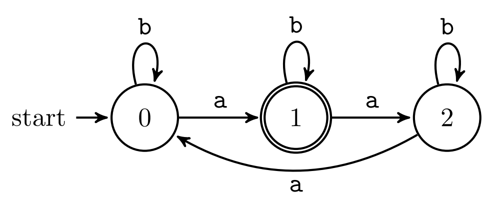
$$\varepsilon \in L_i \text{ iff } \varepsilon \in L'_i \text{ iff } q_i \in F.$$

Furthermore, if $aw \in \Sigma^*$, then it also follows from (4) that, for every i,

$$aw \in L_i \text{ iff } w \in L_{\delta(q_i, a)}.$$

Thus, the vector of languages $\{L_i\}_i$ satisfies the same recursion equations as the vector of languages $\{L_{q_i}(M)\}_i$, proving that they are equal. ⊣

As an example, consider this FSM over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, which accepts exactly the strings with n $\mathtt{a}$'s, where $n \equiv 1 \pmod 3$:



You can check that the 3-SPLE corresponding to this machine is exactly the 3-SPLE from the Exercise in the previous subsection, and that a solution $\{r_0, r_1, r_2\}$ to that system therefore provides a regular expression r_0 matching exactly the strings singled out above.

6.3 An aside: Simpflication of REs

Just as in Section 5, where the construction of FSMs from regular expressions produced very large machines that can be simplified by considering only reachable states, so too here the construction of regular expressions from FSMs can produce large expressions that can be simplified.

7 More closure properties of regular languages

In this section, we show, as a consequence of the equivalence proved in Sections 5 and 6, that regular languages, besides being closed under union, concatenation, and Kleene star, are also closed under intersection, complement, reversal, homomorphism, reverse homorphism, and left and right quotients by an arbitrary language.

7.1 Intersection and Complement

Given

$$M_1 = (Q_1, s_1, F_1, \delta_1) \qquad \text{and} \qquad M_2 = (Q_2, s_2, F_2, \delta_2),$$

we define a machine $M_1 \cap M_2$ as follows:

- $Q = Q_1 \times Q_2$
- $s = (s_1, s_2)$
- $F = \{(q_1, q_2) \mid q_1 \in F_1 \wedge q_2 \in F_2\} = F_1 \times F_2$
- $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$.

It follows that $L(M_1 \cap M_2) = L(M_1) \cap L(M_2)$. We can also define a machine $\overline{M_1}$ as follows:

- $Q = Q_1$
- $s = s_1$
- $F = Q_1 - F_1$
- $\delta = \delta_1$

It follows that $L(\overline{M_1}) = \Sigma^* - L(M_1)$.

7.2 Reversal

We define the reversal of a string w as follows:

$$\begin{aligned} \mathsf{rev}(\varepsilon) &= \varepsilon \\ \mathsf{rev}(aw) &= \mathsf{rev}(w) \cdot (a\varepsilon) \end{aligned}$$

and the reversal of a language L by

$$\mathsf{rev}(L) = \{\mathsf{rev}(w) \mid w \in L\}.$$

It follows by induction on w_1 that

- $\mathsf{rev}(w_1 \cdot w_2) = \mathsf{rev}(w_2) \cdot \mathsf{rev}(w_1)$ for all w_2, and
- $\mathsf{rev}(\mathsf{rev}(w_1)) = w_1$.

It also follows for all languages L_1 and L_2 that

- $\mathsf{rev}(L_1 \cdot L_2) = \mathsf{rev}(L_2) \cdot \mathsf{rev}(L_1)$, and
- $\mathsf{rev}(\mathsf{rev}(L_1)) = L_1$.

Given a regular expression r, we define a regular expression $\mathsf{rev}(r)$ by induction on r:

$$\begin{aligned}
\mathsf{rev}(0) &= 0 \\
\mathsf{rev}(\mathsf{a}) &= \mathsf{a} \\
\mathsf{rev}(r_1 + r_2) &= \mathsf{rev}(r_1) + \mathsf{rev}(r_2) \\
\mathsf{rev}(r_1 r_2) &= \mathsf{rev}(r_2)\mathsf{rev}(r_1) \\
\mathsf{rev}(r^*) &= \mathsf{rev}(r)^*
\end{aligned}$$

The result is:

$$[\![\mathsf{rev}(r)]\!] = \mathsf{rev}([\![r]\!]).$$

It follows that, if R is regular, then $\mathsf{rev}(R)$ is also regular.

7.3 Homomorphisms

A homomorphism is a function $h : \Sigma \to \Sigma^*$. Given such a homomorphism, we can define a function $h^* : \Sigma^* \to \Sigma^*$ by

$$\begin{aligned}
h^*(\varepsilon) &= \varepsilon \\
h^*(aw) &= h(a) \cdot h^*(w)
\end{aligned}$$

The first equation says that h^* preserves ε, and it easily follows by induction that h^* preserves concatenation, in the sense that

$$h^*(w \cdot u) = h^*(w) \cdot h^*(u).$$

Conversely, if $k : \Sigma^* \to \Sigma^*$ is a function that preserves both ε and concatenation, then $k = h^*$ for the homomorphism h defined by

$$h(a) = k(a\varepsilon).$$

Given a homomorphism h and language L, both the image of L under h^*,

$$h^*(L) = \{h^*(w) \mid w \in L\},$$

and the inverse image of L under h^*,

$$(h^*)^{-1}(L) = \{w \mid h^*(w) \in L\},$$

are regular.

First, given a string $w \in \Sigma^*$, we can define, by recursion on w, a regular expression $\overline{w}$ that matches only w:

$$\begin{aligned}
\overline{\varepsilon} &= 1 \\
\overline{a\varepsilon} &= a \\
\overline{aw} &= a\overline{w} \quad (w \neq \varepsilon).
\end{aligned}$$

Then, for an arbitrary regular expression r, we define a regular expression, $h(r)$, by recursion as follows:

$$\begin{aligned}
h(0) &= 0 \\
h(\mathtt{a}) &= \overline{h(\mathtt{a})} \\
h(r_1 + r_2) &= h(r_1) + h(r_2) \\
h(r_1 r_2) &= h(r_1)h(r_2) \\
h(r_1^*) &= h(r_1)^*.
\end{aligned}$$

The result is

$$[\![h(r)]\!] = h([\![r]\!]),$$

showing that the homomorphic image of a regular set is regular.

Now let $M_1 = (Q_1, s_1, F_1, \delta_1)$ be an FSM and define a machine $h^{-1}(M_1)$ as follows:

- $Q = Q_1$
- $s = s_1$
- $F = F_1$
- $\delta(q, a) = \delta_1^*(q, h(a))$.

The result is

$$L(h^{-1}(M_1)) = (h^*)^{-1}(L(M_1)),$$

showing that the inverse homomorphic image of a regular set is also regular.

7.4 Quotients

Given two languages L_1 and L_2, we define the right and left quotients of L_1 by L_2, respectively, as follows:

$$\begin{aligned}
L_1/L_2 &= \{w \mid \exists w_2(w_2 \in L_2 \wedge w \cdot w_2 \in L_1)\} \\
L_2\backslash L_1 &= \{w \mid \exists w_2(w_2 \in L_2 \wedge w_2 \cdot w \in L_1)\}.
\end{aligned}$$

We show that if L_1 is regular, then both of these languages are regular for an arbitrary language L_2.

First, let $M_1 = (Q_1, s_1, F_1, \delta_1)$ be an FSM accepting the language L_1. We define a machine M_1/L_2 as follows:

- $Q = Q_1$
- $s = s_1$
- $F = \{q \mid \exists w(w \in L_2 \wedge \delta_1^*(q, w) \in F_1)\}$
- $\delta = \delta_1$

It follows that $L(M_1/L_2) = L(M_1)/L_2$, and thus that L_1/L_2 is regular.

Next, we prove the following relationship between right and left quotients:

$$\mathsf{rev}(L_2 \backslash L_1) = \mathsf{rev}(L_1)/\mathsf{rev}(L_2).$$

It follows, by applying rev to both sides and using the fact that $\mathsf{rev}(\mathsf{rev}(L)) = L$, that

$$L_2 \backslash L_1 = \mathsf{rev}(\mathsf{rev}(L_1)/\mathsf{rev}(L_2)),$$

and therefore that $L_2 \backslash L_1$ is regular if L_1 is.

8 Variants of FSMs

In this section, we look at two variants of the notion of FSM that seem to add additional power to the model, thereby making it easier to construct machines accepting particular languages, and possibly also increasing the set of languages accepted. In each case, however, we see by means of a reduction that the FSM variant defines the same set of languages as the basic FSMs, i.e., the regular langauges.

8.1 Nondeterministic FSMs

In an ordinary—or, as we say, *deterministic*—FSM, we have a single start state, and every transition from a state of the machine on a letter of the alphabet leads to a unique state. In a *nondeterministic* FSM, or NFSM, we allow ourselves a *set* of start states and a *set* of transitions from a state of the machine on a letter of the alphabet.

DEFINITION 8.1 (Nondeterministic Finite State Machine) A *Nondeterministic Finite State Machine* is a tuple (Q, S, F, δ), where

- Q is a finite set (the *states*)
- $S \subseteq Q$ (the *start states*)
- $F \subseteq Q$ (the *final states*)

- $\delta : Q \times \Sigma \to \mathsf{Pow}(\Sigma)$ (the *transition function*)

Given a state $q \in Q$ and a letter $a \in \Sigma$, we have $\delta(q,a) \subseteq Q$. If $q' \in \delta(q,a)$ we say that there is a transition from q to q' on a; there may be many such transitions from q on a, or there may be none. We can draw an NFSM the same way that we draw an FSM, except that multiple states can be labeled "start" and we can have many (or no) transitions with the same label leaving a particular state.

To define acceptance of a string by an NFSM, we define $\delta^* : Q \times \Sigma \to \mathsf{Pow}(Q)$ by recursion as follows:

$$\begin{aligned}
\delta^*(q,\varepsilon) &= \{q\} \\
\delta^*(q,aw) &= \bigcup\{\delta^*(q',w) \mid q' \in \delta(q,a)\} \\
&= \{q'' \mid \exists q'(q' \in \delta(q,a) \wedge q'' \in \delta^*(q,w))\}.
\end{aligned}$$

We then define

$$L(M) = \{w \mid \exists s \in S\ \delta^*(s,w) \sqcap F\};$$

that is, a string w is accepted by M if there is a start state $s \in S$ such that $\delta^*(s,w)$ contains a final state.

How should we think of NFSMs as machines? We already have some experience with this in connection with our constructions of the concatenation and star machines from Section 5. We can think of the machine as forking copies of itself whenever there are multiple transition on an input letter, with one copy for each possible transition, and destroying such copies if there are no transitions; a string will be accepted by the system if it is accepted by one of the copies. Alternately, we can think of the machine as choosing, in some unobservable manner from the transitions that are available, the next state that it goes to when reading an input letter. In this way, we can only say what the *possible* states of the machine will be after reading a string, and the machine accepts a string if it is *possible* for it to be in a final state after reading the entire string.

NFSMs, despite their extra flexibility, are equivalent to FSMs as language acceptors. In one direction, clearly every FSM can be considered an NFSM for which every $\delta(q,a)$ is a singleton. In the other direction, we can simulate an NFSM, $M_1 = (Q_1, S_1, F_1, \delta_1)$, using an FSM, $\mathsf{Pow}(M_1)$, whose states are subsets of Q_1; we define the machine $\mathsf{Pow}(M_1)$ as follows:

- $Q = \mathsf{Pow}(Q_1)$
- $s = S_1$
- $F = F_1$
- $\delta(X,a) = \bigcup\{\delta_1(q,a) \mid q \in X\} = \{q' \mid \exists q \in X(q' \in \delta_1(q,a))\}$.

The result is that $L(M_1) = L(\mathsf{Pow}(M_1))$, completing the equivalence and showing that NFSMs accept only regular sets.

As an example of the usefulness of NFSMs, consider this alternate proof that regular sets are closed under reversal. Let $M_1 = (Q_1, s_1, F_1, \delta_1)$ be a FSM; we will construct an NFSM, $\mathsf{rev}(M_1)$, that accepts $\mathsf{rev}(L(M_1))$. This machine is defined as follows:

- $Q = Q_1$
- $S = F_1$
- $F = \{s_1\}$
- $\delta(q, a) = \{q' \mid \delta_1(q', a) = q\}$

That is, we just take the machine M_1 and run it nondeterministically in reverse! We start the machine in the final states of M_1, reverse the direction of all of the transitions in δ_1, and accept the string if it is possible to end up in M_1's start state, s_1.

8.2 Adding ε-moves

We can gain even more flexibility in defining machines if we can allow our machines the ability to switch states spontaneously, without reading any input symbols. We call such transitions ε-moves. Here is the definition.

Definition 8.2 (EFSM) A *Nondeterministic Finite State Machine with ε-moves* is a tuple $(Q, S, F, \delta, \varepsilon)$, where

- Q is a finite set (the *states*)
- $S \subseteq Q$ (the *start states*)
- $F \subseteq Q$ (the *final states*)
- $\delta : Q \times \Sigma \to \mathsf{Pow}(Q)$ (the *transition function*)
- $\varepsilon : Q \to \mathsf{Pow}(Q)$ (the *ε-transition function*)

If $q' \in \varepsilon(q)$, we say that the machine can make an ε-transition from q to q'.

- Since the machine is nondeterministic, it doesn't *have* to make an available ε-transition; it can stay where it is.
- The machine can make any number of ε-transitions in a row, all without consuming any characters from the input, if such transitions are possible.

Given a set of states $X \subseteq Q$, we define the ε-closure of X, written X^ε, by the following rules:

$$\frac{q \in X}{q \in X^\varepsilon} \qquad \frac{q \in X^\varepsilon \qquad q' \in \varepsilon(q)}{q' \in X^\varepsilon}\ .$$

A set of states $X \subseteq Q$ is called *ε-closed* if $X^\varepsilon = X$.

We extend the orignal function $\delta : Q \times \Sigma \to \mathsf{Pow}(Q)$ on states to a function $\hat{\delta} : \mathsf{Pow}(Q) \times \Sigma \to \mathsf{Pow}(Q)$ on sets of states by

$$\hat{\delta}(X, a) = \bigcup \{\delta(q, a) \mid q \in X\},$$

and then extend this further to a function $\hat{\delta}^* : \mathsf{Pow}(Q) \times \Sigma^* \to \mathsf{Pow}(Q)$ on strings by induction as follows:

$$\begin{aligned} \hat{\delta}^*(X, \varepsilon) &= X \\ \hat{\delta}^*(X, aw) &= \hat{\delta}^*(\hat{\delta}(X, a)^\varepsilon, w) \end{aligned}$$

It is easy to see by induction that if X is ε-closed, then so is $\hat{\delta}^*(X, w)$ for all strings w. Finally, we define the language accepted by an EFSM:

$$L(M) = \{w \mid \hat{\delta}^*(S^\varepsilon, w) \sqcap F\}.$$

As with NFSMs, we can show the equivalence of EFSMs and FSMs by simulating any EFSM, $M_1 = (Q_1, S_1, F_1, \delta_1, \varepsilon_1)$, using an FSM, $\mathsf{Pow}^\varepsilon(M_1)$ as follows:

- $Q = \{X \mid X \subseteq Q_1 \wedge X^\varepsilon = X\}$
- $S = S_1^\varepsilon$
- $F = \{X \in Q \mid X \sqcap F_1\}$
- $\delta(X, a) = \hat{\delta}(X, a)^\varepsilon$

The states of this machine are thus the ε-closed sets of states of M_1. The start state is ε-closed, as is the result of each transition, so this is indeed an FSM. The result is that $L(M_1) = L(\mathsf{Pow}^\varepsilon(M_1))$.

9 From RE to FSM, Redux

In this section, I will describe three additional constructions that produce machines from regular expressions; the first produces an EFSM, the second produces an NFSM, and the third produces an FSM.

9.1 A standard construction

Our first construction produces an EFSM from any regular expression r, and parallels the recursive construction we gave in Section 5. The machines $M(0)$ and $M(a)$ there can be viewed as EFSMs, so we will just have to show how to form the union, concatenation, and star of EFSMs.

Given disjoint machines $M_1 = (Q_1, S_1, F_1, \delta_1, \varepsilon_1)$ and $M_2 = (Q_2, S_2, F_2, \delta_2, \varepsilon_2)$, we define the machine $M_1 \cup M_2$ as follows:

- $Q = Q_1 \cup Q_2$

- $S = S_1 \cup S_2$
- $F = F_1 \cup F_2$
- $\delta = \delta_1 \cup \delta_2$
- $\varepsilon = \varepsilon_1 \cup \varepsilon_2$.

This is an EFSM and satisfies $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$.

Given the same M_1 and M_2, we define the machine $M_1 \cdot M_2$ as follows:

- $Q = Q_1 \cup Q_2$
- $S = S_1$
- $F = F_2$
- $\delta = \delta_1 \cup \delta_2$
- $\varepsilon = \varepsilon_1 \cup \varepsilon_2 \cup \{(q_1, q_2) \mid q_1 \in F_1 \wedge q_2 \in S_2\}$.

This is an EFSM and satisfies $L(M_1 \cdot M_2) = L(M_1) \cdot L(M_2)$.

Given M_1 as above, we let q_0 be a new state not already in Q_1 and define the machine M_1^* as follows:

- $Q = Q_1 \cup \{q_0\}$
- $S = \{q_0\}$
- $F = F_1 \cup \{q_0\}$
- $\delta = \delta_1 \cup \{(q_0, a, q)) \mid a \in \Sigma \wedge \exists s(s \in S_1 \wedge q \in \delta(s, a))\}$
- $\varepsilon = \varepsilon_1 \cup \{(q, s) \mid q \in F_1 \wedge s \in S_1\}$.

This is an EFSM and satisfies $L(M_1^*) = L(M_1)^*$.

9.2 A construction of Glushkov

Let's call an NFSM $M = (Q, S, F, \delta)$ *standard* if Q is a subset of the natural numbers, $F = \{0\}$, and all states in Q are reachable from S using δ. A standard machine can be specified using only the pair (S, δ), since F is determined and Q can be recovered from δ. Let's write $\#M = 1 + \max(S)$, which is the smallest natural number that is greater than all states of M.

Our second construction, due to Victor Glushkov—and, independently, to Robert McNaughton and Hisao Yamada—produces, for any standard NFSM $M = (S, \delta)$, regular expression r, and natural number $n \geq \#M$, a standard machine $r \cdot M@n$ such that $L(r \cdot M@n) = [\![r]\!] \cdot L(M)$ and any state q of $r \cdot M@n$ that is not a state of M satisfies $q \geq n$. Here are the constructions.

Empty. The machine $0 \cdot M@n$ is $(\emptyset, \emptyset)$.

Letter. The machine $\mathtt{a} \cdot M@n$ is $(\{n\}, \delta \cup \{(n, \mathtt{a}, S)\})$.

Union. Let $M_2 = (S_2, \delta_2)$ be the machine $r_2 \cdot M@n$, and let $(S_1, \delta_1) = r_1 \cdot M@\#M_2$. Then the machine $(r_1 + r_2) \cdot M@n$ is $(S_1 \cup S_2, \delta_1 \cup \delta_2)$.

Concatenation. Let $M_2 = (S_2, \delta_2)$ be the machine $r_2 \cdot M@n$. Then the machine $(r_1 \cdot r_2) \cdot M@n$ is $r_1 \cdot M_2@\#M_2$.

Star. Let S_1 be the smallest set such that $(S_1, \delta_1) = r_1 \cdot (S_1 \cup S, \delta)@n$. Then the machine $r_1^* \cdot M@n$ is (S_1, δ_1).

9.3 A construction of Brzozowski

10 Beyond Regularity

10.1 The Pumping Lemma

In this section, we prove a basic result that allows us to show that certain languages are *not* regular.

10.2 The Myhill-Nerode Theorem

In this section, we prove a characterization of regular languages that gives us an even more powerful—and theoretically completely general—method for proving that languages are not regular, which moreover has the very useful side-effect of giving us a method to construct the *minimal* FSM accepting a given regular language. The method defines, for any language L, an equivalence relation $\equiv_L$ on Σ^*, and then constructs a "machine" M whose states, Q, are the equivalence classes of $\equiv_L$. The result is that L is regular iff Q is *finite*, in which case M is an FSM, $L(M) = L$, and M the minimum number of states of any machine accepting L.

10.3 Context-Free Grammars

10.4 Push-Down Automata

10.5 Other topics