EE595: Algebraic Coding Theory	Sep 12th, 2012

Lectures 4 & 5: Shannon Capacity of a Union of Graphs

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1 Introduction

In the previous lectures, we have defined the Shannon Capacity of a graph, $\Theta(G)$. This lecture is concerned with the Shannon Capacity of the disjoint union $G + H$ of two graphs G and H. The union of two graphs is, informally, what you get when you just place one graph next to the other.

In 1956, Shannon conjectured [1] that $\Theta(G + H) = \Theta(G) + \Theta(H)$. As we will see, this is a very natural conjecture that assumes that when two completely disjoint alphabets are used for zero-error communication, there is no benefit in jointly coding over the two alphabets.

Surprisingly, it was disproved 42 years later by N. Alon [3]. We will present the simpler construction from [4].

The requirement is the construction of a counterexample: two graphs G and H such that the Shannon capacity $\Theta(G+H)$ of their disjoint union is strictly bigger than the sum of their capacities, *i.e.*, $\Theta(G + H) > \Theta(G) + \Theta(H)$. In addition to a specific construction, we develop a technique to upper bound the sizes of independent sets. In the previous lecture we saw that the Lovasz function is an efficiently computable bound on the sizes of independent sets of a graph and its powers. Here we will see how a *dimension argument* can be used to obtain a different bound on independent sets. One key property of the proof is the use of dimension arguments over two different fields, one of even and one of odd characteristic.

2 Review

Recall the definitions:

The strong product of two graphs, G and H, denoted by $G \times H$ is what you get when you draw G and then next to it draw H. If G and H are not disjoint, we first make them disjoint. Formally:

Definition 1 ***(Strong product of graphs)*** *$G + H$ is a graph with*

$$V(G \times H) = \{(v_i, v_j) : v_i \in V(G), v_j \in V(H)\},$$

and

$$E(G \times H) = \{((v_i, v_j), (v_k.v_l)) : [v_i = v_k \vee (v_i, v_k) \in E(G)] \wedge [v_j = v_l \vee (v_j, v_l) \in E(H)]\}.$$

Definition 2 ***(Shannon Capacity of a graph)*** *The capacity, $\Theta(G)$, of a graph G is*

$$\Theta(G) = \sup_k \{\alpha(G^k)^{\frac{1}{k}}, k = 1, 2, \ldots\},$$

where $\alpha(G)$ denotes the independence number (i.e., the cardinality of the maximum independent set) of G, and G^k denotes the k-th strong product of G with itself.

3 Shannon Capacity of a Union of Graphs

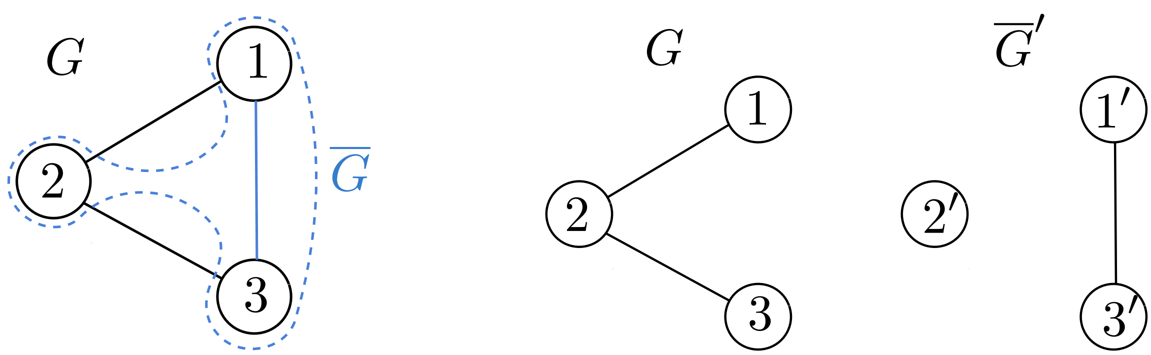
Definition 3 ***((Disjoint) Union of graphs)*** *Let G and H be two graphs. Their (disjoint) union, $G + H$, is the graph whose vertex set is the disjoint union of the vertex sets of G and H and whose edge set is the disjoint union of the edge sets of G and H. More formally,*

$$V(G+H) = V(G) \cup V(H') \quad \textit{and} \quad E(G+H) = E(G) \cup E(H'),$$

where

$$H' = \begin{cases} H, & \textit{if } H \textit{ and } G \textit{ are disjoint graphs,} \\ \textit{a copy of } H, & \textit{if } H \textit{ and } G \textit{ are } \underline{\textit{not}} \textit{ disjoint.} \end{cases} \tag{1}$$

Example 4 *Consider the graph G depicted in Figure 1a, with $V(G) = \{1,2,3\}$ and $E(G) = \{(1,2),(2,3)\}$. G's complement, $\overline{G}$, consists of the same set of vertices and the single edge missing from G, i.e., $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{(1,3)\}$. G and $\overline{G}$ are not disjoint. To construct the (disjoint) union, $G + \overline{G}$, we first make a copy $\overline{G}'$ of $\overline{G}$: $V(\overline{G}') = \{1',2',3'\}$ and $E(\overline{G}') = \{(1',3')\}$. Then, $G + \overline{G}$ (Figure 1b) is the concatenation of G and $\overline{G}'$: $V(G+\overline{G}) = V(G) \cup V(\overline{G}') = \{1,2,3,1',2',3'\}$ and $E(G+\overline{G}) = E(G) \cup E(\overline{G}') = \{(1,2),(2,3),(1',3')\}$.*



(a) G and $\overline{G}$.　　(b) The disjoint union, $G + \overline{G}$.

Figure 1: Example of the disjoint union of a graph G and its complement $\overline{G}$.

The maximum independent set of the union of two graphs is the union of the two maximum independent sets of the individual graphs. As a result, the independence number of $G + H$ equals the sum of the independence numbers of G and H: $\alpha(G + H) = \alpha(G) + \alpha(H)$. Extending this observation, it is rather intuitive that the capacity $\Theta(G + H)$ of the union should be at least as big as the sum of the individual capacities $\Theta(G)$ and $\Theta(H)$ for any two graphs G and H.

This says that if one is allowed to use two alphabets, they can always use them in isolation to obtain rate $\Theta(G) + \Theta(H)$. In his original paper [1], Shannon proves this intuitive fact:

$$\Theta(G + H) \geq \Theta(G) + \Theta(H). \tag{2}$$

Shannon conjectured the equally intuitive fact that cooperating across two disjoint alphabets cannot help. Formally, that (2) holds with strict equality for all pairs of graphs G and H. Surprisingly, however, in 1998 N. Alon constructed specific examples of graphs G and H and proved that the inequality is strict, *i.e.*, $\Theta(G + H) > \Theta(G) + \Theta(H)$.

Theorem 5 *There exist graphs G and H such that*

$$\Theta(G + H) > \Theta(G) + \Theta(H). \tag{3}$$

This lecture is dedicated to the tools and ideas deployed to construct examples of such graphs and show that they satisfy (3). We construct a graph G and use its complement $\overline{G}$ as the second graph H. In other words we construct a specific graph G such that

$$\Theta(G + \overline{G}) > \Theta(G) + \Theta(\overline{G}). \tag{4}$$

The proof consists of three parts. The first, is a lower bound L on $\Theta(G + \overline{G})$ that holds for all graphs G. The second is the development of an algebraic tool that allows upper bounding $\Theta(G)$. This machinery can be seen as a generalization of orthogonal representations used in the Lovasz θ function. The third part encapsulates the construction of the counter example G and upper bounding the sum $\Theta(G) + \Theta(\overline{G})$ utilizing the previous tool in a sophisticated manner. The outcome of the last step, is an upper bound U on $\Theta(G) + \Theta(\overline{G})$ such that $U < L$, which proves the existence of a gap between $\Theta(G + \overline{G})$ and $\Theta(G) + \Theta(\overline{G})$, and essentially Theorem 5. The approach can be summarized as follows:

$$\left.\begin{array}{rcll} \Theta(G + \overline{G}) & \geq & L, & \text{(for all } G\text{'s)} \\ \Theta(G) + \Theta(\overline{G}) & \leq & U, & \text{(for special } G) \end{array}\right\} \quad \overset{U<L}{\Longrightarrow} \quad \exists G : \Theta(G + \overline{G}) < \Theta(G) + \Theta(\overline{G}). \tag{5}$$

Lemma 6 *Consider a graph G with n vertices. Then, $\Theta(G + \overline{G}) \geq \sqrt{2n}$.*

Proof. G and $\overline{G}$ are not disjoint. To construct their disjoint union, $G + \overline{G}$, we create a copy $\overline{G}'$ of $\overline{G}$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(\overline{G}') = \{v'_1, v'_2, \ldots, v'_n\}$ be the sets of vertices of G and $\overline{G}'$, respectively. By the definition of disjoint union, $V\left(G + \overline{G}\right) = V\left(G\right) \cup V\left(\overline{G}'\right)$. Since $V(G)$ and $V(\overline{G}')$ are disjoint sets,

$$|V(G + \overline{G})| = |V(G)| + |V(\overline{G}')| = n + n = 2n. \tag{6}$$

Consider the strong product of $G+\overline{G}$ with itself, denoted by $(G+\overline{G})^2$. The product has $4n^2$ vertices, one for each (ordered) pair of vertices in $V(G + \overline{G})$. Let $\mathcal{I} \subseteq V\left((G + \overline{G})^2\right)$ be the following set:

$$\mathcal{I} = \left\{(v_1, v'_1), (v_2, v'_2), \ldots, (v_n, v'_n), (v'_1, v_1), (v'_2, v_2), \ldots, (v'_n, v_n)\right\}. \tag{7}$$

We claim that there are no edges among the vertices in $\mathcal{I}$. There are four kind of vertex-pairs in $\mathcal{I}$:

- $\Big((v_i', v_i), (v_j, v_j')\Big)$ and $\Big((v_i, v_i'), (v_j', v_j)\Big)$: In $G + \overline{G}$, there is no edge between v_i' and v_j, for all i, j. By Definition 1, it is then straightforward to see that there will be no edge between nodes (v_i', v_i) and (v_j, v_j'), or (v_i, v_i') and (v_j', v_j) in $(G + \overline{G})^2$.

- $\Big((v_i, v_i'), (v_j, v_j')\Big)$, for $i \neq j$:

  - If $(v_i, v_j) \notin E(G)$, then $(v_i, v_j) \notin E(G + \overline{G})$, which, by Definition 1, guarantees that $\Big((v_i, v_i'), (v_j, v_j')\Big) \notin E\left((G + \overline{G})^2\right)$. On the other hand,
  - if $(v_i, v_j) \in E(G)$, then $(v_i', v_j') \notin E(\overline{G}')$, implying that $(v_i', v_j') \notin E(G + \overline{G})$. Again through Definition 1, it follows that $\Big((v_i, v_i'), (v_j, v_j')\Big) \notin E\left((G + \overline{G})^2\right)$.

- $\Big((v_i', v_i), (v_j', v_j)\Big)$ can be treated as the previous case.

The absence of edges between any two nodes of $\mathcal{I}$, implies that the latter is an independent set. Its cardinality is $|\mathcal{I}| = 2n$. Therefore, the maximum independent set in $(G + \overline{G})^2$ will be at least as large, or equivalently

$$\alpha\left((G + \overline{G})^2\right) \geq 2n. \tag{8}$$

The desired lower bound on the capacity of $G + \overline{G}$ immediately follows the definition of $\Theta(\cdot)$:

$$\Theta\left(G + \overline{G}\right) = \sup_k \{\alpha\left((G + \overline{G})^k\right)^{\frac{1}{k}}, k = 1, 2, \ldots\} \geq \alpha\left((G + \overline{G})^2\right)^{\frac{1}{2}} \geq \sqrt{2n}, \tag{9}$$

which completes the proof. ■

In the previous lecture, we demonstrated the *Orthogonal Representation* of a graph, a tool exploited by Lovász to develop upper bounds on the Shannon capacity of a graph, $\Theta(G)$. We introduce an extension of that idea, the *Functional Representation of a graph*, that will be subsequently utilized towards the same ends.

Definition 7 ***(Functional Representation of a graph)*** *Let $\mathbb{F}$ be a field. A Functional Representation, $\mathcal{F}$, of a graph G is:*

1. *A ground set $\mathcal{X}$,*
2. *an element $c_v \in \mathcal{X}$, for each vertex $v \in V(G)$, and*
3. *a function $f_v : \mathcal{X} \to \mathbb{F}$ for each vertex $v \in V(G)$, such that*
   - *(a) $f_v(c_v) \neq 0$, $\forall v \in V(G)$, and*
   - *(b) $f_v(c_u) = 0$, if $(u, v) \notin E(G)$.*

Observation 8 *The Orthogonal Representation of a graph is a special case of a Functional Representation.*

Proof. An Orthogonal Representation of a graph G is a mapping $\rho : V(G) \to \mathbb{R}^n$, such that $\rho(v)^T\rho(u) = 0$, if $(u, v) \in E(G)$. Define the following Functional Representation, $\mathcal{F}$: Let $\mathbb{F} = \mathbb{R}$, and let the ground set $\mathcal{X}$ be the set of vectors assigned to the vertices $V(G)$ by the Orthogonal Representation. For each $v \in V(G)$, there exists an element $c_v = \rho(v) \in \mathcal{X}$ that corresponds to v. Finally, let $f_v(c_u) \doteq \rho(v)^T c_u = \rho(v)^T\rho(u)$. It remains to verify that functions $f_v(\cdot)$ satisfy both requirements $(3a)$ and $(3b)$. By the definition of an Orthogonal Representation it is guaranteed that:

(a) $f_v(c_v) = \rho(v)^T\rho(v) > 0$, $\forall v \in V(G)$, and

(b) $f_v(c_u) = \rho(v)^T\rho(u) = 0$, if $(u, v) \notin E(G)$,

which completes the proof. ■

Note that a Functional Representation comprises n functions $f_v : \mathcal{X} \to \mathbb{F}$, one for each vertex $v \in V(G)$. A function $f_v(c_u)$ maps each element $c_u \in \mathcal{X}$ to an element in $\mathbb{F}$. From another perspective, the entire set $\mathcal{X}$ is mapped under $f_v(\cdot)$ to a vector in $\mathbb{F}^{|\mathcal{X}|}$. In the sequel, we use $\mathbf{f}_v$, to refer to the vector in $\mathbb{F}^{|\mathcal{X}|}$ obtained by calculating $f_v(\cdot)$ over all elements in $\mathcal{X}$. There are n such vectors, one for each $v \in V(G)$, lying in a subspace of $\mathbb{F}^{|\mathcal{X}|}$.

Definition 9 ***(Dimension of a Functional Representation)*** *The dimension $dim\mathcal{F}$ of a Functional Representation $\mathcal{F}$, is the dimension of the subspace generated by the n $\mathbf{f}_v$'s corresponding to the vertices in $V(G)$.*

Theorem 10 *Let G be a graph and $\mathcal{F}$ be a Functional Representation of G. Then, the Shannon Capacity of G is upper bounded by the dimension of $\mathcal{F}$, i.e.,*

$$\Theta(G) \leq dim\mathcal{F}. \tag{10}$$

For this reason this bounding method is using what is called a *dimension argument*. This is a nice linear algebra technique: say we want to bound the size of a set and we can associate each set element to a vector in some vector space. If we can show that the obtained vectors are linearly independent, the cardinality of the set is at most the dimension of the vector space.

To prove Theorem 10 we rely on the following two lemmas:

Lemma 11 *If $\mathcal{F}$ is a Functional Representation of G, $\alpha(G) \leq dim\mathcal{F}$.*

Lemma 12 *If $\mathcal{F}_G$ and $\mathcal{F}_H$ are Functional Representations of graphs G and H, respectively, over the same field $\mathbb{F}$, the strong product $G \times H$ has a Functional Representation $\mathcal{F}_{G\times H}$ over $\mathbb{F}$ such that*

$$dim\mathcal{F}_{G\times H} \leq dim\mathcal{F}_G \cdot dim\mathcal{F}_H. \tag{11}$$

An immediate consequence of Lemma 12 is that the k-th strong product of G with itself, G^k, has a Functional Representation, $\mathcal{F}_{G^k}$, such that $\dim\mathcal{F}_{G^k} = (\dim\mathcal{F}_G)^k$. In conjunction with Lemma 11, it leads to Theorem 10:

$$\Theta(G) = \sup_k\{\alpha(G^k)^{\frac{1}{k}}\} \leq \sup_k\{(\dim\mathcal{F}_{G^k})^{\frac{1}{k}}\} = \sup_k\{(\dim\mathcal{F}_G)^{k\frac{1}{k}}\} = \dim\mathcal{F}. \tag{12}$$

The proof of Lemma 12 is omitted. The proof of Lemma 11 follows.

Proof. **(Lemma 11)** Let $\mathcal{I} \subseteq V(G)$ be a maximum independent set in G (*i.e.*, $|\mathcal{I}| = \alpha(G)$). Here is the dimension argument technique in action: We want to bound the size of this set. We will map each element (each $v \in \mathcal{I}$) to a vector $\mathbf{f}_v$. It suffices to show that these vectors are linearly independent.

Let $t_v \in \mathbb{F}$, $\forall v \in \mathcal{I}$. Recall your kindergarden definition: Vectors $\mathbf{f}_v$, for $v \in \mathcal{I}$ are linearly independent if an only if the following condition holds:

$$\sum_{v \in \mathcal{I}} t_v \cdot \mathbf{f}_v = \mathbf{0} \quad \Rightarrow \quad t_v = 0, \quad \forall v \in \mathcal{I}. \tag{13}$$

The sum-equality is a compact expression for $|V(G)|$ distinct equalities:

$$\sum_{v \in \mathcal{I}} t_v \cdot \mathbf{f}_v = \mathbf{0} \quad \Leftrightarrow \quad \sum_{v \in \mathcal{I}} t_v \cdot f_v(x) = 0, \forall x \in \mathcal{X}. \tag{14}$$

Let u be a vertex in $\mathcal{I}$ and c_u the corresponing element in $\mathcal{X}$. For each $v \in \mathcal{I}$, we have:

$$f_v(c_u) = \begin{cases} \neq 0, & \text{if } v = u, \\ 0, & \text{if } v \neq u \wedge (u,v) \notin E(G), \end{cases} \tag{15}$$

Observe that it cannot be the case that $v \neq u \wedge (u,v) \in E(G)$, because both v and u belong to $\mathcal{I}$, which is (the maximum) independent set. Therefore,

$$\sum_{v \in \mathcal{I}} t_v \cdot f_v(c_u) = t_u f_u(c_u) + \sum_{v \in \mathcal{I} \setminus \{u\}} t_v \cdot \underbrace{f_v(c_u)}_{=0} \tag{16}$$

$$= t_u f_u(c_u), \tag{17}$$

where, $f_u(c_u) \neq 0$. Therefore, $\sum_{u \in \mathcal{I}} t_v \cdot f_v(c_u) \Rightarrow t_v = 0$. This is true for any $u \in \mathcal{I}$, implying, through (14), that condition (13) holds. We have found at least $\alpha(G)$ linearly independent vectors in $\mathcal{F}$. Hence, $\alpha(G) \leq \dim \mathcal{F}$. ∎

3.1 The counter example

In this section, we demonstrate a specific graph G and provide a Functional Representation for both G and its complement $\overline{G}$. Based on the two Functional Representations we will eventually upper bound $\Theta(G) + \Theta(\overline{G})$ by a quantity U such that $U < L$, concluding the proof of Theorem 5.

We construct G as follows:

Let s be a positive integer. Let $V(G)$ be the set of all 3-subsets of $\{1, \ldots, s\}$. In other words, each vertex in G, corresponds to an unordered triplet:

$$V(G) = \{v_A : A \subseteq \{1, \ldots, s\} \wedge |A| = 3\}. \tag{18}$$

Finally, two vertices $v_A, v_B \in V(G)$ are connected with an edge if and only if the corresponding sets A and B share exactly one common element, *i.e.*,

$$(v_A, v_B) \in E(G) \Leftrightarrow |A \cap B| = 1. \tag{19}$$

Example 13 *Figure 2 depicts the above graph G for $s = 5$. It contains $\binom{5}{3} = 10$ vertices, each corresponding to a 3-subset of $\{1, \ldots, 5\}$. v_A and v_B are connected if and only if $|A \cap B| = 1$. For instance, vertex $v_A = v_{(1,2,3)}$ is connected to vertices v_B, $B \in \{(\mathbf{1}, 4, 5), (\mathbf{2}, 4, 5), (\mathbf{3}, 4, 5)\}$. On the contrary, $v_{(1,2,3)}$ and $v_{(1,2,5)}$ are not connected, since the corresponding subsets have two elements in common.*

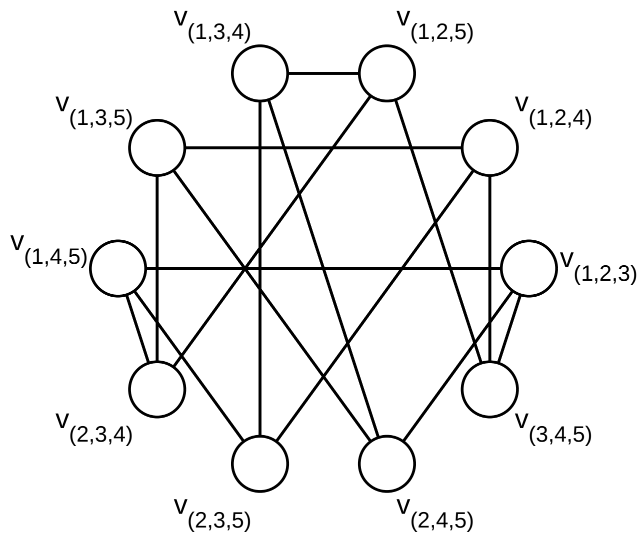


Figure 2: Realization of the counter-example construction for $s = 5$. Each vertex corresponds to a 3-subsets of $\{1, \ldots, 5\}$. Edges connect nodes whose corresponding subsets have exactly one common element.

We construct a Functional Representation $\mathcal{F}$ for G:

1. Let the ground set be the set of all binary vectors of length s, *i.e.*, $\mathcal{X} = \mathbb{F}_2^s$.

2. Assign to each vertex v_A the *characteristic vector* of the set A. Recall that the *characteristic vector* of $A \subseteq \{1, \ldots, s\}$ is a binary vector $\mathbf{c}_A$ of length s, such that

$$\mathbf{c}_A(i) = \begin{cases} 1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A, \end{cases} \quad 1 \leq i \leq s. \tag{20}$$

   Note that the characteristic element assigned to each vertex v_A, denoted in the sequel by $\mathbf{c}_{v_A}$, is indeed an element of the ground set $\mathcal{X}$.

3. Define a function $f_{v_A} : \mathcal{X} \to \mathbb{F}_2$, for each vertex $v_A \in V(G)$, as follows:

$$f_{v_A}(\mathbf{x}) = \left(\sum_{i \in A} x_i \right) \mod 2, \tag{21}$$

   where $\mathbf{x} \in \mathcal{X}$.

To verify that the functions f_{v_A} satisfy the necessary conditions, first observe that if c_{v_B} is the characteristic element of $v_B \in V(G)$, then

$$f_{v_A}(c_{v_B}) = \left(\sum_{i\in A} c_{v_B}\right) \mod 2 = \left(\sum_{i\in A} \mathbf{c}_B(i)\right) \mod 2 = |A \cap B| \mod 2. \tag{22}$$

Consequently,

$$f_{v_A}(c_{v_B}) = \begin{cases} 1, & \text{if } |A\cap B| = 3 \quad\quad \Leftrightarrow A = B, i.e, c_{v_B} = c_{v_A}, \\ 0, & \text{if } |A\cap B| = 2 \text{ or } 0 \quad \Leftrightarrow A \neq B, (v_A, v_B) \notin E(G), \\ 1, & \text{if } |A\cap B| = 1 \quad\quad \Leftrightarrow A \neq B, (v_A, v_B) \in E(G). \end{cases} \tag{23}$$

The first two cases verify that f_{v_A} comply with the requirements of the definition of a Functional Representation (the third case is indifferent):

(a) $f_{v_A}(c_{v_A}) = 1 \neq 0$, $\forall v_A \in V(G)$, and

(b) $f_{v_A}(c_{v_B}) = 0$ if $(v_A, v_B) \notin E(G)$, $\forall v_A, v_B \in V(G)$.

Example 14 *We demonstrate the above Functional Representation $\mathcal{F}$ for the graph G of the Example 13 ($s = 5$). Figure 3 depicts the graph, G. The characteristic vector $\mathbf{c}_A$ of A is marked next to vertex v_A. For instance, vertex $v_{(1,2,3)}$ is assigned vector [11100], the characteristic of $(1, 2, 3)$. The ground set $\mathcal{X}$ contains all 2^5 binary vectors of length $s = 5$. The vectors assigned to the n nodes are only a subset of $\mathcal{X}$. The function corresponding to $v_{(1,2,3)}$ is $f_{v_{(1,2,3)}}(\mathbf{x}) = (x_1+x_2+x_3) \mod 2$. Figure 4 depicts how that function is calculated over the entire ground set $\mathcal{X}$. For instance, when calculated on [11100], the characteristic element of $v_{(1,2,3)}$, it outputs $f_{v_{(1,2,3)}}([11100]) = (1 + 1 + 1) \mod 2 = 1 \neq 0$. Similarly, when calculated on vector [11010], the characteristic element of $v_{(1,2,4)}$ with which $v_{(1,2,3)}$ is not connected, it outputs $f_{v_{(1,2,3)}}([11010]) = (1+1+0) \mod 2 = 0$. The output of $f_{v_{(1,2,3)}}(\cdot)$ calculated over the entire set $\mathcal{X}$ forms a binary vector $\mathbf{f}_{v_{(1,2,3)}}$ of length $|\mathcal{X}| = 2^s$. There are n such vectors, one corresponding to each vertex, that lie in a subspace of $\mathbb{F}^{|\mathcal{X}|} = \mathbb{F}_2^{2^5}$. The dimension of this subspace is $dim\mathcal{F}$.*

We now need to bound the dimension of our Functional Representation $\mathcal{F}$, which will serve as an upper bound on $\Theta(G)$. Recall that we denote by $\mathbf{f}_{v_A}$ the vector formed by calculating $f_{v_A}(\cdot)$ over the entire ground set.

The $n = \binom{s}{3}$ $\mathbf{f}_{v_A}$'s lie in a subspace of $\mathbb{F}^{|\mathcal{X}|} = \mathbb{F}_2^{2^s}$. Now this is an awfully large dimension and we want to find a way to show that they lie in a much lower dimensional subspace. These are $\binom{s}{3}$ vectors so if they were linearly independent their dimension would be $\binom{s}{3}$. This is still too weak to yield our result so we need to argue that they lie in lower dimensions (in fact, s).

To achieve this we define s simple auxiliary functions $I_i(\mathbf{x})$, $1 \leq i \leq s$:

$$I_i(\mathbf{x}) = x_i, \tag{24}$$

i.e., $I_i(\cdot)$ takes an input vector and outputs its i-th coordinate. As for the functions $f_{v_A}(\cdot)$, we can calculate $I_i(\mathbf{x})$ over the entire ground set $\mathcal{X}$ and denote the output 2^s-vector by $\mathbf{I}_i$. Observe that

$$\mathbf{f}_{v_A} = \left(\sum_{i\in A} \mathbf{I}_i\right) \mod 2. \tag{25}$$

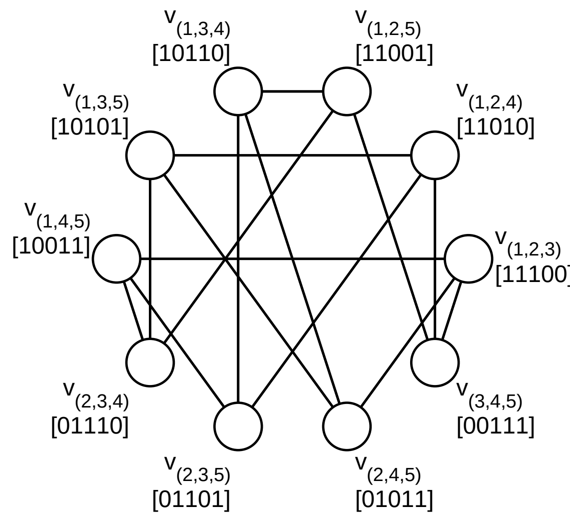


Figure 3: The graph G of Example 13 ($s = 5$) along with the characteristic element assigned to each vertex by the Functional Representation $\mathcal{F}$.

$$f_{v_{(123)}} \left(\begin{array}{c} \uparrow \\ |\mathcal{X}| = 2^s \\ \downarrow \end{array} \underbrace{\begin{bmatrix} 00000 \\ 00001 \\ \vdots \\ 11100 \\ \vdots \\ 11010 \\ \vdots \\ 00111 \\ \vdots \end{bmatrix}}_{\mathcal{X}} \begin{array}{l} \\ \\ \\ \leftarrow c_{v_{(1,2,3)}} \\ \\ \leftarrow c_{v_{(1,2,4)}} \\ \\ \leftarrow c_{v_{(3,4,5)}} \\ \\ \end{array} \right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 3 \\ \vdots \\ 2 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \mod 2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = \mathbf{f}_{v_{(123)}}$$

Figure 4: Evaluating $f_{v_{(123)}}$ over elements of the ground set $\mathcal{X}$. The outputs for the entire ground set forms a 2^s-vector $\mathbf{f}_{v_{(123)}}$.

In words, $\mathbf{f}_{v_A}$ can be written as the sum over the field, of $|A| = 3$ vectors: $\mathbf{I}_i$, $i \in A \subseteq \{1, \ldots, s\}$. All $\mathbf{f}_{v_A}$'s can be expressed as a linear combination of at most s vectors, $\{\mathbf{I}_1, \ldots, \mathbf{I}_s\}$. Thus,

$$\dim \mathcal{F} \leq s. \tag{26}$$

By Theorem 10,

$$\Theta(G) \leq \dim \mathcal{F} \leq s. \tag{27}$$

Walking down the same path, we construct a Functional Representation $\overline{\mathcal{F}}$ for $\overline{G}$ in order to upper bound $\Theta(\overline{G})$ and ultimately the sum $\Theta(G) + \Theta(\overline{G})$. Consider the following $\overline{\mathcal{F}}$:

1. Let the ground set be the set of all vectors of length s with elements in $\mathbb{F}_5$, *i.e.*, $\overline{\mathcal{X}} = \mathbb{F}_5^s$. Note that we have deliberately selected a different field than the one used in $\mathcal{F}$. There is a subtle reason on which we will elaborate later on.

2. As before, assign to each vertex $v_A \in V(\overline{G})$, the *characteristic vector*, $\mathbf{c}_A$ of A. Note that the characteristic element assigned to each vertex v_A, denoted in the sequel by $\mathbf{c}_{v_A}$, is indeed an element of the ground set $\overline{\mathcal{X}} = \mathbb{F}_5^s$.

3. Finally, define a function $\overline{f}_{v_A} : \overline{\mathcal{X}} \to \mathbb{F}_5$ for each vertex $v_A \in V(\overline{G})$, as follows:

$$\overline{f}_{v_A}(\mathbf{x}) = \left(\sum_{i \in A} x_i - 1 \right) \mod 5, \tag{28}$$

   where $\mathbf{x} \in \overline{\mathcal{X}}$.

To verify that the functions $\overline{f}_{v_A}$ satisfy the necessary conditions, first observe that if c_{v_B} is the characteristic element of $v_B \in V(G)$, then

$$\overline{f}_{v_A}(c_{v_B}) = \left(\sum_{i \in A} c_{v_B} - 1 \right) \mod 5 = \left(\sum_{i \in A} \mathbf{c}_B(i) - 1 \right) \mod 5 = (|A \cap B| - 1) \mod 5, \tag{29}$$

where $0 \leq |A \cap B| \leq 3$. Furthermore, $\overline{G}$ being the complement of G:

$$|A \cap B| = 1 \Leftrightarrow (v_A, v_B) \notin E(\overline{G}). \tag{30}$$

Taking the above into consideration, we discern the following cases:

$$\overline{f}_{v_A}(c_{v_B}) = \begin{cases} 2, & \text{if } |A \cap B| = 3 \quad \Leftrightarrow A = B, i.e, c_{v_B} = c_{v_A}, \\ 1 \text{ (or } 4), & \text{if } |A \cap B| = 2 \text{ (or } 0) \quad \Leftrightarrow A \neq B, (v_A, v_B) \in E(\overline{G}), \\ 0, & \text{if } |A \cap B| = 1 \quad \Leftrightarrow A \neq B, (v_A, v_B) \notin E(\overline{G}). \end{cases} \tag{31}$$

By the first and third cases, the only ones playing a role in the definition of Functional Representation:

(a) $\overline{f}_{v_A}(c_{v_A}) = 2 \neq 0$, $\forall v_A \in V(\overline{G})$, and

(b) $\overline{f}_{v_A}(c_{v_B}) = 0$, if $(v_A, v_B) \notin E(\overline{G})$, $\forall v_A, v_B \in V(\overline{G})$.

As far as the dimension of $\overline{\mathcal{F}}$ is concerned, observe that $\overline{\mathbf{f}}_{v_A}$'s can be written as

$$\overline{\mathbf{f}}_{v_A} = \left(\sum_{i \in A} \mathbf{I}_i - \mathbf{1} \right) \mod 5, \tag{32}$$

where $\mathbf{1} \in \mathbb{F}_5^{2^s}$ is the all ones vector. In other words, the entire set of $\overline{\mathbf{f}}_{v_A}$'s can be expressed in terms of $s+1$ vectors, namely $\mathbf{I}_1, \ldots, \mathbf{I}_s$, and $\mathbf{1}$, implying that

$$\dim \overline{\mathcal{F}} \leq s + 1. \tag{33}$$

By Theorem 10,

$$\Theta(\overline{G}) \leq \dim \overline{\mathcal{F}} \leq s + 1. \tag{34}$$

Combining (27) and (34) we obtain an upper bound U_G on $\Theta(G) + \Theta(\overline{G})$ for the specific graph G:

$$\Theta(G) + \Theta(\overline{G}) \leq 2s + 1 \doteq U_G. \tag{35}$$

On the other hand, exploiting Lemma 6, and recalling that G has $n = \binom{s}{3}$ vertices, we obtain

$$\Theta(G + \overline{G}) \geq \sqrt{2\binom{s}{3}} \doteq L_G. \tag{36}$$

It is clear that there exists a choice of s such that $U_G < L_G$, concluding the proof of the main theorem, Theorem 5.

3.2 Concluding remark

In proving Theorem 5, we initially developed a lower bound on $\Theta(G + \overline{G})$ (Lemma 6):

$$\Theta(G + \overline{G}) \geq \sqrt{2n} \doteq L. \tag{37}$$

Subseqently, we demonstrated a specific graph G, and upper bounded $\Theta(G) + \Theta(\overline{G})$ by

$$\Theta(G) + \Theta(\overline{G}) \leq U \doteq \dim \mathcal{F} + \dim \overline{\mathcal{F}}, \tag{38}$$

where $\mathcal{F}$ and $\overline{\mathcal{F}}$ were Functional Representation's of G and $\overline{G}$, respectively, deliberately defined over different fields. The final result was obtained immediately by the fact that for the constructed G, ($\overline{G}$, $\mathcal{F}$, and $\overline{\mathcal{F}}$) the obtained upper bound U was strictly smaller than L.

We conclude the lecture by showing that, irrespectively of G, had $\mathcal{F}$ and $\overline{\mathcal{F}}$ been defined over the same field, it would have been impossible to reach an upper bound $U < L = \sqrt{2n}$. Consider the following line of arguments:

1. G has n vertices. It follows that $G \times \overline{G}$ contains an independent set of size n, and hence

$$\alpha(G \times \overline{G}) \geq n. \tag{39}$$

2. By definition of $\Theta(\cdot)$, $\alpha(G \times \overline{G})$ is a lower bound on $\Theta(G \times \overline{G})$. Therefore,

$$\Theta(G \times \overline{G}) \geq \alpha(G \times \overline{G}) \geq n. \tag{40}$$

3. By Theorem 10, if $\mathcal{F}'$ is a Functional Representation of a graph G', then $\Theta(G') \leq \dim\mathcal{F}'$. Therefore, if $G \times \overline{G}$ has even one Functional Representation with dimension $n' < n$, we would have $\Theta(G \times \overline{G}) \leq n' < n$, which would contradict (40). We conclude, that for all possible Functional Representation's $\mathcal{F}_{G\times\overline{G}}$ of $G \times \overline{G}$,

$$n \leq \dim\mathcal{F}_{G\times\overline{G}}. \tag{41}$$

4. On the other hand (Lemma 12), if $\mathcal{F}$ and $\overline{\mathcal{F}}$ are Functional Representation's of G and $\overline{G}$, respectively, over the same field, then there exists an Functional Representation $\mathcal{F}_{G\times\overline{G}}$ of $G \times \overline{G}$, such that

$$\dim\mathcal{F}_{G\times\overline{G}} \leq \dim\mathcal{F} \cdot \dim\overline{\mathcal{F}}. \tag{42}$$

5. It is straightforward that for any Functional Representation's $\mathcal{F}$ and $\overline{\mathcal{F}}$, of G and $\overline{G}$, respectively, over the same field, we have

$$n \leq \dim\mathcal{F} \cdot \dim\overline{\mathcal{F}}. \tag{43}$$

   Otherwise, we would violate the conclusion of step 3.

6. Applying square root on both sides of (43) and exploiting the relation between the arithmetic and the geometric mean of two positive numbers, we have:

$$\sqrt{n} \leq \sqrt{\dim\mathcal{F} \cdot \dim\overline{\mathcal{F}}} \leq \frac{1}{2}\left(\dim\mathcal{F} + \dim\overline{\mathcal{F}}\right) \tag{44}$$

$$\Rightarrow \quad 2\sqrt{n} \leq \dim\mathcal{F} + \dim\overline{\mathcal{F}}. \tag{45}$$

7. The last step implies that the best upper bound on $\Theta(G) + \Theta(\overline{G})$ we could hope to get is $U = 2\sqrt{n}$. However, $2\sqrt{n} = U \not< L = \sqrt{2n}$, and hence we cannot obtain the desired gap.

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