Topics in Mathematical Physics

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Chapter 1

Gauge Theory & Differential Cohomology

This chapter will focus on generalising Maxwell's theory of electrodynamics to 'higher-form gauge fields'. Just as the 1-form electromagnetic potential couples to the worldlines of charged particles by integration, one can think of coupling higher form ($p > 1$) gauge fields to be integrated over worldvolumes of spatially extended objects called 'p-branes'. A question for consistent physical theories is whether we just include branes of arbitrary dimension and shape. The story gets extremely complicated, and it turns out that classifying these objects requires the use of various topological invariants. For example, it is conjectured that in type II string theory branes are classified by 'twisted K-theory'. We won't get quite that far, but we will discuss the method of differential cohomology when the scenario is somewhat simpler. This chapter mainly follows Greg W. Moore's 2023 TASI lectures and some relevant sections in his notes [MS25].

1.1 Principal Bundles

Since gauge theory in the general context will be needed for parts of this chapter and others, it makes sense to formulate it clearly here. The exposition in terms of principal bundles makes the idea behind gauge theory a lot more concrete. A complete treatment of the mathematics is found in the classic [KN63], we will give a brief overview of some relevant material. This full machinery is in fact necessary when studying the subject in more advanced settings, where the geometry and topology of these spaces becomes more relevant. We first define the concept of a torsor, an object that appears a lot in physics yet is rarely mentioned in early studies.

Definition 1.1.1 (G-Torsor)**.** *A set X is a G-**torsor**, or principal homogenous space, if it is equipped with a free and transitive action $\triangleleft : X \times G \to X$. That is*

1. $\forall x, y \in X \quad \exists g \in G : y = x \triangleleft g$ *and*
2. $x \triangleleft g = x \implies g = 1_G$.

In other words any element of X can be reached with some $g \in G$ and there are no fixed points. Combined this can be written as, $\forall x, y \in X$, $\exists! g \in G : y = x \triangleleft g$. This implies that the underlying

sets of X and G are isomorphic. The isomorphism is not canonical however, it's obtained by first choosing an $x_0 \in X$ and identifying $g \in G$ with $x \cdot g$. In this way a G-torsor is just like G, but we've 'forgotten' the identity. This construction abstracts the concept of having quantities one can compare via differences/quotients, with no natural way to add/multiply them. A classic example of this in electromagnetism. Voltages form a $\mathbb{R}$-torsor, but picking a ground (picking an origin), promotes this to a group. The potential differences are actually a group since they are differences in the $\mathbb{R}$-torsor of voltages, but the voltages themselves have no additive structure. For a full discussion on this see [Joh09].

Definition 1.1.2. *Given a topological group, G, and topological space X, a* ***principal G-bundle*** *is a continuous surjection $\pi : P \to X$ such that :*

1. *$\pi^{-1}(x) = P_x$ are G torsors and P has a continuous fibre preserving right action :*

$$\forall p \in P, \quad \pi(p \lhd g) = \pi(p) \quad \forall g \in G$$

2. *$\pi : P \to X$ is locally trivial : $\forall x \in X$, $\exists U \ni x$ and a G-equivariant homeomorphism, φ_U, such that the following diagram commutes.*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl] \\
& U &
\end{tikzcd}$$

This is the formal definition of the concept of having a continuous family of G-torsors over a space, and exactly the necessary construction to begin talking about gauge theories. As a first example considering the quantum theory of a charged particle. The born rule says that global changes of phase do not change observables. Hence, there is a global $U(1)$ symmetry $\psi \mapsto e^{i\theta}\psi$. One can further show that this imposes the conservation of electric charge. We can further demand the system is invariant under local $U(1)$ symmetries, which would correspond to some kind of "$U(1)$-valued function $g(x) = e^{i\theta(x)}$". Imposing this condition forces the electromagnetic field itself to appear. To formalise this we need a copy of $U(1)$ at all points in space, in order to define the action locally. The key detail is that we have to forget the identity element of the group and treat these copies as $U(1)$-torsors instead, in order to capture the notion of gauge redundancy.

From now on our principal G-bundles will be smooth with G a lie group. Just replace the relevant maps/manifolds above with smooth ones. Our base space will usually be a spacetime M of dimension n unless specified otherwise. The group G tends to be called the **(local) gauge group** by physicists and the **structure group** by mathematicians.

Definition 1.1.3. *A* ***bundle map*** *between two principal G-bundles $\pi : P \to M$ and $\pi' : P' \to M$ is a diffeomorphism $f : P \to P'$ such that:*

1. *f is G-equivariant : $f(p \lhd g) = f(p) \lhd g \quad \forall p \in M, g \in G$*

2. *f is fibre preserving : $\pi' \circ f = \pi$ so the following diagram commutes :*

$$\begin{tikzcd}
P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\
& M &
\end{tikzcd}$$

It turns out that all bundle maps over a manifold M are automorphisms. We can consider the collection of all principal G-bundles over M and the automorphisms between them as having the structure of a groupoid, denoted $\mathrm{Prin}_G(M)$. In the language of category theory a groupoid is just a category in which all morphisms are isomorphisms. For this chapter we will restrict to the case $G = U(1)$, which is actually equivalent to considering $\mathbb{C}^\times$ vector bundles with a Hermitian metric.

Definition 1.1.4. *The **vertical subspace** of a principal G-bundle $\pi : P \to M$ at a point $p \in P$ is defined as $T_pV := \ker \mathrm{d}\pi_p$.*

Central to gauge theory is the concept of a connection. This has an abstract description as an assignment of a complimentary subspace H_pP such that $T_pP = V_pP \oplus H_pP$. This actually turns out to be equivalent to kind of one form defined on all of P satisfying certain conditions. In gauge theory it is this approach that is more applicable.

Definition 1.1.5. *A connection, $\omega \in \Omega^1(P;\mathfrak{g})$, is a Lie algebra valued 1-form satisfying:*

1. $\omega_p \circ \mathrm{d}_e\sigma_p = \mathrm{id}_{\mathfrak{g}}, \quad \forall p \in P, g \in G$, *where $\sigma_p : G \ni g \longmapsto p \lhd g \in P$ is the orbit map*
2. $((\lhd g)^\star\omega)_p(X_p) = \mathrm{Ad}_{g^{-1}}(\omega_p(X_p)) \quad \forall g \in G\ \forall X_p \in T_pP$

$$\begin{tikzcd}
\mathfrak{g} \arrow[d, "\mathrm{Ad}_{g^{-1}}"'] & T_pP \arrow[l, "\omega_p"'] \arrow[dl, "(\lhd g^\star\omega)_p"] \\
\mathfrak{g} &
\end{tikzcd}$$

This is still slightly too unwieldy to make use of. In a principal bundle we can pick a local section, which allows us to write the 1-form on the bundle as an honest local 1-form on spacetime.

Definition 1.1.6. *Given a principal G-bundle, $\pi : P \longrightarrow M$, we define the following:*

1. *A **choice of gauge** is a local section $\sigma : U \to P$.*
2. *A **gauge transformation** is a bundle map $u : P \to P$.*

It can be shown that every principal bundle map (gauge transformation) is actually also an automorphism. We can then get a local representation of the connection as a spacetime 1-form on U by simply pulling back along a local section.

Definition 1.1.7. *The **Yang-Mills field** or **gauge potential**, $\mathcal{A}|_U \in \Omega^1(U;\mathfrak{g})$, is defined locally by $\mathcal{A}|_U := \sigma^\star\omega$ in a fixed gauge $\sigma \in \Gamma(U)$.*

When moving to local data it can be easier to think in terms of group valued functions. This is possible due the existence of a local trivialisation φ. Consider a principal G-bundle as well as an open cover $\{U_\alpha\}_{\alpha\in I}$ of M. Recall that there exists trivialisation so that we can define a diffeomorphism locally,

$$\begin{aligned}
\pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times G \\
p &\longmapsto (\pi(p), \varphi_\alpha(p)),
\end{aligned}$$

where $p \in \pi^{-1}(U_\alpha)$ and $\varphi_\alpha \coloneqq \mathrm{proj}_2 \circ \varphi|_{U_\alpha}$ to stay consistent with the notation from definition 1.1.2. Now we can define **transition functions** on overlaps $U_\alpha \cap U_\beta$,

$$\phi_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow G$$
$$\phi_{\beta\alpha}(\pi(p)) \coloneqq \varphi_\beta(p) \cdot \varphi_\alpha(p)^{-1},$$

where $\cdot$ denotes the group multiplication.

Proposition 1.1.1. *The maps $\phi_{\beta\alpha}$ are well-defined G-valued functions which only depend on $\pi(p)$, not the choice of point $p \in \pi^{-1}(p)$.*

Proof. Firstly, recall that the fibres are G-torsors here. In particular, given $x \in M$ and $p, p' \in \pi^{-1}(x)$, there exists a unique element $g \in G$ such that $p \triangleleft g = p'$. To show that the maps ψ are well-defined we need to show that they do not depend on the point in the fibre. In an overlap $U_\alpha \cap U_\beta$ we have:

$$\phi_{\beta\alpha}(\pi(p')) = \phi_{\beta\alpha}(\pi(p \triangleleft g)) = \varphi_\beta(p \triangleleft g) \cdot \varphi_\alpha(p \triangleleft g)^{-1}.$$

By definition the maps φ_γ are G-equivariant,

$$\phi_{\beta\alpha}(\pi(p \triangleleft g)) = \varphi_\beta(p) \triangleleft g \cdot (\varphi_\alpha(p) \triangleleft g)^{-1},$$

and then by properties of the inverse of a group action:

$$\phi_{\beta\alpha}(\pi(p \triangleleft g)) = \varphi_\beta(p) \triangleleft g \cdot g^{-1} \triangleright \varphi_\alpha(p)^{-1} = \phi_{\beta\alpha}(\pi(p))$$

On the other hand we know, $\pi(p') = \pi(p \triangleleft g) = \pi(p)$, so we have the desired result:

$$\phi_{\beta\alpha} \circ \pi(p) = \phi_{\beta\alpha} \circ \pi(p')$$

□

It's trivial to check that these maps satisfy the so-called **cocycle condition** on triple overlaps:

$$\phi_{\beta_\alpha}(x) = \phi_{\beta\gamma}(x) \cdot \phi_{\gamma\alpha}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma \tag{1.1}$$

Now it turns out that we can actually construct a fibre bundle knowing only these local trivialisations.

Proposition 1.1.2 (Fibre bundle reconstruction)**.** *Let M be a manifold and $\{U_\alpha\}_{\alpha \in I}$ an open cover of M such that there exist G-valued maps $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ satisfying the cocycle condition. Consider a disjoint union over the cover,*

$$N = \coprod_{\alpha \in I} U_\alpha \times G = \{(\alpha; x, g) : \alpha \in I, x \in U_\alpha, g \in G\}.$$

Using the G-valued maps we can define an equivalence relation on this space such that,

$$(\alpha; x, g) \sim (\beta; x, \phi_{\beta\alpha}(x) \cdot g) \quad \forall x \in U_\alpha \cap U_\beta,\ g \in G.$$

Now there exists a principal G-bundle, $\pi : N/\sim \longrightarrow M$ with $\pi(\alpha; x, g) = x$.

Furthermore, the above construction is actually unique up to isomorphism [KN63]. So a principal G-bundle can be equivalently thought of in terms of this local data.

Remark 1.1.1. *In local coordinates, x^μ and y^α_β on the base space and the Lie algebra, we can write the connection as $\mathcal{A}^\alpha_{\mu\beta}(x)$. Given two choices of gauge, $\sigma_i : U_i \to P \quad i \in \{1,2\} : U_1 \cap U_2 \neq \varnothing$, we expect it to transform correctly so we can stitch the fields together. Indeed, given the local gauge transformation $g(x) \in C^\infty(U_1 \cap U_2, G)$, we arrive at the transformation law $\mathcal{A}|_{U_2} = g^{-1}\mathcal{A}|_{U_1} g + g^{-1}\mathrm{d}g$.*

In general, they may not be a globally defined potential that can be stitched together over M. In fact there may not be even a local potential. Fixing a gauge is also an arbitrary choice, even though they may be related by gauge transformations. Since these potentials are not physical in that they can't be measured, it makes sense to try to construct observable 'gauge invariant' quantities.

Definition 1.1.8. *The **curvature**, $\mathcal{F}_\omega \in \Omega^2(P;\mathfrak{g})$, or **field strength** is defined as*

$$\mathcal{F}_\omega = \mathrm{d}\omega + \frac{1}{2}\omega \wedge \omega. \tag{1.2}$$

Note that the wedge product here is defined with the commutator on the Lie algebra, hence the square of $\mathfrak{g}$-valued one forms is not zero in general.

Remark 1.1.2. *Again once a gauge is fixed we have a potential and can write the curvature in terms of its components.*

$$\mathcal{F} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

This expression is gauge invariant in the sense that it transforms as $\mathcal{F}' = g^{-1}\mathcal{F}g$, in contrast to the Yang-Mills field which has an extra term.

The above description gives a nice interpretation of gauge fields in physics. These are often called gauge bosons in the quantum theory. But, we have yet to say anything about matter. To do so one must recall the definition of an associated bundle.

Definition 1.1.9. *Consider a principal G-bundle, $\pi : P \longrightarrow M$, with a faithful left action $\rho : G \times F \longrightarrow F$ for a smooth manifold F. Denote the left action as $\rho(g, f) \coloneqq g \triangleright f$. We can equip $P \times F$ with a right G-action by,*

$$(p, f) \triangleleft g \coloneqq (p \triangleleft g, g^{-1} \triangleright f),$$

*and then take the quotient $P \times_\rho F \coloneqq (P \times F)/G$. This can be turned into a fibre bundle when equipped with the surjection $\pi([(p, f)]) = p$. The fibre bundle $(P \times_\rho F)/G$ is called the **associated bundle** to P with fibre F.*

One can define a vector bundle using this construction too. Consider a representation (left action) $\rho : G \longrightarrow GL(n;\mathbb{C})$ and a principal G-bundle. Then the associated fibre bundle with fibre $\mathbb{C}^n$ is a **complex vector bundle** of rank n. Then replace $\mathbb{C}$ with $\mathbb{R}$ to get the definition of a real vector bundle. When considering matter particles, the idea is that the fibre should be taken to be a representation of some group. Then matter fields are sections of the associated bundle. The full details are not spelled out here, we restrict our attention to bosonic fields for the majority of this chapter.

It also turns out that there is information in the topological non-triviality of a bundle which escapes the curvature. When attempting to distinguish such bundles, the notion of a characteristic class can help. These have various definitions depending on the context and level of abstraction, we will stick with a simple first definition of the Chern class here.

Definition 1.1.10. *The* ***total Chern class*** *for a complex vector bundle, E, is defined for any curvature Ω as,*

$$c(E) = \det\left(I + \frac{i\Omega}{2\pi}\right), \tag{1.3}$$

where the determinant gives a series expansion:

$$c(E) = 1 + c_1(E) + \ldots$$

It can be shown that,

$$c_1(E) = \frac{i}{2\pi}\mathrm{Tr}(\Omega). \tag{1.4}$$

The corresponding class in cohomology $[c_1(E)] \in H^2(M;\mathbb{Z})$ is the ***first Chern class*** *of the bundle.*

The Chern class is well-defined for vector bundles in the sense that is does not depend on the choice of the connection. In our case, what we can do is work with the associated vector bundle $\mathrm{ad}(P)$ to the principal bundle P, the fibres of which are copies of the Lie algebra. The left action induces a connection on the associated vector bundle, and this is what we take the curvature of. In the case of $U(1)$, which is what we will restrict too for most of the chapter, the Lie algebra is ismorphic to $\mathbb{R}$. Note that there is a choice with factors of i, and there is a disagreement between physics and maths conventions. We will drop the factor of i in this definition towards the end of this chapter in attempt to match the literature.

1.2 Maxwell Theory

Classical electrodynamics, in the absence of magnetic currents, can be formalised as gauge theory. Spacetime is a smooth manifold M and $\pi : P \to M$ is a principal $U(1)$-bundle. Let the potential, $\mathcal{A}$, be a local Yang-Mill's field on M. Note that, in the case of $U(1)$, the connection and curvature are abelian since the Lie algebra is one dimensional. Then we can define the Bianchi identity as

$$\mathrm{d}\mathcal{F} = 0. \tag{1.5}$$

This is in fact an actual mathematical identity when the electromagnetic field is modelled as the curvature of a $U(1)$-bundle. Note however, that when the force cannot be modelled as a line bundle connection, this is may become an imposed condition. In this sense we'll now forget the bundle structure momentarily and just think in terms of local spacetime forms. Now, **assuming** the Bianchi identity, we can 'solve' for $\mathcal{F}$ using the Poincaré lemma. Then the equation of motion is derived from an appropriate Lagrangian. We define the current as a one form $\mathfrak{J} \in \Omega^1(M)$ for the source of the field and an action as:

$$S[\mathcal{A}] = \int_M -\frac{1}{2}\mathcal{F} \wedge \star\mathcal{F} + \mathcal{A} \wedge \star\mathfrak{J} \tag{1.6}$$

Note that we do not make the coupling constant explicit. We can vary this action with respect to $\mathcal{A}$ using the Gâteaux derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t} S[\mathcal{A} + t\delta\mathcal{A}]\bigg|_{t=0} = 0. \tag{1.7}$$

Looking at just the integral we have,

$$\begin{aligned}&\int_M -\frac{1}{2}\Big[\mathrm{d}(\mathcal{A} + t\,\delta\mathcal{A}) \wedge \star\,\mathrm{d}(\mathcal{A} + t\,\delta\mathcal{A})\Big] + (\mathcal{A} + t\,\delta\mathcal{A}) \wedge \star\mathfrak{J}\\ &= \int_M -\frac{1}{2}\Big[\mathrm{d}\mathcal{A} \wedge \star\,\mathrm{d}\mathcal{A} + 2t\,\mathrm{d}\mathcal{A} \wedge \star\,\mathrm{d}(\delta\mathcal{A}) + t^2\,\mathrm{d}(\delta\mathcal{A}) \wedge \star\,\mathrm{d}(\delta\mathcal{A})\Big] + t\,\delta\mathcal{A} \wedge \star\mathfrak{J}.\end{aligned}$$

Here we have used the symmetry of the inner product $\langle -, - \rangle = - \wedge \star -$ in the second line. Differentiating and evaluating at $t = 0$ we just have

$$\int_M \langle \mathrm{d}(\delta\mathcal{A}), \mathrm{d}\mathcal{A} \rangle - \langle \delta\mathcal{A}, \mathfrak{J} \rangle = 0.$$

Now we can swap the position of d and replace it with the co-differential $\mathrm{d}^\dagger$ using the defining property $\langle \mathrm{d}\alpha, \beta \rangle = \langle \alpha, \mathrm{d}^\dagger\beta \rangle$.

$$\int_M \langle \delta\mathcal{A}, \mathrm{d}^\dagger\mathrm{d}\mathcal{A} - \mathfrak{J} \rangle = 0$$

This has to hold $\forall \delta\mathcal{A}$ and since $\langle -, - \rangle$ is non-degenerate we have,

$$\mathrm{d}^\dagger\mathcal{F} = \mathfrak{J} \iff \mathrm{d} \star \mathcal{F} = \star\mathfrak{J}. \tag{1.8}$$

Now it's not always true that $\mathcal{F} = \mathrm{d}\mathcal{A}$, even locally, and the obstruction to this is given by the de Rham Cohomology $H^2_{\mathrm{dR}}(M)$.

1.3 Coupling to Particles

If we want to talk about the classical dynamics of a free particle on spacetime we can define an action on its worldline $\mathcal{W}_1$,

$$S[x(\tau)] = \int_{\mathcal{W}_1} T\mathrm{d}s = \int_{\mathcal{W}_1} T\sqrt{-g(\dot{x}, \dot{x})}\mathrm{d}\tau. \tag{1.9}$$

T is the mass of the particle, τ is appropriate parameterisation of the worldline $x(\tau)$, s is the arc length, and $g(-, -)$ is the metric. Minimising the action $\delta S = 0$ results in the following equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(T\frac{\mathrm{d}x_\mu}{\mathrm{d}s}) = 0, \tag{1.10}$$

which can be shown to be equivalent to the geodesic equations

$$\ddot{x}^\mu + \Gamma^\mu_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu = 0.$$

So this is just the motion of a particle in the absence of forces. If we assume the particle now has an electric charge, q_e, we want to couple the particle to a background field by modifying the action to

$$S = \int_{\mathcal{W}_1} T\mathrm{d}s + \int_{\mathcal{W}_1} q_e\mathcal{A}. \tag{1.11}$$

This results in the expected equations of motion, the Lorentz force law,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(T\frac{\mathrm{d}x_\mu}{\mathrm{d}s}) = q_e\mathcal{F}_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{1.12}$$

When generalising to further to consider the quantum theory of the charged particle we expect there to be a path integral with a term $\exp(\frac{i}{\hbar}S)$ weighting each trajectory, meaning

$$\exp(i\int_{\mathcal{W}_1} T\mathrm{d}s + i\int_{\mathcal{W}_1} q_e\mathcal{A}), \tag{1.13}$$

where we've set $\hbar = 1$. We can instead write this as,

$$\exp(i\int_{\mathcal{W}_1} T\mathrm{d}s)\cdot\chi(\mathcal{W}_1), \tag{1.14}$$

where $\chi(\mathcal{W}_1) \in U(1)$ is some factor depending on the worldline. If there exists a potential then we can write this as in 1.13, but as discussed earlier this isn't always the case. Instead, we can slightly modify the properties of such a $U(1)$ factor to account for more general topologies. We know from Stokes' theorem that if $\mathcal{W}_1$ is the boundary of some other surface, $\mathcal{B}_2 = \partial\mathcal{W}_1$, then

$$\int_{\partial\mathcal{B}_2}\mathcal{A} = \int_{\mathcal{B}_2}\mathrm{d}\mathcal{A}, \tag{1.15}$$

where $\mathrm{d}\mathcal{A} = \mathcal{F}$. In that case we can write the $U(1)$ factor as,

$$\chi(\mathcal{W}_1) = \exp(i\int_{\mathcal{B}_2}\mathcal{F}). \tag{1.16}$$

This expression is more useful than the form in terms of the potential since it allows for more topologically non-trivial field strengths. We can make this a formal definition.

Definition 1.3.1 (Differential Character of degree two)**.** *A Cheeger-Simons differential character of degree two is a group homomorphism $\chi : Z_1(M) \to U(1)$ such that $\exists\mathcal{F} \in \Omega^2(M)$ satisfying*

$$\exists\mathcal{B}_2 \in C_2(M) : \partial\mathcal{B}_2 = \mathcal{W}_1 \implies \chi(\mathcal{W}_1) = \exp(i\int_{\mathcal{B}_2}\mathcal{F})$$

In the above we're using the infinite dimensional abelian group $Z_1(M)$ of cycles on M. We want actions to add if particles don't interact with each other, i.e. $\chi(\mathcal{W}_1 \sqcup \mathcal{W}_1') = \chi(\mathcal{W}_1)\cdot\chi(W_1')$, implying we need group homomorphism to respect this behaviour. Note that such a surface bounding a cycle may not always exist, the condition merely says that if this is possible then it will obey the formula. Furthermore, the set of all characters can be endowed with an abelian group structure,

$$(\chi_1\cdot\chi_2)(\mathcal{W}_1) \coloneqq \chi_1(\mathcal{W}_1)\cdot\chi_2(\mathcal{W}_1), \tag{1.17}$$

which brings us to the next definition.

Definition 1.3.2 (Differential Cohomology in degree two). *The Differential Cohomology group of degree two, denoted $\check{H}^2(M)$, is the set of all degree two differential characters, $\chi \in \mathrm{Hom}(Z_1(M), U(1))$, equipped with an abelian group structure.*

This carries all the necessary gauge invariant information of the electromagnetic field. In fact, it is more general than $\mathcal{F}$, and we can consider the field to be **defined** by its differential character instead of its curvature. This approach is similar to that of Wilson lines and holonomy. In particular degree two Cheeger-Simons characters are in one to one correspondence with the holonomy of a connection on principal bundle.

1.4 Charge Quantisation

The above formulation leads to the fact that fluxes must be **quantised**. This can be seen by considering a cycle $\mathcal{W}_1 \in Z_1(M)$ such that it is the boundary of not one but two surfaces; $\partial\mathcal{B}_2 = \mathcal{W}_1$ and $\partial\mathcal{B}_2' = \mathcal{W}_1$ with $\Sigma = \mathcal{B}_2 \cup \bar{\mathcal{B}}_2'$ the combined 2-cycle. Note the opposite orientation on one term to ensure that $\partial\Sigma = 0$. Now we can take the character using either surface, but we'd expect them to give the same result since we're evaluating with a common cycle $\mathcal{W}_1$.

$$\begin{aligned}\chi(\mathcal{W}_1) = \exp i \int_{\mathcal{B}_2} \mathcal{F} &= \exp i \int_{\mathcal{B}_2'} \mathcal{F} \\ \implies \exp(i \int_{\mathcal{B}_2} \mathcal{F} &- i \int_{\mathcal{B}_2'} \mathcal{F}) = 1 \\ \implies \exp i &\int_{\Sigma} \mathcal{F} = 1\end{aligned}$$

This implies that $\mathcal{F}$ has 2π-integral periods, $\mathcal{F} \in \Omega^2_{\mathbb{Z}}(M)$. Note that Moore uses $\Omega^2_{\mathbb{Z}'}(M)$ with $\mathbb{Z}' := 2\pi\mathbb{Z}$, but we make an abuse of notation by omitting the prime. We'll also be sloppy in distinguishing between $\mathbb{R}/\mathbb{Z}$ as an additive group and $U(1)$. The process above is a generalisation of Dirac's charge quantisation argument in the following sense. Dirac supposed the existence of magnetic monopoles, these would be characterised by postulating a 'magnetic charge'. Then the second of Maxwell's equations is,

$$\mathrm{d}\mathcal{F} = \mathfrak{I}_m, \tag{1.18}$$

giving an explicit electric-magnetic duality. One can then propose a magnetic monopole field defined by,

$$\mathcal{F} = q_m \mathrm{d}\mu_g^2 \in \Omega^2(M), \tag{1.19}$$

where q_m is the magnetic charge of the monopole and $\mathrm{d}\mu_g^2$ is the volume two form on the unit two sphere. Note that the space we're working over is now $\mathbb{R}^3 - \{0\}$, where the monopole is treated as if 'at' the origin. Consider a loop, $\mathcal{C}_1$, around the source of the magnetic monopole such that we have a 2-cycle $D^+ \cup D^-$ with $\partial D^+ = \mathcal{C}_1 = -\partial D^-$. Then we can evaluate the characters,

$$\exp i \int_{\mathcal{C}_1} q_e \mathcal{A} = \exp i \int_{D^+} q_e \mathcal{F} = \exp i \int_{D^-} q_e \mathcal{F}, \tag{1.20}$$

where q_e is the electric charge. Then by the same argument as before we obtain,

$$q_e \mathcal{F} \in \Omega^2_{\mathbb{Z}}(M) \implies q_e q_m \in 2\pi\mathbb{Z}. \tag{1.21}$$

It's remarkable that the existence of just one magnetic monopole implies the quantisation of both electric and magnetic charge. That being said there is, as of now, no experimental evidence magnetic monopoles exist in nature.

1.5 Generalising to Branes

In theoretical physics and condensed matter there are often other kinds of objects called branes. These are, very roughly speaking, spatially extended objects of a given dimension. Suppose that these objects are charged and dynamical, then they ought to couple to generalised electromagnetic fields of higher degrees. This is referred to as p-form electrodynamics (or p-form gauge theory in the general non-abelian context). We write $\mathcal{W}_{p+1}$ for the worldvolume that a p-brane traces out in n-dimensional spacetime. We can couple these branes to higher dimensional analogues of Maxwell fields by introducing,

$$\mathfrak{I}_e := q_e \eta(\mathcal{W}_p \hookrightarrow M_n), \tag{1.22}$$

where $\eta(\mathcal{W}_p \hookrightarrow M_n)$ is an $(n-p+1)$-delta function representative of the Poincaré dual to $\mathcal{W}_p$. There are technicalities in making this idea rigorous and we won't dwell on them. The point is that $\mathfrak{I}_e$ has support only on the brane's worldvolume. Now we need to define higher form Maxwell fields that couple to these charges.

Definition 1.5.1 (Generalised Maxwell Theory)**.** *A generalised degree k Maxwell theory has a curvature $\mathcal{F} \in \Omega^k(M)$ with sources $\mathfrak{I}_m \in \Omega^{k+1}(M)$ & $\mathfrak{I}_e \in \Omega^{n-k-1}$ satisfying the equations of motion*

1. $\mathrm{d} \star \mathcal{F} = \mathfrak{I}_e$
2. $\mathrm{d}\mathcal{F} = \mathfrak{I}_m$

Note that by definition the currents are trivialised by $\star\mathcal{F}$ and $\mathcal{F}$ respectively. Now immediately face the issue of losing the interpretation of the Maxwell field as the curvature of a $U(1)$-bundle. This is due to the fact that $\mathrm{d}\mathcal{F}$ is no longer closed, which is forbidden by the Bianchi identity if it was an honest curvature form. Furthermore, we can no longer write $\mathcal{F} = \mathrm{d}\mathcal{A}$ even locally, at least where the support of $\mathfrak{I}$ is, since the Poincaré lemma no longer holds.

When Dirac faced this issue he came up with a solution in which one excises the region where the magnetic charge is defined, $\mathbb{R}^3 - \{0\}$ for the monopole case. In particular, we have a field configuration $\mathcal{F}$ which is closed everywhere except the brane, where it's trivialised by $\mathfrak{I}$. With this interpretation we can formalise the classification of electric charges by considering **relative cohomology**. Alternatively, if one wants to get the full global picture, the correct method is to introduce 'twisted' differential cohomology. Had Dirac gone this route, he may have discovered the generalisation of bundles to gerbes. In this chapter we'll present the former approach, if one wants to be precise we are restricting attention to 'ordinary differential cohomology' as opposed to the twisted variant described in [Fre00].

Definition 1.5.2. *Given a pair $\iota : S \hookrightarrow X$ we construct the **relative de Rham cohomology** as follows.*

1. *The cochains are defined as* $\Omega^k(X,S) \coloneqq \Omega^k(X) \oplus \Omega^{k-1}(S)$

2. *Equip the complex with a differential*

$$\begin{aligned} \mathrm{d} : \Omega^k(X,S) &\longrightarrow \Omega^{k+1}(X,S) \\ (\omega,\sigma) &\mapsto (\mathrm{d}\omega, \iota^\star\omega - \mathrm{d}\sigma). \end{aligned}$$

3. *The degree k relative cohomology of the pair (X,S) is then* $H^k_{\mathrm{dR}}(X,S) \coloneqq \ker \mathrm{d}_k / \operatorname{Im} \mathrm{d}_{k-1}$.

Where we used the same notation as for the de Rham differential, but hopefully this won't cause confusion. It can be seen that $\mathrm{d}^2 = 0$ since,

$$\mathrm{d}^2(\omega,\sigma) = (\mathrm{d}^2\omega, \iota^\star \mathrm{d}\omega - \mathrm{d}\iota^\star\omega + \mathrm{d}^2\sigma),$$

and remembering that the pullback commutes with the exterior derivative. Furthermore, a form being closed with respect to this new differential implies that $\mathrm{d}\omega = 0$, and $\iota^\star\omega - \mathrm{d}\sigma = 0 \implies \omega|_S = \mathrm{d}\sigma$. This is exactly the relation we want when the current is trivialised only a subspace.

Proposition 1.5.1. *There is a short exact sequence in cochains given by:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{k-1}(S) & \hookrightarrow & \Omega^k(X,S) & \twoheadrightarrow & \Omega^k(X) & \longrightarrow & 0 \\ & & \sigma & \longmapsto & (0,\sigma) & & & & \\ & & & & (\omega,\sigma) & \longmapsto & \omega & & \end{array}$$

From Homological algebra it's deduced that this induces a long exact sequence in cohomology.

$$\bullet \longrightarrow H^{k-1}(S) \longrightarrow H^k(X,S) \longrightarrow H^k(X) \longrightarrow H^k(S) \longrightarrow \bullet$$

When discussing p-branes with an electric current, the relevant subspace to consider is $\iota : M^{-\mathfrak{J}_e} \hookrightarrow M$, where $M^{-\mathfrak{J}_e} \coloneqq M - \operatorname{Supp}(\mathfrak{J}_e)$. In this case $(\mathfrak{J}_e, \star\mathfrak{F}) \in \Omega^{n-p+1}(M)$ is an exact form since,

$$\mathrm{d}(\star\mathfrak{F}, 0) = (\mathrm{d}\star\mathfrak{F}, \iota^\star\star\mathfrak{F} - 0) = (\mathfrak{J}_e, \star\mathfrak{F}|_{M^{-\mathfrak{J}_e}}).$$

On the other hand $(\mathfrak{J}_e, 0)$ is closed since $\mathrm{d}\mathfrak{J}_e = 0$, and it is trivialised by $\star\mathfrak{F}$ on $\operatorname{supp}(\mathfrak{J}_e)$. Hence, it defines a potentially non-trivial relative cohomology class. Using Prop 1.5.1 we obtain the following long exact sequence :

$$\bullet \longrightarrow H^{n-k}(M_n) \xrightarrow{\iota^\star} H^{n-k}(M_n^{-\mathfrak{J}_e}) \xrightarrow{\delta} H^{n-k+1}(M_n, M_n^{-\mathfrak{J}_e}) \xrightarrow{\psi} H^{n-k+1}(M_n) \longrightarrow \bullet$$

Definition 1.5.3 (Charge group)**.** *The electric charge group is defined from the above long exact sequence as* $Q_e \coloneqq \ker\psi$.

We know that $\ker\psi = \operatorname{Im}\delta$, and we can use the first isomorphism theorem to deduce that,

$$\operatorname{Im}\delta = H^{n-k}(M_n^{-\mathfrak{J}_e}) / \ker\delta,$$

but we can use exactness again on $\iota^\star$ giving

$$Q_e = H^{n-k}(M_n^{-\mathfrak{I}_e})/\iota^\star H^{n-k}(M_n). \tag{1.23}$$

For a general compact manifold M we expect that the charge vanishes since there is 'nowhere for the field lines to go'. Consider for example the cylinder, $S^1 \times \mathbb{R}$, and the torus $S^1 \times S^1$. Field lines on a torus can wrap around and self intersect, failing to be well-defined and therefore forcing the charge to vanish. For the cylinder on the other there is no issue since they can extend off to infinity. Furthermore, the relative cohomology of $M^{-\mathfrak{I}_e}$ is isomorphic to the compactly supported cohomology of M, which formalises the idea of charge being measured by 'flux at infinity'. A complete overview for the intuition behind this is given in [AO06].

1.6 Differential Cohomology

We can generalise these differential characters to arbitrary degrees. In degree two there is a natural interpretation as the classification of $U(1)$-bundles with connection since the field strength can be viewed as the curvature of a connection. For higher form gauge fields we lose this interpretation, but differential cohomology still classifies something 'bundle-like'.

Definition 1.6.1 (Differential Cohomology)**.**

1. *A degree k* ***Cheeger-Simons differential character*** *is a group homomorphism $\chi \in \mathrm{Hom}(Z_{k-1}(M), U(1))$ such that $\exists \mathcal{F} \in \Omega^k(M)$ satisfying*

$$\exists \mathcal{B}_k \in C_k(M) : \partial \mathcal{B}_k = \mathcal{W}_{k-1} \implies \chi(\mathcal{W}_{k-1}) = \exp i \int_{\mathcal{B}_k} \mathcal{F}.$$

2. *Then the* ***differential cohomology group*** *in degree k, $\check{H}^k(M)$, is the abelian group of degree k differential characters equipped with the usual multiplication.*

Point particle, or 0-brane, coupling is described by bundles with a one form $\mathcal{A}$-field as connection. 1-branes, say strings, couple to an analogous two form B-field associated to a so-called 'gerbe'. So the interpretation is that if differential cohomology classifies these high form Maxwell theories, then we expect that it also classifies these gerbes and higher generalisations. This is in fact true, and we'll touch on this in chapter 4, but an introductory yet rigorous account is given in [Par24].

Let's try to examine how this relates to the physics of gauge potentials. Suppose we have a vanishing field strength $\mathcal{F} \equiv 0$, and the corresponding character $\chi \in \mathrm{Hom}(Z_{k-1}, U(1))$ with two homologous cycles $\Sigma_{k-1}, \Sigma'_{k-1} \in Z_{k-1}$ such that $\exists \mathcal{B}_k \in C_k$ with $\partial \mathcal{B}_k = \Sigma_{k-1} - \Sigma'_{k-1}$. Then we have,

$$\chi(\Sigma_{k-1})/\chi(\Sigma'_{k-1}) = \exp i \int_{\mathcal{B}_2} \mathcal{F} = 1,$$

so χ only depends on the cohomology class of the chain. This means can restrict the map to $\chi \in \mathrm{Hom}(H_{k-1}(M), U(1))$. Now we can apply the universal coefficient theorem on cohomology,

$$\mathrm{Hom}(H_{k-1}(M), U(1)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(M), U(1)) \cong H^{k-1}(M; U(1)), \tag{1.24}$$

but $\mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(M), U(1)) = 0$ since $U(1)$ is divisible.

Definition 1.6.2. *A **flat character** satisfies $\mathcal{F} \equiv 0$. The group of flat characters is isomorphic to $H^{k-1}(M, \mathbb{R}/\mathbb{Z})$ where we write $\check{\phi}$ for the character corresponding $\phi \in H^{k-1}(M, \mathbb{R}/\mathbb{Z})$.*

In electromagnetism when there's a topologically trivial connection then we always have an $\mathcal{A} \in \Omega^1(M)$ such that $\mathcal{F} \equiv \mathrm{d}\mathcal{A}$ globally. Moving up to characters we say:

Definition 1.6.3. *A **topologically trivial character**, $\chi_{\mathcal{A}}$, for a globally defined $\mathcal{A} \in \Omega^{k-1}(M)$ has the form*

$$\chi_{\mathcal{A}}(\mathcal{W}_{k-1}) = \exp i \int_{\mathcal{W}_{k-1}} \mathcal{A}$$

In electromagnetism this means that if we have $\Omega^1(M) \ni \mathcal{A} \mapsto \mathcal{A} + \alpha = \mathcal{A}'$ for some $\alpha \in \Omega^1(M)$ then $\mathcal{F} = \mathrm{d}\mathcal{A} \implies \mathcal{F} = \mathrm{d}\mathcal{A}'$; a gauge redundancy. Likewise for two degree k topologically trivial characters $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{A}'}$ with $\mathcal{A} - \mathcal{A}' = \alpha \in \Omega^{k-1}(M)$ we have,

$$\chi_{\mathcal{A}}/\chi_{\mathcal{A}'}(\mathcal{W}_{k-1}) = \exp i \int_{\mathcal{W}_{k-1}} \alpha = 1 \implies \alpha \in \Omega^{k-1}_{\mathbb{Z}}(M),$$

so the gauge transformation must have integral periods. We can go the other way and suppose to start that $\alpha \in \Omega^{k-1}_{\mathbb{Z}}(M)$. Clearly by a similar argument the transformed $\mathcal{A}' = \mathcal{A} + \alpha$ will stay the same, so we can actually identify gauge transformations with those $(k-1)$-forms with integral periods. We could also consider the subgroup of $(k-1)$-forms with **vanishing** periods altogether. These forms are necessarily not only closed but also exact; $\alpha = \mathrm{d}\beta$ with $\beta \in \Omega^{k-2}(M)$.

Definition 1.6.4. *In degree k differential cohomology :*

1. *The group of **gauge transformations** is isomorphic to $\Omega^{k-1}_{\mathbb{Z}}(M)$.*
2. *The subgroup of **small** gauge transformations are those with vanishing periods.*

Since it's clear that the topologically trivial characters are unchanged by these gauge transformations we can say the following.

Remark 1.6.1. *The group of topologically trivial characters in degree k differential cohomology can be identified with $\Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M)$.*

1.6.1 Interpretation in Low Degrees

Consider the degree 2 case. The space of connections $\mathcal{A}(P)$ over a principal $U(1)$-bundle is an affine space for $\Omega^1(M)$. Recall that an affine space is to a vector space what a torsor is to a group. Picking a base point $\omega_0 \in \Omega^1(M)$ lets us write any other connection as $\omega = \omega_0 + i\alpha$, where $\alpha \in \Omega^1(M)$ is a globally well-defined one form. We want to consider the space of connections which are gauge inequivalent in $\mathcal{A}/\mathcal{G}$ to eliminate redundancy. If we use the fact that $U(1)$-bundles with connection are classified topologically by the first Chern class, then picking a base point amounts to picking a Chern class. As discussed previously we can write,

$$(\mathcal{A}/\mathcal{G})_x \cong \Omega^1/\Omega^1_{\mathbb{Z}}(M), \tag{1.25}$$

for $x = c_1(P) \in H^2(M;\mathbb{Z})$ the first Chern class. The isomorphism is not canonical since it required a choice of base point. Degree two cohomology classifies $U(1)$-bundles with connection, giving the following decomposition:

$$\check{H}^2(M) \cong \coprod_{x \in H^2(M,\mathbb{Z})} (\mathcal{A}/\mathcal{G})_x \cong \frac{\Omega^1(M)}{\Omega^1_{\mathbb{Z}}(M)} \oplus H^2(M;\mathbb{Z}) \tag{1.26}$$

Now also suppose M is a compact and choose a Riemannian metric. Then we can further refine this by using the Hodge decomposition,

$$\Omega^1(M) \cong \mathcal{H}^1 \oplus \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^\dagger, \tag{1.27}$$

where $\mathcal{H}^1$ is the space of harmonic forms.

Proposition 1.6.1. *Harmonic forms are closed :*

$$(\mathrm{d} + \mathrm{d}^\dagger)^2 \gamma := \triangle \gamma = 0 \implies \mathrm{d}\gamma = 0$$

Proof. Start by applying $\langle -, \gamma \rangle$ to both sides :

$$\triangle \gamma = 0 \implies \langle \triangle \gamma, \gamma \rangle = 0$$
$$\langle (\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d})\gamma, \gamma \rangle = \langle \mathrm{d}\mathrm{d}^\dagger \gamma, \gamma \rangle + \langle \mathrm{d}^\dagger \mathrm{d}\gamma, \gamma \rangle = 0$$

Now we use the adjoint property of the inner product to get,

$$\langle \mathrm{d}^\dagger \gamma, \mathrm{d}^\dagger \gamma \rangle + \langle \mathrm{d}\gamma, \mathrm{d}\gamma \rangle = 0$$
$$\implies ||\mathrm{d}^\dagger \gamma||^2 + ||\mathrm{d}\gamma||^2 = 0,$$

so we have both $\mathrm{d}\gamma = 0$ as desired and also $\mathrm{d}^\dagger \gamma = 0$. $\square$

We'd like to decompose the space of gauge transformations, namely the space of one forms with 2π-integral periods.

Proposition 1.6.2.

$$\Omega^k_{\mathbb{Z}}(M) \cong \mathcal{H}^k_{\mathbb{Z}} \oplus \operatorname{Im} \mathrm{d} \tag{1.28}$$

Proof. Start with the usual Hodge decomposition

$$\Omega^k(M) = \mathcal{H}^k \oplus \operatorname{Im} \mathrm{d} \oplus \operatorname{Im} \mathrm{d}^\dagger$$
$$\omega \in \Omega^k(M) \implies \omega = \alpha + \mathrm{d}\beta + \mathrm{d}^\dagger \gamma$$

Where $\alpha \in \mathcal{H}^k(M)$, $\beta \in \Omega^{k-1}(M)$, $\gamma \in \Omega^{k+1}(M)$. We know that gauge transformations are closed hence,

$$\Omega^k_{\mathbb{Z}}(M) \subseteq \ker d \implies \mathrm{d}\omega = \mathrm{d}\alpha + \mathrm{d}^2\beta + \mathrm{d} \circ \mathrm{d}^\dagger \gamma = 0.$$

Furthermore, $\mathrm{d}^2 = 0$ and we know from Pop.1.6.1 that $\mathrm{d}\alpha = 0$ since harmonic forms are closed. Now just apply $\langle -, \gamma \rangle$,

$$\mathrm{d} \circ \mathrm{d}^\dagger \gamma = 0$$
$$\implies \langle \mathrm{d} \circ \mathrm{d}^\dagger \gamma, \gamma \rangle = 0$$
$$\implies \langle \mathrm{d}^\dagger \gamma, \mathrm{d}^\dagger \gamma \rangle = 0,$$

which by non-degeneracy implies $\mathrm{d}^\dagger \gamma = 0$ giving the desired result. $\square$

Now we can decompose the differential cohomology even further, in particular we have

$$\check{H}^2(M) \cong \frac{\mathcal{H}^1(M)}{\mathcal{H}^1_{\mathbb{Z}}(M)} \oplus \operatorname{Im} \mathrm{d}^\dagger \oplus H^2(M;\mathbb{Z}). \tag{1.29}$$

This looks even worse, but it can be massaged slightly. Note that by Hodge de Rham theory we have,

$$\mathcal{H}^1(M) \cong H^1_{\mathrm{dR}}(M) \cong \mathbb{R}^{b_1}, \tag{1.30}$$

which leaves us finally with :

$$\check{H}^2(M) \cong U(1)^{b_1} \oplus H^2(M;\mathbb{Z}) \oplus \operatorname{Im} \mathrm{d}^\dagger \tag{1.31}$$

So differential cohomology in dimension two consists of the following components:

1. The space of topologically trivial flat Wilson lines is the torus $U(1)^{b_1}$.
2. The topological sector is specified by $H^2(M;\mathbb{Z})$, which is finitely generated if M is compact.
3. The infinite dimensional vector space $\operatorname{Im} \mathrm{d}^\dagger$ of oscillator modes.

It's crucial to note that $\mathcal{F} = 0$ does not imply that the character is topologically trivial. It is in this sense that there is inherently more gauge invariant information encoded in $\check{H}^\bullet(M)$ than in just the field strength. There are configurations where $\mathcal{F} = 0$ but the holonomy of the connection is non-zero, for example the Aharonov-Bohm effect. Recall the space of flat characters is specified by $H^1(M;\mathbb{R}/\mathbb{Z})$. The torus $U(1)^{b_1}$ is the connected component of the identity of this group, i.e. the space of Wilson lines for a topologically trivial flat field. Note also that writing this decomposition is equivalent to gauge fixing. Specifically,

$$\alpha \in \operatorname{Im} \mathrm{d}^\dagger \implies \mathrm{d}^\dagger \alpha = \partial^\mu \alpha_\mu = 0,$$

which is the **Landau gauge**. The above decomposition actually holds for arbitrary degrees, as in swapping $2 \leftrightarrow k$ and $b_1 \leftrightarrow b_{k-1}$. In lower degrees we have immediate interpretations. It's clear that $\check{H}^0(\mathrm{pt}) \cong \mathbb{Z}$ since all the other groups vanish apart from the characteristic class $H^0(\mathrm{pt};\mathbb{Z})$. Likewise, $\check{H}^1(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}$ is the topologically trivial flat fields. For any M we also have the following:

Proposition 1.6.3. *A degree one Cheeger-Simons character, $\chi_f \in \check{H}^1(M)$, is a specified by a $U(1)$-valued function, f, with field strength given by*

$$\mathcal{F} = -i f^{-1} \mathrm{d} f.$$

In other words we have $\check{H}^1(M) \cong C^\infty(M, U(1))$.

Proof. A degree 0 cycle is just an integral sum of points $\mathcal{W}_0 = \sum_i n_i p_i : n \in \mathbb{Z}$. Define the character on a chain as,

$$\chi_f(\mathcal{W}_0) := \prod_i f(p_i)^{n_i}.$$

Now consider a 1-cycle which is exact, so $Z_1(M) \ni \gamma : \partial\gamma = p - q$. Then by definition we have,

$$\chi_f(\partial\gamma) = f(p)f(q)^{-1} = \exp i \int_\gamma \mathcal{F}.$$

Also note that $\mathrm{d}\log(f) = f^{-1}\mathrm{d}f$ and

$$\int_\gamma \mathrm{d}\log(f) = \int_{\partial\gamma} \log(f) = \log(f(p)) - \log(f(q)).$$

Then we can exponentiate to obtain,

$$\exp\int_\gamma \mathrm{d}\log(f) = \exp i \int_\gamma -if^{-1}\mathrm{d}f = f(p)f(q)^{-1} = \chi_f(\partial\gamma),$$

from which we conclude that the field strength is of the form $\mathcal{F}(\chi_f) = -if^{-1}\mathrm{d}f \in \Omega^1_{\mathbb{Z}}(M)$. $\square$

The above classification corresponds to a periodic scalar field in physics, which is often written as a local logarithm $f = e^{i\varphi}$. Moving up two degrees, in type II string theory there is a two form B-field which couples to the worldsheets of strings. This does indeed define a class in $\check{H}^3(M)$ as the step up from particles. To fully explore this one has to introduce 'gerbes' which are a higher generalisation of bundles classified by the theory of differential cohomology. For higher degrees of k we can conjecture that this classifies degree k generalised Maxwell fields for branes. This is a good idea, but it turns out to be wrong. For example in type II string theory there are so-called Ramond-Ramond fields that couple to branes. It's conjectured [MS06] that the correct setup for describing these fields should be twisted K-theory. That being said, this formalism is still extremely useful with wide applications in Physics. We haven't scratched the surface here, consult [MS25] and references therein for a detailed account.

1.7 The Dancing Hexagon

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \Omega^{\bullet-1}/\Omega^{\bullet-1}_{\mathbb{Z}}(M) \arrow[rr, "\mathrm{d}"] \arrow[dr] & & \Omega^{\bullet}_{\mathbb{Z}}(M) \arrow[ur] \arrow[dr, "{[-/2\pi]}"] & \\
H^{\bullet-1}(M;\mathbb{R}) \arrow[ur, "\mathrm{rep}"] \arrow[dr, "\sim"'] & & \check{H}^{\bullet}(M) \arrow[ur, "\mathcal{F}(-)"] \arrow[dr, "c"] & & H^{\bullet}(M;\mathbb{R}) \\
& H^{\bullet-1}(M;U(1)) \arrow[rr, "\beta"'] \arrow[ur] & & H^{\bullet}(M;\mathbb{Z}) \arrow[ur, "(-)\otimes\mathbb{R}"'] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

There is a diagram of intertwined exact sequences given above which characterises differential cohomology. Specifically the diagonals are exact sequences as well as the upper and lower edges. Furthermore, the interior squares and triangles are all commuting. Apart from looking pretty, it is useful for computing the differential cohomology based on information on the ordinary cohomologies and differential forms on the space. On top of this, any two such differential cohomology theories which 'fit in the middle' of this diagram are naturally isomorphic, it is in this sense that the diagram characterises these theories. It is often referred to as the 'Dancing Hexagon' since Jim Simons helped choreograph a dance about it, and the diagram was first discovered by him and Sullivan in [SS07]. In this section we'll give a sketch of the proof that Cheeger-Simons characters satisfy the hexagon. For a full technical proof see [Par24]. First let's have a look at the right-hand side of the exact sequence going north-east.

$$\check{H}^{\bullet}(M) \xrightarrow{\mathcal{F}} \Omega^{\bullet}_{\mathbb{Z}}(M) \longrightarrow 0 \tag{1.32}$$

The field strength map, $\mathcal{F} : \check{H}^{\bullet}(M) \to \Omega^{\bullet}_{\mathbb{Z}}(M)$, is simply defined by associating the differential character χ to the 2-form in its definition. This map can be shown to be surjective [Par24] and so 1.32 is indeed exact. Furthermore, we claim the kernel of this map is given be $H^{\bullet-1}(M;U(1)))$. This was shown already in the previous section 1.6, and lead to 1.32. In particular those characters with vanishing field strength are maps $H_{k-1}(M) \to U(1)$, but the universal coefficient theorem and the divisibility of $U(1)$ give the isomorphism. To summarise, this shows exactness of the upward right pointing sequence :

$$
\begin{tikzcd}
& (\chi, \mathcal{F}) \arrow[r, mapsto] & \mathcal{F} & \\
0 \arrow[r] & H^{\bullet-1}(M; U(1)) \arrow[r, hook] & \check{H}^{\bullet}(M) \arrow[r, two heads] & \Omega^{\bullet}_{\mathbb{Z}}(M) \arrow[r] & 0 \\
& \phi \arrow[r, mapsto] & (-\check{\phi}, 0) &
\end{tikzcd}
$$

Note that we there's an awkward minus sign in the definition of the injective map, this is necessary later to commute the bottom triangle. Also from here on we may write the characters as pairs, $(\chi, \mathcal{F})$, which is standard notation. Next we need a homomorphism β that goes along the bottom of the hexagon, this is defined as the Bockstein map.

$$
\begin{aligned}
\beta : H^{\bullet-1}(M; \mathbb{R}/\mathbb{Z}) &\longrightarrow H^{\bullet}(M; \mathbb{Z}) \\
[c] &\longmapsto [\mathrm{d}c]
\end{aligned}
$$

One can check this is well-defined as follows. Consider a subcomplex $A \subset B$ and the quotient B/A. We can define a differential on the quotient by $\mathrm{d}(b + A) = \mathrm{d}b$. Suppose $b + A \in \ker \mathrm{d}$ then we have $\mathrm{d}b \in A$, with $\mathrm{d}b = s$ not necessarily 0. Suppose $[b] = [b']$ with $s' = \mathrm{d}b$ then $b - b' = t \in A \implies \mathrm{d}(s - s') = \mathrm{d}t \in A$, so $[s]$ is well-defined in the cohomology of A. This is the Bockstein map $\beta : H^k(B/A) \ni [b] \to [\mathrm{d}b] \in H^{k+1}(A)$. Now just take $A = C^{\bullet}(M) \otimes \mathbb{Z}$ and $B = C^{\bullet}(M) \otimes \mathbb{R}$ so that $B/A = C^{\bullet}(M) \otimes \mathbb{R}/\mathbb{Z}$. From homological algebra one can show that this is the connecting homomorphism for a long exact sequence in cohomology arising from a short exact sequence of coefficients, $\mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}$. In particular, it follows that the truncated sequence below is exact:

$$
H^{\bullet-1}(M; \mathbb{R}) \xrightarrow{\ \sim\ } \check{H}^{\bullet-1}(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{\ \beta\ } H^{\bullet}(M; \mathbb{Z}) \xrightarrow{\ \otimes\mathbb{R}\ } H^{\bullet}(M; \mathbb{R})
$$

The outer two maps in the above are actually isomorphisms see [Spa81]. By de Rham's theorem we also know that $H^{\bullet}(M; \mathbb{R}) \cong H^{\bullet}_{\mathrm{dR}}(M)$. Looking at the downward diagonal exact sequence we need to define a map which is called topological class, in degree two it's the first Chern class of the bundle. In general, it takes the differential character and maps it to an integer cohomology class $\chi \mapsto c(\chi) \in H^{\bullet}(M; \mathbb{Z})$. When considering degree two, it is a measure of how topologically non-trivial the bundle is, and is in general a surjection. In particular, if $c \neq 0$ then it's not possible to write $\chi = \chi_{\mathcal{A}}$ for some global one form $\mathcal{A} \in \Omega^{\bullet-1}(M)$. Furthermore, from the discussion in 1.6 the kernel of this map is exactly the topologically trivial characters, in other words the downwards diagonal sequence is exact :

$$
\begin{tikzcd}
& (\chi, \mathcal{F}) \arrow[r, mapsto] & c(\chi) & \\
0 \arrow[r] & \Omega^{\bullet-1}/\Omega^{\bullet-1}_{\mathbb{Z}}(M) \arrow[r, hook] & \check{H}^{\bullet}(M) \arrow[r, two heads] & H^{\bullet}(M; \mathbb{Z}) \arrow[r] & 0 \\
& \mathcal{A} \arrow[r, mapsto] & (\chi_{\mathcal{A}}, \mathrm{d}\mathcal{A}) &
\end{tikzcd}
$$

Note that the commutativity of the right square just says that $[\mathcal{F}/2\pi] = [c(\chi)]$. In degree two this is clear by definition of the first Chern class, for higher degrees consult [Par24]. It turns out that the topological class of a character has the form,

$$c(\chi) = -\beta(\chi), \tag{1.33}$$

which does seem to come out of nowhere. We discuss it slightly in chapter 4 as a consequence of Prop 4.2.4. Using this result along with the minus sign involved in the map $\phi \mapsto -\check{\phi}$ gives commutativity of the lower triangle. Next we consider the square on the left. Define the representative map as :

$$\begin{aligned} H^{\bullet-1}(M;\mathbb{R}) &\xrightarrow{\text{rep}} \Omega^{\bullet-1}/\Omega^{\bullet-1}_{\mathbb{Z}}(M) \\ [\mathcal{A}] &\longmapsto \mathcal{A} + \Omega^{\bullet-1}_{\mathbb{Z}}(M) \end{aligned}$$

The map is well-defined since any exact form is also closed. One can then check that the left square commutes. It's clear that the upper triangle commutes since $\mathcal{F} = \mathrm{d}\mathcal{A}$ for a topologically trivial character. The upper sequence can also be shown to be exact by similar arguments in traditional de Rham cohomology.

1.8 The Čech Model

When discovering cohomology for the first time we usually construct the singular cohomology as a dual to the singular homology. Then we find there are often other ways to construct cohomology theories which satisfy the same properties necessary and therefore give the same invariants, namely the Eilenberg-Steenrod axioms. Some of these other constructions may have advantages (like being easier to compute) and disadvantages (like being harder to conceptualise), and are therefore worth studying.

Similarly, in differential cohomology we can construct specific models which have all the necessary properties, yet serve different purposes. We'll focus on the Čech model since it is most fitting in the physics context. The idea is to consider the fields defined on open covers of the underlying manifold. Then the description of differential cohomology is analogous to the Čech cohomology of sheaves, defined in Chapter 4. We'll assume we have a good/Leray open cover $\{U_\alpha\}$. This means that,

$$U_{\alpha_1\ldots\alpha_l} \coloneqq U_{\alpha_1} \cap \ldots \cap U_{\alpha_l},$$

is contractible for all l. It turns out that such a cover always exists on manifolds, so this isn't such a big constraint. Consider a globally well-defined k-form $\mathcal{F} \in \Omega^k(M;\mathbb{R})$. Inclusions of the open sets into the cover induce restrictions on the field. On the open cover itself for example we define,

$$\mathcal{F}_\alpha \coloneqq \mathcal{F}|_{U_\alpha} = \mathrm{d}\mathcal{A}^{[k-1]}_\alpha \quad \mathcal{A} \in \Omega^{k-1}(U_\alpha).$$

On intersections $U_{\alpha\beta}$ we require that $\mathcal{F}_\alpha = \mathcal{F}_\beta$ so we get,

$$\mathrm{d}(\mathcal{A}^{[k-1]}_\alpha - \mathcal{A}^{[k-1]}_\beta) = 0 \implies \mathcal{A}^{[k-1]}_\alpha - \mathcal{A}^{[k-1]}_\beta = \mathrm{d}\mathcal{A}^{[k-2]}_{\alpha\beta}.$$

Here we've used the fact that intersections are contractible in a Leray cover along with the Poincaré lemma to guarantee the existence of $\mathcal{A}^{[k-2]}_{\alpha\beta}$. One can iterate this process until completion to get a 'tower of fields' :

$$\{\mathcal{F}_{\alpha_1}, \mathcal{A}^{[k-1]}_{\alpha_1}, \mathcal{A}^{[k-2]}_{\alpha_1\alpha_2} ... \mathcal{A}^{[0]}_{\alpha_1...\alpha_l}, n_{\alpha_1...\alpha_l}\} \tag{1.34}$$

When the process terminates we arrive at a Čech cocycle condition $n_{\alpha_1...\alpha_l} = \text{const}$. One must impose the condition that this constant is integral to get a cocycle in differential cohomology, the corresponding class in integral cohomology is the topological class of the differential character. One issue with this formulation is there a lot of gauge redundancy, at each level we can transform by a sequence of forms $(\lambda^{[k-1]}_{\alpha}, \lambda^{[k-2]}_{\alpha\beta}, \ldots)$ such that,

$$\begin{aligned}
&\mathcal{A}^{[k-1]}_{\alpha} \longrightarrow \mathcal{A}^{[k-1]}_{\alpha} + \mathrm{d}\lambda^{[k-1]}_{\alpha} \\
&\mathcal{A}^{[k-2]}_{\alpha\beta} \longrightarrow \mathcal{A}^{[k-2]}_{\alpha\beta} + (\lambda^{[k-1]}_{\alpha} - \lambda^{[k-1]}_{\beta}) + \mathrm{d}\lambda^{[k-2]}_{\alpha\beta},
\end{aligned}$$

and so on. One advantage of this model is that it's concrete with an immediate definition for the topological class. We can also work locally with spacetime differential forms, as opposed to abstract cochains. There are other issues, however, with trying to find out where the global nature/holonomy enters the picture for example. As mentioned earlier there are various other models which complement this one, which won't be discussed here for the moment.

Chapter 2

Topological QFT

Quantum field theory as a subject has been incredibly elusive. In particular finding a well-defined mathematical structure that axiomatises and encapsulates all the physics of field theory is open problem. One may try to use the traditional path integral approach, but it seems this poses issues. For example, it is known that there exist field theories which have different Lagrangian yet result in the same observables, even worse there exist theories one can write which actually have no Lagrangian description such as the $(2, 0)-$theory. It is the view of some mathematicians and physicists that a better framework is needed for field theory which can capture all this kind of information. One approach is that of 'Functorial QFT', and topological QFT is a toy model that originated this direction of thinking. This chapter will be based on the first lecture given by Greg W. Moore at TASI 2023 and some material from part 1 of his notes [MS25].

2.1 Bordism

In a certain sense, a lot of fundamental physics aims to predict the future. One often finds themselves with initial data and aims to describe how this data propagates into some final state over time. In classical mechanics this done by solving some ordinary differential equation with initial conditions, or in the classical field theory setting one has partial differential equations with boundary conditions. Even in quantum mechanics the goal is the same, even though we can't say how things change in time with certainty we can still prescribe 'probability amplitudes' to processes to give the likelihood of some final state occurring. This can be done using the path integral which can even be generalised to the case of quantum field theory.

This path integral can be a tricky thing however, being extremely difficult to compute or sometimes failing to be well-defined. On top of this the traditional Lagrangian approach seems to be incomplete in certain settings, failing to a capture all of the physics. The motivation for topological quantum field theory is to abstract away the finicky computational aspect and try to deduce information from properties that the **result** of the path integral ought to have. We start by asserting that physics takes place on a smooth n-dimensional manifold which represents spacetime. Our initial data is then defined over an $(n-1)$-dimensional spatial slice, and likewise the final data should live on an $(n-1)$-dimensional space. We won't impose that these initial and final spaces are connected however, but we do require that these spaces arise as the common boundary of a connecting/interpolating manifold between them.

Finally, we need to label these boundaries in terms of whether they are the initial (ingoing) or final (outgoing) spaces. This can be achieved by using collar neighbourhoods. Think of an initial manifold of a circle S^1 which traces out a tube like shape when propagating. We can think of $\{S^1 \times [0,\epsilon) \quad : \epsilon > 0 \in \mathbb{R}\}$ as an infinitesimal collar which we wrap around the spacetime such that $S^1 \times \{0\}$ is identified with the initial space, hence the name collar neighbourhood. Likewise, a similar construction will be needed for outgoing spaces.

Definition 2.1.1 (Bordism)**.** *An n-dimensional **bordism from** N_0 to N_1 of closed manifolds is a triple (θ_0, M, θ_1) such that :*

1. *M is a manifold with boundary and $\partial M = (\partial M)_0 \sqcup (\partial M)_1$*
2. *$\theta_0 : N_0 \times [0,\epsilon) \longrightarrow M$ an embedding such that $\theta_0 : N_0 \times \{0\} \longrightarrow (\partial M)_0$ is a diffeomorphism.*
3. *$\theta_1 : N_1 \times (1-\epsilon, 1] \longrightarrow M$ an embedding such that $\theta_1 : N_1 \times \{1\} \longrightarrow (\partial M)_1$ is a diffeomorphism.*

A closed manifold means compact and without boundary, which is a restrictive condition. The full treatment of the non-compact case will complicate matters and so will not be addressed in this introduction. We can also take the disjoint union of bordisms in the obvious way as simply the disjoint union of the underlying manifolds. We can also consider gluing two bordisms (N_0, M, N_1) and (N_0', M', N_1') when they have common boundary, $N_1 = N_0'$ for example. Note that we impose that two bordisms are the same if there is a diffeomorphism between them which preserves the ingoing and outgoing labels. Formally,

Definition 2.1.2. *A **diffeomorphism between bordisms***

$$\begin{tikzcd}
N_0 \arrow[r, bend left=50, "{(\theta_0,M,\theta_1)}"{name=U}] \arrow[r, bend right=50, "{(\theta_0',M',\theta_1')}"'{name=D}] & N_1 \arrow[from=U, to=D, "\psi"]
\end{tikzcd}$$

is a diffeomorphism $\psi : M \longrightarrow M'$ such that the following diagrams commute.

$$\begin{tikzcd}
 & M \arrow[dd, "\psi"] \\
N_0 \times [0,\epsilon) \arrow[ur, "\theta_0"] \arrow[dr, "\theta_0'"'] & \\
 & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[dd, "\psi"'] & \\
 & (1-\epsilon, 1] \times N_1 \arrow[ul, "\theta_1"'] \arrow[dl, "\theta_1'"] \\
M' &
\end{tikzcd}$$

It's here where the topological part of the QFT comes in, there's no metric or distance so we can deform these bordisms like rubber. Furthermore, we can take disjoint unions of initial/final spaces $N \coprod N'$ where $N \coprod \varnothing$ and $N \coprod N' \cong N' \coprod N$. All of this structure combined forms a symmetric monoidal/tensor category, the full definition of which will not be treated here, see [Mac71] for details.

Definition 2.1.3. *The symmetric monoidal category* $\mathrm{Bord}_{\langle n-1,n\rangle}$ *has objects as closed* $(n-1)$*-manifolds and morphisms as bordisms between them. The composition of morphisms is the gluing of bordisms, and the identity for a manifold* N *is* $N \times [0,1]$*. The monoidal product is the disjoint union* $-\coprod-$ *and the monoidal unit is the empty manifold* $\varnothing$*.*

Now we have the space we're working over, but we have yet to define fields. We'd like a rule which assigns to each object in $\mathrm{Bord}_{\langle n-1,n\rangle}$ a 'vector space of states' which describes the configuration of the system at that point in time. Note that we use the word 'state' somewhat inaccurately. In quantum mechanics we sometimes say that a state is an element of the Hilbert $\psi \in H$. This is refined when we remember that we can scale the state by any non-zero complex number $\lambda \in \mathbb{C} - \{0\}$ without changing the observable from Born's rule. We ought to make the identification $\psi \sim \lambda\psi$ so that the space of states is actually $H/\sim = \mathbb{P}H$ the projective Hilbert space. This is further complicated when we consider combined systems where we have entangled states, the proper definition would be in terms of density matrices. Regardless, we may abusively refer to this assigned vector space as the state space anyway. Recall that we consider vector spaces as a category as well with the obvious composition and identities.

Definition 2.1.4. *The category* $\mathrm{Vect}_{\mathbb{C}}$ *has vector spaces over* $\mathbb{C}$ *as objects and* $\mathbb{C}$*-linear maps between them as morphisms. It is a symmetric monoidal/tensor category when equipped with the tensor product* $-\otimes-$ *and* $\mathbb{C}$ *as the monoidal unit.*

Here we chose the field $\mathbb{C}$, but the discussion holds in general. Given a bordism between two manifolds, thought of as initial final spaces, we'd like to further assign a linear map between the associated initial and final vector spaces. This is what can be thought of as the time evolution obtained from doing the path integral. It also makes sense that if we evaluate on one bordism and then another, the result should be the same as evaluating on the single glued bordism. This captures the 'locality' of the path integral. Finally remember that the bordism category has a product, the disjoint union. We need the assignment to be compatible with this product, so the evaluation on disjoint unions gets mapped to the tensor product of the evaluations on each component. This is somewhat unnatural since it might seem to make more sense to use the direct sum of vector spaces. The reason is that in quantum theory dictates that when there are two Hilbert spaces H_1 and H_2, the combined Hilbert space ought to be $H_1 \otimes H_2$. This is the natural way to describe entanglement for example. All of the above can be summarised neatly in the following definition due to Atiyah [Ati88].

Definition 2.1.5. *An n-dimensional* ***Topological Quantum Field Theory*** *is a monoidal/tensor functor,* Z*, from the category of bordisms,* $\mathrm{Bord}_{\langle n,n-1\rangle}$*, to the category,* $\mathrm{Vect}_{\mathbb{C}}$*, of vector spaces. Explicitly we have the following :*

1. *Manifolds of codimension one are assigned vector spaces :*

$$Z : \mathrm{ObBord}_{\langle n,n-1\rangle} \ni N \longmapsto Z(N) \in \mathrm{ObVect}_{\mathbb{C}}$$

2. *Bordisms between manifolds are assigned linear maps between the associated vector spaces:*

$$[M : N_0 \to N_1] \longmapsto [Z(M) : Z(N_0) \longrightarrow Z(N_1)]$$

3. *The assignment is functorial:*

$$Z(\mathrm{id}_N) = \mathrm{id}_{Z(N)} \quad \& \quad Z(M' \circ M) = Z(M') \circ Z(M)$$

4. The functor is monoidal:

$$Z(\varnothing) \cong \mathbb{C} \quad \& \quad Z(N \coprod N') \cong Z(N) \otimes Z(N')$$

Not that the full definition of a monoidal categories and functors between them require coherence conditions on the products. We do not list them here, see [Mac71]for details. This definition is also slightly different from Atiyah's original description, but the idea is the same. These are referred to as the Atiyah-Segal axioms, due to the joint contribution.

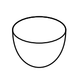
2.2 2D TQFT

At a first glance these axioms seem fairly trivial. They don't have any information about spin structures or principal bundles, and even lack a metric. Surprisingly however, a lot of information can be extracted even in low dimensional cases. Consider the category of two-dimensional bordisms. In the description in the last section we made no mention of orientation, hence we could talk about unoriented bordisms as well. Here we'll assume that all manifolds are orientable, in particular the bordisms themselves will have an orientation as well as the incoming and outgoing states. Also, we will try to reserve N for $n-1$-manifolds and M for n-manifolds. We make the distinction between incoming/outgoing based on whether their orientation agrees/disagrees with the bordism. In this case the objects are closed one dimensional manifolds, but there are only two connected ones up to diffeomorphism, namely the empty manifold and S^1. So any object in is just the disjoint union of finitely many circles and empty manifolds.

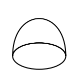
Now the morphisms between these objects are two-dimensional surfaces such that circles are the incoming and outgoing boundaries. It turns out that any such bordism can be described in terms of four elementary maps.

Proposition 2.2.1. *Any two-dimensional bordism can be formed from the composition and disjoint unions of the following:*

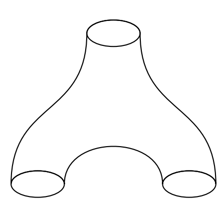
1. *Cup* $\quad \eta : \varnothing \longrightarrow S^1$



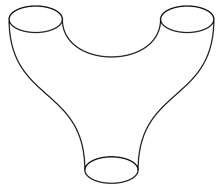
2. *Cap* $\quad \epsilon : S^1 \longrightarrow \varnothing$



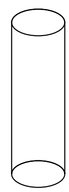
3. *Pair of pants* $\quad \mu : S^1 \coprod S^1 \longrightarrow S^1$



4. *Co-pair of pants* $\delta : S^1 \longrightarrow S^1 \coprod S^1$



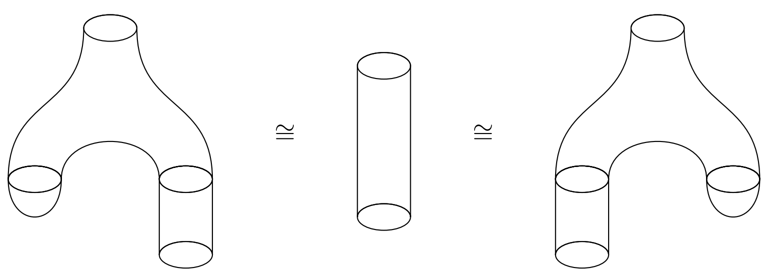
5. *Cylinder* $\mathrm{id}_{S^1} : S^1 \longrightarrow S^1$



Note that we've omitted the boundary data which specifies the incoming and outgoing states. From here on we will make sure incoming states are on the bottom and outgoing states are above. When thinking in terms of diagrams, the above proposition seems fairly clear. It was known to physicists as folk theorem for a while. This means that we can cut up any bordism into more basic components and compute the QFT on these segments. The full amplitude can then be found by just composing the individual linear maps, thanks to functoriality. It's natural to wonder if the choice of cutting makes a difference in the final answer, ideally it shouldn't if we want a consistent theory. This is called the sewing theorem which was proved formally, and more generally, using Morse theory in [MS06].

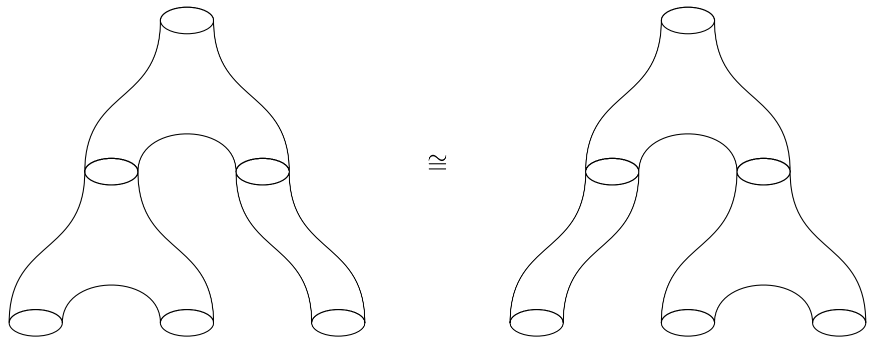
Proposition 2.2.2. *A 2D TQFT is specified by vector space $\mathcal{H} = Z(S^1)$ and linear maps $Z(\mu)$, $Z(\delta)$, $Z(\eta)$, $Z(\epsilon)$, which are called the (co)multiplication and (co)unit maps. They satisfy the following properties:*

1. *$Z(\eta) : \mathbb{C} \longrightarrow \mathcal{H}$ is a unit:*



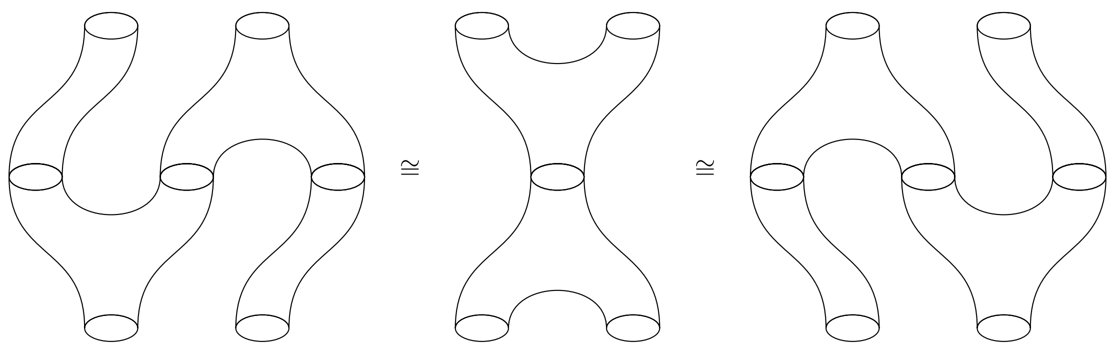
$$Z(\mu) \circ (Z(\eta) \otimes \mathrm{id}_{\mathcal{H}}) \quad \cong \quad \mathrm{id}_{\mathcal{H}} \quad \cong \quad Z(\mu) \circ (\mathrm{id}_{\mathcal{H}} \otimes Z(\eta))$$

2. *The multiplication $Z(\mu) : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$ is associative :*



$Z(\mu) \circ (Z(\mu) \otimes \mathrm{id}_{\mathcal{H}}) \qquad \cong \qquad Z(\mu) \circ (\mathrm{id}_{\mathcal{H}} \otimes Z(\mu))$

The above digrams flipped upside down hold for the counit $Z(\epsilon) : \mathcal{H} \longrightarrow \mathbb{C}$ and the comultiplication $Z(\delta) : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ too, though we haven't included them here. Lastly these maps are compatible in that they also satisfy:

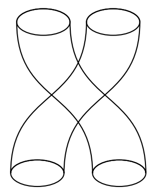


$(\mathrm{id}_{\mathcal{H}} \otimes Z(\mu)) \circ (Z(\delta) \otimes \mathrm{id}_{\mathcal{H}}) \qquad \cong \qquad Z(\delta) \circ Z(\mu) \qquad \cong \qquad (Z(\mu) \otimes \mathrm{id}_{\mathcal{H}}) \circ (\mathrm{id}_{\mathcal{H}} \otimes Z(\delta)$

The interpretation of maps $\mathbb{C} \longrightarrow Z(N)$ is that they pick out a state in $\mathcal{H}$, in this case it picks an element which behaves like a unit with respect to the multiplication. We can identify it with the image of $1 \in \mathbb{C}$, explicitly we write $Z(\eta) : \mathbb{C} \ni \alpha \mapsto 1_{\mathcal{H}}\alpha \in \mathcal{H}$. Again remember we're using the word state somewhat loosely here, traditionally we don't have a way to multiply states and hence think of 'units' with respect to a product. In fact, usually multiplication is defined for **operators** on Hilbert spaces. This alludes to a duality called the 'state-operator correspondence' which can be made explicit when moving to **conformal** field theories. In this case we introduce a metric but quotient by conformal transformations, as opposed to plain diffeomorphisms. One key observation is that the punctured complex plane, $\mathbb{C} - \{0\}$, is actually conformal equivalent to a cylinder, for example. In that case 'inserting an operator' at the origin is the same as producing a state. Anyway, the full story is best discussed in detail elsewhere.

Likewise, the counit $Z(\epsilon)$ takes a state and returns a complex number, think of this as the trace $T(\psi) = Z(\epsilon)(\psi), \quad \psi \in \mathcal{H}$. We also have a bilinear map on pairs states that we can think of as multiplication, $Z(\mu)$. Note that we can interchange the two incoming circles by a diffeomorphism action, so this multiplication is actually commutative. If we're being proper we should introduce the 'twist map'.

Remark 2.2.1. *The **twist map** $\theta : Z(N_1) \otimes Z(N_2) \longrightarrow Z(N_2) \otimes Z(N_1)$ is defined by the TQFT evaluated on the following bordism:*



It is defined by $\theta : a \otimes b \mapsto b \otimes a$, and is trivial in the two-dimensional case since there's only one non-trivial state space apart from $\mathbb{C}$. In particular θ is an isomorphism and $Z(\mu) \circ \theta = Z(\mu)$.

Furthermore, from looking at the topology of the associated bordism we can deduce that the unit map indeed behaves like a unit for the multiplication. Associativity can also be seen by simply looking at the appropriate bordsim and stretching/squeezing it to take the right shape. As long as we use diffeomorphism actions, and take care of keeping track of incoming/outgoing states, the manipulations are fairly trivial. Lastly the dual and counit/comultiplication maps clearly obey similar identities. There is also a final compatibility condition that is also satisfied which gives a special algebraic structure to these maps.

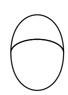
By evaluating the TQFT on various bordisms we can extract diffeomorphism invariants of the underlying manifold. For example, we can treat a closed 2-dimensional manifold as a bordism from the empty to set to the empty set. Two such bordisms are diffeomorphic if and only if they are diffeomorphic as closed manifolds, since there is no collar neighbourhood conditions to satisfy. It's clear that the TQFT applied to one such manifold, M, is a topological invariant of the manifold by functoriality; explicitly isomorphisms get mapped to isomorphisms. Moreover, we know that,

$$Z(\varnothing) = \mathbb{C} \implies Z(M) : \mathbb{C} \longrightarrow \mathbb{C},$$

and that $\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{C}}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, so the linear map in this case is just a complex number. By analogy with physics we call this the partition function. Traditionally the meaning of the partition function in QFT is ambiguous, but here it gains a clean geometric interpretation.

Definition 2.2.1. *The **partition function** of a closed n-manifold, M, is the complex number $Z(M) \in \mathbb{C}$ where we treat the M as a bordism between empty sets.*

Remark 2.2.2. *As an example consider the two sphere S^2 in 2D TQFT :*



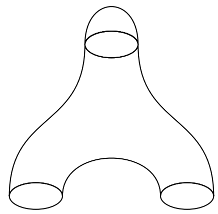
Using functoriality and cutting the sphere along the equator we have $Z(S^2) = Z(\epsilon \circ \eta) = Z(\epsilon) \circ Z(\eta)$. As discussed earlier we know that $Z(\eta)(1) = 1_{\mathcal{H}}$ defines a unit and $Z(\epsilon) = T$ is a map into $\mathbb{C}$, so we have $T(1_{\mathcal{H}}) \in \mathbb{C}$.

Another bordism of significant importance is the pair of pants. Applying the TQFT to it gives a multiplication operation $Z(\mu)(a \otimes b) := a \cdot b \in \mathcal{H}$ on $\mathcal{H} \otimes \mathcal{H}$. We can consider composing the pair of paints with the cap bordism and then applying the TQFT,

$$Z(\epsilon \circ \mu)(a \otimes b) = Z(\epsilon) \circ Z(\mu)(a \otimes b) = T(a \cdot b), \tag{2.1}$$

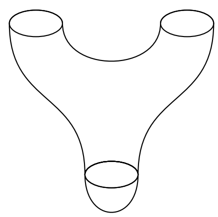
where we have used the earlier definition of the linear map $T : \mathcal{H} \longrightarrow \mathbb{C}$

Remark 2.2.3. *The operation $Z(\epsilon \circ \mu) := \beta$ defines a symmetric bilinear form $\beta : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathbb{C}$. This is the TQFT evaluated on the bordism below.*



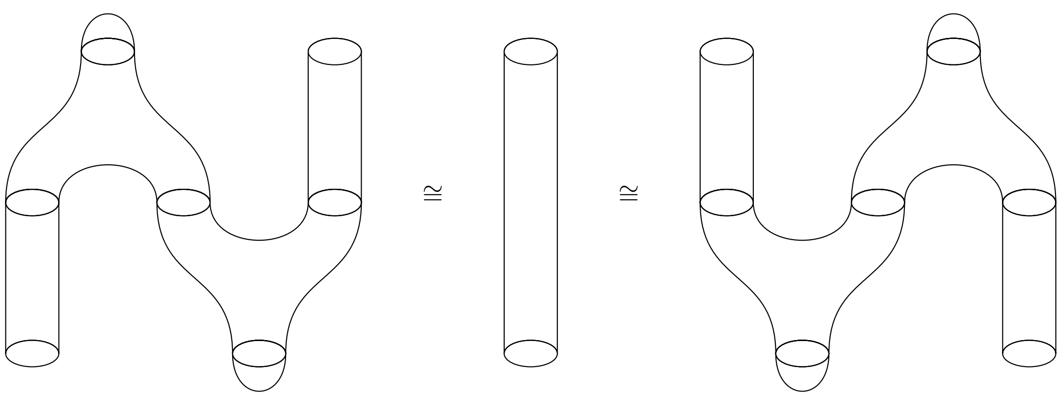
There's a similar map we can define using the dual co-multiplication and unit maps.

Remark 2.2.4. *The operation $Z(\delta \circ \eta) := \tilde{\beta}$ defines a symmetric bilinear map $\tilde{\beta} : \mathbb{C} \longrightarrow \mathcal{H} \otimes \mathcal{H}$. Pictorially this is just the TQFT applied to the diagram below:*



Proposition 2.2.3. *For a closed $n-1$-manifold, N, we have $Z(N)^\star \cong Z(\bar{N})$. Here $\bar{N}$ indicates the same manifold but with the opposite orientation. In particular $Z(N)$ is finite dimensional.*

Proof. Consider the equivalent bordisms below, these are the snake relations.



We can break up the diagrams by considering mapping into the middle three circles and then mapping out again.

$$\left[\mathcal{H} \cong \mathcal{H} \otimes \mathbb{C} \xrightarrow{\mathrm{id}_{\mathcal{H}} \otimes \tilde{\beta}} \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \xrightarrow{\beta \otimes \mathrm{id}_{\mathcal{H}}} \mathbb{C} \otimes \mathcal{H} \cong \mathcal{H}\right] \cong \mathrm{id}_{\mathcal{H}}$$

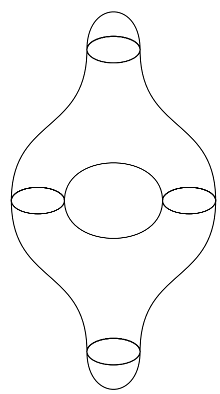
But by the snake relations this composition has to equal the identity map, in other words the two maps are inverses to each other. It follows from this that β is non-degenerate $\implies Z(\bar{N}) = \mathcal{H} \cong \mathcal{H}^\star = Z(N)^\star \implies \mathcal{H}$ is finite dimensional.

□

This is quite surprising. From just simple consistency conditions on what a sensible field theory should obey, we're able to deduce that the space of states must be finite dimensional. It's also quite unfortunate since it means these axioms actually **don't** describe a lot of topological field theories in physics! Nonetheless, it is still highly applicable even in this base form, see [MS25] for a full discussion. The above examples demonstrate the power of this approach, we can treat calculate algebraic quantities from basic topological properties of surfaces. In two dimensions these processes are particularly intuitive, and just amounts to deforming diagrams. Now that we know that the space of states is finite dimensional, is there any way to deduce what this number is?

Proposition 2.2.4. *For a closed and connected $n-1$-manifold we have $Z(N \times S^1) \cong \dim_{\mathbb{C}} \mathcal{H}$.*

Proof. Consider the composition of the bilinear form with its inverse :



In the case of 2 dimensions we have a torus which we can chop up nicely, immediately giving the result:

$$Z(\epsilon \circ \mu \circ \delta \circ \eta) = \beta \circ \tilde{\beta} = T(\mathrm{id}_{\mathcal{H}}) = \dim_{\mathbb{C}} \mathcal{H}$$

□

2.3 Frobenius Algebras are 2D-TQFTs

We've seen that 2D-TQFT has a very algebraic nature, does it correspond to some kind of known structure? Recall that an algebra is a vector space equipped with a bilinear map $A \times A \to A$ which is thought of as the product. If an algebra comes equipped with a unit element with respect to the multiplication we say it is unital. Furthermore, we can consider dual maps going in the opposite direction by flipping the arrows, giving so-called co-units and co-multiplications respectively. An algebra with this collection of compatible date has a special name.

Definition 2.3.1. *A* ***Frobenius Algebra*** *is a tuple $(A, \mu, \delta, \eta, \epsilon)$ where A is a $\mathbb{C}$-vector space equipped with linear maps $(\mu, \delta, \eta, \epsilon)$:*

1. *Multiplication* $\mu : A \otimes A \to A$
2. *Comultiplication* $\delta : A \to A \otimes A$
3. *Unit* $\eta : \mathbb{C} \to A$
4. *Counit* $\epsilon : A \to \mathbb{C}$

which satisfy the following commutative diagrams :

1. *Unital and Counital :*

$$\begin{tikzcd}
 & A & \\
A \otimes A \arrow[ur, "\mu"] & & A \otimes A \arrow[ul, "\mu"'] \\
\mathbb{C} \otimes A \arrow[u, "\eta\otimes\mathrm{id}"] & A \arrow[l, "\cong"] \arrow[r, "\cong"'] \arrow[uu, "\mathrm{id}"] & A \otimes \mathbb{C} \arrow[u, "\mathrm{id}\otimes\eta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "\delta"'] \arrow[dr, "\delta"] \arrow[dd, "\mathrm{id}"] & \\
A \otimes A \arrow[d, "\delta\otimes\mathrm{id}"'] & & A \otimes A \arrow[d, "\mathrm{id}\otimes\delta"] \\
\mathbb{C} \otimes A \arrow[r, "\cong"'] & A & A \otimes \mathbb{C} \arrow[l, "\cong"]
\end{tikzcd}$$

2. *Associativity and Coassociativity :*

$$\begin{tikzcd}
A & A \otimes A \arrow[l, "\mu"'] \\
A \otimes A \arrow[u, "\mu"] & A \otimes A \otimes A \arrow[l, "\mu\otimes\mathrm{id}"] \arrow[u, "\mathrm{id}\otimes\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[d, "\mathrm{id}\otimes\delta"'] & A \arrow[l, "\delta"'] \arrow[d, "\delta"] \\
A \otimes A \otimes A & A \otimes A \arrow[l, "\delta\otimes\mathrm{id}"]
\end{tikzcd}$$

3. *The Frobenius condition :*

$$\begin{tikzcd}
 & A \otimes A & \\
A \otimes A \otimes A \arrow[ur, "\mu\otimes 1_A"] & A \arrow[u, "\delta"] & A \otimes A \otimes A \arrow[ul, "1_A\otimes\mu"'] \\
 & A \otimes A \arrow[ul, "1_A\otimes\delta"] \arrow[u, "\mu"] \arrow[ur, "\delta\otimes 1_A"'] &
\end{tikzcd}$$

The naming of the maps is no coincidence, it's clear that the commutative diagrams here correspond exactly to the relations in 2.2.2, with the addition of further being commutative. This fact can be formalised by saying that there is an equivalence of categories between two-dimensional topological quantum field theories and commutative Frobenius algebras. So, remarkably, doing topological quantum field theory in two dimensions is the same studying commutative Frobenius algebras. The full proof is a statement on the equivalence of categories, which has not been spelled out in detail here, see [Abr96].

What about higher dimensions? It can get quite hard to visualise bordisms, and deciding how to cut them up is difficult. In these kinds of situations, proving an algebraic equivalence like the above can be incredibly useful. It effectively turns difficult geometric problems into potentially simpler algebraic manipulations. Another way to generalise the discussion is to consider 'extended TQFT'. In this case one allows for boundaries in the $n-1$-manifolds, which leads to various complications. In particular one wants to enforce locality not only in the 'time direction' but also along each spatial one. The treatment involves higher category theory, where for example boundaries are morphisms of morphisms. Another way to extend these ideas is the notion of framed bordisms. In this case one can decorate the manifolds with orientation, spin structures, metrics and so on. Clearly these are far more relevant for the physical universe, which requires such things to exist. A lot of progress in these directions is attributed to Lurie [Lur09], who outlined a proof of the so-called 'cobordism hypothesis'.

Chapter 3

Symplectic Reduction & Quantisation

Hamiltonian mechanics has a clean formulation in terms of symplectic geometry. In a system with symmetries one has a natural procedure to eliminate redundant degrees of freedom and work on geometric space of reduced dimension. On the other hand one can quantise a Hamiltonian system by 'promoting' observable functions to Hermitian operators. This process can be tricky, especially on reduced spaces with complicated geometry. It turns out that performing quantisation first and then reducing is actually equivalent to going the other way. Furthermore, the former is often an easier process. In this sense 'symplectic reduction commutes with quantisation'. This chapter will introduce the symplectic formulation of classical mechanics as well the ideas behind reduction. One of the mathematical tools needed to make the above discussion precise is BRST cohomology, which has wide applications in quantum field theories and string theories. We will also briefly introduce some aspects of geometric quantisation.

3.1 Hamiltonian Mechanics

Hamiltonian mechanics has an elegant geometric formulation in terms of symplectic geometry. It's useful to understand this first.

Definition 3.1.1 (Symplectic form)**.** *A symplectic form on a manifold M is 2-from $\omega \in \Omega^2(M)$ such that :*

1. $\mathrm{d}\omega = 0$
2. *ω is non-degenerate*

Definition 3.1.2. *(Hamiltonian) A Hamiltonian system is a triple, (M, ω, H), where $H \in C^\infty(M)$ is the Hamiltonian or the energy of the system.*

Theorem 3.1.1. *A symplectic manifold is always even dimensional.*

Proof. Since $\dim T_pM = \dim M \quad \forall p \in M$ it suffices to show that $\dim T_pM = 2k$, $k \in \mathbb{N}$. In a chart, U, ω_{ij} is an antisymmetric matrix so we have $\omega_{ij} = -\omega_{ji}$. Then properties of the determinant give $\det \omega_{ij} = \det \omega_{ji} = (-1)^n \det \omega_{ij}$. Non degeneracy of a matrix is equivalent to the condition that the determinant is non-zero, so $n = 2k$. $\square$

In physics, we can consider a particle confined to move on a configuration space, Q, which is an n-dimensional manifold. The classic example is a free particle moving on $\mathbb{R}^n$, but this generalises to any manifold and any number of particles, the configuration space would have to be upped to an nm-dimensional manifold for m particles with n degrees of freedom. For simplicity, we'll stick to the single particle case, where for each degree of freedom we have an associated momentum. The natural phase space is then the cotangent bundle $T^\star Q$ of dimension $2n$. Around any point in the cotangent bundle we always have local coordinates which we denote (x^μ, p_μ). In general there always exists a 'tautological one-form' $\theta = p_\mu \mathrm{d}x^\mu$, using this one has a natural definition for the symplectic form.

Definition 3.1.3. *The* ***canonical*** *symplectic form is defined through the tautological one form, θ, by $\omega = -\mathrm{d}\theta$.*

Note that there is actually sign convention chosen here. The minus sign is inserted here so that the symplectic form locally looks like $\omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu$, which will turn out to give the correct form of Hamilton's equations through definition 3.1.4. One could instead use the convention $\omega = \mathrm{d}\theta$, which would result in $\omega = \mathrm{d}p_\mu \wedge \mathrm{d}x^\mu$. The latter choice seems more natural at first, but then one has to introduce a minus sign in definition 3.1.4. We will stick with the former convention.

Definition 3.1.4. *The Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ is defined by $\iota_X \omega = \mathrm{d}H$.*

This defines a set of ordinary differential equations, namely Hamilton's equations of motion, the solutions of which give the integral curves of X_H. The time evolution of a state to a time $t \in \mathbb{R}$ is then given by the flow $\phi_t : M \to M$. As discussed earlier the sign choices are such that this definition coincides with the usual equations of motion. Now the states in classical mechanics can be represented in a coordinate chart as (x^μ, p_μ) for $\mu \in \{1, \dots, n\}$. One of the defining properties of classical mechanics is the fact that observable quantities are smooth functions. For example, we have a given observable as the Hamiltonian itself, which defines the total energy of the system. We may also consider the position and momentum observables as simply the coordinate functions of the points in configuration and momentum space.

Definition 3.1.5. *The set of* ***classical observables*** *for a Hamiltonian system is $C^\infty(M)$.*

There often special functions which are preserved along the Hamiltonian flow. These are **conserved quantities** which are 'constant in time' if one considers the Hamiltonian as defining the time evolution of a system.

Definition 3.1.6. *Given a Hamiltonian flow, $\phi_t : M \longrightarrow M$, a function $f \in C^\infty(M)$ is a conserved quantity if*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_t) = 0. \tag{3.1}$$

On the other hand one can consider shifting along directions in phase space which leave the system invariant. These would be described locally by vector fields which preserve the Hamiltonian and respect the symplectic structure.

Definition 3.1.7. *A vector field $X \in \mathfrak{X}(M)$ is a (Hamiltonian) symmetry for the system if*

1. $X(H) = 0$
2. $\mathcal{L}_X \omega = 0$

It turns out that there is a correspondence between local symmetries and global conserved quantities. In fact, every local symmetry defines a global conserved quantity under a mild assumption.

Theorem 3.1.2 (Noether)**.** *If $X \in \mathfrak{X}(M)$ is a symmetry and $H^1_{\mathrm{dR}}(M) = 0$ then $\exists f \in C^\infty(M)$ such that*

$$\iota_X \omega = \mathrm{d}f, \tag{3.2}$$

where f is a conserved quantity which is constant along Hamiltonian flow.

Proof. To show such an f exists we just expand the Lie derivative using Cartan's magic formula,

$$\mathcal{L}_X \omega = \iota_X \circ \mathrm{d}\omega + \mathrm{d} \circ \iota_X \omega = 0,$$

but we know that $\mathrm{d}\omega = 0$ since ω is symplectic. Then remaining is

$$\mathrm{d} \circ \iota_X \omega = 0$$

which implies, using the triviality of $H^1_{\mathrm{dR}}(M)$, that $\exists f \in C^\infty(M)$ satisfying $\iota_X \omega = \mathrm{d}f$. Now it remains to show that f is conserved along flows generated by X. We just have to apply the interior product successively to the symplectic form :

$$\begin{gathered} \iota_X \circ \iota_{X_H} \omega = \iota_X \circ \mathrm{d}H = X(H) = 0 \\ -\iota_{X_H} \circ \iota_X \omega = -\iota_{X_H} \circ \mathrm{d}f = 0 \implies X_H(f) = 0. \end{gathered}$$

□

Note the conserved quantity $f \in C^\infty(M)$ is only unique up to a constant function. The notation X_f may be used for symmetry associated to the conserved quantity f. In general the symplectic two form actually defines a map :

$$\begin{aligned} \flat : \mathfrak{X}(M) &\longrightarrow \Omega^1(M) \\ X_f &\mapsto \iota_{X_f} \omega = \mathrm{d}f. \end{aligned}$$

This is also an isomorphism thanks to the non-degeneracy of the symplectic form, with inverse given by :

$$\begin{aligned} \natural : \Omega^1(M) &\longrightarrow \mathfrak{X}(M) \\ X_f &= \omega^{-1}(-, \mathrm{d}f) \end{aligned}$$

In components, we have,

$$X_f^i = \omega^{ij} \partial_j f,$$

where ω^{ij} is the matrix inverse of ω_{ij}. Now the set of all observables can be given more structure by equipping it with a bilinear product using the symplectic form.

Definition 3.1.8. *The* ***Poisson bracket*** *of $f, g \in C^\infty(M)$ is a bilinear map :*

$$\begin{aligned}\{-,-\} : C^\infty(M) \times C^\infty(M) &\longrightarrow C^\infty(M)\\ (f,g) &\longmapsto \{f,g\} = \omega(X_f, X_g) = \omega^{-1}(\mathrm{d}g, \mathrm{d}f)\end{aligned}$$

This bracket is alternating, which is immediate from the anti-symmetry of the symplectic form. It also satisfies a Jacobi identity which can be deduced using the fact that symplectic form is closed. Furthermore, the bracket can be thought of as a derivation on the algebra of smooth functions with pointwise multiplication :

$$X_f \cdot g = \mathrm{d}g(X_f) = \iota_{X_f}\omega(X_g) = \{f,g\} = \omega(X_f, X_g)$$

$$X_f \cdot (gh) = (X_f \cdot g)h + gX_f \cdot h$$

In this sense we have the following :

Definition 3.1.9. *The* ***Classical algebra of observables*** *is the* ***Poisson algebra*** $(C^\infty(M), \{-,-\})$. *It satisfies :*

1. *Alternating property :*
$$\{f,f\} = 0$$
2. *Jacobi identity :*
$$\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$$
3. *Poisson derivation :*
$$\{f,gh\} = \{f,g\}h + g\{f,h\}$$

Note that the algebra of observables is equipped with an associative product in the form of the usual multiplication of smooth functions. On top of this it has the Poisson bracket which has a derivation on it using the adjoint action, meaning that properties (1) & (2) make it into a **Lie algebra**. In particular, it is not associative, the failure of which is the Jacobi identity. If we further add the condition that the adjoint action is a derivation on the **associative** product of smooth functions, then we obtain (3) which is the full structure of a Poisson algebra. We can define the time evolution of observables using the Poisson bracket too. Hamilton's equations of motion are equivalent to,

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \{f, H\}, \tag{3.3}$$

where f can be swapped for x^i and p_i to get the familiar form of the equations of motion. In this way we can say that an arbitrary observable is a conserved quantity if it Poisson commutes with the Hamiltonian $\{f, H\} = 0$.

3.2 Symplectic Reduction

We can make the discussion of symmetries more formal be **defining** them as special kinds of Lie group actions on the manifold. These are the continuous symmetries that occur in physics so often.

Definition 3.2.1. *A* ***Lie group action*** *on a manifold M is a smooth map,*

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, m) &\mapsto g \triangleright m, \end{aligned}$$

which induces a diffeomorphism on M by:

$$\begin{aligned} \psi : G &\longrightarrow \mathrm{Diff}(M) \\ g &\mapsto \psi_g \end{aligned}$$

A general Lie group action is not restrictive enough for our purposes. We have an extra symplectic structure on our smooth manifold which should be compatible with the action naturally.

Definition 3.2.2. *A Lie group action is* ***symplectic*** *if it preserves the symplectic form*

$$\psi_g^\star \omega = \omega \quad \forall g \in G$$

Furthermore, we have a Hamiltonian on the symplectic manifold which ought to be compatible with the group action too. Given the group action we define the orbit map φ_p,

$$\begin{aligned} \varphi_p : G &\longrightarrow M \\ g &\mapsto g \triangleright p, \end{aligned}$$

which can then be differentiated to produce the associated vector field:

$$\begin{aligned} \mathrm{d}_e \varphi_p : \mathfrak{g} &\longrightarrow \mathfrak{X}(M) \\ \sigma &\mapsto X_p^\sigma \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}(e^{-t\sigma} \triangleright p)|_{t=0}. \end{aligned}$$

This defines how symmetries in the sense of vector fields on M arise from the Lie group itself. Now we can formalise the concept of a momentum, or charge, associated to a given symmetry,

Definition 3.2.3. *The* ***momentum map*** $\mu : M \longrightarrow \mathfrak{g}^\star$ *satisfies:*

$$\mathrm{d}\langle \mu, \sigma \rangle = \iota_{X^\sigma} \omega \quad \forall p \in M, \, \forall \sigma \in \mathfrak{g}$$

Here $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g}^\star \to \mathbb{R}$ is the natural pairing between the Lie algebra and its dual. In particular $\langle \mu, \sigma \rangle \in C^\infty(M)$ such that,

$$\mu^\sigma(x) \coloneqq \langle \mu, \sigma \rangle(x) = \langle \mu(x), \sigma \rangle \quad \forall x \in M, \tag{3.4}$$

where the functions μ^σ can be thought of as the components of the momentum map.

Definition 3.2.4. *A symplectic action ψ_g for a group $G \ni g$ with a map $\mu : M \to \mathfrak{g}^\star$ is a* ***Hamiltonian action*** *if the following conditions hold :*

1. *μ is a momentum map.*
$$\mathrm{d}\langle \mu, \sigma \rangle = \iota_{X^\sigma}\omega \quad \forall \sigma \in \mathfrak{g}$$
2. *μ is G-equivariant.*
$$\mu \circ \psi_g = \mathrm{Ad}_g^\star \circ \mu \quad \forall g \in G$$

When a dynamical system has symmetries/conserved quantities there are subspaces in the phase space which are invariant under the action. For example consider a field in $\mathbb{R}^3$ with an axially symmetric potential and Hamiltonian. There is an $SO(2)$ group action on the space which leaves circles of a given radius fixed. In a cylindrical chart, (R, ϕ, z), we'd say that ϕ is an ignorable coordinate. By Noether's theorem this symmetry generates a conserved quantity, the axial angular momentum L_z. The momentum map in this case is just the function $\mu(x,p) = L_z$, hence the naming.

Then it makes sense to reduce the space down to a subspace in which the dynamics are non-trivial. One can consider the preimage of the momentum map $\mu^{-1}(l_z)$ for a given fixed value of l_z and quotient out by this space. In the configuration space this can be visualised as cutting into $\mathbb{R}^3$ for a given value of ϕ and collapsing the surrounding space along the orbit circles of $SO(2)$, resulting in a two-dimensional (R, z) plane. Note the axial momentum p_ϕ is also fixed by the choice of l_z; leaving a 4-dimensional phase space from the original 6-dimensional one. This idea can be generalised to arbitrary Hamiltonian systems and is called symplectic reduction [MW74].

Theorem 3.2.1 (Marsden-Weinstein)**.** *Consider a Hamiltonian system, (M, ω, H), and a momentum map $\mu : M \to \mathfrak{g}^\star$ for a Hamiltonian action of G. Furthermore, assume the action is free and proper with $\lambda \in \mathfrak{g}^\star$ a regular value. The quotient $\tilde{M}_\lambda := \mu^{-1}(\lambda)/G_\lambda$ is a smooth manifold of dimension $\dim M - 2\dim G$ and inherits a unique symplectic structure by $\pi^\star\tilde{\omega}_\lambda = \iota^\star\omega$. Here $\pi : \mu^{-1}(\lambda) \longrightarrow \tilde{M}_\lambda$ is the quotient map and $\iota : \mu^{-1}(\lambda) \longrightarrow M$ is the embedding of the orbit space.*

Note that reduction gives a smooth manifold only if the group action is free. In the previous example the $SO(2)$ action is not free on the z-axis, and hence gives a singularity along it. These kinds of spaces are called **orbifolds**. In general this process of symplectic reduction is quite delicate, and the quotient space might be complicated and singular. Despite these problems, it is already an incredibly powerful tool in Hamiltonian mechanics.

3.3 Quantisation

The above was all to do with classical mechanics. One hopes for a formal way to quantise this system to get a theory of quantum mechanics. The idea is to take the Poisson algebra of observables and promote it to a Lie algebra of quantum operators which act on a Hilbert space, $\mathcal{H}$, of wavefunctions. The Poisson bracket for classical observables is now promoted to the Lie bracket for quantum operators, which is just the commutator of linear maps. This idea dates back to Dirac and is essentially the concept of 'first quantisation'. Let's write down at least what's expected of a quantisation procedure.

1. An $\mathbb{R}$-linear map $\mathcal{Q} : C^\infty(M) \ni f \longmapsto \hat{f} \in \mathcal{A} \subseteq \mathrm{End}(\mathcal{H})$
$$\mathcal{Q}(\lambda f + g) = \lambda\mathcal{Q}(f) + \mathcal{Q}(g)$$

2. The map is ‘unital’:
$$\mathcal{Q} : C^\infty(M) \ni 1 \longmapsto \mathrm{id}_{\mathcal{H}} \in \mathcal{A}$$
3. Operators in the image are hermitian:
$$\mathcal{Q}(f)^\star = \mathcal{Q}(f)$$
4. The Dirac quantisation condition holds:
$$\mathcal{Q}\{f, g\} = \frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$$

Usually the Hilbert space is thought of as the space of wavefunctions, properly defining this requires some functional analysis, which won’t be treated here. The parameter $\hbar$ is a small real number constant. We can define a classical limit in the sense that $\hbar \to 0$ makes all commutators trivial, and hence observables behave like classical commuting quantities. It’s also postulated that operators are hermitian to ensure that the observables, or eigenvalues, are real. The Dirac quantisation condition is particurly nice in the Heisenberg picture of quantum mechanics. Recall that time evolution is goverend by eq. (3.3). Applying the operator to this equation we immediately obtain,

$$\frac{\mathrm{d}}{\mathrm{d}t} A = \frac{i}{\hbar}[A, \hat{H}], \tag{3.5}$$

where $A \in \mathcal{A}$ is a time independant operator and $\hat{H} = \mathcal{Q}(H)$ is the quantum Hamiltonian operator. Consider $T^\star\mathbb{R}$ equipped with the standard coordinate chart and symplectic form $\mathrm{d}x \wedge \mathrm{d}p$. We can compute the Poisson bracket easily as $\{x, p\} = 1$. Then applying the Dirac quantisation condition and the unital property,

$$\mathcal{Q}\{x, p\} = \mathcal{Q}(1) = \mathrm{id}_{\mathcal{H}} = \frac{i}{\hbar}[\hat{x}, \hat{p}], \tag{3.6}$$

immediately gives the canonical commutation relations. So it does seem to make sense that the above conditions are reasonable for a quantisation operator.Now let’s try to give an explicit construction. Suppose we have an exact symplectic form $\omega = \mathrm{d}\vartheta$ for some globally well-defined ‘potential’ $\vartheta \in \Omega^1(M)$.

Definition 3.3.1. *The* ***Kostant–Souriau (K-S) prequantum operator*** *is defined by,*

$$\mathcal{Q}(f) \coloneqq -i\hbar\Big(X_f - \frac{1}{i\hbar}\vartheta(X_f)\Big) + f, \tag{3.7}$$

where $\omega = \mathrm{d}\vartheta$ and $f \in C^\infty(M)$.

We can make this definition more general by allowing ϑ not to be well-defined globally, but demanding that the $i\omega$ is the curvature of a line a bundle with connection (L, ∇). This can be done only if $\omega/\hbar$ has 2π-integral periods; the **Bohr-Somerfield condition**. In fact, this takes us all the way back to chapter 1, where we would say that ω defines a class in differential cohomology. Then the formula becomes [Hal13],

$$\mathcal{Q}(f) \coloneqq -i\hbar\nabla_{X_f} + f, \tag{3.8}$$

where now we can recover definition 3.3.1 if we consider the covariant derivative with respect to a local potential ϑ.

Theorem 3.3.1. *The K-S operator satisfies the Dirac quantisation condition:*

$$-i\hbar\mathcal{Q}\{f,g\} = [\mathcal{Q}(f),\mathcal{Q}(g)]$$

This seems quite well and good, except it doesn't work. In general there is no way to define such a map on the whole space of classical observables; the content of the **Groenewold–Van Hove theorem**. One has to restrict to a smaller set of 'complete' observables. The above procedure is referred to as **prequantisation**, indicating that there are still more steps to be carried out. Firstly, when we assume that phase space is the cotangent bundle of some configuration space, $M = T^\star Q$, we have a splitting of coordinates into position and momentum. Then we can demand that the wavefunctions only depend on one such set of quantities. In the above formalism we can treat wavefunctions as the sections of the line bundle, under appropriate normalisation conditions.

In general, this process is called **polarisation**, and is not so trivial as in the special case of the cotangent bundle. Finally, one more step called **metaplectic correction** must be performed. A full description is not treated here, consult [Hal13] for a detailed exposition. It's quite evident that this process can be convoluted, if it exists. This can be made even worse by considering complicated reduced spaces, which may be singular. On the other hand, if the reduced space arises from a simpler space which we know how to deal with, one could hope to quantise the simpler case and then reduce the space down.

3.4 BRST Cohomology

Our discussion will focus on the case of first extracting a reduced algebra of **classical observables** on the quotient, but it can be extended to the reduction of quantum observables too. First, we need an algebraic interlude. Given a Lie group, G, one can attempt to compute the cohomology of the underlying manifold using traditional methods from algebraic and differential topology. In particular one can take the associated de Rham complex of the manifold and compute its cohomology. It turns out there is a more efficient method, making use of the extra Lie structure, by effectively linearising the problem to the Lie algebra level. This is the motivation for Lie algebra cohomology.

Consider a representation (ρ, V) of a Lie algebra $\mathfrak{g}$, or equivalently a $\mathfrak{g}$-module. Note that all Lie groups/algebras discussed here are finite dimensional, and we're working over $\mathbb{R}$, but the field can be general. The main reference for this section have been these readable lecture notes [FO06], but there are also standard textbook references such as [Kim93].

Definition 3.4.1. *The* ***Chevalley-Eileneberg*** *complex consists of k-chains in*

$$C^k(\mathfrak{g}; V) \coloneqq \mathrm{Hom}(\wedge^k\mathfrak{g}, V) \cong \wedge^k\mathfrak{g}^\star \otimes V,$$

equipped with a differential $\mathrm{d} : \wedge^k\mathfrak{g}^\star \otimes V \longrightarrow \wedge^{k+1}\mathfrak{g}^\star \otimes V$ *satisfying*

1. $\forall v \in V;\quad \mathrm{d}v(x) = \rho(x)v \quad \forall x \in \mathfrak{g}$
2. $\forall \alpha \in \mathfrak{g}^\star;\quad \mathrm{d}\alpha(x,y) = -\alpha([x,y]) \quad \forall x,y \in \mathfrak{g}$
3. $\forall \alpha,\beta \in \wedge^\bullet\mathfrak{g}^\star;\quad \mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge\mathrm{d}\beta$
4. $\forall \omega\otimes v \in \wedge^\bullet\mathfrak{g}^\star\otimes V;\quad \mathrm{d}(\omega\otimes v) = \mathrm{d}\omega\otimes v + (-1)^{|\omega|}\omega\wedge\mathrm{d}v$

The differential is defined first on pure vectors, then dual lie algebra elements, and finally generalise to the full complex obeying the Leibniz rules. In (3) and (4) we've only defined the action on pure tensors, but it's easily linearly extended. A perhaps cleaner way of defining the differential is to use the definition of lie bracket directly. Note that the map,

$$\begin{aligned} [-,-] : \mathfrak{g} \wedge \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x \wedge y &\mapsto [x, y], \end{aligned}$$

induces a dual map,

$$\begin{aligned} \mathrm{Hom}(-, \mathbb{R})([-,-]) \coloneqq \mathrm{d} : \mathfrak{g}^\star &\longrightarrow \mathfrak{g}^\star \wedge \mathfrak{g}^\star \\ \alpha &\mapsto -\alpha \circ [-,-], \end{aligned}$$

from the definition of the dual space and the contravariant Hom functor. Since the application is just pre-composition, this is equivalent to (2) in the previous definition up to a minus sign to ensure that the operator is nilpotent. The differential on V is just defined as the application of the representation to an element of the lie algebra, $\rho \in \mathrm{Hom}(\mathfrak{g}, GL(V))$ by $\mathrm{d} : V \to \mathfrak{g}^\star \otimes V$. Both of these can then be extended to higher degree elements by using relevant graded Leibniz rules (3) and (4). Nonetheless, it can be simpler to introduce bases of $\mathfrak{g}$ and $\mathfrak{g}^\star$ to get a concrete formula.

Definition 3.4.2. *Given bases $\{\tau_a\}$ of $\mathfrak{g}$ and $\{\sigma^a\}$ of $\mathfrak{g}^\star$ we define the following operators:*

1. *The **ghost** operator $c^a = \sigma^a \wedge (-)$*
2. *The **anti-ghost** operator $b_a = \iota_{\tau_a}(-)$*

Proposition 3.4.1. *The classical **BRST operator** or **Chevalley-Eilenberg** has the form,*

$$\mathrm{d} = c^i \rho(\tau_i) - \frac{1}{2} f^i_{jk} c^j c^k b_i, \tag{3.9}$$

in a given basis $\{\tau_i\}$ & $\{\sigma^i\}$ with structure constants $[\tau_i, \tau_j] = f^k_{ij} \tau_k$.

It can be shown that the above is equivalent to the earlier definition 3.4.1.

Proposition 3.4.2. *The BRST operator is actually is a differential in the homological algebra sense, i.e.* $\mathrm{d}^2 = 0$.

Proof. It follows from the Lie algebra homomorphism property,

$$\rho([x, y]) = [\rho(x), \rho(y)] = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x),$$

that the differential is nilpotent on V. In particular in degree one,

$$\begin{aligned} \mathrm{d} \circ \mathrm{d}v(x, y) &= \rho(y)\mathrm{d}v(x) - \rho(x)\mathrm{d}v(y) - \mathrm{d}v([x, y]) \\ &= \rho(y)\rho(x)v - \rho(x)\rho(y)v + \rho([x, y]) = 0, \end{aligned}$$

which can then be inductively extended. Similarly, the operator is nilpotent on $\mathfrak{g}^\star$ by the Jacobi identity, which can then also be inductively extended to arbitrary degrees. □

Our reason for defining this tool is to get a 'reduction of observables'. Say we have a momentum map with 0 a regular value. On the unreduced space we have a Poisson algebra of observables $C^\infty(M)$, it's natural to ask if we can deduce somehow the observables on the quotient, $C^\infty(\tilde{M})$. For our purposes, in the case of symplectic reduction, one takes $\mathfrak{g}$ to be the Lie algebra associated to a Hamiltonian action $G \triangleright M$. Then the representation is defined by the action of the associated fundamental vector field on a function. We can consider the cohomology of the associated complex as above. It can be shown that we have the following result,

$$H^0(\mathfrak{g}; C^\infty(\mu^{-1}\{0\})) \cong C^\infty(\tilde{M}), \tag{3.10}$$

which is pretty nice. But to use this we would already have to know the algebra of observables on $\mu^{-1}\{0\}$. Ideally one would like to deduce the reduced algebra starting with $C^\infty(M)$ itself. To do this we need to introduce another definition.

Definition 3.4.3. *Consider a $\mathfrak{g}$-module V. A **projective resolution** of V is a terminating chain complex $(K^\bullet, \delta)$ of $\mathfrak{g}$-modules with $H^i(K^\bullet) = V$ if $i = 0$ and $H^i(K^\bullet) = 0$ otherwise.*

$$\cdots \longrightarrow K^2 \longrightarrow K^1 \longrightarrow K^0 \longrightarrow 0$$

We can extend this complex by one factor and turn it into a fully exact sequence by considering the map

$$\epsilon : K^0 \longrightarrow K^0/\operatorname{Im}\delta \subset V.$$

Now we have a so-called left resolution,

$$\cdots \xrightarrow{\delta} K^2 \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^0 \xrightarrow{\epsilon} V \longrightarrow 0,$$

which we can then tensor with $\wedge^p\mathfrak{g}^\star$ to get another projective resolution, but this time for $\wedge^p\mathfrak{g}^\star \otimes V$:

$$\cdots \xrightarrow{\delta} \wedge^p\mathfrak{g}^\star \otimes K^2 \xrightarrow{\delta} \wedge^p\mathfrak{g}^\star \otimes K^1 \xrightarrow{\delta} \wedge^p\mathfrak{g}^\star \otimes K^0 \xrightarrow{\epsilon} \wedge^p\mathfrak{g}^\star \otimes V \longrightarrow 0$$

Note we've used the same notation for the induced maps as for the original ones. This is just because the induced maps simply ignore the Lie algebra part of the product. We know the above is a resolution since it's a homological algebra fact that the tensor product is right exact. But the for each W with $\wedge^p\mathfrak{g}^\star \otimes K$ we can define the Chevalley-Eilenberg complex as well. Stacking these on top of each other we obtain the BRST complex:

$$\begin{tikzcd}
 & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \\
\bullet \arrow[r] & \wedge^{p-1}\mathfrak{g}^\star \otimes K^{q+1} \arrow[r, "\delta"] \arrow[d, "\mathrm{d}"] & \wedge^{p-1}\mathfrak{g}^\star \otimes K^{q} \arrow[r, "\delta"] \arrow[d, "\mathrm{d}"] & \wedge^{p-1}\mathfrak{g}^\star \otimes K^{q-1} \arrow[r] \arrow[d, "\mathrm{d}"] & \bullet \\
\bullet \arrow[r] & \wedge^{p}\mathfrak{g}^\star \otimes K^{q+1} \arrow[r, "\delta"] \arrow[d, "\mathrm{d}"] & \wedge^{p}\mathfrak{g}^\star \otimes K^{q} \arrow[r, "\delta"] \arrow[d, "\mathrm{d}"] & \wedge^{p}\mathfrak{g}^\star \otimes K^{q-1} \arrow[r] \arrow[d, "\mathrm{d}"] & \bullet \\
\bullet \arrow[r] & \wedge^{p+1}\mathfrak{g}^\star \otimes K^{q+1} \arrow[r, "\delta"] \arrow[d] & \wedge^{p+1}\mathfrak{g}^\star \otimes K^{q} \arrow[r, "\delta"] \arrow[d] & \wedge^{p+1}\mathfrak{g}^\star \otimes K^{q-1} \arrow[r] \arrow[d] & \bullet \\
 & \bullet & \bullet & \bullet &
\end{tikzcd}$$

Here there is the horizontal differential defined from the resolution, and the vertical differential defined as the Chevalley-Eileneberg operator. Now we can combine these two differentials into a new operator $D = \mathrm{d} + (-1)^p\delta$ acting on $C^{p,q} := \wedge^p\mathfrak{g}^\star \otimes K^q$. The reason for the minus sign is that it ensures nilpotency,

$$\begin{aligned} D^2 &= (\mathrm{d} + (-1)^{p+1}\delta) \circ (\mathrm{d} + (-1)^p\delta) \\ &= \mathrm{d}\circ\mathrm{d} + \delta\circ\delta + (-1)^p\mathrm{d}\circ\delta - (-1)^p\delta\circ\mathrm{d} = 0, \end{aligned}$$

by enforcing anti-commutativity. This differential does not preserve the bidegree of the double complex since it is a map $D : C^{p,q} \longrightarrow C^{p+1,q} \oplus C^{p,q-1}$. One can construct a complex with differential D which **does** preserve a total degree $p - q$ for $C^{p,q}$. So, finally we have a graded complex $\mathcal{C}^n = \bigoplus_{p-q=n} C^{p,q}$ with differential $D = \mathrm{d} + (-1)^p\delta$. Using the above we can compute the cohomology and we have the general result:

$$H^n(\mathcal{C}) \cong H^n(\mathfrak{g}; V) \tag{3.11}$$

All of the above is immediately applicable to the case of in symplectic geometry. $\mu^{-1}\{0\}$ is a closed embedded submanifold of M. This means it can be defined as the vanishing locus smooth functions on M. In particular,

$$\mathcal{I} := \{f \in C^\infty(M) | f \circ \mu^{-1}\{0\} = 0\},$$

is an ideal $\mathcal{I} \lhd C^\infty(M)$. This is often called the **vanishing ideal**. Now pick a basis $\{\tau_a\}$ of $\mathfrak{g}$. Then the components of the momentum map, $\mu_a(x) := \mu(\tau_a)(x) \in \mathbb{R}^k$ generate the ideal since we assumed 0 is a regular value. Then we have an isomorphism:

$$C^\infty(\mu^{-1}\{0\})) \cong C^\infty(M)/\mathcal{I},$$

Now we would like to find a resolution for quotient by the vanishing ideal in terms of the full algebra. It turns out the correct tool for this is the **Koszul complex**.

$$K^\bullet = \wedge^\bullet\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\ \delta\ } C^\infty(M) \longrightarrow 0$$

The differential δ is first defined on $\mathfrak{g} \otimes C^\infty(M)$ by,

$$\delta(\tau_a \otimes f) = \delta(\mu_a) \otimes f + \tau_a \otimes \delta(f),$$

where $\delta(\tau_a) = \mu_a$ and $\delta(f) = 0 \, \forall f \in C^\infty(M)$. So the above expression reduces to,

$$\delta(\tau_a \otimes f) = \mu_a f,$$

on pure tensors. This can be linearly extended to all of $\mathfrak{g} \otimes C^\infty(M)$ in the usual fashion. The differential acts on a full pure element of $\mathfrak{g} \wedge \mathfrak{g} \otimes C^\infty(M)$ by,

$$\delta(x \wedge y \otimes f) = y \otimes \mu(x)y - x \otimes \mu(y)f,$$

or as an **odd** derivation. The above complex does indeed satisfy $\delta^2 = 0$ and the cohomology is trivial in all degrees apart from zero:

$$H^k(\wedge^\bullet \mathfrak{g} \otimes C^\infty(M)) = \begin{cases} C^\infty(\mu^{-1}\{0\}) & k = 0 \\ 0 & k > 0 \end{cases}$$

As earlier we can augment this complex by its cohomology to obtain the **Koszul resolution**:

$$\wedge^\bullet \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\ \delta\ } C^\infty(M) \longrightarrow C^\infty(M)/\mathcal{I} \cong C^\infty(\mu^{-1}\{0\}) \longrightarrow 0$$

Here the penultimate map is just the quotient projection. Now we can construct the double complex $C^{p,q} := \wedge^p \mathfrak{g}^\star \otimes \wedge^q \mathfrak{g} \otimes C^\infty(M)$.

Definition 3.4.4. *The **BRST complex** has cochains,*

$$\mathcal{C}^n := \bigoplus_{p-q=n} \wedge^p \mathfrak{g}^\star \otimes \wedge^q \mathfrak{g} \otimes C^\infty(M),$$

*equipped with the differential $D = \mathrm{d} + (-1)^p \delta$. The degree of the complex is the **ghost number**.*

Then one can compute the cohomology of the associated the BRST complex and obtain the following result:

Theorem 3.4.1. *There is an isomorphism of vector spaces:*

$$H^n(\mathcal{C}^\bullet) \cong H^n(\mathfrak{g}) \otimes C^\infty(\tilde{M})$$

In particular, we have,

$$H^0(\mathcal{C}^\bullet) \cong C^\infty(\tilde{M}),$$

finally. While the isomorphism of vector spaces is useful, we'd like to be able to have the result extend to the **algebra** of observables on the reduced space. The first step in doing this would be to define a Poisson algebra structure on $H^n(\mathcal{C}^\bullet)$. This process can in fact be done, and such a theorem can be formulated precisely. We won't discuss it for now, but see [Kim93] for details. When doing quantum theory, one goal is to deduce the quantum algebra associated to the reduced phase space. On the other hand we can also define a 'quantum BRST' to get a reduced operator algebra from the original. When quantising a system in general, the reduced space can be quite difficult to work with. In some cases it may be easier to actually first quantise the original space, and then perform

BRST on the quantum algebra of observables. For this to be a valid procedure, one hopes that the following diagram commutes.

$$\begin{tikzcd}
(C^\infty(M), \omega) \arrow[r, "\mathcal{Q}"] \arrow[d, "\mathrm{BRST}"'] & (\mathcal{A}, [\cdot, \cdot]) \arrow[d, "\mathrm{BRST}"] \\
(C^\infty(\tilde{M}), \tilde{\omega}) \arrow[r, "\mathcal{Q}"'] & (\tilde{\mathcal{A}}, [\cdot, \cdot])
\end{tikzcd}$$

Chapter 4

Branes as Sheaves

One is often first introduced to branes in the context of submanifolds. Branes are not just subspaces however, they have gauge fields living on them, and the submanifold is only one part of the data. One then hopes that branes can be described by bundles over the associated submanifold, however, such a picture breaks down at short distances and a more abstract formulation is necessary. It is conjectured that D-branes in particular can be formalised using the machinery of sheaves. In the B-Model for example, open string spectra between D-branes can be computed efficiently in this language. This chapter will give a light introduction to some basic sheaf theory and properties of the B-model. We will be following Eric Sharpe's lectures [Sha03] as well parts of U. Bruzzo's lectures [Bru03] for some algebraic geometric background.

4.1 Sheaves

In string theories one needs to impose boundary conditions on the endpoints of open strings. One such way is to prescribe a submanifold which the strings endpoints are confined to move on, this is the Dirichlet boundary condition. When considering the quantum theory, roughly speaking, the spectrum of states of open strings between two D-branes forms a chain complex and the cohomology of this gives the physically relevant states. It turns out that these branes ought to be dynamical and have gauge fields defined over them, so the idea is to have bundles defined over submanifolds within some bulk space. This is one way in which sheaves enter the picture, though it's important to realize that this only an approximate description. In general branes don't have such a neat geometric realisation, this is where the mathematical machinery shows its worth. In this introduction we'll restrict our attention to the case where we **can** assume the geometric picture, though the point of this formalism is that it can be extended to account for more general cases.

Sheaves are ways of characterising when local data can be constructed into global data on a larger space. The idea is to assign a certain kind of object locally (over each open subset) such that they can be restricted to arbitrarily small subsets. A quick definition is as follows.

Definition 4.1.1. *To a topological space* (X, τ_X) *one defines the category* $\mathrm{Op}(X)$ *with objects as* *open sets* $\mathrm{ObOp}(X) = \tau_X$ *and morphisms as inclusions :*

$$\mathrm{Hom}_{\mathrm{Op(X)}}(U, V) = \begin{cases} U \hookrightarrow V & \text{if} \quad U \subseteq V \\ \varnothing & \text{else} \end{cases}$$

Definition 4.1.2. *A* ***presheaf*** *on a topological space* X *is a contravariant functor* $\mathcal{F} : \mathrm{Op}(X) \longrightarrow \mathrm{AbGrp}$ *into the category of abelian groups. In particular, each open set gets assigned an abelian group and each inclusion* $\iota : U \hookrightarrow V$ *is assigned a restriction* $\mathcal{F}(\iota) \coloneqq \mathrm{r}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$*.*

$$\begin{tikzcd}
\mathcal{F}(U) & \mathcal{F}(V) \arrow[l, "\mathrm{r}_U^V"'] \\
U \arrow[u, "\mathcal{F}"] \arrow[r, hook] & V \arrow[u, "\mathcal{F}"']
\end{tikzcd}$$

Explicitly $\forall U \in \tau_X$ *we have* $\mathrm{r}_U^U = \mathrm{id}_{\mathcal{F}(U)}$ *and*

$$\forall U, V, W \in \tau_X : U \hookrightarrow V \hookrightarrow W \implies \mathrm{r}_U^V \circ \mathrm{r}_V^W = \mathrm{r}_U^W.$$

We say elements of the presheaf, $\mathcal{F}$, over an open subset, U, are **local sections** $s \in \mathcal{F}(U)$. A sheaf is construction which only considers those presheaves with local sections that obey certain 'gluing properties'.

Definition 4.1.3. *A* ***sheaf*** *is a presheaf,* $\mathcal{F}$*, such that for all open covers* $\{U_i\}_{i\in I} : U_i \subseteq U \quad \& \quad \forall i \in I$ *the following conditions hold:*

1. *Locality :*
$$s, t \in \mathcal{F}(U) : \mathrm{r}_{U_i}^U \circ s = \mathrm{r}_{U_i}^U \circ t \quad \forall i \in I \implies s = t$$

2. *Gluing :*
$$s_i \in \mathcal{F}(U_i) : \mathrm{r}_{U_i \cap U_j}^U \circ s_i = \mathrm{r}_{U_i \cap U_j}^U \circ s_j \quad \forall i, j \in I \implies \exists s \in \mathcal{F}(U) : \mathrm{r}_{U_i}^U \circ s = s_i \quad \forall i \in I$$

Definition 4.1.4. *A* ***morphism of sheaves*** *over a space* X *is a natural transformation* $\eta : \mathcal{F} \to \mathcal{F}'$*. In particular the following diagram commutes* $\forall U, V \in \tau_X$*:*

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\eta_V"] \arrow[d, "\mathrm{r}_U^V"'] & \mathcal{F}'(V) \arrow[d, "\mathrm{r}_U^{V\prime}"] \\
\mathcal{F}(U) \arrow[r, "\eta_U"'] & \mathcal{F}'(U)
\end{tikzcd}$$

Note that mathematicians like to consider more abstract generalisations of sheaves on arbitrary categories. We won't trouble ourselves with such things here since it's not relevant to this limited discussion. There will be a lack of rigour in this chapter since there a lot of the objects defined need the full machinery of algebraic geometry to understand. Again we won't trouble ourselves with this, even though the subject is extremely relevant, for obvious reasons. A physicist looking for an entry point is directed to [Bru03]. A classic example of a sheaf is the set of smooth functions over a manifold or the set of sections over a vector bundle. If we consider the above as sheaves of abelian groups, then we are forgetting their module structure and only talking about their additive properties. Here we assigned abelian groups to each open set, but this can be swapped for another object like a ring. A sheaf of modules has a sheaf of rings which act on a sheaf of abelian groups, analogous to how a module is an abelian group with a ring action. For example the sheaf with coefficients in the abelian group of sections of a smooth vector bundle can be acted on by the sheaf with coefficients in the ring of smooth functions. We say sheaves arising as modules of holomorphic sections of a complex vector bundle have the property of being **locally free**. This won't be defined fully here, since the full discussion won't be relevant for this introduction. So here we ignore the difference between a locally free sheaf and a sheaf arising from holomorphic sections. On top of this we may not distinguish between the vector bundle and the associated sheaf of sections.

Definition 4.1.5. *Given a map $\iota : X \longrightarrow Y$ and a sheaf $\mathcal{F}$ on X, we can define the* ***pushforward sheaf*** *on Y by :*

$$(\iota_\star \mathcal{F})(U) = \mathcal{F}(\iota^{-1}(U)) \quad \forall U \in \tau_Y$$

Consider the case where $X = x$ is a point, so the map is just the inclusion $\iota : x \hookrightarrow Y$. Define a sheaf on x which is just assigns to it the complex numbers $\mathcal{F}(x) = \mathbb{C}$, or equivalently the sheaf associated to a complex line bundle on the point. Then the pushforward sheaf is characterised in this case by :

$$(\iota_\star \mathcal{F})(U) = \begin{cases} \mathbb{C} & x \in U \\ \{0\} & x \notin U \end{cases}$$

This follows from the fact that preimage of $U \in \tau_Y$ is empty unless $x \in U$. So then we deduce that $(\iota_\star \mathcal{F})(U) = \mathbb{C}$ only if $x \in U$. Furthermore, this sheaf is supported (non-zero) only at the point $x \in U$ if Y is Hausdorff. This is due to the gluing property, explicitly if $x \in U \in \tau_Y : x \neq y \implies \exists y \in V \in \tau_Y : U \cap V = \varnothing$. But then $y \notin U \implies \mathcal{F}(V) = \{0\}$. This is an example of a **skyscraper sheaf**, since it looks something like a Dirac delta distribution at a point.

In general with a submanifold $\iota : S \hookrightarrow Y$ and a vector bundle on E on S, the pushforward sheaf $\iota_\star E$ only has support on S. This is one way to formalise the idea of a vector bundle living only on a submanifold, and is sometimes called a torsion sheaf.

4.2 Sheaf Cohomology

Just like in singular cohomology where one defines complexes and exact sequences of abelian groups/modules, one can do the same for sheaves. There are a lot of subtleties when determining whether local constructions in sheaf theory can be extended to global ones. These are important, but we won't cover them here since it will delay getting to the physics. Following the approach of [Sha03] we won't be completely rigorous.

Definition 4.2.1. *Given a space X and a sheaf morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ we can define the following associated presheaves :*

1. *The kernel presheaf :*
$$U \longmapsto \ker[\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)] \quad \forall U \in \tau_X$$
2. *The image presheaf :*
$$U \longmapsto \mathrm{Im}[\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)] \quad \forall U \in \tau_X$$

The kernel presheaf is actually a sheaf but the image presheaf is not. There is a construction that turns presheaves into sheaves, aptly named sheafification. We won't go into this here, but will implicitly assume its use when necessary in the upcoming constructions.

Definition 4.2.2. *A sequence of sheaves, $\mathcal{F}^i$, and sheaf morphisms, ϕ_i, is **exact** at $\mathcal{F}^k$ if $\mathrm{Im}\,\phi_{k-1} = \ker\phi_k$. The sequence is an **exact sequence** if it is exact $\forall k$.*

We can define sheaf cohomology on a topological space by considering a chain complex of sheaves equipped with an appropriate coboundary operator. Recall that a good open cover U_i is an open cover such that all intersections are contractible, and that this always exists on a manifold. Our chains will be defined as sections on open sets, overlaps, overlaps of overlaps and so on. The differential effectively measures how much these local sections fail to patch together.

Definition 4.2.3. *The **Sheaf Cohomology** for space X with an open cover $\{U_i\}$ and sheaf $\mathcal{F}$ of abelian groups is defined as follows :*

1. *Cochains are elements of*
$$C^k(\mathcal{F}) \coloneqq \coprod_{i_a \neq i_b} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_k}) \coloneqq \coprod_{i_a \neq i_b} \mathcal{F}(U_{i_o \ldots i_k})$$
2. *The differential $\delta : C^k(\mathcal{F}) \to C^{k+1}(\mathcal{F})$ is defined as :*
$$(\delta s)_{i_0 \ldots i_k} = \sum_{j=0}^{k+1} (-1)^j \mathrm{r}^U_{U_{i_0 \ldots i_k}} s_{i_0 \ldots \hat{i_j} \ldots i_{k+1}}$$
3. *The sheaf cohomology is the cohomology of this complex :*
$$H^k(X; \mathcal{F}) \coloneqq \ker\delta_k / \mathrm{Im}\,\delta_{k-1}$$

Note again that there we're glossing over technicalities. Strictly speaking what we defined above was Čech cohomology for a sheaf in a given open cover. One has to take a 'direct limit' to actually get the sheaf cohomology.

Proposition 4.2.1. *Degree zero sheaf cohomology defines global sections.*
$$H^0(M; \mathcal{F}) \coloneqq \Gamma(X, \mathcal{F})$$

Proof. This follows immediately from the definition of coboundary operator. Consider a degree zero cochain $\sigma_\alpha \in \mathcal{F}(U_\alpha)$. To be closed under δ means,
$$\delta\sigma = \mathrm{r}^\alpha_{\alpha\beta}\sigma - \mathrm{r}^\beta_{\alpha\beta}\sigma = 0,$$
but the gluing axiom for sections over a sheaf implies that σ_α can be stitched into a section over all of X. This is precisely the definition of what we call **global sections** over X. □

The classification of principal $U(1)$-bundles can be shown quickly using sheaves.

Proposition 4.2.2. *Principal U(1)-bundles over a manifold, X, are classified by $H^1(X;\mathfrak{H}^\times)$, where $\mathfrak{H}^\times$ is the sheaf of nowhere vanishing holomorphic functions.*

Proof. First we'll show that $H^1(X;\mathfrak{H}^\times)$ classifies principal $U(1)$-bundles. Note that the sheaf $\mathfrak{H}^\times$ is equivalent to the sheaf of $U(1)$ valued functions on X. A degree one cochain is just a collection of $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathbb{C}$ defined on overlaps. We say that this a cocycle if it is δ closed on triple overlaps, note we are using multiplicative notation,

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}(x) = 1 \quad \forall x \in U_{\alpha\beta\gamma}.$$

This is exactly the cocycle condition for transition functions defining a principal $U(1)$-bundle, Furthermore we have that two cochains $\{g_{\alpha\beta}\}$, $\{h_{\alpha\beta}\}$ are equivalent in cohomology if they differ by an exact cochain. In degree one we have,

$$h_{\alpha\beta}g_{\alpha\beta}^{-1} = \phi_\alpha\phi_\beta^{-1} \implies h_{\alpha\beta} = \phi_\alpha g_{\alpha\beta}\phi_\beta^{-1},$$

for some 0-cochain ϕ_α. But this actually the same condition as for ϕ to define an isomorphism between two bundles in terms of their local trivialisations. $\square$

We've glossed over some technicalities with direct limits again, but a proof including those can be found in [Par24]. In homological algebra we often make use of long exact sequences induced by short exact sequences, indeed there are similar results for sheaf cohomology.

Proposition 4.2.3. *A map of sheaves $\mathcal{F} \longrightarrow \mathcal{G}$ over a space X induces a map on sheaf cohomology $H^1(X;\mathcal{G}) \longrightarrow H^1(X;\mathcal{F})$. Furthermore, a short exact sequence of sheaves,*

$$0 \longrightarrow \mathcal{F}_1 \hookrightarrow \mathcal{F}_2 \twoheadrightarrow \mathcal{F}_3 \longrightarrow 0,$$

induces a long exact sequence in sheaf cohomology :

$$\bullet \longrightarrow H^k(X;\mathcal{F}_1) \longrightarrow H^k(X;\mathcal{F}_2) \longrightarrow H^k(X;\mathcal{F}_3) \longrightarrow H^{k+1}(X;\mathcal{F}_1) \longrightarrow \bullet$$

Proposition 4.2.4. *There is a bijection between principal $U(1)$-bundles and elements of the second integral cohomology $H^1(X;\mathbb{Z})$*

$$H^1(X;\mathfrak{H}^\times) \cong H^2(X;\mathbb{Z}) \tag{4.1}$$

Proof. Start by considering the following short exact sequence :

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow C^\infty_{\mathbb{R}} \xrightarrow{e^{2\pi i(-)}} \mathfrak{H}^\times \longrightarrow 0$$

Here $\underline{\mathbb{Z}}$ is the constant sheaf of integers, $C^\infty_{\mathbb{R}}$ is the sheaf of real valued smooth functions and $\mathfrak{H}^\times$ is the sheaf of nowhere vanishing holomorphic functions. The exponential is defined in the usual way on smooth functions and one can check that this does in fact induce a well-defined surjective sheaf map. Then we have the following induced long exact sequence :

$$\bullet \longrightarrow H^k(X;\mathbb{Z}) \longrightarrow H^k(X;C^\infty_{\mathbb{R}}) \longrightarrow H^k(X;\mathfrak{H}^\times) \longrightarrow H^{k+1}(X;\mathbb{Z}) \longrightarrow \bullet$$

Note that $H^k(X; C^\infty_{\mathbb{R}}) = 0 \ \forall k > 0$. This is due to the existence of partitions of unity over manifolds, as a consequence of being Hausdorff and second countable. This leaves short exact sequences :

$$0 \longrightarrow H^k(X; \mathfrak{H}^\times) \xrightarrow{\sim} H^{k+1}(X; \mathbb{Z}) \longrightarrow 0$$

In particular, we have the desired isomorphism for $k = 1$ by exactness. $\square$

This relates back to chapter 1 in the sense of classifying higher versions of $U(1)$-bundles. For example, we also have,

$$H^2(X; \mathfrak{H}^\times) \cong H^3(X; \mathbb{Z}),$$

where now this corresponds to the classification of 2-form B-fields on gerbes in string theory. Recall in chapter 1 we neglected to define the topological class of a differential character when discussing the Dancing Hexagon. Here it has the interpretation as the image of the connecting Bockstein map in the long exact sequence. This gives an idea of as to where the map in 1.33 comes from, up to a minus sign.

In ordinary cohomology we have extension groups $\mathrm{Ext}^1_{\mathbb{Z}}(A; B)$ arising from the failure of cohomology to be the literal hom-dual of homology. In sheaf cohomology we actually have two different kinds of Ext groups, local and global. The process of computing global Ext requires first finding local Ext and then using a spectral sequence. As always, details will be omitted to give a succinct definition.

Definition 4.2.4. *A **coherent sheaf** is a sheaf $\mathcal{F}$ such that there exists a **free resolution**. In particular there should exist locally free sheaves $\mathcal{G}_i$ such that,*

$$0 \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{G}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

is an exact sequence of sheaves for some $n \in \mathbb{N}$.

Holomorphic vector bundles, skyscraper sheaves, and pushforwards of vector bundles are all examples of coherent sheaves. The intuition is that coherent sheaves can be acted upon by an appropriate ring of algebraic functions over the space. For example the set of sections of a holomorphic vector bundle can be acted upon by pointwise multiplication of a holomorphic function. Non-examples include $\mathfrak{H}^\times$ and $\underline{\mathbb{Z}}$, the nowhere zero holomorphic functions and the constant sheaf of integers. In the first case multiplication by the constant zero function, or any other somewhere zero holomorphic function, clearly destroys the sheaf. Similarly, any non-constant or noninteger function breaks the integer sheaf. Now we suppose that D-branes are always modelled by coherent sheaves. Consider two such coherent sheaves $\mathcal{S}$ and $\mathcal{T}$ over a space X. We'd like to compute the so-called **local Ext**, denoted $\underline{\mathcal{E}x\tau}_X(\mathcal{S}; \mathcal{T})$. Start with the free resolution for $\mathcal{S}$.

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{S} \longrightarrow 0$$

Then apply $\underline{\mathrm{Hom}}(-, \mathcal{T})$ to the sequence (apart from the right-most term):

$$0 \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{T}) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{T}) \longrightarrow \cdots \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F}_n, \mathcal{T}) \longrightarrow 0$$

Now we define the local extension groups, $\underline{Ext}^q_X(\mathcal{S};\mathcal{T})$, to be the degree q cohomology of the above complex. There also seems to be arbitrary choice of free resolution involved; while this is not unique it is also true that the local Ext doesn't depend on this choice. Given the local Ext groups we can then compute the **global Ext** groups, denoted $\mathrm{Ext}^n_X(\mathcal{S};\mathcal{T})$, using the **local-to-global** spectral sequence:

$$E_2^{p,q} : H^p(X; \underline{Ext}^q_X(\mathcal{S};\mathcal{T})) \implies \mathrm{Ext}^{p+q}_X(\mathcal{S};\mathcal{T}) \tag{4.2}$$

The spectral sequence can be thought of as a collection double complexes equipped with differentials $\mathrm{d}_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$. Then the cohomology of this complex with respect to the differential defines the next complex in the sequence $E_{r+1}^{p,q}$. The process is iterated until it terminates, giving the desired group. This whole fiasco seems rather long, but the examples discussed here will lead to a lot of simplifications. In practice one can avoid explicitly defining the differentials and pray that the sequence is simple, as we will see in the coming examples.

4.3 B-Model

An attempt will be made to review some basic properties of the B-model in string theory. This by no means a full discussion, see [Von05]. In string theories we tend to consider compactifications of the form, $\mathbb{R}^{1,3} \times X$, where X is some small compact space. We can simplify the model this way and only consider physics on X without the complications of time. The dynamics of a string can be formulated as a 'non-linear sigma model', which means we have maps $\phi : \Sigma \to X$ from a two-dimensional surface into the target space X. It can be thought of as embedding a two-dimensional string worldsheet into the space, analogous to a one dimensional worldline of a point particle. The target space will be taken to be a complex three-dimensional Kähler manifold. This basically means it's a real manifold of dimension $2n = 6$ with a complex structure compatible with the metric. Recall that a complex structure is, roughly, something that decomposes the tangent space into holomorphic and anti-holomorphic parts. We're allowed to effectively 'work on' the worldsheet of the string by setting the so-called string coupling, g_s, to zero. Even further simplifications arise by allowing worldsheet supersymmetry, or additional fermionic degrees of freedom on the string. It turns out that to get the most simplification without trivial dynamics we need $N = (2,2)$-supersymmetry. One can show that, in the B-model, only constant maps of the worldsheet contribute. This seems like too much of a simplification, but the dynamics are not entirely trivial due to the fermionic degrees of freedom and 'topological twisting'.

When D-branes are treated as boundary conditions, the endpoints of the string are confined to the associated submanifold. Classically, the endpoint of an open string intersects a brane at a point. Furthermore, when quantising, this point behaves like a charged particle in the sense that it couples to a $U(1)$ gauge field living on the brane. These are called **Chan Paton factors**. In general one can have multiple branes overlapping on the same manifold, in this case the gauge field turns out to be $U(N)$ for N intersecting branes. In the B-model it can also be deduced that the submanifold associated to the brane has to be holomorphically embedded. In particular, only even dimensional submanifolds are allowed [Asp04]. The compact space is often required to have six dimensions to give the correct ten dimensional spacetime predicted by string theory, in such cases we are limited to 0, 2, 4, and 6 dimensional D-branes. Now recall that only constant worldsheet maps are considered. So, we don't really have a picture of a string wobbling about while attached to branes, instead it's just a point with fermionic degrees of freedom. It's intuitively clear that such a string cannot 'connect' spatially separated submanifolds. This implies that the spectrum of open string states between two branes ought to be non-zero only if the underlying manifolds have a non-trivial 'intersection'. This is one example of a result that can be proved if we assume the following proposition is true.

Proposition 4.3.1. *Given two branes $\mathcal{F}$ and $\mathcal{G}$ on X we conjecture that the spectrum of open string states joining them is computed by:*

$$\mathcal{H}\{\mathcal{F} \rightleftharpoons \mathcal{G}\} = \bigoplus_n \mathrm{Ext}^n_X(\mathcal{F};\mathcal{G}) \tag{4.3}$$

Consider two $D=6$-branes $\mathcal{F}$ and $\mathcal{G}$ which wrap the whole space X. In this case the associated sheaves are exactly holomorphic vector bundles over all of X. Here the interpretation is that these are the gauge bundles associated to the brane. The goal is to compute $\mathrm{Ext}^n_X(\mathcal{F};\mathcal{G})$. So start with a free resolution of $\mathcal{F}$, which is trivial in this case since $\mathcal{F}$ is a bundle over the whole of X :

$$0 \longrightarrow \mathcal{F} \xrightarrow{\quad\sim\quad} \mathcal{F} \longrightarrow 0$$

Then applying $\underline{\mathrm{Hom}}(-,\mathcal{G})$ we have :

$$0 \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) \longrightarrow 0$$

Taking the cohomology gives the local Ext :

$$\underline{\mathcal{E}xt}^n_X(\mathcal{F};\mathcal{G}) = \begin{cases} \mathcal{F}^\vee \otimes \mathcal{G} & n=0 \\ 0 & n>0 \end{cases}$$

Note that $\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) \cong \mathcal{F}^\vee \otimes \mathcal{G}$, where $\mathcal{F}^\vee$ is the dual bundle. Then we can compute global Ext using the spectral sequence,

$$E_2^{p,q=0} : H^p(X; \underline{\mathcal{E}xt}^0_X(\mathcal{F};\mathcal{G})) \implies \mathrm{Ext}^p_X(\mathcal{F};\mathcal{G}),$$

where we've used that the sequence terminates for $q>0$ since the local Ext is concentrated in degree zero. Furthermore, the differentials $\mathrm{d}_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are trivial for $r>0$. Finally, this leaves us with the spectrum of open string states :

$$\mathcal{H}\{\mathcal{F} \rightleftharpoons \mathcal{G}\} = \bigoplus_n H^n(X;\mathcal{F}^\vee \otimes \mathcal{G}) \tag{4.4}$$

This does indeed agree with the alternative computation done independently by Witten in [Wit03]. Now let's consider another case where the base manifold is $X = \mathbb{C}^3$ with two $D=0$-branes stacked at a point $\iota : p \hookrightarrow \mathbb{C}^3$. Recall that these can be modelled by skyscraper sheaves $\mathfrak{H}_p := \iota_\star \mathfrak{H}$ where $\mathfrak{H}$ is sheaf associated to the trivial holomorphic line bundle. In special cases, there is a Koszul resolution which one can use to determine the locally-free resolution of a sheaf, analogous to the discussion earlier in section 3.4.

$$0 \longrightarrow \textstyle\bigwedge^3 \mathfrak{H} \longrightarrow \bigwedge^2 \mathfrak{H} \longrightarrow \bigwedge^1 \mathfrak{H} \longrightarrow \bigwedge^0 \mathfrak{H} \longrightarrow \mathfrak{H}_p \longrightarrow 0$$

Coordinate expressions for the differentials can be derived after using the identifications, $\wedge^2\mathfrak{H} \cong \mathfrak{H}^3 \cong \wedge^1\mathfrak{H}$, and $\wedge^3\mathfrak{H} \cong \mathfrak{H} \cong \wedge^0\mathfrak{H}$. Here we write $\mathfrak{H}^3 := \mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H}$.

$$0 \longrightarrow \mathfrak{H} \xrightarrow{\begin{bmatrix} x \\ -y \\ z \end{bmatrix}} \mathfrak{H}^3 \xrightarrow{\begin{bmatrix} 0 & -z & -y \\ -z & 0 & x \\ y & x & 0 \end{bmatrix}} \mathfrak{H}^3 \xrightarrow{\begin{bmatrix} x & y & z \end{bmatrix}} \mathfrak{H} \longrightarrow \mathfrak{H}_p \longrightarrow 0 \tag{4.5}$$

Now take Homs:

$$0 \longrightarrow \underline{\mathrm{Hom}}(\mathfrak{H}, \mathfrak{H}_p) \longrightarrow \underline{\mathrm{Hom}}(\mathfrak{H}^3, \mathfrak{H}_p) \longrightarrow \underline{\mathrm{Hom}}(\mathfrak{H}^3, \mathfrak{H}_p) \longrightarrow \underline{\mathrm{Hom}}(\mathfrak{H}, \mathfrak{H}_p) \longrightarrow 0$$

The maps above are defined as the Hom duals of the functions in 4.5, which is just pre-composition $f^\vee := (-) \circ f$. Note that $\underline{\mathrm{Hom}}(\mathfrak{H}, \mathfrak{H}_p) = \mathfrak{H}_p$ and $\underline{\mathrm{Hom}}(\mathfrak{H}^3, \mathfrak{H}_p) = \mathfrak{H}_p^3$. We can simplify the process by letting $p = 0$, so the branes are sitting at the origin, without loss of generality. In that case the maps in 4.5 are the zero, since the sheaf is supported only at the origin $x = y = z = 0$. Then the dual maps are identically zero as well, meaning the cohomology of the complex is trivial. In particular, we have:

$$\underline{Ext}^n_{\mathbb{C}^3}(\mathfrak{H}_p; \mathfrak{H}_p) = \begin{cases} \mathfrak{H}_p & n = 0 \\ \mathfrak{H}_p^3 & n = 1 \\ \mathfrak{H}_p^3 & n = 2 \\ \mathfrak{H}_p & n = 3 \end{cases} \tag{4.6}$$

Now the last step is to use the local to global spectral sequence. For skyscraper sheaves we use the fact that the cohomology is concentrated in degree zero.

$$H^n(X; \mathfrak{H}_p) = \begin{cases} \mathbb{C} & n = 0 \\ 0 & n > 0 \end{cases}$$

We use the additivity property of $H^0(-; \mathfrak{H}_p)$ to also work out that:

$$H^n(\mathbb{C}^3; \mathfrak{H}_p) = \begin{cases} \mathbb{C}^3 & n = 0 \\ 0 & n > 0 \end{cases}$$

Then once again the spectral sequence becomes trivial,

$$E_2^{i=0,j} : H^0(\mathbb{C}^3; \underline{Ext}^j_{\mathbb{C}^3}(\mathfrak{H}_p; \mathfrak{H}_p)) \implies \mathrm{Ext}^{0+j}_{\mathbb{C}^3}(\mathfrak{H}_p; \mathfrak{H}_p), \tag{4.7}$$

where $E_2^{i,j} = 0$ for $i > 0$ by the fact the cohomology is concentrated in degree zero. Since the differentials, $\mathrm{d}_{r=2} : E_2^{i,j} \to E_2^{i+2,j-2+1}$, map between different i they are trivial for $r > 0$. This means we can read off that,

$$H^0(\mathbb{C}^3; \underline{Ext}^j_{\mathbb{C}^3}(\mathfrak{H}_p; \mathfrak{H}_p)) \cong \mathrm{Ext}^j_{\mathbb{C}^3}(\mathfrak{H}_p; \mathfrak{H}_p).$$

Finally, the resulting space of states can be read off:

$$\mathrm{Ext}^n_{\mathbb{C}^3}(\mathfrak{H}_p;\mathfrak{H}_p)=\begin{cases}\mathbb{C} & n=0\\ \mathbb{C}^3 & n=1\\ \mathbb{C}^3 & n=2\\ \mathbb{C} & n=3\end{cases}\implies \mathcal{H}\{\mathfrak{H}_p \rightleftharpoons \mathfrak{H}_p\}\cong \mathbb{C}\oplus\mathbb{C}^3\oplus\mathbb{C}^3\oplus\mathbb{C} \tag{4.8}$$

Great, but what does this mean? We've been treating D-branes as submanifolds with gauge fields on them. Remember that this interpretation is only true in the approximate case when the size of the space is large compared to the size of the brane. We'll assume we're in such a regime and proceed. In the classical picture strings have endpoints fixed to a rigid D-brane submanifold, suppose a string begins and ends on the same brane. Then when quantising the dynamics of the string there appears a photon-like excitation corresponding to a $U(1)$-gauge field. Furthermore, if the brane has p-dimensions than this field is described a generalised p-form Maxwell theory similar to chapter 1. These are the Chan-Paton factors mentioned earlier.

Furthermore, suppose the brane of dimension p is embedded in an n-dimensional space. Then in the quantum theory one obtains $n-p$ massless scalar fields, the same number of directions transverse to the brane. These fields describe small movements of the brane itself in directions orthogonal to it; meaning the quantum theory of strings on the brane determines the geometry of the brane itself. If these scalar fields are dynamical, then the geometry of the brane itself must be dynamical. In particular the expectation values of these scalar fields describe the fuzzy position of the brane in space [Pol07]. This is similar to the story of the spin-2 graviton emerging from the quantisation of closed string states.

In this case we have open strings connected to a point-like $D0$-brane which loops back to itself. Equivalently this is two $D0$-branes stacked on each other at the origin with strings connecting them. Since the brane 0-dimensional, its normal bundle is the whole space; it can wiggle in any direction in $\mathbb{C}^3$. So $\mathrm{Ext}^1_{\mathbb{C}^3}(\mathfrak{H}_p;\mathfrak{H}_p)$ describes the $6-0=6$ scalar fields associated to these directions in space. There is a sense in which the existence of a brane breaks translational symmetry in space. This symmetry breaking causes 'Goldstone bosons' or Higgs fields corresponding to the transverse directions. On the other hand $\mathrm{Ext}^0_{\mathbb{C}^3}(\mathfrak{H}_p;\mathfrak{H}_p)=\mathbb{C}$ just says that there is one gauge field on the brane. This follows since the fields on a point are just a choice of vector in the associated fibre, which in this case is just $\mathbb{C}$. Note that the above interpretation of Ext groups is true only if the boundary conditions on either end of the open string states are the same, say for example a string looping back to the same brane.

Mathematically there is relationship between different Ext groups called **Serre duality**:

$$\dim \mathrm{Ext}^p_X(\mathcal{F};\mathcal{G})=\dim \mathrm{Ext}^{n-p}_X(\mathcal{G};\mathcal{F}\otimes K_X) \tag{4.9}$$

Here K_X is the canonical line bundle, or the top exterior power of the holomorphic cotangent bundle. In our situation the physical interpretation is that $\mathrm{Ext}^2_{\mathbb{C}^3}(\mathfrak{H}_p;\mathfrak{H}_p)$ and $\mathrm{Ext}^3_{\mathbb{C}^3}(\mathfrak{H}_p;\mathfrak{H}_p)$ are dual to the degree 0 and 2 groups from before, where take $\mathcal{F}=\mathcal{G}=\mathfrak{H}_p$ in the above expression. Note that strings here considered to be oriented, which is just a choice of direction along the length. The dual groups can be thought of as describing antiparticles for open strings with the opposite orientation at a point on the brane. This is far from the whole picture, however. In general when considering open string spectra between **different** branes there are complications. It turns out that, due to the **Freed-Witten anomaly**, one has to change the picture we had for the pushforward sheaf defining the brane. Consider a brane, $\iota_\star\mathcal{F}$, wrapping a submanifold $\iota: S\hookrightarrow X$. One imposes that $\iota_\star\mathcal{F}$ is the pushforward of the sheaf of sections of $\mathcal{F}\otimes\sqrt{K_S^\vee}$, as opposed to the usual vector bundle $\mathcal{F}$. Here $\sqrt{K_S^\vee}$ is the 'square root' of the dual canonical line bundle on S. It is a fact that such a square root object is a bundle if and only if the normal bundle to S admits a 'spin structure'. At the start of these notes, in chapter 1, it was mentioned that K-theory is needed to understand the full picture of branes. The above argument is one example which has a nice interpretation using such a framework. The details are quite technical, and won't be described here for the moment. In our simple examples, it turns out that the extra factor cancels out anyway. In particular, we have restricted our attention to branes over the same submanifold; when considering branes on distinct submanifolds this modification becomes extremely important.

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